Lyapunov Stability

Contents

Consider a linear time-varying system

$$\dot{x} = A(t)x, \quad , x(t_0) = x_0 \in \mathbb{R}^n \tag{1}$$

with the assumption that $A(t)$ is piecewise continuous and nonsingular for all $t \geq t_0$. From linear system theory, we know its solution is given by

$$x(t; t_0, x_0) = \Phi(t, t_0) x_0$$

where $\Phi(t, t_0)$ is called state transition matrix satisfies

$$\frac{\partial}{\partial t} \Phi(t, t_0) = A(t) \Phi(t, t_0), \quad \forall t \geq t_0$$

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From the perspective of Lyapunov definition:

Theorem: The equilibrium of LTV System (1) is

- stable iff $c(t_0) \triangleq \sup_{t \geq t_0} \|\Phi(t, t_0)\| < \infty$ and u.s. iff there exists $c_0 = \sup_{t_0 \geq 0} \|c(t_0)\| < \infty$
- g.a.s. iff it is stable and

$$\lim_{t \to \infty} \|\Phi(t, t_0)\| = 0 \quad \forall t_0 \geq 0$$

- g.u.a.s iff there exist positive constants α and β such that

$$\|\Phi(t, t_0)\| \leq \alpha e^{-\beta(t - t_0)}, \quad \forall t \geq t_0 \geq 0$$

Note: For Linear system g.u.a.s = g.e.s, and all the properties hold globally.

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From the perspective of Lyapunov equation:

Thereom: Assume that the elements of $A(t)$ are *uniformly bounded*, the equilibrium of System (1) is g.u.a.s(g.e.s) iff, for any given symmetric p.d. *continuous and bounded* matrix $Q(t)$, there is a continuously differentiable bounded p.d. symmetric matrix P satisfies the Lyapunov equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

$V(t,x) = x^\top P(t)x$ is the Lyapunov function verifying the condition for g.e.s.

The complete proof can be found in Khali's nonlinear system Section 4.6.

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From the perspective of $A(t)$:Hurwitz of $A(t)$ indicates stable?

Counterexample:

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

The eigenvalues of $A(t)$ for each fixed t,

$$\lambda(A(t)) = -0.25 \pm j0.5\sqrt{1.75}$$

have negative real parts and are also independent of t. Despite this the equilibrium at origin is unstable because

$$\Phi(t, 0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

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From the perspective of $A(t)$:

Theorem: Let element of $A(t)$ of the LTV System (1) be differentiable and bounded functions of time, and there is a positive constant σ such that

$$\mathrm{Re}\{\mathrm{eig}(A(t)\} \leq -\sigma, \quad \forall t \geq 0.$$

Then the equilibrium at origin is said to be g.u.a.s (g.e.s), if

$$||\dot{A}(t)|| \in \mathcal{L}_2.$$

The complete proof can be found in Ioannou's Robust Adaptive Control Section 3.4.

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Example: Recall the LTV system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - e^{-t}x_1 + u$$

with $u = -x_1$

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\mathrm{e}^{-t} - 1 & -1 \end{pmatrix}$$

and

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Consider the non-autonomous system

$$\dot{x} = f(x, t)$$

assume f is $\mathcal{C}^1$ w.r.t x. Then, in the neighborhood of equilibrium, f has a Taylor expansion that can be write as

$$x = A(t)x + g(x, t)$$

where

$$A(t) = \bigtriangledown f|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

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Lyapunov Indirect Method

Theorem: Consider the equilibrium $x_e = 0$ of the nonlinear system $\dot{x} = f(t, x)$ and the equilibrium $z_e = 0$ of the LTV system $\dot{z} = A(t)z$, where $A(t)$ represents the Jacobian matrix of $f(t, x)$ evaluated at origin. Assume $A(t)$ is uniformly bounded and the remaining term $g(t, x)$ satisfies

$$\lim_{|x| \to 0} \sup_{t \geq 0} \frac{|g(t, x)|}{|x|} = 0$$

Then the following statements are true:

- z_e is u.a.s (e.s), then x_e is l.u.a.s (l.e.s)
- z_e is unstable, then x_e is unstable

This is also known as *First Method of Lyapunov.*

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Example: Consider again our pendulum system

$$\dot{x}_1 = x_2$$

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with Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a \cos x_1 & -b \end{bmatrix}$$

For equilibrium at $(\pi, 0)$, we have

$$A' = \left. \frac{\partial f}{\partial x} \right|_{x_1 = \pi, x_2 = 0} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

whose eigenvalue $\lambda_{1,2} = -\frac{1}{2} b \pm \frac{1}{2}\sqrt{b^2 + 4a}$. For all $a > 0, b \geq 0$, there is one eigenvalue in the right half plane, hence, the equilibrium at $(\pi, 0)$ is unstable.

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In the absence of friction, i.e. $b = 0$, we have A is marginally stable, one may wonder, *whether this can indicate the stable (in the sense of Lyapunov) of the original nonlinear system?*

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Lyapunov Indirect Method

Unfortunately, if z_e is stable or uniformly stable, NO conclusion can be drawn about x_e.

Example: Consider the scalar system

$$\dot{x} = ax^3$$

Linearizing the system around origin yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.3ax^2\right|_{x=0} = 0$$

This is one eigenvalue lies on the imaginary axis. However, we know the system's stability property depends on the value of a.

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Converse Theorem

Theorem: Consider the non-autonomous system

$$\dot{x} = f(t, x), \quad x(0) = x_0 \in \mathbb{R}^n$$

with $x_e = 0$ and f is Lipschitz continuous in x and piecewise continuous in t for all $x \in \mathcal{D} := \{x \in \mathbb{R}^n | \|x\| < r\}$ and $t \geq 0$.

If origin is e.s, then there exist a $\mathcal{C}^1$ function $V(t, x)$ and positive constants c_1, c_2, c_3, c_4 such that

$$c_1\|x\|^2 \leq V(t, x) \leq c_2\|x\|^2$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3\|x\|^2$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|, \quad \forall x \in \mathcal{D}, \forall t \geq 0$$

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$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\|, \quad \forall x \in \mathcal{D}, \forall t \geq 0$$

Example: Consider a system

$$\dot{z} = f(z), \quad z(0) = z_0 \in \mathbb{R}^m$$

$$\dot{x} = Ax + Bz, \quad x(0) = x_0 \in \mathbb{R}^n$$

with $z = 0$ is an exponentially stable equilibrium of $z-$subsystem, A is Hurwitz and B is a finite constant matrix. Analyze the stability property of the closed-loop system. Prove the origin of the overall system is also e.s.

Summary

Four types of system:

$$\dot{x} = f(t,x), \quad \dot{x} = f(x), \quad \dot{x} = A(t)x, \quad \dot{x} = Ax$$

Three stability properties:

i) stable (u.s)

ii) a.s (u.a.s, g.u.a.s,)

iii) e.s. (g.e.s)

Two Lyapunov function criteria:

a) p.d(p.s.d.)

b) n.d.(n.s.d)

plus radially unbounded

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Contents

- LTV systems
- Lyapunov Indirect Method
- Converse Theorem
- Other Stability Concepts(Supplementary)

Input-Output stability

Consider an LTI system described by the convolution of two functions $u, h : \mathcal{R}^+ \to \mathcal{R}$ defined as*

$$y(t) = u * h \triangleq \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t u(t-\tau)h(\tau)d\tau$$

We say above system is $\mathcal{L}_p$ stable if $u \in \mathcal{L}_p \Rightarrow y \in \mathcal{L}_p$ and $\|y\|_p \leq c\|u\|_p$ for some constant $c \geq 0$ and any $u \in \mathcal{L}_p$. When $p = \infty, \mathcal{L}_p$ stability, i.e., $\mathcal{L}_\infty$ stability, is also referred to as bounded-input bounded-output (BIBO) stability.

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*Let $H(s)$ be the Laplace transform of the I/O operator $h(\cdot).H(s)$ is called the transfer function and $h(t)$ is the impulse response of the system.

Input-Output stability

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Input-to-State Stability

Consider the system

$$\dot{x} = f(t, x, u)$$

where $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$.

Definition :The system above is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$$

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Suppose the unforced system

$$\dot{x} = f(t, x, 0)$$

has certain stability property at the origin $x = 0$. What can we say about the behavior of the system in the presence of a bounded input $u(t)$?

Lemma : Suppose $f(t, x, u)$ is continuously differentiable and globally Lipschitz in (x, u), uniformly in t. If the unforced system has a globally exponentially stable equilibrium point at the origin $x = 0$, then the forced system is input-to-state stable.

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Exercise:

$$\ddot{x} + 2\dot{x}^3 + 2x = 0$$