Model Reference Adaptive Control II

Contents

Unnormalized Adaptive Laws

Consider the plant with $n^* = 1$, the control law

$$\dot{\omega}_1 = F\omega_1 + gu_p, \quad \omega_1(0) = 0,$$

$$\dot{\omega}_2 = F\omega_2 + gy_p, \quad \omega_2(0) = 0,$$

$$u_p = \theta^\top \omega$$

where

- $\omega = \left[\omega_1^\top \omega_2^\top, y_p, r\right]^\top \in \mathbb{R}^{2n}$
- $\theta(t)$ is the estimate of $\theta^* = \left[\theta_1^{*\top}, \theta_2^{*\top}, \theta_3^*, c_0^*\right]^\top$, calculated from the matching equation
- (F, g) is the state-space realization of $\frac{\alpha(s)}{\Lambda(s)}$, i.e.,

$$(sI - F)^{-1} g = \frac{\alpha(s)}{\Lambda(s)}$$

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Augmented state-space

$$\dot{Y}_c = A_0 Y_c + B_c \theta^{*\top} \omega + B_c(u_p - \theta^{*\top}\omega), \quad Y_c(0) = Y_0$$

$$y_p = C_c^\top Y_c$$

where we have added and subtracted the desired input $B_c \theta^{*\top}\omega$

and $Y_c = \left[x_p^\top, \omega_1^\top, \omega_2^\top\right]^\top \in \mathbb{R}^{n_p+2n-2}$

$$A_0 = \begin{bmatrix} A_p & 0 & 0 \\ 0 & F & 0 \\ gC_p^\top & 0 & F \end{bmatrix}, \quad B_c = \begin{bmatrix} B_p \\ g \\ 0 \end{bmatrix},$$

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Absorb the term $B_c\theta^{*\top}\omega$ into homogeneous part

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is an Hurwitz matrix. Moreover, there exists nonminimal state space representation for reference model

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$$\begin{aligned}\dot{e} &= A_c e + B_c \left(u_p - \theta^{*\top}\omega\right), \\ e_1 &= C_c^\top e.\end{aligned}$$

Since $C_c^\top (sI - A_c)^{-1} B_c c_0^* = W_m(s)$, we have

$$e_1 = W_m(s)\rho^* \left(u_p - \theta^{*\top}\omega\right)$$

where $\rho^* = \frac{1}{c_0^*}$.

Now, we can use above PM to generate a wide class of adaptive laws for estimating θ^* by using the results of parameter estimation.

Drawbacks: r might not be a signal can be arbitrarily designed.

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Alternatively, rewrite the tracking error equation

$$\dot{e} = A_c e + \bar{B}_c \rho^* \tilde{\theta}^\top \omega,$$

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where $\bar{B}_c = B_c c_0^*$, Note

$$W_m(s) = C_c^\top (sI - A_c)^{-1} \bar{B}_c$$

is SPR and A_c is Hurwitz, it motivates the Lyapunov function

$$V(\tilde{\theta}, e) = \frac{e^\top P_c e}{2} + \frac{\tilde{\theta}^\top \Gamma^{-1} \tilde{\theta}}{2} |\rho^*|$$

where $\Gamma^\top = \Gamma > 0$, $P_c = P_c^\top > 0$ and satisfies equations

$$P_c A_c + A_c^\top P_c = -qq^\top - v_c L_c$$

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The time derivative

$$\dot{V} = -\frac{e^\top q q^\top e}{2} - \frac{v_c}{2} e^\top L_c e + e^\top P_c \bar{B}_c \rho^* \tilde{\theta}^\top \omega + \tilde{\theta}^\top \Gamma^{-1} \dot{\tilde{\theta}} \, |\rho^*|$$

Since $e^\top P_c \bar{B}_c = e^\top C_c = e_1$ and $\rho^* = |\rho^*| \operatorname{sgn}(\rho^*)$, choosing

$$\dot{\tilde{\theta}} = \dot{\theta} = -\Gamma e_1 \omega \operatorname{sgn}(\rho^*)$$

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$$\dot{V} = -\frac{e^\top q q^\top e}{2} - \frac{v_c}{2} e^\top L_c e \le 0$$

implies

- $e, \tilde{\theta} \in \mathcal{L}_\infty$, $x_p, y_p, \omega_1, \omega_2 \in \mathcal{L}_\infty$, $u_p = \theta^\top \omega \in \mathcal{L}_\infty$
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Summary: Direct unnormalized MRAC for $n^* = 1$

Scheme $u_p = \theta^\top \omega$

$$\dot{\omega}_1 = F\omega_1 + gu_p, \quad \omega_1(0) = \omega_{10} \in \mathbb{R}^{n-1}$$

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$$\dot{\theta} = -\Gamma e_1 \omega \operatorname{sgn}\left(\frac{k_p}{k_m}\right) \quad \theta(0) = \theta_0 \in \mathbb{R}^{2n}$$

Properties:

(i) All signals in the closed-loop plant are bounded, and $e_1 \to 0$ asymptotically for any $r \in \mathcal{L}_\infty$.

(ii) If r is S.R. of order $2n, \dot{r} \in \mathcal{L}_\infty$, and $Z_p(s), R_p(s)$ are relatively coprime, then $|\tilde{\theta}| \to 0, e_1 \to 0$ exponentially fast.

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Direct unnormalized MRAC for $n^* = 2$

Problem : $W_m(s)$ is no longer SPR. Recall the error equation

$$\dot{e} = A_c e + \bar{B}_c \rho^* \tilde{\theta}^\top \omega = A_c e + \bar{B}_c (s + p_0) \rho^* \tilde{\theta}^\top \phi,$$

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with $\phi = \frac{1}{s+p_0}\omega$, $p_0 > 0$. In transfer function form, we have

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Now, we can apply the same adaptive law

$$\dot{\tilde{\theta}} = \dot{\theta} = -\Gamma e_1 \phi \operatorname{sgn}(\rho^*)$$

Scheme:

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$$\dot{\phi} = -p_0\phi + \omega, \qquad \phi(0) = 0 \in \mathbb{R}^{2n}$$

$$u_p = \theta^\top \omega + \dot{\theta}^\top \phi, \qquad \omega = \left[\omega_1^\top, \omega_1^\top, y_p, r\right]^\top,$$

Remark: When the relative degree is greater than 2 the procedure is very similar, but the control law becomes more complex, refer to Robust Adaptive Law Section 6.4.

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u_p &= \theta^\top \omega + \dot{\theta}^\top \phi, \quad & \omega &= \left[\omega_1^\top, \omega_1^\top, y_p, r\right]^\top,
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Recall control law

$$u_p = \theta_1^\top(t)\frac{\alpha(s)}{\Lambda(s)}u_p + \theta_2^\top(t)\frac{\alpha(s)}{\Lambda(s)}y_p + \theta_3(t)y_p + c_0(t)r$$

and the error equation

$$e_1 = W_m(s)\rho^*\left(u_p - \theta^{*\top}\omega\right)$$

where $\rho^* = \frac{1}{c_0^*} = \frac{k_p}{k_m}, \theta^* = \left[\theta_1^{*\top}, \theta_2^{*\top}, \theta_3^*, c_0^*\right]^\top$. Regardless of the relative degree, express it as a standard B-SPM

$$e_1 = \rho^*\left(\theta^{*\top}\phi + u_f\right)$$

where $\phi = -W_m(s)\omega, u_f = W_m(s)u_p$.

Use the result of Parameter Estimation, we have

$$\dot{\theta} = \Gamma \varepsilon \phi \operatorname{sgn}\left(\frac{k_p}{k_m}\right), \qquad \dot{\rho} = \gamma \varepsilon \xi$$

with $\varepsilon = \frac{e_1 - \rho\xi}{m_s^2}, m_s^2 = 1 + \phi^\top \phi + u_f^2, \xi = \theta^\top \phi + u_f, \phi = -W_m(s)\omega, u_f = W_m(s)u_p$.

The MRAC scheme applied to the plant with arbitrary n^* has the following properties:

(i) All signals are uniformly bounded and the tracking error e_1 converges to zero as $t \to \infty$.

(iii) If the reference input signal r is sufficiently rich of order $2n, \dot{r} \in \mathcal{L}_\infty$, and R_p, Z_p are coprime, the tracking error e_1 and parameter error $\tilde{\theta} = \theta - \theta^*$ converge to zero exponentially fast.

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$$\dot{\theta} = \Gamma \varepsilon \phi \,\mathrm{sgn}\left(\frac{k_p}{k_m}\right), \qquad \dot{\rho} = \gamma \varepsilon \xi$$

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as follows: Because $\rho^* = \frac{1}{c_0^*}$ and $\theta^{*\top}\omega = \theta_0^{*\top}\omega_0 + c_0^* r$ where $\theta_0^* = \left[\theta_1^{*\top}, \theta_2^{*\top}, \theta_3^*\right]^\top$ and $\omega_0 = \left[\omega_1^\top, \omega_2^\top, y_p\right]^\top$, we have

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Note that: the adaptive laws based on the linear model generate estimates of c_0^* only, without any knowledge of the $\operatorname{sgn}(\rho^*)$ or lower bound for $|\rho^*|$. It turns out, however, that in the stability analysis of the MRAC schemes whose adaptive laws are based on linear model, $1/c_0(t)$ is required to be bounded.

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This can be guaranteed by modifying the adaptive laws for $c_0(t)$ using projection, so that

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$$y_p = \frac{k_p\,(s+b_0)}{(s^2+a_1 s+a_0)} u_p$$

where $k_p > 0, b_0 > 0$, and k_p, b_0, a_1, a_0 are unknown constants.

The reference model is chosen as

$$y_m = \frac{3}{s+3} r$$

Q1) Design a direct MRAC such that $y_p \to y_m$ as $t \to \infty$.

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Contents

In the case of $n^* = 1$, control law admits the same form

$$\dot{\omega}_1 = F\omega_1 + gu_p, \quad \omega_1(0) = 0 \in \mathbb{R}^{n-1}$$

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where $\omega = \left[\omega_1^\top, \omega_2^\top, y_p, r\right]^\top \in \mathbb{R}^{2n}$.

Rewrite the plant polynomials as

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For *indirect* methods, the relation mapping the desired controller parameter θ^* and the plant parameter θ_p^* is critical. Recall the matching equation

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we can obtain the algebraic equations

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The adaptive law for generating $\hat{k}_p, \hat{b}_p, \hat{a}_p, \hat{a}_{n-1}$ is constructed by considering the parametric model

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Express the error equation PM in terms of $\tilde{\theta}_p$:

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where $C_c^\top(sI - A_c)^{-1}B_c = \frac{1}{k_m}W_m(s)$. Motivating the Lyapunov-like function

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By rendering $\dot{V}$ as

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The simplicity of Indirect MRAC with normalized adaptive law arises from the fact that the control and adaptive laws are designed independently.

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with $b_m = k_p$. Form the expression into a SPM

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$$z = \frac{s^n}{\Lambda_p(s)} y_p, \quad \phi = \left[\frac{\alpha_m^\top(s)}{\Lambda_p(s)} u_p, -\frac{\alpha_{n-1}^\top(s)}{\Lambda_p(s)} y_p \right]^\top$$

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$$z = \frac{s^n}{\Lambda_p(s)} y_p, \quad \phi = \left[\frac{\alpha_m^\top(s)}{\Lambda_p(s)} u_p, -\frac{\alpha_{n-1}^\top(s)}{\Lambda_p(s)} y_p \right]^\top$$

$$\theta_p^* = [b_m, \ldots, b_0, a_{n-1}, \ldots, a_0]^\top$$

Step II: Control law

$$u_p = \theta_1^\top \frac{\alpha(s)}{\Lambda(s)} u_p + \theta_2^\top \frac{\alpha(s)}{\Lambda(s)} y_p + \theta_3 y_p + c_0 r = \theta^\top \omega$$

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$$\omega = \left[\frac{\alpha^\top(s)}{\Lambda(s)} u_p, \frac{\alpha^\top(s)}{\Lambda(s)} y_p, y_p, r \right]^\top, \quad \alpha(s) = \alpha_{n-2}(s)$$

and $\theta = \left[\theta_1^\top, \theta_2^\top, \theta_3, c_0\right]^\top$ is calculated using the mapping $\theta(t) = f\left(\theta_p(t)\right)$, which obtained from the matching equation

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Step III: Adaptive Law, e.g. the gradient algorithm

$$\dot{\theta}_p = \mathrm{Pr}(\Gamma \varepsilon \phi)$$

$$\varepsilon = \frac{z - \theta_p^T \phi}{m_s^2}, \quad m_s^2 = 1 + \phi^T \phi$$

where $\mathrm{Pr}(\cdot)$ guarantees namely $\hat{b}_m = \hat{k}_p$, satisfies $\hat{b}_m \,\mathrm{sgn}\,(k_p) \geq k_0 > 0, \forall t \geq 0$, provided $\hat{b}_m(0) \,\mathrm{sgn}\,(k_p) \geq k_0 > 0$. The $\mathrm{Pr}(\cdot)$ operator does not affect the other elements of θ_p.

Property: The indirect MRAC with normalized gradient-based adaptive law guarantees that all signals are bounded and the tracking error $e_1 = y_p - y_m$ converges to zero as $t \to \infty$.

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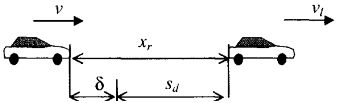


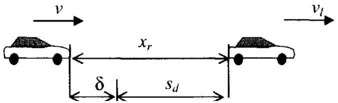
Figure: car following

The ACC system regulates the following vehicle's speed v towards the leading vehicle's speed v_l and maintains the intervehicle distance x_r close to the desired spacing s_d., that is given by

$$s_d = s_0 + hv$$

where

- s_0 is a fixed safety intervehicle spacing
- h is constant time headway



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$$v_r \to 0, \quad \delta \to 0 \quad \text{as } t \to \infty$$

where $v_r := v_l - v$ is the speed error or relative speed and $\delta := x_r - s_d$ is the separation or spacing error.

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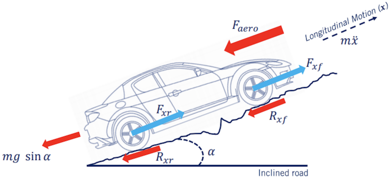

$$m\ddot{x} = F_{xf} + F_{xr} - F_{aero} - R_{xf} - R_{xr} - mg\sin\alpha$$

where

F_{xg}, F_{xr}: front and rear tire force

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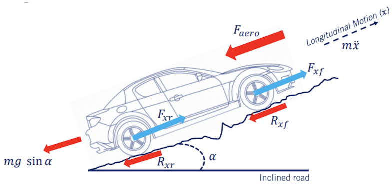

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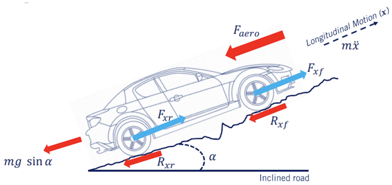

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Simplification is necessary and feasible because:

- the human driver can drive the vehicle without having in mind such a complex nonlinear system
- the brake/throttle commands are low frequency inputs

Step 0 Simplified first-order model:

$$\dot{v} = -av + bu + d$$

where

- v is the longitudinal speed and u is the brake/throttle command
- d is the modeling uncertainty
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$$v_m = \frac{a_m}{s + a_m}\,(v_l + k\delta)$$

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$$V = \frac{e^2}{2} + \sum_{i=1}^{3} \frac{b}{2\gamma_i}\tilde{k}_i^2$$

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where $\dot{k}_3^* = \frac{a\dot{v}_l - \dot{d}}{b}$ is a bounded disturbance term.

- $e \in \mathcal{L}_\infty$, $k_i \in \mathcal{L}_\infty$, which in turn implies that all the other signals in the closed-loop system are bounded
- if v_l and d are constants, $e \in \mathcal{L}_2$ and e, δ converge to zero with time.

Note that, the above adaptive control scheme is designed for the throttle subsystem. For the brake subsystem another controller is developed.

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Step VII Simulation verification

- Leading vehicle with constant speed
- Leading vehicle with slowly change speed
- Consider constant disturbance
- Consider slowly time-varying disturbance
- Consider measurement noise
- ...