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Robust Model Predictive Control

Saša V. Raković
School of Automation, Beijing Institute of Technology, Beijing, China

Abstract

Model predictive control (MPC) is indisputably one of the rare modern control techniques that has significantly affected control engineering practice due to its natural ability to systematically handle constraints and optimize performance. Robust MPC is an improved form of MPC that is intrinsically robust in the face of uncertainty. The main goal of robust MPC is to devise an optimization-based control synthesis method that accounts for the intricate interactions of the uncertainty with the system, constraints, and performance criteria in a theoretically rigorous and computationally tractable way.

Keywords

Model predictive control · Robust model predictive control · Robust optimal control · Robust stability

Introduction

Robust MPC synthesis (Raković 1233) is best seen as a repetitive decision-making process, in which the underlying decision-making process is a robust optimal control (ROC) problem. The ROC problem is formulated in such a way so as to guarantee that all possible predictions of the controlled state and control actions sequences satisfy constraints and that the "worst-case" cost is minimized. The decision variable in the underlying ROC problem is a control policy, which is a sequence of control laws and which enables different control actions at different predicted states. Robust MPC makes use of the solution to the associated ROC problem in a recursive manner in order to implement the feedback control law, which is, in fact, equal to the first control law of an optimal control policy. A theoretically exact robust MPC can be obtained either by employing, in a repetitive fashion, the dynamic programming solution of the ROC problem or by solving online, in a recursive manner, corresponding infinite–dimensional minimaximization. The associated computational complexity has considerably hampered practical applicability of exact robust MPC, and it has made robust MPC an active research field with the prime challenge to devise design methods that adequately handle the effects of the uncertainty and yet are computationally plausible.

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Recent research in MPC has witnessed advances across areas of robust, stochastic, and adaptive MPC. Research in these areas has led to a plethora of MPC algorithms that address specific forms of uncertainty affecting the system under consideration. It is somewhat an unfortunate fact that a number of recent proposals blur the boundaries between robust, stochastic, and adaptive MPC syntheses, which strictly speaking belong to distinctly different mathematical classes. Failing to recognize inevitable heterogeneity of MPC under uncertainty syntheses exhibits potential misunderstanding of appropriate approach to handling different types of uncertainty. The information available about the uncertainty dictates directly the theoretically correct class of MPC under uncertainty.

Understanding MPC Under Uncertainty

The uncertainty types underpinning robust, stochastic, and adaptive MPC are illustrated by a scalar discrete time system

$$x^+ = x + u + w \text{ with } w \in \mathbb{W} := \{-1, 0, 1\},$$

where x is the state, u is the control, w is the uncertainty, and x^+ is the successor state.

Robust MPC is concerned with the set–membership uncertainty, for which the underlying available information on the uncertainty is that at any time instance $w_k \in \mathbb{W} = \{-1, 0, 1\}$. When at any time instance only state x_k is known, state feedback control laws $u_k(\cdot)$ with values $u_k(x_k)$ are structurally permissible, and this information pattern leads to a *minimax* MPC problem. Likewise, when at any time instance both state x_k and uncertainty w_k are known, state–uncertainty feedback control laws $u_k(\cdot, \cdot)$ with values $u_k(x_k, w_k)$ are structurally permissible, and this information pattern results in a *maximin* MPC problem. To illustrate the difference between the minimax and maximin cases, consider the distance function of x^+ from the origin. The optimal minimax distance is attained by control law $u(x) = -x$, and it is equal to 1 for all x, while the optimal maximin distance is attained by control law $u(x, w) = -x - w$ and it is equal to 0 for all x.

Stochastic MPC considers the case in which a probabilistic information about uncertainty is provided. Typically, uncertainty process is assumed to be stationary and uncorrelated, and stochastic characteristics of the underlying uncertainty are known. For our example, it could be given that $p(w = -1) = 10^{-9}$, $p(w = 1) = 10^{-9}$ and $p(w = 0) = 1 - 2 \cdot 10^{-9}$. In this case, the underlying controlled state process is itself a stochastic process whose essential characteristics can be propagated compatibly with the system equation, employed control functions that should utilize informationally consistent stochastic characteristics of the controlled state process, and prior probabilistic postulates on the initial state. It would be very odd to neglect available stochastic information about uncertainty and implement robust MPC instead of stochastic MPC as well as to compare robust and stochastic MPC, as these belong to mathematically incomparable categories.

In *adaptive* MPC, for a simplified but frequently considered setting, the model of the system, i.e., uncertainty, is unknown and constant. In turn, an implied information about uncertainty is that it actually takes on one of the three possible values $w = -1$, or $w = 0$, or $w = 1$. Any control feedback $u(x)$ allows for learning the value of uncertainty, which is easily deduced from the value of the successor state $w = x^+ - (x + u(x))$. Thus, model uncertainty, in this simplified example, vanishes after one step, and the system becomes purely deterministic after one step. It does not seem appropriate to employ either robust or stochastic MPC as control strategies for adaptive MPC.

Uncertainty Effect on Robust MPC Synthesis

We now dissect the uncertainty effect on robust MPC in a verbose mode and then formally introduce exact robust MPC.

The uncertainty type: The considered uncertainty type is the set–membership uncertainty. Traditionally, robust MPC treats the minimax case in which, at any time instance k, the state x_k is known, while the values of the current and future uncertainty w_{k+i} are not known but are guaranteed to belong to a known uncertainty constraint set.

The control policy: Under minimax information pattern, state feedback control functions $u_k(\cdot)$, with values $u_k(x_k)$ at states x_k, are structurally permissible and theoretically correct. Thus, within the context of robust MPC, the use of a control policy, which is a sequence of state feedback control functions, is structurally permissible and theoretically flawless.

The generalized state and control predictions: When x and $u(x)$ are the current state and control, the successor state x^+ can take any value in the set of possible successor states, where each possible successor state arises due to a particular uncertainty realization. This set-valued nature of possible successor states propagates through predictions. Consequently, it is necessary to employ generalized predictions in terms of sets of possible states and sets of related control actions.

The controlled dynamics under uncertainty: The controlled dynamics under uncertainty is the dynamics of sets of possible states, which is the controlled set-to-set–dynamics.

The robust constraint satisfaction: The robust constraint satisfaction refers to the requirement that the actual constraints are satisfied for all possible states and associated control actions arising due to all possible uncertainty realizations.

The "worst-case" performance: It is natural to minimize the "worst-case" performance over all uncertainty realizations. The minimization of the "worst-case" performance is done by a selection of a suitable control policy, and it can be also attained by minimization of an adequate, generalized cost function.

The generalized state and control origins: In minimax case of our example, only the stabilization of the state set $\{-1, 0, 1\}$ is possible, and keeping states therein requires control actions from the control set $\{-1, 0, 1\}$. In robust MPC, the state and control origins need to be considered in a generalized sense as suitably defined state and control sets.

Exact Robust MPC: Contemporary Setting

The system: The most common setting in robust MPC considers the control systems modeled, in discrete time, by:

$$x^+ = f(x, u, w), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, and $x^+ \in \mathbb{R}^n$ are, respectively, the current state, control, uncertainty, and the successor state, while $f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is the state transition map assumed to be continuous.

The constraints: The system variables x, u, and w are subject to hard constraints:

$$(x, u, w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W}. \tag{2}$$

The state and control constraint sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are assumed to be closed, while the uncertainty constraint set $\mathbb{W} \subset \mathbb{R}^p$ is assumed to be compact.

The control policy: The use of a control policy

$$\Pi_{N-1} := \{\pi_0(\cdot), \pi_1(\cdot), \ldots, \pi_{N-1}(\cdot)\}, \tag{3}$$

where N is the prediction horizon and each $\pi_k(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is a control law, is structurally permissible and desirable.

The generalized state and control predictions: The interaction of the uncertainty with the system is captured by invoking the maps $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ specified, for any subset X of $\mathbb{R}^n$ and any control function $\kappa(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$, by:

$$F(X, \kappa) := \{f(x, \kappa(x), w) : x \in X, \ w \in \mathbb{W}\}$$
$$\text{and } G(X, \kappa) := \{\kappa(x) : x \in X\}. \tag{4}$$

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The state and control predictions are set-valued and, for each relevant k, obey the relations:

$$X_{k+1} = F(X_k, \pi_k) \text{ and } U_k = G(X_k, \pi_k), \text{ with } X_0 := \{x\}. \tag{5}$$

The set sequences $\mathbf{X}_N := \{X_0, X_1, \ldots, X_{N-1}, X_N\}$ and $\mathbf{U}_{N-1} := \{U_0, U_1, \ldots, U_{N-1}\}$ represent the predicted sets of possible states and related control actions, and these set sequences are commonly known as the state and control tubes.

The robust constraint satisfaction: The robust constraint satisfaction reduces to the conditions that, for all $k = 0, 1, \ldots, N-1$, the set inclusions

$$X_k \subseteq \mathbb{X} \text{ and } U_k \subseteq \mathbb{U} \tag{6}$$

hold true and that the set of possible states X_N at the prediction time instance N satisfies the set inclusion:

$$X_N \subseteq \mathbb{X}_f, \tag{7}$$

where $\mathbb{X}_f \subseteq \mathbb{X}$ is a suitable terminal constraint set.

The terminal constraint set: The terminal constraint set is obtained by considering the uncertain dynamics:

$$x^+ = f(x, \kappa_f(x), w) \tag{8}$$

controlled by a local control function $\kappa_f(\cdot)$. The design of a control law $\kappa_f(\cdot)$ is usually performed offline, and the terminal constraint set $\mathbb{X}_f$ accounts locally for the state and control constraints. The terminal constraint set $\mathbb{X}_f$ is assumed to be closed and robust positively invariant for the dynamics (8) and constraint sets (2). Thus, the set $\mathbb{X}_f$ and a local control function $\kappa_f(\cdot)$ satisfy:

$$F(\mathbb{X}_f, \kappa_f) \subseteq \mathbb{X}_f \subseteq \mathbb{X} \text{ and } \mathbb{U}_f := G(\mathbb{X}_f, \kappa_f) \subseteq \mathbb{U}. \tag{9}$$

The best choice for $\mathbb{X}_f$ is the maximal robust positively invariant set for the dynamics (8) and constraint sets (2).

The generalized origin: The most natural candidate for the generalized origin is a minimal robust positively invariant set for the dynamics (8) and constraint sets (2). This set is entirely determined by the associated state set–dynamics

$$X^+ = F(X, \kappa_f). \tag{10}$$

The minimal robust positively invariant set is compact and well defined in the case when the local control function $\kappa_f(\cdot)$ ensures that the corresponding map $F(\cdot, \kappa_f)$ is a contraction over the space of compact subsets of $\mathbb{X}_f$ (Artstein and Raković 2008), in which case it is the unique solution to the fixed point set equation:

$$X = F(X, \kappa_f), \tag{11}$$

and it is an exponentially stable attractor for the state set–dynamics (10) with the basin of attraction being the space of compact subsets of $\mathbb{X}_f$. Thus, the usual $(0, 0)$ fixed-point pair is replaced by the fixed-point pair of sets $(\mathbb{X}_\mathcal{O}, \mathbb{U}_\mathcal{O})$ satisfying:

$$\mathbb{X}_\mathcal{O} = F(\mathbb{X}_\mathcal{O}, \kappa_f) \subseteq \mathbb{X}_f \text{ and } \mathbb{U}_\mathcal{O} := G(\mathbb{X}_\mathcal{O}, \kappa_f) \subseteq \mathbb{U}_f. \tag{12}$$

The generalized cost function: The stage cost function $\ell(\cdot, \cdot) : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_+$ is continuous and adequately lower bounded with respect to the generalized origin $\mathbb{X}_\mathcal{O}$ so that, for all $x \in \mathbb{X}$ and all $u \in \mathbb{U}$,

$$\alpha_1(\text{dist}(\mathbb{X}_\mathcal{O}, x)) \le \ell(x, u), \tag{13}$$

where $\alpha_1(\cdot)$ is a $\mathcal{K}_\infty$–class function and $\text{dist}(\mathbb{X}_\mathcal{O}, \cdot)$ is the distance function from the set $\mathbb{X}_\mathcal{O}$. The consideration of the generalized origin requires the condition that for all $x \in \mathbb{X}_\mathcal{O}$ the use of local control function $\kappa_f(\cdot)$ is "free of charge" with respect to $\ell(\cdot, \cdot)$, i.e., that for all $x \in \mathbb{X}_\mathcal{O}$ we have:

$$\ell(x, \kappa_f(x)) = 0. \tag{14}$$

The terminal cost function $V_f(\cdot)$ is assumed to be continuous and adequately upper bounded with respect to the generalized origin $\mathbb{X}_\mathcal{O}$ so that, for all $x \in \mathbb{X}_f$,

$$V_f(x) \leq \alpha_2(\mathrm{dist}(\mathbb{X}_{\mathcal{O}}, x)), \tag{15}$$

where, as above, $\alpha_2(\cdot)$ is a $\mathcal{K}_\infty$–class function. In addition, the terminal cost function $V_f(\cdot)$ satisfies locally a usual condition for robust stabilization, so that, for all $x \in \mathbb{X}_f$ and all $w \in \mathbb{W}$,

$$V_f(f(x, \kappa_f(x), w)) - V_f(x) \leq -\ell(x, \kappa_f(x)). \tag{16}$$

The cost function $V_N(\cdot,\cdot,\cdot)$ is defined, for all $x \in \mathbb{X}$, all Π_{N-1} and all $\mathbf{w}_{N-1} := \{w_0, w_1, \ldots, w_{N-1}\}$, by

$$V_N(x, \Pi_{N-1}, \mathbf{w}_{N-1}) := \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N), \tag{17}$$

where $u_k := \pi_k(x_k)$ and $x_k := x_k(x, \Pi_{N-1}, \mathbf{w}_{N-1})$ denote the solution of (1) when the initial state is x, control policy is Π_{N-1} and uncertainty realization is $\mathbf{w}_{N-1}$.

The exact ROC: The exact ROC problem $\mathbb{P}_N(x)$, $x \in \mathbb{X}$, takes the form of an infinite–dimensional minimaximization:

$$\begin{aligned} J_N(x, \Pi_{N-1}) &:= \max_{\mathbf{w}_{N-1} \in \mathbb{W}^N} V_N(x, \Pi_{N-1}, \mathbf{w}_{N-1}), \\ V_N^0(x) &:= \min_{\Pi_{N-1} \in \mathbf{\Pi}_{N-1}(x)} J_N(x, \Pi_{N-1}), \\ \Pi_{N-1}^0(x) &\in \arg \min_{\Pi_{N-1} \in \mathbf{\Pi}_{N-1}(x)} J_N(x, \Pi_{N-1}), \end{aligned} \tag{18}$$

where $\mathbf{\Pi}_{N-1}(x)$ denotes the set of the constraint admissible control policies defined, for all $x \in \mathbb{X}$, by:

$$\mathbf{\Pi}_{N-1}(x) := \{\Pi_{N-1} : \text{conditions (5)–(7) hold}\}. \tag{19}$$

With the intended scope of this article in mind, it is assumed that the infinite–dimensional minimaximization is well posed and that the value function $V_N^0(\cdot)$ admits an upper bound in terms of a suitable $\mathcal{K}_\infty$–class function as specified in (23).

The value function $V_N^0(\cdot)$ might admit nonunique optimal control policies, so that $\Pi_{N-1}^0(\cdot)$ represents a selection from the set of optimal control policies. The effective domain $\mathcal{X}_N$ of the value function $V_N^0(\cdot)$ and optimal control policy $\Pi_{N-1}^0(\cdot)$,

$$\mathcal{X}_N := \{x \in \mathbb{R}^n : \mathbf{\Pi}_{N-1}(x) \neq \emptyset\} \tag{20}$$

is known in the literature as the N–step minimax controllable set to a target set $\mathbb{X}_f$, and it is such that $\mathbb{X}_f \subseteq \mathcal{X}_N \subseteq \mathbb{X}$.

The exact robust MPC: The exact robust MPC implements values of the control law $\pi_0^0(\cdot)$. The control law $\pi_0^0(\cdot)$ is well defined for all $x \in \mathcal{X}_N$, and it induces the controlled uncertain dynamics specified, for all $x \in \mathcal{X}_N$, by:

$$x^+ \in \mathcal{F}(x),\ \mathcal{F}(x) := \{f(x, \pi_0^0(x), w) : w \in \mathbb{W}\}. \tag{21}$$

The robust MPC law $\pi_0^0(\cdot)$ renders the N–step minimax controllable set $\mathcal{X}_N$ robust positively invariant so that, for all $x \in \mathcal{X}_N$,

$$\mathcal{F}(x) \subseteq \mathcal{X}_N \subseteq \mathbb{X} \text{ and } \pi_0^0(x) \in \mathbb{U}. \tag{22}$$

Furthermore, the associated value function $V_N^0(\cdot) : \mathcal{X}_N \to \mathbb{R}_+$ is, by construction, a Lyapunov certificate verifying the robust asymptotic stability of the generalized origin $\mathbb{X}_{\mathcal{O}}$ for the controlled uncertain dynamics (21) with the basin of attraction being equal to the N–step minimax controllable set $\mathcal{X}_N$. More precisely, for all $x \in \mathcal{X}_N$, it holds that:

$$\alpha_1(\mathrm{dist}(\mathbb{X}_{\mathcal{O}}, x)) \leq V_N^0(x) \leq \alpha_3(\mathrm{dist}(\mathbb{X}_{\mathcal{O}}, x)), \tag{23}$$

where $\alpha_3(\cdot)$ is postulated $\mathcal{K}_\infty$–class function, while for all $x \in \mathcal{X}_N$ and all $x^+ \in \mathcal{F}(x)$, it holds that:

$$V_N^0(x^+) - V_N^0(x) \leq -\alpha_1(\mathrm{dist}(\mathbb{X}_{\mathcal{O}}, x)). \tag{24}$$

Under reasonable conditions, exact robust MPC induces strong structural properties, but its computational complexity is overwhelming. However, in the preceding formulation, uncertainty effects have been "dissected," and the "basic building blocks" employed for exact robust MPC have been identified.

Alternative Approaches to Robust MPC

Inherent Robustness of MPC

An approach to robust MPC can be derived by attempting to utilize inherent robustness of conventional MPC, see, e.g., Limon et al. (2009) and related references. Such approach is an indirect approach to robust MPC, and it is computationally feasible. This approach considers a nominal MPC possibly applied to modified model, constraints, and cost, and when possible it makes use of robustness of its stability properties. The approach might be well suited for problems with neither state nor terminal constraints. However, the robustness properties can be very limited in terms of the magnitude of tolerable uncertainty. In addition, the related robustness analysis is unnecessarily conservative and geometrically insensitive, i.e., it is of very little value for optimal state trajectories on, or near, the boundaries of the state constraints. Major drawbacks of such an approach, however, stem from the facts that the stability property of the conventional MPC might fail to be robust (Grimm et al. 2004) and that the optimal control of constrained discrete time systems can be a fragile process itself (Raković 2009).

Scenario-Based Approaches for Robustness

The approach based on utilization of scenarios of uncertainty realizations is particularly very well suited for the case of linear systems with convex cost functions subject to convex constraints and when the uncertainty set is specified as the convex hull of finitely many points (Scokaert and Mayne 1998). In this approach, with each of finitely many extreme uncertainty realizations, a pair of "extreme" state and control sequences is associated. The collection of all pairs of such "extreme" state and control sequences is optimized subject to dynamical constraints (which are deterministic since uncertainty realizations are postulated), causality constraints, and state, control, and terminal constraints. This approach converts the infinite–dimensional minimaximization into an equivalent finite–dimensional minimaximization, but it remains computationally intractable due to an exponential increase of the number of extreme uncertainty realizations with respect to the prediction horizon.

Some recent proposals consider a limited number of uncertainty realizations, and related scenarios are constructed by allowing uncertainty to vary over a shorter horizon, while uncertainty is considered constant over the remainder of the prediction horizon. Such approaches enhance computational aspects and can be used in more general settings, see, e.g., Lucia et al. (2012) and related references. Theoretically, it is difficult to justify the postulated structural restriction on the scenarios of uncertainty realizations. Furthermore, it is highly unlikely for such approaches to guarantee a priori any of the desirable structural properties, as provided by alternative robust MPC methods.

A very promising approach is to deploy scenario-based optimization for robust MPC (Calafiore and Campi 2006; Calafiore and Fagaino 2013), in which uncertainty is sampled in a traditional manner. These approaches provide probabilistic guarantees of structural properties that are strictly weaker than the absolute ones guaranteed by robust MPC.

Parameterized Predictions and Control Policy

The parameterization of state and control predictions as well as control policy has led to major advances in robust MPC. The core simplification in these approaches is the use of finite–dimensional parameterization of control policy. The control policy is parameterized so as to allow for the utilization of both the least conservative parameterized state and control predictions and a range of simpler, but sensible, cost functions. The computational complexity of the exact state and control tubes is tackled by employing implicit representations of the predicted sets of possible states and control actions. Alternatively, simpler outer-bounding approximations of the exact state and control tubes are deployed, and the exact set–dynamics of the state and control tubes (5) is relaxed to

$$\{x_0\} \subseteq X_0, \text{ and, } F(X_k, \pi_k) \subseteq X_{k+1}$$
$$\text{and } G(X_k, \pi_k) \subseteq U_k.$$

The terminal constraint set $\mathbb{X}_f$ is deployed as computationally feasible and "relaxed form" of the generalized state origin $\mathbb{X}_{\mathcal{O}}$. The performance requirements are carefully prioritized, and modified when necessary, and expressed in terms of the cost functions that do not require intractable minimax optimization but still ensure that the associated value function verifies the robust stability and attractivity of the generalized origin $\mathbb{X}_{\mathcal{O}}$ or the terminal constraint set $\mathbb{X}_f$.

Tube MPC

A new paradigm in robust MPC that has effectively defined modern robust MPC is tube MPC. Tube MPC has been pioneered and advanced primarily through joint and individual research activities of David Q. Mayne and Saša V. Raković. For completeness, we provide a brief overview of tube MPC for linear systems subject to bounded additive uncertainty

$$x^+ = Ax + Bu + w. \tag{25}$$

In what follows, $\oplus$ denotes the Minkowski set addition, while $\ominus$ denotes the Pontryagin set difference.

Rigid Tube MPC

In rigid tube MPC, introduced and discussed in detail in Mayne et al. (2005), state and control predictions are parameterized as

$$x_k = z_k + s_k \text{ and } u_k = v_k + Ks_k, \tag{26}$$

and state feedback control laws of related control policy are

$$\pi_k(x_k, z_k, v_k) = v_k + K(x_k - z_k). \tag{27}$$

The sets forming state and control tubes are parameterized as

$$X_k = z_k \oplus S \text{ and } U_k = v_k \oplus KS. \tag{28}$$

The state tube cross-sectional shape set S is assumed to be robust positively invariant set for $s^+ = (A + BK)s + w$, i.e.,

$$(A + BK)S \oplus \mathbb{W} \subseteq S. \tag{29}$$

Rigid tube MPC is computationally efficient since it reduces to a conventional MPC for a virtual deterministic linear system

$$z_{k+1} = Az_k + Bv_k \tag{30}$$

subject to modified constraints

$$z_k \in \mathbb{Z} := \mathbb{X} \ominus S \text{ and } v_k \in \mathbb{V} := \mathbb{U} \ominus KS, \tag{31}$$

and suitable terminal constraints $z_N \in \mathbb{Z}_f$ as well as uncertainty free stage and terminal cost functions $\ell(\cdot, \cdot)$ and $V_f(\cdot)$ with values $\ell(z_k, v_k)$ and $V_f(z_N)$. Rigid tube MPC enables a flexible selection of free initial condition of the virtual system

$$x - z_0 \in S \tag{32}$$

at any state x. This condition proves to be a key ingredient in establishing robust exponential stability of the set S for uncertain dynamics controlled by rigid tube MPC $u_0^0(x) = v_0^0(x) + K(x - z_0^0(x))$, based on the exponential stability of the origin for the virtual system $z^+(x) = Az_0^0(x) + Bv_0^0(x)$.

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Homothetic Tube MPC

Homothetic tube MPC (Raković et al. 2012, 2013) relaxes rigidity of state and control tubes parameterization in (28) by deploying state and control tubes whose terms are parameterized as

$$X_k = z_k \oplus \alpha_k S \text{ and } U_k = v_k \oplus \alpha_k KS, \tag{33}$$

where nonnegative scalars $\alpha_k \in \mathbb{R}_{\geq 0}$ are referred to as the state and control tubes scaling factors. The exact set–dynamics of the state tubes is utilized, and it is expressed as

$$Az_k + Bv_k + \alpha_k(A + BK)S \oplus \mathbb{W}$$
$$\subseteq z_{k+1} \oplus \alpha_{k+1}S, \tag{34}$$

while state and control constraints take form

$$z_k \oplus \alpha_k S \subseteq \mathbb{X} \text{ and } v_k \oplus \alpha_k KS \subseteq \mathbb{U}, \quad (35)$$

and adequate terminal constraints $(z_N, \alpha_N) \in \Omega_f$ as well as uncertainty free stage and terminal cost functions $\ell(\cdot,\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ with values $\ell(z_k, v_k, \alpha_k)$ and $V_f(z_N, \alpha_N)$. The selection of z_0 and α_0 is governed by

$$x - z_0 \in \alpha_0 S. \quad (36)$$

Homothetic tube MPC, a detailed summary of which can be found in Raković et al. (2012, 2013), enlarges domain of attraction and improves attractivity properties compared to rigid tube MPC.

Elastic Tube MPC

Elastic tube MPC (Raković et al. 2016) further relaxes rigidity and homotheticity of tubes parameterizations in (28) and (33). In elastic tube MPC, the state tube cross-sectional shape set is, in fact, a value of a set-valued function $S(\cdot)$ given, for all $a \in \mathbb{R}^q_{\geq 0}$, by

$$S(a) := \{x \ : \ Cx \leq a\}, \quad (37)$$

where a suitable matrix $C \in \mathbb{R}^{q \times n}$ is constructed offline. The terms of state and control tubes are parameterized as

$$X_k = z_k \oplus S(a_k) \text{ and } U_k = v_k \oplus KS(a_k), \quad (38)$$

where nonnegative variables a_k are referred to as the state and control tubes elasticity factors. The exact set–dynamics of the state tubes is given by

$$\begin{aligned} &Az_k + Bv_k + (A + BK)S(a_k) \oplus \mathbb{W} \\ &\subseteq z_{k+1} \oplus S(a_{k+1}), \end{aligned} \quad (39)$$

while state and control constraints take form

$$z_k \oplus S(a_k) \subseteq \mathbb{X} \text{ and } v_k \oplus KS(a_k) \subseteq \mathbb{U}, \quad (40)$$

and suitable terminal constraints $(z_N, a_N) \in \Omega_f$ as well as uncertainty free stage and terminal cost functions $\ell(\cdot,\cdot,\cdot)$ and $V_f(\cdot,\cdot)$ with values $\ell(z_k, v_k, a_k)$ and $V_f(z_N, a_N)$. To ensure computational practicability, guaranteed and convex reformulations of set–dynamics (39) and state and control constraints (40) are utilized, while z_0 and a_0 satisfy

$$x - z_0 \in S(a_0). \quad (41)$$

Elastic tube MPC enlarges domain of attraction and improves attractivity properties relative to both the homothetic and rigid tube MPC. See Raković et al. (2016) for more details about elastic tube MPC.

Tube MPC with Time-Varying Cross Sections

Robust MPC based on constraint tightening proposed in Chisci et al. (2001) can be seen as a tube MPC with time-varying cross sections, since related sets of possible state and controls take the form

$$X_k = z_k \oplus S_k \text{ and } U_k = v_k \oplus KS_k, \quad (42)$$

where

$$S_{k+1} = (A + BK)S_k \oplus \mathbb{W}. \quad (43)$$

As proposed in Chisci et al. (2001), this form of tube MPC utilizes virtual system (30) and time-varying state and control constraints

$$z_k \in \mathbb{Z}_k := \mathbb{X} \ominus S_k \text{ and } v_k \in \mathbb{V}_k := \mathbb{U} \ominus KS_k, \quad (44)$$

and terminal constraints $z_N \oplus S_N \subseteq \mathbb{X}_f$ where $\mathbb{X}_f$ is the maximal robust positively invariant set for $s^+ = (A + BK)s + w$ and constraints $(s, Ks) \in \mathbb{X} \times \mathbb{U}$ and $w \in \mathbb{W}$. The proposal of Chisci et al. (2001) minimizes the cost in terms of "control perturbations" $c_k = v_k + Kz_k$, and it enforces a restrictive constraint on the initial states $z_0 = x$, i.e., $S_0 = \{0\}$, which prevents robust exponential stability to be established. However, this proposal can be modified to include selection of initial condition of the virtual system via $x - z_0 \in \mathbb{X}_f$ in analogy to (32), which would enable robust exponential stability to be verified.

Tube MPC with Optimized Time-Varying Cross Sections

A further contribution to robust MPC is made in Löfberg (2003) and Goulart et al. (2006), where the so-called affine in the past disturbances control policy is utilized. Within the context of tube MPC, the related parameterizations of predicted states and controls are

$$x_k = z_k + \sum_{j=1}^{k} T_{(j,k)} w_j \text{ and}$$
$$u_k = v_k + \sum_{j=1}^{k} M_{(j,k)} w_j, \tag{45}$$

where matrices $T_{(j,k)}$ and $M_{(j,k)}$ are decision variables and the related sets of possible states and controls take the form

$$X_k = z_k \oplus \bigoplus_{j=1}^{k} T_{(j,k)} \mathbb{W} \text{ and}$$
$$U_k = v_k \oplus \bigoplus_{j=1}^{k} M_{(j,k)} \mathbb{W}. \tag{46}$$

The state tube dynamics is governed by deterministic z–dynamics (30) and matrix-valued $T_{(j,k)}$–dynamics

$$T_{(j,k+1)} = AT_{(j,k)} + BM_{(j,k)} \text{ with } T_{(j,j)} = I. \tag{47}$$

The state and control constrains are given by

$$z_k \oplus \bigoplus_{j=1}^{k} T_{(j,k)} \mathbb{W} \subseteq \mathbb{X} \text{ and}$$
$$v_k \oplus \bigoplus_{j=1}^{k} M_{(j,k)} \mathbb{W} \subseteq \mathbb{U}, \tag{48}$$

and terminal constraints are $z_N \oplus \oplus_{j=1}^{N} T_{(j,N)} \mathbb{W} \subseteq \mathbb{X}_f$. The polyhedral state, control, and terminal constraints can be efficiently handled by using support functions and duality. A range of uncertainty free cost functions can be utilized to design a robust exponentially stable tube MPC via efficient online optimization of the uncertainty free states z_k and controls v_k and collections of matrices $T_{(j,k)}$ and $M_{(j,k)}$.

Parameterized Tube MPC

The current state of the art in robust MPC is parameterized tube MPC (Raković 2012; Raković et al. 2012), which employs superposition and convex decomposition for predictions of states and controls

$$x_k = \sum_{j=0}^{k} x_{(j,k)} \text{ and } u_k = \sum_{j=0}^{k} u_{(j,k)}, \text{ with} \tag{49}$$

$$x_{(j,k)} = \sum_{i=0}^{q} \lambda_i x_{(i,j,k)} \text{ and } u_{(j,k)} = \sum_{i=0}^{q} \lambda_i u_{(i,j,k)}, \tag{50}$$

via partial extreme states and controls $x_{(0,k)}$, $x_{(i,j,k)}$, $u_{(0,k)}$, and $u_{(i,j,k)}$, and where $\lambda_i = \lambda_{(i,j,j)}$ are convex multipliers. The sets of possible states and controls take the form

$$X_k = \bigoplus_{j=0}^{k} X_{(j,k)} \text{ and } U_k = \bigoplus_{j=0}^{k} U_{(j,k)}, \tag{51}$$

where partial state and control tubes cross sections are given as $X_{(0,k)} = \{x_{(0,k)}\}$, $X_{(j,k)} = \operatorname{convh}(\{x_{(1,j,k)}, x_{(2,j,k)}, \ldots, x_{(q,j,k)}\})$, $U_{(0,k)} = \{u_{(0,k)}\}$, and $U_{(j,k)} = \operatorname{convh}(\{u_{(1,j,k)}, u_{(2,j,k)}, \ldots, u_{(q,j,k)}\})$. The exact set–dynamics of the state tubes is governed by deterministic extreme partial states dynamics specified by

$$x_{(0,k+1)} = Ax_{(0,k)} + Bu_{(0,k)} \text{ with } x_{(0,0)} = x \text{ and}$$
$$x_{(i,j,k+1)} = Ax_{(i,j,k)} + Bu_{(i,j,k)} \text{ with } x_{(i,j,j)} = \overline{w}_i, \tag{52}$$

where $\mathbb{W} = \operatorname{convh}(\{\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_q\})$. The state and control constrains are given by

$$\bigoplus_{j=0}^{k} X_{(j,k)} \subseteq \mathbb{X} \text{ and } \bigoplus_{j=0}^{k} U_{(j,k)} \subseteq \mathbb{U}, \tag{53}$$

and terminal constraints are $\bigoplus_{j=0}^{N} X_{(j,N)} \subseteq \mathbb{X}_f$. The polyhedral state, control, and terminal constraints can be efficiently handled by using support functions without need to perform any of Minkowski set additions and convex hull operations utilized for parameterization and analysis. A range of uncertainty free cost functions can be utilized to design a robust exponentially stable tube MPC via efficient online optimization of the sequences of extreme partial states $x_{(0,k)}$ and $x_{(i,j,k)}$ and extreme partial controls $u_{(0,k)}$ and $u_{(i,j,k)}$. Parameterized tube MPC outperforms proposals of Löfberg (2003) and Goulart et al. (2006), as it uses separable nonlinear state feedback control functions.

Closing Remarks and Recommended Reading

The exact robust MPC synthesis has reached a remarkable degree of theoretical maturity in the general setting. Furthermore, a variety of rather sophisticated robust MPC synthesis methods, which are both computationally efficient and theoretically sound, has been developed for the frequently encountered linear–polyhedral case. The further advances in the robust MPC field should be driven by the utilization of more structured types and models of the uncertainty. Given a tremendous progress in robust MPC and its successful utilization across a number of control engineering areas, a comprehensive survey of this field is definitely due.

The comprehensive monograph (Rawlings and Mayne 2009) provides an in-depth systematic exposure to MPC and robust MPC, and it is also a rich source of relevant references. The invaluable overview of the theory and computations of the maximal and minimal robust positively invariant sets can be found in Kolmanovsky and Gilbert (1998) and Artstein and Raković (2008). The important work Scokaert and Mayne (1998) points out the theoretical benefits of the use of the control policy, but it also indicates indirectly the computational impracticability of the associated feedback minimax robust MPC. The early tube MPC synthesis (Langson et al. 2004; Mayne et al. 2005) represent key steps forward in robust MPC for the linear–polyhedral setting. Homothetic and elastic tube MPC syntheses (Raković et al. 2012, 2013, 2016) provide a number of improvements relative to the first generation of tube MPC (Mayne et al. 2005), and these methods have a high potential to effectively handle the parametric uncertainty of the matrix pair (A, B). The current state of the art in the linear–polytopic setting is reached by parameterized tube MPC (Raković 2012; Raković et al. 2012). The output feedback robust MPC synthesis in the linear–polyhedral setting can be handled with direct extensions of the tube MPC syntheses (Mayne et al. 2006, 2009).

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