# Nonlinear Differential Equations

- Nonlinear Differential Equations
- Existence and uniqueness of solutions
- Taylor-Model Based Integrators
- Runge-Kutta Integrators
- Stability Analysis
- Multi-Step Methods

# Contents

# Problem Formulation

The focus of this lecture is on ordinary differential equations (ODEs),

$$\forall t \in [0,T], \qquad \dot{x}(t) = f(t, x(t)) \quad \text{with} \quad x(0) = x_0 \ .$$

Here, $x : [0,T] \to \mathbb{R}^n$ is the state trajectory.

**Assumptions:**

- The function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ may be nonlinear.
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# Explicit solution

- In general: no explicit solution possible
- But it some special cases, we can solve the nonlinear differential equation by using the concept of separation of variables.

**Seperation of variables:**

- Assumption: $f$ is scalar separable; that is,

$$f(t,x) = f_1(x) f_2(t) \ .$$

- Strategy: integrate the equation

$$\frac{\dot{x}(t)}{f_1(x(t))} = f_2(t) \ ,$$

with respect to $t$ on both sides and eliminate $x(t)$.

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# Example: quadratic differential equation

**Nonlinear ODE:**

$$\dot{x}(t) = -x^2(t) \quad \text{with} \quad x(0) = 1 \; .$$

**Separation of variables:**

$$-\frac{\dot{x}(t)}{x(t)^2} = 1 \qquad \stackrel{\text{integrate}}{\Longrightarrow} \qquad \frac{1}{x(t)} - \frac{1}{x(0)} = t$$

**Elimination of $x(t)$:**

$$x(t) = \frac{1}{1+t} \quad \text{for all} \quad t \geq 0 \; .$$

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# Example: Gauss' differential equation

**ODE:**

$$\dot{x}(t) = -t x(t) \quad \text{with} \quad x(0) = 1 \ .$$

**Separation of variables:**

$$\frac{\dot{x}(t)}{x(t)} = -t \qquad \Longrightarrow \qquad \log(x(t)) = -\frac{1}{2} t^2$$

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# Integral Form

The ordinary differential equation (ODE)

$$\forall t \in [0,T], \qquad \dot{x}(t) = f(x(t)) \quad \text{with} \quad x(0) = x_0 \ .$$

can be equivalently be written in its integral form

$$\forall t \in [0,T], \qquad x(t) = x_0 + \int_0^t f(x(s))\,\mathrm{d}s \ .$$

# Lipschitz continuity

**Recall:**

- A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called (globally) Lipschitz continuous, if there exist a constant $L < \infty$ with

$$\forall x, y \in \mathbb{R}, \qquad \|f(x) - f(y)\| \leq L\|x - y\| \; .$$

# Existence and Uniqueness

**Theorem (Picard-Lindelöf):**

- If $f$ is globally Lipschitz continuous, the ODE has a unique solution.

**Proof:** (main idea, rough sketch only)

1) Start with <u>any</u> continuous function $y_1 : [0, T] \to \mathbb{R}$ and iterate

$$y_{i+1}(t) = x_0 + \int_0^t f(y_i(s))\,\mathrm{d}s \qquad \text{[Picard iteration]}$$

2) Show that $y_1, y_2, y_3, \ldots$ is a Cauchy sequence, $y^* = \lim_{k\to\infty} y_i$.

3) Conclude that the (unique) limit point $y^*$ satisfies the ODE.

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## Some technical details

- Define $\Delta(t) = \max_{s \in [0,t]} |y_2(s) - y_1(s)|$.
- If $|y_{i+1}(t) - y_i(t)| \leq \frac{(tL)^{i-1}}{(i-1)!}\Delta(t)$, then

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\begin{aligned}
|y_{i+2}(t) - y_{i+1}(t)| &\leq L \left| \int_0^t [y_{i+1}(\tau) - y_i(\tau)] \, \mathrm{d}\tau \right| \\
&\leq \int_0^t L \frac{(\tau L)^{i-1}}{(i-1)!} \Delta(t) \, \mathrm{d}\tau = \frac{(tL)^i}{i!} \Delta(t) .
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- Thus, we have

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# Example: Linear ODEs

- Linear ODE: $\dot{x}(t) = Ax(t)$, $A \in \mathbb{R}^{n \times n}$, with $x(0) = x_0 \in \mathbb{R}^n$.
- Picard iteration:

$$\begin{aligned} y_1(t) &= x_0 \\ y_2(t) &= x_0 + tAx_0 \\ y_3(t) &= x_0 + tAx_0 + \tfrac{t^2}{2}A^2x_0 \\ &\vdots \end{aligned}$$

- Take the limit to get explicit solution

$$x(t) = e^{At}x_0 = \sum_{i=0}^{\infty} \frac{1}{i!}[tA]^i x_0 \ .$$

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# Examples for nonlinear ODEs

- The ODE $\dot{x}(t) = x(t)^2$, with $x(0) = 1$ has the explicit solution

$$x(t) = \frac{1}{1-t} \quad \text{for} \quad t < 1$$

  Why does the solution not exist for $t \geq 1$?

- The ODE $\dot{x}(t) = 2\sqrt{x}$, with $x(0) = 0$ has more than one solution,

$$\text{for example} \quad x(t) = 0 \quad \text{and} \quad x(t) = t^2 \ .$$

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Why is there more than one solution?

# Gronwall's Lemma

Let $f$ be globally Lipschitz continuous and

$$\dot{x}(t) = f(x(t)) \qquad x(0) = x_0 \tag{1}$$

$$\dot{z}(t) = f(z(t)) \qquad z(0) = z_0 \; . \tag{2}$$

Then we have

$$\|x(t) - z(t)\| \leq \|x_0 - z_0\| e^{Lt} \; .$$

# Proof of Gronwall's Lemma

Main idea: start the Picard iteration

$$y_{i+1}(t) = z_0 + \int_0^t f(y_i(s))\,\mathrm{d}s \quad \text{at} \quad y_1(t) = x(t)\ .$$

The first iteration is given by

$$y_2(t) \ = \ z_0 + \int_0^t f(x(s))\,\mathrm{d}s \ = \ x(t) + [z_0 - x_0] = x(t) + e_1(t)$$

with

$$e_1(t) \quad = \quad y_2(t) - x(t) \qquad \text{and} \qquad \|e_1(t)\| \quad \leq \quad \|x_0 - z_0\|$$

# Proof of Gronwall's Lemma

1. Use the first iteration as induction start (previous slide).
2. The induction assumption is that

$$\begin{aligned} y_{k+1}(t) &= x(t) + e_k(t) \\ \text{and} \quad \|e_k(t)\| &\le \|x_0 - z_0\| \sum_{i=0}^{k-1} \frac{L^i t^i}{i!} \end{aligned}$$

3. Induction step: set $e_{k+1}(t) = y_{k+2}(t) - x(t)$ and work out

$$\begin{aligned} \|e_{k+1}(t)\| &= \left\| z_0 + \int_0^t f(x(s) + e_k(s))\,\mathrm{d}s - x(t) \right\| \\ &\le \|x_0 - z_0\| + \left\| \int_0^t f(x(s) + e_k(s)) - f(x(s))\,\mathrm{d}s \right\| \\ &\le \|x_0 - z_0\| \left[ 1 + \int_0^t \sum_{i=0}^{k-1} L \frac{L^i s^i}{i!}\,\mathrm{d}s \right] = \|x_0 - z_0\| \sum_{i=0}^{k} \frac{L^i t^i}{i!}. \end{aligned}$$

# Proof of Gronwall's Lemma

Summary:

- We know that the Picard iteration

$$y_{k+1}(t) = z_0 + \int_0^t f(y_k(s))\,\mathrm{d}s \quad \text{at} \quad y_1(t) = x(t)\;.$$

  converges and the limit is given by

$$\lim_{k\to\infty} y_k(t) = z(t)$$

- The corresponding error is bounded by

$$\begin{aligned} \|z(t) - x(t)\| &= \lim_{k\to\infty} \|y_k(t) - x(t)\| = \lim_{k\to\infty} \|e_{k-1}(t)\| \\ &\leq \|x_0 - z_0\| \sum_{i=0}^{\infty} \frac{L^i t^i}{i!} = \|x_0 - z_0\| e^{Lt}\;. \end{aligned} \tag{3}$$

# Conditioning of Differential Equations

- The factor $e^{Lt}$ can be interpreted as a **global** upper bound on the condition number of a differential equation
- In general, for large $t$, predictions are impossible: "butterfly effect".
- BUT: Gronwall's lemma has no information about the stability properties of the differential equation
- For some differential equations, a **local** analysis yields better bounds and potentially indicates *local stability*

# First Order Variational Analysis

- Consider the differential equations

$$\dot{x}(t) = f(x(t)) \qquad \text{with} \qquad x(0) = x_0$$

$$\dot{z}(t) = f(z(t)) \qquad \text{with} \qquad z(0) = z_0$$

  for a continuously differentiable right-hand $f$.

- The linear matrix differential equation

$$\dot{X}(t) = \frac{\partial f(x(t))}{\partial x} X(t) \quad \text{with} \quad X(0) = I$$

  is called the first order variational differential equation.

- It yields the first order Taylor approximation

$$z(t) = x(t) + X(t)(z_0 - x_0) + \mathbf{o}\left(\|z_0 - x_0\|\right)$$

  Unfortunately: in general only valid for finite $t \leq T < \infty$ !

# Details

Introduce the shorthands

$$e(t) = z(t) - x(t) - X(t)(z_0 - x_0) \quad \text{and} \quad A(t) = \frac{\partial f(x(t))}{\partial x}$$

We have

$$\begin{aligned} \dot{e}(t) &= f(z(t)) - f(x(t)) - \dot{X}(t)(z_0 - x_0) \\ &= A(t)e(t) + o(\|e(t)\|) + o(\|X(t)\|\|x_0 - z_0\|) \\ \text{with} \quad e(0) &= 0\,. \end{aligned}$$

We can use a Picard iteration (or variant of Gronwall's lemma) to show that $\|e(t)\| = o(\|x_0 - z_0\|)$ for all $t \leq T < \infty$. (Exercise!)

# Steady-States

A point $x_0 \in \mathbb{R}^n$ is called a steady-state (or critical point) of $f$ if

$$f(x_0) = 0 \; .$$

If $f$ is continuously differentiable, the ODE

$$\dot{z}(t) = f(z(t)) \qquad \text{with} \qquad z(0) = z_0$$

can be analyzed in a local neighborhood of $x_0$. We have

$$A = \frac{\partial f(x_0)}{\partial x_0} \quad \text{and} \quad X(t) = e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} (tA)^i \; .$$

# Local Stability Analysis

- If the eigenvalues of $A$ have all strictly negative real parts, then

$$\lim_{t \to \infty} e^{At} \;=\; 0 \;.$$

- In this case, we have

$$\forall t \in [0, \infty), \quad z(t) = e^{At} z_0 + o(\|z_0\|) \qquad \text{and} \qquad \lim_{t \to \infty} z(t) = 0$$

  for sufficiently small $\|z_0\|$. This implies *local asymptotic stability*.

- A similar local stability analysis via first order variational analysis is possible in the neighborhood of periodic orbits (Floquet theory).

# Contents

# Taylor expansion of ODEs

A Taylor expansion of the solution $x(t)$ can be constructed recursively:

- $x(t_0) = x_0$
- $\dot{x}(t_0) = f(t_0, x_0)$
- $\ddot{x}(t_0) = \frac{\partial}{\partial t} f(t, x(t))\Big|_{t=t_0} = f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)$
- and so on ...
- Finally, $x(t) =$

  $x_0 + f(t_0, x_0)(t - t_0) + \frac{(t-t_0)^2}{2}\left[f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)\right] + \dots$

  for small $t$.

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- $x(t_0) = x_0$
- $\dot{x}(t_0) = f(t_0, x_0)$
- $\ddot{x}(t_0) = \frac{\partial}{\partial t} f(t, x(t))\Big|_{t=t_0} = f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)$
- and so on ...
- Finally, $x(t) =$
  $x_0 + f(t_0, x_0)(t - t_0) + \frac{(t-t_0)^2}{2}\left[f_t(t_0, x_0) + f_x(t_0, x_0) f(t_0, x_0)\right] + \ldots$
  for small $t$.

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# Taylor expansion of ODEs

A general Taylor expansion can be computed by consecutive differentiation:

1. Set $\phi_0(t,x) = x$.
2. For $r = 0 : s-1$

   set $\phi_{r+1}(t,x) = \left(\frac{\partial}{\partial t}\phi_r(t,x)\right) + \left(\frac{\partial}{\partial x}\phi_r(t,x)\right) f(t,x)$.
3. Return the Taylor expansion

$$x(t) = \sum_{i=0}^{s} \frac{1}{i!}\phi_i(t_0,x_0)(t-t_0)^i + \mathbf{O}((t-t_0)^{s+1}) \ .$$

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# Integration Algorithm (Constant Step-Size)

**Input:**

- The right-hand side function $f$ and an initial value $x_0$.
- Order $s$ and constant step-size $h = T/N$; set $i = 0$ and $y_0 = x_0$.

**Repeat:** (until $i = N$)

- Compute $y_{i+1} = \sum_{k=0}^{s} \frac{1}{k!} \phi_k(t_i, y_i) h^k$
- Compute $t_{i+1} = t_i + h$ and set $i \leftarrow i + 1$.

**Theorem:**

- If $f$ is globally Lipschitz continuous and smooth, then

$$\forall i \in \{0, \ldots, N\}, \qquad y_i = x(t_i) + \mathbf{O}(h^s)\ .$$

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$$\forall i \in \{0, \ldots, N\}, \qquad y_i = x(t_i) + \mathbf{O}(h^s) \ .$$

# Proof (main idea)

1. Since $f$ is globally Lipschitz, the solution $x$ of the ODE exists.
2. Since $f$ is smooth, the functions $\phi_0, \phi_1, \ldots, \phi_s$ are smooth, too.
3. We already know that $x(t) = \sum_{k=0}^{s} \frac{1}{k!}\phi_k(x_0)h^k + \mathbf{O}(h^{s+1})$.
4. Show by induction (use Gronwall's lemma for a clean proof) that

$$y_i = x(ih) + i \cdot \mathbf{O}(h^{s+1}) = x(ih) + \frac{T}{h} \cdot \mathbf{O}(h^{s+1}) = x(ih) + \mathbf{O}(h^s).$$

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# Contents

# Limitations of Taylor model based integrators

1. Taylor model based intgration is easy to implement, but
   - we need to evaluate derivatives of $f$
   - it is not the most efficient scheme for obtaining convergence order $s$.
2. Runge-Kutta integrators compute an approximation $y \approx x(h)$ by evaluating $f$ at more than one point, but don't evaluate derivatives.

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# Explicit Runge Kutta method (constant step-size)

**Initialization:**

- Set $h = T/N$, $t_0 = 0$, $i = 0$, and $y_0 = x_0$.

**Repeat:** (until $i = N$)

- Compute $t_{i+1} = t_i + h$.
- Compute $k_r = f(t_i + h\gamma_r, y_i + \sum_{j=1}^{r-1} h\alpha_{r,j}k_j)$ for $r = 1, \ldots, s$.
- Set $y_{i+1} = y_i + h\sum_{r=1}^{s} \beta_r k_r$ and then $i \leftarrow i + 1$.

**Output:**

- Time grid $[t_1, t_2, \ldots, t_N]$ and state trajectory $y_0, y_1, y_2, \ldots, y_N$.

# Consistency conditions

**Main idea:**

- Choose the coefficients $\alpha_{r,j}$, $\beta_r$, and $\gamma_r$ such that

$$\forall r \in \{1, \ldots, q\}, \quad \left. \frac{\partial^r y_{i+1}}{\partial h^r} \right|_{h=0} = \Phi_r(y_i) \ .$$

# Example 1: Euler's method

- For $s = 1$, the Runge-Kutta method takes the form

$$\begin{aligned} k_1 &= f(t_i, y_i) \\ y_{i+1} &= y_i + h\beta_1 k_1 = y_i + h\beta_1 f(t_i, y_i) \end{aligned} \quad (4)$$

- We have

$$\left.\frac{\partial y_{i+1}}{\partial h}\right|_{h=0} = \left|\frac{\partial}{\partial h}\left(y_i + h\beta_1 f(t_i, y_i)\right)\right|_{h=0} = \beta_1 f(t_i, y_i) \quad (5)$$

and

$$\phi_1(t, x) = f(t, x) \quad (6)$$

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$$\phi_1(t, x) = f(t, x) \tag{6}$$

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- The equation

$$\left.\frac{\partial y_{i+1}}{\partial h}\right|_{h=0} = \phi_1(t_i, y_i) \qquad \overset{(5),(6)}{\Longleftrightarrow} \qquad \beta_1 f(t_i, y_i) = f(t_i, y_i)$$

  is satisfied for $\beta_1 = 1$.

- Result: Euler's method

$$y_{i+1} = y_i + hf(t_i, y_i)\ .$$

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$$y_{i+1} = y_i + hf(t_i, y_i) \ .$$

# Example 2: Heun's method

- Heun's method is given by the coefficient scheme

$$
\begin{array}{c|cc}
\gamma_1 & 0 & \\
\gamma_2 & \alpha_{2,1} & 0 \\
\hline
 & \beta_1 & \beta_2
\end{array}
\quad = \quad
\begin{array}{c|cc}
0 & 0 & \\
1 & 1 & 0 \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

- The corresponding method can be written as

$$
\begin{aligned}
k_1 &= f(t_i, y_i) \\
k_2 &= f(t_i + h, y_i + hk_1) \\
y_{i+1} &= y_i + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right).
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# Example 3: RK 4

- A very elegant method of order $4$ is given by the scheme

$$\begin{aligned} k_1 &= f(t_i, y_i) \\ k_2 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \\ k_3 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right) \\ k_4 &= f(t_i + h, y_i + hk_3) \\ y_{i+1} &= y_i + h\left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right) . \end{aligned}$$

  This method is called the <u>classical</u> Runge Kutta method.

# Step-size control

## Main idea

- User input: local error tolerance $\mathrm{TOL}$ and absolute tolerance $\mathrm{ATOL}$.
- Compute two approximations

$$y_{n+1} = x(y_n, h) + \mathbf{O}(h^{r+1}) \quad \text{and} \quad z_{n+1} = x(y_n, h) + \mathbf{O}(h^{r+2})$$

  with different local convergence orders, where $x(\cdot, h)$ denotes the exact solution of

$$\forall t \in [0, h], \quad \dot{x}(x_0, t) = f(x(x_0, t)) \quad \text{with} \quad x(x_0, 0) = x_0$$

  in dependence on $x_0$.

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in dependence on $x_0$.

# Step-size control

## Main idea (continued)

- Compute componentwise error estimates

$$e_j = \frac{\|y_{n+1,j} - z_{n+1,j}\|}{\mathrm{TOL}|y_{n,j}| + \mathrm{ATOL}}$$

- Determine a new step-size $h \to \rho \frac{h}{\sqrt[r+1]{\|e\|}}$, e.g. with $\rho = 0.9$.
- Accept the step if $\|e\| \leq 1$ (usually $\infty$-norm); reject otherwise.

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# Example: Fehlberg's Method

The Runge-Kutta-Fehlberg method is given by the tableau

$$
\begin{array}{c|ccc}
0 & & & \\
\frac{1}{2} & \frac{1}{2} & & \\
1 & \frac{1}{256} & \frac{255}{256} & \\
\hline
 & \frac{1}{512} & \frac{255}{256} & \frac{1}{512} \\
\hline
 & \frac{1}{256} & \frac{255}{256} &
\end{array}
$$

The two steps have consistency orders $1$ and $2$, respectively.

# Contents

# Linear Test Problem

An important test problem for numerical integration schemes is given by the linear ODE

$$\dot{x}(t) = f(x(t)) \qquad \text{with} \qquad f(x) = \lambda x$$

for a parameter $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) < 0$.

- Write the ODE integrator step in the form $y_{k+1} = I(f, h, y_k)$
- If the integrator satisfies

  $$\forall h > 0, \quad \|I(f, h, y)\|_2 \leq \|y\|_2$$

  for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) < 0$, then the integrator is called A-stable.

# Stability of Runge-Kutta Integrators

The integrator step of a Runge-Kutta scheme is given by

$$\begin{aligned} k &= \lambda(y \cdot \mathbf{1} + hAk) \\ I(f,h,y) &= y + hb^{\mathsf{T}}k \end{aligned}$$

It can be written in the form

$$\begin{aligned} I(f,h,y) &= R(h\lambda)y \\ \text{with} \quad R(\alpha) &= 1 + \alpha b^{\mathsf{T}}(I - \alpha A)^{-1}\mathbf{1} \end{aligned}$$

The method is A-stable if $|R(\alpha)| \leq 1$ for all $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) \leq 0$.

Notice: there is no A-stable explicit Runge-Kutta integrator!

# Examples

- The explicit Euler method, $y_{k+1} = y_k + hf(y_k)$, satisfies

$$R(\alpha) = 1 + \alpha \ .$$

  This method is clearly not A-stable.

- The implicit Euler method, $y_{k+1} = y_k + hf(y_{k+1})$, satisfies

$$R(\alpha) = 1 + \frac{\alpha}{1-\alpha} = \frac{1}{1-\alpha} \ .$$

  This is an example for an A-stable method.

# Examples

- The trapezoidal scheme (order $2$) is given by

$$\begin{aligned} k_1 &= f(y_k) \\ k_2 &= f\left(y_k + \frac{1}{2}hk_1 + \frac{1}{2}hk_2\right) \\ y_{k+1} &= y_k + \frac{h}{2}k_1 + \frac{h}{2}k_2 \end{aligned}$$

- The method is A-stable, because we have

$$\begin{aligned} R(\alpha) &= 1 + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\alpha}{2} & 1 - \frac{\alpha}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{2+\alpha}{2-\alpha} \end{aligned}$$

# L-Stability

- Notice that $A$-stable methods do not always perform well for very "stiff" test differential equations; that is, for $\mathrm{Re}(\lambda) \ll -1$.
- A method is called L-stable if
  1. it is A-stable and
  2. we additionally have $\lim_{\alpha \to -\infty} R(\alpha) = 0$.

**Examples**

- The implicit Euler method, $R(\alpha) = \frac{1}{1-\alpha}$, is L-stable.
- The trapezoidal rule, $R(\alpha) = \frac{2+\alpha}{2-\alpha}$, is A-stable but not L-stable.

# Contents

# Multi-Step Methods

- Taylor model based integration and Runge-Kutta methods construct $y_k$ in dependence on $y_{k-1}$.
- This is in contrast to multi-step methods, where $y_k$ may depend on $y_{k-1}, y_{k-2}, \ldots, y_{k-m}$.
- Of special interest are the so-called linear multi-step methods, which have the form

$$\sum_{i=0}^{m} \alpha_{m-i} y_{k-m} = h \sum_{i=0}^{m} \beta_{m-i} f(y_{k-i}) \qquad \text{with} \qquad a_R = 1$$

- The method is explicit if $\beta_m = 0$. Otherwise, it is implicit.

# Multi-Step Methods

- Linear Multi-Step Methods (LMMs) can be constructed by starting with a numerical integration formula,

  $$y_{k-s} + \int_{t-sh}^{t} f(x(\tau))\,\mathrm{d}\tau \;=\; y_{k-s} + h\sum_{i=0}^{m} \beta_{m-i} f(x(t-ih)) + O(h^{q+1}).$$

  Thus, if $y_{k-i} \approx x(t-ih)$, we have

  $$y_k = y_{k-s} + h\sum_{i=0}^{m} \beta_{m-i} f(y_{k-i}) \qquad \Longrightarrow \qquad y_k \approx x(t)\ .$$

  The consistency order $q$ of this approximation depends merely on the accuracy of the numerical integration formula.

# Examples

- Adams-Bashforth formulas (explicit)

$$
\begin{aligned}
s=1, m=1: \qquad y_k &= y_{k-1} + hf(y_{k-1}) \\
s=1, m=2: \qquad y_k &= y_{k-1} + h\left(\frac{3}{2}f(y_{k-1}) - \frac{1}{2}f(y_{k-2})\right) \\
&\vdots
\end{aligned}
$$

- Adams-Moulton formulas (implicit)

$$
\begin{aligned}
s=1, m=0: \qquad y_k &= y_{k-1} + hf(y_k) \\
s=1, m=1: \qquad y_k &= y_{k-1} + h\left(\frac{1}{2}f(y_k) + \frac{1}{2}f(y_{k-1})\right) \\
&\vdots
\end{aligned}
$$

# More Examples

- Nyström's method ($s = 2$)

$$y_k = y_{k-2} + 2hf(y_{k-1})$$

- Milne-Simpson method ($s = 2$)

$$y_k = y_{k-2} + h\left(\frac{1}{3}f(y_k) + \frac{4}{3}f(y_{k-1}) + \frac{1}{3}f(y_{k-2})\right)$$

- It's easy to come up with more methods: any numerical integration formula yields yet another LMM.

# Backward Differencing Formulas (BDF)

- Instead of numerical integration, we can also use numerical differentiation to derive LMMs. This yields BDF methods.
- Examples:

$$\begin{aligned} y_k - y_{k-1} &= hf(y_k) \\ y_k - \frac{4}{3}y_{k-1} + \frac{1}{3}y_{k-2} &= \frac{2}{3}hf(y_k) \\ y_k - \frac{18}{11}y_{k-1} + \frac{9}{11}y_{k-2} - \frac{2}{11}y_{k-2} &= \frac{6}{11}hf(y_k) \\ &\vdots \end{aligned}$$

- Exercise: combine numerical differentiation and integration to derive even more LMMs...

# Dahlquist's Theorem

**Theorem.** Let $f$ be globally Lipschitz. The iterates of the LMM

$$\sum_{i=0}^{m} \alpha_{m-i} y_{k-m} = h \sum_{i=0}^{m} \beta_{m-i} f(y_{k-i}) \qquad \text{with} \qquad a_R = 1$$

converge to the solution of the ODE, $\dot{x}(t) = f(x(t))$ for $h \to 0$, if

1. the initial iterates $y_0, y_1, \ldots, y_{m-1}$ converge,
2. the method is consistent ($=$ has at least consistency order $1$),

$$\sum_{i=0}^{m} \alpha_i = 0 \quad \text{and} \quad \sum_{i=0}^{m} i\alpha_i + \sum_{i=0}^{m} \beta_i = 0, \quad \text{and}$$

3. the roots $\lambda_j \in \mathbb{C}$ of the first characteristic polynomial

$$\rho(\lambda) = \sum_{i=0}^{m} \alpha_i \lambda^i$$

satisfy $|\lambda_j| \leq 1$ and all multiple roots satisfy $|\lambda_j| < 1$.