Sobolev Spaces

- Weak Derivatives
- Mollifications
- Sobolev Spaces
- Hilbert-Sobolev Spaces

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Linear Functionals

Basic Notation

- Let $L^1_{\mathrm{loc}}(\mathbb{R})$ denote the set of locally integrable functions.
- The support of a function $\phi : \mathbb{R} \to \mathbb{R}$ will be denoted by

$$\mathrm{Supp}(\Phi) \stackrel{\mathrm{def}}{=} \mathrm{cl}\left(\{x \in \mathbb{R} \mid \phi(x) \neq 0\}\right) \ .$$

- $C_0^\infty(\mathbb{R})$ denotes the set of smooth functions with compact support.

Linear Functional

- Every $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ can be associated with a linear functional

$$\Lambda_f : C_0^\infty(\mathbb{R}) \to \mathbb{R} \qquad \text{given by} \qquad \Lambda_f(\phi) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}} f\phi \, \mathrm{d}x \ .$$

Partial Integration

Main idea

- If $f \in C^1(\mathbb{R})$ is continuously differentiable, then

$$\Lambda_{f'}(\phi) \;=\; \int_{\mathbb{R}} f'\phi \,\mathrm{d}x \;=\; -\int_{\mathbb{R}} f\phi' \,\mathrm{d}x$$

  for all $\phi \in C_0^\infty(\mathbb{R})$.

- Interesting observation: if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ we can still define

$$\Lambda_{f'}(\phi) \;\stackrel{\mathrm{def}}{=}\; -\int_{\mathbb{R}} f\phi' \,\mathrm{d}x \;.$$

  The derivative f' does not need to exist!

Weak Derivatives

Higher order derivatives:

- If $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ we define

$$\Lambda_{D^k f}(\phi) \stackrel{\mathrm{def}}{=} (-1)^k \int_{\mathbb{R}} f D^k \phi \, \mathrm{d}x \ ,$$

  where $D^k\phi$ denotes the k-th derivative of ϕ; for any $k \in \mathbb{N}$.

Definition

- If there exists a function $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ with $\Lambda_{D^k f} = \Lambda_g$, we say that g is the k-th weak derivative of f.

Weak Derivatives

Examples

- The function $f(x) = |x|$ has a weak derivative, because

$$\Lambda_f(\phi) = \int_{\mathbb{R}} |x|\phi'(x)\,\mathrm{d}x = \int_0^{\infty} \phi(x)\,\mathrm{d}x - \int_{-\infty}^{0} \phi(x)\,\mathrm{d}x = \Lambda_g(\phi)$$

  for $g(x) = \mathrm{sgn}(x)$.

- The function $f(x) = \mathrm{sgn}(x)$ has no weak derivative. (Exercise!)

- The weak derivative of the indicator of the rational numbers $\mathbb{Q}$,

$$f(x) = I_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise,} \end{cases}$$

  is given by $g(x) = 0$.

Multivariate Case

General Notation

- For an open set $\Omega \subseteq \mathbb{R}^n$, define

$$D^\alpha \phi \stackrel{\text{def}}{=} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi \quad \text{and} \quad |\alpha| \stackrel{\text{def}}{=} \sum_{i=1}^n \alpha_i$$

  for all $\phi \in C_0^\infty(\Omega)$ and $\alpha \in \mathbb{N}^n$.

- If there exists a function $g \in L^1_{\text{loc}}(\Omega)$ with

$$\forall \phi \in C_c^\infty(\Omega), \qquad \int_\Omega f D^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g \phi \, \mathrm{d}x \ ,$$

  we say that g is the α-th weak derivative of f and write "$g = D^\alpha f$".

Norms

Notation

- We use the notation

$$\forall \phi \in C_0^\infty(\Omega), \qquad \|\phi\|_{C^N} \stackrel{\text{def}}{=} \max_{0 \le |\alpha| \le N} \sup_x \|D^\alpha \phi(x)\|_\infty$$

  to denote the N-th order supremum norm on $C_0^\infty(\Omega)$.

Upper Bounds

- Let $D^\alpha f = g \in L^1_{\text{loc}}(\Omega)$ and $\phi \in C_0^\infty(\Omega)$ be given and let K be a compact set with $\text{Supp}(\phi) \subseteq K \subseteq \Omega$. Then

$$|\Lambda_{D^\alpha f}(\phi)| = \left| \int_K g D^\alpha \phi \, \mathrm{d}x \right| \le \underbrace{\left(\int_K |g| \, \mathrm{d}x \right)}_{= \|g\|_{L^1(K)}} \cdot \|\phi\|_{C^{|\alpha|}} .$$

Basic Properties of Weak Derivatives

Theorem

- Let $f \in L^1_{\mathrm{loc}}(\Omega)$ admit weak derivatives up to order $|\alpha| \leq N$.

  1. If $g_1, g_2 \in L^1_{\mathrm{loc}}(\Omega)$ are weak α-th derivatives of f, then $g_1(x) = g_2(x)$ for almost all $x \in \Omega$.
  2. If $\alpha, \beta \in \mathbb{N}^n$ satisfy $|\alpha| + |\beta| \leq N$, then

  $$D^\alpha (D^\beta f) = D^\beta (D^\alpha f) \; .$$

  3. Let $f_n, g_n \in L^1_{\mathrm{loc}}(\Omega)$ be convergent sequences with $D^\alpha f_n = g_n$. Then

  $$\left(f = \lim_{n \to \infty} f_n \quad \text{and} \quad g = \lim_{n \to \infty} g_n \right) \qquad \Longrightarrow \qquad g = D^\alpha f \; .$$

Basic Properties of Weak Derivatives

Proof (sketches only)

1. $0 = \int_\Omega (g_1 - g_2)\phi \,\mathrm{d}x$ for all $\phi \in C_0^\infty(\Omega)$ implies $g_1 = g_2$ a.e..

2. For every $\phi \in C_0^\infty(\Omega)$ we have

$$(-1)^{|\beta|} \int_\Omega D^\alpha f(D^\beta \phi) \;=\; (-1)^{|\alpha+\beta|} \int_\Omega f(D^{\alpha+\beta}\phi)\,.$$

   The latter expression is invariant w.r.t. commuting α and β.

3. For every test function $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \int g\phi \,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega g_n \phi \,\mathrm{d}x \\ &= \lim_{n\to\infty} (-1)^\alpha \int_\Omega f_n D^\alpha \phi \,\mathrm{d}x \;=\; (-1)^\alpha \int_\Omega f D^\alpha \phi \,\mathrm{d}x. \end{aligned}$$

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Mollifiers

Standard Mollifier

- The standard mollifier function on $\mathbb{R}^n$ is defined by

$$S(x) \stackrel{\text{def}}{=} \begin{cases} S_0 \exp\left(-\frac{1}{1-\|x\|_2^2}\right) & \text{if } \|x\|_2 < 1 \\ 0 & \text{otherwise,} \end{cases}$$

  where $S_0 > 0$ is constant and such that $\int_{\mathbb{R}^n} S(x)\,\mathrm{d}x = 1$.

- Its associated scaled version (with scaling $\epsilon > 0$) is denoted by

$$S_\epsilon(x) \stackrel{\text{def}}{=} \epsilon^{-n} S\left(\epsilon^{-1}x\right) \quad \Longrightarrow \quad \int_{\mathbb{R}^n} S_\epsilon(x)\,\mathrm{d}x \;=\; 1\;.$$

- The function S_ϵ is smooth, non-negative, and symmetric, $S_\epsilon(-x) = S_\epsilon(x)$. Its support is the ϵ-disc in $\mathbb{R}^n$.

Mollifications

Notation

- For any $\epsilon > 0$ we introduce the notation

$$B_\epsilon(x) \stackrel{\text{def}}{=} \{ \, y \in \mathbb{R}^n \mid \|y - x\| < \epsilon \, \} \ .$$

- For an open set $O \subseteq \mathbb{R}^n$ we write

$$O_\epsilon \stackrel{\text{def}}{=} \{ \, x \in \mathbb{R}^n \mid \mathrm{cl}(B_\epsilon(x)) \subseteq O \, \} \ .$$

Definition

- The convolution of a function $f \in L^p_{\text{loc}}(\Omega)$, $1 \le p < \infty$, with S_ϵ,

$$f_\epsilon = S_\epsilon * f \quad \text{with} \quad f_\epsilon(x) \stackrel{\text{def}}{=} \int_\Omega S_\epsilon(x-y) f(y) \, \mathrm{d}y,$$

  is called the mollification of f.

Basic Approximation Theorem

Theorem

- The set $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, $1 \leq p < \infty$. This means that for every $f \in L^p(\Omega)$ one can find a sequence of functions $f_1, f_2, \ldots \in C_0^\infty(\Omega)$ with $\lim_{k\to\infty} \|f - f_k\|_{L^p} = 0$.

Proof.

- It is sufficient to show the claim for $0 \leq f(x) < \infty$.
- We can always approximate f with a step function: e.g., set

$$\begin{aligned} M_{\epsilon,k} &\stackrel{\text{def}}{=} \{x \in \Omega \mid k\epsilon \leq f(x) \leq (k+1)\epsilon\} \\ \text{and} \quad \sigma_\epsilon(x) &\stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \left[\inf_{x \in M_{\epsilon,k}} f(x)\right] I_{M_{\epsilon,k}}(x) \end{aligned}$$

Basic Approximation Theorem

Proof (continued).

- The above construction is such that $0 \le \sigma_\epsilon \le f \le \sigma_\epsilon + \epsilon$.
- We can refine the step function, e.g., by setting $\tilde{\sigma}_k \stackrel{\text{def}}{=} \sigma_{2^{-k}}$.
- Then we have $0 \le \tilde{\sigma}_1 \le \tilde{\sigma}_2 \le \ldots \le \tilde{\sigma}_k \le f$ and $f = \lim\limits_{k\to\infty} \tilde{\sigma}_k$.
- Now another really technical step: we set

$$s_k \stackrel{\text{def}}{=} \begin{cases} \min\{k, \tilde{\sigma}_k\} & \text{if } \|x\| \le k \\ 0 & \text{otherwise} \end{cases}$$

  such that s_k is a finite superposition of characteristic functions of bounded measurable sets $A \subseteq \Omega$ and $\lim_{k\to\infty} \|f - s_k\|_{L^p} = 0$.
- Summary so far: it is sufficient to approximate characteristic functions in $L^p(\Omega)$ with functions in $C_0^\infty(\Omega)$.

Basic Approximation Theorem

Proof (continued).

- Recall that any bounded measurable sets $A \subseteq \Omega$ can be approximated by an open bounded set $A \subseteq O \subseteq \Omega$ such that $\|I_A - I_O\|_{L^p} < \epsilon$ for given $\epsilon > 0$.
- Moreover, we can approximate I_O with a smooth function $0 \leq \varphi_k \leq I_O$ with $\lim_{k\to\infty} \varphi_k = I_0$ such that $\|I_O - \varphi_k\|_{L^p} \to 0$.
- For instance, we can set $\varphi_k \stackrel{\text{def}}{=} S_{2^{-(k+1)}} * I_{O_{2^{-k}}}$ for $k \gg 1$; recalling that S_ϵ denotes the standard mollifier (see above).
- By collecting all the above argument, the proof is complete.

Mollifications

Theorem

Let $f_\epsilon = S_\epsilon * f$ denote the mollification of $f \in L^p_{\mathrm{loc}}(\Omega)$.

1. We have $f_\epsilon \in C_0^\infty(\Omega_\epsilon)$.
2. If $f \in C^0(\Omega)$, then $f_\epsilon \to f$ uniformly on any compact $K \subseteq \Omega$.
3. If $f \in L^p(\Omega)$, then $\|f_\epsilon\|_{L^p(\Omega_\epsilon)} \ \le \ \|f\|_{L^p(\Omega_\epsilon)}$ for any $1 \le p < \infty$.
4. For $f \in L^p_{\mathrm{loc}}(\Omega)$, we have $f_\epsilon \to f$ in $L^p_{\mathrm{loc}}(\Omega)$.
5. The operator $S_\epsilon * \cdot$ commutes with weak differentiation,

$$\text{on } \Omega_\epsilon : \qquad D^\alpha(S_\epsilon * f) \ = \ S_\epsilon * (D^\alpha f),$$

   as long as $f \in L^1_{\mathrm{loc}}(\Omega)$ admits a weak derivative $D^\alpha f$ for $\alpha \in \mathbb{N}^n$.

Mollifications

Proof (sketches only)

1. Very easy. (Exercise!)

2. For every $x \in \Omega_\epsilon$ we have an estimate of the form

$$
\begin{aligned}
|f(x) - f_\epsilon(x)| &\leq \int_\Omega S_\epsilon(x-y)|f(y)-f(x)|\,\mathrm{d}y \\
&\leq \sup_{y \in \Omega} |f(x) - f(y)| \quad \text{s.t.} \quad \|x-y\| \leq \epsilon\ .
\end{aligned}
$$

   If f is continuous, we get the convergence statement for $\epsilon \to 0$.

Mollifications

Proof (sketches only)

3. For $p = 1$, the proof is relatively easy, since

$$\begin{aligned}
\|f_\epsilon\|_{L^1(\Omega_\epsilon)} &= \int_{\Omega_\epsilon} \left| \int_{\mathbb{R}^n} S_\epsilon(x-y) f(y)\,\mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_{\Omega_\epsilon} \int_{\mathbb{R}^n} S_\epsilon(z) |f(x-z)|\,\mathrm{d}z\,\mathrm{d}x \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} \left(\int_{\Omega_\epsilon} |f(x-z)|\,\mathrm{d}x \right) S_\epsilon(z)\,\mathrm{d}z \\
&\leq \|f\|_{L^1(\Omega)} \int_{\mathbb{R}^n} S_\epsilon(z)\,\mathrm{d}z = \|f\|_{L^1(\Omega)} \ .
\end{aligned}$$

Exercise: show this also for $1 < p < \infty$. (Hint: Hölder's inequality)

Mollifications

Proof (sketches only)

4. We use the notation $g_\epsilon = S_\epsilon g$ and $f_\epsilon = S_\epsilon f$ and fix a $\delta > 0$. Next, take a continuous function g such that

$$\|f - g\|_{L^p(\Omega)} \leq \frac{\delta}{3} \,.$$

Due to third statement (see previous slide), we have

$$\|f_\epsilon - g_\epsilon\|_{L^p(\Omega_\epsilon)} \;\leq\; \|f - g\|_{L^p(\Omega_\epsilon)} \;\leq\; \frac{\delta}{3} \,.$$

Consequently,

$$\|f - f_\epsilon\|_{L^p(\Omega_\epsilon)} \leq \frac{2\epsilon}{3} + \|g - g_\epsilon\|_{L^p(\Omega_\epsilon)}$$

Use the second statement (see above) to conclude the proof.

Mollifications

Proof (sketches only)

5. Since S_ϵ can be regarded as a smooth test function, we find

$$\begin{aligned}(D^\alpha(S_\epsilon * f))(x) &= D_x^\alpha \int_\Omega S_\epsilon(x-y)f(y)\,\mathrm{d}y \\ &= (-1)^{|\alpha|}\int_\Omega D_y^\alpha S_\epsilon(x-y)f(y)\,\mathrm{d}y \\ &= \int_\Omega S_\epsilon(x-y)D_y^\alpha f(y)\,\mathrm{d}y \\ &= (S_\epsilon * (D^\alpha f))(x)\end{aligned}$$

for all $x \in \Omega_\epsilon$. Consequently, mollification and weak differentiation commute.

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Sobolev Spaces

Notation

- We use the notation $W^{k,p}(\Omega)$, $1 \le p < \infty$, to denote the set of locally integrable functions $f : \Omega \to \mathbb{R}$ such that, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$, the weak derivative $D^\alpha f$ exists and belongs to $L^p(\Omega)$.

Norm

- We define the following norm for all $f \in W^{k,p}(\Omega)$:

$$\|f\|_{W^{k,p}(\Omega)} \overset{\text{def}}{=} \begin{cases} \left(\int_\Omega \sum_{|\alpha| \le k} |D^\alpha f|^p \, \mathrm{d}x \right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup}_\Omega |D^\alpha f| & \text{if } p = \infty \ . \end{cases}$$

Definition

- The pair $(W^{k,p}, \| \cdot \|_{W^{k,p}(\Omega)})$ is called a Sobolev space.

Sobolev Spaces

Theorem

- Every Sobolev space, $W^{k,p}(\Omega)$, is a Banach space.

Proof

- First notice that for $f, g \in W^{k,p}(\Omega)$ and $a, b \in \mathbb{R}$, we have

$$D^\alpha(af + bg) = aD^\alpha f + bD^\alpha g \in L^p(\Omega)$$

  Thus, $W^{k,p}(\Omega)$ is a vector space.

- We also need to check that $\|\cdot\|_{W^{k,p}}$ is a norm. The absolute homogeniety and the positive definiteness axiom are easy to check. The triangle inequality follows from Minkowski's inequality. (Exercise!)

Sobolev Spaces

Proof (continued)

- It remains to check that $W^{k,p}(\Omega)$ is complete. Let $f_1, f_2, \ldots \in W^{k,p}(\Omega)$ be a Cauchy sequence in $W^{k,p}(\Omega)$. This implies that there exist limit functions $f, f_\alpha \in L^p(\Omega)$ with

$$\lim_{i\to\infty} \|f_i - f\|_{L^p(\Omega)} \; = \; 0 \qquad \text{and} \qquad \lim_{i\to\infty} \|D^\alpha f_i - f_\alpha\|_{L^p(\Omega)} \; = \; 0$$

  for all α with $|\alpha| \le k$. Next it follows from the basic properties (see Slides 2-9 and 2-10) of the weak derivative that $f_\alpha = D^\alpha f$, which yields completeness.

Sobolev Spaces

Theorem

- Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f \in W^{k,p}(\Omega)$, with $1 \leq p < \infty$. Then there exists a sequence of functions $f_i \in C^\infty(\Omega)$ such that $\|f - f_i\|_{W^{k,p}} \to 0$ for $n \to \infty$.

Proof (very rough sketch...).

- The proof is very similar to the above proof, where we had shown that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$. The main idea is to construct suitable approximation of u by using mollification. (Details: Exercise!).

Sobolev Spaces

Important Remark:

- The above theorem shows that $C^\infty(\Omega)$ is dense in $W^{k,p}$, but this does not imply that $C_0^\infty(\Omega)$ still also has this property.
- Instead we introduce the following definition.

Definition:

- The subspace $W_0^{k,p}(\Omega) \subseteq W^{k,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. This means that $f \in W_0^{k,p}(\Omega)$ if and only if there exists a sequence of functions $f_k \in C_0^\infty(\Omega)$ with $\|f - f_k\|_{W^{k,p}} \to 0$.

Sobolev Spaces

More remarks:

- Since $W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$ it is itself a Banach space with the same norm.

Intuition:

- One can think of $W_0^{1,p}(\Omega)$ as a space of functions which vanish along the boundary $\partial\Omega$.
- Similarly, one can think of $W_0^{k,p}(\Omega)$ as a space of functions f for which $D^\alpha f$ vanishes along the boundary $\partial\Omega$ for $|\alpha| \leq k-1$.
- BUT: keep in mind that $\partial\Omega$ is a set of Lebesgue measure zero. If we want to make this precise, we need trace operators (not discussed in detail in this lecture).

Properties of Weak Derivatives

Theorem

- Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leq p < \infty$, $|\alpha| \leq k$. If $f, g \in W^{k,p}(\Omega)$, then
  1. the restriction of f to any open set $O \subseteq \Omega$ is in $W^{k,p}(O)$,
  2. $D^\alpha f \in W^{k-|\alpha|}(\Omega)$,
  3. if $h \in C^k(\Omega)$, then $fh \in W^{k,p}(\Omega)$; also there exists a $C(h, \Omega) < \infty$ with
  $$\|fh\|_{W^{k,p}} \;\leq\; C(h,\Omega) \cdot \|f\|_{W^{k,p}} \;.$$
  4. if $\varphi : O \to \Omega$ is C^k diffeomorphism between open sets whose Jacobian has a uniformly bounded inverse, then $f \circ \varphi \in W^{k,p}(O)$.

Proof: Exercise!

Advanced Topics

Remarks

- It is not difficult to prove that on open interval domains $\Omega \subseteq \mathbb{R}^n$ there exists for every $f \in W^{1,1}(\Omega)$ an absolutely continuous function $\tilde{f}$ such that $\tilde{f} = f$ and $\tilde{f}' = D_{x_1} f$ almost everywhere in Ω.
- If the boundary of the open domain $\Omega \subseteq \mathbb{R}^n$ is sufficiently regular (often one assumes that $\partial\Omega$ is Lipschitz), one can find for every $f \in W^{1,p}(\Omega)$ a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ such that $Ef = f$ on Ω and $Ef = 0$ and an open set $\tilde{\Omega} \supset \mathrm{cl}(\Omega)$.
- There is a long list of Sobolev embedding theorems around. For instance, if $p > n$ then every $f \in W^{1,p}(\mathbb{R}^n)$ is Hölder continuous (possibly after modification on a set of measure 0).

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Hilbert space

A vector space H with inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ is called a (real) Hilbert space if for all $x, y \in H$ and all $a, b \in \mathbb{R}$:

1. Symmetry: $\langle y, x \rangle = \langle x, y \rangle$.
2. Linearity: $\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$.
3. Positivity: $\langle x, x \rangle \geq 0$ such that $\|x\|_H = \sqrt{\langle x, x \rangle}$ is a norm and such that $(H, \|\cdot\|)$ complete.

Remark

- By construction: every Hilbert space is also a Banach space.

Cauchy-Schwarz Inequality

In any Hilbert space we have

$$\langle x, y\rangle^2 \ \le\ \langle x, x\rangle \langle y, y\rangle \ =\ \|x\|_H^2 \|y\|_H^2$$

Proof We may assume $y \neq 0$. Next,

$$\begin{aligned} \|x\|_H^2 &= \left\| \frac{\langle x, y\rangle}{\langle y, y\rangle} y + x - \frac{\langle x, y\rangle}{\langle y, y\rangle} y \right\|_H^2 \\ &= \left| \frac{\langle x, y\rangle}{\langle y, y\rangle} \right|^2 \|y\|_H^2 + \left\| x - \frac{\langle x, y\rangle}{\langle y, y\rangle} y \right\|_H^2 \ \ge\ \frac{\langle x, y\rangle^2}{\langle y, y\rangle} \end{aligned}$$

implies the Cauchy-Schwarz inequality.

Cauchy-Schwarz Inequality

In any Hilbert space we have

$$\langle x, y \rangle^2 \;\le\; \langle x, x \rangle \langle y, y \rangle \;=\; \|x\|_H^2 \|y\|_H^2$$

Proof We may assume $y \neq 0$. Next,

$$\begin{aligned}
\|x\|_H^2 &= \left\| \frac{\langle x, y \rangle}{\langle y, y \rangle} y + x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\|_H^2 \\
&= \left| \frac{\langle x, y \rangle}{\langle y, y \rangle} \right|^2 \|y\|_H^2 + \left\| x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\|_H^2 \;\ge\; \frac{\langle x, y \rangle^2}{\langle y, y \rangle}
\end{aligned}$$

implies the Cauchy-Schwarz inequality.

The L^2 Space

Definition

- The set $L^2(\Omega)$ can be equipped with its associated L^2-scalar product,

$$\langle f, g\rangle_{L^2} \overset{\text{def}}{=} \int_\Omega f(x)g(x)\,\mathrm{d}x\ .$$

  It is simply called THE L^2-space.

Theorem

- The L^2-space is a Hilbert space.
- If $f \in L^2(\Omega)$ and $\langle f, \varphi\rangle_{L^2} = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $f(x) = 0$ for almost every $x \in \Omega$.

The L^2 Space

Proof (sketch).

- The L^2-scalar product satisfies Hilbert's inner product axioms.
- Let $f \in L^2(\Omega)$ be given and $\langle f, \varphi \rangle_{L^2} = 0$ for all $\varphi \in C_0^\infty(\Omega)$. Since C_0^∞ is dense in $L^2(\Omega)$, there exists a sequence of functions $f_k \in C_0^\infty(\Omega)$ with $\|f_k - f\|_{L^2} \to 0$. Thus,

$$\|f\|_{L^2}^2 \quad = \quad \langle f, f \rangle_{L^2} \; = \; \langle f, f - f_k \rangle_{L^2} \; \le \; \|f\|_{L^2} \|f - f_k\|_{L^2} \to 0$$

and, consequently, $f = 0$ a.e. in Ω.

Hilbert-Sobolev Spaces

- The case $p = 2$ is of particular relevance. We use the notation

$$H^k(\Omega) \stackrel{\text{def}}{=} W^{k,2}(\Omega) .$$

- This is because $H^k(\Omega)$ can be equipped with inner product

$$\langle f, g \rangle_{H^k} \stackrel{\text{def}}{=} \sum_{|\alpha| \le k} \langle D^\alpha f, D^\alpha g \rangle_{L^2}$$

- It follows immediately that $H^k(\Omega)$ is a Hilbert space (and a Banach space). It is called the Hilbert-Sobolev space of order k.
- The subspace $H_0^k(\Omega) \stackrel{\text{def}}{=} W_0^{k,2}(\Omega)$ is a Hilbert space, too.

Dual Spaces

- Let $(H, \langle \cdot, \cdot \rangle)$ be a general Hilbert space norm $\|x\|_H = \sqrt{\langle x, x \rangle}$.
- For every $x \in H$ an associated linear functional is given by

$$\forall y \in \mathbb{H}, \qquad \Lambda_x(y) \stackrel{\text{def}}{=} \langle x, y \rangle \ .$$

- Its norm is defined as

$$\forall x \in H, \qquad \|\Lambda_x\|_{H^*} \stackrel{\text{def}}{=} \sup_{y \in H} \frac{|\Lambda_x(y)|}{\|y\|_H}$$

- For every given $x \in H$, the functional Λ_x is bounded, since

$$\|\Lambda_x\|_{H^*} = \sup_{y \in H} \frac{|\langle x, y \rangle|}{\|y\|_H} \leq \|x\|_H \ ,$$

where we have used the Cauchy-Schwarz inequality.

Dual Spaces

- Question: can every bounded linear functional on H be constructed as on the previous slide?
- Before we give an answer, let us introduce the following definition.

Definition

- The set of bounded linear functionals on H, denoted by H^*, is called the dual space of H. It is equipped with the norm

$$\forall \Lambda \in H^*, \qquad \|\Lambda\|_{H^*} \stackrel{\text{def}}{=} \sup_{y \in H} \frac{|\Lambda(y)|}{\|y\|_H} .$$

Frechét Riesz' Theorem

Theorem

- Let H be a Hilbert space with dual space H^*, as defined above. The space $(H^*, \|\cdot\|_{H^*})$ is a Banach space. Moreover, there exists for every $\Lambda \in H^*$ a unique $x \in H$ such that

$$\forall y \in H, \quad \Lambda(y) = \langle y, x\rangle_H \qquad \text{and} \qquad \|\Lambda\|_{H^*} = \|x\|_H \ .$$

Proof.

- Let us first show that H^* is complete: if $\Lambda_1, \Lambda_2, \ldots \in H^*$ is a Cauchy sequence, there exists for every $\epsilon > 0$ a $N \in \mathbb{N}$ such that $\|\Lambda_i - \Lambda_j\|_{H^*} < \epsilon$ for all $i, j \geq N$ and, consequently,

$$|\Lambda_i(y) - \Lambda_j(y)| \ \leq \ \|\Lambda_i - \Lambda_j\|_{H^*}\|y\|_H \ \leq \ \epsilon\|y\| \ .$$

Frechét Riesz' Theorem

Proof (continued).

Thus, $\Lambda_1(y), \Lambda_2(y), \ldots \in \mathbb{R}$ is a Cauchy sequence. As such, $\Lambda(y) \stackrel{\text{def}}{=} \lim_{k\to\infty} \Lambda_k(y)$ exists and is a linear functional on H. Since

$$|\Lambda(y)| \;=\; \lim_{k\to\infty} |\Lambda_k(y)| \;\leq\; \limsup_k \|\Lambda_k\|_{H^*}\|y\|,$$

the functional Λ is bounded and, consequently, in H^*. Thus, the dual space H^* is complete.

- Our next goal is to show that the map $J : H \to H^*$ given by $x \to \Lambda_x$ is bijective. First, it is clearly injective since

$$\forall y \in H, \quad \Lambda_x(y) - \Lambda_{x'}(y) = \langle x - x', y\rangle = 0 \qquad \Longrightarrow \qquad x = x' \,.$$

Frechét Riesz' Theorem

Proof (continued).

- In order to proceed, let us analyze the quadratic functional

$$\forall y \in H, \quad f(y) \stackrel{\text{def}}{=} \|y\|_H^2 - 2\Lambda(y) \ .$$

Notice that f is bounded from below, since

$$f(y) \ \geq \ \|y\|_H^2 - 2|\Lambda(y)| \ \geq \ \|y\|_H^2 - 2\|\Lambda\|_{H^*}\|y\| \ \geq \ -\|\Lambda\|_{H^*}^2 \ .$$

Thus, $\alpha \stackrel{\text{def}}{=} \inf_{y \in H} f(y)$ exists. Let $x_1, x_2, \ldots \in H$ be a minimizing sequence. Thus, we find (see next slide)

Frechét Riesz' Theorem

Proof (continued).

$$
\begin{aligned}
\|x_k - x_l\|_H^2 &= 2\|x_k\|_H^2 + 2\|x_l\|_H^2 - \|x_k + x_l\|_H^2 \\
&= 2\|x_k\|_H^2 - 4\Lambda(x_k) + 2\|x_l|_H^2 - 4\Lambda(x_l) \\
& \quad -4\left\|\frac{x_k + x_l}{2}\right\|_H^2 + 8\Lambda\left(\frac{x_k + x_l}{2}\right) \\
&= 2f(x_k) + 2f(x_l) - 4f\left(\frac{x_k + x_l}{2}\right) \\
&\leq 2f(x_k) + 2f(x_l) - 4\alpha \ \rightarrow \ 0 \quad (\text{for } k, l \rightarrow \infty \ ).
\end{aligned}
$$

- Thus, $x_1, x_2, \ldots \in H$ is a Cauchy sequence and $x = \lim_{k \to \infty} x_k$ exists.

Frechét Riesz' Theorem

Proof (continued).

- In summary, since f is continuous, x is a minimizer of f,

$$f(x) = \lim_{k \to \infty} f(x_k) = \alpha \; .$$

- Because $f(x) \leq f(x + ty)$ for all $t \in \mathbb{R}$, $y \in H$, we find

$$\begin{aligned} & \|x\|_H^2 - 2\Lambda(x) \;\leq\; \|x + ty\|_H^2 - 2\Lambda(x + ty) \\ \Longrightarrow \quad & 0 \;\leq\; 2t\langle x, y\rangle - 2t\Lambda(y) + t^2\|y\|_H^2 \\ \Longrightarrow \quad & \forall y \in H, \quad \langle x, y\rangle = \Lambda(y) \; . \end{aligned}$$

Moreover, since J is injective, x is unique; and J is bijective.

Frechét Riesz' Theorem

Proof (continued).

- Our last step is to show that $\|\Lambda\|_{H^*} = \|x\|_H$. It follows from

$$\|\Lambda\|_{H^*} = \sup_{y\in H} \frac{\Lambda(y)}{\|y\|_H} = \sup_{y\in H} \frac{\langle y, x\rangle}{\|y\|_H} \leq \|x\|_H$$

and $$\|x\|_H^2 = \Lambda(x) \leq \|\Lambda\|_{H^*}\|x\|_H \, .$$

This completes the proof.

Remark

- The map $J : H \to H^*$ in the above proof turns out to be an isometric isomorphism between the spaces H and H^*.

Distributions

Motivation

- The Frechét-Riesz representation theorem is the basis for many existence theorems in PDE theory.
- Our next goal is to understand the dual Hilbert-Sobolev spaces

$$H^{-k} \stackrel{\text{def}}{=} \left(H_0^k\right)^* .$$

Definition

- Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A linear functional $\Lambda : C_0^\infty(\Omega) \to \mathbb{R}$ is called a distribution, if there exists for every compact $K \subseteq \Omega$ an integer $N \geq 0$ and a constant C such that $|\Lambda(\phi)| \leq C\|\phi\|_{C^N}$ for every $\phi \in C^\infty(\Omega)$ with support contained in K.

Distributions

Remark

- In the above definition N and C may depend on K.
- If N does not depend on K, we say that Λ has finite order.
- The smallest such integer N is called the order of the distribution.

Example

- Let $a \in \Omega$ be a given point. The linear functional

$$\forall \phi \in C_0^\infty(\Omega), \qquad \Lambda(\phi) \;=\; \phi(a)$$

is called the Dirac distribution at a. It has the order $N = 0$.

Distributional Derivative

Definition

- The α-th derivative ($\alpha \in \mathbb{N}^n$) of a distribution Λ is defined by

$$\forall \phi \in C_0^\infty(\Omega), \qquad D^\alpha \Lambda(\phi) \overset{\text{def}}{=} (-1)^{|\alpha|} \Lambda(D^\alpha \phi) \; .$$

Remarks

- Don't mix up distributional and weak derivatives. Distributional derivatives are more general. They always exist!
- The derivative $D^\alpha \Lambda$ is itself a distribution. But, because

$$|D^\alpha \Lambda| \; = \; |\Lambda(D^\alpha \phi)| \; \leq \; C \|D^\alpha \phi\|_{C^N} \; \leq \; C \|\phi\|_{C^{N+|\alpha|}} \; ,$$

we need to replace N by $N + |\alpha|$.

The space $H^{-1}(\Omega)$.

Theorem

- Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For every element $\Lambda \in H^{-1}(\Omega)$, one can find functions $f_0, f_1, \ldots, f_n \in L^2(\Omega)$ such that

$$\Lambda = \Lambda_{f_0} + \sum_{i=1}^{n} D^1_{x_i} \Lambda_{f_i}$$

and $$\|\Lambda\|_{H^{-1}} = \left(\sum_{i=0}^{n} \int_\Omega f_i^2 \, \mathrm{d}x \right)^{\frac{1}{2}} .$$

Proof

- The Frechét Riesz representation theorem ensures that for every $\Lambda \in H^{-1}(\Omega)$, there exist a (even unique!) $g \in H_0^1(\Omega)$ such that

$$\begin{array}{rrcl} \forall \phi \in H_0^{-1}(\Omega), & \Lambda(\phi) & = & \langle g, \phi \rangle_{H^1} \\ \text{and} & \|\Lambda\|_{H^{-1}} & = & \|g\|_{H^1} \ . \end{array}$$

With $f_0 \stackrel{\text{def}}{=} g$ and $f_i \stackrel{\text{def}}{=} -D^1_{x_i} g$, the first relation becomes

$$\begin{aligned} \Lambda(\phi) &= \int_\Omega \left[g\phi + \sum_{i=1}^n D^1_{x_i} g D^1_{x_i} \phi \right] \mathrm{d}x \\ &= \Lambda_{f_0}(\phi) - \sum_{i=1}^n \Lambda_{f_i}(D^1_{x_i}\phi) \ = \ \Lambda_{f_0}(\phi) + \sum_{i=1}^n D^1_{x_i} \Lambda_{f_i}(\phi) \end{aligned}$$

and the second relation yields $\|\Lambda\|_{H^{-1}} = \left(\sum_{i=0}^n \|f_i\|^2_{L^2}\right)^{\frac{1}{2}}$.

The space $H^{-1}(\Omega)$.

Remarks

- The relation $\Lambda = \Lambda_{f_0} + \sum_{i=1}^{n} D^1_{x_i}\Lambda_{f_i}$ from the above theorem is sometimes also written in the sloppy but intuitive form

$$f = f_0 + \sum_{i=1}^{n} \partial_{x_i} f_i \ .$$

  Here, f is regarded as a representative for an element of $H^{-1}(\Omega)$ and the terms $\partial_{x_i} f_i$ are interpreted as distributional derivatives.

- The above theorem can, of course, be generalized to other Hilbert-Sobolev spaces H^{-k}, too. (Exercise!)

Poincare's Inequality

Motivation

- Recall that the set $H_0^1(\Omega)$ is equipped with the norm

$$\|f\|_{H^1} \;=\; \langle f, f\rangle_{H^1}^{\frac{1}{2}} \;=\; \left(\int_\Omega f^2 + |Df|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

- Question: could we, instead, also use the "simplified" map

$$\|f\|_0 \;\stackrel{\text{def}}{=}\; \left(\int_\Omega |Df|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

  to define a norm on $H_0^1(\Omega)$?

- It turns out that if Ω is bounded in some direction then $\|\cdot\|_0$ is indeed a norm on H_0^1 that is equivalent to the original norm $\|\cdot\|_{H^1}$. This is a consequence of Poincare's inequality, which is introduced next.

Poincare's Inequality

Theorem

- Let $\Omega \subseteq \mathbb{R}^n$ be an open set that is bounded in at least one direction.

  Then there is a constant $C < \infty$ such that

$$\forall f \in H_0^1(\Omega), \qquad \int_\Omega f^2 \,\mathrm{d}x \;\leq\; C \int_\Omega |Df|^2 \,\mathrm{d}x \;.$$

Proof

- Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we may assume $f \in C_0^\infty(\Omega)$.
- We may assume w.l.o.g. that $0 < x_n < a < \infty$ for all $x \in \Omega$.
- We'll write $x = (x', x_n)$ with $x' = (x_1, x_2, \ldots, x_{n-1})$.

Poincare's Inequality

Proof (continued)

- An application of triangle- and the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
|f(x', x_n)| &= \left| \int_0^{x_n} D_{x_n} f(x', s)\, \mathrm{d}s \right| \\
&\leq \int_0^a 1 \cdot |D_{x_n} f(x', s)|\, \mathrm{d}s \\
&\leq \sqrt{a} \left(\int_0^a |D_{x_n} f(x', s)|^2\, \mathrm{d}s \right)^{\frac{1}{2}} .
\end{aligned}
$$

  Consequently, we have

$$
|f|^2 \;\leq\; a \int_0^a |D_{x_n} f(x', s)|^2\, \mathrm{d}s \quad \Longrightarrow \quad \int_\Omega |f|^2\, \mathrm{d}x \;\leq\; a^2 \int_\Omega |D_{x_n} f|^2\, \mathrm{d}x
$$

- Use $|D_{x_n} f| \leq |Df|$ and set $C = a^2$.

Poincare's Inequality

Discussion

- Back to our question: can we use the map

$$\|f\|_0 \stackrel{\text{def}}{=} \left(\int_\Omega |Df|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

  to define a norm on $H_0^1(\Omega)$?

- Poincare: "yes, if Ω is bounded in at least one direction".

- The equivalence of the two norms follows from

$$\|f\|_0 \le \|f\|_{H^1} \le \sqrt{1+C}\, \|f\|_0 .$$

Lax-Milgram Lemma

- The following lemma by Lax and Milgram is a very practical generalization of the representation theorem by Frechét and Riesz.

Lemma

- Let $a : H \times H \to \mathbb{R}$ be a bilinear form on the Hilbert space H with

$$\forall x, y \in H, \quad a(x,x) \ \geq \ \alpha \|x\|_H^2 \quad \text{and} \quad |a(y,x)| \leq \beta \|x\|_H \|y\|_H$$

for constant $\alpha, \beta \in (0, \infty)$. Then there exists for every given $\Lambda \in H^*$ a unique $x \in H$ such that

$$\forall y \in H, \qquad \Lambda(y) = a(y,x) \qquad \text{and} \qquad \|x\|_H \leq \alpha^{-1} \|\Lambda\|_{H^*} \ .$$

Lax-Milgram Lemma

Proof

- For every given $x \in H$, the map $\Gamma_x(\cdot) \stackrel{\text{def}}{=} a(\cdot, x)$ satisfies $\Gamma_x \in H^*$, because Γ_x is linear and bounded:

$$\sup_{y \in H} \frac{|\Gamma_x(y)|}{\|y\|_H} \;=\; \sup_{y \in H} \frac{|a(y,x)|}{\|y\|_H} \;\leq\; \beta \|x\|_H \; .$$

- Thus, Frechét-Riesz implies that there exists an element $Ax \in H$ with

$$\forall y \in H, \qquad \Gamma_x(y) = \langle y, Ax \rangle_H \qquad \text{and} \qquad \|Ax\|_H \;=\; \|\Gamma_x\|_{H^*} \; .$$

- Because we assume that a is also linear in its second argument, the map $x \to Ax$ is linear. It is also bounded, because

$$\forall x \in H, \quad \|Ax\|_H \;=\; \sup_{y \in H} \frac{|\Gamma_x(y)|}{\|y\|_H} \;=\; \sup_{y \in H} \frac{|a(y,x)|}{\|y\|_H} \;\leq\; \beta \|x\|_H \; .$$

Lax-Milgram Lemma

Proof (continued)

- For any given $\Lambda \in H^*$ we can find a unique $b \in H$ with

$$\forall y \in H, \qquad \Lambda(y) = \langle y, b \rangle_H \ .$$

- Collect the above relations:

$$\begin{aligned} \forall y \in H, \quad a(y,x) = \Lambda(y) \qquad & \Longleftrightarrow \qquad \forall y \in H, \quad \langle y, Ax \rangle_H = \langle y, b \rangle_H \\ & \Longleftrightarrow \qquad Ax = b \ . \end{aligned}$$

- Next goal: show that $Ax = b$ has a unique solution $x \in H$.
- Plan: choose a smalle $\delta > 0$ and analyze the "gradient method"

$$x^+ = x - \delta(Ax - b) \ .$$

Lax-Milgram Lemma

Proof (continued)

- In detail, the "gradient method operator" $T : H \to H$, given by

$$Tx \stackrel{\text{def}}{=} x - \delta(Ax - b)$$

  is linear and turns out to be contractive on H. This means that we'll be able to apply Banach's fixed point theorem.

- The above plan works out, because we have the estimate

$$\begin{aligned} \|x - \delta Ax\|_H^2 &\leq \|x\|_H^2 - 2\delta a(w, w) + \delta^2 \|Ax\|_H^2 \\ &\leq (1 - 2\delta\alpha + \delta^2\beta^2)\|x\|_H^2 \ . \end{aligned}$$

  For sufficiently small $0 < \delta \ll 1$ this is a strict contraction.

Lax-Milgram Lemma

Proof (continued)

- In summary, Banach's fixed point theorem implies the existence of a unique x with $Tx = x$. This is equivalent to $Ax = b$ and, in turn,

$$\forall y \in H, \quad a(y,x) = \Lambda(y) \; .$$

- If we substitute $x = y$, we further find

$$\begin{array}{rrcl} & \alpha \|x\|_H^2 & \leq & a(x,x) \; = \; \Lambda(x) \; \leq \; \|\Lambda\|_{H^*} \|x\|_H \\ \Longrightarrow & \|x\|_H & \leq & \alpha^{-1} \|\Lambda\|_{H^*} \; . \end{array}$$

  This concludes our proof.

Weak Convergence

Definition

- Let H be a real Hilbert space. We say that a sequence $x_1, x_2, \ldots \in H$ converges weakly to $x \in H$ if

$$\forall y \in H, \qquad \lim_{k \to \infty} \langle y, x_k \rangle \ = \ \langle y, x \rangle \ .$$

Theorem (Variant/consequence of Banach-Alaoglu theorem)

- Every bounded sequence in a Hilbert space H contains a weakly convergent subsequence.
- Every bounded, closed and convex subset $C \subseteq X$ is weakly sequentially compact.

Remark: usually this result is proven in a more context of reflexive Banach spaces, but this goes beyond this lecture.