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Characteristics Optimization via Compressed Sensing in Quantum State Estimation

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Abstract—**Compressed sensing (CS) has been proved efficiently in reducing the required number of measurements in the quantum state estimation. In this paper, the relationship between the optimal number of measurement settings and the error of quantum state estimation via compressed sensing are studied by means of analyzing the lower bound of measurement settings and upper bound of error of quantum state estimation. Based on the results obtained by the CS theory, we analyze the above mentioned characteristics of three observation matrices and give the performance comparisons through the numerical simulations. The obtained optimal conditions and error bounds in the state estimation process can provide theoretical instructions for the selection of the minimum number of measurement settings in the quantum state estimation.**

I. Introduction

The state of an n-qubit quantum system can be described as a density matrix $\boldsymbol{\rho}$ with the dimension size $d \times d$ in a Hilbert space, where $d = 2^n$. The density matrix $\boldsymbol{\rho}$ is a Hermitian matrix whose elements present the measurement probabilities in each possible state. The quantum state can't be measured directly [1]. However, the quantum state information can be obtained by the projection measurements of the quantum state. One can use the measurement data of the projection to estimate the density matrix. The number of parameters $d \times d$ of $\boldsymbol{\rho}$ implies that at least $O(d^2)$ measurement settings are needed to estimate all elements of $\boldsymbol{\rho}$ correctly. For example, to estimate the $\boldsymbol{\rho}$ of an 8-qubit system, it needs at least $d \times d = 2^8 \times 2^8 = 65536$ measurement settings. The post processing of these data may take several weeks in MATLAB on the computer with 2 cores of 2.4 GHz Intel Xeon E5-2407 CPUs. This is obviously impractical [2]. The compressed sensing theory was proposed by Donoho and Candes in 2006 [3-4,18]. The core idea of compressed sensing is if a signal is sparse in some orthogonal space, then the signal can be reconstructed accurately from a small number of measurements, which means the signal can be decomposed to a vector that contains only a few nonzero entries in a specified basis. Furthermore, the compressed sensing theory can be extended from sparse vector recovery to low rank matrix recovery [6-7], because the singular value vector of a low rank matrix is sparse. As a matter of fact, the quantum states that people are often interested in are pure states or nearly pure states [8]. More precisely, for most useful quantum states, the row of $\boldsymbol{\rho}$ is essentially supported on an r-dimensional space, which means the density matrix $\boldsymbol{\rho}$ is close (in a given norm) to a matrix with rank less than or equal to r, where $r \sim O(1)$ is a small positive integer. Such states are very common in many physical systems. For example, a pure state subject to a local noise process [8]. In such a case, the compressed sensing theory can be applied to the quantum state estimation. This is feasible because the rank of the density matrix is low or nearly low, which satisfies the requirements of compressed sensing for the signal recovery. On the other hand, several algorithms have been developed for the quantum state estimation. Smith [9] summarized the least square (LS) problem and solved it by using MATLAB toolbox. Liu [10] converted the estimation problem to the Dantzig format and used a convex optimization technique to solve it. Li [11] applied the Alternating Direction Method of Multipliers (ADMM) algorithm to quantum state estimation, and reached a solution with good accuracy.

In this paper, we study the reduction of the number of measurement settings via CS. Based on the existing compressed sensing theoretical results we analyze and summarize the relationship between the optimal number of measurement settings and the error bounds. We also study the precondition of different theorems and provide the theoretical instructions for the selection of the minimum number of measurement settings in the quantum state estimation.

This paper is organized as follows. In Section II, we analyze and summarize the relationship between the optimal number of measurement settings and the error bounds in different preconditions for quantum state estimation via CS. In Section III, we use numerical simulations to verify the theoretical optimal minimum number of measurement settings. Finally the conclusion is summarized in Section IV.

II. Analysis of Optimal Number of Measurement Settings and Error Bounds

For a low rank density matrix, people can reduce the number of measurement settings (or measurement rate) in

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quantum state estimation. For an r-rank matrix, the number of unknown parameters dropped from the order of $d \times d$ to $r \times d$, that is $O(rd)$. Then it is reasonable to guess that $O(rd)$ measurement settings could possibly suffice to estimate the quantum state. In recent years, compressed sensing is utilized to reduce the number of measurement settings in quantum state estimation [12] [8]. Based on the prior information that the rank of the density matrix is low, the quantum state estimation can be viewed as a process of compressed sensing reconstruction of solving a convex optimization problem:

$$\min \|\boldsymbol{\sigma}\|_* \quad \text{s.t.} \quad \mathbf{A} \cdot \text{vec}(\boldsymbol{\sigma}) = \mathbf{A} \cdot \text{vec}(\boldsymbol{\rho}) \tag{1}$$

where $\|\cdot\|_*$ is the nuclear norm and equals to the sum of singular values. $\mathbf{A}$ is called as measurement matrix or sampling operator. M denotes the number of measurement settings, and it represents the number of observables. $\mathbf{A}$ is a $M \times d^2$ matrix. It's a linear map that takes values from $d \times d$ to $M \times 1$, $M \square d^2$, vec(·) represents the transformation from a matrix to a vector by stacking the matrix's columns in order on the top of one another.

If one wants to solve the problem (1) successfully by means of compressed sensing for low rank matrices, there are two theories need to be considered: 1) rank Restricted Isometry property (RIP) [5]. 2) Dual Certification [12]. There are some research results of these two theories which can be applied into the quantum state estimation.

A. Rank Restricted Isometry property (RIP)

The rank RIP was proposed by Recht [13]. It is a property of measurement matrix to guarantee that $O(rd \log d)$ measurement settings will be sufficient to reconstruct the density matrix [13] exactly if the measurement matrix is a nearly isometric random matrix. Under special cases, for example, if the measurement matrix is Gaussian ensemble, then only $O(rd)$ measurement settings will be enough to estimate the density matrix accurately [14].

The geometrical interpretation of rank RIP is: whether the manifold of rank-r matrices in $C^{d \times d}$ can be mapped into a M - dimensional space, with low distortion in the 2_norm, where M is far less than d^2. Such a linear map A is said to satisfy rank RIP. The r-restricted isometry constant δ_r is used to describe the maximum distortion degree in the map, which is the smallest number let $(1-\delta_r)\|\mathbf{X}\|_F \le \|\mathrm{A}(\mathbf{X})\|_F \le (1+\delta_r)\|\mathbf{X}\|_F$ hold for all matrices $\mathbf{X}$ of rank at most $r, \delta_r \in (0,1)$, in which $\|\cdot\|_F$ is the Frobenius norm. Similarly, δ_{2r} is the smallest number for all matrices of rank at most 2r, By definition if $r \le r'$, $\delta_r \le \delta_{r'}$, which means $\delta_r \le \delta_{2r}$. The rank RIP gives that if the measurement matrix satisfies rank RIP and the r-restricted isometry constant δ_r is small enough, e.g. $\delta_{5r} < 1/10$, then the solution of the convex optimization problem (1) is unique and equal to the density matrix.

In fact, it is a very difficult process to prove whether the map constituted by a measurement matrix satisfies rank RIP or not. People have not yet found a suitable measurement matrix which satisfies rank RIP. For this reason, nearly isometric random matrix was proposed which satisfies rank RIP with very high probability.

In CS, there are two ingredients for a measurement matrix $\mathbf{A}$ to be a nearly isometric random matrix. The first one is it must be isometric in expectation. Specifically, for any $d \times d$ matrix to be estimated X, there is $\mathbf{E}(\|\mathbf{A} \cdot \text{vec}(X)\|_F^2) = \|X\|_F^2$. The second, the probability of large distortions of length must be exponentially small, which means for all X and $0 < \varepsilon < 1$ [13], there is

$$\mathbf{P}\left(\left|\|\mathbf{A} \cdot \text{vec}(X)\|_F^2 - \|X\|_F^2\right| \ge \varepsilon \|X\|_F^2\right) \le 2\exp\left(-\frac{M}{2}\left(\frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{3}\right)\right)$$

There are some examples of nearly isometric random matrix that have been proved: the ensemble with **independent identically distributed** (short for i.i.d.) Gaussian entries: $\mathbf{A}_{ij} \sim \mathrm{N}(0, \frac{1}{M})$, entries sampled from an i.i.d. symmetric Bernoulli distribution:

$$\mathbf{A}_{ij} = \begin{cases} \sqrt{\frac{1}{M}} & \text{with probability } \frac{1}{2} \\ -\sqrt{\frac{1}{M}} & \text{with probability } \frac{1}{2} \end{cases} \tag{2}$$

In addition, it is demonstrated that when the measurement matrix $\mathbf{A}$ is a nearly isometric random matrix and fix $\delta \in (0,1)$, there exist positive constants c_0 and c_1 depending only on δ such that if the number of measurement settings satisfies [13]:

$$M \ge 4c_0 rd \ln d \tag{3}$$

Then, the measurement matrix $\mathbf{A}$ satisfies rank RIP, and the least probability P is:

$$\mathrm{P} \ge 1\text{-}\exp(\text{-}c_1 \cdot M) \tag{4}$$

It can be observed from (3) that the lower bound of the number of measurement settings is $4c_0 rd \log d$ when the measurement matrix is nearly isometric, which satisfies rank RIP with the probability near to 1.

Furthermore, the upper bound of errors in problem (1) can be found when nearly isometric random matrices are adopted as measurement matrices. Denote $\hat{\boldsymbol{\rho}}$ is the optimal solution of (1). The density matrix $\boldsymbol{\rho}$ is low rank or nearly low rank. The error can be described in the Frobenius norm form $\|\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}\|_F$. We set $\boldsymbol{\rho}_r$ to be the matrix of rank r that that best approximates $\boldsymbol{\rho}$ in nuclear norm. That is, $\boldsymbol{\rho}_r$ is the optimal solution of the problem: $\arg\min_{\boldsymbol{\rho}_r} \|\boldsymbol{\rho} - \boldsymbol{\rho}_r\|$ s.t. $\text{rank}(\boldsymbol{\rho}_r) \le r$. If the measurement

matrix $\mathbf{A}$ satisfies rank RIP with the *r*-restricted isometry constant $\delta_{5r} < 1/10$, and consider the noise during the process of measurement, that means the constraint becomes $\left\|\mathbf{A}\cdot\mathrm{vec}(\boldsymbol{\sigma})-\mathbf{A}\cdot\mathrm{vec}(\boldsymbol{\rho})\right\|_F \le \varepsilon$, and the upper bound of errors can be written as [15]:

$$\left\|\hat{\boldsymbol{\rho}}-\boldsymbol{\rho}\right\|_F \le c_0\cdot\frac{\left\|\boldsymbol{\rho}-\boldsymbol{\rho}_r\right\|_*}{\sqrt{r}}+c_1\cdot\varepsilon \tag{5}$$

where c_0 and c_1 are two small constants.

The right side of (5) is the sum of two terms: an approximation term $c_0\left\|\boldsymbol{\rho}-\boldsymbol{\rho}_r\right\|_*/\sqrt{r}$, and a measurement-error term $c_1\cdot\varepsilon$ which is proportional to the noise level ε. If $\boldsymbol{\rho}$ is pure state or nearly pure state, then $\boldsymbol{\rho}=\boldsymbol{\rho}_r$, and the approximation term becomes zero. The measurement-error term is also zero if perfect noiseless data is used. According to (5), it's easy to see that if pure state or nearly pure state and noiseless measurement are considered, the two terms of (5) will vanish and the optimal solution of (1) is exactly equal to the density matrix.

In particular, when the measurement matrix $\mathbf{A}$ is a Gaussian measurement ensemble, lower bound of the number of measurement settings (3) can be further reduced. That is, if each "row" $\mathbf{A}_i, 1\le i\le M$ contain the i.i.d. $\mathrm{N}(0,\frac{1}{M})$ entries, and

$$\mathbf{A}_i \sim \mathrm{N}(0,\frac{1}{M}) \tag{6}$$

where the $\mathbf{A}_i$'s are independent from each other.

In this case, fix a constant δ, $\delta\in(0,1)$, there exists a constant D, and lower bound of the number of measurement settings becomes [14]:

$$M \ge D\cdot r\cdot d \tag{7}$$

where $\mathrm{D}\ge 2\ln 36\sqrt{2}-9$ [7], then the measurement matrix satisfies rank RIP with isometric constant $\delta_r\le\delta$, and the probability can achieve:

$$\mathrm{P}\ge 1\text{-}c\cdot\exp(\text{-}d\cdot M) \tag{8}$$

where c is a positive constant.

People can use different estimators (methods for reconstructing the unknown density matrix) to reconstruct the density matrix. In this paper, we use the estimator of matrix LASSO:

$$\min_{\rho}\ \left\|\boldsymbol{\rho}\right\|_*+\frac{\mathrm{u}}{2}\left\|y-\mathbf{A}\cdot\mathrm{vec}(\boldsymbol{\rho})\right\|_F^2 \tag{9}$$

where $y=\mathbf{A}\,\mathrm{vec}(\boldsymbol{\rho})$ is the output of measurement procedure, and y is an observation vector; $\mathrm{u}>0$ is a constant, and $\left\|\cdot\right\|_F$ is the Frobenius norm.

Let $\hat{\boldsymbol{\rho}}$ be the solution to the problem (9). If the isometric constant of the measurement matrix $\mathbf{A}$ satisfies $\delta_{4r}\le(3\sqrt{2}-1)/17$ and $\left\|\mathbf{A}^*(y-\mathbf{A}\cdot\mathrm{vec}(\boldsymbol{\rho}))\right\|\le\mu/2$, where μ usually set as $16\sqrt{d\cdot\varepsilon}$; ε is the noise in (5); $\left\|\cdot\right\|$ is the operator norm which equals to the largest singular value. Then, upper bound of error becomes:

$$\left\|\hat{\boldsymbol{\rho}}-\boldsymbol{\rho}\right\|_F^2 \le c_1 r\mu^2 \tag{10}$$

From the above analysis, one can find that if the measurement matrix $\mathbf{A}$ satisfies rank RIP with a small isometric constant, the density matrix can be reconstructed accurately. If the measurement matrix is nearly isometric random matrix which means satisfies rank RIP with very high probability, then a certain number of measurement settings can be ensured to reconstruct the density matrix with the probability of approaching 1. The use of rank RIP can be very convenient for the error analysis. On the other hand, rank RIP is a very restrictive property. It can be sufficient to analyze the lower bound of the number of measurement settings and the upper bound of error, but it is difficult to verify whether a measurement matrix satisfies rank RIP. In practical applications, people can only use the measurement matrices that have already been proved to satisfy rank RIP or nearly satisfy rank RIP. But such matrices are few, and what can be used in quantum systems is less. People want to have a simple theory to determine whether a matrix can be used as a measurement matrix. The dual certification provides a method of constructing measurement matrix.

B. Dual Certification

In fact, the measurement matrix $\mathbf{A}$ is an incomplete basis which is made of a set of non collinear vectors $\{\mathbf{A}_i, 1\le i\le M\}$ in d^2 dimensional space. In other words, the measurement matrix can be constructed by a subset of a complete basis. There is a method to construct a measurement matrix: choose a random subset of size M from a complete orthonormal basis $\{\omega_a\}_{a=1}^{d^2}$ in $d\times d$ dimensional space, and then 'vectorize' the subset as $\mathbf{A}_i=\mathrm{vec}(\omega_a), i=1,\ldots M$ to form a measurement matrix, where ω_a belongs to the subset. According to Gross's research [16], if the number of measurement setting satisfies:

$$M \ge v(1+\beta)rd\ln^2 d \tag{11}$$

then the solution of the optimization problem (1) is unique and equal to the density matrix $\boldsymbol{\rho}$ with reconstruction probability at least

$$P \ge 1-d^{-\beta} \tag{12}$$

where β is a positive constant which ensures a balance between the lower bound of the number of measurement settings and the probability of recovering the density

matrix exactly. v is $\boldsymbol{\rho}$'s coherence with respect to the orthonormal basis. The definition of coherence v is the minimum value that let

$$\max_a \left\|\omega_a\right\|^2 \le v\frac{1}{d} \tag{13}$$

hold. Obviously, $v \in [1, d]$. v has the minimum value when the singular values are uniformly distributed and equals to the maximum value when the singular values focus on one dimension.

Initially, the matrices used as measurement matrix are the Gauss random matrix, Bernoulli random matrix, partial Fourier matrix etc.. In quantum state estimation, people usually use Pauli matrices to construct measurement matrix which is of the form $w(k) = \otimes_{i=1}^{n} w_i$, where $w_i \in \{\mathrm{I}, \sigma_x, \sigma_y, \sigma_z\}$, $k \in \{1, 2 \cdots d^2\}$. $\otimes$ is the Kronecker product [8]. The process of constructing the measurement matrix is as follows: choose M integers $A_1, A_2 \cdots A_M \in [1, d^2]$ at random, then one has:

$$\mathbf{A} = \begin{pmatrix} \mathrm{vec}(w(A_1))^T \\ \mathrm{vec}(w(A_2))^T \\ \vdots \\ \mathrm{vec}(w(A_M))^T \end{pmatrix} / \sqrt{d} \tag{14}$$

Does the measurement matrix constructed by Pauli matrices satisfy the rank RIP? This is a problem can't be solved until 2010, Liu [10] proved that if the number of measurement settings satisfies

$$M \sim O(rd \log^6 d) \tag{15}$$

then the measurement matrix in (14) satisfies the rank RIP with the probability approaching to 1. For a $w(k)$, $\sqrt{d} \cdot w(k)$ is a unitary matrix. The singular value decomposition of $\sqrt{d} \cdot w(k)$ is $\sqrt{d} \cdot w = (\sqrt{d} \cdot w) \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}^{\dagger}$, in which the superscript † indicates conjugate transpose.

Replace ω_a with $w(k)$ in (13), one has $v = 1$. According to the condition (11), for the Pauli measurements if the number of measurement settings satisfies:

$$M \ge r(1+\beta) d \ln^2 d \tag{16}$$

Then the solution of (1) is unique and equal to $\boldsymbol{\rho}$ with probability at least $1 - d^{-\beta}$.

It can be seen from (15) and (16) that, for Pauli measurement, the minimal M of dual certification is less than the minimal M of rank RIP. But it is very difficult to use the dual certification theory to do individual error analysis. In quantum state estimation via compressed sensing by using Pauli measurements, people usually use the dual certification theory to explore the number of measurement settings of the lower error bound, and use the rank RIP theory to analyze the upper bound of error. It can be proved that the upper bound of error when Pauli measurements are used is [12]:

$$\left\|\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}\right\|_* \le c_0 \cdot ru + c_1 \cdot \left\|\boldsymbol{\rho} - \boldsymbol{\rho}_r\right\|_* \tag{17}$$

where c_0 and c_1 are positive constants depending only on isometric constants.

The upper bound of error in quantum state estimation via CS describes the maximum distortion between the reconstructed density matrix and the true density matrix. While in a specific reconstruction, the reconstruction performance is evaluated by the normalized error which defines as

$$\text{error} = \frac{\left\|\boldsymbol{\rho}^* - \hat{\boldsymbol{\rho}}\right\|_F^2}{\left\|\boldsymbol{\rho}^*\right\|_F^2} \tag{18}$$

where $\hat{\boldsymbol{\rho}}$ is the estimated state and $\boldsymbol{\rho}^*$ is the true state. In numerical simulation, $\boldsymbol{\rho}^*$ is generated form normalized Wishart random matrices with form as [18]:

$$\boldsymbol{\rho}^* = \frac{\Psi_r \cdot \Psi_r{}^{\dagger}}{\mathrm{tr}(\Psi_r \cdot \Psi_r{}^{\dagger})} \tag{19}$$

where Ψ_r is a complex $d \times r$ matrix with i.i.d. complex random Gaussian entries. The density matrix constructed by (19) satisfies the constraint that the trace of density matrix equal to 1.

Define the measurement rate as

$$\eta = M / d^2 \tag{20}$$

which indicates that the smaller the η is, the smaller the number of measurement settings is.

C. Numercial Analysis

Now we analyze the relationship between the measurement rate and the probability of exact reconstruction by using three different measurement matrices: measurement matrix constructed by 1) Pauli matrices (14); 2) nearly isometric random matrix (2); 3) Gaussian measurement ensemble (6). It can be observed from (16), (12) and (3), (4) and (7), (8) that the greater the number of measurement settings M is, the higher the exactly reconstruction probability is. We set the exactly reconstruction probability to 0.99 which is a fair probability to say close to 1 to explore the minimum measurement rates under different measurement matrices.

1) Measurement matrix constructed by Pauli matrices. For example, to a quantum system with the

qubit $n=5$ $d=2^n=32$, set $r=1$, $\beta=1.32878$, $1-d^{-\beta}=0.99$, and $\mathrm{P}\geq 0.99$. According to (16), (20) the measurement rate η can be calculated as:

$$\eta=\frac{(1+\beta)rd\ln^2 d}{d^2}\approx 0.87 \tag{21}$$

Equation (21) shows that, with reconstruction probability greater than 0.99, η is required at least 0.87 to reconstruct the density matrix exactly. Similarly, when $n=6, n=7$, η is required at least 0.57 and 0.36 to reconstruct the density matrix.

2) Nearly isometric random matrix. According to (3), (20), (4), and set $c_0=c_1=1$, $r=1$, when $n=5$, 6 and 7, respectively, it can be calculate that, $\eta=\frac{4rd\ln d}{d^2}$, the measurement rate at least be 0.43, 0.26, 0.15 to reconstruct the density matrix with probability greater than 0.99.

3) Gaussian measurement ensemble. According to (7), (20), (8), when $n=5$, 6 and 7, respectively, it can be calculate that, $\eta=\frac{Drd}{d^2}$ the measurement rate at least be 0.28, 0.14, 0.07 to reconstruct the density matrix exactly with reconstruction probability greater than 0.99.

We can conclude that to obtain same probability to reconstruct the density matrix exactly, the least measurement rate in three measurement matrices is Gaussian measurement matrix, and the most is the measurement matrix constructed by Pauli matrices. The minimum measurement rates to reconstruct the density matrix exactly with probability greater than 0.99 are showed in Table I.

TABLE I. MINIMUM MEASUREMENT RATE WITH RECONSTRUCTON PROBABILITY GREATERE THAN 0.99

Measurement Matrix	Measurement Rate		
	n = 5	*n= 6*	*n = 7*
Pauli matrices	0.87	0.57	0.36
Nearly isometric random matrix	0.43	0.26	0.15
Gaussian measurement ensemble	0.28	0.14	0.07

III. NUMERICAL SIMULATION

In this Section, we carry out the experiment to validate the theoretical lower bounds of the number of measurement settings M when $n=6$ under three different measurement matrices by using ADMM algorithm. It's also to validate the lower bound of measurement rate using $M=\eta\cdot d^2$ derived by (20). ADMM algorithm [11] is an optimization algorithm which can solve the quantum state estimation via CS problem efficiently and accurately. The theoretical lower bounds were presented in (3), (7) and (11). Furthermore, we answer the question: in order to get an fixed estimation accuracy of 95% in practice, how many measurement settings exactly are required?

The measurement matrices we choose are: 1) measurement matrix constructed by Pauli matrices; 2) Entries sampled from an i.i.d. symmetric Bernoulli distribution shown in (2) which is a nearly isometric random matrix; 3) Gaussian measurement ensemble shown in (6). The measurement rate increases from $\eta=0.05$ to $\eta=0.5$ and the incremental step size is $\Delta\eta=0.05$. Under each measurement rate, the algorithm runs the measurement and reconstruction 3 times and the estimation error is the mean value of the 3 normalized estimation errors, and 100 iterations in every reconstruction.

Fig. 1 depicts the normalized estimation errors with different measurement rates under 3 measurement matrices in which the star red line, circle black line and cross blue line represent the measurement matrix constructed by Pauli matrices, nearly isometric random measurement matrix, and Gaussian measurement ensemble measurement matrix, respectively. Blue dash-dotted line represents the normalized estimation error is 0.05. If the normalized estimation error is greater than 1, we record it as 1.

Fig. 2 is the exact reconstruction probability with different measurement rates under 3 measurement matrices in the case $n=6$. The exact reconstruction means the normalized estimation error equals to zero. The star red line, circle black line and cross blue line represent the measurement matrix constructed by Pauli matrices, nearly isometric random measurement matrix, and Gaussian measurement ensemble measurement matrix, respectively.

1) It can be seen from Table I that, for the measurement matrix constructed by Pauli matrices, when the measurement rate η is 0.57, the density matrix can be constructed exactly with probability close to 1; for the nearly isometric random measurement matrix, theoretically, the density matrix can be reconstructed exactly with probability nearly 1 when η is about 0.26; for the Gaussian measurement ensemble measurement matrix, theoretically, the density matrix can be reconstructed exactly with probability nearly 1 when η is about 0.14

2) One can observe from the star red line of Fig. 1 that for the measurement matrix constructed by Pauli matrices, error is nearly zero when η is about 0.35, which means about 35% of measurements would be enough to reconstruct the density matrix exactly; for the nearly isometric random measurement matrix, the circle line of Fig. 2 shows that error is nearly zero when η is about 0.3. for the Gaussian measurement ensemble measurement matrix, the cross blue line in Figs. 1 and 2 indicate that the measurement rate η need to be about 0.15 to make sure the error approach to zero in practice.

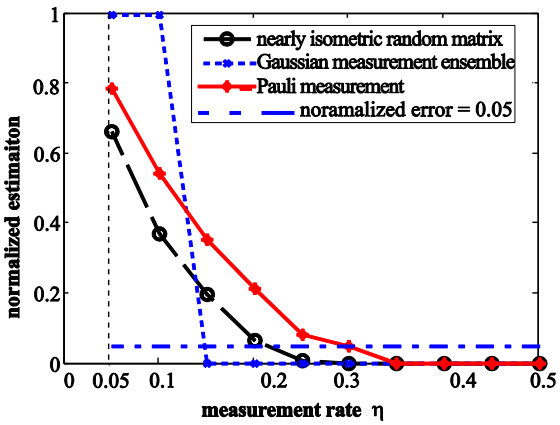

Fig. 1 Normalized estimation errors with different measurement rates.

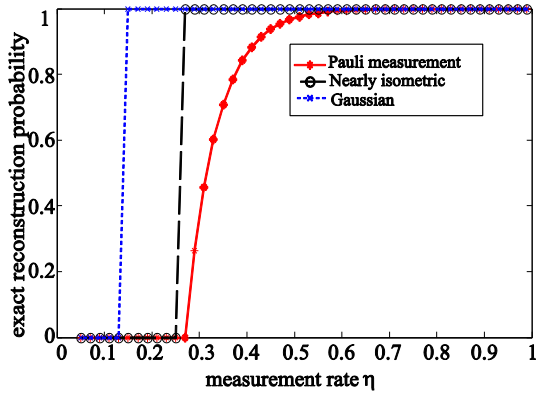

Fig. 2 Reconstruction probabilities with different measurement rates.

We notice that the lower bound of the measurement rate in the experiments is a little higher than theoretical lower bounds except Pauli measurement. There are two possible reasons. The first is the limited number of iterations make the solution of the algorithm do not converge to the density matrix. The second is the algorithm itself. In the process of calculating the density matrix, the algorithm solves the low rank matrix and the conjugate symmetric matrix separately, which makes the algorithm converge only to a neighborhood of the density matrix. The reason of Pauli measurement is that, due to Fig. 2, the exact reconstruction probability of Pauli measurement is less sensitive to the number of measurement rate which makes the theoretical bound is a rough estimation.

It can be observed from Figs. 1 and 2 that for ADMM algorithm with 100 iterations, the density can be reconstructed with $\text{error} \le 0.05$, and the measurement rate is about 0.22 under using nearly isometric random matrix, and about 0.14 under using Gaussian measurement ensemble, and about 0.30 under using Pauli measurement. That means when $n = 6$, about 901, 573 and 1228 measurement settings are required to achieve 95% estimation accuracy for nearly isometric random matrix, Gaussian measurement ensemble, and Pauli measurement, respectively.

IV. Conclusion

In this paper, we studied the reduction of the required number of measurement settings via CS in quantum state estimation. Based on the existing CS theoretical results, the relationship between the optimal number of measurement settings and the error bounds were analyzed and summarized. We studied the precondition of different theorems, furthermore, we validated the lower bound of the measurement settings by the mean of numerical experiments. The research results in this paper can provide an instruction for the selection of the minimum number of measurement settings in the quantum state estimation.

References

[1] S. Cong, S. Kuang "Review of state estimation methods in quantum systems". *Control and Decision*, 23(2), pp.121-126, 2008.

[2] H. Häffner, W. Hänsel, C.F Roos., et al. "Scalable multiparticle entanglement of trapped ions, " Nature, 438 (7068), pp. 643-646, 2005

[3] D.L Donoho. "Compressed sensing.," IEEE Transactions on Information Theory, 52(4), pp. 1289-1306, 2006.

[4] E.J. Candes, T. Tao, "Decoding by linear programming,", IEEE Transactions on Information Theory, 51 (12), pp. 4203-4215, 2005.

[5] E.J. Candès, "The restricted isometry property and its implications for compressed sensing", Comptes Rendus Mathematique, 346(9), pp. 589-592, 2008.

[6] E.J. Candès., B. Recht. "Exact matrix completion via convex optimization," Foundations of Computational Mathematics, 9 (6), pp. 717-772, 2009.

[7] M. Fazel, E. Candes, B. Recht, et al. "Compressed sensing and robust recovery of low rank matrices," IEEE 42nd Asilomar Conference on Signals, Systems and Computers, pp. 1043-1047, 2008.

[8] D. Gross, Y.K Liu., S.T. Flammia, et al. "Quantum state tomography via compressed sensing," Physical review letters, 105(15): 150401, 2010.

[9] A. Smith, C.A. Riofrío., B.E .Anderson, et al. "Quantum state tomography by continuous measurement and compressed sensing," Physical Review A, 87(3): 030102, 2013.

[10] Y.K. Liu, "Universal low-rank matrix recovery from Pauli measurements," Advances in Neural Information Processing Systems. pp. 1638-1646, 2011.

[11] K. Li, S.Cong, "A Robust Compressive Quantum State Tomography Algorithm Using ADMM" World Congress. 19(1), pp. 6878-6883,2014,.

[12] S.T. Flammia., D. Gross, Y.K. Liu, et al. "Quantum tomography via compressed sensing: error bounds, sample complexity and efficient estimators," New Journal of Physics, 14(9), 095022, 2012

[13] B. Recht, M. Fazel, P.A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," SIAM review, 52(3), pp. 471-501, 2010.

[14] E.J Candes., Y. Plan, "Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements," IEEE Transactions on Information Theory, 57(4), pp. 2342-2359, 2011.

[15] M. Fazel, E. Candes, B. Recht, et al. "Compressed sensing and robust recovery of low rank matrices," IEEE 42nd Asilomar Conference on Signals, Systems and Computers, pp. 1043-1047, 2008.

[16] D. Gross, "Recovering low-rank matrices from few coefficients in any basis," IEEE Transactions on Information Theory, 57(3), pp. 1548-1566, 2011.

[17] K. Życzkowski, K.A. Penson, I. Nechita, et al. "Generating random density matrices," Journal of Mathematical Physics, 52(6): 062201,2011.

[18] Foucart, Simon, and H. Rauhut. A mathematical introduction to compressive sensing, Boston: Birkhäuser, 2013.