Supplementary materials for
New description of the scaling evolution of
the cosmological magneto-hydrodynamic system

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Appendix. Dissipation coefficients

Here we list the values of dissipation coefficients adopted in the letter.

Electric conductivity. The electric conductivity in the early universe before electrons become massive are approximated as [1]

$$\sigma_Y = \frac{6^4\zeta(3)^2\pi^{-3}}{\frac{\pi^2}{8}+\frac{20}{3}+\frac{2}{3}}\frac{a(T)T}{g'^2\ln g'^{-1}}, \qquad T \gg T_{\rm EW}, \quad (1)$$

$$\sigma = \frac{12^4\zeta(3)^2\pi^{-3}N_{\rm leptons}}{3\pi^2+32N_{\rm species}}\frac{a(T)T}{e^2\ln e^{-1}}, \quad m_e \ll T_{\rm EW}, \quad (2)$$

where $e = 0.30$ and $g' = 0.36$ are the coupling constant of the (hyper-)electromagnetic fields. $m_e = 0.5\,\mathrm{MeV}$ is the electron mass, and $T_{\rm EW} \sim 0.1\,\mathrm{TeV}$ is the electroweak scale. $N_{\rm leptons}, N_{\rm species}$ count the number of relevant charge careers, which are summarized in Table 1. At lower temperatures, nonrelativistic electron-ion scattering is modeled by the Spitzer theory [2]

$$\sigma = \frac{4\pi e^2 a(T)n_{e,\rm p}(T)}{\nu_{ei,\rm p}(T)m_e} = \frac{4\pi a(T)T^{\frac{3}{2}}}{e^2 m_e^{\frac{1}{2}}\ln\Lambda_{ei}}, \qquad T \ll m_e, \quad (3)$$

where $n_{e,\rm p}$ is the physical electron number density, ν_{ei} is the electron-ion collision frequency, and $\ln\Lambda_{ei} \simeq 20$ is the Coulomb logarithm.

Shear viscosity. The shear viscosity in the early universe above $T > m_e$ are estimated in Ref. [1], neglecting contribution from neutrinos. When the mean free path of neutrinos is shorter than the characteristic scales of the system, they contribute to the shear viscosity as well. In that case, the dimensionful Fermi constant $G_F = 1.17\times10^{-5}\,\mathrm{GeV}^{-2}$ comes into the expression, introducing non-trivial T dependence [3, 4]. Neutrinos eventually decouple from the fluid before electrons become massive.

$$\eta = \frac{\frac{45}{2\pi^2}\left(\frac{5}{2}\right)^3\zeta(5)^2\left(\frac{12}{\pi}\right)^5\cdot\frac{3}{2}}{9\pi^2+224\left(5+\frac{1}{2}\right)}\frac{1}{g'^4\ln g'^{-1}}\frac{1}{g_{\rm fluid}a(T)T}, \qquad T \gg T_{\rm EW}, \quad (4)$$

$$\eta = \frac{\frac{45}{2\pi^2}\left(\frac{5}{2}\right)^3\zeta(5)^2\left(\frac{12}{\pi}\right)^5 N_{\rm leptons}}{9\pi^2+224N_{\rm species}}\frac{1}{e^4\ln e^{-1}}\frac{1}{g_{\rm fluid}a(T)T}, \qquad m_e \ll T \ll T_{\rm EW}, \quad (5)$$

$$\eta = \frac{1}{5}\frac{g_\nu}{g_{\rm fluid}}l_{\nu\rm mfp}, \qquad \text{when neutrinos contribute to shear viscosity}, \quad (6)$$

where the neutrino mean free path is [4]

$$l_{\nu\rm mfp} = \frac{1}{a(T)G_{\rm F}^2T^5}. \quad (7)$$

When electrons become massive, nonrelativistic ion-ion scattering [2] and scattering between neutral hydrogens [5] are relevant before and after recombination, respectively. Similarly to neutrinos, photons contribute to the shear viscosity when their mean free path is shorter than the characteristic scales of the system.

$$\eta = \frac{T}{a(T)\nu_{ii,\rm p}(T)m_p} = \frac{T^{\frac{5}{2}}}{a(T)e^4 n_{i,\rm p}(T)m_p^{\frac{1}{2}}\ln\Lambda_{ii}}, \qquad T_{\rm rec} \ll T \ll m_e, \quad (8)$$

$$\eta = \frac{T^{\frac{1}{2}}}{\pi a_{\rm B}^2 a(T)n_{H,\rm p}(T)m_p^{\frac{1}{2}}}, \qquad T \ll T_{\rm rec}, \quad (9)$$

$$\eta = \frac{1}{5}\frac{g_\gamma}{g_{\rm fluid}}l_{\gamma\rm mfp}, \qquad \text{when photons contribute to shear viscosity}, \quad (10)$$

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where the mean free path of photons is

$$l_{\gamma\mathrm{mfp}} = \frac{a(T)^2}{\sigma_{\mathrm{T}} n_e(T)}. \tag{12}$$

In the above expressions, $m_p = 1\,\mathrm{GeV}$ is the proton mass, ν_{ii} is the ion-ion collision frequency, $\ln\Lambda_{ii} \simeq 20$ is the Coulomb logarithm, $a_{\mathrm{B}} := 4\pi/(e^2 m_e)$ is the Bohr radius, and $\sigma_{\mathrm{T}} := (8\pi/3)(e^2/4\pi m_e)^2$ is the Thomson cross section.

Drag force. When the mean free paths of particles, *i.e.,* photons and neutrinos, are larger than the coherence length, the particles act as a homogeneous background on which the fluid is dissipated via drag (friction) term, $-\alpha\boldsymbol{v}$. The drag coefficients due to neutrinos and photons are given by [5]

$$\alpha_\nu = \frac{g_\nu}{g_{\mathrm{fluid}} - 5.25} l_{\nu\mathrm{mfp}}^{-1}, \tag{13}$$

$$\alpha_\gamma = \frac{4}{3}\frac{\rho_\gamma}{\rho_{\mathrm{b}}} l_{\gamma\mathrm{mfp}}^{-1}. \tag{14}$$

Also, when the hydrogen atoms decouple from the plasma because of their neutralness, they account for the ambipolar drag [5, 6],

$$\alpha_{\mathrm{amb}} = 2\pi \left(\frac{e^2 a_{\mathrm{B}}^3}{m_p}\right)^{\frac{1}{2}} a(T) n_{H,p}. \tag{15}$$

In the matter dominated era, the Hubble friction should also be taken into account,

$$\alpha_{\mathrm{Hubble}} := aH. \tag{16}$$

References

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Table of degrees of freedom

T	T_0- 0.1 meV$-$	$T_{\rm rec}, T_{\gamma \rm dec}-$ 1 eV$-$	$m_e, T_{\nu \rm dec}-$ 1 MeV$-$	$m_\mu, m_u, m_d,$ $m_s, T_{\rm QCD}-$ 0.1 GeV$-$	m_τ, m_c, m_b $-$ 1 GeV$-$	$m_t, m_W,$ $T_{\rm EW}-$ 0.1 TeV$-$
Relativistic contents of the plasma	$-$	γ	$\cdots,$ e, ν	$\cdots,$ μ, u, d, s, g	$\cdots,$ τ, c, b	$\cdots,$ t, H, W, Z
g	3.36	3.36				
g_s	3.91	3.91	10.75	61.75	86.25	106.75
$g_{\rm fluid}$	0	2				
$N_{\rm leptons}$	0	0	1	2	3	(3/2)
$N_{\rm species}$	0	0	1	4	20/3	$(5+1/2)$

Table 1: The values of $N_{\rm leptons}, N_{\rm species}$ from Ref. [1] as well as the values of $g, g_s, g_{\rm fluid}$, as functions of the radiation temperature T. Both of $N_{\rm leptons}, N_{\rm species}$ count the number of fermions but are weighted by their lepton charges and electromagnetic charges. The symmetric phase belongs to a different regime but substitution of the values in the table into Eq. (2) with replacing the coupling coefficient $e \to g'$ reproduces Eq. (1) [1].