Adaptive Pole Placement Control II

Contents

Adaptive Luenberger observer

Consider the LTI SISO plant

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = C^\top x$$

In the case A, B, C are known, the *Luenberger observer* is in the form of

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \quad \hat{x}(0) = \hat{x}_0$$
$$\hat{y} = C^\top \hat{x}$$

where K is chosen such that $A - KC^\top$ is Hurwitz, guarantees that $\hat{x} \to x$ exponentially fast. The existence of K is ensured by the observability of pair $(A, C^\top)$

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Idea: $$(A, B, C) \to G(s) \to \hat{G}(s) \to (\hat{A}, \hat{B}, \hat{C})$$

mapping of the $2n$ estimated parameters of $G(s)$ to the $n^2 + 2n$ parameters of A, B, C is not uniqueunless (A, B, C) is in a observer canonical form, i.e., the plant is represented as

$$\dot{x}_o = \left[-a_p \mid \frac{I_{n-1}}{0} \right] x_o + b_\rho u$$

$$y = [1, 0, \ldots, 0] x_o$$

where $a_p = [a_{n-1}, a_{n-2}, \ldots, a_0]^\top$ and $b_p = [b_{n-1}, b_{n-2}, \ldots, b_0]^\top$ are the coefficients of the transfer function

$$G(s) = \frac{y(s)}{u(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0 s}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0 s}$$

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Then the adaptive observer is given by

$$\dot{\hat{x}} = \hat{A}(t)\hat{x} + \hat{b}_p(t)u + K(t)(y - \hat{y}), \quad \hat{x}(0) = \hat{x}_0$$

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where $\hat{x}$ is the estimate of x_o and

$$\hat{A}(t) = \left[-\hat{a}_p(t) \mid \frac{I_{n-1}}{0} \right], \quad K(t) = a^* - \hat{a}_p(t)$$

$\hat{a}_p(t)$ and $\hat{b}_p(t)$ are the estimates of the vectors a_p and b_p, respectively. $a^* \in \mathcal{R}^n$ is chosen so that

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Adaptive Luenberger observer

Theorem: The adaptive Luenberger observer with gradient-based algorithm guarantees the following properties:

(i) If choose $u \in \mathcal{L}_\infty$ and A is Hurwitz, all signals are bounded.

(ii) Furthermore, if choose u is sufficiently rich of order $2n$, then the state observation error $|\hat{x} - x_o|$ and the parameter estimation error $\tilde{\theta}$ converge to zero exponentially fast.

Brief Proof. (i) The observer equation may be written as

$$\dot{\hat{x}} = A^* \hat{x} + \hat{b}_p(t) u + \left(\hat{A}(t) - A^* \right) x_o$$

(ii) The state observation error $\tilde{x} = \hat{x} - x_o$ satisfies

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Contents

APPC via state space approach

The plant

$$y_p = G_p(s)u_p, \quad G_p(s) = \frac{Z_p(s)}{R_p(s)}$$

where $G_p(s)$ is proper and $R_p(s)$ is a monic polynomial.

The control objective is to choose u_p so that the closed-loop poles are assigned to those of a monic Hurwitz polynomial $A^*(s)$ and $y_p \to y_m$

and Assumptions are the same as before

P1. $R_p(s)$ is a monic polynomial whose degree n is known.

P2. $Z_p(s), R_p(s)$ are coprime and degree $(Z_p) < n$.

P3. $Q_m(s)y_m = 0$ and $Q_m(s)$ is assumed to be coprime with $Z_p(s)$.

APPC via state space approach

Step 1. PPC for known parameters We start by considering the expression

$$e_1 = \frac{Z_p(s)}{R_p(s)} u_p - y_m$$

for the tracking error. Filtering each side of with $\frac{Q_m(s)}{Q_1(s)}$, where $Q_1(s)$ is an arbitrary monic Hurwitz polynomial of degree q, and using $Q_m(s) y_m = 0$, we obtain

$$e_1 = \frac{Z_p Q_1}{R_p Q_m} \bar{u}_p, \quad \bar{u}_p = \frac{Q_m}{Q_1} u_p$$

In this way, have converted the tracking problem into the regulation problem of choosing $\bar{u}_p$ to regulate e_1 to zero.

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Let (A, B, C) be a state-space realization of error equation in the observer canonical form, i.e.

$$\dot{e} = Ae + B\bar{u}_p, \quad e_1 = C^\top e$$

where $A = \left[-\theta_1^* \mid \frac{I_{n+q-1}}{0}\right], \quad B = \theta_2^*, \quad C = [1, 0, \ldots, 0]^\top$ and $\theta_1^*, \theta_2^* \in \mathcal{R}^{n+q}$ are the coefficient vectors of the polynomials $R_p(s)Q_m(s) - s^{n+q}$ and $Z_p(s)Q_1(s)$, respectively.

Note, because R_pQ_m, Z_p are coprime, any possible zero-pole cancellation between $Q_1(s)$ and $R_p(s)Q_m(s)$ will occur in $\mathbb{R}[s] < 0$ due to $Q_1(s)$ being Hurwitz, which implies that (A, B) is always stabilizable.

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APPC via state space approach

Consider the feedback control law

$$\bar{u}_p = -K_c \hat{e}, \quad u_p = \frac{Q_1}{Q_m} \bar{u}_p$$

where $\hat{e}$ is the state of the full-order Luenberger observer

$$\dot{\hat{e}} = A\hat{e} + B\bar{u}_p - K_o \left(C^\top \hat{e} - e_1 \right)$$

and K_c and K_o are solutions to the polynomial equations

$$\det (sI - A + BK_c) = A_c^*(s)$$

$$\det \left(sI - A + K_o C^\top \right) = A_o^*(s)$$

where A_c^* and A_o^* are given monic Hurwitz polynomials of degree $n + q$.

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APPC via state space approach

The design of A_c^* and A_o^* polynomials

- the roots of $A_c^*(s) = 0$ represent the desired pole locations of the transfer function of the closed-loop plant
- the roots of $A_o^*(s) = 0$ are equal to the poles of the observer dynamics

The existence of K_c and K_o

- The existence of K_c follows from the controllability of (A, B)
- The existence of K_o follows from the observability of (A, C).

Note, because of the special canonical observable form,

$K_o = \alpha_o^* - \theta_1^*$, where α_o^* is the coefficient vector of $A_o^*(s)$.

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Theorem: If Assumptions P1-P3 hold, Consider the system

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where $G_p(s)$ is proper and $R_p(s)$ is a monic polynomial. The control law

$$u_p = \frac{Q_1}{Q_m}\bar{u}_p, \quad \bar{u}_p = -K_c\hat{e}$$

$$\dot{\hat{e}} = A\hat{e} + B\bar{u}_p - K_o\left(C^\top\hat{e} - e_1\right)$$

guarantees that all signals in the closed-loop plant are bounded and e_1 converges to zero exponentially fast.

APPC via state space approach

Step 2. Estimation of plant parameters The adaptive law for estimating the plant parameters is given by, for instance

$$\dot{\theta}_p = \Gamma \varepsilon \phi$$

$$\varepsilon = \frac{z - \theta^\top \phi}{m_s^2}, \quad m_s^2 = 1 + \phi^\top \phi$$

z, ϕ and $\theta_p = \left[\theta_a^\top, \theta_b^\top\right]^\top$ are as defined same as ones given in polynomial approach.

APPC via state space approach

Step 3. Adaptive control law Using the CE approach, the adaptive control law is given by

$$\dot{\hat{e}} = \hat{A}\hat{e} + \hat{B}\bar{u}_p - \hat{K}_o\left(c^\top \hat{e} - e_1\right)$$

$$\bar{u}_p = -\hat{K}_c\hat{e}, \quad u_p = \tfrac{Q_1(s)}{Q_m(s)}\bar{u}_p$$

where

$$\hat{A}(t) = \left[-\theta_1(t) \mid \frac{I_{n+q-1}}{0}\right], \quad \hat{B}(t) = \theta_2(t)$$

$\theta_1(t)$ and $\theta_2(t)$ are the coefficient vectors of the polynomials

$$\hat{R}_p(s,t)Q_m(s) - s^{n+q} = \left(s^n + \theta_a^\top(t)\alpha_{n-1}(s)\right)Q_m(s) - s^{n+q}$$

$$\hat{Z}_p(s,t)Q_1(s) = \theta_b^\top(t)\alpha_{n-1}(s)Q_1(s)$$

respectively.

APPC via state space approach

and

$$\hat{K}_o(t) = \alpha_0^* - \theta_1(t)$$

α_0^* is the coefficient vector of $A_o^*(s)$; and $\hat{K}_c(t)$ is calculated at each time t by solving the polynomial equation

$$\det\left(sI - \hat{A} + \hat{B}\hat{K}_c\right) = A_c^*(s)$$

Note, the stabilizability problem arises in adaptive control law, where for the calculation of $\hat{K}_c$ to be possible the pair $(\hat{A}(t), \hat{B}(t))$ has to be controllable at each time t and *for implementation purposes strongly controllable.*

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APPC via state space approach

Theorem Assume that the polynomials $\hat{Z}_p, \hat{R}_p Q_m$ are strongly coprime at each time t. Then all the signals in the closed-loop APPC scheme via state-space approach are uniformly bounded, and the tracking error $e_1 \to 0$ asymptotically .

Example We consider the same scalar plant

$$y_p = \frac{b}{s+a} u_p$$

where a and b are unknown constants with $b \neq 0$. The input u_p is to be chosen so that the poles of the closed-loop plant are placed at the roots of $A^*(s) = (s+1)^2 = 0$, and y_p tracks the reference signal $y_m = 1$.

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LQC

Idea: using an optimization technique to achieve our tracking or regulation objective by *minimizing a certain cost function that reflects the performance of the closed-loop system.*

Unify the regulation or tracking problem of the system as

$$\dot{e} = Ae + B\bar{u}_p$$

$$e_1 = C^\top e$$

where $u_p = \frac{Q_1(s)}{Q_m(s)}\bar{u}_p$, and $\bar{u}_p$ is to be chosen so that $e \in \mathcal{L}_\infty$ and $e_1 \to 0$ as $t \to \infty$.

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LQC

The desired $\bar{u}_p$ to meet this objective is chosen as the one that minimizes the quadratic cost

$$J = \int_0^\infty \left(e_1^2(t) + \lambda \bar{u}_p^2(t) \right) dt$$

where $\lambda > 0$, a weighting coefficient to be designed, penalizes the level of the control input signal. The optimum control input $\bar{u}_p$ that minimizes J is

$$\bar{u}_p = -K_c e, \quad K_c = \lambda^{-1} B^\top P$$

where $P = P^\top$ satisfies the *Riccati Equation*

$$A^\top P + PA - PB\lambda^{-1} B^\top P + CC^\top = 0$$

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Remark:

1. With $\lambda > 0$ and finite, the LQC guarantees $A - BK_c$ is Hurwitz, $e, e_1 \to 0$ *exponentially fast*, and $\bar{u}_p \in \mathcal{L}_\infty$
2. The location of the eigenvalues of $A - BK_c$ depends on the particular choice of λ. In general, there is NO guarantee that one can find a λ so that the closed-loop poles are equal to the roots of the desired polynomial $A^*(s)$

LQC

Consider the state e may not be available for measurement, we need to use

$$\bar{u}_p = -K_c \hat{e}, \quad K_c = \lambda^{-1} B^\top P$$

where $\hat{e}$ is the state of the observer equation

$$\dot{\hat{e}} = A\hat{e} + B\bar{u}_p - K_o \left(C^\top \hat{e} - e_1 \right)$$

where K_o are solutions to the polynomial equations

$$\det \left(sI - A + K_o C^\top \right) = A_o^*(s)$$

where A_o^* is given monic Hurwitz polynomials of degree $n + q$.

LQC

Theorem The LQ control law

$$u_p = \frac{Q_1(s)}{Q_m(s)} \bar{u}_p$$

with

$$\bar{u}_p = -K_c \hat{e}, \quad K_c = \lambda^{-1} B^\top P$$

guarantees that all the eigenvalues of $A - BK_c$ are in $\Re[s] < 0$, all signals in the closed-loop plant are bounded, and $e_1(t) \to 0$ as $t \to \infty$ *exponentially fast.*

ALQC

Use the CE approach to form the adaptive control law

$$\dot{\hat{e}} = \hat{A}\hat{e} + \hat{B}\bar{u}_p - \hat{K}_o\left(c^\top \hat{e} - e_1\right)$$

$$u_p = \frac{Q_1(s)}{Q_m(s)}\bar{u}_p, \quad \bar{u}_p = -\hat{K}_c\hat{e}, \quad \hat{K}_c = \lambda^{-1}\hat{B}^\top P$$

where $\hat{A}, \hat{B}, \hat{K}_o$ are generated using the gradient-based adaptive law (see parameter estimation schemes) and $P(t)$ is calculated by solving the Riccati equation

$$\hat{A}^\top(t)P(t) + P(t)\hat{A}(t) - P(t)\hat{B}(t)\lambda^{-1}\hat{B}^\top(t)P(t) + CC^\top = 0$$

ALQC

Theorem: Assume that the polynomials $\hat{R}_p(s,t)Q_m(s)$ and $\hat{Z}_p(s,t)$ are **strongly coprime** at each time t. Then the ALQC scheme guarantees that all signals in the closed-loop plant are bounded and the tracking error e_1 converges to zero as $t \to \infty$.

Remark 1: the ALQC scheme depends on the solvability of the algebraic Riccati equation. For the solution $P(t) = P^\top(t) > 0$ to exist, the pair $(\hat{A}, \hat{B})$ has to be stabilizable at each time t. A sufficient condition for $(\hat{A}, \hat{B})$ to be stabilizable is that the polynomials $\hat{R}_p(s,t)Q_m(s)$ and $\hat{Z}_p(s,t)$ are coprime at each time t.

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Remark 2: For $P(t)$ to be uniformly bounded, however, we will require $\hat{R}_p(s,t)Q_m(s)$ and $\hat{Z}_p(s,t)$ to be strongly coprime at each time t.

Contents

Loss of stabilizability

Recall scalar example

$$\dot{y} = -ay + bu$$

with indirect APPC law

$$u = -ky, \quad k = \frac{\hat{a} + a_m}{\hat{b}}$$

The system loss the stabilization, when $\hat{b} \to 0$. The solution is a projection operation

$$\dot{\hat{b}} = \begin{cases} \gamma_2 yu & \text{if } |\hat{b}| > b_0 \text{ or if } |\hat{b}| = b_0 \text{ and } \operatorname{sgn}(b)yu \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Modification for ensure stabilizability

a) Parameter Projection Methods: exists a *convex* subset $\mathcal{C}_0$ of the parameter space that is assumed to have the following properties:

(i) The unknown plant parameter vector $\theta_p^* \in \mathcal{C}_0$.

(ii) Every member θ_p of $\mathcal{C}_0$ has a corresponding level of stabilizability greater than ε^* for some known constant $\varepsilon^* > 0$.

Advantage: The projection based on gradient method is simple and does not alter the usual properties of the adaptive law that are used in the stability analysis.

Disadvantage: This approach relies on the rather strong assumption that the set $\mathcal{C}_0$ is known. No procedure has been proposed for constructing such a set $\mathcal{C}_0$ for a general class of plants.

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Projection operation

The gradient algorithm with projection is computed by applying the gradient method to the following minimization problem with constraints:

$$\text{minimize}\, J(\theta)$$

$$s.t. \quad \theta \in S$$

where S is a convex subset of $\mathcal{R}^n$ with smooth boundary almost everywhere. Assume that S is given by

$$S = \{\theta \in \mathcal{R}^n \mid g(\theta) \leq 0\}$$

where $g : \mathcal{R}^n \to \mathcal{R}$ is a smooth function.

Projection operation

The adaptive laws based on the gradient method can be modified to guarantee that $\theta \in S$ by solving the constrained optimization problem given above to obtain

$$\dot{\theta} = \Pr(-\Gamma\nabla J) = \begin{cases} -\Gamma\nabla J & \text{if } \theta \in S^0 \text{ or } \theta \in \delta(S) \\ & \text{and } -(\Gamma\nabla J)^\top \nabla g \leq 0 \\ -\Gamma\nabla J + \Gamma\frac{\nabla g \nabla g^\top}{\nabla g^\top \Gamma \nabla g}\Gamma\nabla J & \text{otherwise} \end{cases}$$

where $\delta(S) = \{\theta \in \mathcal{R}^n \mid g(\theta) = 0\}$ and $S^0 = \{\theta \in \mathcal{R}^n \mid g(\theta) < 0\}$ denote the boundary and the interior, respectively, of S. E.g. gradient algorithm based on the instantaneous cost function:

$\nabla J = -\varepsilon\phi$

Modification for ensure stabilizability

Theorem The gradient adaptive laws and the LS adaptive laws with the projection modifications retain all the properties that are established in the absence of projection and in addition guarantee that $\theta(t) \in S \forall t \geq 0$, provided $\theta(0) \in S$ and $\theta^* \in S$.

Other modification methods FYI:

b) Heuristics methods, such as re-initialization, ignore the undesired value

c) Correction Approach

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