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Real-space renormalization of bond-disordered conductance lattices

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We propose a new real-space renormalization approach for the conductivity of bond-disordered conductance lattices, and investigate two-dimensional square and three-dimensional simple cubic lattices with a binary distribution of conductances, $\rho(\sigma) = p\delta(\sigma - \sigma_1) + (1-p)\delta(\sigma - \sigma_2)$. It is shown that our transformations not only give a good description of the percolation conductivity near the critical point, but lead to an approximation for the lattice conductivity $\bar{\sigma}(p)$ which is superior to the effective-medium approximation for all values of σ_2/σ_1 and p. In particular, the slopes of $\bar{\sigma}(p)$ at $p = 0$ and $p = 1$ are reproduced exactly, and in two dimensions the transformations satisfy the selfdual symmetry of the square lattice. For percolation conduction problems ($\sigma_2/\sigma_1 = 0$) we determine the conductivity exponents t and s, and compare our results with alternative estimates. We also present a simple approximate solution to the renormalization relations which is very accurate for all values of p and produces reasonable rough estimates of t and s.

I. INTRODUCTION

Over the past few years random conductance lattices have received a great deal of attention.[1-22] Particular interest has been focused on the behavior of the conductivity in so-called percolation conduction problems.[1-13] These investigations relate to systems with a fraction of conductances whose value is either zero (nonconducting bonds) or infinite (superconducting bonds). The presence of such bonds defines a percolation problem on the underlying lattice, and the conductivity either vanishes or diverges at the percolation threshold. As only a few exact results are available,[4,13-16] random conductance lattices have mainly been investigated by Monte Carlo techniques[1-6,17] on the one hand, and by a number of analytic approximations[1,2,18-20] on the other hand. It has been demonstrated that outside critical regions the conductivity is well approximated by effective-medium type theories.[1,2,18-20] Recently, however, an increasing number of investigations have concentrated on the behavior of the conductivity near critical points.[1-13,21] The similarity with thermodynamic phase transitions has been noticed, and scaling assumptions for the conductivity have been proposed.[8,9,21] The corresponding critical exponents have been estimated from Monte Carlo simulations performed on large finite lattices,[1-7] and from a number of real-space renormalization approaches.[8-12] Monte Carlo calculations face the problem that statistical fluctuations inhibit an accurate evaluation of the results close to the critical point.[4] The numerical estimates therefore rely on the assumption that the behavior of the conductivity can be described by a constant critical exponent over quite an extended critical region. We shall come back to this point later in the paper. Real-space renormalization procedures, on the other hand, suffer from the serious problem that the approximations involved are very difficult to quantify. Different approaches lead to appreciably different values for the critical exponents, and it does not seem possible to estimate the influence of the various approximations on the accuracy of the results.

In this paper we consider bond-disordered two-dimensional square and three-dimensional simple-cubic lattices with a binary distribution of conductances σ_{ij},

$$\rho(\sigma) = p\delta(\sigma - \sigma_1) + (1-p)\delta(\sigma - \sigma_2). \tag{1}$$

We propose a new real-space renormalization approach which has several advantages over other treatments. In particular, it is not only applicable near a critical point, but for all σ_1 and σ_2 leads to a surprisingly accurate description of the lattice conductivity $\bar{\sigma}(p)$ over the whole range of p values. In the percolation problem our procedure leads to a bond-probability renormalization, which is identical with that proposed recently by Reynolds, Klein, and Stanley,[23] and the two-dimensional version of our approach can be regarded as an exact description of the "iterated Wheatstone bridge," a pseudolattice introduced in a previous paper.[16]

In Sec. II we describe the renormalization procedure and discuss some of its properties. In particular, we show that the resulting approximation for $\bar{\sigma}(p)$ becomes exact in the low-concentration limits $p \to 0$ and $p \to 1$ for all values of σ_1 and σ_2. We furthermore develop a simple approximate solution to the renormalization relations. In Sec. III we present our results for the conductivity of binary bond-disordered lattices and compare them with Monte Carlo simulations and with effective-medium approximations. Particular emphasis is put on percolation conduction problems ($\sigma_2/\sigma_1 = 0$), where we estimate the critical exponents and make comparison with alternative approaches. In Sec.

IV, finally, we investigate our results near critical points in more detail and present some arguments that numerical approaches might underestimate the true exponents in certain cases.

II. REAL-SPACE RENORMALIZATION FOR BOND-DISORDERED CONDUCTANCE LATTICES

A. Transformations

We consider infinite two-dimensional square and three-dimensional simple cubic lattices, the bonds of which represent conductances σ_{ij} that are independently distributed according to a probability density $\rho(\sigma)$ of the form of Eq. (1). A real-space renormalization of the lattice can be described by a transformation W which transforms a lattice of spacing l with a probability density $\rho_n(\sigma)$ into a lattice of spacing bl with probability density $\rho_{n+1}(\sigma) = W\{\rho_n(\sigma)\}$. To define the transformations used in this paper, we consider a partition of the lattice into equivalent cells (clusters) that maintain the symmetry of the lattice. Each cell is then renormalized to a basic unit of the transformed lattice, consisting of one bond in each lattice direction. Most of our calculations are based on the simple cell of Fig. 1(a) and its three-dimensional analog, which both lead to transformations with a rescale factor of $b = 2$. Only in two dimensions we have also investigated the corresponding $b = 3$ transformation.

For the renormalization of $\rho_n(\sigma)$ we propose the following transformation:

$$\rho_{n+1}(\sigma) = W\{\rho_n(\sigma)\} = \int d\sigma_1 \cdots d\sigma_N \, \rho_n(\sigma_1) \cdots \rho_n(\sigma_N) \times \delta(\sigma - g(\sigma_1, \ldots, \sigma_N)), \quad (2)$$

where $g(\sigma_1, \ldots, \sigma_N)$ is the normalized equivalent conductance of a cell, measured between two parallel equipotential surfaces which are perpendicular to a lattice direction. $\sigma_1, \ldots, \sigma_N$ denote those conductances of the cell which are relevant for the determination of g. For the two-dimensional $b = 2$ cell of Fig. 1(a), $g(\sigma_1, \ldots, \sigma_5)$ is thus the conductance of a Wheatstone bridge [Fig. 1(b)], and for the three-dimensional $b = 2$ cell $g(\sigma_1, \ldots, \sigma_{12})$ is the conductance between two diagonally opposite vertices of an octahedron (normalized with a factor of $\frac{1}{2}$).

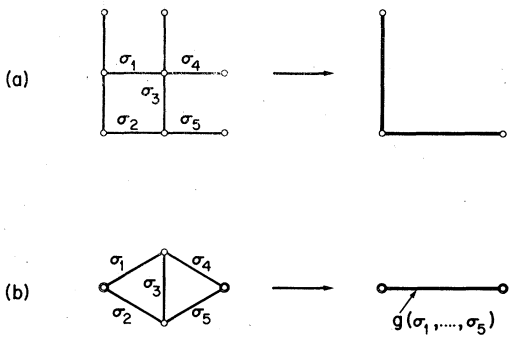

FIG. 1. (a) $b = 2$ renormalization of the square lattice (cf. Fig. 3a of Ref. 23). (b) Schematic representation of the conductance $g(\sigma_1, \cdots, \sigma_5)$ which defines the renormalization of the probability density $\rho(\sigma)$ [see Eq. (2)].

Repeated application of Eq. (2) to an original probability density $\rho_0(\sigma)$ of the form of Eq. (1) will eventually lead to

$$\rho_\infty(\sigma) = \delta(\sigma - \hat{\sigma}), \quad (3)$$

where $\hat{\sigma}(p; \sigma_1, \sigma_2)$ is our renormalization approximation for the lattice conductivity $\bar{\sigma}(p; \sigma_1, \sigma_2)$. In the subsequent sections of this paper we shall evaluate $\hat{\sigma}(p; \sigma_1, \sigma_2)$ by different methods and discuss some of its most important properties.

For the special case of the percolation conduction problem ($\sigma_1 = 1$, $\sigma_2 = 0$) it is often convenient to write[8]

$$\rho_n(\sigma) = (1 - p_n)\delta(\sigma) + p_n \tilde{\rho}_n(\sigma). \quad (4)$$

The renormalization transformation of Eq. (2) is then split up into the two transformations

$$p_{n+1} = R(p_n) \quad (5)$$

and

$$\tilde{\rho}_{n+1}(\sigma) = \tilde{W}\{\tilde{\rho}_n(\sigma)\}, \quad (6)$$

where the kernel of the integral Eq. (6) depends on p_n. The fixed point properties of Eqs. (5) and (6) determine the percolation critical point p_c, the correlation length exponent ν and the conductivity exponent t (see Sec. III B), where we note that p_c and ν are obtained from Eq. (5) alone. A similar treatment of the case $\sigma_1 = \infty$, $\sigma_2 = 1$ yields the second conductivity exponent s.

We observe that our renormalization transformation of the bond probability, Eq. (5), is identical to that proposed recently by Reynolds, Klein and Stanley.[23] Straley[9] also uses the same cells in his renormalization of the percolation conduction problem, but his transformation for $\rho(\sigma)$ is different from ours.

B. Some properties

In Sec. III we shall see that our renormalization group approach leads to an excellent description of the lattice conductivity, outside critical regions as well as near a critical point. Here we want to present some rigorous results. We first prove that for our $b = 2$ transformations in both two and three dimensions $\hat{\sigma}(p; \sigma_1, \sigma_2)$ becomes exact in the

limits $p \to 0$ and $p \to 1$.

We define

$$\langle\sigma\rangle_n \equiv \int d\sigma\, \sigma \rho_n(\sigma)\,, \tag{7}$$

and first consider the limit $p \to 0$. From an analysis of the properties of $g(\sigma_1, \ldots, \sigma_N)$ it is then straightforward to show that

$$\langle\sigma\rangle_n = \sigma_2 + d\sigma_2 \frac{2^{dn}(\sigma_1 - \sigma_2)p}{(2^{dn}-1)\sigma_1 + [(d-1)2^{dn}+1]\sigma_2} + O(p^2)\,, \tag{8}$$

where d is the dimensionality of the lattice. As $\lim_{n\to\infty}\langle\sigma\rangle_n = \hat{\sigma}$ [see Eq. (3)], we immediately obtain

$$\left.\frac{d\hat{\sigma}}{dp}\right|_{p=0} = d\sigma_2 \frac{\sigma_1 - \sigma_2}{\sigma_1 + (d-1)\sigma_2}\,. \tag{9}$$

The limit $p \to 1$ is simply obtained by exchanging σ_1 with σ_2 and p with $(1-p)$ in Eq. (8). The slopes of Eq. (9), and their counterparts at $p = 1$, coincide with those obtained from the effective-medium approximation[1,2] which has been shown[2] to become exact in the low concentration limits.

In two dimensions our transformation furthermore preserves the self-dual symmetry[4,14,16] of the square lattice. This follows from the fact that our $g(\sigma_1, \ldots, \sigma_5)$ [see Eq. (2)] is the equivalent conductance of a Wheatstone bridge which is self-dual. As a consequence our results satisfy all exact relations.[4,14,16] that follow from the selfdual symmetry, as, e.g.,

$$\hat{\sigma}(p; \sigma_1, \sigma_2)\hat{\sigma}(1-p; \sigma_1, \sigma_2) = \sigma_1\sigma_2\,. \tag{10}$$

Equation (10) implies that our renormalization approach reproduces the exact percolation critical point, $p_c = \frac{1}{2}$, and that the two conductivity exponents t and s are equal.

Finally, we note that the corresponding two-dimensional transformations with $b = 3, 4, \ldots$ also preserve self-duality, but do not lead to the exact slopes at $p = 0$ and $p = 1$. The deviations, however, are very small for all values of σ_1 and σ_2.

C. Evaluation of $\hat{\sigma}(p; \sigma_1, \sigma_2)$

If we start with a $\rho_0(\sigma)$ which is the sum of two δ functions [see Eq. (1)] and repeatedly apply our transformation of Eq. (2), each $\rho_n(\sigma)$ also is a sum of δ functions. The number of these δ functions, however, increases very rapidly with increasing n, and it seems impossible to determine $\lim_{n\to\infty}\rho_n(\sigma) = \delta(\sigma - \hat{\sigma})$, i.e., $\hat{\sigma}(p; \sigma_1, \sigma_2)$, exactly. We therefore have to use approximate methods to investigate the properties of $\hat{\sigma}(p; \sigma_1, \sigma_2)$.

For example, we may use the fact that $\hat{\sigma} = \lim_{n\to\infty}\langle\sigma\rangle_n$, where $\langle\sigma\rangle_n$ is defined in Eq. (7) and can be evaluated by Monte Carlo techniques. We have performed such calculations for n up to $n = 10$ in two dimensions and up to $n = 6$ in three dimensions. These values correspond to systems with $5^{10} \approx 9.8\times10^6$ and $12^6 \approx 3.0\times10^6$ bonds, respectively, so that the fluctuations in the Monte Carlo data are already very small and the resulting approximations for $\hat{\sigma}$ should be very accurate. The results of these calculations are presented in Sec. III.

A simple approximation, which replaces $\tilde{\rho}_n(\sigma)$ in Eq. (4) by a δ function after each renormalization step, has been applied to several renormalization transformations[8,10-12] in order to obtain rough estimates for the conductivity exponents. In the following, we propose a modification of this approach which turns out to lead to an excellent approximation for $\hat{\sigma}(p; \sigma_1, \sigma_2)$ for all values of σ_1, σ_2, and p. For simplicity we consider the percolation conduction case $\sigma_1 = 1$, $\sigma_2 = 0$. Our transformation of Eq. (2) transforms a $\rho_n(\sigma)$ of the form

$$\rho_n(\sigma) = (1 - p_n)\delta(\sigma) + p_n\delta(\sigma - \sigma_n) \tag{11}$$

into

$$\rho_{n+1}(\sigma) = (1 - p_{n+1})\delta(\sigma) + \sum_{i=1}^{M} q_i(p_n)\delta(\sigma - g_i(\sigma_n))\,, \tag{12}$$

where p_{n+1} is related to p_n by Eq. (5), and the conductances $q_i(\sigma_n)$ denote the M possible nonzero values of the function $g(\sigma_1, \ldots, \sigma_N)$ when $\sigma_1, \ldots, \sigma_N$ are distributed according to $\rho_n(\sigma)$. The coefficient $q_i(p_n)$ is the probability with which the value $g_i(\sigma_n)$ is assumed. The sum over the M δ functions is then replaced by a single δ function,

$$\rho_{n+1}(\sigma) \approx (1 - p_{n+1})\delta(\sigma) + p_{n+1}\delta(\sigma - \sigma_{n+1})\,, \tag{13}$$

where σ_{n+1} is defined as the *geometric mean* of the conductances $g_i(\sigma_n)$, so that our renormalization transformation is described by

$$\sigma_{n+1} = \exp\ \frac{1}{p_{n+1}} \sum_{i=1}^{M} q_i(p_n) \ln g_i(\sigma_n) \tag{14}$$

and by Eq. (5). A similar approximation can easily be constructed for the case $\sigma_1 = \infty$, $\sigma_2 = 1$; and even for arbitrary σ_1 and σ_2 we can successfully use the same ideas, although here the recursion relation for σ_n becomes slightly more complicated.

In both two and three dimensions the convergence of σ_n to $\sigma_\infty \equiv \hat{\sigma}_{\text{approx}}$ is extremely fast (less than 10 iterations are needed if p is not very close to p_c). Over the whole range of p values, $\hat{\sigma}_{\text{approx}}$ turns out to be an excellent approximation of $\hat{\sigma}$ and therefore of the true lattice conductivity (see Sec. III). It can be argued that the approximation introduced by Eqs. (13) and (14) is somewhat arbitrary and uncontrolled. On the other hand, one can consider Eqs. (5) and (14), or their analogs for general σ_1 and σ_2, as the *definitions* of a renormalization

transformation which, despite its simplicity, leads to reasonable results both for the critical exponents and for the overall behavior of the lattice conductivity. Similar simplified transformations have been considered in Refs. 11 and 12.

Our approximative treatment of the renormalization transformations differs from the usual approach[8,10-12] by the use of the *geometric* instead of the *arithmetic* mean [see Eq. (14)]. One can give general arguments why the *geometric* mean, rather than some other average, should be used (see Sec. VI of Ref. 16), and it can be shown, e.g., that the use of the arithmetic mean in our treatment would not lead to a meaningful approximation of the lattice conductivity $\bar{\sigma}(p;\sigma_1,\sigma_2)$. In all approaches (Refs. 8 and 10–12, and present work) the geometric mean furthermore leads to much better estimates for the critical exponents t and s than the arithmetic mean.

III. RESULTS

A. $\hat{\sigma}(p;\sigma_1,\sigma_2)$ as an approximation of the lattice conductivity

We have applied our $b=2$ renormalization procedures to several two-dimensional square and three-dimensional simple cubic lattices with a binary distribution of conductances [see Eq. (1)]. The resulting approximation of the lattice conductivity, $\hat{\sigma}(p;\sigma_1,\sigma_2)$, has been evaluated by Monte Carlo techniques and by a simple approximate treatment of the renormalization transformations (see Sec. IIC). Our results for $\hat{\sigma}$ can then be compared with those obtained from other approximations for the lattice conductivity, as e.g., the effective-medium theory[1,2] and large-scale numerical simulations.[1,2,24]

Before we present some explicit results, we can summarize our investigations as follows:

(a) For all values of σ_1, σ_2, and p the result of the simple approximate treatment $\hat{\sigma}_{\text{approx}}$ leads to an extremely accurate approximation of $\hat{\sigma}(p;\sigma_1,\sigma_2)$.

(b) $\hat{\sigma}(p;\sigma_1,\sigma_2)$ gives an excellent description of the true lattice conductivity, near critical points as well as outside critical regions. It represents an approximation which is superior to the effective medium theory in all cases where the two can be distinguished.

If σ_1 and σ_2 are comparable, $\hat{\sigma}(p;\sigma_1,\sigma_2)$ and its approximation $\hat{\sigma}_{\text{approx}}$ practically coincide with the effective medium result in both two and three dimensions, which is known[1,2] to be very accurate in these cases. Even if σ_1 and σ_2 differ by one order of magnitude, $\hat{\sigma}$ and $\hat{\sigma}_{\text{approx}}$ agree with the effective medium approximation to within 1% or 2%, and existing numerical simulations[1,2] are hardly accurate enough to favor one of the approximations.

We therefore concentrate on percolation conduction problems, where $\sigma_2/\sigma_1=0$. The p dependence of $\hat{\sigma}(p;1,0)$ is displayed in Fig. 2 for the two-dimensional square lattice and in Fig. 3 for the three-dimensional simple cubic lattice. Comparison is made with the effective medium approximation and with numerical simulations by Kirkpatrick[1,2] and Straley.[24] The percolation critical point p_c is given by the unstable fixed point of Eq. (5), with

$$R(p)=p^2(2+2p-5p^2+2p^3) \tag{15}$$

in two dimensions, and

$$R(p)=p^2(4+8p-14p^2-40p^3+16p^4+288p^5-655p^6+672p^7-376p^8+112p^9-14p^{10}) \tag{16}$$

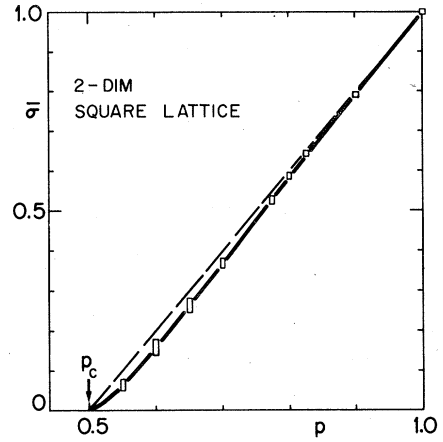

FIG. 2. Bond-percolation conductivity ($\sigma_1=1$, $\sigma_2=0$) for the two-dimensional square lattice. The heavy line represents the approximation $\hat{\sigma}(p;1,0)$ obtained from our $b=2$ renormalization, and is compared with the effective medium result (dashed line) and with Monte Carlo simulations of Kirkpatrick (Ref. 2) and Straley (Ref. 24) (vertical bars).

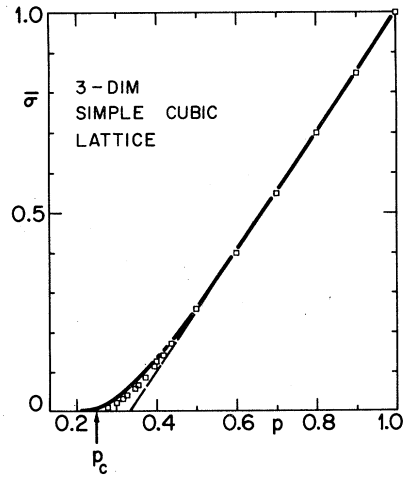

FIG. 3. Same as Fig. 2, but for the three dimensional simple cubic lattice.

in three dimensions. We obtain $p_c = \frac{1}{2}$ in two dimensions and $p_c \approx 0.2085$ in three dimensions.[25] The two-dimensional value is exact (self-duality!) and the three-dimensional value is reasonably close to the exact one of about 0.247.[26] The agreement between $\hat{\sigma}$ and the Monte Carlo data is excellent over the whole region of p values, i.e., $p_c \leq p \leq 1$, and the resulting description of the lattice conductivity is by far superior to that obtained from the effective-medium theory. In particular, the critical behavior, which will be discussed in more detail in the following sections, is represented very well. The small systematic deviations in the critical region of the three-dimensional system are due to the slightly too-low value we obtain for p_c.

For the case $(\sigma_1 = \infty, \sigma_2 = 1)$, $\hat{\sigma}(p; \infty, 1)$ gives a similarly accurate description of the lattice conductivity $\bar{\sigma}$ in the region $0 \leq p \leq p_c$. As $\bar{\sigma}$ diverges at p_c, our too-low p_c value in three dimensions makes the absolute accuracy of $\hat{\sigma}(p; \infty, 1)$ somewhat less spectacular than in the case $\sigma_1 = 1$, $\sigma_2 = 0$. The general behavior, however, is very well represented, especially if compared with the effective medium approximation.

B. Critical exponents

In this section we shall estimate the critical exponents ν, t and s from an analysis of our renormalization transformation near $p = p_c$. The correlation length exponent ν and the two conductivity exponents t and s are defined as follows.[4,8,21,23] As $|p - p_c| \to 0$, the correlation length ξ is assumed to diverge as $\xi \sim |p - p_c|^{-\nu}$ and the lattice conductivity $\bar{\sigma}(p; \sigma_1, \sigma_2)$ is assumed to vary as $\bar{\sigma} \sim (p - p_c)^t$ for the case $\sigma_1 = 1, \sigma_2 = 0$, and as $\bar{\sigma} \sim (p_c - p)^{-s}$ for the case $\sigma_1 = \infty, \sigma_2 = 1$.

The correlation length exponent ν is determined by the transformation of the bond probability, Eq. (5), alone,[8,9,23,27]

$$\nu = \ln b / \ln \lambda_p , \tag{17}$$

where

$$\lambda_p = \left. \frac{dR}{dp} \right|_{p=p_c} , \tag{18}$$

and b is the rescale factor of the transformation. In the case $\sigma_1 = 1, \sigma_2 = 0$ we observe that at $p = p_c$ Eq. (6) has a solution of the form[8]

$$\lambda_t \tilde{\rho}_c(\lambda_t \sigma) = \tilde{W}\{\tilde{\rho}\ (\sigma)\} , \tag{19}$$

and t can be determined from ν and from the eigenvalue λ_t of Eq. (19),

$$t = \nu \ln \lambda_t / \ln b = \ln \lambda_t / \ln \lambda_p . \tag{20}$$

Similarly, s is given by

$$s = -\nu \ln \lambda_s / \ln b = -\ln \lambda_s / \ln \lambda_p , \tag{21}$$

TABLE I. Critical exponents in two dimensions.

	$b = 2$	$b = 3$	Numerical
ν	1.428	1.380	1.33 ± 0.05[a]
t	1.32 ± 0.02	1.33 ± 0.02	1.1 ± 0.1[b] 1.15 ± 0.2[c]
s	1.32 ± 0.02	1.33 ± 0.02	1.15 ± 0.25[c] 1.0 ± 0.1[d]
u	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$[e]

[a] References 5, 28, and 29.
[b] References 3-6.
[c] Reference 7.
[d] Reference 8.
[e] Exact.

where the eigenvalue λ_s is defined analogous to λ_t, but for the case $\sigma_1 = \infty, \sigma_2 = 1$.

For the determination of λ_t we use a method similar to that described by Stinchcombe and Watson.[8] We start with $\tilde{\rho}_0(\sigma) = \delta(\sigma - \sigma_0)$ and use Eq. (6) to generate $\tilde{\rho}_n(\sigma), n = 1, 2, \ldots$. The nth approximation $\lambda_t^{(n)}$ to λ_t is then defined by

$$\int d\sigma\, \lambda_t^{(n)} \tilde{\rho}_n(\lambda_t^{(n)} \sigma) f(\sigma) = \int d\sigma\, \tilde{\rho}_{n+1}(\sigma) f(\sigma) . \tag{22}$$

Stinchcombe and Watson[8] use $f(\sigma) = \sigma$ and estimate λ_t by extrapolation of the sequence $\{\lambda_t^{(n)}\}$. We found it useful to generate several such sequences by using different $f(\sigma)$, e.g., $f(\sigma) = \sigma$, $f(\sigma) = \ln\sigma$ and $f(\sigma) = \sigma^{-1}$. This allows a reasonably accurate extrapolation from only the first two or three members of the sequences. The determination of λ_s is carried out by exactly the same method, but instead of Eq. (6), we have to use the appropriate analogous equation for the case $\sigma_1 = \infty$, $\sigma_2 = 1$. Our estimates for the critical exponents are listed in Tables I and II, and compared with the best numerical estimates from Monte Carlo simulations. In two dimensions we have also analyzed the transformation with rescale factor $b = 3$. The critical expo-

TABLE II. Critical exponents in three dimensions.

	$b = 2$	Numerical
ν	1.031	0.85 ± 0.05[a], 0.95 ± 0.05[b]
t	2.14 ± 0.02	1.70 ± 0.05[c], 1.6 ± 0.1[d]
s	0.76 ± 0.01	0.70 ± 0.05[c], 0.9 ± 0.1[e]
u	0.74 ± 0.01	0.72 ± 0.05[c], 0.67 ± 0.08[e]

[a] References 5 and 28.
[b] Reference 29.
[c] Reference 4.
[d] References 3 and 5.
[e] Reference 6.

nent u which describes the behavior of $\bar{\sigma}$ at $p = p_c$ and near $\sigma_2 = 0$,[4,21]

$$\bar{\sigma}(p_c; \sigma_1, \sigma_2) \sim \sigma_1^{1-u}\sigma_2^{u}, \quad \sigma_2 \to 0, \tag{23}$$

has not been determined independently, but is calculated from t and s according to Straley's relation[4,21]

$$u = t/(t+s). \tag{24}$$

We finally note that within the approximate treatment defined by Eqs. (5) and (14), λ_t is simply given by $\lambda_t^{(0)}$ as calculated from Eq. (22) with $f(\sigma) = \ln\sigma$. The resulting rough estimates for the conductivity exponents, $t = s = 1.26$ ($b = 2$) and $t = s = 1.27$ ($b = 3$) in two dimensions, and $t = 2.10$, $s = 0.71$, and $u = 0.75$ in three dimensions, are already quite close to the values listed in Tables I and II.

In two dimensions our results satisfy the exact relation $t = s$ which implies $u = \frac{1}{2}$. The ν values in both two and three dimensions coincide[25] with those obtained by Reynolds *et al.*[23] and are somewhat higher than the best numerical estimates. A comparison of the conductivity exponents shows that our three-dimensional values for s and u agree well with the numerical estimates, while our values for $t = s$ in two dimensions and for t in three dimensions seem rather high. In the following section, however, we present some arguments which indicate that the corresponding numerical estimates might be too low.

A comparison with exponent values obtained from alternative real-space renormalization approaches is rather confusing. Estimates quoted include 1.13,[8,10] 1.33,[8,10] 1.32,[9] and 1.28 (Ref. 9) for $t = s$ in two dimensions; 2.36,[10] 2.34,[10] and 2.04 (Ref. 9) for t in three dimensions; and 0.46,[10] 0.64,[10] and 0.81 (Ref. 9) for s in three dimensions. The same methods lead to ν values of 1.63,[10] 1.31,[27] 1.32,[9] and 1.22 (Ref. 9) in two dimensions and of 1.22,[8] and 0.94 (Ref. 9) in three dimensions.

IV. CONCLUSIONS AND DISCUSSION

Stinchcombe and Watson[8] have attempted to adjust their critical exponents by estimating the effects of certain approximations made in their renormalization treatments. Nevertheless, it seems impossible to obtain accurate information about the reliability of the results from different real-space renormalization approaches. On the other hand, we believe that the renormalization procedure presented and and analyzed in Sec. II and III has some definite advantages over alternative approaches. For all values of σ_1 and σ_2 it leads to an excellent approximation of the lattice conductivity over the whole range of p values, in critical regions as well as outside. The slopes at $p = 0$ and $p = 1$ are even reproduced exactly, and in two dimensions the transformations preserve the self-dual symmetry of the square lattice. As a consequence, the percolation critical point for the square lattice, $p_c = \frac{1}{2}$, is given exactly, and other rigorous relations, as, e.g., $t = s$ and Eq. (10), are satisfied. In three dimensions the p_c value is still reasonably close to the exact one, so that even in percolation conduction problems the behavior of the conductivity is very well approximated.

With respect to the accuracy of the conductivity exponents t and s we would like to add a few remarks. Numerical estimates of t and s are usually based on fitting Monte Carlo data for $\ln\bar{\sigma}$ vs $\ln|p - p_c|$ in the "critical region" by a straight line. The large fluctuations in the Monte Carlo data close to p_c, however, restrict the useful fitting region to p values with $|p - p_c| \geq 0.05$–0.1.[4] The estimates therefore rely on the assumption that $\bar{\sigma}$ obeys a simple power law over quite an extended critical region. As there exists no theoretical justification of such an assumption, we have investigated the detailed behavior of our approximation $\hat{\sigma}$ as we move away from the critical point. We define "p-dependent exponents" $\hat{t}(p)$ and $\hat{s}(p)$ by

$$\hat{t}(p) = d\ln\hat{\sigma}(p; 1, 0)/d\ln(p - p_c) \tag{25}$$

and

$$\hat{s}(p) = -d\ln\hat{\sigma}(p; \infty, 1)/d\ln(p_c - p) \tag{26}$$

and present the results in Figs. 4 and 5. In two dimensions $\hat{t}$ and $\hat{s}$ are of course symmetric with respect to $p = p_c = \frac{1}{2}$ and decrease quite rapidly with increasing $|p - p_c|$. The decrease is even more pronounced for $\hat{t}$ in three dimensions, while here $\hat{s}$ varies only slowly with increasing $|p - p_c|$. From a fit of $\ln\hat{\sigma}(p; 1, 0)$ vs $\ln(p - p_c)$ by a straight line in a region $0.1 \lesssim p - p_c \lesssim 0.2$ we would therefore expect to obtain t values of approximately 1.2 and

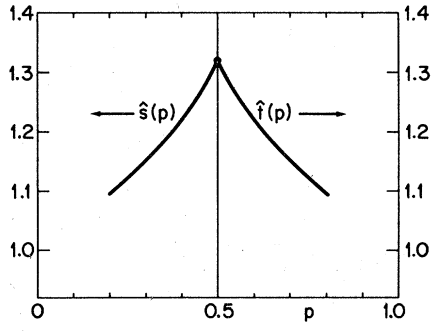

FIG. 4. Plot of the "p-dependent exponents" $\hat{t}(p)$ and $\hat{s}(p)$, defined by Eqs. (25) and (26), for the two-dimensional square lattice.

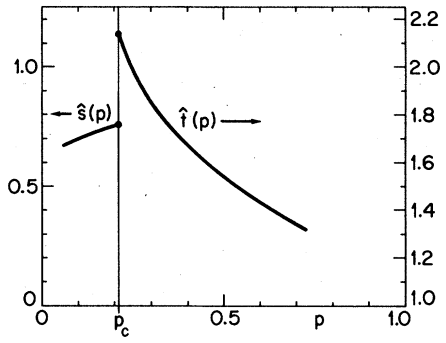

FIG. 5. Same as Fig. 4, but for the three-dimensional simple cubic lattice.

1.9, respectively, in two and three dimensions. These values are appreciably lower than the correct ones of about 1.32 and 2.14 (see Tables I and II). If the true lattice conductivity would show a behavior similar to that of our approximate results, we would therefore expect that the numerical estimates for the exponent t are too small. In this connection we remark that Watson and Leath[22] also obtain a smaller value for t if they enlarge the fitting region in their study of site percolation conduction in two dimensions. From different studies[22,30] the onset of the critical region was estimated to occur at $|p - p_c| \sim 0.1$. Two additional observations seem to support our argument. As we go from the $b = 2$ to the $b = 3$ cell in two dimensions, ν decreases appreciably while t stays practically constant (Table I). The estimates for the s value, finally, are in good agreement with our result in three dimensions (Table II) where $\hat{s}(p)$ varies very slowly in the critical region. These arguments are of course by no means rigorous, as the exact analytic behavior of the conductivity near p_c is not known. We think, however, that this problem deserves further investigation.

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