Stability Theory

Contents

Motivation

Typical non-linearities arise from

- Physical model:

  nonlinear resistence, nonlinear friction...

- Controller:

  nonlinear control law, relay, saturation, dead zone, quantization...

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Consider our motivating case

$$\dot{x}(t) = \theta x(t) + u(t)$$

where θ denotes the unknown parameter. Write it as an SPM

$$\underbrace{\dot{x}(t) - u(t)}_{z(t)} = \theta x(t)$$

Estimation θ by $\hat{\theta}(t)$, then we have

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The estimation error $\tilde{\theta} := \hat{\theta} - \theta$ is reflected by the difference between $z(t)$ and $\hat{z}(t)$. Define a cost function

$$J(\hat{\theta}) = \frac{(z(t) - \hat{z}(t))^2}{2} = \frac{(z(t) - \hat{\theta}x(t))^2}{2}$$

$J(\hat{\theta})$ is convex, the minimization problem is well-posed.

Applying the Gradient method

$$\dot{\hat{\theta}} = -\gamma \bigtriangledown J(\hat{\theta}) = \gamma(z(t) - \hat{\theta}x(t))x(t)$$

Adaptive control law

$$u = -\hat{\theta}x(t) - kx(t)$$

with $k > 0$. Does this work?

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Recall, we have shown in simulation that control law with an adaptive law

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works.

- Which one really works, or both? Which one is better?
- Are they "always" work, or just occasionally? Under what condition?
- Does $\hat{\theta}(t)$ converge to θ? How fast will $x(t)$ go to zero?

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Introduction

The concept of stability is concerned with the investigation and characterization of the behavior of dynamic systems.

So stability analysis is not just about stabilize the system.

The purpose of this lecture is to present some basic definitions and results on stability that are useful for the design and analysis of control systems, *especially for non-linear system.*

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For function of times, we define the $\mathcal{L}_p$ norm

$$\|x\|_p \triangleq \left(\int_0^\infty |x(\tau)|^p d\tau \right)^{1/p}$$

for $p \in [0, \infty)$ and say that $x \in \mathcal{L}_p$ when $\|x\|_p$ exists, i.e. finite.

The $\mathcal{L}_\infty$ norm defined as

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Continuity

Definitions:

Recall : Continuity

1. Piecewise Continuity. A function $f : [0, \infty) \mapsto \mathbb{R}$ is piecewise continuous on $[0, \infty)$ if f is continuous on any finite interval $[t_0, t_1] \in [0, \infty)$ except for a **finite number of points**.

2. Lipschitz . A function $f : [a, b] \mapsto \mathbb{R}$ is Lipschitz on $[a, b]$ if

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2|$$

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Three important negative facts

1. $\lim_{t\to\infty} \dot{f}(t) = 0$ does NOT imply that $f(t)$ has a limit as $t \to \infty$.

counter-example: $f(t) = \sin(\sqrt{1+t})$

2. $\lim_{t\to\infty} f(t) = c$ for some constant $c \in \mathbb{R}$ does NOT imply that $\dot{f}(t) \to 0$ as $t \to \infty$.

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3. $f \in \mathcal{L}_p \cap \mathcal{L}_\infty$ for some $p \in [1, \infty)$ does NOT imply $f \to 0$ as $t \to \infty$.

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Two important positive facts

Lemma

For a scalar-valued function $f(t)$, if it is bounded from below and is nonincreasing, then it has a limit as $t \to \infty$.

Theorem (Barb$\breve{a}$lat's Lemma)

If $f \in \mathcal{L}_p \cap \mathcal{L}_\infty$ and $\dot{f} \in \mathcal{L}_\infty$ for some $p \in [1, \infty)$, then $f \to 0$ as $t \to \infty$.

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System and Equilibrium

Consider system described by differential equations of form

$$\dot{x}(t) = f(t, x), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

where $t_0 \geq 0$, $f : [t_0, \infty) \times \mathcal{B}_r \mapsto \mathbb{R}^n$ with $\mathcal{B}_r := \{x \in \mathbb{R} | |x| < r\}$.

For every $x_0 \in \mathcal{B}_r$, $t_0 \in \mathbb{R}+$, the solution denoted by $x(t; t_0, x_0)$

- exists, if f is piecewise continuous
- exists and *unique*, if f is locally Lipschitz.

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Definition: A state x_e is said to be an equilibrium state of System (1), if $f(t, x_e) \equiv 0$ for all $t \geq t_0$.

Example: 1) $\dot{x} = (x-1)x$; 2) $\dot{x}_1 = x_1 x_2, \dot{x}_2 = x_1^2$

For convenience, we normally shift the equilibrium to the origin via a change of variables: $y := x - x_e$, then

$$\dot{y} = f(t, y + x_e) \triangleq g(t, y)$$

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The system is

- *Autonomous* or time-invariant, if f does not depend on t;

  Non-autonomous or time-varying otherwise.

- Linear if $f(t, x) = A(t)x$; Nonlinear otherwise.

Note: system considered in this lecture is nonlinear by default.

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The properties of the system will said to be true

- Globally, if $r = \infty$ and true for all $x_0 \in \mathbb{R}^n$. Locally, otherwise.
- Uniformly, if true for all $t_0 \geq 0$.

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Linearization

Applying the Taylor series expansion yields the linearization of the nonlinear system (1) about x_e as

$$\dot{x} \approx A\left(x_e\right) x$$

where $A\left(x_e\right)$ is the **Jacobian matrix** of $f(x)$ evaluated at x_e

$$A\left(x_e\right) = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=x_e}$$

Linearization

Note that if A is full rank, then

$$x_e \text{ of linear system} = \text{origin} = x_e \text{ of nonlinear system}$$

Therefore, the **local** behavior of a nonlinear system around equilibria can be studied by linearization. Consider a 2-nd order system with

$$\mathrm{eig}(A) = \{\lambda_1, \lambda_2\}$$

we can use Phase Portraits to study the behaviors the local stability of the nonlinear systems.

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Phase Portrait

1. Stable or unstable node occurs when both λ_1 and λ_2 are **real** and have **the same** sign.

$$\lambda_{1,2} > 0 \qquad\qquad \lambda_{1,2} < 0$$

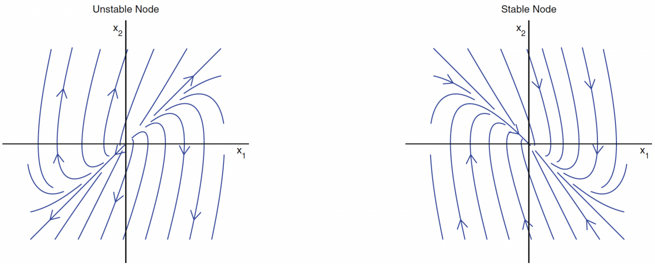


Figure: Phase portrait of a node

Phase Portrait

2. Saddle point occurs when both λ_1 and λ_2 are **real** and have **opposite** signs.

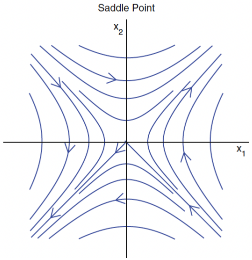


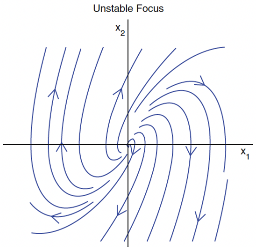
Figure: Phase portrait of a saddle point

The system is always on the verge of instability.

Phase Portrait

3. Stable or unstable focus occurs when occurs when both λ_1 and λ_2 are a complex conjugate pair.

$$\mathrm{Re}(\lambda_{1,2}) > 0 \qquad\qquad \mathrm{Re}(\lambda_{1,2}) < 0$$



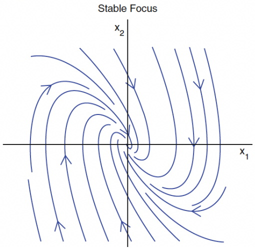


Figure: Phase portrait of a focus

Phase Portrait

4. Center occurs when both λ_1 and λ_2 are purely imaginary.

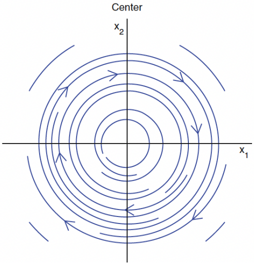


Figure: Phase portrait of a saddle point

All trajectories encircle the center point at the origin with concentric-level curves.

Example: Consider a rotating pendulum without friction

$$\ddot{\theta} + \frac{g}{l}\sin\theta - \omega^2 \sin\theta\cos\theta = 0$$

Setting $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$ yields the state space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 + \omega^2 \sin x_1 \cos x_1 \end{bmatrix}$$

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Meaning:

- when the angular speed ω exceeds a certain value, then pendulum will be suspended by an angle as the centrifugal force exerted on the pendulum is in balance with its weight;
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$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 + \omega^2\left(\cos^2 x_1 - \sin^2 x_1\right) & 0 \end{bmatrix}$$

Jacobian matrix at the equilibrium points :

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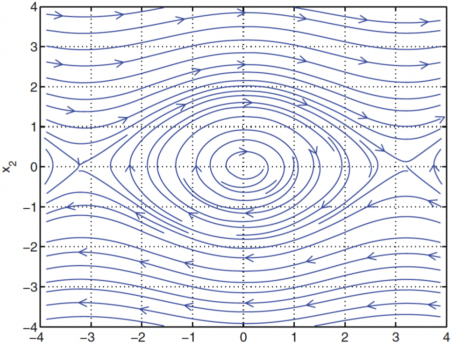
The Phase portrait



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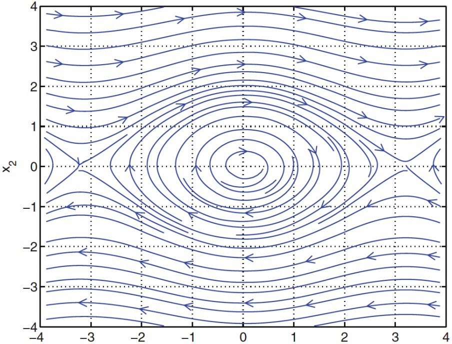

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Linearization can merely provide information on the local stability in a region about an equilibrium point. Global stability of a nonlinear system over its entire solution domain is difficult to analyze in this way.

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The trajectories of the nonlinear solution can exhibit other unpredictable behaviors, for example chaos, finite time escape and limit cycle etc.(See Khalil's Nonlinear Systems for more details.)

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- x_e is **uniformly stable**, if δ does NOT depend on t_0.
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- x_e satisfies the condition ii) is said to be **attractive**.
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$$\dot{x} = -\frac{x-1}{1+t}, \quad x(t_0) = x_0$$

It has a isolated equilibrium $x_e = 1$. Shift the equilibrium to the origin by defining $y := x - 1$, yields

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for all $x_0 \in \mathcal{B}_r$. Constant α is called the rate of convergence.

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with equilibrium $x_e = 0$. Its solution is given by

$$x(t) = \left(\frac{x_0^2}{1 + 2x_0^2 (t - t_0)} \right)^{\frac{1}{2}}$$

For any given $\epsilon > 0$, set $\delta(\epsilon) = \epsilon$, we can show

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Exercise:

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$$\dot{x} = ax, x(0) = x_0 \in \mathbb{R}_+$$

2)

$$\dot{x} = x^2 - 2x, x(0) = x_0$$