Model Reference Adaptive Control

Contents

General Structure

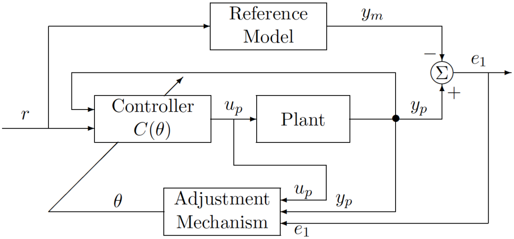

Figure: General Structure of MRAC

Categories:

i) Direct and Indirect MRAC

ii) Adaptive law with and without normalization

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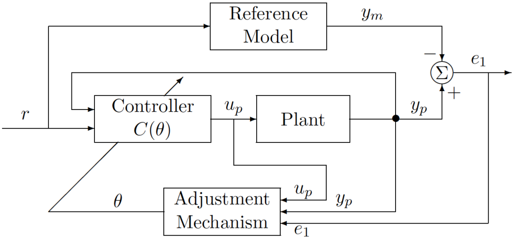

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Adaptive regulation

Example: one parameter case adaptive regulation

Consider

$$\dot{x} = ax + u, \quad x(0) = x_0$$

where a is a constant but unknown. The control objective is find proper input signal u such that $x \to 0$ as $t \to \infty$.

Reference model

$$\dot{x} = -a_m x, \quad a_m > 0$$

Control law

$$u = -k(t)x$$

with $k^* = a_m + a$.

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Adaptive Regulation

Indirect approach:

$$k(t) = a_m + \theta, \quad \dot{\theta} = \Phi(x, u)$$

Direct approach: rewrite the plant as

$$\dot{x} = -a_m x + k^* x + u = -a_m x - \tilde{k} x$$
$$\dot{\tilde{k}} = \dot{k} := \varphi(x)$$

with $\tilde{k} = k - k^*$. Consider a Lyapunov candidate function

$$V(x, \tilde{k}) = \frac{x^2}{2} + \frac{\tilde{k}^2}{2\gamma}$$

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Adaptive Regulation

To generate the adaptive law for k. Taking the time derivative of V as

$$\dot{V} = -a_m x^2 - \tilde{k} x^2 + \frac{\tilde{k}\varphi}{\gamma}$$

Take $\varphi(x) = \gamma x^2$, we have

$$\dot{V} = -a_m x^2$$

indicates

- $x, \tilde{k}, k \in \mathcal{L}_\infty$
- $x \in \mathcal{L}_2, \dot{x} \in \mathcal{L}_\infty$, therefore $x \to 0$ as $t \to \infty$.

Note, we cannot establish that $k \to k^*$.

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Adaptive Tracking

Example 2: Two parameter case adaptive tracking. Consider

$$\dot{x} = -ax + bu$$

a, b unknown, but $sgn(b)$ is assume to be known. Reference model

$$x_m = \frac{b_m}{s + a_m} r$$

It is assumed that a_m, b_m, and r are chosen so that x_m represents the desired state response of the plant. The control goal is

$$x(t) \to x_m(t)$$

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The optimal control law

$$u = -k^* x + l^* r$$

where k^* and l^* verify

$$\frac{x(s)}{r(s)} = \frac{bl^*}{s + a + bk^*} = \frac{b_m}{s + a_m} = \frac{x_m(s)}{r(s)}$$

results in

$$l^* = \frac{b_m}{b}, \quad k^* = \frac{a_m - a}{b}$$

Since a, b unknown, we propose the control law

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Adaptive Tracking

Express the plant equation in terms of k^* and l^*

$$\dot{x} = -a_m x + b_m r + b(k^* x - l^* r + u)$$

Define the error $e := x - x_m$, we have a B-SPM

$$e = \frac{b}{s + a_m} (k^* x - l^* r + u)$$

Alternatively, we can express the error equation as

$$\begin{aligned} \dot{e} &= -a_m e + b(k^* x - l^* r + u) \\ &= -a_m e + b(-\tilde{k} x + \tilde{l} r) \end{aligned}$$

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This motivates the following Lyapunov candidate function for designing the adaptive laws

$$V(e, \tilde{k}, \tilde{l}) = \frac{e^2}{2} + \frac{\tilde{k}^2}{2\gamma_1}|b| + \frac{\tilde{l}^2}{2\gamma_2}|b|$$

where $\gamma_1 > 0, \gamma_2 > 0$. The time derivative is given by

$$\dot{V} = -a_m e^2 - b\tilde{k}ex + bler + \frac{|b|\tilde{k}}{\gamma_1}\dot{\tilde{k}} + \frac{|b|\tilde{l}}{\gamma_2}\dot{\tilde{l}} = -a_m e^2$$

Because $|b| = b\,\mathrm{sgn}(b)$, the indefinite terms disappear if we choose

$$\dot{k} = \gamma_1 ex\,\mathrm{sgn}(b), \quad \dot{l} = -\gamma_2 er\,\mathrm{sgn}(b)$$

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Adaptive Tracking

Properties:

- $e, \tilde{k}, \tilde{l} \in \mathcal{L}_\infty$, $x, u \in \mathcal{L}_\infty$
- $e \in \mathcal{L}_2$ and $\dot{e}, \dot{k}, \dot{l} \in \mathcal{L}_\infty$ if $r \in \mathcal{L}_\infty$
- if r is s.r. of order 2, $e \to 0$, $k \to k^*$, $l \to l^*$ exponentially fast.

Alternatively, we can generate $\hat{a}, \hat{b}$ first, then combined with

$u = kx + lr$ where

$$k(t) = \frac{a_m + \hat{a}(t)}{\hat{b}(t)}, \quad l(t) = \frac{b_m}{\hat{b}(t)}$$

This is a typical indirect MRAC method.

Note, for now, methods presented do not have normalization.

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Contents

Problem Statement

Consider the SISO LTI plant

$$
\begin{aligned}
\dot{x}_p &= A_p x_p + B_p u_p, \quad x_p(0) = x_0 \in \mathbb{R}^{n_p} \\
y_p &= C_p^\top x_p
\end{aligned}
$$

where $y_p, u_p \in \mathbb{R}$ and A_p, B_p, C_p have the appropriate dimensions.

The transfer function of the plant :

$$
y_p = G_p(s) u_p = k_p \frac{Z_p(s)}{R_p(s)} u_p
$$

where $Z_p(s), R_p(s)$ are monic known polynomials, constant k_p is a.k.a. high-frequency gain.

Problem Statement

Reference model

$$\dot{x}_m = A_m x_m + B_m r, \quad x_m(0) = x_{m0} \in \mathbb{R}^{n_m}$$

$$y_m = C_m^\top x_m$$

where $y_m \in \mathbb{R}$ and $r \in \mathbb{R}$ is the reference input.

The transfer function

$$y_m = W_m(s) r$$

is expressed as

$$W_m(s) = k_m \frac{Z_m(s)}{R_m(s)}$$

where $Z_m(s), R_m(s)$ are monic polynomials and k_m is a constant.

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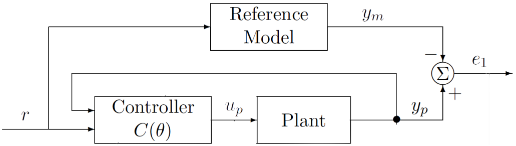
Problem Statement

MRC problem: The MRC objective is to determine the plant input u_p so that all signals(x_p, y_p, u_p) are bounded and the plant output y_p tracks the reference model output y_m as close as possible for any given reference input $r(t)$ which is a uniformly bounded piecewise continuous function of time.



Problem assumptions

Plant :

P1. $Z_p(s)$ is a monic Hurwitz polynomial with degree of m_p.

P2. An upper bound n of the degree n_p of $R_p(s)$ is known.

P3. The relative degree $n^* = n_p - m_p$ of $G_p(s)$ is known.

P4. The sign of the high-frequency gain k_p is known.

Reference model:

M1. $Z_m(s), R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, where $p_m \leq n$

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MRC scheme

In addition, for MRC problem, we assume $G_p(s)$ is known exactly.

A trivial choice for u_p is

$$u_p = \frac{k_m}{k_p} \frac{Z_m(s)}{R_m(s)} \frac{R_p(s)}{Z_p(s)} r$$

which leads to the closed-loop transfer function

$$\frac{y_p}{r} = \frac{k_m}{k_p} \frac{Z_m}{R_m} \frac{R_p}{Z_p} \frac{k_p Z_p}{R_p} = W_m(s)$$

Drawbacks: This control law may involve zero-pole cancellations outside $\mathcal{C}^-$ when $R_p(s)$ is not Hurwitz.

Core design logic : transfer function matching

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Let us consider the feedback control law

$$u_p = \theta_1^{*\top} \frac{\alpha(s)}{\Lambda(s)} u_p + \theta_2^{*\top} \frac{\alpha(s)}{\Lambda(s)} y_p + \theta_3^* y_p + c_0^* r$$

where $\Lambda(s) = \Lambda_0(s) Z_m(s)$ and $\Lambda_0(s)$ is monic, Hurwitz, and of degree $n_0 = n - 1 - q_m$.

$$\begin{aligned} \alpha(s) &\triangleq \alpha_{n-2}(s) = \left[s^{n-2}, s^{n-3}, \ldots, s, 1\right]^\top && \text{for } n \geq 2 \\ \alpha(s) &\triangleq 0 && \text{for } n = 1 \end{aligned}$$

$c_0^*, \theta_3^* \in \mathbb{R}; \theta_1^*, \theta_2^* \in \mathbb{R}^{n-1}$ are constant parameters.

Hence, the controller parameter vector to be designed is

$$\theta^* = \left[\theta_1^{*\top}, \theta_2^{*\top}, \theta_3^*, c_0^*\right]^\top \in \mathbb{R}^{2n}$$

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MRC scheme

We can now meet the control objective if we select θ^* so that the closed-loop poles are stable and the closed-loop transfer function $\frac{y(s)}{r(s)} = G_c(s) = W_m(s)$, i.e., the matching equation

$$\frac{c_0^* k_p Z_p \Lambda}{\left(\Lambda - \theta_1^{*\top} \alpha\right) R_p - k_p Z_p \left(\theta_2^{*\top} \alpha + \theta_3^* \Lambda\right)} = k_m \frac{Z_m}{R_m}$$

is satisfied for all $s \in \mathcal{C}$. Choosing

$$c_0^* = \frac{k_m}{k_p}$$

and using $\Lambda(s) = \Lambda_0(s) Z_m(s)$, the matching equation becomes

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$$\left(\Lambda - \theta_1^{*\top}\alpha\right) R_p - k_p Z_p \left(\theta_2^{*\top}\alpha + \theta_3^* \Lambda\right) = Z_p \Lambda_0 R_m$$

MRC scheme

Equating the coefficients of the powers of s on both sides, we can express the matching equation in terms of the algebraic equation

$$S\bar{\theta}^* = p$$

where $\bar{\theta}^* = \left[\theta_1^{*\top}, \theta_2^{*\top}, \theta_3^*\right]^\top$; S is an $(n + n_p - 1) \times (2n - 1)$ matrix that depends on the coefficients of R_p, k_p, Z_p, and Λ; and p is an $n + n_p - 1$ vector with the coefficients of $\Lambda R_p - Z_p \Lambda_0 R_m$.

Lemma Let the degrees of $R_p, Z_p, \Lambda, \Lambda_0$, and R_m be as specified in assumptions. Then the solution $\bar{\theta}^*$ of $S\bar{\theta}^* = p$ always exists.In addition, if R_p, Z_p are coprime and $n = n_p$, then the solution $\bar{\theta}^*$ is unique.

MRC scheme

Equating the coefficients of the powers of s on both sides, we can express the matching equation in terms of the algebraic equation

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A state-space realization of the control law

$$\dot{\omega}_1 = F\omega_1 + gu_p, \quad \omega_1(0) = 0 \in \mathbb{R}^{n-1}$$

$$\dot{\omega}_2 = F\omega_2 + gy_p, \quad \omega_2(0) = 0 \in \mathbb{R}^{n-1}$$

$$u_p = \theta^{*\top}\omega$$

where $\theta^* = \left[\theta_1^{*\top}, \theta_2^{*\top}, \theta_3^*, c_0^*\right]^T$, $\omega = [\omega_1^\top, \omega_2^\top, y_p, r]^\top$ and

$$F = \begin{bmatrix} -\lambda_{n-2} & -\lambda_{n-3} & -\lambda_{n-4} & \cdots & -\lambda_0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

λ_i are the coefficients of $\Lambda(s)$

$$\Lambda(s) = s^{n-1} + \lambda_{n-2}s^{n-2} + \cdots + \lambda_1 s + \lambda_0 = \det(sI - F)$$

MRC scheme

Example: Let us consider the second-order plant

$$y_p = \frac{-3(s+4)}{s^2 - 3s + 2} u_p$$

and the reference model

$$y_m = \frac{1}{s+1} r$$

1) check assumptions of plant and reference: $n_p = 2.$, $n^* = 1$ is equal to that of the reference model.

2) choose the polynomial $\Lambda(s) = s + 2 = \Lambda_0(s)$ and the control input

$$u_p = \theta_1^* \frac{1}{s+2} u_p + \theta_2^* \frac{1}{s+2} y_p + \theta_3^* y_p + c_0^* r$$

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3) solve θ^* by matching equation

$$\frac{y_p(s)}{r(s)} = \frac{-3c_0^*(s+4)(s+2)}{(s+2-\theta_1^*)(s-1)(s-2)+3(s+4)(\theta_2^*+\theta_3^*(s+2))} = G_c(s)$$

Forcing $G_c(s) = \frac{1}{s+1}$, we have $c_0^* = -\frac{1}{3}$, and the matching equation becomes

$$(\theta_1^* - 3\theta_3^*)s^2 + (-3\theta_1^* - 3\theta_2^* - 18\theta_3^*)s + 2\theta_1^* - 12\theta_2^* - 24\theta_3^* = -8s^2 - 18s - 4$$

4) Implement control law as

$$\dot{\omega}_1 = -2\omega_1 + u_p$$

$$\dot{\omega}_2 = -2\omega_2 + y_p$$

$$u_p = -2\omega_1 - 4\omega_2 + 2y_p - \left(\frac{1}{3}\right)r.$$

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