Lyapunov Stability

Contents

Recall the pendulum equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1 - b x_2$$

has two equilibrium points at

$$x_{e1} = (0\,,\ 0)^\top$$

is (asymptotically) stable, and

$$x_{e2} = (\pi\,,\ 0)^\top$$

is unstable.

The conclusion relies on explicit knowledge of solution or physical meaning of the system, which in general is NOT easy to obtain!

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Motivation

However, the conclusion can also be reached by using the energy concepts. Define the energy of the pendulum as

$$E(x) = \overbrace{mgl(1-\cos x_1)}^{\text{potential energy}} + \overbrace{\frac{1}{2}m(lx_2)^2}^{\text{kinetic energy}}$$
$$= ml^2[a(1-\cos x_1) + \frac{1}{2}x_2^2]$$

with the reference of the potential energy is chosen as $E(0) = 0$, i.e. x_{e1} has no energy.

The derivative of the energy along the trajectory of the system

$$\frac{\mathrm{dE}}{\mathrm{dt}} = -ml^2bx_2^2 \leq 0$$

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The derivative of the energy along the trajectory of the system

$$\dot{E} = -ml^2bx_2^2 \leq 0$$

- if $b = 0$, $E =$ constant $\Longleftrightarrow$ Stable equilibrium
- if $b \neq 0, E$ keeps decreasing until it eventually reaches zero $\Longleftrightarrow$ Asymptotically stable equilibrium

In 1982, Lyapunov showed that certain energy-like function could be used to determine the stability of an equilibrium.

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Autonomous system

Recall: Consider the autonomous system

$$\dot{x} = f(x), \tag{1}$$

where $f(x) : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz map from a domain $\mathcal{D} \subset \mathbb{R}^n$. Without loss of generality, we assume $x_e = \{0\} \in \mathcal{D}$.

Let function $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$(continuous differentiable) function on x. $\dot{V}(x)$ is defined as

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \left[\frac{\partial V}{\partial x_1}, \quad \frac{\partial V}{\partial x_2}, \quad \ldots, \quad \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

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Lyapunov Stability Theorem

Theorem: For the autonomous system defined in (1), let

$V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

i) $V(0) = 0$

ii) $V(x) > 0$ for any $x \in \mathcal{D} - \{0\}$

iii) $\dot{V} \leq 0$ for any $x \in \mathcal{D}$.

Then, the equilibrium at the origin is *stable* in the sense of Lyapunov. Moreover, origin is *asymptotically stable*, if

$$\dot{V} < 0$$

for any $x \in \mathcal{D} - \{0\}$.

Proof can be found in Khalil's *nonlinear systems* Theorem 4.1.

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Example: Consider a first-order system

$$\dot{x} = -g(x)$$

where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0, xg(x) > 0, \forall x \in (-a, 0) \cup (0, a)$$

The system has a isolated equilibrium at $x = 0$. Explicit solution is impossible to obtain. Consider a Lyapunov function

$$V(x) = x^2 \to \dot{V} = -2xg(x)$$

or

$$V(x) = \int_0^x g(y)\mathrm{d}y \to \dot{V} = -g^2(x)$$

Note: The choice of Lyapunov function is not unique.

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Definitions:

1)A function $V : \mathcal{D} \to \mathbb{R}$ is said to be *positive definite* if it satisfies

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2)A function $V(x)$ is said to be *positive semi-definite*, if it just satisfies the weaker condition

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Lyapunov Stability Theorem

Rephrase the Lyapunov's theorem as : *The origin is stable if there is a $\mathcal{C}^1$ and positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semi-definite, and the origin is a.s. if $\dot{V}(x)$ is negative definite.*

For the class of function of quadratic form

$$V(x) = x^\top P x, \quad x \in \mathbb{R}^n$$

where P is a real symmetric matrix. In this case,

$$V(x) \text{ is p.d.} \iff P \text{ is p.d.} \iff \text{eig}(P) > 0$$

Example: $V(x) = ax_1^2 + 2x_1x_3 + ax_2^2 + 4x_2x_3 + ax_3^2$ and $\dot{V}(x) = bx_1^2 + 2x_1x_3 + bx_2^2 + 4x_2x_3 + bx_3^2$, under what condition of a, b, yields a stable origin? an a.s. origin?

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Example: Recall the pendulum system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1 - b x_2$$

with $\mathcal{D} := (-\pi, \pi) \times \mathbb{R}$, consider $x_e = (0,0)^\top$ and candidate Lyapunov function is chosen as $V(x) = ml^2[a(1 - \cos x_1) + \frac{1}{2}x_2^2]$ satisfies $V(x)$ is p.d., and the time derivative of V is given by

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is only n.s.d. One can conclude the origin is *stable*, **but NOT *a.s.*!!!!** However, according to Barbalat's Lemma, we have

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Let's find a Lyapunov function $V(x)$ that would have a n.d. $\dot{V}(x)$.

Try a more general quadratic form

$$V(x) = x^\top P x + a\,(1 - \cos x_1)$$

with

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}, \quad p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

The derivative is given by

$$\begin{aligned} \dot{V}(x) = & a\,(1 - p_{22})\,x_2 \sin x_1 + (p_{11} - p_{12}b)\,x_1 x_2 - a p_{12} x_1 \sin x_1 \\ & + (p_{12} - p_{22}b)\,x_2^2 \end{aligned}$$

By taking $p_{22} = 1$ and $p_{11} = p_{12}b$, we have

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$$\dot{V} = -ap_{12}x_1 \sin x_1 + (p_{12} - b)\, x_2^2$$

Summary the requirements of p.d. and n.d.

$$\begin{cases} p_{12}b > 0, \quad p_{12}b - p_{12}^2 > 0 \\ p_{12} > 0, \quad p_{12} - b < 0 \end{cases}$$

choose $p_{12} = 0.5b$, yields the time derivative

$$\dot{V} = -0.5abx_1 \sin x_1 - 0.5bx_2^2$$

is n.d. for any $x \in \mathcal{D} := \{x \in \mathbb{R}^2 | x_1 \in (-\pi, 0) \cup (0, \pi)\}$. Hence, the origin is *locally* a.s..

This example emphasis, the conditions in the Lyapunov stability theorem are only sufficient!

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Theorem:(Barbashin-Krasovskii) For the autonomous system (1) that well-defined in $\mathbb{R}^n$, let $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

i) $V(x)$ is positive definite

ii) $V(x)$ is *radially unbounded*, that is

$$||x|| \to \infty \quad \text{implies} \quad V(x) \to \infty$$

iii) $\dot{V}(x)$ is negative definite

then the equilibrium at origin is *globally* asymptotically stable.

Example: $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$

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Lemma For the autonomous system (1), $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function, suppose the equilibrium at origin is contained by $\mathcal{D}$. If $V(0) = 0$ and there is a point at x_0 arbitrarily close to the origin such that $V(x_0) > 0$, then we can always construct a set

$$\mathcal{U} := \{x \in \mathcal{B}_r | V(x) > 0\}$$

such that

- $\mathcal{U}$ is non-empty set contained in $\mathcal{B}_r = \{x \in \mathbb{R}^n | ||x|| < r\}$ for some $r > 0$.
- Its boundary is the surface $V(x) = 0$ or the sphere $||x|| = r$

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- $\mathcal{U}$ is non-empty set contained in $\mathcal{B}_r = \{x \in \mathbb{R}^n | ||x|| < r\}$ for some $r > 0$.
- Its boundary is the surface $V(x) = 0$ or the sphere $||x|| = r$

Example : $V(x) = x_1^2 - x_2^2$.

Unstable

Lemma For the autonomous system (1), $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function, suppose the equilibrium at origin is contained by $\mathcal{D}$. If $V(0) = 0$ and there is a point at x_0 arbitrarily close to the origin such that $V(x_0) > 0$, then we can always construct a set

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Theorem:(Chetaev) For the autonomous system (1), $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function, suppose the equilibrium at origin is contained by $\mathcal{D}$, $V(0) = 0$ and there is a point at x_0 arbitrarily close to the origin such that $V(x_0) > 0$, if we can find a set aforementioned $\mathcal{U}$ such that $\dot{V}(x) > 0$ for any $x \in \mathcal{U}$, then the origin is *unstable*.

Note: the simplest case is $\mathcal{U} = \mathcal{B}_r - \{0\}$.

Corollary: For the system (1), if a $\mathcal{C}^1$ function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ is positive definite, but V_1 itself is NOT negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

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Recall the energy-like Lyapunov function of pendulum system whose time derivative

$$\dot{V}(x) = -bx_2^2$$

According to Barbala's lemma, we have x_2 converges to zero.

Notice, however, $\dot{V}$ is n.d. except on the line $x_2 = 0$. In other words, to maintain $\dot{V} = 0$, x_2 has to and *always* equal to 0,

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow -a\sin(x_1) = 0 \Rightarrow x_1 = 0$$

Hence, the system can maintain $\dot{V} = 0$ condition only at the origin, therefore, as $V(x)$ decreasing towards to 0, consequently, $x(t)$ converges to origin, i.e. origin is a.s.

A wrong derivation is $x_2 \to 0 \Rightarrow \dot{x}_2 = 0$!

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LaSalle's Theorem

Definition: A set $\Omega \in \mathbb{R}^n$ is said to be an *invariant* set w.r.t the autonomous system (1), if $x(t_0) \in \Omega \Rightarrow x(t) \in \Omega$ for all $t \in \mathbb{R}$. That is, if a solution belongs to Ω at some instant, then it belongs to Ω for all future and past time, e.g. equilibrium.

Theorem: For the autonomous system (1), Let $V : \mathcal{D} \to \mathbb{R}$ be a p.d. $\mathcal{C}^1$ function, suppose $\dot{V}(x) \leq 0$ for all $x \in \mathcal{D}$. Let

$$E := \{x \in \mathcal{D} | \dot{V}(x) = 0\}$$

and Ω be the *largest invariant set* of E, then all solution starting in $\mathcal{D}$ approaches Ω as $t \to \infty$.

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For the autonomous system (1), Let $V : \mathcal{D} \to \mathbb{R}$ be a p.d. $\mathcal{C}^1$ function on domain $\mathcal{D}$ containing the equilibrium at $x = 0$,

Corollary 1: suppose $\dot{V}(x) \leq -W(x) \leq 0$ for all $x \in \mathcal{D}$ and some positive-semidefinite $W(x)$, then it holds

$$W(x) \to 0 \text{ as } t \to \infty$$

Corollary 2: suppose $\dot{V}(x) \leq 0$ for all $x \in \mathcal{D}$. Let

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The Invariance Principle

Example: Now we are ready to analysis the motivating example

$$\dot{x} = \theta x + u$$

with the control law $u = -kx$ and k is given by the adaptive law

$$\dot{k} = \gamma x^2$$

Taking $x_1 = x$ and $x_2 = k$, the closed-loop system is

$$\begin{aligned} \dot{x}_1 &= -\left(x_2 - \theta\right) x_1 \\ \dot{x}_2 &= \gamma x_1^2 \end{aligned}$$

Step I: Check the type of system and find the equilibrium

$$x_e = [0, c]^\top$$

Step II: Shift equilibrium to the origin by $z_1 = x_1, z_2 = x_2 - c$

$$\begin{aligned}\dot{z}_1 &= -\left(z_2 + c - \theta\right) z_1 \\ \dot{z}_2 &= \gamma z_1^2\end{aligned}$$

Step III: Consider the Lyapunov candidate function

$$V(z) = \frac{1}{2} z_1^2 + \frac{1}{2\gamma} z_2^2$$

and check the positive definiteness (and radially unboundedness)

Step IV: Calculate the time derivative

$$\dot{V} = -(z_2 + c - \theta)z_1^2 + z_2 z_1^2 = -(c - \theta)z_1^2$$

Step V: Draw the conclusion about the equilibrium $x_e = [0, c]^\top$

- $c < \theta \Rightarrow \dot{V} \geq 0 \Rightarrow x_e$ is unstable
- $c = \theta \Rightarrow \dot{V} = 0 \Rightarrow x_e$ is stable
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According to the LaSalle's corollary, we have $x_1 \to 0$ as $t \to \infty$. However, the origin is not the only solution of $\dot{V} = 0$, x_e is not g.a.s!

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The LTI system

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n$$

It has an equilibrium at origin. The equilibrium is isolated iff $\det(A) \neq 0$, otherwise every point in the null space of A

$$\text{null}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$$

is an equilibrium of the system.

From the linear system theorem, we know that

- All Re$\lambda_i < 0$ or Re$\lambda_i = 0$ with associated Jordan block's rank is $1 \iff$ origin is "neutrally" stable or marginally stable
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Consider a quadratic Lyapunov candidate function

$$V(x) = x^\top P x$$

with P is a real symmetric p.d. matrix. The derivative of V along the trajectories of linear system is given by

$$\dot{V} = x^\top P A x + x^\top A^\top P x = -x^\top Q x$$

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Lyapunov Equation

Theorem: A matrix is Hurwitz, that is Re$(\lambda_i) < 0$ *if and only if for any* given p.d. symmetric matrix Q, there exists a p.d. symmetric matrix P that satisfies the Lyapunov equation

$$PA + A^\top P = -Q$$

Moreover, the solution P is unique.

The proof can be found in Khalil's Nonlinear system Theorem 4.6.

Example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad \textbf{with} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Remark: For LTI, a.s. implies g.e.s. for unique equilibrium.

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Exponentially Stable

Theorem: Let $x = 0$ be an equilibrium point for the autonomous system (1) and $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that:

$$k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a$$

$$\dot{V} \leq -k_3\|x\|^a$$

for any $t \geq 0$ and any $x \in \mathcal{D}$, where k_1, k_2, k_3 and a are positive constants. Then the origin is *exponentially stable*. If conditions hold globally, then the origin is g.e.s..

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Finite-Time Stable(Supplementary)

Theorem: Let $x = 0$ be an equilibrium point for the autonomous system (1) and $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ and positive definite function such that:

$$\dot{V} \leq -kV(x)^\alpha$$

for any $t \geq 0$ and any $x \in \mathcal{D}$, where $k > 0$ and $\alpha \in (0, 1)$. Then the origin is *finite-time stable* with the settling time

$$T = \frac{V(0)^{1-\alpha}}{k(1-\alpha)}.$$

If conditions hold globally, then the origin is globally finite time stable.

Lyapunov's Direct Method

Example: Consider the system

$$\dot{x}_1 = x_2 + cx_1\left(x_1^2 + x_2^2\right)$$
$$\dot{x}_2 = -x_1 + cx_2\left(x_1^2 + x_2^2\right)$$

where c is a constant, study the stability property of the system.

Step I: Check the type of system, find the equilibrium $x_e = [0, 0]$.

Step II: Consider a candidate Lyapunov function

$$V(x) = x_1^2 + x_2^2$$

that is positive definite and radially unbounded.

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Step III: Calculate the time derivative

$$\dot{V} = 2c(x_1^2 + x_2^2)^2$$

Step IV: draw the conclusion

- $c = 0 \Rightarrow \dot{V} = 0 \Rightarrow$ origin is stable
- $c < 0 \Rightarrow \dot{V}$ is negative definite $\Rightarrow$ origin is g.a.s, but NOT e.s or finite time convergence.
- $c > 0 \Rightarrow \dot{V} > 0$ for any $x \in \mathcal{U} := \{x \in \mathbb{R}^2 | 0 < ||x|| < r, r > 0\}$ $\Rightarrow$ origin is unstable

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