Lebesgue Integral

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Contents

Sigma Algebras

Let X be a non-empty set. A collection $\mathcal{S}(X)$ of subsets of X is called a σ-algebra on X if

- $S \in \mathcal{S}(X)$ implies $S^{\mathrm{c}} = (X \setminus S) \in \mathcal{S}(X)$, and
- $S_i \in \mathcal{S}(X)$ for all $i \in I$, with I countable, implies $\bigcup_{i \in I} S_i \in \mathcal{S}(X)$.

In words: $\mathcal{S}(X)$ is closed under complements and countable unions.

Sigma Algebras

Basic Properties:

1. We have $X \in \mathcal{S}(X)$,
2. as well as $\varnothing \in \mathcal{S}(X)$, and
3. if I is countable, then $\bigcap_{i \in I} S_i \in \mathcal{S}(X)$.

Proof:

1. $S \in \mathcal{S}(X) \quad \Longrightarrow \quad S^{\mathrm{c}} \in \mathcal{S}(X) \quad \Longrightarrow \quad X = S \cup S^{\mathrm{c}} \in \mathcal{S}(X)$,
2. $X \in \mathcal{S}(X) \quad \Longrightarrow \quad \varnothing = X^{\mathrm{c}} \in \mathcal{S}(X)$, and
3. $\bigcap_{i \in I} S_i = \left(\bigcup_{i \in I} S_i^{\mathrm{c}}\right)^{\mathrm{c}} \in \mathcal{S}(X)$.

Sigma Algebras

Examples:

1. The collection $\{\emptyset, X\}$ is a σ-algebra on X.
2. The power set 2^X is also a σ-algebra on X.
3. For any collection $\mathcal{F} \subseteq 2^X$ one can find a σ-algebra $\overline{\mathcal{F}}$ such that $\mathcal{F} \subseteq \overline{\mathcal{F}} \subseteq 2^X$. The smallest σ-algebra $\overline{\mathcal{F}}$ with this property is called the σ-algebra that is generated by $\mathcal{F}$.
4. The σ-algebra that is generated by $\mathcal{F}$ can alternatively be obtained as the intersection of all σ-algebras that contain $\mathcal{F}$. (Proof: exercise!)

Borel Sigma Algebras

Let (X, d) be a metric space, $\mathcal{F}$ the set of open subsets of X, and $\mathcal{B}(X)$ the σ-algebra that is generated by $\mathcal{F}$.

Definition:

- The collection $\mathcal{B}(X)$ is called the Borel σ-algebra of X.

Properties:

1. $\mathcal{B}(X)$ contains all open subsets of X,
2. $\mathcal{B}(X)$ contains all closed subsets of X, and
3. $\mathcal{B}(X)$ contains countable unions of all of these sets.

Measures

Let X be a non-empty set and $\mathcal{S}(X)$ a σ-algebra on X.

Definition:

A map $\mu : \mathcal{S}(X) \to [0, \infty]$ is called a measure on $\mathcal{S}(X)$ if

1. $\mu(\varnothing) = 0$, and
2. for any countable collection of disjoint sets $S_i \in \mathcal{S}(X)$, $i \in I$, we have

$$\mu\left(\bigcup_{i \in I} S_i\right) = \sum_{i \in I} \mu(S_i)$$

(note: the order of the summation is irrelevant, because $\mu(S_i) \geq 0$)

Measures

Language

- The triple $(X, \mathcal{S}(X), \mu)$ is called a measure space,
- the elements of $\mathcal{S}(X)$ are called measurable sets, and
- a set $S \subseteq \mathbb{R}^n$ is called Borel measurable if $S \in \mathcal{B}(\mathbb{R}^n)$.

Examples

- The function $\mu(S) = 0$ defines a (rather useless) measure
- If $F : \mathbb{R} \to \mathbb{R}$ is a continuous non-increasing function, one can define

$$\forall S \in \mathcal{B}(\mathbb{R}), \quad \mu(S) \; = \; \inf_{a,b} \sum_{j=1}^{\infty} |F(b_j) - F(a_j)| \quad \text{s.t.} \quad \bigcup_{j=1}^{\infty} [a_j, b_j] \supseteq S$$

Measures

Theorem Let $(X, \mathcal{S}(X), \mu)$ be a measure space. Then:

- *Monotonicity:* $S, S' \in \mathcal{S}(X)$ with $S \subseteq S'$ implies $\mu(S) \leq \mu(S')$,
- *Subadditivity:* any countable collection $S_i \in \mathcal{S}(X)$, $i \in I$, satisfies

$$\mu\left(\bigcup_{i \in I} S_i\right) \leq \sum_{i \in I} \mu(S_i)$$

- *Continuity:* if $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots \in \mathcal{S}(X)$, then

$$\mu\left(\bigcup_{i \in \mathbb{N}} S_i\right) = \lim_{i \to \infty} \mu(S_i) \quad \text{and} \quad \mu\left(\bigcap_{i \in \mathbb{N}} S_i^{\mathrm{c}}\right) = \lim_{i \to \infty} \mu(S_i^{\mathrm{c}})$$

Null Sets

Let $(X, \mathcal{S}(X), \mu)$ be a measure space.

Definition

- If $S \in \mathcal{S}(X)$ satisfies $\mu(S) = 0$, then it is called a null set.

Language

- If a property is true for all $x \in X$ except on a null set, we say that this property holds almost everywhere.

Completions

- If $S, S' \in \mathcal{S}(X)$ satisfy $S' \subseteq S$ and $\mu(S) = 0$, then $\mu(S') = 0$.
- The union, $\overline{\mathcal{S}(X)}$, of $\mathcal{S}(X)$ with the set of subsets of all its null sets is called the null set completion of $\mathcal{S}(X)$; μ extends naturally to $\overline{\mathcal{S}(X)}$.

Lebesgue Measures

Let $\mathcal{L}(\mathbb{R}) = \overline{\mathcal{B}(\mathbb{R})}$ be the null-set completion of the Borel set $\mathcal{B}(\mathbb{R})$ with respect to the measure

$$\forall S \in \mathcal{B}(\mathbb{R}), \quad \mu(S) \;=\; \inf_{a,b} \sum_{j=1}^{\infty} |b_j - a_j| \quad \text{s.t.} \quad \bigcup_{j=1}^{\infty} [a_j, b_j] \supseteq S$$

Recall that μ extends to $\mathcal{L}(\mathbb{R})$ by setting $\mu(S) = 0$ for all S that are the subset of a null set, $S \subseteq S'$, $\mu(S') = 0$.

Definition

- The measure μ is called the Lebesgue measure on $\mathcal{L}(\mathbb{R})$.
- The sets $S \in \mathcal{L}(\mathbb{R})$ are called Lebesgue measurable.

Lebesgue Measures

Examples

- The interval $[a, b]$ is Lebesgue measurable, $\mu([a, b]) = |b - a|$.
- For a countable set $S = \{x_1, x_2, x_3, \ldots\}$, $x_i \in \mathbb{R}$, we have $\mu(S) = 0$.
- In particular, $\mu(\mathbb{Q}) = 0$, where $\mathbb{Q}$ denotes the rational numbers.
- We have $\mu(\mathbb{R}) = \infty$.
- We have $\mu(\mathbb{R} \setminus \mathbb{Q}) = \mu(\mathbb{R}) - \mu(\mathbb{Q}) = \infty$.
- There exist sets $S \subseteq \mathbb{R}$ that are not Lebesgue measurable. (Exercise!)

Lebesgue Measures

Multivariate Lebesgue Measure

- The above construction can be extended to $\mathbb{R}^n$.
- We introduce the notation $[a,b] = [a_1,b_1] \times [a_2,b_2] \times [a_n,b_n] \subseteq \mathbb{R}^n$.
- The intervals $[a,b] \in \mathbb{R}^n$ have the volume

$$|b-a| \ = \ \prod_{i=1}^{n} |b_i - a_i| \ .$$

- Next, we can extend our definition of μ by setting

$$\forall S \in \mathcal{B}(\mathbb{R}^n), \quad \mu(S) \ = \ \inf_{a,b} \sum_{j=1}^{\infty} |b^j - a^j| \quad \text{s.t.} \quad \bigcup_{j=1}^{\infty} [a^j, b^j] \supseteq S \ .$$

- The rest of the construction is analogous to the scalar case.

Contents

Simple Functions

The indicator function of a set $S \subseteq \mathbb{R}$ is denoted by

$$I_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise .} \end{cases}$$

Definition

- A function $\phi : \mathbb{R} \to \mathbb{R}$ is called simple, if there exist coefficients $c_1, c_2, \ldots, c_n$ and sets $S_1, S_2, \ldots, S_n \subseteq \mathbb{R}$ such that

$$\phi = \sum_{i=1}^{n} c_i I_{S_i} \ .$$

Measurable Functions

Definition

- A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lebesgue measurable if

$$\forall S \in \mathcal{B}(\mathbb{R}), \qquad f^{-1}(S) \in \mathcal{L}(\mathbb{R}) \ .$$

Examples

- The indicator function I_S is Lebesgue measurable iff $S \in \mathcal{L}(\mathbb{R})$.
- If $S_i \in \mathcal{L}(\mathbb{R})$, $c_i \in \mathbb{R}$, then $\phi = \sum_{i=1}^{n} c_i I_{S_i}$ is Lebesgue measurable.
- All continuous functions are Lebesgue measurable.

Measurable Functions

Properties

- If f, g Lebesgue measurable, then $f+g$, $f*g$, f/g are measurable.
- If f Lebesgue measurable and g **continuous**, then $g \circ f$ is measurable.
- If $f_1, f_2, f_3 \ldots$ is a sequence of Lebesgue measurable functions, then

$$\limsup_{k\to\infty} f_k \qquad \text{and} \qquad \liminf_{k\to\infty} f_k$$

  are Lebesgue measurable.
- If f is measurable, then $f_+(x) = \max\{0, f(x)\}$ is measurable.
- If f is measurable, then $f_-(x) = \min\{0, f(x)\}$ is measurable.

Lebesgue Integral

Let $L_+(\mathbb{R})$ be the set of non-negative Lebesgue measurable functions.

Definition

- For a simple function $\phi = \sum_{i=1}^{n} c_i I_{S_i} \in L_+(\mathbb{R})$, we define

$$\int_{\mathbb{R}} \phi \,\mathrm{d}\mu \;=\; \sum_{i=1}^{n} c_i \mu(S_i)$$

- For measurable function $f \in L_+(\mathbb{R})$, we define

$$\int_{\mathbb{R}} \phi \,\mathrm{d}\mu \;=\; \sup_{\phi \in L^+(\mathbb{R})} \int_{\mathbb{R}} \phi \,\mathrm{d}x \qquad \text{s.t.} \qquad \begin{cases} \phi \le f \\ \phi \text{ is simple} \end{cases}$$

Lebesgue Integral

Definition

- For a general measurable function f, we define

$$\int_{\mathbb{R}} f \,\mathrm{d}\mu \;=\; \int_{\mathbb{R}} f_+ \,\mathrm{d}\mu - \int_{\mathbb{R}} (-f_-) \,\mathrm{d}\mu$$

  whenever $\int_{\mathbb{R}} f_+ \,\mathrm{d}\mu < \infty$ or $\int_{\mathbb{R}} (-f_-) \,\mathrm{d}\mu < \infty$.

- If $\int_{\mathbb{R}} |f| \,\mathrm{d}\mu < \infty$, f is called Lebesgue integrable.

Lebesgue Integral

Let $L^1(\mathbb{R})$ denote the set of Lebesgue integrable functions.

Properties

- If $f, g \in L^1(\mathbb{R})$ and $a, b \in \mathbb{R}$, then $af + bg \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} (af + bg)\,\mathrm{d}\mu \;=\; a \int_{\mathbb{R}} f\,\mathrm{d}\mu + b \int_{\mathbb{R}} g\,\mathrm{d}\mu \;.$$

- if $f, g \in L^1(\mathbb{R})$ and $f \leq g$, then

$$\int_{\mathbb{R}} f\,\mathrm{d}\mu \;\leq\; \int_{\mathbb{R}} g\,\mathrm{d}\mu \;.$$

- if $f, g \in L^1(\mathbb{R})$ and $f(x) = g(x)$ for almost every $x \in \mathbb{R}$, then

$$\int_{\mathbb{R}} f\,\mathrm{d}\mu \;=\; \int_{\mathbb{R}} g\,\mathrm{d}\mu \;.$$

Comments on Notation

- As mentioned, Lebesgue measures can also be defined on $\mathbb{R}^n$.
- The set of the corresponding Lebesgue integrable functions $f : \mathbb{R}^n \to \mathbb{R}$ will be denoted by $L^1(\mathbb{R}^n)$.
- For any exponent $p > 0$, the notation $L^p(\mathbb{R}^n)$ is used to denote the set of function $f : \mathbb{R}^n \to \mathbb{R}$ with $|f|^p \in L^1(\mathbb{R}^n)$.
- For a Lebesgue measurable subset $\Omega \subseteq \mathbb{R}^n$, we say that $f \in L^p(\Omega)$ if $f I_\Omega \in L^p(\mathbb{R}^n)$, where I_Ω denotes the indicator function of Ω. Also:

$$\int_\Omega f \, \mathrm{d}\mu \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f I_\Omega \, \mathrm{d}\mu \ .$$

- If it is clear from the context that "$\mathrm{d}x$" denotes a Lebesgue measure, we simply write

$$\int_\Omega f \, \mathrm{d}x \quad \text{instead of} \quad \int_\Omega f \, \mathrm{d}\mu \ .$$

Beppo Levi's Theorem

Theorem

- Let $f_1, f_2, \ldots \in L^1(\Omega)$ be a monotonically increasing sequence of non-negative functions with

$$\sup_{k \in \mathbb{N}} \int_\Omega f_k(x)\,\mathrm{d}x < \infty\ .$$

Then there exists $f \in L^1(\Omega)$ such that $f_k \to f$ almost everywhere and

$$\lim_{k\to\infty} \int_\Omega f_k(x)\,\mathrm{d}x \;=\; \int_\Omega \lim_{k\to\infty} f_k(x)\,\mathrm{d}x \;=\; \int_\Omega f(x)\,\mathrm{d}x\ .$$

Beppo Levi's Theorem

Proof. (Step 1)

- As the sequence f_k is monotonically increasing, the limit

$$f(x) = \lim_{k \to \infty} f_k(x) = \sup_{k \to \infty} f_k(x)$$

is well defined and a measurable function. Additionally, due to the monotonicity of the Lebesgue integral, we have

$$\gamma \;\stackrel{\mathrm{def}}{=}\; \lim_{k \to \infty} \int_\Omega f_k(x)\,\mathrm{d}x \;<\; \infty\,.$$

Beppo Levi's Theorem

Proof. (Step 2)

- Since we assume $f \geq 0$: for any measurable set $S_i \subseteq \Omega$, we can set $c_i \stackrel{\text{def}}{=} \inf_{x \in S_i} f(x)$ and show that

$$c_i \mu(S_i) \leq \lim_{k \to \infty} \int_{S_i} f_k(x) \, \mathrm{d}x \; = \; \gamma(S_i) \; .$$

Technical details: the case $c_i = 0$ is trivial. So, we may assume $0 < \tilde{c}_i < c_i$. The sets

$$\Sigma_k^{\tilde{c}_i} \; \stackrel{\text{def}}{=} \; \{ \, x \in S_i \mid f_k(x) > \tilde{c}_i \, \}$$

are measurable (since $f_k \in L^1(\Omega)$), monotonically increasing, and

$$\tilde{c}_i \mu(\Sigma_k^{\tilde{c}_i}) \leq \int_{S_i} f_k \, \mathrm{d}x \leq \sup_{k \in \mathbb{N}} \int_{S_i} f_k \, \mathrm{d}x = \gamma(S_i) \; .$$

Since $\cup_{k \in \mathbb{N}} \Sigma_k^{\tilde{c}_i} = S_i$, the claim follows for $\tilde{c}_i \to c_i$.

Beppo Levi's Theorem

Proof. (Step 3)

- If the sets $S_i \subseteq \Omega$ from the previous step are a disjoint partition of Ω,

$$\sum_i c_i \mu(S_i) \ \leq \ \sum_i \lim_{k\to\infty} \int_{S_i} f_k(x)\,\mathrm{d}x \ = \ \lim_{k\to\infty} \int_\Omega f_k(x)\,\mathrm{d}x \ \leq \ \gamma \ .$$

  Since the partition is arbitrary we can take the supremum (see the definition of the Lebesgue integral!) and it follows

$$\gamma \ = \ \lim_{k\to\infty} \int_\Omega f_k \,\mathrm{d}x \ \leq \ \int_\Omega f \,\mathrm{d}x \leq \gamma \ .$$

  This concludes the proof.

Consequences of Beppo Levi's Theorem

- Beppo Levi's theorem can be used to analyze sums of non-negative Lebesgue integrable g_k by setting $f_k = \sum_{i=1}^{k} g_k$.
- Also, if $g_k \in L^1(\Omega)$ is an arbitrary monotone sequence with

$$\sup_{k \in \mathbb{N}} \left| \int_\Omega g_k(x)\,\mathrm{d}x \right| < \infty$$

the limit $g = \lim_{k\to\infty} g_k \in L^1(\Omega)$ is integrable and

$$\lim_{k\to\infty} \int_\Omega g_k(x)\,\mathrm{d}x \;=\; \int_\Omega \lim_{k\to\infty} g_k(x)\,\mathrm{d}x \;=\; \int_\Omega g(x)\,\mathrm{d}x \;.$$

Proof: set $f_k = \pm(g_k - g_1)$ a.e. and apply Beppo-Levi.

Lebesgue Dominated Convergence Theorem

Theorem

- Let $f_1, f_2, \ldots \in L^1(\Omega)$ be a sequence of functions that converges to f almost everywhere. If there exists $g \in L^1(\Omega)$ with a.e. $|f_k| \leq g$ for all $k \in \mathbb{N}$, then $f \in L^1(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega f_k(x)\,\mathrm{d}x \;=\; \int_\Omega \lim_{k\to\infty} f_k(x)\,\mathrm{d}x \;=\; \int_\Omega f(x)\,\mathrm{d}x\ .$$

Proof.

- The functions $h_k(x) \stackrel{\text{def}}{=} \sup_{i\geq k}\{|f_i(x) - f(x)|\}$ (a.e.) are Lebesgue integrable (since $|h_k(x)| \leq 2g(x)$) and are monotonically decreasing to 0. The above variant of Beppo Levi's theorem can be applied to the sequence h_k, which then yields the desired statement.

Contents

Banach Spaces

Definition

- A map $\|\cdot\| : X \to [0,\infty)$ on a real vector space X is a norm, if

$$\begin{aligned} \|x\| &= 0 \quad \Longleftrightarrow \quad x = 0, \\ \|\alpha x\| &= |\alpha|\|x\| \\ \|x+y\| &\leq \|x\| + \|y\| \end{aligned}$$

  for all $x, y \in X$ and all $\alpha \in \mathbb{R}$.

- A normed real vector space X is called (real) Banach space if it is complete. That is, every Cauchy sequence in X has a limit in X.

Banach Spaces

Continuous Function Space

- Let $\Omega \subseteq \mathbb{R}^n$ be a set. We use the notation

$$C^0(\Omega) \stackrel{\text{def}}{=} \{ \, f : \Omega \to \mathbb{R} \;|\; f \text{ continuous on } \Omega \, \} \, .$$

- If Ω is bounded, then $C^0(\mathrm{cl}(\Omega))$ is a Banach space with respect to its associated supremum norm

$$\|f\|_{C^0} \stackrel{\text{def}}{=} \sup_{x \in \mathrm{cl}(\Omega)} |f(x)| \, .$$

- Proof: Exercise!

Banach Spaces

Continuously Differentiable Functions

- Let $\Omega \subseteq \mathbb{R}^n$ be open. For a multi-index $\alpha \in \mathbb{N}^n$, we write $|\alpha| = \sum_{i=1}^n \alpha_i$ and introduce the shorthand

$$D^\alpha f \stackrel{\text{def}}{=} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \, .$$

  Next, set $\overline{\Omega} \stackrel{\text{def}}{=} \mathrm{cl}(\Omega)$ and define

$$C^k(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{ f : \overline{\Omega} \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : \\ D^\alpha f \in C^0(\Omega) \;\text{ and } D^\alpha f \text{ has a} \\ \text{continuous extension to } \overline{\Omega} \end{array} \right\} .$$

- $C^k(\overline{\Omega})$ is a Banach space with norm $\|f\|_{C^k} \stackrel{\text{def}}{=} \sum_{\|\alpha\| \leq k} \|D^\alpha f\|_{C^0}$.

Lebesgue Norms

- The L^p-norm on $L^p(\Omega)$ is given by

$$\|f\|_{L^p} \stackrel{\text{def}}{=} \left(\int_\Omega |f|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

- Notice that $\|f\|_{L^p}$ merely implies $f = 0$ almost everywhere. However, if we write $f \sim g$ if f and g coincide almost everywhere, then $\|\cdot\|_{L^p}$ is positive definite on $L^p(\Omega)/\sim$.
- In practice: we simply write $L^p(\Omega)$ instead of $L^p(\Omega)/\sim$.
- Absolute homogeniety, $\|\alpha f\|_{L^p} = |\alpha|\|f\|_{L^p}$ follows trivially from the above definition.
- The triangle inequality will be established below.

Hölder's Inequality

Theorem

- For $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, we have

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q} \ .$$

Proof.

- As $\log$ is a concave function, $\log(x)'' = -x^{-2} < 0$, we have

$$\log\left(\frac{a}{p} + \frac{b}{q}\right) \ \geq \ \frac{1}{p} \log(a) + \frac{1}{q} \log(b) = \log\left(a^{\frac{1}{p}} b^{\frac{1}{q}}\right)$$

for any $a, b \geq 0$. Thus, also $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.

Hölder's Inequality

Proof (continued)

- We may assume $f, g \geq 0$ and $0 < \|f\|_{L^p}, \|g\|_{L^q} < \infty$. (otherwise the statement is trivial)
- Substitute $a = \frac{f(x)^p}{\|f\|_{L^p}^p}$ and $b = \frac{g(x)^p}{\|g\|_{L^q}^p}$ in the above inequality and integrate on both sides:

$$\int_\Omega \frac{f(x)g(x)}{\|f\|_{L^p}\|g\|_{L^q}}\,\mathrm{d}x \;\leq\; \frac{1}{p} + \frac{1}{q} \;=\; 1\,.$$

It follows that

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}\;.$$

Minkowski's Inequality

Theorem

- For $1 \leq p < \infty$ and $f, g \in L^p(\Omega)$, we have

$$\|f+g\|_{L^p} \ \leq \ \|f\|_{L^p} + \|g\|_{L^q} \ .$$

Proof.

- For $p = 1$, this follows simply from

$$\begin{aligned} \|f+g\|_{L^1} \quad &= \quad \int_\Omega \|f+g\|_1 \, \mathrm{d}x \\ &\leq \quad \int_\Omega (\|f\|_1 + \|g\|_1) \, \mathrm{d}x \ \leq \ \|f\|_{L^1} + \|g\|_{L^1} \ . \end{aligned}$$

Minkowski's Inequality

Proof (continued).

- For $p > 1$, we set $q = \frac{p}{p-1}$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
- If we set $h = |f+g|^{p-1}$ it follows that $h^q = |f+g|^p$ and then

$$\|h\|_{L^q} = \|f+g\|_{L^p}^{\frac{p}{q}} \quad \text{and} \quad |f+g|^p = |f+g|h \leq |fh| + |gh| \,.$$

Thus, Hölder's inequality yields

$$\begin{aligned} \|f+g\|_{L^p}^p &= \int_\Omega |f+g|^p \,\mathrm{d}x \;\leq\; \|fh\|_{L^1} + \|gh\|_{L^1} \\ &\leq (\|f\|_{L^p} + \|g\|_{L^p})\|h\|_{L^q} \\ &\leq (\|f\|_{L^p} + \|g\|_{L^p})\|f+g\|_{L^p}^{\frac{p}{q}} \,. \end{aligned}$$

Due to $p - \frac{p}{q} = 1$, the Minkowski inequality follows.

Fischer-Riesz Theorem

Theorem

- The pair $(L^p(\Omega), \|\cdot\|_{L^p})$ is a Banach space.

Proof.

- $L^p(\Omega)$ is a normed space, but we need to show that it's complete.
- Thus, let $f_1, f_2, \ldots \in L^p(\Omega)$ be a Cauchy sequence and $\epsilon_1, \epsilon_2, \ldots \geq 0$ a sequence that satisfies $\sum_{k=1}^{\infty} \epsilon_k < \infty$. Now, we can find an monotonically increasing index sequence i_k such that

$$\forall i, j \geq i_k, \qquad \|f_i - f_k\|_{L^p} \leq \epsilon_k \ .$$

Set $u_1 = f_{i_1}$ and $u_k = f_{i_k} - f_{i_{k-1}}$. Clearly,

$$\sigma \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \|u_k\|_{L^p} \leq \|f_{i_1}\|_{L^p} + \sum_{k=2}^{\infty} \|f_{i_k} - f_{i_{k-1}}\|_{L^p} < \infty$$

Fischer-Riesz Theorem

Proof (continued).

- Next, set $v_k \stackrel{\text{def}}{=} \sum_{i=1}^{k} |u_k| \in L^p(\Omega)$. Clearly,

$$\|v_k\|_{L^p} \;\leq\; \sum_{i=1}^{k} \|u_k\|_{L^p} \qquad \Longrightarrow \qquad \int_\Omega |v_k|^p \,\mathrm{d}x \;\leq\; \sigma^p \;.$$

  Beppo Levi's theorem implies that $v \stackrel{\text{def}}{=} \lim_{k\to\infty} v_k \in L^p(\Omega)$ and

$$\int_\Omega |v|^p \,\mathrm{d}x \;\leq\; \sigma^p \qquad \Longrightarrow \qquad \|v_k\|_{L^p} \;\leq\; \sigma \;.$$

  Note that this the above implies that the series

$$f(x) = \sum_{k=1}^{\infty} u_k(x)$$

  converges almost everywhere and $|f| \leq v$ implies $f \in L^p(\Omega)$.

Fischer-Riesz Theorem

Proof (continued).

- Since $u_1 + u_2 + \ldots + u_k = f_{i_k}$ (telescope sum), it follows that

  $$|f - f_{i_k}| \leq |f| + |f_{i_k}| \leq 2v \qquad \Longrightarrow \qquad |f - f_{i_k}|^p \leq 2^p v^p \in L^1(\Omega) \; .$$

  Thus, the Lebesgue dominated convergence theorem yields convergence of the subsequence

  $$\lim_{k \to \infty} \int_\Omega |f - f_{i_k}|^p \, \mathrm{d}x \; = \; 0 \; .$$

- The whole sequence also converges to f, since for $i \geq i_k$

  $$\|f_i - f\|_{L^p} \leq \|f_i - f_{i_k}\|_{L^p} + \|f_{i_k} - f\|_{L^p} \leq \epsilon_k + \|f_{i_k} - f\|_{L^p} \to 0 \; .$$

  Thus, $f = \lim_{i \to \infty} f_i$ in $L^p(\Omega)$, which concludes the proof.