Feedback Control

- Open-Loop Control
- Feedback Control
- Proportional Control

Contents

Open-loop Control

Linear control system model

$$\dot{x}(t) = ax(t) + bu(t) \quad \text{and} \quad x(0) = x_0$$

Open-loop control

- First choose $u(t)$ on the control horizon $[0, T]$ (offline)
- Send the input u to the system and use it "blindly".

Open-loop Control

Example

$$\dot{x}(t) = x(t) + u(t) \quad \text{and} \quad x(0) = 1$$

Open-loop control

- If we choose $u(t) = -1$ the system state satisfies

$$\dot{x}(t) = x(t) - 1 \quad \text{with} \quad x(0) = 1 \;,$$

  i.e., we have $x(t) = 1$ for all $t \geq 0$.

Question: would this work in practice ?

Open-loop Control

Problem:

$$\dot{x}(t) = x(t) - 1 \quad \text{and} \quad x(0) = 1 + \epsilon$$

- If there is a small error $\epsilon \neq 0$ in the initial value, we have

  $$x(t) = 1 + \epsilon e^t \ ,$$

  i.e., the error grows exponentially in t.

- Similar problem occur if the dynamic model is not accurate.

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Summary: disadvantages of open-loop control

- If we have small errors in the model (model-plant mismatch) or if we work with inaccurate initial values, the online implementation can be unstable / inaccurate.
- There may be external "disturbances", the control design might have to be adapted online rather than pre-computed offline.
- If we have a very long control horizon, we might not want to plan all our actions "in advance", but rather in dependence on the current "situation".

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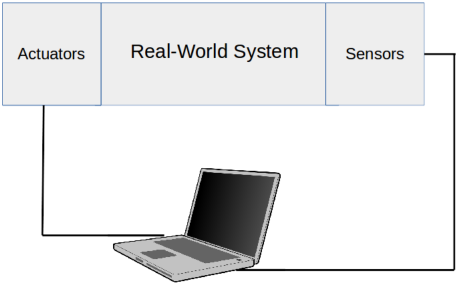
Main idea of feedback control



1. Receive measurement of $x(t)$ from sensor
2. Compute $u(t)$ in dependence on $x(t)$
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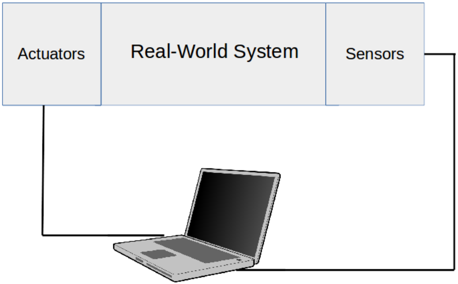

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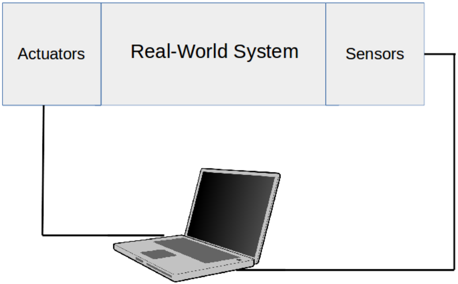

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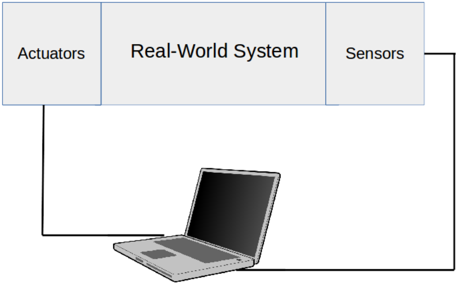

Feedback law

- The map, μ, from the measurement $x(t)$ to a control decision $u(t)$, is called a control law:

$$u(t) = \mu(x(t)) \ .$$

Closed-loop system

If we substitute $u(t) = \mu(x(t))$, we obtain the closed-loop system dynamics:

$$\dot{x}(t) = ax(t) + b\mu(x(t))$$

- If μ is an affine function, this is a linear system.
- If μ is non-affine, we have to solve a so-called nonlinear differential equation for finding $x(t)$.

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Affine feedback (P-control)

- Proportional control is based on affine feedback laws,

$$\mu(y(t)) \;=\; u_{\mathrm{s}} + k(y(t) - x_{\mathrm{s}}) \;=\; u_{\mathrm{s}} + k * e(t)$$

  Here, $y(t) = x(t)$ is called the sensor output.

- Constant $u_{\mathrm{s}} \in \mathbb{R}$ is a control offset
- Constant $x_{\mathrm{s}} \in \mathbb{R}$ is called the set-point (sometimes denoted by y_{ref})
- The function $e(t) = x(t) - x_{\mathrm{s}}$ is called error signal.
- The constant $k \in \mathbb{R}$ is called the *proportional gain*

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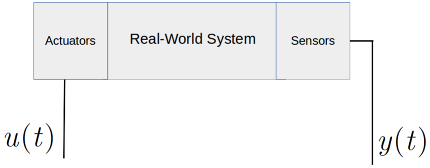
Model-free P-control tuning: Step 1



If the system is stable:

- Play around with the open loop system.
- Adjust $u(t) = u_{\mathsf{s}}$ such that open-loop system satisfies

$$x(T) \approx x_{\mathsf{s}} \qquad \text{after (a possibly long) time } T\,.$$

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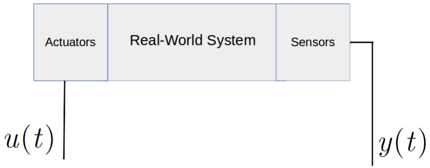

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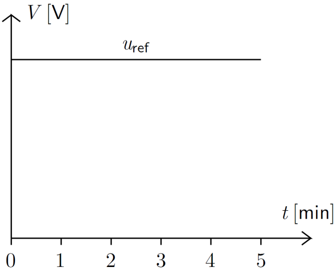

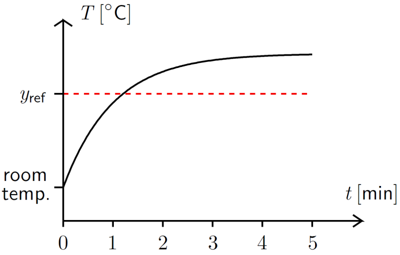

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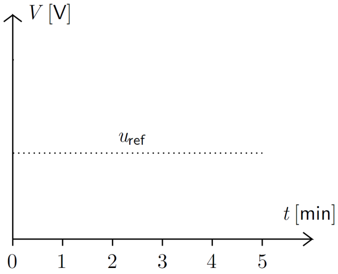

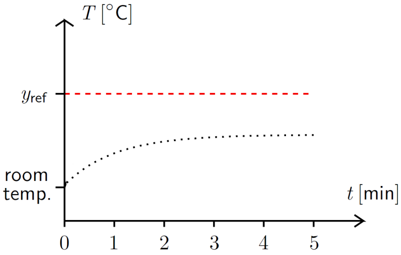

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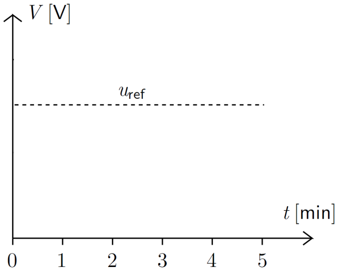
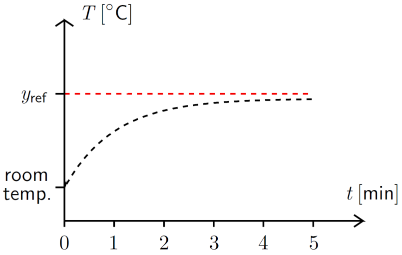
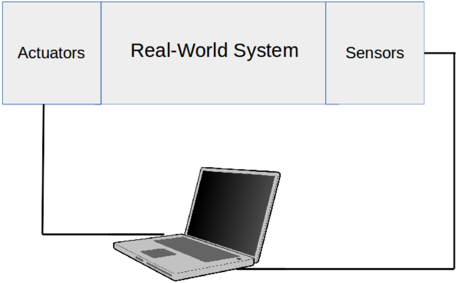




Model-free P-control tuning: Step 2



- Increase / decrease the control gain k and test the closed-loop behavior of the system
- Fine-tuning of u_s and k if needed.

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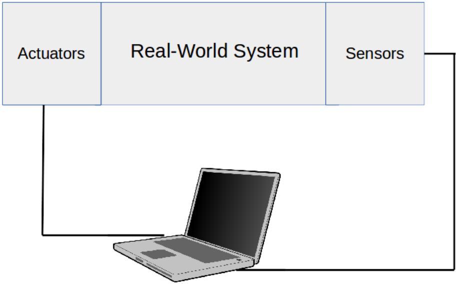

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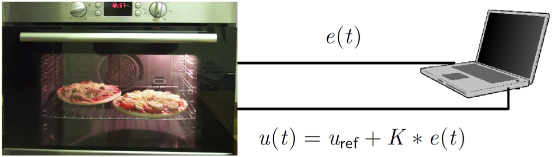

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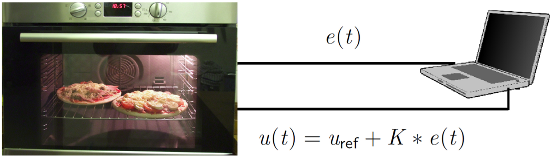
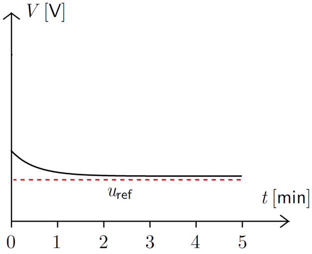
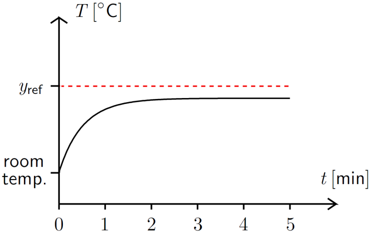
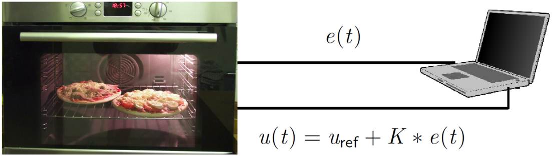


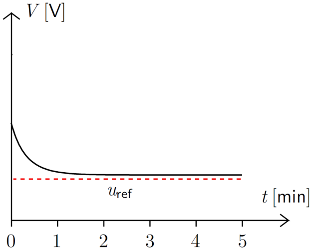


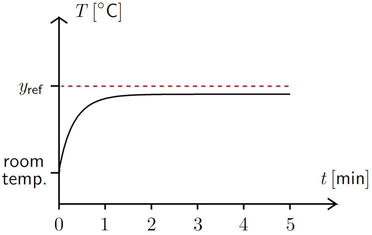


$$k = -0.25 \left[\frac{\mathrm{V}}{^\circ\mathrm{C}}\right]$$

Tuning Step 2







$$k = -0.50 \left[\frac{\text{V}}{^\circ\text{C}}\right]$$

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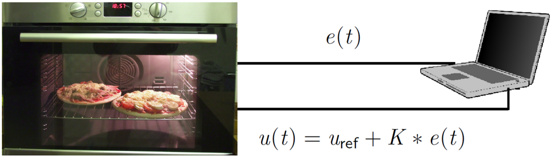

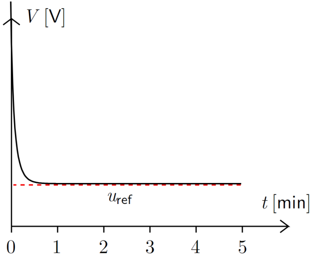

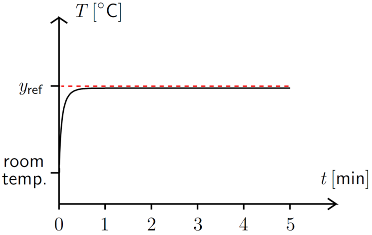

$$k = -2.00 \left[\frac{\mathrm{V}}{{}^\circ\mathrm{C}}\right]$$

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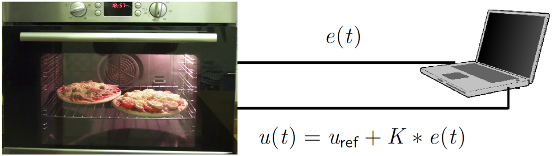
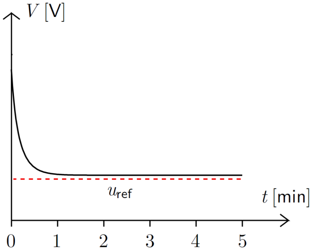
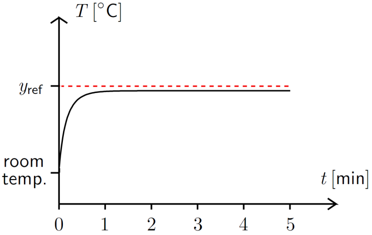


$$u(t) = u_{\text{ref}} + K * e(t)$$





$$k = -1.00 \left[\frac{\text{V}}{^\circ\text{C}}\right]$$

Model-based P-control tuning

- If we have a system model of the form

$$\dot{x}(t) = ax(t) + bu(t)$$

the control design can be done more systematically, i.e., we can design u_{s} and k without "trial-and-error" real-world experiments.

Model-based P-control tuning: Step 1

Model

$$\dot{x}(t) = ax(t) + bu(t)$$

Set-point

- Let us assume that the set point x_{s} is given.
- If $b \neq 0$, the associated steady-state control input can be found by solving

$$0 = ax_{\mathrm{s}} + bu_{\mathrm{s}} \qquad \Leftrightarrow \qquad u_{\mathrm{s}} = -\frac{ax_{\mathrm{s}}}{b} \ .$$

Model-based P-control tuning: Step 2

Closed-loop model

$$
\begin{aligned}
\dot{x}(t) &= ax(t) + b(k(x(t) - x_\mathrm{s}) + u_\mathrm{s}) \\
&= a(x(t) - x_\mathrm{s}) + ax_\mathrm{s} + b(k(x(t) - x_\mathrm{s}) + u_\mathrm{s}) \\
&= (a + bk)(x(t) - x_\mathrm{s}) + \underbrace{ax_\mathrm{s} + bu_\mathrm{s}}_{=0}
\end{aligned}
$$

Explicit solution

- The explicit solution for the closed-loop system state is given by

$$
x(t) = x_\mathrm{s} + (x(0) - x_\mathrm{s})e^{(a+bk)t} \ .
$$

Model-based P-control tuning: Step 2

Closed-loop system state:

$$x(t) = x_{\mathrm{s}} + (x(0) - x_{\mathrm{s}})e^{(a+bk)t} \ .$$

Choice of the proportional gain:

- Choose k such that $a + bk < 0$, i.e., such that

$$\lim_{t \to \infty} x(t) = x_{\mathrm{s}} \ .$$

- Choose $|k|$ not too large, as the actual control input,

$$u(t) = k(x(t) - x_{\mathrm{s}}) + u_{\mathrm{s}} \ ,$$

  may be large if $x(t) - x_{\mathrm{s}}$ is large.

Example: temperature control

Example: The temperature $T(t)$ inside an oven can be modelled as

$$\dot{T}(t) = \frac{u(t)}{C} - \frac{k_{\mathsf{Wall}}}{C}(T(t) - T_{\mathrm{room}}) \,.$$

Notation:

- Supplied power at the heating coil: $u(t) = P_{\mathrm{coil}}(t)$
- room temperature: T_{room}
- heat transition coefficient of the wall: k_{Wall}
- heat capacity of the oven: C

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Set point: We want to bring the temperature to

$$T_{\mathrm{s}} = 200^{\circ}\mathsf{C}\ .$$

Steady-state control input:

- Solve the steady equation

$$0 = \frac{u_{\mathrm{s}}}{C} - \frac{k_{\mathsf{Wall}}}{C}(T_{\mathrm{s}} - T_{\mathrm{room}})\ ,$$

- We find the explicit expression

$$u_{\mathrm{s}} = k_{\mathsf{Wall}}(T_{\mathrm{s}} - T_{\mathrm{room}})\ .$$

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Proportional controller:

$$u(t) = u_{\mathrm{s}} + k(T(t) - T_{\mathrm{s}}) \ .$$

Closed-loop system:

- Closed-loop system has the form

$$\dot{T}(t) = \frac{k - k_{\mathrm{Wall}}}{C} \left[T(t) - T_{\mathrm{s}} \right] \ .$$

- Temperature converges to the desired set-point T_{s}, if $k < k_{\mathrm{Wall}}$,

$$T(t) = T_{\mathrm{s}} + (T(0) - T_{\mathrm{s}}) e^{\frac{k - k_{\mathrm{Wall}}}{C} t} \ .$$

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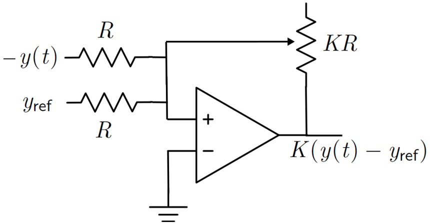
Proportional Control Circuit

- Replace "computer" by electrical circuit (e.g. for our oven):



(here $y(t)$, y_{ref} and $K(y(t) - y_{\text{ref}})$ are voltages)

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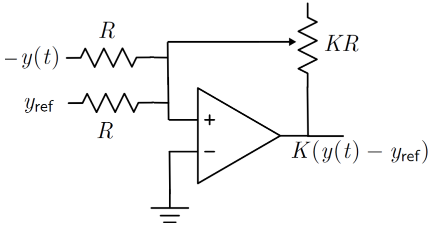

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