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AN ITERATIVE RANK PENALTY METHOD FOR NONCONVEX QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMS*

CHUANGCHUANG SUN† AND RAN DAI†

Abstract. This paper examines the nonconvex quadratically constrained quadratic programming (QCQP) problems using an iterative method. A QCQP problem can be handled as a linear matrix programming problem with a rank-one constraint on the to-be-determined matrix. One of the existing approaches for solving nonconvex QCQPs relaxes the rank one constraint on the unknown matrix into a semidefinite constraint to obtain the bound on the optimal value without finding the exact solution. By reconsidering the rank one matrix, the iterative rank penalty (IRP) method is proposed to gradually approach the rank one constraint. Each iteration of IRP is formulated as a convex problem with semidefinite constraints. Furthermore, an augmented Lagrangian method, called an extended Uzawa algorithm, is developed to solve the sequential problem at each iteration of IRP for improved scalability and computational efficiency. Simulation examples are presented using the proposed method, and comparative results obtained from the other methods are provided and discussed.

Key words. quadratically constrained quadratic programming, semidefinite programming, nonconvex optimization, augmented Lagrangian method, linear convergence

AMS subject classifications. 90C26, 90C22

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1. Introduction.

The general or nonconvex quadratically constrained quadratic programming (QCQP) problem has recently attracted significant interests due to its wide applications. For example, any polynomial programming problems of optimizing a polynomial objective function subject to polynomial inequality constraints can be reformulated as general QCQP problems [10, 17]. In addition, we can find QCQP applications in the areas of max-cut problems [19], production planning [30], signal processing [27], sensor network localizations [8], and optimal power flow [9], just to name a few.

Convexification and relaxation techniques have been commonly used when solving nonconvex optimization problems [1, 28, 20]. Efforts toward solving nonconvex QCQPs have been pursued in two directions, obtaining a bound on the optimal value and finding a feasible solution. For simplicity, the QCQPs discussed below represent general/nonconvex QCQPs. Extensive relaxation methods have been investigated to obtain a bound on the optimal value of a QCQP. The linear relaxation approach introduces extra variables to transform the quadratic objective and constraints into bilinear terms, which is followed by linearization of the bilinears [2, 29]. The final linear formulation reaches a bound on the QCQP optimal value with fast convergence but low accuracy. The semidefinite programming (SDP) relaxation introduces a rank one matrix to replace the quadratic objective and constraints with linear matrix equalities/inequalities. However, the nonlinear rank one constraint on the unknown matrix is substituted by semidefinite relaxation. In general, the SDP relaxation reaches a tighter bound on the optimal value than that obtained from linear relaxation [17].

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†Department of Mechanical and Aerospace Engineering, The Ohio State University, Columbus, OH 43210 (sun.2264@osu.edu, dai.490@osu.edu).

A detailed discussion of various relaxation approaches and the comparison of their relative accuracy is provided in [4].

However, finding a bound on the optimal value of QCQP does not imply generating an optimal solution, not even a feasible one. One of the efforts for obtaining a feasible solution utilizes an iterative linearization approach to gradually improve the objective value [17]. However, this method does not provide any guarantee of convergence. Another approach is to generate randomized samples and solve the QCQP on average of the distribution. However, the randomization approach does not apply to problems with equality constraints and the optimality is not guaranteed. The branch and bound (BNB) method has been frequently utilized to search for the optimal solution of nonconvex problems [11, 3, 15]. Although BNB can lead to a global optimal solution, the searching procedure is time-consuming, especially for large-scale optimization problems. Recent work in [31] proposes that the structure of a QCQP can be changed based on graph theory to obtain a low-rank solution which greatly reduces the gap between the exact solution and the relaxed one. Furthermore, work in [26] generates a series of SDPs to solve polynomial optimization problems, which is applicable to small-scale QCQPs.

After reviewing the literature, we come to a conclusion that a more efficient approach is required to solve QCQP problems. In our previous work of [34, 16], the iterative rank penalty (IRP) method has been proposed to solve homogeneous QCQPs. The IRP method adds a penalty function that approximately represents the rank of the unknown matrix and then finds the unknown rank one matrix through a newly developed alternating minimization algorithm (AMA). The iterations gradually minimize the rank of the unknown matrix until it satisfies the rank one constraint. This paper explores the problem for inhomogeneous QCQPs and focuses on the analysis of convergence conditions. Under these conditions, proof of local convergence to a local optimum at a linear convergence rate is provided based on the relationship between two adjacent iterations. Each iteration of IRP is formulated as a convex optimization problem with semidefinite constraints. To improve the scalability and computational efficiency in solving large-scale convex optimization problems with semidefinite constraints, an extended Uzawa algorithm, based on the augmented Lagrangian method [14, 22], is developed to solve the sequential problem formulated at each iteration of IRP. In addition, convergence to global optimality for the extended Uzawa algorithm is provided.

The special contribution of this paper is a novel iterative approach to solving nonconvex QCQPs and proof of the linear convergence of the iterative approach. Furthermore, the proposed approach is accomplished by solving each iteration via a scalable and computationally efficient extended Uzawa algorithm. In the following, the QCQP formulation is introduced in section 2. The IRP method is discussed in section 3 with a linear convergence rate and local optimality proof. In section 4, the extended Uzawa algorithm for solving large-scale SDPs is introduced and its convergence analysis is discussed. Simulation examples are presented in section 5 to verify the effectiveness of the proposed methods. We conclude the paper with a few remarks in section 6.

2. Problem formulation.

A general QCQP problem can be expressed in the form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T Q_0 \mathbf{x} + \mathbf{a}_0{}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T Q_j \mathbf{x} + \mathbf{a}_j{}^T \mathbf{x} \le c_j \;\; \forall\, j = 1, \ldots, m, \\ & \mathbf{l}_{\mathbf{x}} \le \mathbf{x} \le \mathbf{u}_{\mathbf{x}}, \end{aligned} \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the unknown vector to be determined, $Q_j \in \mathbb{S}^n$, $j = 0, \ldots, m$, is an arbitrary symmetric matrix, $c_j \in \mathbb{R}$, $j = 1, \ldots, m$, and $\mathbf{a}_j \in \mathbb{R}^n$, $j = 0, \ldots, m$. Moreover, $\mathbf{l}_\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u}_\mathbf{x} \in \mathbb{R}^n$ are the lower and upper bounds on $\mathbf{x}$, respectively. Since Q_j, $j = 0, \ldots, m$, are not necessarily positive semidefinite, the problem in (2.2) is generally classified as nonconvex and NP-hard, requiring global search for its optimal solution.

The above QCQP problem with an inhomogeneous quadratic objective and constraints can be transformed into homogeneous ones by introducing a new variable $t \in \mathbb{R}$. It can be verified that

$$\mathbf{x}^T Q_j \mathbf{x} + \mathbf{a}_j^T \mathbf{x} = \begin{bmatrix} \mathbf{x}^T & t \end{bmatrix} \begin{bmatrix} Q_j & \mathbf{a}_j/2 \\ \mathbf{a}_j^T/2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix}, \quad j = 0, \ldots, m,$$

where $t^2 = 1$ is a quadratic constraint as well. Then $\frac{\mathbf{x}}{t}$ will be the solution of the original problem stated in (2.1). In addition, linear constraints in (2.1) can be rewritten in the above quadratic form as well by setting $Q = \mathbf{0}$. The homogeneous QCQP problem is formulated as

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T Q_0 \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T Q_j \mathbf{x} \leq c_j \ \ \forall\, j = 1, \ldots, m. \end{aligned} \tag{2.2}$$

Based on this fact, any inhomogeneous QCQP can be transformed into a homogeneous one. Without loss of generality, the following approach to solving nonconvex QCQP problems focuses on homogeneous QCQPs.

3. An iterative approach to nonconvex QCQPs.

3.1. The lower bound on the optimal value of QCQPs. In order to solve the nonconvex QCQP in (2.2), the semidefinite relaxation method is first introduced to find a tight lower bound on the optimal objective value. By applying an interior point method, the relaxed formulation can be solved via an SDP solver [36]. After introducing a rank one positive semidefinite matrix $X = \mathbf{x}\mathbf{x}^T$, it is straightforward that such a matrix equality constraint is equivalent to the fact that $X \succeq \mathbf{0}$ and $\mathbf{rank}(X) \leq 1$. Consequently, we can gain an equivalent reformulation with the matrix rank constraint in the following:

$$\begin{aligned} \min_{X} \quad & \langle X, Q_0 \rangle \\ \text{s.t.} \quad & \langle X, Q_j \rangle \leq c_j \ \ \forall\, j = 1, \ldots, m, \\ & X \succeq 0,\ \mathbf{rank}(X) \leq 1. \end{aligned} \tag{3.1}$$

By removing the rank constraint, which carries all of the nonlinearity and nonconvexity of the original problem, the relaxed formulation is written as

$$\begin{aligned} \min_{X} \quad & \langle X, Q_0 \rangle \\ \text{s.t.} \quad & \langle X, Q_j \rangle \leq c_j \ \ \forall\, j = 1, \ldots, m, \\ & X \succeq 0, \end{aligned} \tag{3.2}$$

where $\mathbb{S}_+$ denotes a positive semidefinite matrix and $\langle \cdot \rangle$ denotes the inner product of two matrices, i.e., $\langle A, B \rangle = \mathbf{tr}(A^T B)$. The semidefinite constraint relaxes the original formulation in (2.2), which generally yields a tighter lower bound on the optimal value of (2.2) than the one obtained from the linearization relaxation technique [4]. Therefore, by reformulating the problem of (2.2) in the form of (3.2), we obtain a lower bound on the optimal value of (2.2). However, the relaxation method will not yield an optimized solution of the unknown variables $\mathbf{x}$. Compared to the original formulation in (3.2), the only difference of the relaxation approach is that the rank one constraint on matrix X is excluded. In order to obtain the optimal solution of $\mathbf{x}$, we reconsider the rank one constraint on matrix X in (3.1) and propose the IRP approach to gradually reach the constraint.

3.2. An iterative rank penalty method. We first introduce one preparatory lemma.

LEMMA 1 (Corollary 4.3.37 in [24]). *For a given matrix $X \in \mathbb{S}^n$ with eigenvalues in increasing order, denoted as $[\lambda_1^X, \lambda_2^X, \ldots, \lambda_n^X]$, when $m \le n$, one has $\lambda_m^X \le \lambda_{\max}(V^TXV)$ for any $V \in \{V | V \in \mathbb{R}^{n\times m}, V^TV = I_m\}$, where $\lambda_{\max}(\bullet)$ denotes the largest eigenvalue of $\bullet$. Moreover, equality holds when columns of V are the eigenvectors of X associated with the m smallest eigenvalues.*

As described before, the QCQP problem in (2.2) is equivalent to the one formulated in (3.1). Through Lemma 1, problem (3.1) can be reformulated as

$$\begin{array}{cc} \min_{X\in\mathbb{S}^n, V\in\mathbb{R}^{n\times(n-1)}} & \langle X, Q_0\rangle \\ \text{s.t.} & X \in \mathcal{C}, \\ & V^TXV = 0, \\ & V^TV = I_{n-1}, \end{array} \tag{3.3}$$

where $\mathcal{C} = \{X | X \in \mathbb{S}^n, X \succeq 0, \langle X, Q_j\rangle \le c_j \;\forall\, j = 1, \ldots, m\}$ and V are the eigenvectors of X associated with the $n-1$ smallest eigenvalues. For constraint $V^TXV = 0$, we take $\lambda_{n-1}(V^TXV)$ (lower bounded by 0 due to $X \succeq \mathbf{0}$) as a penalty term and approximate (3.3) by

$$\begin{array}{cc} \min_{X,V} & \langle X, Q_0\rangle + w\lambda_{n-1}(V^TXV) \\ \text{s.t.} & X \in \mathcal{C}, \\ & V^TV = I_{n-1}, \end{array} \tag{3.4}$$

where w is the penalty coefficient. Since the objective and constraints of (3.4) have nonconvex terms, solving X and V simultaneously is computationally complicated. The AMA scheme is to alternatively search for one of the two unknown variable sets and has been applied to problems when solving all variables simultaneously is not feasible. In problem (3.4), as solving X and V simultaneously is not straightforward, a new AMA is developed to solve X and V alternatively such that searching each individual variable is computationally favorable. The two alternative steps are stated below with k representing the iterative step index:

$$\begin{aligned} X_k &= \arg\min_{X\in\mathcal{C}} \langle X, Q_0\rangle + w_k\lambda_{n-1}(V_{k-1}^TXV_{k-1}) \\ &= \arg\min_{X\in\mathcal{C}} \langle X_k, Q_0\rangle + w_kr_k \\ &\qquad \text{s.t. } r_kI_{n-1} - V_{k-1}^TX_kV_{k-1} \succeq 0 \text{ and} \end{aligned} \tag{3.5a}$$

$$\begin{aligned} V_k &= \arg\min_{V^TV=I_{n-1}} \lambda_{n-1}(V^TX_kV) \\ &= \text{orthonormal eigenvectors of } X_k \text{ associated with the} \\ &\qquad n-1 \text{ smallest eigenvalues.} \end{aligned} \tag{3.5b}$$

The X_k update in (3.5a) introduces a slack variable, r_k, and a semidefinite constraint, $r_k I_{n-1} - V_{k-1}^T X_k V_{k-1} \succeq 0$, to transform the original nonconvex problem into a convex one. The reformulation of V_k update in (3.5b) is according to Lemma 1. Note that the penalty coefficient w_k can be updated along each iteration to balance between accuracy and convergence, which will be addressed below.

While the V_k update in (3.5b) has a closed form solution, i.e., the one obtained via the eigenvalue decomposition, the X_k update needs to solve an SDP problem in the form of

$$\begin{aligned} \min_{X_k, r_k} \quad & \langle X_k, Q_0 \rangle + w_k r_k \\ \text{s.t.} \quad & \langle X_k, Q_j \rangle \le c_j \ \ \forall\, j = 1, \ldots, m, \\ & r_k I_{n-1} - V_{k-1}^T X_k V_{k-1} \succeq 0, \ X_k \succeq 0, \end{aligned} \tag{3.6}$$

where $w_k > 0$ is the penalty coefficient for r_k in the kth iteration and V_{k-1} are the orthonormal eigenvectors corresponding to the $n-1$ smallest eigenvalues of X_{k-1} solved at the previous iteration $k-1$. By considering the approximate rank function in the objective, we are trying to optimize the original objective function and at the same time minimize the newly introduced parameter r_k such that when $r_k = 0$, the rank one constraint on X is satisfied. Meanwhile, since X_k is constrained to be positive semidefinite, the term $V_{k-1}^T X_k V_{k-1}$ is positive semidefinite as well, which implies that the value of r_k is nonnegative in order to satisfy $r_k I_{n-1} - V_{k-1}^T X_k V_{k-1} \succeq 0$ in (3.6). The above approach is repeated until $r_k \le \epsilon$, where ϵ is a small threshold for stopping criteria. Once the rank one matrix X is obtained, the optimal solution of x is determined by $\mathbf{x} = \sqrt{\lambda_n}\mathbf{v}$, where $\lambda_n \in \mathbb{R}$ is the largest eigenvalue of X and $\mathbf{v} \in \mathbb{R}^n$ is the corresponding normal eigenvector. The IRP algorithm is summarized below.

Algorithm 1: Iterative Rank Penalty Method
Input: Parameters Q_0, Q_j, c_j, $j = 1, \ldots, m$, $w_0 > 0$, $\alpha > 1, w_{\max}$, ϵ,
Output: Unknown rank one matrix X and unknown state vector $\mathbf{x}$
begin
1. **initialize** Set $k = 0$, solve (3.2) to find X_0, and obtain V_0 from X_0; set $k = k + 1$
2. **while** $r_k > \epsilon$
3. Solve problem (3.6) and obtain X_k, r_k using the extended Uzawa algorithm in section 4[1]
4. From X_k, find its orthonormal eigenvectors V_k[2]
5. $k = k + 1$, $w_k = w_{k-1} * \alpha$
6. **end while**
7. Find $\mathbf{x}$ from X

end

Although the linear approximation based nonlinear programming (NLP) method [6] can be applied to solve nonconvex QCQP problems, linearization of nonlinear terms may change the feasibility zone and lead to an infeasible problem. However, the IRP introduces a penalty term and a slack variable constraint in (3.5), which

[1]If the optimum pair (X_k, r_k) is not unique for iteration k, the one with the smallest r_k is selected.

[2]If X_k has repeated eigenvalues, an orthonormal basis from the eigenspace is randomly selected as columns of V_{k-1}.

makes the feasibility remain unchanged along the iterations. Specifically, the two sets of constraints $\langle X_k, Q_j\rangle \le c_j \ \forall\ j = 1, \dots, m$ and $X_k \succeq 0$ in the subproblem (3.6) are automatically feasible due to the feasibility assumption of the original problem. For another constraint $r_k I_{n-1} - V_{k-1}^T X_k V_{k-1} \succeq 0$, as r_k is a nonnegative variable to be determined, there always exists a large enough r_k to guarantee its feasibility.

Compared to the linear approximation based NLP method, the proposed IRP develops a completely different computational framework. Each iteration in IRP adds a semidefinite constraint and a weighted penalty term in the objective function, which retains the convex nature of the original objective function, and the other constraints remained unchanged. In other words, by adding an evolving semidefinite constraint on the relaxed problem (3.2) at each iteration, IRP aims to gradually reach the rank one constraint on the unknown matrix X. Unlike the linearization based algorithm, original constraints in each iteration of IRP are not dependent on solution from the last iteration. Thus, the iteration will not easily stuck at an infeasible point, especially for QCQPs with equality quadratic constraints. Furthermore, instead of using a random initial guess, the IRP algorithm above intuitively has an initial guess from the semidefinite relaxation expressed in (3.2). Except for the newly introduced variable r_k, there are no additionally introduced unknown variables in the formulation. This simple procedure can be easily implemented for any nonconvex QCQP problems. Further analysis of the local convergence properties of IRP with a linear rate is discussed below.

3.3. Local convergence of IRP algorithm. In the following, we will prove local convergence of the IRP in (3.5) to a local optimum (X^*, V^*) of (3.3). To simplify the notation, we define $f(X, V) : \mathbb{S}^n \times \mathbb{R}^{n\times(n-1)} \to \mathbb{R} := \langle X, Q_0\rangle + w_k \lambda_{n-1}(V^T X V)$. In addition, two maps $p(V) : \mathbb{R}^{n\times(n-1)} \to \mathbb{S}^n$ and $q(X) : \mathbb{S}^n \to \mathbb{R}^{n\times(n-1)}$ are defined to represent the compact form of (3.5a) and (3.5b), respectively. Then we have

$$X_k = p(V_{k-1}), \quad V_k = q(X_k). \tag{3.7}$$

Before we discuss the C^1 continuity of $p(V)$ and $q(X)$, we define $\mathbf{X} \in \mathbb{R}^{n^2} := \mathrm{vec}(X)$, where $\mathbf{X}$ is the vectorization of X by stacking the columns of X on top of one another. Similarly, $\mathbf{V} \in \mathbb{R}^{n(n-1)} := \mathrm{vec}(V)$ is defined accordingly. Then the following lemma states the conditions under which $p(V)$ and $q(X)$ are continuous and differentiable in some neighborhood of (X^*, V^*).

ASSUMPTION 2. *The SDP relaxation* (3.2) *is feasible and bounded, i.e.,* $\forall X_k$, $|\langle X_k, Q_0\rangle| \le M < +\infty$.

LEMMA 3 (Theorem 2.1 in [7]). *Assume the conditions below are satisfied:*

(i) $f(X, V)$ *is twice differentiable in some neighborhood of* (X^*, V^*).

(ii) $A = f_{\boldsymbol{XX}}(X, V) = \frac{\partial^2 f}{\partial \boldsymbol{X}^2}$ *and* $C = f_{\boldsymbol{VV}}(X, V) = \frac{\partial^2 f}{\partial \boldsymbol{V}^2}$ *at* (X^*, V^*) *are both positive definite for all* $\{w_k\}$*; denote* $B = f_{\boldsymbol{XV}}(X, V) = \frac{\partial^2 f}{\partial \boldsymbol{X} \partial \boldsymbol{V}}$.

(iii) $X^* \in p(V^*)$ *and* $V^* \in q(X^*)$*. Then in some neighborhood of* (X^*, V^*), $p(V)$ *and* $q(X)$ *are continuously differentiable.*

Remark. Condition (iii) in Lemma 3 can be guaranteed by assuming that when Assumption 2 holds, $\forall X \in \{X | X \in \mathcal{C}, (V^*)^T X V^* = 0\}$, $\langle X, Q_0\rangle \ge \langle X^*, Q_0\rangle$ and $\forall X \in \{X | X \in \mathcal{C}, (V^*)^T X V^* \ne 0$, $w_k \ge \frac{2M}{\lambda_{n-1}((V^*)^T X V^*)}$, with M defined in Assumption 2. In other words, although w_k is varying, as long as it is large enough, condition (iii) will be satisfied; i.e., X^* will be the optimum of the X-update subproblem.

From (3.7), we define two maps $g : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ and $h : \mathbb{R}^{n(n-1)} \to \mathbb{R}^{n(n-1)}$ and then get

$$\mathbf{X}_k = g(\mathbf{X}_{k-1}) = p(\mathbf{V}_{k-1}) = p(q(\mathbf{X}_{k-1})),$$
$$\mathbf{V}_k = h(\mathbf{V}_{k-1}) = q(\mathbf{X}_k) = q(p(\mathbf{V}_{k-1})).$$

LEMMA 4. *With the conditions in Lemma* 3 *satisfied, the following two statements are equivalent:*

(i) $f''(\boldsymbol{X}^*, \boldsymbol{V}^*) = \left[\begin{smallmatrix} A & B \\ B^T & C \end{smallmatrix}\right]$ *is positive definite;*

(ii) $\rho(g'(\boldsymbol{X}^*)) = \rho(h'(\boldsymbol{V}^*)) < 1$, *where* $\rho(\bullet)$ *is the spectral radius of matrix* $\bullet$.

Proof. Beginning with the Schur complement,

$$\begin{aligned} \text{(i)} &\iff A, C \succ \mathbf{0}, C - B^T A^{-1} B \succ \mathbf{0} \\ &\iff C^{-1/2}(C - B^T A^{-1} B) C^{-1/2} \succ \mathbf{0} \\ &\iff I - C^{-1/2} B^T A^{-1} B C^{-1/2} \succ \mathbf{0} \\ &\iff \rho(C^{-1/2} B^T A^{-1} B C^{-1/2}) < 1 \\ &\iff \rho(C^{-1} B^T A^{-1} B) = \rho(A^{-1} B C^{-1} B^T) < 1, \end{aligned} \tag{3.8}$$

where the last equality comes from $\rho(PQ) = \rho(QP)$ for P and Q in appropriate dimensions. Differentiating (3.7) leads to $g'(\mathbf{X}^*) = p'q'(\mathbf{X}^*)$. Then differentiating $\frac{\partial f(\mathbf{X}, q(\mathbf{X}))}{\partial \mathbf{V}} = 0$ with respect to $\mathbf{X}$ and substituting $\mathbf{X}$ by $\mathbf{X}^*$ leads to

$$f_{\mathbf{V}\mathbf{V}} q'(\mathbf{X}^*) + B^T = C q'(\mathbf{X}^*) + B^T = 0.$$

Similarly, one has

$$f_{\mathbf{X}\mathbf{X}} p'(\mathbf{V}^*) + B = A p'(\mathbf{V}^*) + B = 0.$$

Combining the above three equations gives

$$g'(\mathbf{X}^*) = A^{-1} B C^{-1} B^T$$

and

$$h'(\mathbf{V}^*) = C^{-1} B^T A^{-1} B.$$

Based on (3.8), it then ends the proof. ∎

PROPOSITION 5 (local convergence). *Suppose* $(\boldsymbol{X}^*, \boldsymbol{V}^*)$ *is a local minimum of* (3.3) *satisfying the conditions in Lemma* 3. *Assume the Hessian of* $f(\boldsymbol{X}, \boldsymbol{V})$ *at* $(\boldsymbol{X}^*, \boldsymbol{V}^*)$ *is positive definite; then for any starting point* $(\boldsymbol{X}_k, \boldsymbol{V}_k)$ *in some neighborhood of* $(\boldsymbol{X}^*, \boldsymbol{V}^*)$, *the sequence* $\{\boldsymbol{X}_k, \boldsymbol{V}_k\}$ *generated by Algorithm* 1 *will converge to* $(\boldsymbol{X}^*, \boldsymbol{V}^*)$ *Q-linearly.*

Proof. Lemma 4 indicates that if the Hessian of the cost function, $f(\mathbf{X}, \mathbf{V})$, is positive definite at $(\mathbf{X}^*, \mathbf{V}^*)$, we have $\rho = \rho(g'(\mathbf{X}^*)) = \rho(h'(\mathbf{V}^*)) < 1$. For $\rho < 1$, there exists $\delta > 0$ such that $\rho + 2\delta < 1$. Theorem 3.8 in [32] shows that there exists a matrix norm $\| \bullet \|_\delta$ (depending on δ) such that

$$\|g'(\mathbf{X}^*)(\mathbf{X} - \mathbf{X}^*)\|_\delta \le (\rho + \delta)\|\mathbf{X} - \mathbf{X}^*\|_\delta. \tag{3.9}$$

Since $g'(\mathbf{X})$ is continuous in the neighborhood of $\mathbf{X}^*$, i.e., $N(\mathbf{X}^*) = \{\mathbf{X} | \|\mathbf{X} - \mathbf{X}^*\|_\delta \le \varepsilon\}$, then for any $\mathbf{X} \in N(\mathbf{X}^*)$, we have

$$\|g'(\mathbf{X})(\mathbf{X} - \mathbf{X}^*)\|_\delta \le (\rho + 2\delta)\|\mathbf{X} - \mathbf{X}^*\|_\delta. \tag{3.10}$$

Then, starting with $\mathbf{X}^* = g(\mathbf{X}^*)$, we have

$$
\begin{aligned}
||\mathbf{X}_{k+1} - \mathbf{X}^*\|_\delta &= \|g(\mathbf{X}_k) - g(\mathbf{X}^*)\|_\delta = \left\| \int_{\mathbf{X}^*}^{\mathbf{X}_k} g'(\mathbf{X}) d\mathbf{X} \right\|_\delta \\
&= \left\| \int_0^1 g'(t(\mathbf{X}_k - \mathbf{X}^*) + \mathbf{X}^*)(\mathbf{X}_k - \mathbf{X}^*) dt \right\|_\delta \\
&\leq \int_0^1 \left\| g'(t(\mathbf{X}_k - \mathbf{X}^*) + \mathbf{X}^*)(\mathbf{X}_k - \mathbf{X}^*) \right\|_\delta dt \\
&\leq (\rho + 2\delta) \|\mathbf{X}_k - \mathbf{X}^*\|_\delta, \qquad (3.11)
\end{aligned}
$$

where the third equality follows the transformation $\mathbf{X} := t(\mathbf{X}_k - \mathbf{X}^*) + \mathbf{X}^*$ with $0 \leq t \leq 1$. We will show below why the last inequality holds.

As $0 \leq t \leq 1$ and $\mathbf{X}_k \in N(\mathbf{X}^*)$, they indicate that $t(\mathbf{X}_k - \mathbf{X}^*) + \mathbf{X}^* \in N(\mathbf{X}^*)$. From (3.10), we have

$$
\|g'(t(\mathbf{X}_k - \mathbf{X}^*) + \mathbf{X}^*)(t(\mathbf{X}_k - \mathbf{X}^*))\|_\delta \leq (\rho + 2\delta) \|t(\mathbf{X}_k - \mathbf{X}^*)\|_\delta. \qquad (3.12)
$$

For $0 < t \leq 1$, by canceling the parameter t from both sides, it leads to

$$
\|g'(t(\mathbf{X}_k - \mathbf{X}^*) + \mathbf{X}^*)(\mathbf{X}_k - \mathbf{X}^*)\|_\delta \leq (\rho + 2\delta) \|\mathbf{X}_k - \mathbf{X}^*)\|_\delta. \qquad (3.13)
$$

On the other hand, for $t = 0$, (3.13) is also true according to (3.9). Then we end by showing that (3.13) is true for $0 \leq t \leq 1$ and it is exactly the last inequality in (3.11). The same scheme can be applied to prove the similar result of $\{\mathbf{V}_k\}$. As a result, we arrive at the conclusion that $\{\mathbf{X}_k, \mathbf{V}_k\}$ generated by Algorithm 1 will converge to $(\mathbf{X}^*, \mathbf{V}^*)$ Q-linearly. ∎

4. An extended Uzawa algorithm. Each iteration of the IRP method requires one to solve a convex optimization problem with linear matrix inequalities (LMIs). For large-scale QCQPs, the performance of the convex optimization solver determines the scalability and computational capability of the proposed IRP method. However, existing convex optimization solvers, such as SeDuMi [33] and SDPT3 [35], are time-consuming and not applicable to large-scale convex problems, especially to problems with a dimension larger than 100 and multiple LMIs. Therefore, an extended Uzawa algorithm is developed here to solve the convex optimization problem at each iteration of IRP.

The Uzawa algorithm was originally introduced to solve concave problems [21]. When strong duality holds for a primal-dual problem, the optimal solution is the saddle point of the Lagrangian. Therefore, the Uzawa algorithm is applied to iteratively approach the saddle point of the Lagrangian. Work in [12] has applied the Uzawa algorithm to matrix completion problems with linear scalar/vector constraints. An extended Uzawa algorithm is developed here for convex problems with both scalar and LMI constraints.

To make it general, we consider the following convex optimization problem:

$$
\begin{aligned}
\min_{x} \quad & f_0(\mathbf{x}) \\
\text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \ i = 1, \dots, N, \\
& \mathcal{A}_j(\mathbf{x}) \preceq \mathbf{0}, \ j = 1, \dots, J,
\end{aligned}
\qquad (4.1)
$$

where $\mathbf{x} \in \mathbb{R}^n$ are the unknown variables, $f_i(\mathbf{x}) \in \mathbb{R}$, $i = 0, \dots, N$, are convex functions, and $\mathcal{A}_j(\mathbf{x}) \in \mathbb{S}^{n_j}$, $j = 1, \dots, J$, are LMIs. For simplicity, we define $\mathcal{F}(\mathbf{x}) \in$

$\mathbb{R}^{l\times l} := \mathbf{diag}(f_1(\mathbf{x}),\ldots,f_N(\mathbf{x}),\mathcal{A}_1(\mathbf{x}),\ldots,\mathcal{A}_J(\mathbf{x}))$. Without loss of generality, we assume that $\mathcal{F}(x)$ is symmetric. In addition, as linear equality constraints can be written as a pair of linear inequality constraints, they are omitted in the formulation. The Lagrangian function for (4.1) is given by $\mathcal{L}(\mathbf{x},S) = f_0(\mathbf{x}) + \langle S, \mathcal{F}(\mathbf{x})\rangle$, where $S \in \mathbb{R}^{l\times l}$ is the dual matrix variable. For a convex problem in (4.1) satisfying Slater's condition, the strong duality holds such that the primal-dual optimal pair, $(\mathbf{x}^*, S^*)$, has the relationship such that $\sup_S \inf_{\mathbf{x}} \mathcal{L}(\mathbf{x},S) = \mathcal{L}(\mathbf{x}^*,S^*) = \inf_{\mathbf{x}} \sup_S \mathcal{L}(\mathbf{x},S)$.

Consequently, the saddle point of the Lagrangian is the optimal pair, $(\mathbf{x}^*, S^*)$, which can be determined via the Uzawa algorithm. As the initial value of Lagrangian multipliers is trivial for a convex problem, $S_0 = \mathbf{0}$ is selected as the starting point and the iteration procedure of the extended Uzawa algorithm is formulated as

$$\begin{aligned} \mathbf{x}_h &= \arg\min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, S_{h-1}), \\ S_h &= [S_{h-1} + \delta_h \mathcal{F}(x_h)]_+, \end{aligned} \tag{4.2}$$

where $\delta_h > 0$ is the step size at iteration h and the operator $[\bullet]_+$ is defined according to the data type of $\bullet$. For a vector $s \in \mathbb{R}^l$, $[s_i]_+ = \max(0, s_i)$, $i = 1,\ldots,l$, while for a matrix $S \in \mathbb{S}^l$ with eigenvalues $\lambda \in \mathbb{R}^l$ and corresponding eigenvectors $V \in \mathbb{R}^{l\times l}$, $[S]_+ = V\mathbf{diag}(\max(\lambda,0))V^T$, where $\max(\lambda,0)$ will replace negative value elements in λ with zeros. The following efforts focus on the convergence proof of the extended Uzawa algorithm stated in (4.2). The convergence proof of the Uzawa algorithm for convex problems with linear scalar/vector constraints can be found in [12, 13]. However, the statement below focuses on proving the convergence of the extended Uzawa algorithm developed to solve more general convex problems, including LMI constraints.

ASSUMPTION 6. *$\mathcal{F}(\mathbf{x})$ is Lipschitz continuous. Namely, for any $\mathbf{x}_1$ and $\mathbf{x}_2$, the inequality $\|\mathcal{F}(\mathbf{x}_1) - \mathcal{F}(\mathbf{x}_2)\|_F \leq L(\mathcal{F})\|\mathbf{x}_1 - \mathbf{x}_2\|_F$ holds for a nonnegative constant $L(\mathcal{F})$.*

ASSUMPTION 7. *The objective function $f_0(\mathbf{x})$ is strongly convex. As a result, there exists a positive constant $\xi > 0$ for a strictly convex $f_0(\mathbf{x})$ such that $\langle \nabla f_0(\mathbf{x}_h) - \nabla f_0(\mathbf{x}^*), \mathbf{x}_h - \mathbf{x}^*\rangle \geq \xi\|\mathbf{x}_h - \mathbf{x}^*\|_F^2$* [5].

We first establish a preparatory lemma for the convergence proof.

LEMMA 8. *Let $(\mathbf{x}^*, S^*)$ be an optimal primal-dual pair for* (4.1)*; then, for each $\delta \geq 0$, we have $S^* = [S^* + \delta\mathcal{F}(\mathbf{x}^*)]_+$.*

Proof. As $(\mathbf{x}^*, S^*)$ is an optimal primal-dual pair, $\langle S^*, \mathcal{F}(\mathbf{x}^*)\rangle = 0$, $S^* \succeq \mathbf{0}$, and $\mathcal{F}(\mathbf{x}^*) \preceq \mathbf{0}$. Additionally, based on the fact that matrices $\mathcal{F}(\mathbf{x}^*)$ and S^* are symmetric, $S^*\mathcal{F}(\mathbf{x}^*)$ is diagonalizable and can be written as

$$S^*\mathcal{F}(\mathbf{x}^*) = \mathcal{F}(\mathbf{x}^*)S^* = Q\Lambda_{S^*}\Lambda_{\mathcal{F}(\mathbf{x}^*)}Q^T = \mathbf{0},$$

where $\Lambda_{S^*}\Lambda_{\mathcal{F}(\mathbf{x}^*)} = \mathbf{0}$, $\Lambda_{\mathcal{F}(\mathbf{x}^*)} \preceq \mathbf{0}$, and $\Lambda_{S^*} \succeq \mathbf{0}$ [18]. Consequently, we can get

$$[S^* + \delta\mathcal{F}(\mathbf{x}^*)]_+ = Q[\Lambda_{S^*} + \delta\Lambda_{\mathcal{F}(\mathbf{x}^*)}]_+ Q^T = S^*.$$

When $\mathcal{F}(\mathbf{x}^*)$ is a scalar constraint, Lemma 8 degenerates to $\lambda^* = [\lambda^* + \delta\mathcal{F}(\mathbf{x}^*)]_+$, where $\lambda^* \in \mathbb{R}$ is the dual variable of $\mathcal{F}(\mathbf{x}^*)$. ∎

PROPOSITION 9. *Assuming problem* (4.1) *satisfies Assumptions* 6 *and* 7 *and the step size δ_h in* (4.2) *satisfies $0 < \mathrm{inf}\delta_\mathrm{h} \leq \mathrm{sup}\delta_\mathrm{h} < 2\xi/\|\mathrm{L}(\mathcal{F})\|^2$, where $\xi \in \mathbb{R}$ and*

$L(\mathcal{F}) \in \mathbb{R}$ are the parameters in the two aforementioned assumptions, respectively, then the sequence obtained from (4.2) *will converge to the global optimum of the convex problem* (4.1) *when its strong duality holds.*

Proof. At each iteration h of the extended Uzawa algorithm defined in (4.2), the solution $\mathbf{x}_h$ minimizes $\mathcal{L}(\mathbf{x}, S_{h-1}) = f(\mathbf{x}) + \langle S_{h-1}, \mathcal{F}(\mathbf{x})\rangle$; then the first order optimality condition is expressed as

$$\nabla f_0(\mathbf{x}_h) + \left[\langle S_{h-1}, \tfrac{\partial \mathcal{F}}{\partial(\mathbf{x}_h^1)}\rangle, \ldots, \langle S_{h-1}, \tfrac{\partial \mathcal{F}}{\partial(\mathbf{x}_h^n)}\rangle\right]^T = 0, \tag{4.3}$$

where the superscript denotes the index of vector $\mathbf{x}$. Moreover, as f_i, $i = 1, \ldots, N$, is a convex function and $\mathcal{A}_j$, $j = 1, \ldots, J$, is an LMI, the first order condition holds for the constructed $\mathcal{F}$ in the form

$$\mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{x}_h) \succeq \sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial(\mathbf{x}_h^i)}(\mathbf{x}^i - \mathbf{x}_h^i). \tag{4.4}$$

From (4.3) and (4.4), one can get

$$\langle \nabla f_0(\mathbf{x}_h), \mathbf{x} - \mathbf{x}_h\rangle + \langle S_{h-1}, \mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{x}_h)\rangle \geq 0. \tag{4.5}$$

At the optimal point, the inequality relationship stated in (4.5) will be satisfied as well when substituting $(\mathbf{x}_h, S_{h-1})$ by the primal-dual pair $(\mathbf{x}^*, S^*)$, which is expressed as

$$\langle \nabla f_0(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^*\rangle + \langle S^*, \mathcal{F}(\mathbf{x}) - \mathcal{F}(\mathbf{x}^*)\rangle \geq 0. \tag{4.6}$$

Since the inequality relationships in (4.5) and (4.6) hold for all $\mathbf{x}$, substituting $\mathbf{x}$ in (4.5) by $\mathbf{x}^*$ and (4.6) by $\mathbf{x}_h$ and then adding them gives

$$\langle \nabla f_0(\mathbf{x}_h) - \nabla f_0(\mathbf{x}^*), \mathbf{x}_h - \mathbf{x}^*\rangle + \langle S_{h-1} - S^*, \mathcal{F}(\mathbf{x}_h) - \mathcal{F}(\mathbf{x}^*)\rangle \leq 0. \tag{4.7}$$

From Assumption 7,

$$\langle \nabla f_0(\mathbf{x}_h) - \nabla f_0(\mathbf{x}^*), \mathbf{x}_h - \mathbf{x}^*\rangle \geq \xi \|\mathbf{x}_h - \mathbf{x}^*\|_F^2. \tag{4.8}$$

Combining (4.7) and (4.8) gives

$$\langle S_{h-1} - S^*, \mathcal{F}(\mathbf{x}_h) - \mathcal{F}(\mathbf{x}^*)\rangle \leq -\langle \nabla f_0(\mathbf{x}_h) - \nabla f_0(\mathbf{x}^*), \mathbf{x}_h - \mathbf{x}^*\rangle \leq -\xi\|\mathbf{x}_h - \mathbf{x}^*\|_F^2.$$

Since S_h in the extended Uzawa algorithm is updated via $S_h = [S_{h-1} + \delta_h \mathcal{F}(\mathbf{x}_h)]_+$, and Lemma 8 holds for δ_h such that $S^* = [S^* + \delta_h \mathcal{F}(\mathbf{x}^*)]_+$, then subtracting S^* from S_h gives

$$\|S_h - S^*\|_F^2 = \|[S_{h-1} + \delta_h \mathcal{F}(\mathbf{x}_h)]_+ - [S^* + \delta_h \mathcal{F}(\mathbf{x}^*)]_+\|_F^2. \tag{4.9}$$

The upper bound on $\|S_h - S^*\|_F^2$ is determined by

$$\begin{aligned} \|S_h - S^*\|_F^2 &\leq \|S_{h-1} - S^* + \delta_h(\mathcal{F}(\mathbf{x}_h) - \mathcal{F}(\mathbf{x}^*))\|_F^2 \\ &\leq \|S_{h-1} - S^*\|_F^2 + 2\delta_h\langle S_{h-1} - S^*, \mathcal{F}(\mathbf{x}_h) - \mathcal{F}(\mathbf{x}^*)\rangle \\ &\quad + \delta_h^2\|\mathcal{F}(\mathbf{x}_h) - \mathcal{F}(\mathbf{x}^*)\|_F^2 \\ &\leq \|S_{h-1} - S^*\|_F^2 - (2\xi\delta_h - \delta_h^2 L(\mathcal{F})^2)\|\mathbf{x}_h - \mathbf{x}^*\|_F^2. \end{aligned} \tag{4.10}$$

As it is assumed that $0 < \inf\delta_h \leq \sup\delta_h < 2\xi/\|L(\mathcal{F})\|^2$, there exists a $\beta > 0$ such that $2\xi\delta_h - \delta_h^2 L(\mathcal{F})^2 \geq \beta$ for all $h > 1$. Then (4.10) becomes

$$\|S_h - S^*\|_F^2 \leq \|S_{h-1} - S^*\|_F^2 - \beta\|\mathbf{x}_h - \mathbf{x}^*\|_F^2, \tag{4.11}$$

and thus the proposition is proved. ∎

Considering the aforementioned sequential problem formulated in (3.6), the extended Uzawa algorithm is applied to solve each iteration of IRP. As stated in problem (3.6), X_k and r_k are the unknown variables to be solved at iteration step k of IRP. Note that to satisfy the strong convexity assumption (Assumption 7) for the extended Uzawa algorithm, the objective in the IRP sequential problems is represented by a proximal strongly convex objective function, $J' = \langle X_k, Q_0\rangle + w_k r_k + \frac{1}{2\tau}(\|X_k\|_F^2 + r_k^2)$, with $\tau > 0$ being a relatively large number. The revised formulation of (3.6) using J' as the objective function satisfies Assumptions 6 and 7. With the regularization terms $\frac{1}{2\tau}(\|X\|_F^2 + r_k^2)$ added to the subproblem, we consider the objective in problem (3.1) as $\langle X, Q_0\rangle + \frac{1}{2\tau}\|X\|_F^2$ instead of $\langle X, Q_0\rangle$ and the penalty for subproblem (3.6) as $w_k r_k + \frac{1}{2\tau} r_k^2$ instead of $w_k r_k$. Since $r_k \to 0$ is proved for the IRP algorithm, $w_k r_k + \frac{1}{2\tau} r_k^2$ and $w_k r_k$ are equivalent at $r_k \to 0$ and the approximation only comes from $\frac{1}{2\tau}\|X\|_F^2$. With any constant τ assigned to the subproblem sequence, the proof of linear convergence rate for the IRP algorithm still applies, as adding these regularization terms will not violate the conditions stated in Proposition 5.

According to the extended Uzawa algorithm, at each iteration step h the unknown variables and Lagrangian multipliers in problem (3.6) are updated through (4.2). Moreover, due to the fact that in the subproblem the objective is quadratic and the constraints are linear, the primal update in (4.2) is to solve a simple linear system, and the dual update is also of closed form. Noteworthily, the initial values of all variables are set as zeros in the appropriate dimension, as the initial setting is trivial for a convex problem.

5. Simulation. To verify the feasibility and efficiency of the proposed IRP method and the extended Uzawa algorithm, two types of simulation examples are provided. The first one solves the mixed-boolean quadratic programming problems using the proposed IRP method where the sequential problem at each iteration is solved via the extended Uzawa algorithm. The second one applies the IRP method in an optimal attitude control problem to verify the effectiveness of IRP in real applications. All of the simulation is run on a desktop computer with a 3.50 GHz processor and a 16 GB RAM.

5.1. Mixed-boolean quadratic programming. In this subsection, the proposed IRP method is applied to solve mixed-boolean quadratic programming problems formulated as

$$\begin{aligned} \min \quad & \mathbf{x}^T Q_0 \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^T Q_i \mathbf{x} + c_i \le 0 \ \ \forall\, i = 1, \ldots, m, \\ & \mathbf{x}_j \in \{1, -1\} \ \ \forall\, j \in N_I,\ \mathbf{x}_l \le \mathbf{x}_k \le \mathbf{x}_u \ \ \forall\, k \notin N_I, \end{aligned} \tag{5.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, m is the number of inequality constraints, N_I is the index set of the integer variables, and $\mathbf{x}_l$ and $\mathbf{x}_u$ are the lower and upper bounds of the continuous variables, respectively. The matrices Q_0 and Q_i, $i = 1, \ldots, m$, are randomly generated, and they are not necessarily positive semidefinite. Since the bivalent constraint on an integer variable, $\mathbf{x}_j$, can be expressed as a quadratic equality constraint in the form of $(\mathbf{x}_j + 1)(\mathbf{x}_j - 1) = 0$, $j \in N_I$, problem (5.1) is converted into a nonconvex QCQP problem which can be solved by the proposed IRP method. The parameters in IRP are set as $n = 50$, $\alpha = 2$, and $\epsilon = 1e-5$.

The comparative results are obtained from the commercial optimization software, Tomlab mixed-integer nonlinear programming solver "minlpBB," which utilizes branch and bound to search for optimal solutions [23]. 50 random cases are generated and solved via both IRP and "minlpBB." For each case, the objective values obtained from both methods are recorded in Figure 1. After comparison, the objective value obtained from IRP is always smaller than the corresponding one computed from "minlpBB" for all of the 50 cases. These facts validate the advantages of IRP in solving nonconvex QCQP problems. Furthermore, the value of r_k, representing the second largest eigenvalue of the unknown matrix X, at each iteration is demonstrated in Figure 2 for one case. As r_k converges to a number close to zero within nine iterations, Figure 2 verifies the convergence of the IRP method to a rank one matrix. The other cases also yield zero r_k at the convergence point. To save space, the values of r_k for the other cases are not displayed here.

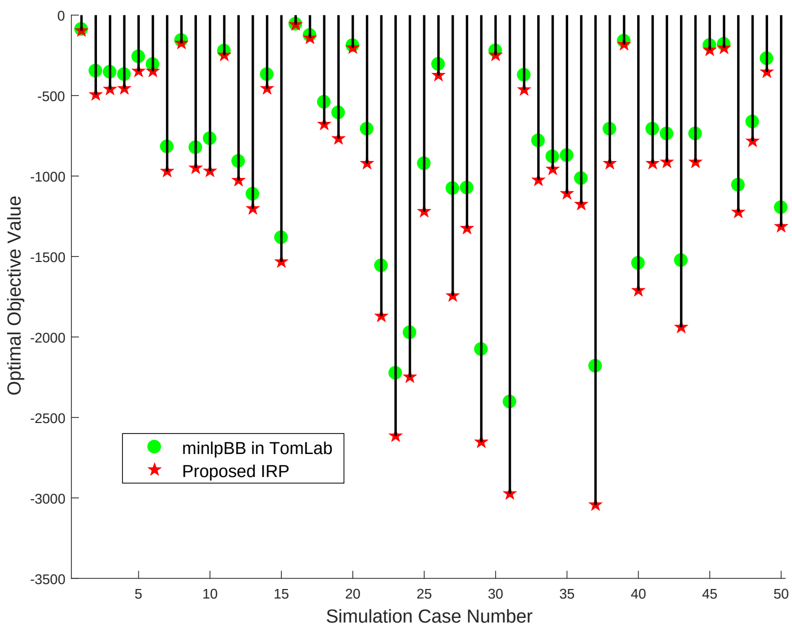

FIG. 1. *Comparative results between minlpBB and IRP for* 50 *cases.*

For each iteration of the IRP method, it will take the extended Uzawa algorithm an average of 4.6 seconds to solve the convex problem formulated in (3.6) for the 50 cases discussed above. Thus, the overall computation time using the combined IRP and extended Uzawa algorithms is around 27 seconds. However, it takes significantly increased computational time, around 10 times longer, to find the solution of each iteration of the IRP method using the "SeDuMi" convex optimization solver. Furthermore, the relative error, $\left|\frac{J(X_S^*)-J(X_U^*)}{J(X_S^*)}\right|$, between objective values from "SeDuMi," denoted by $J(X_S^*)$, and the extended Uzawa algorithm, denoted by $J(X_U^*)$, averages 0.41% for all 50 cases. The average computation time of the "minlpBB" solver is 2.1 seconds for the 50 cases. Moreover, we implement the three algorithms in problem (5.1) at different scales, ranging from $n = 10$ to $n = 150$. Figure 3 indicates that IRP with each iteration solved via Uzawa is much more efficient than IRP with each iteration solved via SeDuMi. Furthermore, the IRP-Uzawa algorithm is able to solve large-scale QCQPs, e.g., $n > 100$, while IRP-SeDuMi is not applicable. Although the existing "minlpBB" solver is faster than IRP-Uzawa for small-scale problems, e.g.,

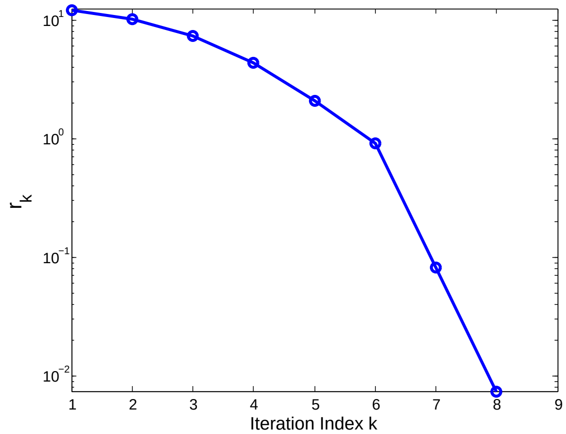

FIG. 2. *Value of r_k at each iteration for one selected case.*

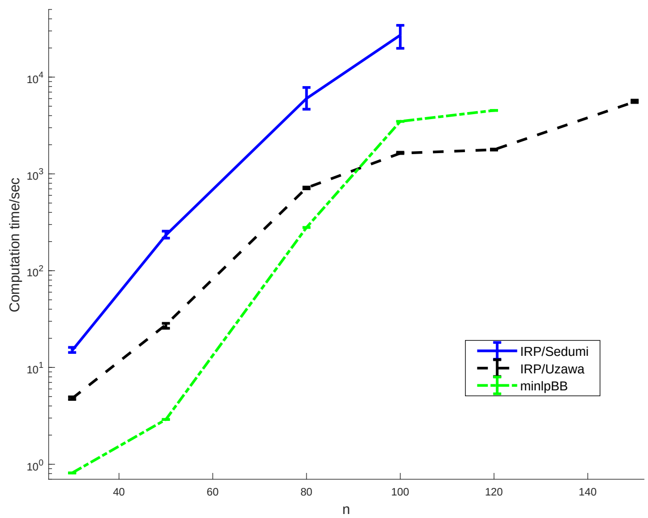

FIG. 3. *Computational time from three methods for problems of different scales.*

$n < 100$, IRP-Uzawa is also more efficient than "minlpBB" for larger-scale problems and "minlpBB" is not able to solve problems with $n > 150$ using a standard desktop.

5.2. Optimal attitude control. To further verify the effectiveness and feasibility of IRP in real applications, the optimal attitude control problem for spacecraft is considered here. The objective of the optimal attitude control problem is to find the optimal control torque $\mathbf{u} \in \mathbb{R}^3$ to maneuver the orientation of spacecraft with minimum control efforts while satisfying a set of constraints over time interval $t \in [t_0, t_f]$. The constraints include boundary conditions, rotational dynamics, unit quaternion kinematics, and attitude forbidden and mandatory zones.

The rotational dynamics of a rigid body is expressed as

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{u}, \tag{5.2}$$

where $\mathbf{J} = \mathbf{diag}(J_1, J_2, J_3)$ represents the moment of inertia matrix of the spacecraft in the body frame, $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3$ denotes the spacecraft angular velocity in the body frame, and $\mathbf{q} = [q_1, q_2, q_3, q_4]^T \in \mathbb{R}^4$, $\|\mathbf{q}\| = 1$ denotes the attitude in unit quaternions. The initial and terminal conditions on angular velocity and the attitude are assumed to be given. In addition, the magnitude of the control torque and the angular velocity is constrained by $|\mathbf{u}| \le \beta_u$, $|\boldsymbol{\omega}| \le \beta_\omega \, \forall \, t \in [t_0, t_f]$. The unit quaternion kinematics is determined by

$$\dot{\mathbf{q}} = \frac{1}{2}\boldsymbol{\Omega}\mathbf{q}, \tag{5.3}$$

where

$$\Omega(t) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}.$$

The spacecraft attitude during the orientation is constrained by a set of forbidden and mandatory zones. Work in [25] has formulated this type of attitude constraints as quadratic inequalities in the form of

$$\mathbf{q}^T M_f \mathbf{q} = \mathbf{q}^T \begin{bmatrix} A & b \\ b^T & d \end{bmatrix} \mathbf{q} \le 0, \tag{5.4}$$

where

$$\begin{aligned} A &= \mathbf{x}_F \mathbf{y}^T + \mathbf{x}_F \mathbf{y}^T - (\mathbf{x}_F^T \mathbf{y} + \cos\theta_F)\mathbf{I}_3, \\ b &= -\mathbf{x}_F \times \mathbf{y}, \;\; d = \mathbf{x}_F^T \mathbf{y} - \cos\theta_F. \end{aligned} \tag{5.5}$$

The above constraint prevents vector $\mathbf{x}_F$ from the forbidden zone with boresight vector $\mathbf{y}$ and constrained angle θ_F. Similar quadratic inequalities for mandatory zones can be established to constrain an angle between vectors $\mathbf{x}_M$ and $\mathbf{y}$ within a desired angle θ_M.

In summary, the optimal control problem to minimize the total control efforts for spacecraft reorientation subject to constraints can be formulated as

$$\begin{aligned} \min_{\mathbf{u},\omega,\mathbf{q}} \quad & \int_{t_0}^{t_f} \mathbf{u}^T \mathbf{u} \, dt \\ \text{s.t.} \quad & \mathbf{J}\dot{\boldsymbol{\omega}}(t) = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{u}, \\ & \dot{\mathbf{q}}(t) = \frac{1}{2}\boldsymbol{\Omega}(t)\mathbf{q}(t), \; \|\mathbf{q}\| = 1, \\ & \mathbf{q}^T M_{f_l} \mathbf{q} \le 0, \, l = 1, \dots, n', \\ & \mathbf{q}^T M_{m_s} \mathbf{q} \ge 0, \, s = 1, \dots, h', \\ & |\mathbf{u}_i| \le \beta_{u_i}, \, |\boldsymbol{\omega}_i| \le \beta_{\omega_i}, \, i = 1, 2, 3, \\ & \boldsymbol{\omega}(t_0) = \boldsymbol{\omega}_0, \, \boldsymbol{\omega}(t_f) = \boldsymbol{\omega}_f, \, \mathbf{q}(t_0) = \mathbf{q}_0, \, \mathbf{q}(t_f) = \mathbf{q}_f, \end{aligned} \tag{5.6}$$

where $\boldsymbol{\omega}_0$ and $\mathbf{q}_0$ represent initial angular velocity and attitude orientation, respectively, $\boldsymbol{\omega}_f$ and $\mathbf{q}_f$ represent final angular velocity and attitude orientation, respectively, and n' and h' represent the number of forbidden zones and mandatory zones, respectively.

By utilizing the discretization technique, the optimal attitude control problem formulated as an NLP problem in (5.6) can be transformed into a nonconvex QCQP problem in the form of (2.2). One simulation result is demonstrated here to reorient the spacecraft with minimum total control efforts while preventing its telescope pointing vector from the three forbidden zones and keeping the antenna vector in the mandatory zone within $t \in [0, 20]$ seconds. Three forbidden zones are randomly selected without overlapping each other but may overlap with the mandatory zone. In addition, both initial and terminal attitude are properly selected to prevent violation of the attitude constraints. The spacecraft is assumed to carry a light-sensitive telescope with a fixed boresight vector $\mathbf{y}_t$, defined as $\mathbf{y}_t = [0, 1, 0]^T$, while the boresight vector of the antenna is set as $\mathbf{y}_a = [0, 0, 1]^T$, both in the spacecraft body frame. The other simulation parameters are given in Table 1.

TABLE 1
Simulation parameters for the optimal attitude control problem.

Parameter	Value
J	diag[54,63,59] kg· m^2
$\lvert\omega_i\rvert, i = 1, 2, 3$	$\leq$ 0.3 rad/s
$\lvert u_i\rvert, i = 1, 2, 3$	$\leq$ 3 rad/s^2
Initial attitude $\mathbf{q}_0$	$[0.82,0.52,–0.12,–0.23]^T$
Terminal attitude $\mathbf{q}_f$	$[0.275386,–0.51,–0.78,–0.24]^T$
Mandatory zone 1	$\mathbf{x}_{M_1} = [-0.81, 0.55, -0.19]^T$, $\theta_{M_1} = 70°$
Forbidden zone 1	$\mathbf{x}_{F_1} = [0, -1, 0]^T$, $\theta_{F_1} = 40°$
Forbidden zone 2	$\mathbf{x}_{F_2} = [0, 0.82, 0.57]^T$, $\theta_{F_2} = 30°$
Forbidden zone 3	$\mathbf{x}_{F_3} = [-0.12, -0.14, -0.98]^T$, $\theta_{F_3} = 20°$

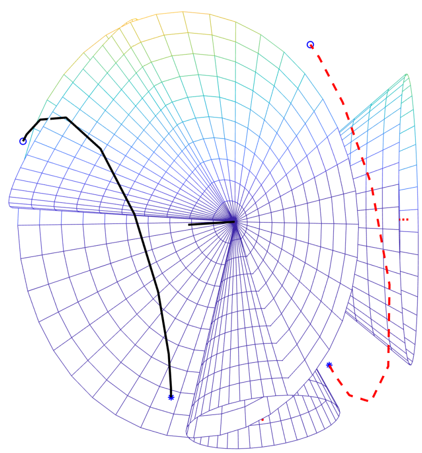

FIG. 4. *Trajectory of the telescope pointing vector (dashed line) and the antenna point vector (solid line) in three-dimensional space. The star and circle represent the initial and terminal orientations, respectively. The cone with the boresight vector in a solid line represents the mandatory zone, and the other three with the boresight vectors in dotted lines are the forbidden zones.*

Figure 4 presents the trajectories of the telescope pointing vector and the antenna pointing vector in the constrained three-dimensional space with 10 discrete nodes. The value of r_k at each iteration is provided in Figure 5, which indicates that r_k converges

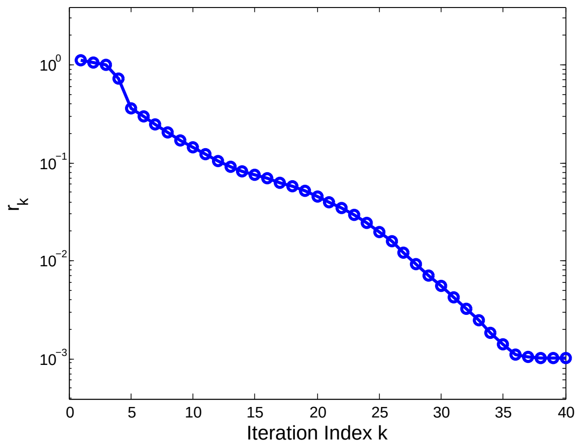

FIG. 5. *Value of r_k at each iteration for the attitude control problem.*

to zero within a few iterations. Furthermore, we find comparative results solved via the commercial NLP solver, SNOPT [23]. Depending on the initial guess of the unknown variables, the NLP solver cannot guarantee a convergent solution. When a group of initial guesses is randomly generated, the convergent solutions from the NLP solver lead to two sets of objective values, 31.79 and 76.92. However, the objective value found from IRP is 23.72, which reduces 25.39% compared to the smallest objective value obtained from an NLP solver. This simulation example verifies the feasibility of implementing IRP in a real optimal control problem. It takes the IRP 781.6 seconds to generate the optimal solution, while the NLP solver takes an average of 8.4 seconds for convergent cases.

6. Conclusions. This paper proposes an iterative rank penalty (IRP) method to solve general/nonconvex quadratically constrained quadratic programming (QCQP) problems. The subproblem at each iteration of IRP is formulated as a semidefinite programming (SDP) problem, and an extended Uzawa algorithm, based on the augmented Lagrangian method, is developed to improve the scalability and computational efficiency in solving large-scale SDPs at each iteration of IRP. Theoretical analysis on the convergence of the proposed IRP method and the extended Uzawa algorithm is discussed. The effectiveness and improved performance of the proposed approach are verified by different types of simulation examples.

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