Ordinary Differential Equations

- Ordinary Differential Equations
- Existence and uniqueness of solutions
- Gronwall's Lemma

Contents

Problem Formulation

The focus of this lecture is on ordinary differential equations (ODEs),

$$\text{a.e. } t \in \mathbb{R}, \qquad \dot{x}(t) = f(t, x(t)) \quad \text{with} \quad x(0) = x_0 \ .$$

Here, $x \in W^{1,1}_{\text{loc}}(\mathbb{R})^n$ denotes the state trajectory.

Remarks:

- The function $f \in L^1_{\text{loc}}(\mathbb{R}; C^0(\mathbb{R}^n))^n$ is locally integrable and continuous in x, but sometimes stronger assumptions are needed.
- "$\dot{x}$" denotes a weak derivative of the state $x \in W^{1,1}_{\text{loc}}(\mathbb{R})^n$.
- The initial value $x_0 \in \mathbb{R}^n$ is given (at least in this lecture; later more).

Explicit solution attempt

- In general: no explicit solution possible
- But it some special cases, we can solve a nonlinear differential equation by using the concept of separation of variables.

Seperation of variables:

- Assumption: f is scalar separable; that is,

$$f(t,x) = f_1(x)f_2(t) \ .$$

- Strategy: integrate the equation

$$\frac{\dot{x}(t)}{f_1(x(t))} = f_2(t) \ ,$$

with respect to t on both sides and eliminate $x(t)$.

Example: quadratic differential equation

Nonlinear ODE:

$$\dot{x}(t) = -x^2(t) \quad \text{with} \quad x(0) = 1 \; .$$

Separation of variables:

$$-\frac{\dot{x}(t)}{x(t)^2} = 1 \qquad \overset{\text{integrate}}{\Longrightarrow} \qquad \frac{1}{x(t)} - \frac{1}{x(0)} = t$$

Elimination of $x(t)$:

$$x(t) = \frac{1}{1+t} \quad \text{for all} \quad t \geq 0 \; .$$

Example: Gauss' differential equation

ODE:

$$\dot{x}(t) = -tx(t) \quad \text{with} \quad x(0) = 1 \; .$$

Separation of variables:

$$\frac{\dot{x}(t)}{x(t)} = -t \qquad \Longrightarrow \qquad \log(x(t)) = -\frac{1}{2}t^2$$

Elimination of $x(t)$:

$$x(t) = e^{-\frac{t^2}{2}} \; ,$$

Contents

Integral Form

Definition

- Assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n)^n$ is continuous in x. A function $x \in W^{1,1}((0,T))^n$ is called a weak solution of

$$\text{a.e. } t \in [0,T], \qquad \dot{x}(t) = f(t, x(t)) \quad \text{with} \quad x(0) = x_0 \ . \tag{1}$$

  on the interval $[0,T]$ if we have

$$-\int_0^T \dot{\phi}(t)^\intercal x(t) \mathrm{d}t - \phi(0)^\intercal x_0 \ = \ \int_0^T \phi(t)^\intercal f(t, x(t)) \, \mathrm{d}t$$

  for all test functions with $\phi \in C^\infty([0,T])^n$ with $\phi(T) = 0$.

Theorem

- A function $x \in W^{1,1}((0,T))^n$ is a weak solution of (1) if and only if

$$\text{a.e. } t \in [0,T], \qquad x(t) = x_0 + \int_0^t f(s, x(s)) \, \mathrm{d}s \ .$$

Integral Form

Proof.

- As a consequence of the Lebesgue dominated convergence theorem,

$$F(t) \overset{\mathrm{def}}{=} \int_0^t f(s, x(s))\,\mathrm{d}s$$

  is continuous on $[0, T]$ for any given $x \in W^{1,1}((0,T))^n$.

- Consequently, since $F(0) = 0$, we have

$$\int_0^T \phi(t)^\intercal f(t, x(t))\,\mathrm{d}t \;=\; -\int_0^T \dot{\phi}(t)^\intercal F(t)\,\mathrm{d}t$$

  for all $\phi \in C^\infty([0,T])^n$ with $\phi(T) = 0$.

Integral Form

Proof.

- Moreover, all $\phi \in C^\infty([0,T])^n$ with $\phi(T) = 0$ satisfy

$$0 = \phi(0) + \int_0^T \dot{\phi}(t)\,\mathrm{d}t \quad \Longrightarrow \quad \phi(0)^\intercal x_0 = -\int_0^T \dot{\phi}(t)^\intercal x_0\,\mathrm{d}t$$

- Thus, if x is a weak solution, then

$$\begin{aligned} & -\int_0^T \dot{\phi}(t)^\intercal x\,\mathrm{d}t + \int_0^T \dot{\phi}(t)^\intercal x_0\,\mathrm{d}t = -\int_0^T \dot{\phi}(t)^\intercal F(t)\,\mathrm{d}t \\ \Longleftrightarrow \quad & 0 \;=\; \langle x - x_0 - F, \dot{\phi}\rangle_{L^2} \end{aligned}$$

  for all $\phi \in C^\infty([0,T])^n$ with $\phi(T) = 0$.

- Note that there exists for every $\xi \in C^\infty([0,T])^n$ a $\phi \in C^\infty([0,T])^n$ with $\phi(T) = 0$ and $\dot{\phi} = \xi$.

Integral Form

Proof.

- Thus, in summary we find that

$$
\begin{array}{rrcl}
 & \forall \xi \in C^\infty([0,T])^n, \qquad 0 & = & \langle x - x_0 - F, \xi \rangle_{L^2} \\
\Longleftrightarrow & \text{a.e. } t \in [0,T], \qquad 0 & = & x(t) - x_0 - F(t) \\
\Longleftrightarrow & \text{a.e. } t \in [0,T], \qquad x(t) & = & x_0 + \int_0^t f(s, x(s))\, \mathrm{d}s \; .
\end{array}
$$

- This completes the proof, as the above derivation can be reversed.

Uniform Lipschitz continuity

In the following, we are interested in potentially time-varying right-hand functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, which have the following two properties:

1. The function f satisfies $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n)^n$.
2. The function f is **uniformly** Lipschitz continuous in x. This means that there exist a constant $L < \infty$ with

$$\forall t \in \mathbb{R},\ \forall x, y \in \mathbb{R}^n, \qquad \|f(t,x) - f(t,y)\| \leq L\|x - y\| \ .$$

   Here, "uniformly" means that L does not depend on t, x, or y.

Existence and Uniqueness

Theorem (Caratheodory + Picard-Lindelöf):

- If $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n)^n$ is uniformly Lipschitz continuous in x, the ODE has a unique weak solution $x \in W^{1,1}([0,T])^n$ for all $0 < T < \infty$.

Proof: (outline of the main idea)

1) Start with <u>any</u> continuous function $y_1 \in C^0([0,T])$ and iterate

$$y_{i+1}(t) = x_0 + \int_0^t f(s, y_i(s))\,\mathrm{d}s \qquad \text{[Picard iteration]}$$

2) Show that $y_1, y_2, y_3, \ldots$ is a Cauchy sequence, $y^* = \lim_{k\to\infty} y_i$.

3) Conclude that the (unique) limit point y^* satisfies the ODE.

Existence and Uniqueness

Proof: (details)

- Since $y_1 \in C^0([0,T])^n$ while $f \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^n)^n$ is continuous in x, the Picard iterates satisfy $y_2, y_3, \ldots \in C^0([0,T])^n$.
- Let us define $\Delta(t) = \max_{s \in [0,t]} |y_2(s) - y_1(s)|$.
- Claim: we have

  $$|y_{i+1}(t) - y_i(t)| \ \leq \ \frac{(tL)^{i-1}}{(i-1)!} \Delta(t)$$

  for all $i \in \mathbb{N}_{\geq 1}$.
- Note that the claim is correct for $i = 1$ (induction start).

Existence and Uniqueness

Proof: (continued)

- Next, suppose the claim holds for i (induction assumption), then

$$\begin{aligned} \|y_{i+2}(t) - y_{i+1}(t)\| &= \int_0^t \left(f(s, y_{i+1}(s)) - f(s, y_i(s)) \right) \mathrm{d}s \\ &\leq \int_0^t L \, \|y_{i+1}(s) - y_i(s)\| \, \mathrm{d}s \\ &\leq \int_0^t L \frac{(sL)^{i-1}}{(i-1)!} \Delta(t) \, \mathrm{d}s \; = \; \frac{(tL)^i}{i!} \Delta(t) \; . \end{aligned}$$

  This corresponds to our induction step, proving the claim.

Existence and Uniqueness

Proof: (details)

- By using the above result, we find

$$\begin{aligned} |y_n(t) - y_m(t)| &\leq \sum_{i=n}^{m-1} |y_{i+1}(t) - y_i(t)| \leq \sum_{i=n}^{m-1} \frac{(tL)^{i-1}}{(i-1)!} \Delta(t) \\ &\leq \frac{(tL)^{n-1}}{(n-1)!} e^{L|t|} \Delta(t) \; . \end{aligned}$$

Thus, $y_1, y_2, \ldots$ is a Cauchy sequence in the Banach space $C^0([0,T])^n$. Consequently, the limit $x = \lim_{k \to \infty} y_k$ exists and satisfies

$$\forall t \in [0,T], \qquad x(t) = x_0 + \int_0^t f(s, x(s)) \, \mathrm{d}s \; .$$

Existence and Uniqueness

Proof: (details)

- According to Banach's fixed point theorem the limit function x is unique and also continuous, $x \in C^0([0,T])^n$.
- As x is continuous, it is also Lebesgue integrable, $x \in L^1((0,T))^n$.
- From the equation

$$\forall t \in [0,T], \qquad x(t) = x_0 + \int_0^t f(s, x(s))\,\mathrm{d}s$$

  it follows that $\dot{x}(t) \stackrel{\text{def}}{=} f(t, x(t))$ is a weak derivative of x that is itself in $L^1((0,T))^n$, since we assume that $f \in L^1_{\text{loc}}(\mathbb{R}; C^0(\mathbb{R}^n))^n$.
- In summary, we have $x \in W^{1,1}((0,T))^n$, which completes our proof.

Example: Linear ODEs

- Linear ODE: $\dot{x}(t) = Ax(t)$, $A \in \mathbb{R}^{n \times n}$, with $x(0) = x_0 \in \mathbb{R}^n$.
- Picard iteration:

$$\begin{aligned} y_1(t) &= x_0 \\ y_2(t) &= x_0 + tAx_0 \\ y_3(t) &= x_0 + tAx_0 + \tfrac{t^2}{2}A^2x_0 \\ &\vdots \end{aligned}$$

- Take the limit to get explicit solution

$$x(t) = e^{At}x_0 \qquad \text{with} \qquad e^{At} \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \frac{1}{i!}[tA]^i x_0 \ .$$

Examples for nonlinear ODEs

- The ODE $\dot{x}(t) = x(t)^2$, with $x(0) = 1$ has the explicit solution

$$x(t) = \frac{1}{1-t} \quad \text{for} \quad t < 1$$

  Why does the solution not exist for $t \geq 1$?

- The ODE $\dot{x}(t) = 2\sqrt{x}$, with $x(0) = 0$ has more than one solution,

$$\text{for example} \quad x(t) = 0 \quad \text{and} \quad x(t) = t^2 \ .$$

  Why is there more than one solution?

Contents

Gronwall's Lemma

Lemma

- Let $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfy the differential inequality

$$\text{a.e. } t \geq t_0, \qquad w(t) \leq a \int_{t_0}^{t} w(s)\,\mathrm{d}s + b$$

  for given constants $a, b \geq 0$, $t_0 \in \mathbb{R}$. Then we have $w(t) \leq e^{a(t-t_0)}b$ for almost every $t \geq t_0$.

Gronwall's Lemma

Proof

- The function $\psi(t) \stackrel{\text{def}}{=} a \int_{t_0}^{t} w(s)\,\mathrm{d}s + b$ is absolutely continuous and satisfies (a.e.) $\psi'(t) = aw(t) \leq a\psi(t)$. This implies

$$(e^{-at}\psi(t))' \;=\; e^{-at}(\psi'(t) - a\psi(t)) \leq 0,$$

Thus, $e^{-at}\psi(t)$ is monotonously decreasing and, consequently,

$$\text{a.e. } t \geq t_0, \quad e^{-at}w(t) \;\leq\; e^{-at}\psi(t) \;\leq\; e^{-at_0}\psi(t_0) \;=\; be^{-at_0}\,.$$

From the latter inequality we conclude that (a.e.) $w(t) \leq e^{a(t-t_0)}b$.

Consequence of Gronwall's Lemma

Theorem

- Let $f, g \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n)^n$ be uniformly Lipschitz continuous in x with Lipschitz constant $L < \infty$ such that

$$\varepsilon(t) \;\overset{\mathrm{def}}{=}\; \sup_{x \in \mathbb{R}^n} \mathrm{ess}\, \|f(t,x) - g(t,x)\| \;<\; \infty \,.$$

If $x, z \in W^{1,1}([0,T])^n$ satisfy

$$\begin{aligned} \dot{x}(t) &= f(t,x(t)) \qquad & x(t_0) &= x_0 \\ \dot{z}(t) &= f(t,z(t)) \qquad & z(t_0) &= z_0 \,, \end{aligned}$$

then we have

$$\|x(t) - z(t)\| \;\leq\; e^{L(t-t_0)} \left(\|x_0 - z_0\| + \int_{t_0}^{t} \varepsilon(s)\, \mathrm{d}s \right) \,.$$

Consequence of Gronwall's Lemma

Proof

- The difference function $e(t) \stackrel{\text{def}}{=} x(t) - z(t)$ satisfies

$$
\begin{aligned}
e(t) \;&=\; \int_{t_0}^{t} \left(f(s, x(s)) - f(s, z(s)) \right) \, \mathrm{d}s \\
&\quad + \int_{t_0}^{t} \left(f(s, z(s)) - f(s, z(s)) \right) \, \mathrm{d}s + (x_0 - z_0) \; .
\end{aligned}
$$

  This implies

$$
\| e(t) \| \;\leq\; L \int_{t_0}^{t} \| e(s) \| \, \mathrm{d}s + \int_{t_0}^{t} \varepsilon(s) \, \mathrm{d}s + \| x_0 - z_0 \| \; .
$$

  An application of Gronwall's lemma yields the desired upper bound.

Conditioning of Differential Equations

- The factor e^{Lt} can be interpreted as a **global** upper bound on the condition number of a differential equation.
- In general, for large t, predictions are impossible: "butterfly effect".
- BUT: Gronwall's lemma has no information about the stability properties of the differential equation for $t \to \infty$.
- For some differential equations, a **local** analysis yields better bounds and potentially indicates *local stability*.

First Order Variational Analysis

- Consider the differential equations

$$\dot{x}(t) = f(x(t)) \qquad \text{with} \qquad x(0) = x_0$$

$$\dot{z}(t) = f(z(t)) \qquad \text{with} \qquad z(0) = z_0$$

  for a continuously differentiable right-hand f with bounded Jacobian.

- The linear matrix differential equation

$$\dot{X}(t) = \frac{\partial f(x(t))}{\partial x} X(t) \quad \text{with} \quad X(0) = I$$

  is called the first order variational differential equation.

- It yields the first order Taylor approximation

$$z(t) = x(t) + X(t)(z_0 - x_0) + \mathbf{o}\left(\|z_0 - x_0\|\right)$$

  Unfortunately: in general only valid for finite $t \leq T < \infty$!

Quick Proof (sketch only)

- Since f has a bounded Jacobian, the first order variational differential equation has a unique solution X.
- Next, introduce the shorthands

$$e(t) = z(t) - x(t) - X(t)(z_0 - x_0) \quad \text{and} \quad A(t) = \frac{\partial f(x(t))}{\partial x}$$

We have

$$\begin{aligned} \dot{e}(t) &= f(z(t)) - f(x(t)) - \dot{X}(t)(z_0 - x_0) \\ &= A(t)e(t) + o(\|e(t)\|) + o(\|X(t)\|\|x_0 - z_0\|) \\ \text{with} \quad e(0) &= 0 \, . \end{aligned}$$

Gronwall's lemma yields $\|e(t)\| = o(\|x_0 - z_0\|)$ for all $t \leq T < \infty$.

Steady-States

A point $x_0 \in \mathbb{R}^n$ is called a steady-state (or critical point) of f if

$$f(x_0) = 0 \; .$$

If f is continuously differentiable, the ODE

$$\dot{z}(t) = f(z(t)) \qquad \text{with} \qquad z(0) = z_0$$

can be analyzed in a local neighborhood of x_0. We have

$$A = \frac{\partial f(x_0)}{\partial x_0} \quad \text{and} \quad X(t) = e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!}(tA)^i \; .$$

Local Stability Analysis

- If the eigenvalues of A have all strictly negative real parts, then

$$\lim_{t\to\infty} e^{At} \;=\; 0 \; .$$

- In this case, we have (Exercise: prove this!)

$$\forall t \in [0,\infty), \quad z(t) = e^{At} z_0 + o(\|z_0\|) \qquad \text{and} \qquad \lim_{t\to\infty} z(t) = 0$$

  for sufficiently small $\|z_0\|$. This implies *local asymptotic stability*.

- A similar local stability analysis via first order variational analysis is possible in the neighborhood of periodic orbits (Floquet theory).