Linear Time-Invariant Control Systems

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Linear Time-Invariant Control Systems

Let $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and $b \in \mathbb{R}^{n_x}$ be given. The differential equation

$$\dot{x}(t) \;=\; Ax(t) + Bu(t) + b \quad \text{with} \quad x(0) = x_0$$

is called a linear time-invariant (LTI) control system in standard form.

- $x : \mathbb{R} \to \mathbb{R}^{n_x}$ is the state trajectory
- $u : \mathbb{R} \to \mathbb{R}^{n_u}$ is the control input
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Linear Input-Output Systems

Assumption: we can observe a linear combination of the states,

$$y(t) = Cx(t) + d \; .$$

The matrix $C \in \mathbb{R}^{n_y \times n_x}$ and $d \in \mathbb{R}^{n_y}$ are given.

- $y : \mathbb{R} \to \mathbb{R}^{n_y}$ is the output function
- The corresponding system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + b \quad \text{with} \quad x(0) = x_0 \\ y(t) &= Cx(t) + d \end{aligned}$$

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Steady states

For a constant control input, $u(t) = u_{\text{ref}}$, the system

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Set-points

If the goal is to bring the system to the set-point y_{ref}, we would like to adjust u_{ref} such that

$$y_{\mathsf{ref}} = Cx_{\mathsf{ref}} + d = -CA^{-1}\left(Bu_{\mathsf{ref}} + b\right) + d \; .$$

If $n_y = n_u$ and $CA^{-1}B$ invertible, we find

$$u_{\mathsf{ref}} \quad = \quad \left[CA^{-1}B\right]^{-1}\left(d - CA^{-1}b - y_{\mathsf{ref}}\right)$$

$$\text{and} \quad x_{\mathsf{ref}} \quad = \quad -A^{-1}\left(Bu_{\mathsf{ref}} + b\right) \; .$$

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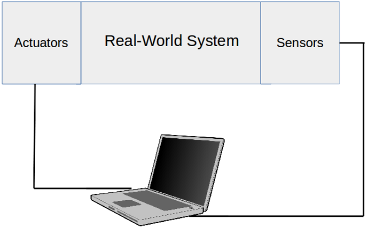
Proportional control



Recall: P-control law given by $(K \in \mathbb{R}^{n_u \times n_y})$

$$u(t) = \; u_{\mathsf{ref}} + K(y(t) - y_{\mathsf{ref}})$$

Proportional control of linear input output systems

Overview:

$$\begin{aligned} y(t) &= Cx(t) + d && \text{output function} \\ u(t) &= u_{\mathsf{ref}} + K(y(t) - y_{\mathsf{ref}}) && \text{proportional control law} \\ \dot{x}(t) &= Ax(t) + Bu(t) + b && \text{linear control system (model)} \end{aligned}$$

Closed-loop dynamics:

$$\begin{aligned} \dot{x}(t) &= \underbrace{(A + BKC)}_{=A_{\mathsf{cl}}} x(t) + \underbrace{(b + B(u_{\mathsf{ref}} + K(d - y_{\mathsf{ref}})))}_{=b_{\mathsf{cl}}} \\ x(0) &= x_0 \, . \end{aligned}$$

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Closed-Loop Trajectories

Closed-loop dynamics:

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- x_{ref} is a steady-state (by construction)
- If A_{cl} is invertible, explicit solution given by

$$x(t) = e^{A_{\mathsf{cl}} t}(x_0 - x_{\mathsf{ref}}) + x_{\mathsf{ref}}$$

Limit behavior

- If the eigenvalues of the matrix $A_{\mathsf{cl}} = A + BKC$ have negative real parts,

$$\lim_{t \to \infty} e^{A_{\mathsf{cl}} t} \to 0 \ ,$$

  the system is called *asymptotically stable*.

- The closed loop trajectory of asymptotically stable system satisfies

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{A_{\mathsf{cl}} t}(x_0 - x_{\mathsf{ref}}) + x_{\mathsf{ref}} = x_{\mathsf{ref}} \ .$$

Tuning the proportional gain

- We would like to choose the proportional gain K such that all eigenvalues of

$$A_{\mathsf{cl}} = A + BKC$$

have negative real-parts.

- Idea: if we have single-input-single-output system, we can scatter-plot

$$\mathsf{Re}\left(\mathsf{eig}(A + BKC)\right)$$

as a function of K.

- We will later about more systematic methods for choosing K...

Example

- Consider the case

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad C = (1\ 0)$$

- The eigenvalues of $A_{\text{cl}} = A + BKC$ are

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 3 + K}$$

- Thus, for $K < -3$ the closed-loop system is asymptotically stable.
- For $K < -3.25$ the closed-loop response "oscillates" ($\text{Im}(\lambda) \neq 0$)

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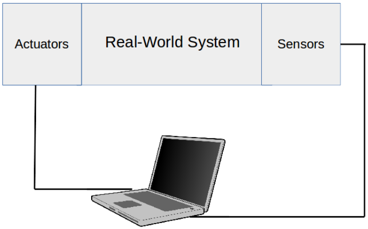
PD control



Recall: PD-control law given by $(K, K_\mathrm{D} \in \mathbb{R}^{n_u \times n_y})$

$$u(t) = u_{\text{ref}} + K(y(t) - y_{\text{ref}}) + K_\mathrm{D}\dot{y}(t)$$

Reformulation

Overview:

$$\begin{aligned} y(t) &= Cx(t) + d && \text{output function} \\ u(t) &= u_{\mathsf{ref}} + K(y(t) - y_{\mathsf{ref}}) + K_{\mathrm{D}}\dot{y}(t) && \text{PD control law} \\ \dot{x}(t) &= Ax(t) + Bu(t) + b && \text{linear control system (model)} \end{aligned}$$

Substitution:

$$\begin{aligned} u(t) - u_{\mathsf{ref}} &= KC(x(t) - x_{\mathsf{ref}}) + K_{\mathrm{D}}C\left(Ax(t) + Bu(t) + b\right) \\ &= \left[I - K_{\mathrm{D}}CB\right]^{-1}\left(KC + K_{\mathrm{D}}CA\right)\left(x(t) - x_{\mathsf{ref}}\right) \end{aligned}$$

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PD Closed-Loop Response

Explicit solution for the PD closed-loop response trajectory is given by

$$x(t) = e^{A_{\mathsf{cl}} t}(x_0 - x_{\mathsf{ref}}) + x_{\mathsf{ref}}$$

with

$$A_{\mathsf{cl}} = A + B \left[I - K_{\mathrm{D}} CB \right]^{-1} \left(KC + K_{\mathrm{D}} CA \right)$$

Remark: In practice, we often have $CB = 0$. In this case the closed-loop response matrix is given by

$$A_{\mathsf{cl}} = A + BKC + BK_{\mathrm{D}}CA \,.$$

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- If $K < -1$ and $K_{\mathrm{D}} < 0$, we have asymptotic stability
- If we set $K = -1 - \frac{K_{\mathrm{D}}^2}{4}$, $\lambda_{1,2} = \frac{K_{\mathrm{D}}}{2}$ can be tuned by choosing K_{D}

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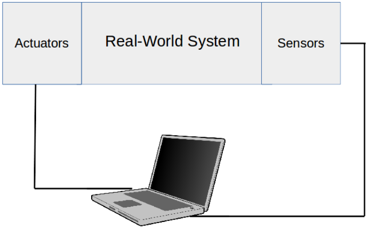

Recall: PI-control law given by $(K, K_\mathrm{I} \in \mathbb{R}^{n_u \times n_y})$

$$u(t) = \; u_{\mathsf{ref}} + K(y(t) - y_{\mathsf{ref}}) + K_\mathrm{I} \int_0^t (y(\tau) - y_{\mathsf{ref}})\,\mathrm{d}\tau$$

Reformulation

Main idea: Introduce the auxiliary state

$$z(t) = \begin{pmatrix} x(t) - x_{\mathsf{ref}} \\ \int_0^t \left[x(\tau) - x_{\mathsf{ref}}\right] \, \mathrm{d}\tau \end{pmatrix}$$

Closed-loop differential equation:

$$\dot{z}(t) = \underbrace{\begin{pmatrix} A + BKC & BK_{\mathsf{I}}C \\ I & 0 \end{pmatrix}}_{=A_{\mathsf{cl}}} z(t) \quad \text{with} \quad z(0) = \begin{pmatrix} x_0 - x_{\mathsf{ref}} \\ 0 \end{pmatrix}$$

Explicit solution: $x(t) = x_{\mathsf{ref}} + (1\,0)\, e^{A_{\mathsf{cl}} t} z(0)$.

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