Parameter Estimation III

Contents

- Parameter Estimation for DPM

- Parameter Estimation for B-SPM

- Adaptive Observer

Strictly Positive Real

Theorem: Assume that a rational function $G(s)$ of the complex variable $s = \sigma + j\omega$ is real for real s and is not identically zero for all s. Let n^* be the relative degree of $G(s) = \frac{Z(s)}{R(s)}$. Then, $G(s)$ is SPR if and only if

(i) $|n^*| \leq 1$ and $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$

(ii) $\mathrm{Re}[G(j\omega)] > 0, \forall \omega \in (-\infty, \infty)$

(iii) (a) When $n^* = 1, \lim_{|\omega| \to \infty} \omega^2 \,\mathrm{Re}[G(j\omega)] > 0$

(b) When $n^* = -1, \lim_{|\omega| \to \infty} \frac{G(j\omega)}{j\omega} > 0.$

Example: $G(s) = \frac{1}{s+\alpha}$

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Corollary:

(i) $G(s)$ is SPR if and only if $1/G(s)$ is SPR.

(ii) If $G(s)$ is SPR, then, $|n^*| \leq 1$, and the zeros and poles of $G(s)$ lie in $\mathrm{Re}[s] < 0$

Example: (i) $G_1(s) = \frac{s-1}{(s+2)^2}$ (ii) $G_2(s) = \frac{1}{(s+2)^2}$

(iii) $G_3(s) = \frac{s+3}{(s+1)(s+2)}$

For $G_3(s)$, we have that

$$\mathrm{Re}\left[G_3(j\omega)\right] = \frac{6}{\left(2-\omega^2\right)^2 + 9\omega^2} > 0, \quad \forall \omega \in (-\infty, \infty)$$

which together with the stability of $G_3(s)$ implies that $G_3(s)$ is PR. However $G_3(s)$ violates (iii)(a) of Theorem, it is not SPR.

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LKY

Lemma: (Lefschetz-Kalman-Yakubovich (LKY) Lemma) Given a stable matrix A, a vector B such that (A, B) is controllable, a vector C and a scalar $d \geq 0$, the transfer function defined by

$$G(s) = d + C^\top (sI - A)^{-1} B$$

is SPR if and only if for any positive definite matrix L, there exist a symmetric positive definite matrix P, a scalar $\nu > 0$ and a vector q such that

$$A^\top P + PA = -qq^\top - \nu L$$
$$PB - C = \pm q\sqrt{2d}$$

LKY

Example: Consider the system

$$y = G(s)u$$

where $G(s) = \frac{s+3}{(s+1)(s+2)}$. We would like to verify whether $G(s)$ is SPR by using LKY. The system has the state space representation

$$\dot{x} = Ax + Bu$$

$$y = C^\top x$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The controllable requirement is relaxed by the next lemma.

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MKY

Lemma: (Meyer-Kalman-Yakubovich (MKY) Lemma) Given a stable matrix A, vectors B, C and a scalar $d \geq 0$, we have the following: If

$$G(s) = d + C^\top (sI - A)^{-1} B$$

is SPR, then for any given $L = L^\top > 0$, there exists a scalar $\nu > 0$, a vector q and a $P = P^\top > 0$ such that

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Note, for strictly proper transfer function $G(s)$, we have $d = 0$, that is $PB = C$.

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SPR-Lyapunov Design for DPM

Consider

$$z = W(s)\theta^{*\top}\psi$$

Since θ^* is a constant vector, the DPM may be written as

$$z = W(s)L(s)\left[\theta^{*\top}\phi\right]$$

where $\phi = L^{-1}(s)\psi, L(s)$ is chosen so that $L^{-1}(s)$ is a proper stable transfer function, and $W(s)L(s)$ is a proper SPR transfer function. We form the normalized estimation error

$$\varepsilon = z - \hat{z} - W(s)L(s)\left[\varepsilon n_s^2\right] = W(s)L(s)\left[-\tilde{\theta}^\top\phi - \varepsilon n_s^2\right]$$

where n_s is designed so that $\frac{\phi}{m_s} \in \mathcal{L}_\infty$ for $m_s^2 = 1 + n_s^2$.

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SPR-Lyapunov Design for DPM

Since $W(s)L(s)$ is strictly proper, there exists a Hurwitz A_c

$$\dot{e} = A_c e + b_c \left(-\tilde{\theta}^\top \phi - \varepsilon n_s^2\right)$$
$$\varepsilon = c_c^\top e$$

where $W(s)L(s) = c_c^\top (sI - A_c)^{-1} b_c$. According to MKY lemma, there exist matrices $P_c = P_c^\top > 0, L_c = L_c^\top > 0$, a vector q, and a scalar $v > 0$ such that

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The adaptive law for θ is then can be generated using the

Lyapunov-like function

$$V = \frac{e^\top P_c e}{2} + \frac{\tilde{\theta}^\top \Gamma^{-1} \tilde{\theta}}{2}$$

where $\Gamma = \Gamma^\top > 0$. The time derivative

$$\dot{V} = -\frac{1}{2} e^\top q q^\top e - \frac{v}{2} e^\top L_c e + e^\top P_c b_c \left(-\tilde{\theta}^T \phi - \varepsilon n_s^2 \right) + \tilde{\theta}^\top \Gamma^{-1} \dot{\tilde{\theta}}$$

Since $e^\top P_c b_c = e^\top c_c = \varepsilon$, it follows that by choosing $\dot{\tilde{\theta}} = \dot{\theta}$ as

$$\dot{\theta} = \Gamma \varepsilon \phi$$

we get

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SPR-Lyapunov Design for DPM

Convergence Properties: The gradient-based adaptive law for DPM guarantees that

(i) $\varepsilon, \theta \in \mathcal{L}_\infty$ and $\varepsilon, \varepsilon n_s, \dot{\theta} \in \mathcal{L}_2$ independent of the boundedness properties of ϕ.

(ii) If $n_s, \phi, \dot{\phi} \in \mathcal{L}_\infty$ and ϕ is PE, then $\theta(t) \to \theta^*$ exponentially fast.

Proof can be found in Robust Adaptive Control Section 4.8.

Example Consider the previous plant:

$$y = \frac{b_1 s + b_0}{s^2 + 3s + 2} u$$

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$$y = \frac{1}{(s+1)(s+2)} \theta^{*\top} \psi$$

where $\theta^* = [b_1, b_0]^\top, \psi = [\dot{u}, u]^\top$. We then choose $L(s) = s + 2$ so that $W(s)L(s) = \frac{1}{s+1}$ is SPR and rewrite PM as

$$y = \frac{1}{s+1} \theta^{*\top} \phi, \quad \phi = \left[\frac{s}{s+2} u, \frac{1}{s+2} u \right]^\top$$

Next, apply the adaptive law

$$\dot{\theta} = \Gamma \varepsilon \phi$$

where $\varepsilon = y - \frac{1}{s+1} \left(\theta^\top \phi + \varepsilon n_s^2 \right), n_s = \alpha \phi^\top \phi, \alpha > 0$.

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Contents

Gradient-based for B-SPM

Consider the B-SPM

$$z = \rho^*(\theta^{*\top}\phi + z_0)$$

where z, z_0 are available for measuring and ρ^*, θ^* are unknown parameters. But the sign of ρ^* is assumed to be known.

The estimation error is generated as

$$\varepsilon = \frac{z - \hat{z}}{m_s^2}, \quad \hat{z} = \rho(t)\left(\theta(t)^\top \phi + z_0\right)$$

where $\rho(t), \theta(t)$ are the estimates of ρ^*, θ^*, respectively, at time t and where m_s is designed to bound ϕ, z_0 from above. An example of m_s with this property is $m_s^2 = 1 + \phi^\top\phi + z_0^2$.

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Gradient-based for B-SPM

Let us consider the cost

$$J(\rho,\theta) = \frac{\varepsilon^2 m_s^2}{2} = \frac{\left(z - \rho\xi - \rho^*\theta^\top\phi - \rho^* z_0 + \rho^*\xi\right)^2}{2m_s^2}$$

where $\xi = \theta^\top\phi + z_0$ is available for measurement. Applying the gradient method yields

$$\dot{\theta} = -\Gamma_1 \nabla J_\theta = \Gamma_1 \varepsilon \rho^* \phi, \quad \dot{\rho} = -\gamma \nabla J_\rho = \gamma\varepsilon\xi$$

where $\Gamma_1 = \Gamma_1^\top > 0, \gamma > 0$ are the adaptive gains. We bypass the unknown ρ^* by employing the equality

$$\Gamma_1 \rho^* = \Gamma_1 |\rho^*| \operatorname{sgn}(\rho^*) = \Gamma \operatorname{sgn}(\rho^*)$$

where $\Gamma = \Gamma_1 |\rho^*|$.

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Gradient-based for B-SPM

The adaptive laws for θ, ρ now be written as

$$\dot{\theta} = \Gamma \varepsilon \phi \operatorname{sgn}(\rho^*) \quad \theta(0) = \theta_0 \in \mathbb{R}^{n-1}$$

$$\dot{\rho} = \gamma \varepsilon \xi, \qquad \rho(0) = \rho_0 \in \mathbb{R}$$

Theorem: The gradient-based adaptive law for B-SPM guarantees that

(i) $\varepsilon, \varepsilon m_s, \dot{\theta}, \dot{\rho} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\theta, \rho \in \mathcal{L}_\infty$

(ii) If $\frac{\xi}{m_s} \in \mathcal{L}_2$, then $\rho(t) \to \bar{\rho}$ as $t \to \infty$, where $\bar{\rho}$ is a constant.

(iii) If $\frac{\xi}{m_s} \in \mathcal{L}_2$ and $\frac{\phi}{m_s}$ is PE, then $\theta(t) \to \theta^*$ as $t \to \infty$.

For unknown $\operatorname{sgn}(\rho)$ case, refer to Robust Adaptive Control 4.5 Nussbaum Gain.

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Theorem: The gradient-based adaptive law for B-SPM

guarantees that

(i) $\varepsilon, \varepsilon m_s, \dot{\theta}, \dot{\rho} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\theta, \rho \in \mathcal{L}_\infty$

(ii) If $\frac{\xi}{m_s} \in \mathcal{L}_2$, then $\rho(t) \to \bar{\rho}$ as $t \to \infty$, where $\bar{\rho}$ is a constant.

(iii) If $\frac{\xi}{m_s} \in \mathcal{L}_2$ and $\frac{\phi}{m_s}$ is PE, then $\theta(t) \to \theta^*$ as $t \to \infty$.

For unknown $\operatorname{sgn}(\rho)$ case, refer to Robust Adaptive Control 4.5

Nussbaum Gain.

Gradient-based for B-SPM

The adaptive laws for θ, ρ now be written as

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Summary

SPM

- Gradient-based for instantaneous cost function
- Gradient-based for integral cost function
- Recursive LS and Modified (Projected) recursive LS
- Pure LS and Modified(Resetting) pure LS

DPM

- SPR-Lyapunov designed

B-SPM

- Gradient-based for instantaneous cost function

Contents in the sequel are supplementary

Contents

- Parameter Estimation for DPM
- Parameter Estimation for B-SPM
- Adaptive Observer

Adaptive Luenberger observer

Consider the LTI SISO plant

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = C^\top x$$

In the case A, B, C are known, the *Luenberger observer* is in the form of

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \quad \hat{x}(0) = \hat{x}_0$$
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where K is chosen such that $A - KC^\top$ is Hurwitz, guarantees that $\hat{x} \to x$ exponentially fast. The existence of K is ensured by the observability of pair $(A, C^\top)$

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Idea: $(A, B, C) \to G(s) \to \hat{G}(s) \to (\hat{A}, \hat{B}, \hat{C})$

mapping of the $2n$ estimated parameters of $G(s)$ to the $n^2 + 2n$ parameters of A, B, C is not uniqueunless (A, B, C) is in a observer canonical form, i.e., the plant is represented as

$$\dot{x}_o = \left[-a_p \mid \frac{I_{n-1}}{0} \right] x_o + b_\rho u$$

$$y = [1, 0, \ldots, 0] x_o$$

where $a_p = [a_{n-1}, a_{n-2}, \ldots, a_0]^\top$ and $b_p = [b_{n-1}, b_{n-2}, \ldots, b_0]^\top$ are the coefficients of the transfer function

$$G(s) = \frac{y(s)}{u(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0 s}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0 s}$$

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Then the adaptive observer is given by

$$\dot{\hat{x}} = \hat{A}(t)\hat{x} + \hat{b}_p(t)u + K(t)(y - \hat{y}), \quad \hat{x}(0) = \hat{x}_0$$
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where $\hat{x}$ is the estimate of x_o and

$$\hat{A}(t) = \left[-\hat{a}_p(t) \mid \frac{I_{n-1}}{0} \right], \quad K(t) = a^* - \hat{a}_p(t)$$

$\hat{a}_p(t)$ and $\hat{b}_p(t)$ are the estimates of the vectors a_p and b_p, respectively. $a^* \in \mathcal{R}^n$ is chosen so that

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Theorem: The adaptive Luenberger observer with gradient-based algorithm guarantees the following properties:

(i) If choose $u \in \mathcal{L}_\infty$ and A is a stable matrix, all signals are bounded.

(ii) Furthermore, if choose u is sufficiently rich of order $2n$, then the state observation error $|\hat{x} - x_o|$ and the parameter estimation error $\tilde{\theta}$ converge to zero exponentially fast.

Brief Proof. (i) The observer equation may be written as

$$\dot{\hat{x}} = A^* \hat{x} + \hat{b}_p(t) u + \left(\hat{A}(t) - A^* \right) x_o$$

(ii) The state observation error $\tilde{x} = \hat{x} - x_o$ satisfies

$$\dot{\tilde{x}} = A^* \tilde{x} + \tilde{b}_p u - \tilde{a}_p y$$

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