Linear Time-Invariant Systems

- Introduction
- Matrix Exponentials
- Construction of Solutions

Contents

Linear Time-Invariant Differential Equations

Let $A \in \mathbb{R}^{n_x \times n_x}$ and $b \in \mathbb{R}^{n_x}$ be given. The differential equation

$$\dot{x}(t) \;=\; Ax(t) + b \quad \text{with} \quad x(0) = x_0$$

is called a linear time-invariant (LTI) system in standard form.

- $x : \mathbb{R} \to \mathbb{R}^{n_x}$ is called state trajectory,
- $x_0 \in \mathbb{R}^{n_x}$ is called the initial value,
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Example: harmonic oscillator

A linear differential equation with $b = 0$,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is called a harmonic oscillator. Componentwise notation:

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & -x_1(t) \end{array} \right\} \quad \text{with} \quad \left\{ \begin{array}{l} x_1(0) = 0 \\ x_2(0) = 1 \end{array} \right\}$$

A solution trajectory is given by

$$x(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

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Eliminating Higher Order Derivatives

Higher order linear differential equation (LDE) have the form

$$\frac{\partial^m}{\partial t^m} y(t) \; = \; \sum_{i=0}^{m-1} D_i \frac{\partial^i}{\partial t^i} y(t) + d \quad \text{with} \quad \frac{\partial^i}{\partial t^i} y(0) = y_i$$

for $i \in \{0, \ldots, m-1\}$.

- The matrices $D_0, D_1, \ldots, D_{m-1} \in \mathbb{R}^{n_y \times n_y}$ are given,
- the vector $d \in \mathbb{R}^{n_y}$ is called the offset,
- and $y_0, y_1, \ldots, y_{m-1} \in \mathbb{R}^{n_y}$ denote initial values of the derivatives.

Question: Can we write this LDE in standard form?

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Eliminating Higher Order Derivatives

Solution strategy: introduce a new state

$$x(t) = \left[y(t)^\intercal, \dot{y}(t)^\intercal, \ddot{y}(t)^\intercal, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} y(t)^\intercal \right]^\intercal$$

and define

$$A = \begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ D_0 & D_1 & \ldots & D_{m-2} & D_{m-1} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ d \end{pmatrix} .$$

Don't forget the initial value: $x_0 = \left[y_0^\intercal, y_1^\intercal, y_2^\intercal, \ldots, y_{m-1}^\intercal \right]^\intercal$.

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Example: recursive-integrator

A "recursive intergrator" is obtained for the choice

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \quad \text{with} \quad b = 0 \quad \text{and} \quad x_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

- The last component satisfies $\dot{x}_{n_x}(t) = 0$, i.e., we have $x_{n_x}(t) = 1$.
- Next, $\dot{x}_{i-1}(t) = x_i(t)$ for all $i \in \{2, \ldots, n_x\}$ (= recursive integration)
- Solution can be written explicitly as $x_{n_x - i}(t) = \frac{t^i}{i!}$.

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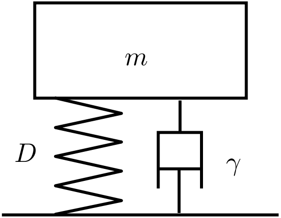
Example: passive spring-damper systems



- spring force is $F_{\mathsf{s}}(t) = -Ds(t)$,
- drag force $F_{\mathrm{d}}(t) = -\gamma v(t) = -\gamma \dot{s}(t)$
- Newton's laws of motion

  $m\ddot{s}(t) = -Ds(t) - \gamma \dot{s}(t)$

In order to write this system in standard form, we regard the velocity $v(t) = \dot{s}(t)$ as a differential state:

$$\begin{aligned} \dot{s}(t) &= v(t) \\ \dot{v}(t) &= -\frac{D}{m}s(t) - \frac{\gamma}{m}v(t) \end{aligned}$$

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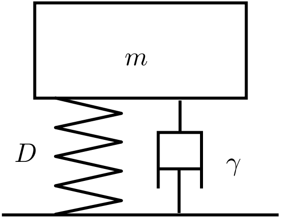

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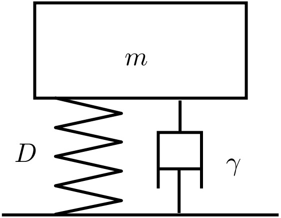

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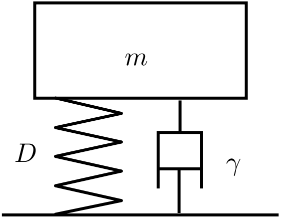

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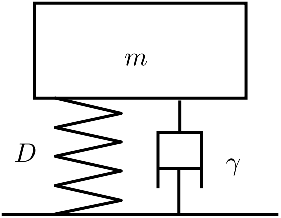

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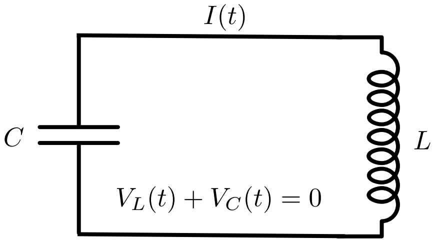
can be written in the standard form $\dot{x}(t) = Ax(t) + b$ with

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{D}{m} & -\frac{\gamma}{m} \end{pmatrix} \quad \text{and} \quad b = 0 \; ,$$

where $x = [s, v]^{\mathsf{T}}$.

Example: electrical resonant circuits



- denote with $I(t)$ the current in the circuit,
- voltage at the inductor: $V_L(t) = L\dot{I}(t)$,
- voltage $V_C(t)$ at the capacitor satisfies $I(t) = C\dot{V}_C(t)$,
- Kirchhoff's voltage law: $V_C(t) + V_L(t) = 0$

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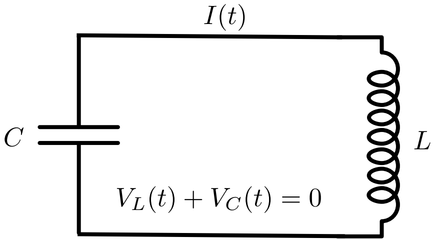

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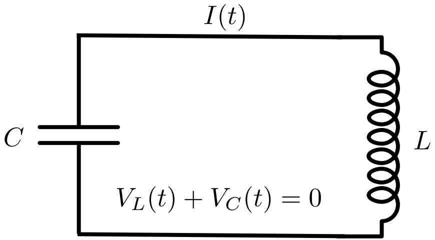

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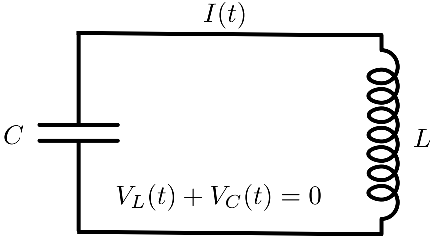

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Substituting these physical relations gives

$$I(t) = C\dot{V}_C(t) = -C\dot{V}_L(t) = -LC\ddot{I}(t) \qquad \Leftrightarrow \qquad \ddot{I}(t) = -\frac{1}{LC}I(t)\ .$$

This can be written as a differential equation in standard form:

$$\begin{array}{rcl} \dot{I}(t) & = & \frac{1}{L}V_L(t) \\ \dot{V}_L(t) & = & -\frac{1}{C}I(t) \end{array} , \qquad \text{i.e.,} \qquad A = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{pmatrix}$$

and $b = 0$. The states $x(t) = (I(t), V_L(t))^\mathsf{T}$ are the current and the voltage at the inductor.

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For a matrix $A \in \mathbb{R}^{n \times n}$ we define the exponential via its Taylor series,

$$X(t) = e^{tA} = \sum_{i=0}^{\infty} \frac{1}{i!} [tA]^i \; .$$

- The sum on the right-hand side converges uniformly.
- Syntax in JULIA / MATLAB:

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Properties of matrix exponentials

Define $X(t) = e^{tA}$.

- We have $X(0) = I$.
- The time derivative of X satisfies $\dot{X}(t) = A \cdot X(t)$.
- $X(t)$ commutes with A, i.e., $AX(t) = X(t)A$.
- If $A \cdot B = B \cdot A$, then $e^{A+B} = e^A \cdot e^B$.
- **But in general** $e^{A+B} \neq e^A \cdot e^B$ **!!!**
- $X(t_1) \cdot X(t_2) = X(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}$.
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Matrix exponentials for diagonalizable matrices

Let A be diagonalizable, $A = TDT^{-1}$ with $D = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_{n_x}\right)$, and T invertible.

- The matrix exponential function $X(t) = e^{At}$ can be written as

$$\begin{aligned} X(t) &= \sum_{i=0}^{\infty} \frac{1}{i!} \left(TDT^{-1}\right)^i t^i \\ &= T \left(\sum_{i=0}^{\infty} \frac{1}{i!} D^i t^i \right) T^{-1} \\ &= T \mathrm{diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right) T^{-1} \ , \end{aligned}$$

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- The diagonal "modes" are easy to analyze:

$$e^{\lambda_i t} = e^{\sigma_i t}\left(\cos(\omega_i t) + \sin(\omega_i t)\sqrt{-1}\right) \ ,$$

  - $\sigma_i = \mathrm{Re}(\lambda_i)$ can be interpreted as a exponential growth/decay factor.
  - $\omega_i = \mathrm{Im}(\lambda_i)$ can be interpreted as oscillation frequency

Jordan normal form

What if A is not diagonalizable? Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

- In this case, we can only find a Jordan normal form

$$A = T\,(D + N)\,T^{-1} \ ,$$

D is diagonal, T invertible, N nil-potent, $N^m = 0$, $DN = ND$.

- The function $X(t) = e^{At}$ can then be written as

$$X(t) = Te^{t(D+N)}T^{-1} = Te^{tD}e^{tN}T^{-1} \ ,$$

since D and N commute and $e^{tN} = \sum_{i=0}^{m-1} \frac{1}{i!} N^i t^i$.

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Example: recursive integrator

The recursive integrator system is already in Jordan normal form,

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Thus, the function $X(t) = e^{tA}$ takes the form

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Uniqueness of Solutions

If we have two solutions $x_1, x_2 : \mathbb{R} \to \mathbb{R}^{n_x}$, then $y = x_1 - x_2$ satisfies

$$\dot{y}(t) = Ay(t) \quad \text{with} \quad y(0) = 0 \; .$$

The auxiliary function $v(t) = e^{-At}y(t)$ satisfies

$$\begin{aligned} \dot{v}(t) &= -Ae^{-At}y(t) + e^{-At}Ay(t) = -Ae^{-At}y(t) + Ae^{-At}y(t) = 0 \\ v(0) &= 0 \; , \end{aligned}$$

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A state $x_{\mathsf{s}} \in \mathbb{R}^{n_x}$ is called a steady-state if

$$\dot{x}(t) = Ax(t) + b \quad \text{with} \quad x(0) = x_{\mathsf{s}}$$

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Next, we show that the function

$$x(t) = X(t)x_0 + Y(t)b = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\, b \,\mathrm{d}\tau$$

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