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On Stabilizing Terminal Costs and Regions for Configuration-Constrained Tube MPC

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***Abstract*—This paper introduces a novel class of terminal regions and cost functions for tube model predictive control (TMPC). Our focus is on polytopic configuration-constrained TMPC schemes, which offer flexibility by introducing a significant amount of variables to model the shape of the propagated sets. This flexibility, however, comes with a challenge, namely, to enforce stability efficiently. To address this challenge, we propose tailored terminal regions and cost functions enabling efficient and stable TMPC implementations, without relying on regularity assumptions about the control system or configuration templates. Numerical case studies demonstrate the effectiveness and performance of the proposed control scheme.**

***Index Terms*—Robust Control, Model Predictive Control, Convex Optimization**

I. Introduction

TMPC is a modern form of robust MPC [2], [14], focusing on optimizing robust forward invariant tubes and their control laws. In the simplest case, these are *rigid* sets (RTMPC) [9], [10], whose centers are optimized online. Other methods, such as Homothetic TMPC (HTMPC) [7], [11], Elastic TMPC (ETMPC) [4], [12], Configuration-Constrained Tube MPC (CCTMPC) [18], TMPC based on system level synthesis [16], as well as ellipsoidal [19], and generic TMPC schemes based on set propagation [6] all optimize tubes online, differing in their particular parameterization schemes.

TMPC formulations, while enforcing recursive feasibility and stability, rely heavily on the particular choice of set and feedback parameterization [6], [14]. Recursive feasibility can be established simply by enforcing that the last set in the predicted tube is a robust control invariant (RCI) set—as it defines a constant tube that is feasible over infinite horizons. Enforcing stability, in a suitable sense, is often more challenging despite the existence of generic approaches based on set-dissipativity for constructing terminal ingredients [17].

TMPC methods reformulate the underlying optimization problem as a higher-dimensional *nominal* MPC problem by parameterizing sets and control laws. To develop stabilizing TMPC schemes, one can leverage techniques for nominal MPC, such as using terminal costs and regions [8], [14] or sufficiently long horizons [5]. Terminal regions and costs can often be used to find stabilizing prediction horizons [3].

Despite these efforts, two challenges arise in the context of stabilizing TMPC. First, the inherent uncertainty in the system needs to be accounted for when defining stability. Second, and more critically, an accurate set representation can lead to a large-scale optimization problem, rendering traditional formulations ineffective. For instance, an accurate polytopic parameterization for a low-dimensional system may involve hundreds of facets, resulting in large-scale optimal control problems. Stabilizing schemes, such as [15], relying on control invariant terminal sets are often unsuitable for such problems due to their computational cost, unless implicit parameterizations are used [1], [13].

In summary, modern TMPC methods, such as CCTMPC, introduce a significant number of states for parameterizing the tube, posing a non-trivial challenge in enforcing stability. This paper aims to address this challenge by proposing a solution tailored specifically to CCTMPC, which encapsulates other parameterizations like RTMPC (in both explicit and implicit forms) and HTMPC as special cases [18].

Main Contributions and Overview

Section II reviews configuration-constrained polytopes and their relevance to invariance (Propositions 1 and 2). It also introduces the TMPC formulation and analyzes its optimal RCI sets (Proposition 3). The main contribution, detailed in Section III, consists in novel self-optimizing terminal ingredients for stabilizing CCTMPC (Theorem 1). Section IV focuses on their efficient implementation, which is provably stable without regularity assumptions such as the existence of a robust contractive simple polytope (Corollary 1). This results in a flexible TMPC scheme for systems with additive and multiplicative uncertainty. Section V presents a numerical example and Section VI concludes the paper.

II. Configuration-Constrained Tube MPC

A. Configuration Triples

Let $F \in \mathbb{R}^{\mathsf{f}\times n}$ be a given matrix, whose rows are the normals of the facets of a given polytope in $\mathbb{R}^n$. This means that $Fx \leq 0$ implies[1] $x = 0$. In the following, we denote by

$$P(y) \ := \ \{x \in \mathbb{R}^n \mid Fx \leq y\} \ ,$$

[1] In the sequel, inequalities between vectors are defined componentwise.

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the polytope associated with $y \in \mathbb{R}^{\mathsf{f}}$.

Proposition 1: For all $y \in \mathbb{R}^{\mathsf{f}}$ with $P(y) \neq \varnothing$, there exist matrices $E \in \mathbb{R}^{\mathsf{e}\times\mathsf{f}}$ and $V_1, V_2, \ldots, V_{\mathsf{v}} \in \mathbb{R}^{n\times\mathsf{f}}$ such that

$$P(z) \ = \ \mathrm{convh}(\{V_1 z, V_2 z, \ldots, V_{\mathsf{v}} z\}) \quad \iff \quad Ez \leq 0$$

and $Ey \leq 0$. Here, v denotes the number of vertices of $P(y)$.

Proof: The existence of such matrices follows from [18, Thm. 2], if $P(y)$ is entirely simple [18, Def. 4]. To show the statement holds if $P(y)$ is not entirely simple, recall that the set $\{y \in \mathbb{R}^{\mathsf{f}} \mid P(y) \text{ is not entirely simple}\}$ has Lebesgue measure zero. In particular, its complement is dense in $\mathbb{Y} := \{y \in \mathbb{R}^{\mathsf{f}} \mid P(y) \neq \varnothing\}$ [18, Cor. 3]. Moreover, for a given F, there exist finitely many vertex configurations. Thus, we can construct a sequence $(y_k)_{k\in\mathbb{N}}$ converging to y such that each $P(y_k)$ is entirely simple and all share the same configuration. Let $\tilde{E} \in \mathbb{R}^{\tilde{\mathsf{e}}\times\mathsf{f}}$ and $V_1, \ldots, V_{\tilde{\mathsf{v}}} \in \mathbb{R}^{n\times\mathsf{f}}$ be such that

$$P(z) \ = \ \mathrm{convh}(\{V_1 z, \ldots, V_{\tilde{\mathsf{v}}} z\}) \quad \iff \quad \tilde{E}z \leq 0$$

and $Ey_k \leq 0$ for all $k \in \mathbb{N}$. In the limit as $k \to \infty$ we have

$$P(y) \ = \ \mathrm{convh}(\{V_1 y, \ldots, V_{\tilde{\mathsf{v}}} y\}).$$

and $\tilde{E}y \leq 0$. Now, we may have $\tilde{\mathsf{v}} > \mathsf{v}$ due to redundant vertices, i.e. $V_i y = V_j y$ for some indices (i, j). But, such redundant vertex matrices can be directly eliminated. Finally, to ensure that the conclusion of the proposition holds, we construct E by augmenting $\tilde{E}z \leq 0$ with constraints

$$(V_i - V_j)z \leq 0 \quad \text{and} \quad (V_j - V_i)z \leq 0$$

for all redundant vertex index pairs (i, j) removed. Thus, the statement is true even if $P(y)$ is not entirely simple. ∎

Remark 1: For entirely simple polytopes the matrix E can be constructed following the considerations in [18, Sec. 3.5]. In particular, one can set

$$E^{\intercal} = ((FV_1 - \mathbb{I}_{\mathsf{f}})^{\intercal}, \ldots, (FV_{\mathsf{v}} - \mathbb{I}_{\mathsf{f}})^{\intercal})$$

with $\mathbb{I}_{\mathsf{f}} \in \mathbb{R}^{\mathsf{f}\times\mathsf{f}}$ being the identity matrix, and then remove redundancies from $Ey \leq 0$ using an LP solver.

If (F, E, V) with $V = (V_1, \ldots, V_{\mathsf{v}})$ satisfies the condition from Proposition 1 we call it a configuration triplet. Every $P(z)$ with $Ez \leq 0$ has the same face configuration. Thus, we can say $Ez \leq 0$ is a configuration constraint as it freezes the face configuration of the representable polytopes.

Remark 2: Choosing (F, E, V), or equivalently $P(y)$ in Proposition 1, enables direct tuning of the tradeoff between conservativeness and complexity in polytopic computations. For instance, choosing $P(y)$ as a simplex with $\mathsf{v} = \mathsf{f} = n + 1$ yields a scalable polytopic template with reduced accuracy. Also note that choosing $P(y)$ from a polytope class with known vertex configuration avoids costly vertex enumeration.

Remark 3: If $P(y)$ in Proposition 1 is entirely simple, the convex cone $\{y \in \mathbb{R}^{\mathsf{f}} \mid Ey \leq 0\}$ has a non-empty interior [18, Thm. 1]. While almost every polytope is entirely simple [18, Cor. 3], we do not impose this regularity condition.

B. Configuration-Constrained Polytopic Tubes

This paper is concerned with uncertain systems of the form

$$x^+ = Ax + Bu + w \tag{1}$$

with state $x \in \mathbb{X}$, control $u \in \mathbb{U}$, and disturbance $w \in \mathbb{W}$. Throughout this paper, $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are assumed to be closed and convex state and control constraint sets.

Remark 4: All results in this paper can be extended, by means of [18, Cor. 4], to systems with parametric uncertainty of the form $(A, B) \in \mathrm{convh}(\{(\mathsf{A}_1, \mathsf{B}_1), \ldots, (\mathsf{A}_{\mathsf{q}}, \mathsf{B}_{\mathsf{q}})\})$.

The set transition map of (1), defined for all $X \subseteq \mathbb{R}^n$, and all feedback laws $\mu \in \mathcal{U}$, is given by

$$f(X, \mu) \ := \ \{Ax + B\mu(x) + w \mid x \in X,\, w \in \mathbb{W}\} \,.$$

Here, $\mathcal{U} := \{\mu \mid \mu : \mathbb{R}^n \to \mathbb{U}\}$ is the set of feasible feedback control laws. Additionally, we recall that

1) a set $X \subseteq \mathbb{R}^n$ is robust control invariant (RCI) if
$$\exists \mu \in \mathcal{U} : \quad f(X, \mu) \subseteq X, \quad \text{and}$$
2) a sequence $X_0, X_1, \ldots \subseteq \mathbb{R}^n$ is called a robust forward invariant tube (RFIT) if
$$\forall k \in \mathbb{N},\ \exists \mu_k \in \mathcal{U} : \quad f(X_k, \mu_k) \subseteq X_{k+1}.$$

Configuration-constrained polytopic tubes can be uniquely characterized by a convex inequality.[2]

Proposition 2: Let (F, E, V) be a configuration triplet. The sequence $P(y_0), P(y_1), \ldots$ with parameter vectors $y_0, y_1, \ldots \in \{y \in \mathbb{R}^{\mathsf{f}} \mid Ey \leq 0\}$ is an RFIT if and only if there exist control inputs $v_{k,1}, \ldots, v_{k,\mathsf{v}} \in \mathbb{U}$ such that

$$\forall k \in \mathbb{N}, \qquad F\left(AV_i y_k + Bv_{k,i}\right) + c \ \leq \ y_{k+1}$$

holds for all $i \in \{1, \ldots, \mathsf{v}\}$. Here, the vector $c \in \mathbb{R}^{\mathsf{f}}$ is given by $c_j := \max_{w\in\mathbb{W}} F_j w$ for all $j \in \{1, \ldots, \mathsf{f}\}$.

Proof: Follows from Proposition 1 with [18, Cor. 4]. ∎

Let $S_i := e_i^{\intercal} \otimes \mathbb{I}_m$ with e_i the i-th unit vector in $\mathbb{R}^{\mathsf{v}}$ and $\otimes$ the Kronecker product. If $v = (v_1^{\intercal}, \ldots, v_{\mathsf{v}}^{\intercal})^{\intercal}$, we can recover each vertex control input as $v_i = S_i v$ for all $i \in \{1, \ldots, \mathsf{v}\}$.

C. Optimal Invariant Polytopes

Optimal invariant sets can be computed, in the context of TMPC, by choosing a stage cost $L(X, \mu)$ mapping a set X and policy $\mu \in \mathcal{U}$ to a scalar. Since we are interested in sets $X = P(y)$, we focus on stage costs that can be written as

$$L(P(y), \mu) \ = \ \ell(y, v) \quad \text{with} \quad S_i v \ = \ \mu(V_i y) \,,$$

for all $i \in \{1, \ldots, \mathsf{v}\}$. Throughout, $\ell : \mathbb{R}^{\mathsf{f}} \times \mathbb{R}^{\mathsf{v}m} \to \mathbb{R}$ is assumed to be a strictly convex function. An optimal RCI polytope $P(y_{\mathsf{s}})$ with vertex control inputs $S_i v_{\mathsf{s}}$ for all $i \in \{1, \ldots, \mathsf{v}\}$, can be constructed with a minimizer $(y_{\mathsf{s}}, v_{\mathsf{s}})$ of

$$\min_{y,v} \ \ell(y, v) \quad \text{s.t.} \begin{cases} \forall i \in \{1, \ldots, \mathsf{v}\}, \\ F(AV_i y + BS_i v) + c \leq y \quad \mid \quad \overline{\lambda}_i \\ Ey \leq 0,\ S_i v \in \mathbb{U},\ V_i y \in \mathbb{X}. \end{cases} \tag{2}$$

Here, $\overline{\lambda}_1, \ldots, \overline{\lambda}_{\mathsf{v}} \in \mathbb{R}^{\mathsf{f}}$ denote the multipliers of the indicated linear inequalities.

[2] If (1) has parametric uncertainty, the inequality must be enforced for all vertices $(\mathsf{A}_1, \mathsf{B}_1), \ldots, (\mathsf{A}_{\mathsf{q}}, \mathsf{B}_{\mathsf{q}})$ of the uncertainty set (see Remark 4).

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D. Tube MPC via Cost-to-travel Functions

The parameterized Configuration-Constrained Tube MPC problem (compare [18, Eq. (45)]) is given by

$$\begin{aligned}\min_{y,v}\ & \ell_0(y_0)+\sum_{k=0}^{N-1}\ell(y_k,v_k)+\ell_N(y_N)\\ \text{s.t.}\ & \begin{cases}\forall k\in\{0,\ldots,N-1\},\ \forall i\in\{1,\ldots,\mathsf{v}\},\\ F(AV_iy_k+BS_iv_k)+c\leq y_{k+1}\\ Ey_k\leq 0,\ S_iv_k\in\mathbb{U},\ V_iy_k\in\mathbb{X},\\ Fx_0\leq y_0,\ y_N\in\mathbb{T}.\end{cases}\end{aligned}\tag{3}$$

Here, ℓ_0 and ℓ_N denote, respectively, the initial and terminal cost. Moreover, $\mathbb{T}\subseteq\mathbb{R}^{\mathsf{f}}$ denotes a convex terminal region.

The cost-to-travel function $V:\mathbb{R}^{\mathsf{f}}\times\mathbb{R}^{\mathsf{f}}\to\mathbb{R}\cup\{-\infty,\infty\}$ associated to the CCTMPC problem (3) is [18, Sect. 5.2]

$$\begin{aligned}V(y,y^+)=&\min_v\ \ell(y,v)\\ &\text{s.t.}\ \begin{cases}\forall i\in\{1,\ldots,\mathsf{v}\},\\ F(AV_iy+S_iBv)+c\leq y^+\\ Ey\leq 0,\ S_iv\in\mathbb{U},\ V_iy\in\mathbb{X}\end{cases}\end{aligned}\tag{4}$$

with $V(y,y^+)=\infty$ when (4) is infeasible. Next, we consider the convex optimization problem

$$\min_{y,y^+}\ V(y,y^+)\quad\text{s.t.}\quad y=y^+\mid\lambda_{\mathsf{s}},\tag{5}$$

with $\lambda_{\mathsf{s}}\in\mathbb{R}^{\mathsf{f}}$ denoting the multiplier associated to $y=y^+$. We assume throughout that (5) admits a unique minimizer $(y_{\mathsf{s}},y_{\mathsf{s}})$ and dual solution λ_{s} such that strong duality holds.[3]

Proposition 3: The pair $(y_{\mathsf{s}},y_{\mathsf{s}})$ is a minimizer of (5) if and only if $(y_{\mathsf{s}},v_{\mathsf{s}})$ minimizes (2). Moreover, the multipliers $\overline{\lambda_1},\ldots,\overline{\lambda_{\mathsf{v}}}$ in (2) and λ_{s} in (5) satisfy $\lambda_{\mathsf{s}}=-\sum_{i=1}^{\mathsf{v}}\overline{\lambda}_i$.

Proof: The first claim follows directly from the definition of V. For the second claim, the Lagrangian functions of (2) and (5)—using (4), are respectively given by

$$\begin{aligned}\Lambda_1(y,v,\overline{\lambda},\gamma):=\ & g\left(y,v,\overline{\lambda},\gamma\right)\\ &+\sum_{i=1}^{\mathsf{v}}\overline{\lambda}_i^{\mathsf{T}}FAV_iy-\sum_{i=1}^{\mathsf{v}}\overline{\lambda}_i^{\mathsf{T}}y\end{aligned}\tag{6}$$

$$\begin{aligned}\Lambda_2(y,y^+,v,\overline{\lambda},\lambda_{\mathsf{s}},\gamma):=\ & g\left(y,v,\overline{\lambda},\gamma\right)+\lambda_{\mathsf{s}}^{\mathsf{T}}(y-y^+)\\ &+\sum_{i=1}^{\mathsf{v}}\overline{\lambda}_i^{\mathsf{T}}FAV_iy-\sum_{i=1}^{\mathsf{v}}\overline{\lambda}_i^{\mathsf{T}}y^+\ .\end{aligned}\tag{7}$$

Here, the shorthand $h:\mathbb{R}^{\mathsf{f}}\times\mathbb{R}^{m\mathsf{v}}\to\mathbb{R}$ is defined such that

$$h(y,v)\leq 0\iff\begin{cases}\forall i\in\{1,\ldots,\mathsf{v}\}\\ Ey\leq 0,\ S_iv\in\mathbb{U},\ V_iy\in\mathbb{X}\ ,\end{cases}$$

while $g:\mathbb{R}^{\mathsf{f}}\times\mathbb{R}^{m\mathsf{v}}\times\mathbb{R}^{\mathsf{vf}}\times\mathbb{R}\to\mathbb{R}$ is such that

$$g\left(y,v,\overline{\lambda},\gamma\right)=\ell(y,v)+\gamma h(y,v)+\sum_{i=1}^{\mathsf{v}}\overline{\lambda}_i^{\mathsf{T}}(FBS_iv+c)\ .$$

A direct comparison of (6) and (7) shows that Λ_1 and Λ_2 are equivalent whenever $\lambda_s+\sum_{i=1}^{\mathsf{v}}\overline{\lambda}_i=0$, as claimed. ■

[3]Strong duality holds if ℓ is bounded below and radially unbounded and if (2) admits a strictly feasible point (Slater's constraint qualification). If (2) is a convex QP with ℓ bounded below, no constraint qualification is needed.

Finally, we recall [18, see Sect. 5.2] that (3) is equivalent to

$$\begin{aligned}\min_{y,v}\ & \ell_0(y_0)+\sum_{k=0}^{N-1}V(y_k,y_{k+1})+\ell_N(y_N)\\ \text{s.t.}\ & Fx_0\leq y_0,\quad y_N\in\mathbb{T}\ .\end{aligned}\tag{8}$$

III. Convergent Sequences of Configuration-Constrained Polytopes

This section introduces a novel construction of terminal sets for CCTMPC. It relies on the existence of a sequence of polytopes contracting exponentially to an RCI set. It does not require a robust contractive polytope to exist (compare [18]).

A. Exponentially Convergent Sequences of Polytopes

One way to enforce recursive feasibility of TMPC is to steer the sequence $P(y_0),\ldots P(y_N)$ to the optimal RCI set,

$$X_{\mathsf{s}}=P(y_{\mathsf{s}}),$$

induced by (2). Such constraint is, in general, restrictive, as it requires enforcing $y_N=y_{\mathsf{s}}$. We propose, instead, to use a forward invariant sequence

$$P(y_N),\ P(y_{N+1}),\ \ldots\quad\text{with}\quad \lim\nolimits_{k\to\infty}P(y_{N+k})=P(y_{\mathsf{s}})\ .$$

The key idea is to introduce an exponentially convergent sequence of parameters and vertex control inputs of the form

$$y_{N+k}\ :=\ y_{\mathsf{s}}+\beta^k(y_N-y_{\mathsf{s}})\tag{9}$$

$$\text{and}\qquad v_{N+k}\ :=\ v_{\mathsf{s}}+\beta^k(v_N-v_{\mathsf{s}})\tag{10}$$

for $k\in\mathbb{N}$, where $\beta\in[0,1)$ is a contraction parameter. These sequences depend on y_N and v_N, whose value can be chosen by the controller. To set up this construction, we let $\mathbb{S}$ be the set of pairs $(y,v)\in\mathbb{R}^{\mathsf{f}}\times\mathbb{R}^{\mathsf{v}m}$ satisfying

$$\begin{aligned}&F(AV_i(y-y_{\mathsf{s}})+BS_i(v-v_{\mathsf{s}}))\leq\beta(y-y_{\mathsf{s}}),\\ &Ey\leq 0,\ S_iv\in\mathbb{U},\ \text{and}\ V_iy\in\mathbb{X}\end{aligned}\tag{11}$$

for all $i\in\{1,\ldots,\mathsf{v}\}$ and any given $(y_{\mathsf{s}},v_{\mathsf{s}})$. Notice that, since we assume (2) has a minimizer, we have $(y_{\mathsf{s}},v_{\mathsf{s}})\in\mathbb{S}$, and $\mathbb{S}$ is nonempty *by construction*. The relevance of this condition is clarified by the following technical lemma.

Lemma 1: Let $(y_{\mathsf{s}},v_{\mathsf{s}})$ denote an optimal solution of (2) and let $\beta\in[0,1)$ be given. If $(y_N,v_N)\in\mathbb{S}$ holds, then (9) and (10) satisfy the recursive feasibility conditions

$$\forall i\in\{1,\ldots,\mathsf{v}\},\qquad\begin{cases}F(AV_iy_k+BS_iv_k)+c\leq y_{k+1}\\ Ey_k\leq 0,\ S_iv_k\in\mathbb{U},\ V_iy_k\in\mathbb{X}\end{cases}$$

for all $k\in\{N,N+1,N+2,\ldots\}$.

Proof: Since $(y_{\mathsf{s}},v_{\mathsf{s}})$ solves (2) and since we can multiply both sides of the first inequality in (11) by β^k, the inequalities

$$\begin{aligned}F(AV_iy_{\mathsf{s}}+BS_iv_{\mathsf{s}})+c\ &\leq\ y_{\mathsf{s}}\\ F(AV_i(\beta^k(y-y_{\mathsf{s}}))+BS_i(\beta^k(v-v_{\mathsf{s}})))\ &\leq\ \beta^{k+1}(y-y_{\mathsf{s}})\end{aligned}$$

hold. Substituting the sequences (9) and (10) and adding the above inequalities yields the recursive inequality

$$\forall k\geq N,\qquad F(AV_iy_k+BS_iv_k)+c\ \leq\ y_{k+1}.$$

Moreover, from (5), it follows that $Ey_{\mathsf{s}}\leq 0$, $V_iy_{\mathsf{s}}\in\mathbb{X}$, and $S_iv_{\mathsf{s}}\in\mathbb{U}$ hold for all $i\in\{1,\ldots,\mathsf{v}\}$. These, together with the remaining conditions in (11) imply the remaining recursivity relations recalling that $\mathbb{X}$ and $\mathbb{U}$ are convex. ■

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B. Terminal Cost and Region

The main contribution of this paper consists of proposing and analyzing the terminal cost function

$$\ell_N(y) := \min_v \sum_{k=0}^{\infty} s_k(y,v) \quad \text{s.t.} \quad (y,v) \in \mathbb{S}$$

together with its implicitly defined terminal region

$$\mathbb{T} := \left\{ y \in \mathbb{R}^{\mathsf{f}} \;\middle|\; \ell_N(y) \le \infty \right\} \subseteq \mathbb{R}^{\mathsf{f}} .$$

Here, s_k is a shorthand for the difference

$$s_k(y,v) := \ell\left(y_{\mathsf{s}} + \beta^k(y-y_{\mathsf{s}}), v_{\mathsf{s}} + \beta^k(v - v_{\mathsf{s}})\right) - \ell(y_{\mathsf{s}}, v_{\mathsf{s}}) .$$

The terminal region $\mathbb{T}$ is equal to the projection of $\mathbb{S}$ onto $\mathbb{R}^{\mathsf{f}}$. Consequently it is nonempty, since $(y_{\mathsf{s}}, v_{\mathsf{s}}) \in \mathbb{S}$.

Remark 5: The sum $\sum_{k=0}^{\infty} s_k(y,v)$ is convergent, since ℓ is convex and thus locally Lipschitz continuous. Moreover, it can be worked out explicitly for various stage costs. For example, let ℓ be strictly convex with $\ell(z) = z^{\intercal} H z$—stacking its arguments into $z \in \mathbb{R}^{\mathsf{f}\mathsf{v}m}$—for some matrix $H \succeq 0$. Expanding the products in s_k, using the geometric series formula on each term, and collecting terms, yields

$$\sum_{k=0}^{\infty} s_k(y,v) = \frac{\ell(y+\beta y_{\mathsf{s}}, v + \beta v_{\mathsf{s}}) - \ell(y_{\mathsf{s}} + \beta y_{\mathsf{s}}, v_{\mathsf{s}} + \beta v_{\mathsf{s}})}{1-\beta^2} .$$

One can show, using a similar argument, that whenever ℓ is a convex polynomial, the sum can be found in closed-form. The next sections show how ℓ_N, $\mathbb{T}$, and the initial cost

$$\ell_0(y) := \lambda_{\mathsf{s}}^{\intercal} y ,$$

lead to an asymptotically stable CCTMPC scheme.

C. Construction of a Lyapunov function

The following analysis shows that the rotated cost-to-travel function R together with the rotated end cost M, defined by

$$\begin{aligned} R(y,y^+) &:= V(y,y^+) + \lambda_{\mathsf{s}}^{\intercal}(y-y^+) - V(y_{\mathsf{s}}, y_{\mathsf{s}}), \\ M(y) &:= \ell_N(y) + \lambda_{\mathsf{s}}^{\intercal}(y - y_{\mathsf{s}}) \end{aligned}$$

satisfy a Lyapunov condition. This is summarized next.

Theorem 1: The functions R and M are both non-negative and satisfy Lyapunov's descend condition,

$$\forall y \in \mathbb{R}^{\mathsf{f}}, \quad \min_{y^+} \left\{ R(y,y^+) + M(y^+) \right\} \;\le\; M(y).$$

Proof: By weak duality in (5), we have $R(y,y^+) \ge 0$. Let y_{N+k} and v_{N+k} be defined as in (9) and (10) for arbitrary $y_N \in \mathbb{T}$ and let

$$v_N := v_N^{\star}(y_N) \in \underset{v}{\operatorname{argmin}} \sum_{k=0}^{\infty} s_k(y_N, v) \quad \text{s.t.} \quad (y_N, v) \in \mathbb{S} .$$

In the following, we use the notation

$$r(y,u,y^+) \;:=\; \ell(y,u) + \lambda_{\mathsf{s}}^{\intercal}(y-y^+) - \ell(y_{\mathsf{s}}, v_{\mathsf{s}})$$

for the rotated stage cost function. Due to Lemma 1, we have

$$0 \;\le\; R(y_{N+k}, y_{N+k+1}) \;\le\; r(y_{N+k}, v_{N+k}, y_{N+k+1}) \quad (12)$$

for all $k \in \mathbb{N}$. Next, we have

$$\begin{aligned} 0 &\overset{(12)}{\le} \sum_{k=0}^{\infty} r(y_{N+k}, v_{N+k}, y_{N+k+1}) \\ &= \sum_{k=0}^{\infty} s_k(y_N, v_N) + \sum_{k=0}^{\infty} (\beta^k - \beta^{k+1}) \lambda_{\mathsf{s}}^{\intercal}(y_N - y_{\mathsf{s}}) \\ &= \ell_N(y_N) + \lambda_{\mathsf{s}}^{\intercal}(y_N - y_{\mathsf{s}}) = M(y_N) . \end{aligned}$$

Since this holds for all $y_N \in \mathbb{T}$ and since $M(y_N) = \infty$ for $y_N \notin \mathbb{T}$, it follows that M is nonnegative. Moreover, since M is defined by an infinite sum, it satisfies

$$\begin{aligned} M(y_N) &\ge r(y_N, v_N^{\star}(y_N), y_{N+1}) + M(y_{N+1}) \\ &\overset{(12)}{\ge} R(y_N, y_{N+1}) + M(y_{N+1}) \\ &\ge \min_{y^+} \left\{ R(y_N, y^+) + M(y^+) \right\} \end{aligned}$$

for all $y_N \in \mathbb{T}$. This yields the theorem's statement after setting $y = y_N$, recalling that $M(y) = \infty$ for $y \notin \mathbb{T}$. ∎

IV. Stabilizing Tube MPC

The stabilizing CCTMPC variant [18] requires an entirely simple contractive polytope to exist and be precomputed. In contrast, the next section introduces a stabilizing variant of CCTMPC with self-optimizing terminal ingredients and without such regularity assumptions. This added flexibility can result in a larger region of attraction, compare Section V.

A. CCTMPC with Self-Optimizing Terminal Set

The proposed CCTMPC formulation can be written as

$$\begin{aligned} &\min_{y,v} \; \lambda_{\mathsf{s}}^{\intercal} y_0 + \sum_{k=0}^{N-1} \ell(y_k, v_k) + \sum_{k=0}^{\infty} s_k(y_N, v_N) \\ &\text{s.t.} \; \begin{cases} \forall k \in \{0,\ldots,N-1\},\; \forall i \in \{1,\ldots,\mathsf{v}\}, \\ F(AV_i y_k + BS_i v_k) + c \le y_{k+1} \\ E y_k \le 0,\; S_i v_k \in \mathbb{U},\; V_i y_k \in \mathbb{X},\; Fx \le y_0 \\ F(AV_i(y_N - y_{\mathsf{s}}) + BS_i(v_N - v_{\mathsf{s}})) \le \beta(y_N - y_{\mathsf{s}}) \\ S_i v_N \in \mathbb{U},\; E y_N \le 0,\; V_i y_N \in \mathbb{X} . \end{cases} \end{aligned} \quad (13)$$

Notice that the last polytope $P(y_N)$ is optimized subject to the constraint $(y_N, v_N) \in \mathbb{S}$ using ℓ_N as a terminal cost.

B. Controller Implementation

The strictly convex program (13) has $(N+1)(\mathsf{f} + \mathsf{v}m)$ variables and $(N+1)(\mathsf{f}\mathsf{v} + \mathsf{e} + \mathsf{v}n_{\mathbb{U}} + \mathsf{v}n_{\mathbb{X}}) + \mathsf{f}$ constraints. Here, $n_{\mathbb{U}}$ and $n_{\mathbb{X}}$ are the number of constraints encoding $u \in \mathbb{U}$ and $x \in \mathbb{X}$, respectively. In particular, if $\mathbb{U}$ and $\mathbb{X}$ are polyhedra and ℓ is a strictly convex quadratic form, (13) is a strictly convex QP (see Remark 5). This formulation has one less variable and two fewer constraints than the original CCTMPC scheme, compare [18, Rmk. 8 and Eq. (38)]. Its design is also simpler requiring the solution of a single convex program in $\mathsf{f} + \mathsf{v}m$ variables and $\mathsf{f}\mathsf{v} + \mathsf{e} + \mathsf{v}n_{\mathbb{U}} + \mathsf{v}n_{\mathbb{X}}$ constraints to compute y_{s}. The scheme in [18] requires solving an additional convex program of at least the same complexity for computing a contractive polytope and one LP in one variable and $\mathsf{f}+1$ constraints for the contraction rate.

Let $(y^\star(x), v^\star(x))$ be the parametric minimizer of (13). The MPC feedback law $\mu_{\mathrm{MPC}} : \mathbb{X} \to \mathbb{U}$ can be recovered by solving, for example, a strictly convex QP such as

$$\mu_{\mathrm{MPC}}(x) \in \underset{u\in\mathbb{U}}{\operatorname{argmin}}\ q(u) \quad \text{s.t. } Ax + Bu \in P\left(y_1^\star(x)\right) \quad (14)$$

for any suitable penalty $q : \mathbb{R}^m \to \mathbb{R}$. By construction, (14) is always feasible. The corresponding closed-loop system is

$$x_{k+1} = Ax_k + B\mu_{\mathrm{MPC}}(x_k) + w_k\ . \quad (15)$$

The state trajectory $x_0, x_1, \dots$ depends on the sequence $w_0, w_1, \dots \in \mathbb{W}$ whose value is unknown until the control input has been computed. This must be properly accounted for while analyzing the stability of the closed-loop system.

C. Stability Analysis

The following corollary of Theorem 1 establishes stability of the closed-loop sequence (15) in the enclosure sense [17, Def. 8]. The statement holds for any strictly convex ℓ, without any regularity assumptions on A and B.

Corollary 1: Let (13) be feasible for $x = x_0$. Then, (13) is feasible for all state measurements $x = x_k$ with $k \in \mathbb{N}$ and $x_k \in P(y_0^\star(x_k))$ for all $k \in \mathbb{N}$. Moreover the sequence $y^\star(x_0), y^\star(x_1), \dots$ is asymptotically stable. In particular,

$$\lim_{k\to\infty} y^\star(x_k) \ = \ [y_{\mathsf{s}}, \dots, y_{\mathsf{s}}].$$

Proof: The statement of this corollary follows directly from Theorem 1, because Problem (13) is equivalent to

$$\min_{y} \sum_{k=0}^{N-1} R(y_k, y_{k+1}) + M(y_N) \quad \text{s.t.} \quad Fx \le y_0, \quad (16)$$

recalling that M satisfies a Lyapunov descent inequality and that due to the strict convexity of ℓ, the minimizer $(y_{\mathsf{s}}, y_{\mathsf{s}})$ of (5) is unique and consequently R and M are positive definite. The detailed argument can be found in [18, Thm. 4]: although M differs, the proof follows the same steps. ∎

V. Numerical Illustration

We consider a discretized[4] triple integrator given by

$$A = \begin{pmatrix} 1 & h & {}^1\!/_2\, h^2 \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} {}^1\!/_6\, h^3 \\ {}^1\!/_2\, h^2 \\ h \end{pmatrix}$$

with $h = {}^1\!/_3$. The state and control constraints are given by

$$\mathbb{X} = [-2, 11.5] \times [-6, 6.5] \times [-6.5, 6.1] \quad \text{and} \quad \mathbb{U} = [-10, 10],$$

while the additive uncertainty lies in the set

$$\mathbb{W} = \left\{ \begin{pmatrix} h & {}^1\!/_2\, h^2 & {}^1\!/_6\, h^3 \\ 0 & h & {}^1\!/_2\, h^2 \\ 0 & 0 & h \end{pmatrix} w \ \middle|\ w \in [-{}^1\!/_{15}, {}^1\!/_{15}]^3 \right\} .$$

We introduce a discretization of the 2-sphere which provides one way to set up (F, E, V) in $\mathbb{R}^3$. Consider the index set

[4]One can add the time discretization error as an extra uncertain input. In this case, the discretization is exact for piecewise constant inputs.

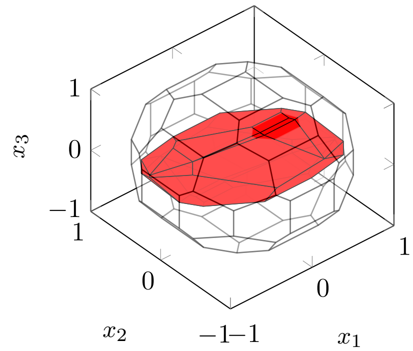

Fig. 1. Polytopes $P(y_{\mathsf{s}})$ (red) and $P((1,\dots,1)^\mathsf{T})$ (white).

$$\Delta_p = \left\{ \alpha \in \mathbb{Z}^2 \ \middle|\ \begin{matrix} \alpha_1 + 4|\alpha_2| \le 4p \\ \text{with} \quad \alpha_1 \ \ge 1 \end{matrix} \right\} \cup \{pe_2, -pe_2\}$$

with $e_2 = (0,1)^\mathsf{T}$, and parameter $p \in \mathbb{N}$, as well as the map

$$\Phi(\alpha) = \frac{\pi}{2p} \begin{cases} \left(\dfrac{p\alpha_1}{p - |\alpha_2|}, \alpha_2 \right)^\mathsf{T} & \text{if } \alpha_1 \ge 1 \\ \alpha & \text{otherwise.} \end{cases}$$

We define $F \in \mathbb{R}^{\mathsf{f}\times 3}$ such that its rows F_i satisfy

$$\{F_1^\mathsf{T}, F_2^\mathsf{T}, \dots, F_\mathsf{f}^\mathsf{T}\} = \left\{ \begin{pmatrix} \cos(\phi_1)\cos(\phi_2) \\ \cos(\phi_2)\sin(\phi_1) \\ \sin(\phi_2) \end{pmatrix} \ \middle|\ \begin{matrix} \exists \alpha \in \Delta_p : \\ \phi = \Phi(\alpha) \end{matrix} \right\} .$$

$P((1,\dots,1)^\mathsf{T})$ is simple with $\mathsf{f} = |\Phi| = 4p^2 + 2$ facets, $\mathsf{v} = 8p^2$ vertices, and $\mathsf{e} = 12p^2$ edges. Its vertices $v_1, \dots, v_\mathsf{v}$ and their configuration are known by construction.[5] Thus, matrices $V_1, \dots, V_\mathsf{v} \in \mathbb{R}^{3\times\mathsf{f}}$ with $v_i = V_i(1,\dots,1)^\mathsf{T}$ for all $i \in \{1,\dots,\mathsf{v}\}$ can be computed directly. In particular, for $p = 3$ we have $\mathsf{f} = 38$, $\mathsf{v} = 72$, and $\mathsf{e} = 108$. We computed $E \in \mathbb{R}^{\mathsf{e}\times\mathsf{f}}$ per Remark 1. The cost ℓ is chosen as in [18]; i.e.,

$$\begin{aligned} \ell(y, v) := \quad & \left\|\overline{V}y\right\|_\mathsf{Q}^2 + \sum_{i=1}^{\mathsf{v}} \left\|\left(V_i - \overline{V}\right) y\right\|_\mathsf{S}^2 \\ & + \left\|\overline{S}v\right\|_\mathsf{R}^2 + \sum_{i=1}^{\mathsf{v}} \left\|(S_i - \overline{S})v\right\|_\mathsf{T}^2\ , \end{aligned}$$

$$\text{with} \quad \overline{V} := \frac{1}{\mathsf{v}} \sum_{i=1}^{\mathsf{v}} V_i \quad \text{and} \quad \overline{S} := \frac{1}{\mathsf{v}} \sum_{i=1}^{\mathsf{v}} S_i\ .$$

with $\mathsf{Q} = \mathbb{I}_3$, $\mathsf{R} = 1$, $\mathsf{S} = 10\mathbb{I}_3$, and $\mathsf{T} = 1$. The optimal RCI set $P(y_{\mathsf{s}})$ is not simple, it has 50 vertices and 36 facets (see Fig. 1). We implemented (13) for $N \in \{1,\dots,10\}$ with $\beta = 0.975$. The scheme requires solving a QP with $110(N+1)$ variables and $3060N + 3098$ constraints at each time. Sequence (15) with $x_0 = (3.5, 3.5, 3)^\mathsf{T}$ was computed using the same sequence $w_1, w_2, \dots$, and μ_{MPC} from (14) with $q(u) = u^2$. Figure 2 shows $P(y_{\mathsf{s}})$ and $P(y_0^\star(x_k))$ for $k \in \{0, 1, 9\}$ and $N \in \{1, 3, 5\}$. The enclosures $P(y_0^\star(x_k))$ converge to $P(y_{\mathsf{s}})$, as guaranteed by Corollary 1.

[5]$\mathrm{convh}(\{F_1^\mathsf{T}, \dots, F_\mathsf{f}^\mathsf{T}\})$ is, for any p, a polytopic simplicial 2-sphere with octahedral symmetry. Its facet configuration can be easily deduced over Δ_p. The vertex configuration for $P((1,\dots,1)^\mathsf{T})$ is deduced by duality.

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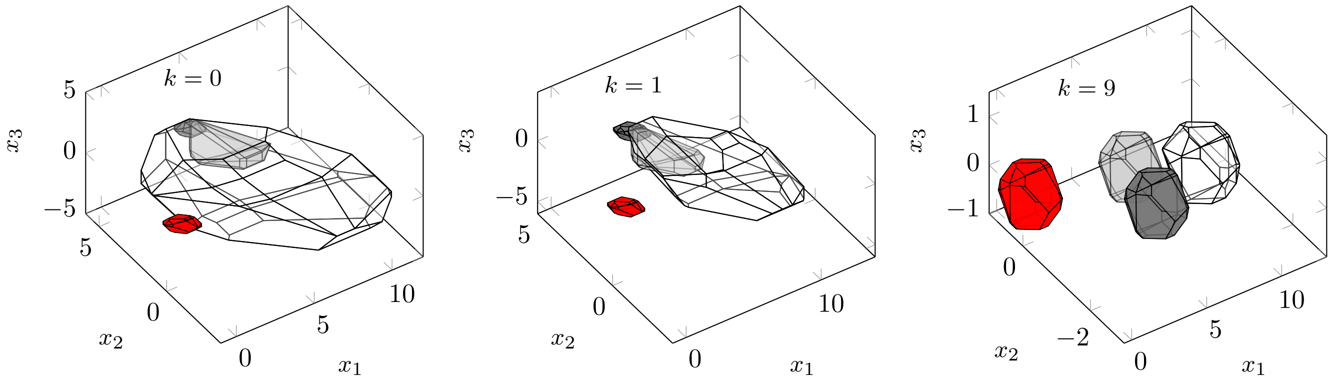

Fig. 2. Enclosures $\boldsymbol{P(y^\star(x_k))}$ for $\boldsymbol{N = 1}$ (off-white), $\boldsymbol{N = 3}$ (light gray), $\boldsymbol{N = 5}$ (dark gray), and optimal RCI polytope $\boldsymbol{P(y_s)}$ (red).

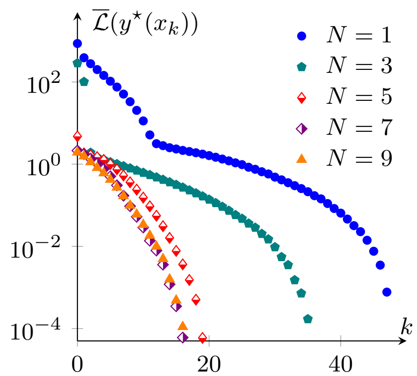

Fig. 3. Scaled Lyapunov function $\boldsymbol{\overline{\mathcal{L}}(y^\star(x_k))}$ for $\boldsymbol{N \in \{1, 3 \ldots, 9\}}$.

Figure 3 shows the scaled Lyapunov function,

$$\overline{\mathcal{L}}(y) := \frac{\mathcal{L}(y)}{R_\infty} \quad \text{with} \quad \mathcal{L}(y) = \sum_{k=0}^{N-1} R(y_k, y_{k+1}) + M(y_N)$$

and $R_\infty = \sum_{k=0}^{\infty} R(y_k^\star(x_0), y_{k+1}^\star(x_0))$, evaluated along (15). The scaling R_∞ was computed by adding from $k = 0$ up to $k = 500$ to achieve a relative error of 10^{-6}. The decrease in $\overline{\mathcal{L}}$ illustrates the stability of the controller, irrespective of N—as predicted by Corollary 1. The suboptimality of the infinite-horizon cost approximation is also shown since $\mathcal{L}(y^\star(x_0)) \approx 860R_\infty$ for $N = 1$ and $\mathcal{L}(y^\star(x_0)) \approx 2R_\infty$ for $N \geq 7$. Polytopic inner and outer approximations of the region of attraction (ROA) were computed by maximizing $c_i^\mathsf{T} x$ over the feasible set of the TMPC problem with c_i^T chosen as F_i with $p = 20$. The ROA of the proposed formulation was consistently larger with respect to [18] (for the same β). For example, for $N = 1$, the volume of the ROA for this CCTMPC is at least 8% larger with respect to the method in [18].

VI. Conclusions

This paper introduced a stabilizing CCTMPC method that optimizes its own terminal ingredients. The formulation does not require regularity assumptions such as the existence of a robust contractive set, as required in [18]. Instead, it relies on a recursively feasible and exponentially convergent polytopic sequence whose initial point is optimized online. As a result, the offline design phase of this controller is simpler since the terminal cost is a *convergent* infinite sum along the (parameters of) the above sequence which can, for many practical choices of the stage cost, be obtained in closed form (see Remark 5). This novel CCTMPC method results in a more flexible tube MPC scheme for linear systems with both additive and multiplicative uncertainty.

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