Adaptive Control of Nonlinear Systems

Contents

Two categories:

- The nonlinear systems whose nonlinear functions are known and whose unknown parameters appear linearly
- The unknown nonlinear functions are assumed to be approximated with unknown parameters multiplying known basis functions

Purpose: give a flavor of the complexity of controlling nonlinear systems with unknown parameters and/or unknown nonlinearities.

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Feedback Linearization

Idea: using change of coordinates and feedback control to *cancel all or most of the nonlinear terms* so that the system behaves as a linear or partially linear system.

Let us consider the nonlinear system in affine with input

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n, u, y \in \mathcal{R}$, and f, g, h are smooth (differentiable infinitely many times) nonlinear functions.

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Feedback Linearization

Differentiate y w.r.t time to obtain

$$\dot{y} = \frac{\partial h}{\partial x}(x) f(x) + \frac{\partial h}{\partial x}(x) g(x) u$$

where

$$\frac{\partial h}{\partial x} f = \frac{\partial h}{\partial x_1} f_1 + \cdots + \frac{\partial h}{\partial x_n} f_n \triangleq L_f h$$

and $L_f h$ is known as the Lie derivative.

Relative Degree for nonlinear systems:

- If $\frac{\partial h}{\partial x}(x_0)\, g(x_0) \neq 0$ at some point x_0, then we say that the system (1) has relative degree 1 at x_0.

Remark: In LTI, this means the system has one pole more than zeros.

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Relative Degree for nonlinear systems:

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- If $\frac{\partial h}{\partial x}(x)g(x) = 0\ \forall x \in B_{x_0}$, then we take the second derivative of y to obtain

$$\ddot{y} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} f\right) f + \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} f\right) gu$$

  If $\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}(x)f(x)\right) g(x)\Big|_{x=x_0} \neq 0$, then (1) is said to have relative degree 2 at x_0.

Remark: the relative degree is equal to the number of times the output has to be differentiated until the input appears.

Feedback Linearization

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Feedback Linearization

Lie derivative notation:

$$
\begin{aligned}
L_f^0 h &\triangleq h, \\
L_f^1 h &\triangleq L_f h \triangleq \frac{\partial h}{\partial x} f, \\
L_f^2 h &\triangleq L_f \left(L_f h\right) = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} f\right) \cdot f \\
L_f^3 h &\triangleq L_f \left(L_f^2 h\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} f\right) \cdot f\right) \cdot f \\
&\;\;\vdots
\end{aligned}
$$

i.e.,

$$
L_f^{i+1} h \triangleq L_f \left(L_f^i h\right) = \frac{\partial \left(L_f^i h\right)}{\partial x} \cdot f, \quad i = 0, 1, 2, 3, \ldots
$$

Feedback Linearization

In addition, we define

$$L_g L_f h \triangleq \frac{\partial (L_f h)}{\partial x} \cdot g$$

Nonlinear relative degree definition: the SISO nonlinear system (1) has relative degree ρ at a point x_0 if

(i) $L_g L_f^i h(x) = 0, \ \forall x \in B_{x_0}$, where B_{x_0} is some neighborhood of $x_0 \forall i = 1, 2, 3, \ldots \rho - 2$

(ii) $L_g L_f^{\rho-1} h(x_0) \neq 0$

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Remark: the relative degree is equal to the number of times the output has to be differentiated until the input appears.

Feedback Linearization

Assume that the system (1) has relative degree $\rho = n$ at x, and consider the transformation

$$z_i = y^{(i-1)} = L_f^{i-1} h(x), i = 1, 2 ... n$$

It follows that

$$\dot{z}_{i-1} = z_i, i = 2, ... n$$

$$\dot{z}_n = L_f^n h(x) + \left(L_g L_f^{n-1} h(x) \right) u$$

$$y = z_1$$

which is known as the canonical or normal form of the system with no zero dynamics.

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Feedback Linearization

The feedback control law

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left[v - L_f^n h(x) \right]$$

where $v \in \mathcal{R}$ is a new input, leads to the new LTI system

$$\dot{z} = Az + Bv, \quad y = C^T z$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Feedback Linearization

Since system

$$\dot{z} = Az + Bv, \quad y = C^T z,$$

is an observable and controllable LTI system, the input $v = K_c \hat{z}$ can be easily selected to meet regulation or tracking objectives for the plant output y. In this case the control law

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left[K_c \hat{z} - L_f^n h(x) \right]$$

cancels all the nonlinearities via feedback and forces the closed-loop system to behave as an LTI system.

Remark : An essential condition is the accuracy of the plant model.

Feedback Linearization

If the system (1) has relative degree $\rho < n$, then the change of coordinates

$$z_1 = y, \quad z_2 = \dot{y}, \quad \dots, \quad z_\rho = y^{(\rho-1)} = L_f^{(\rho-1)} h(x)$$

will lead to ρ dimension subsystem

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{\rho-1} &= z_\rho, \\
\dot{z}_\rho &= L_f^\rho h(x) + \left(L_g L_f^{\rho-1} h(x) \right) u
\end{aligned}$$

Feedback Linearization

Additional $n - \rho$ states are defined as

$$z_{\rho+1} = h_{\rho+1}(x)$$

$$\vdots$$

$$z_n = h_n(x)$$

with $h_i(x)$ satisfying $\frac{\partial h_i(x)}{\partial x} g(x) = 0$, to obtain the additional states

$$\dot{z}_{\rho+1} = \frac{\partial h_{\rho+1}(x)}{\partial x} \cdot f(x) \triangleq \varphi_{\rho+1}(z)$$

$$\vdots$$

$$\dot{z}_n = \frac{\partial h_n(x)}{\partial x} \cdot f(x) \triangleq \varphi_n(z)$$

Feedback Linearization

If we use the feedback control law

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left[v - L_f^{\rho} h(x) \right]$$

the overall system

$$\dot{z}_i = z_{i+1}, \quad i = 1, 2, ...\rho - 1$$

$$\dot{z}_\rho = v$$

$$\dot{z}_j = \varphi_i(z), \quad j = \rho + 1, ...n$$

$$y = z_1$$

In this case the input v may be chosen to drive the output y and states $z_1, \ldots, z_\rho$ to zero or meet some tracking objective for y.

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Feedback Linearization

Note, the choice of the input v, however, may not guarantee that the states $z_{\rho+1}, \ldots, z_n$ are bounded even when $z_1, \ldots, z_\rho$ are driven to zero.

When $z_1 = z_2 = \cdots = z_\rho = 0$, the dynamics

$$\begin{aligned} \dot{z}_{\rho+1} &= \varphi_{\rho+1}(0, \ldots, 0, z_{\rho+1}, \ldots, z_n) \\ &\vdots \\ \dot{z}_n &= \varphi_n(0, \ldots, 0, z_{\rho+1}, \ldots, z_n) \end{aligned}$$

are called the zero dynamics of (1). If the equilibrium of zero dynamics is asymptotically stable, the system (1) is said to be minimum-phase.

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Feedback Linearization

I/O feedback linearization :The process of using feedback to transform the nonlinear system to a linear system from the input v to the output y . In the case of $\rho = n$, the system (1) is linearized without zero dynamics, and this process is called *full-state feedback linearization.*

Example Consider the third-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + u, \\ \dot{x}_3 &= -u \\ y &= x_1 \end{aligned}$$

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$$\dot{x}_1 = x_2$$

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Feedback Linearization

Example Consider the controlled van der Pol equation

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + \varepsilon \left(1 - x_1^2\right) x_2 + u, \\
y &= x_2,
\end{aligned}$$

where $\varepsilon > 0$ is a constant. The first derivative of y is

$$\dot{y} = -x_1 + \varepsilon \left(1 - x_1^2\right) x_2 + u.$$

Contents

CLF

As shown in the case of adaptive laws based on the SPR-Lyapunov synthesis approach, Lyapunov functions are also used to design adaptive laws in addition to analysis. Along this spirit, Lyapunov functions can be used to design stabilizing feedback control laws. Consider the nonlinear system

$$\dot{x} = f(x, u), \tag{2}$$

where $x \in \mathcal{R}^n, u \in \mathcal{R}$, and $f(0,0) = 0$. We want to choose the control input $u = q(x)$ with $q(0) = 0$ so that the equilibrium $x_e = 0$ of the closed-loop system

$$\dot{x} = f(x, q(x))$$

is global asymptotically stable.

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CLF

The CLF method involves the selection of a function $V(x)$ as a Lyapunov candidate. We therefore need to find $q(x)$ so that

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x, q(x)) \leq -Q(x)$$

where $Q(x)$ is a positive definite function.

The Lyapunov function which can serve this purpose is referred to as the *control Lyapunov function (CLF)*.

Remark: It is obvious that the success of the approach depends very much on the choice of $V(x)$.

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CLF

Definition: A smooth positive definite and radially unbounded function $V(x)$ is called a CLF for System (2) if

$$\inf_u \left\{ \frac{\partial V}{\partial x}(x) f(x,u) \right\} \leq -Q(x) \quad \forall x \neq 0$$

for some positive definite function $Q(x)$.

Example: Consider the system

$$\dot{x}_1 = x_2 - x_1$$

$$\dot{x}_2 = \sin x_1 \cos x_2 - x_2^3 + u$$

$$y = x_1$$

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Backstepping

Let us consider a first-order nonlinear system

$$\dot{x} = x^3 u$$

which is uncontrollable at $x = 0$. We consider $V(x) = \frac{x^2}{2}$ as a potential CLF. Then $\dot{V} = x^4 u$, and for

$$u = -x^2$$

we obtain

$$\dot{V} = -x^6 \leq -Q(x) = -x^6.$$

Therefore, $V(x)$ is a CLF, and the equilibrium $x_e = 0$ described by $\dot{x} = -x^5$ is a.s. The choice of the CLF in this case was easy.

Backstepping

Let us now consider the second-order nonlinear system

$$\dot{x}_1 = x_1^3 x_2,$$
$$\dot{x}_2 = u$$

Comparing with previous first order system, we see that x_2 took the place of u. If x_2 was the control variable, then $x_2 = -x_1^2$ would guarantee that x_1 converges to zero. Let us denote this desired value of x_2 as $x_{2d} \triangleq q(x_1) = -x_1^2$ and define the error between x_2 and x_{2d} as

$$z = x_2 - x_{2d} = x_2 - q(x_1).$$

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Backstepping

Then the second-order system may be represented as

$$\dot{x}_1 = x_1^3 \left(z + q\left(x_1\right)\right), \quad z = x_2 - q\left(x_1\right)$$
$$\dot{x}_2 = u$$

If we treat z as the new state, we obtain the representation

$$\dot{x}_1 = x_1^3 \left(z + q\left(x_1\right)\right)$$
$$\dot{z} = u - \dot{q}(x_1)$$

This new representation is obtained by using $q(x)$ as a feedback control loop while we "back step" $q(x)$ through the integrator. This process motivated the name backstepping.

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Backstepping

By augmenting the CLF for the system $\dot{x}_1 = x_1^3 u$ with a quadratic term in the error variable z, i.e.,

$$V_a(x, z) = V\left(x_1\right) + \frac{z^2}{2}$$

where $V\left(x_1\right) = \frac{x_1^2}{2}$. Then

$$\dot{V}_a = x_1^4 z + x_1^4 q\left(x_1\right) + zu - z\dot{q}\left(x_1\right)$$

where $q\left(x_1\right) = -x_1^2$. Hence

$$\dot{V}_a = -x_1^6 + z\left[x_1^4 - \dot{q}\left(x_1\right) + u\right]$$

Backstepping

Choosing

$$u = \dot{q}\,(x_1) - x_1^4 - z$$

we have

$$\dot{V}_a = -x_1^6 - z^2 \leq -\left(x_1^6 + z^2\right) = -Q(x,z),$$

which implies that the equilibrium $x_{1e} = 0, z_e = 0$ of the closed-loop system

$$\dot{x}_1 = x_1^3\left(z - x_1^2\right),$$

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is asymptotically stable.

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Since

$$\dot{q}\left(x_1\right) = -\frac{dx_1^2}{dt} = -2x_1\dot{x}_1 = -2x_1^4\left(z - x_1^2\right),$$

the control law can be implemented as

$$\begin{aligned} u &= -2x_1^4\left(z - x_1^2\right) - x_1^4 - z \\ &= -x_2 - x_1^2 - x_1^4 - 2x_1^4 x_2 \end{aligned}$$

Remark: The above example demonstrates that if we know the CLF of the system, we can generate the CLF of the same system augmented with a pure integrator.

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Backstepping

Extend to the more general nonlinear system

$$\begin{aligned}\dot{x}_1 &= f(x_1) + g(x_1)\,x_2, \quad f(0)=0,\\ \dot{x}_2 &= u,\end{aligned} \tag{3}$$

where $f(0)=0; x_1 \in \mathcal{R}^n; x_2, u \in \mathcal{R}.$, satisfies :

Assumption 1 Consider the system $\dot{x}_1 = f(x_1) + g(x_1)\,u$, there exists a $u = q_1(x_1), q_1(0) = 0$ which satisfies

$$\frac{\partial V}{\partial x_1}(x_1)\left(f(x_1) + g(x_1)\,q_1(x_1)\right) \le -Q(x_1) < 0$$

for some smooth, p.d., r.u. $V : \mathcal{R}^n \to \mathcal{R}$, and a p.d. $Q : \mathcal{R}^n \to \mathcal{R}$, i.e., $V(x)$ is a CLF for the feedback x_1-system.

Backstepping

Extend to the more general nonlinear system

$$\begin{aligned} \dot{x}_1 &= f\left(x_1\right) + g\left(x_1\right) x_2, \quad f(0) = 0, \\ \dot{x}_2 &= u, \end{aligned} \tag{3}$$

where $f(0) = 0; x_1 \in \mathcal{R}^n; x_2, u \in \mathcal{R}.$, satisfies :

Assumption 1 Consider the system $\dot{x}_1 = f\left(x_1\right) + g\left(x_1\right) u$, there exists a $u = q_1\left(x_1\right), q_1(0) = 0$ which satisfies

$$\frac{\partial V}{\partial x_1}\left(x_1\right)\left(f\left(x_1\right) + g\left(x_1\right) q_1\left(x_1\right)\right) \leq -Q\left(x_1\right) < 0$$

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Backstepping

Lemma Consider the nonlinear $(n+1)$-order system (3) which satisfies the Assumption 1. Then the function

$$V_a\left(x_1, x_2\right) = V\left(x_1\right) + \frac{z^2}{2}$$

where $z = x_2 - q_1\left(x_1\right)$ and $q_1\left(x_1\right)$ is defined in Assumption 1, is a CLF for the system (3) in the sense that there exists a feedback control law

$$u = q_2\left(x_1, x_2\right)$$

which guarantees that the equilibrium $x_{1e} = 0, x_{2e} = 0$ is g.a.s..

Backstepping

Furthermore, one choice of such a feedback control law is

$$q_2 = -c\left(x_2 - q_1\left(x_1\right)\right) + \frac{\partial q_1}{\partial x_1}\left(x_1\right)\left(f\left(x_1\right) + g\left(x_1\right)x_2\right) - \frac{\partial V}{\partial x_1}\left(x_1\right)g\left(x_1\right)$$

where c > 0 is an arbitrary constant.

Proof: We treat x_2 as the virtual control with the desired value of $q_1\left(x_1\right)$. The error variable $z = x_2 - q_1\left(x_1\right)$ is used to transform the system to

$$\begin{aligned}\dot{x}_1 &= f\left(x_1\right) + g\left(x_1\right)\left[q_1\left(x_1\right) + z\right] \\ \dot{z} &= u - \frac{\partial q_1}{\partial x_1}\left(x_1\right)\left[f\left(x_1\right) + g\left(x_1\right)\left(q_1\left(x_1\right) + z\right)\right]\end{aligned}$$

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Backstepping

Then $\dot{V}_a$ can be calculated as

$$\begin{aligned}\dot{V}_a =& \frac{\partial V}{\partial x_1}(x_1)\left(f(x_1) + g(x_1)(q_1(x_1) + z)\right) \\ & + z\left(u - \frac{\partial q_1}{\partial x_1}(x_1)\left(f(x_1) + g(x_1)(q_1(x_1) + z)\right)\right) \\ \leq & - Q(x_1) + z\left(u - \frac{\partial q_1}{\partial x_1}\left(f + g(q_1 + z)\right) + \frac{\partial V}{\partial x_1} g\right)\end{aligned}$$

where the last inequality is obtained by using Assumption 1.

Choosing $u = q_2$ aforementioned, it follows that

$$\dot{V}_a \leq -Q(x_1) - cz^2 \leq -Q_a(x_1, z)$$

where $Q_a(x_1, z) = Q(x_1) + cz^2$ is positive definite in $\mathcal{R}^{n+1}$

Backstepping

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Backstepping

Remark: The choice of the control law q_2 is not unique, and in some cases it may not be the desirable one since it cancels all the nonlinear terms in the expression of V_a inside the brackets.

Remark: The backstepping method may be further extended to

$$
\begin{aligned}
\dot{x}_1 &= f(x_1) + g(x_1)\, x_2, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_{k-1} &= x_k, \\
\dot{x}_k &= u,
\end{aligned}
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where f, g satisfy Assumption 1.

Backstepping

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