Parametric Models

- Models for Dynamic Systems
- Plant Parametric Models

Contents

- Models for Dynamic Systems

- Plant Parametric Models

State-Space Models for Dynamic Systems

State-Space Models:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

$$y(t) = g(x(t), u(t), t)$$

where

- t is the time variable
- $x(t)$ is an n-dimensional vector with real elements that denotes the state of the system
- $u(t)$ is an r-dimensional vector with real elements that denotes the input variable or control input of the system

State-Space Models for Dynamic Systems

State-Space Models:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

$$y(t) = g(x(t), u(t), t)$$

where

- t is the time variable
- $x(t)$ is an n-dimensional vector with real elements that denotes the state of the system
- $u(t)$ is an r-dimensional vector with real elements that denotes the input variable or control input of the system

State-Space Models for Dynamic Systems

State-Space Models:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

$$y(t) = g(x(t), u(t), t)$$

where

- $y(t)$ is an l-dimensional vector with real elements that denotes the output variables that can be measured
- f, g are real vector valued functions
- $x(t_0)$ denotes the value of $x(t)$ at the initial time $t = t_0 \geq 0$

Note, n is the dimension of the state x, called the order of the system

State-Space Models for Dynamic Systems

State-Space Models:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

$$y(t) = g(x(t), u(t), t)$$

where

- $y(t)$ is an l-dimensional vector with real elements that denotes the output variables that can be measured
- f, g are real vector valued functions
- $x(t_0)$ denotes the value of $x(t)$ at the initial time $t = t_0 \geq 0$

Note, n is the dimension of the state x, called the order of the system

State-Space Models for Dynamic Systems

LTV system: when f, g are linear functions of x and u

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$

$$y = C^\top(t)x + D(t)u$$

where $A(t) \in \mathcal{R}^{n\times n}, B(t) \in \mathcal{R}^{n\times r}, C(t) \in \mathcal{R}^{n\times l}$, $D(t) \in \mathcal{R}^{l\times r}$. If in addition to being linear, f, g do not depend on time t, we have

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$y = C^\top x + Du$$

where A, B, C, and D are matrices of the same dimension as above but with constant elements.

State-Space Models for Dynamic Systems

LTV system: when f, g are linear functions of x and u

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$

$$y = C^\top(t)x + D(t)u$$

where $A(t) \in \mathcal{R}^{n\times n}, B(t) \in \mathcal{R}^{n\times r}, C(t) \in \mathcal{R}^{n\times l}$, $D(t) \in \mathcal{R}^{l\times r}$. If in addition to being linear, f, g do not depend on time t, we have

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$y = C^\top x + Du$$

where A, B, C, and D are matrices of the same dimension as above but with constant elements.

State-Space Models for Dynamic Systems

LTV system: when f, g are linear functions of x and u

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$
$$y = C^\top(t)x + D(t)u$$

where $A(t) \in \mathcal{R}^{n\times n}, B(t) \in \mathcal{R}^{n\times r}, C(t) \in \mathcal{R}^{n\times l}$, $D(t) \in \mathcal{R}^{l\times r}$. If in addition to being linear, f, g do not depend on time t, we have

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$
$$y = C^\top x + Du$$

where A, B, C, and D are matrices of the same dimension as above but with constant elements.

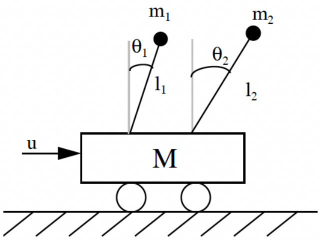

Figure 2.1 Cart with two inverted pendulums

Example: Let us consider the cart with the two inverted pendulums shown in Figure 2.1, where M is the mass of the cart, m_1 and m_2 are the masses of the bobs, and l_1 and l_2 are the lengths of the pendulums, respectively.

Example: Using Newton's law and assuming small angular deviations of $|\theta_1|, |\theta_2|$, the equations of motions are given by

$$
\begin{aligned}
M\dot{v} &= -m_1 g\theta_1 - m_2 g\theta_2 + u \\
m_1\left(\dot{v} + l_1\ddot{\theta}_1\right) &= m_1 g\theta_1 \\
m_2\left(\dot{v} + l_2\ddot{\theta}_2\right) &= m_2 g\theta_2
\end{aligned}
$$

where v is the velocity of the cart, u is an external force, and g is the acceleration due to gravity. Assume that $m_1 = m_2 = 1$ kg and $M = 10m_1$. If we now let $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_1 - \theta_2$, $x_4 = \dot{\theta}_1 - \dot{\theta}_2$ be the state variables, write the dynamic of the system into the following form:

$$\dot{x} = Ax + Bu$$

Example: Using Newton's law and assuming small angular deviations of $|\theta_1|, |\theta_2|$, the equations of motions are given by

$$\begin{aligned} M\dot{v} &= -m_1 g\theta_1 - m_2 g\theta_2 + u \\ m_1\left(\dot{v} + l_1\ddot{\theta}_1\right) &= m_1 g\theta_1 \\ m_2\left(\dot{v} + l_2\ddot{\theta}_2\right) &= m_2 g\theta_2 \end{aligned}$$

where v is the velocity of the cart, u is an external force, and g is the acceleration due to gravity. Assume that $m_1 = m_2 = 1$ kg and $M = 10m_1$. If we now let $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_1 - \theta_2$, $x_4 = \dot{\theta}_1 - \dot{\theta}_2$ be the state variables, write the dynamic of the system into the following form:

$$\dot{x} = Ax + Bu$$

Let $x = [x_1, x_2, x_3, x_4]^\top$, we have the 4th order system is described by

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1.2\alpha_1 & 0 & -0.1\alpha_1 & 0 \\ 0 & 0 & 0 & 1 \\ 1.2\left(\alpha_1 - \alpha_2\right) & 0 & \alpha_2 - 0.1\left(\alpha_1 - \alpha_2\right) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \beta_1 \\ 0 \\ \beta_1 - \beta_2 \end{bmatrix}$$

and $\alpha_1 = \frac{g}{l_1}, \alpha_2 = \frac{g}{l_2}, \beta_1 = -\frac{0.1}{l_1}$, and $\beta_2 = -\frac{0.1}{l_2}$.

Solutions

For LTV system:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau$$

with $\Phi(t, t_0)$ is transition matrix, satisfies

$$\frac{\partial \Phi\left(t, t_0\right)}{\partial t} = A(t)\Phi\left(t, t_0\right), \quad \Phi\left(t_0, t_0\right) = I$$

For LTI system, it holds $\Phi(t, t_0) = e^{A(t-t_0)}$ and the solution $x(t), y(t)$ are given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = C^{\top}x(t) + Du(t)$$

Note: In this course, we usually assume D matrix is zero.

Solutions

For LTV system:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau$$

with $\Phi(t, t_0)$ is transition matrix, satisfies

$$\frac{\partial \Phi\left(t, t_0\right)}{\partial t} = A(t)\Phi\left(t, t_0\right), \quad \Phi\left(t_0, t_0\right) = I$$

For LTI system, it holds $\Phi(t, t_0) = e^{A(t-t_0)}$ and the solution $x(t), y(t)$ are given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = C^{\top}x(t) + Du(t)$$

Note: In this course, we usually assume D matrix is zero.

Solutions

For LTV system:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau$$

with $\Phi(t, t_0)$ is transition matrix, satisfies

$$\frac{\partial \Phi\left(t, t_0\right)}{\partial t} = A(t)\Phi\left(t, t_0\right), \quad \Phi\left(t_0, t_0\right) = I$$

For LTI system, it holds $\Phi(t, t_0) = e^{A(t-t_0)}$ and the solution

$x(t), y(t)$ are given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = C^\top x(t) + Du(t)$$

Note: In this course, we usually assume D matrix is zero.

Solutions

For LTV system:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau$$

with $\Phi(t, t_0)$ is transition matrix, satisfies

$$\frac{\partial \Phi\left(t, t_0\right)}{\partial t} = A(t)\Phi\left(t, t_0\right), \quad \Phi\left(t_0, t_0\right) = I$$

For LTI system, it holds $\Phi(t, t_0) = e^{A(t-t_0)}$ and the solution

$x(t), y(t)$ are given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = C^{\top}x(t) + Du(t)$$

Note: In this course, we usually assume D matrix is zero.

Input-Output Model

Consider a system described by the nth-order differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = b_m u^{(m)}(t) + \cdots + b_0 u(t)$$

where

$y^{(i)}(t) \triangleq \frac{d^i}{dt^i} y(t)$, and $u^{(i)}(t) \triangleq \frac{d^i}{dt^i} u(t)$;

$u(t)$ is the input variable, and $y(t)$ is the output variable;

the coefficients $a_i, b_j, i = 0, 1 \ldots, n-1, j = 0, 1, \ldots, m$ are constants, and n and m are constant integers.

Note, above equation can only describe single-input-single-output (SISO) system.

Input-Output Model

Consider a system described by the nth-order differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = b_m u^{(m)}(t) + \cdots + b_0 u(t)$$

where

$y^{(i)}(t) \triangleq \frac{d^i}{dt^i} y(t)$, and $u^{(i)}(t) \triangleq \frac{d^i}{dt^i} u(t)$;

$u(t)$ is the input variable, and $y(t)$ is the output variable;

the coefficients $a_i, b_j, i = 0, 1 \ldots, n-1, j = 0, 1, \ldots, m$ are constants, and n and m are constant integers.

Note, above equation can only describe single-input-single-output (SISO) system.

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Definitions:

1. We say that $G(s)$ is proper, if $n \geq m$; strictly proper if $n > m$; and biproper if $n = m$.
2. The relative degree n^* of $G(s)$ is defined as $n^* = n - m$.
3. Coprime

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

Definitions:

1. We say that $G(s)$ is proper, if $n \geq m$; strictly proper if $n > m$; and biproper if $n = m$.
2. The relative degree n^* of $G(s)$ is defined as $n^* = n - m$.
3. Coprime

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Definitions:

1. We say that $G(s)$ is proper, if $n \geq m$; strictly proper if $n > m$; and biproper if $n = m$.
2. The relative degree n^* of $G(s)$ is defined as $n^* = n - m$.
3. Coprime

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Definitions:

1. We say that $G(s)$ is proper, if $n \geq m$; strictly proper if $n > m$; and biproper if $n = m$.
2. The relative degree n^* of $G(s)$ is defined as $n^* = n - m$.
3. Coprime

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} := \frac{Z(s)}{P(s)}$$

Definitions:

1. Characteristic equation $P(s) = 0$
2. Roots of $P(s) = 0$ are poles of system $= \mathrm{eig}[\mathrm{A}]$
3. Roots of $Z(s) = 0$ are zeros of system
4. Minimum phase and Hurwitz

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} := \frac{Z(s)}{P(s)}$$

Definitions:

1. Characteristic equation $P(s) = 0$
2. Roots of $P(s) = 0$ are poles of system $= \mathrm{eig}[\mathrm{A}]$
3. Roots of $Z(s) = 0$ are zeros of system
4. Minimum phase and Hurwitz

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} := \frac{Z(s)}{P(s)}$$

Definitions:

1. Characteristic equation $P(s) = 0$
2. Roots of $P(s) = 0$ are poles of system $= \mathrm{eig}[\mathrm{A}]$
3. Roots of $Z(s) = 0$ are zeros of system
4. Minimum phase and Hurwitz

Input-Output Model

To obtain the transfer function, we take the Laplace transform on both sides of the equation and assume zero initial conditions

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} := \frac{Z(s)}{P(s)}$$

Definitions:

1. Characteristic equation $P(s) = 0$
2. Roots of $P(s) = 0$ are poles of system $= \mathrm{eig}[\mathrm{A}]$
3. Roots of $Z(s) = 0$ are zeros of system
4. Minimum phase and Hurwitz

Relation between SS and TF

SS representation

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$y = C^\top x + Du$$

Transfer function

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

From a SS representation to a transfer function:

$$G(s) = C^\top (sI - A)^{-1} B + D$$

From a transfer function description to a SS representation, is not as straightforward and not *Unique*.

Relation between SS and TF

SS representation

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$y = C^\top x + Du$$

Transfer function

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

From a SS representation to a transfer function:

$$G(s) = C^\top (sI - A)^{-1} B + D$$

From a transfer function description to a SS representation, is not as straightforward and not *Unique*.

Relation between SS and TF

SS representation

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$y = C^\top x + Du$$

Transfer function

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

From a SS representation to a transfer function:

$$G(s) = C^\top (sI - A)^{-1} B + D$$

From a transfer function description to a SS representation, is not as straightforward and not *Unique*.

Relation between SS and TF

For instance, consider a 4th order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \frac{Z(s)}{Z(s)}.$$

Notice that, by multiplying by $Z(s)/Z(s)$, we do not change the transfer function, $G(s)$. Equating the numerator and denominator polynomials yields

$$Y(s) = \left[b_3 s^3 + b_2 s^2 + b_1 s + b_0 \right] Z(s)$$

and

$$U(s) = \left[s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \right] Z(s)$$

Relation between SS and TF

For instance, consider a 4th order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \frac{Z(s)}{Z(s)}.$$

Notice that, by multiplying by $Z(s)/Z(s)$, we do not change the transfer function, $G(s)$. Equating the numerator and denominator polynomials yields

$$Y(s) = \left[b_3 s^3 + b_2 s^2 + b_1 s + b_0 \right] Z(s)$$

and

$$U(s) = \left[s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \right] Z(s)$$

Define the four state variables as follows:

$$
\begin{aligned}
x_1(t) &= z(t) \\
x_2(t) &= \dot{x}_1(t) = \dot{z}(t) \\
x_3(t) &= \dot{x}_2(t) = \dddot{z}(t) \\
x_4(t) &= \dot{x}_3(t) = \dddot{z}(t)
\end{aligned}
$$

Then we obtain

$$\dot{x}_4(t) = -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t)$$

and the corresponding output equation is

$$y(t) = b_0 x_1(t) + b_1 x_2(t) + b_2 x_3(t) + b_3 x_4(t).$$

Define the four state variables as follows:

$$
\begin{aligned}
x_1(t) &= z(t) \\
x_2(t) &= \dot{x}_1(t) = \dot{z}(t) \\
x_3(t) &= \dot{x}_2(t) = \ddot{z}(t) \\
x_4(t) &= \dot{x}_3(t) = \dddot{z}(t)
\end{aligned}
$$

Then we obtain

$$\dot{x}_4(t) = -a_0 x_1(t) - a_1 x_2(t) - a_2 x_3(t) - a_3 x_4(t) + u(t)$$

and the corresponding output equation is

$$y(t) = b_0 x_1(t) + b_1 x_2(t) + b_2 x_3(t) + b_3 x_4(t).$$

Exercise: Consider a fourth order dynamics

$$y^{(4)} - 1.1\,(\alpha_1 + \alpha_2)\,y^{(2)} + 1.2\alpha_1\alpha_2 y = \beta_1 u^{(2)} - \alpha_1\beta_2 u$$

find its transfer function and ss presentation.

Canonical form

For transfer function

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

controllable canonical form

$$\dot{x}_c = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots 0 \end{bmatrix} x_c$$

Canonical form

or in the observer form

$$\dot{x} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ b_m \\ \vdots \\ b_0 \end{bmatrix} u$$

$$y = [1, 0, \ldots, 0]x$$

Contents

- Models for Dynamic Systems
- Plant Parametric Models

Linear Parametric Model

Consider an n-th order differential equation given by

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + \cdots + b_0 u$$

where a_i and b_i are unknown parameters characterizing the plant.

IF the system can be written in the compact form of

$$z = \theta^\top \phi \tag{1}$$

where θ is a vector contains all the unknown parameter, z and ϕ are signal available for measurement, then we refer (1) as *Linear Static Parametric Model* (SPM)

Linear Parametric Model

Consider an n-th order differential equation given by

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = b_{n-1} u^{(n-1)} + b_{n-2} u^{(n-2)} + \cdots + b_0 u$$

where a_i and b_i are unknown parameters characterizing the plant.

IF the system can be written in the compact form of

$$z = \theta^\top \phi \tag{1}$$

where θ is a vector contains all the unknown parameter, z and ϕ are signal available for measurement, then we refer (1) as *Linear Static Parametric Model* (SPM)

Linear Parametric Model

Consider an n-th order differential equation given by

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = b_{n-1} u^{(n-1)} + b_{n-2} u^{(n-2)} + \cdots + b_0 u$$

where a_i and b_i are unknown parameters characterizing the plant.

IF the system can be written in the compact form of

$$z = W(s)(\theta^\top \phi) \qquad (2)$$

where θ is a vector contains all the unknown parameter, z and ϕ are signal available for measurement, $W(s)$ **is a known stable proper transfer function** then we refer (2) as *Linear Dynamic Parametric Model* (DPM)

Linear Parametric Model

For SPM and DPM, given the measurements $z(t), \phi(t)$, we can design on-line estimation algorithm that estimates θ with $\hat{\theta}(t)$ at each time.

The linearity is important for accurate estimation!

Example: Consider a first-order system

$$\dot{x} = -x + ax + bu$$

where a, b are unknown parameters.

How about $\dot{x} = ax + bu$? What if $\dot{x}$ is not available?

Linear Parametric Model

For SPM and DPM, given the measurements $z(t), \phi(t)$, we can design on-line estimation algorithm that estimates θ with $\hat{\theta}(t)$ at each time.

The linearity is important for accurate estimation!

Example: Consider a first-order system

$$\dot{x} = -x + ax + bu$$

where a, b are unknown parameters.

How about $\dot{x} = ax + bu$? What if $\dot{x}$ is not available?

Linear Parametric Model

For SPM and DPM, given the measurements $z(t), \phi(t)$, we can design on-line estimation algorithm that estimates θ with $\hat{\theta}(t)$ at each time.

The linearity is important for accurate estimation!

Example: Consider a first-order system

$$\dot{x} = -x + ax + bu$$

where a, b are unknown parameters.

How about $\dot{x} = ax + bu$? What if $\dot{x}$ is not available?

Bilinear Parametric Model

A special case of nonlinear in the parameters models for which convergence results exist is

$$z = \rho(\theta^\top \phi + z_1) \tag{3}$$

and

$$z = W(s)\rho(\theta^\top \phi + z_1) \tag{4}$$

where θ, ρ are unknown parameters, z, z_1 and ϕ are signals available for measurement, $W(s)$ is a known stable proper transfer function then we refer (3) and (4) as *Bilinear Static Parametric Model* (B-SPM) and *Bilinear Dynamic Parametric Model*(B-DPM), respectively.

Bilinear Parametric Model

Example: Consider the mass-spring-dashpot system shown in Figure 2.1 .Using Newton's law, we obtain the differential equation that describes the dynamics of the system as

$$M\ddot{x} = u - kx - f\dot{x}$$

Let us assume that the mass M, damping coefficient f and spring-coefficient k are the constant unknown parameters that we want to estimate online, write the parametric model of the system.

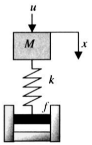

Figure 2.1. *Mass–spring–dashpot system.*

Bilinear Parametric Model

Example: Consider the second ARMA model

$$y(k+4) = -a_1 y(k+3) - a_2 y(k+2) + b_1 u(k+1) + b_2 u(k)$$

where $y(k)$ and $u(k)$ are available signals. Express the model into a form of SPM.

What if $a_1 \neq 0$ but is very small, and the rest of unknown parameters are extremely large, can you write the model into a B-SPM form so that the parameter estimation will be in similar range?

Bilinear Parametric Model

Example: Consider the second ARMA model

$$y(k+4) = -a_1 y(k+3) - a_2 y(k+2) + b_1 u(k+1) + b_2 u(k)$$

where $y(k)$ and $u(k)$ are available signals. Express the model into a form of SPM.

What if $a_1 \neq 0$ but is very small, and the rest of unknown parameters are extremely large, can you write the model into a B-SPM form so that the parameter estimation will be in similar range?

Bilinear Parametric Model

Generalization: Consider a general SISO LTI system with transfer function equals to

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Given $m \leq n$ and only $y(t)$ and $u(t)$ are available for measurements. Express the model into a form of SPM.

$$z = \frac{1}{\Lambda(s)}y^{(n)} = \frac{s^n}{\Lambda(s)}y$$

$$\theta^* = [b_m, \ldots, b_0, a_{n-1}, \ldots, a_0]^T \in \mathcal{R}^{n+m+1}$$

$$\phi = \left[\frac{s^m}{\Lambda(s)}u, \ldots, \frac{1}{\Lambda(s)}u, -\frac{s^{n-1}}{\Lambda(s)}y, \ldots, -\frac{1}{\Lambda(s)}y\right]^T$$

Bilinear Parametric Model

Generalization: Consider a general SISO LTI system with transfer function equals to

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

Given $m \leq n$ and only $y(t)$ and $u(t)$ are available for measurements. Express the model into a form of SPM.

$$z = \frac{1}{\Lambda(s)} y^{(n)} = \frac{s^n}{\Lambda(s)} y$$

$$\theta^* = [b_m, \ldots, b_0, a_{n-1}, \ldots, a_0]^T \in \mathcal{R}^{n+m+1}$$

$$\phi = \left[\frac{s^m}{\Lambda(s)} u, \ldots, \frac{1}{\Lambda(s)} u, -\frac{s^{n-1}}{\Lambda(s)} y, \ldots, -\frac{1}{\Lambda(s)} y \right]^T$$

Summary:

- System Modeling: SS and Transfer function
- Four parametric models