Parameter Estimation

Contents

Parameter Estimation

$$\text{Input} \overset{\theta^*}{\Longleftrightarrow} \text{Output}$$

θ^*: model parameters that determine the structure of plant, and hence, determine the relation between inputs and outputs. In turn, model parameters can be deduced by observing the system's response to certain inputs. These techniques are often referred to as parameter estimation techniques,contain three main steps:

1. Select an appropriate parameterization of the plant model
2. Construct an adaptive law updating the parameter estimates
3. Establish conditions that guarantee the convergence of estimation error

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One Parameter Case

Consider the first-order plant model

$$y = \frac{a}{s+2}u$$

where a is the only unknown parameter. y and u available for measurement.

1) Parametric Model: SPM

$$z = \theta^* \phi$$

with

$$z = y, \quad \theta^* = a, \quad \phi = \frac{1}{s+2}u$$

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One Parameter Case

2) Adaptive Law for $\theta(t)$:

Estimation model

$$\hat{z} = \theta(t)\phi$$

Error signal

$$\varepsilon = \frac{z - \hat{z}}{m_s^2} = \frac{(\theta^* - \theta)\phi}{m_s^2}$$

m_s^2 is a normalizing signal

$$m_s^2 = 1 + \alpha\phi^2$$

with $\alpha \geq 0$. This property of m_s is used to establish the boundedness of the estimated parameters even when ϕ is not guaranteed to be bounded. Note, ε is an available signal.

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2) Adaptive Law for $\theta(t)$:

Intuitively, we can generate $\theta(t)$ by:

$$\theta(t) = \frac{z(t)}{\phi(t)}$$

However, when $\phi(t)$ is close to zero, it may lead to erroneous parameter estimates.

Consider a quadratic cost criterion

$$J(\theta) = \frac{\varepsilon^2 m_s^2}{2} = \frac{(z - \theta\phi)^2}{2m_s^2}$$

minimize J with respect to θ using the gradient method to obtain

$$\dot{\theta} = -\gamma \bigtriangledown J = \gamma\varepsilon\phi, \quad \theta(0) = \theta_0$$

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3) Convergence analysis

Define $\tilde{\theta}(t) := \theta(t) - \theta^*$, yields

$$\dot{\tilde{\theta}} = -\gamma \frac{\phi^2}{m_s^2}\tilde{\theta}, \quad \tilde{\theta}(0) = \tilde{\theta}_0$$

Consider a Lyapunov candidate function

$$V(\tilde{\theta}) = \frac{\tilde{\theta}^2}{2\gamma}$$

that is p.d., radially unbounded (and decrescent). The time derivative is

$$\dot{V} = -\frac{\phi^2}{m_s^2}\tilde{\theta}^2 = -\varepsilon^2 m_s^2 \leq 0$$

$\dot{V}$ is negative semidefinite, only guarantees u.s., i.e. $\tilde{\theta}(t), \theta(t) \in \mathcal{L}_\infty$, What else?

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GUAS

Definition: A continuous function $\varphi : [0, r] \mapsto \mathcal{R}^+$(or a continuous function $\varphi : [0, \infty) \mapsto \mathcal{R}^+$)is said to belong to class $\mathcal{K}$, i.e., $\varphi \in \mathcal{K}$ if (i) $\varphi(0) = 0$ (ii) φ is strictly increasing on $[0, r]$(or on $[0, \infty)$).

eg: The function $\varphi(|x|) = \frac{x^2}{1+x^2}$.

Definition: A function $V(t, x) : \mathcal{R}^+ \times \mathcal{B}(r) \mapsto \mathcal{R}$ with $V(t, 0) = 0 \forall t \in \mathcal{R}^+$is said to be decrescent if there exists $\varphi \in \mathcal{K}$ such that $|V(t, x)| \leq \varphi(|x|) \quad \forall t \geq 0$ and $\forall x \in \mathcal{B}(r)$ for some $r > 0$.

eg : $V(t, x) = \frac{1}{1+t}x^2$ is decrescent because $V(t, x) = \frac{1}{1+t}x^2 \leq x^2 \forall t \in \mathcal{R}^+$but $V(t, x) = tx^2$ is not.

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GUAS

$x_e = 0$ is an equilibrium point and $\dot{V}$ is

$$\dot{V} = \frac{\partial V}{\partial t} + (\nabla V)^\top f(t, x)$$

Theorem : Suppose there exists a positive definite $\mathcal{C}^1$ function $V(t, x) : \mathcal{R}^+ \times B(r) \mapsto \mathcal{R}$ for some $r > 0$ and $V(t, 0) = 0, \forall t \in \mathcal{R}^+$. Then the following statements are true:

- If $\dot{V} \leq 0$, then $x_e = 0$ is stable.
- If V is decrescent and $\dot{V} \leq 0$, then $x_e = 0$ is u.s.
- If V is decrescent and $\dot{V} < 0$, then x_e is u.a.s.
- If V is decrescent, radially unbounded and $\dot{V} < 0$, then x_e is g.u.a.s.

3) Convergence analysis

Let us now integrate both sides of previous inequality, we have

$$V(t) - V(0) = -\int_0^t \varepsilon^2(\tau) m_s^2(\tau) d\tau$$

It indicates that

- $\varepsilon m_s, \varepsilon \in \mathcal{L}_2$(since $m_s^2 > 1$) , $\varepsilon m_s, \varepsilon \in \mathcal{L}_\infty$
- $\dot{\theta} = \gamma \varepsilon m_s \frac{\phi}{m_s} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$
- If $\dot{\phi}$ is bounded, we have $\ddot{\theta}(t) \in \mathcal{L}_\infty$, therefore $\dot{\theta}(t)$ converges to zero as time goes to infinity.

However, the above properties still do not guarantee that $\theta(t) \to \theta^*$ as $t \to \infty$.

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Recall

$$\dot{\tilde{\theta}} = -\gamma \frac{\phi^2}{m_s^2}\tilde{\theta}, \quad \tilde{\theta}(0) = \tilde{\theta}_0$$

with time solution admits the form

$$\tilde{\theta}(t) = e^{-\gamma \int_0^t \frac{\phi^2(\tau)}{m_s^2(\tau)} d\tau}\tilde{\theta}_0$$

Take $\alpha = 0$, i.e. $m_s^2 = 1$, consider e.g. i) $\phi(t) = 0$ ii) $\phi(t) = \mathrm{e}^{-\frac{t}{2}}$ iii) $\phi(t) = \frac{1}{\sqrt{1+t}}$, we can have s., but not g.u.a.s.

If there exist $\alpha_0 > 0$ such that

$$\int_0^t \frac{\phi^2(\tau)}{m_s^2(\tau)} d\tau \geq \alpha_0 t$$

then we can easily conclude that $\tilde{\theta}(t) \to 0$ exponentially fast. This is referred to as *Persistent Excitation (PE)* condition.

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Estimation Algorithm Summary

The parameter estimation algorithm for estimating the constant a of the scalar LTI system

$$y = \frac{a}{s+2}u$$

now can be summarized as

$$\begin{aligned}
&\hat{a} = \theta(t) \\
&\dot{\theta} = \gamma\varepsilon\phi, \quad \theta(0) = \theta_0, \\
&\varepsilon = \tfrac{(z-\hat{z})}{m_s^2}, \quad \hat{z} = \theta\phi, \\
&z = y, \quad \phi = \tfrac{1}{s+2}u, \quad m_s^2 = 1 + \alpha\phi^2, \quad \alpha > 0
\end{aligned}$$

$\tilde{\theta}(t) \to 0$ exponentially fast if $\frac{\phi}{m_s}$ is a persistently exciting signal.

Two Parameters Case

Consider the plant model

$$y = \frac{b}{s+a} u$$

where a, b are unknown parameters, assume $\dot{y}, y$ and u are available for measuring.

1) Parametric Model : SPM

$$z = \theta^{*\top} \phi$$

with $z = \dot{y}$, $\theta^* = [a, b]^\top$ and $\phi = [-y, u]^\top$.

Two Parameters Case

2) Adaptive Law for $\theta(t)$:

Estimation model

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Error signal

$$\varepsilon = \frac{z - \hat{z}}{m_s^2} = \frac{z - \theta^\top \phi}{m_s^2}$$

$m_s^2 = 1 + \alpha \phi^\top \phi, \alpha \geq 0$ is a normalizing signal to ensure $\frac{\phi}{m_s} \in \mathcal{L}_\infty$.Similarly, using the gradient method to minimize the cost function

$$J(\theta) = \frac{\varepsilon^2 m_s^2}{2} = \frac{(z - \theta_1 \phi_1 - \theta_2 \phi_2)^2}{2 m_s^2}$$

yields,

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2) Adaptive Law for $\theta(t)$:

$$\dot{\theta} = -\Gamma \nabla J = \Gamma \varepsilon \phi, \quad \theta(0) = \theta_0 \in \mathbb{R}^2$$

with $\Gamma > 0$ is the adaptive gain. e.g. $\Gamma = \gamma I$ with $\gamma > 0$.

3) Convergence analysis:

$$\dot{\tilde{\theta}} = \Gamma \phi \varepsilon = -\Gamma \frac{\phi \phi^T}{m_s^2} \tilde{\theta}, \qquad \tilde{\theta}(0) = \theta_0 - \theta^* := \tilde{\theta}_0$$

Clearly, as a LTV system, the convergence property of $\tilde{\theta}$ (that is the stability property of the equilibrium at origin) will very much depend on the property of time-varying matrix $\frac{\phi \phi^T}{m_s^2}$.

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3)Convergence analysis:

Recall the time solution of the error system

$$\tilde{\theta}(t) = e^{-\Gamma \int_0^t \frac{\phi(\tau)\phi^\top(\tau)}{m_s^2(\tau)} d\tau} \tilde{\theta}_0$$

One can image, similarly, if there exists $\alpha_0 > 0$ such that the PE condition

$$\int_0^t \frac{\phi(\tau)\phi^\top(\tau)}{m_s^2(\tau)} \mathrm{d}\tau \geq \alpha_0 \mathrm{t}$$

is verified, then we can also have $\tilde{\theta}(t) \to 0$ exponentially fast.

In one parameter case, we have constant input guarantees the exponentially converge. How about in two parameters case?

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3)Convergence analysis: For simplicity, assume $a > 0$, i.e. plant is stable, and choose $m_s^2 = 1$, $u = c_0 > 0$, then at steady state

$$y(t) = \frac{bc_0}{a} := c_1$$

and

$$-\frac{\Gamma \phi \phi^T}{m_s^2} = -\gamma \begin{bmatrix} c_1^2 & -c_0 c_1 \\ -c_0 c_1 & c_0^2 \end{bmatrix} \triangleq A$$

where A is a constant matrix with eigenvalues $0, -\gamma(c_0^2 + c_1^2)$, which only implies *stable in the sense of Lyapunov.* Therefore, we cannot satisfies the PE condition.

What if we choose $u(t) = e^{-t}, u(t) = \frac{1}{1+t}, u(t) = \sin(t)$?

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The parameter estimation algorithm for estimating the constant a of the scalar LTI system $y = \frac{b}{s+a}u$ now can be summarized as

$$\hat{a} = \theta_1, \quad \hat{b} = \theta_2$$

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$\tilde{\theta}(t) \to 0$ exponentially fast if $\frac{\phi}{m_s}$ is a persistently exciting signal.

Given that $\phi(t) = [1, \frac{b}{s+a}]^\top u(t)$, the question that comes up next is : **how to choose u to guarantee that ϕ has the appropriate properties that imply exponential stability for $\tilde{\theta}_e = 0$.**

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Persistence Excitation

Theorem: The vector $\phi \in \mathbb{R}^n$ is **P.E.** with level α_0 if it satisfies

$$\int_t^{t+T_0} \phi(\tau)\phi^\top(\tau)d\tau \geq \alpha_0 T_0 I$$

for some $\alpha_0 > 0, T_0 > 0$ and for any $t \geq 0$.

Remark:

- $\phi(t)\phi(t)^\top$ is always positive semidefinite at any specific time instant, hence the PE condition requires that its integral over any interval of time of length T_0 is a positive definite matrix.
- The gradient-based adaptive algorithm for SPM guarantees that $\theta(t)$ converges exponentially to θ^*, if $\phi(t)$ is bounded and PE.

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$$\int_t^{t+T_0} \phi(\tau)\phi^\top(\tau)d\tau \geq \alpha_0 T_0 I$$

for some $\alpha_0 > 0, T_0 > 0$ and for any $t \geq 0$.

Remark:

- $\phi(t)\phi(t)^\top$ is always positive semidefinite at any specific time instant, hence the PE condition requires that its integral over any interval of time of length T_0 is a positive definite matrix.
- The gradient-based adaptive algorithm for SPM guarantees that $\theta(t)$ converges exponentially to θ^*, if $\phi(t)$ is bounded and PE.

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Example: Let us consider the following signal

$$y = A\sin(\omega t + \varphi)$$

with a known frequency ω but an unknown phase φ and unknown amplitude A. Our objective is to use the knowledge of ω and y to estimate A and φ.

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$$z = \theta^{*\top}\phi$$

where $\theta^* = [A\cos\varphi, A\sin\varphi]^\top$ and $\phi = [\sin\omega t, \cos\omega t]^\top$.

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2) Gradient-based Adaptive Law

$$\dot{\theta} = \Gamma \varepsilon \phi, \quad \theta(0) = \theta_0,$$
$$\varepsilon = \frac{(z-\hat{z})}{m_s^2}, \quad \hat{z} = \theta^\top \phi, \quad m_s^2 = 1 + \alpha \phi^\top \phi, \quad \alpha \geq 0.$$

Since $\phi \in \mathcal{L}_\infty$, the normalizing signal may be taken to be equal to zero, i.e., $\alpha = 0$. The adaptive gain may be chosen as $\Gamma = \gamma I$ for some $\gamma > 0$, leading to

$$\dot{\theta}_1 = \gamma \varepsilon \sin \omega t, \quad \dot{\theta}_2 = \gamma \varepsilon \cos \omega t$$

3) Convergence Analysis: Previously, we have shown that

i) $\tilde{\theta}, \theta \in \mathcal{L}_\infty$ and $\varepsilon, \dot{\theta} \in \mathcal{L}_\infty \cap \mathcal{L}_2$

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3) Convergence Analysis: Check the PE of ϕ by

$$\frac{1}{T_0}\int_t^{t+T_0} \phi(\tau)\phi^\top(\tau)d\tau = \frac{1}{T_0}S\left(t, T_0\right)$$

where

$$S\left(t, T_0\right) \triangleq \begin{bmatrix} \frac{T_0}{2} - \frac{\sin 2\omega(t+T_0) - \sin 2\omega t}{4\omega} & -\frac{\cos 2\omega(t+T_0) - \cos 2\omega t}{4\omega} \\ -\frac{\cos 2\omega(t+T_0) - \cos 2\omega t}{4\omega} & \frac{T_0}{2} + \frac{\sin 2\omega(t+T_0) - \sin 2\omega t}{4\omega} \end{bmatrix}$$

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Finally, the estimate of amplitude and phase is calculated by

$$\hat{A}(t) = \sqrt{\theta_1^2(t) + \theta_2^2(t)}, \quad \hat{\varphi} = \cos^{-1}\left(\frac{\theta_1(t)}{\hat{A}(t)}\right)$$

Clearly, the calculation of $\hat{\varphi}(t)$ at each time t is possible provided $\hat{A}(t) \neq 0$. This implies that $\theta(t)$ should not go through origin, which is something that cannot be guaranteed by the gradient-based adaptive law. Luckily, we know that θ converges to θ^* exponentially fast, and θ^* cannot be both equal to zero (otherwise $y \equiv 0$). Hence, we can calculate $\hat{\varphi}$ after some time T or we can eliminate the "bad" situation where $\|\hat{A}(t)\| = 0$.

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Sufficiently Rich

Recall the two parameter case:

$$\dot{y} = -ay + bu, \quad y(0) = y_0$$

where a, b are unknown and y, u are available for measuring at each time t. We would like to estimate a, b by properly processing y, u. This is possible if the input and output data contains sufficient information about unknown parameter, for instance

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$$y(t) = e^{-at}\left(y_0 - \frac{b}{a}c_0\right) + \frac{b}{a}c_0$$

carries information about the zero frequency gain $\frac{b}{a}$ of the plant only, which is not sufficient to determine a, b uniquely.

3. for $u = \sin\omega_0 t$, plant's steady-state response is given by

$$y_{ss}(t) = A\sin(\omega_0 t + \varphi)$$

where

$$A = \frac{|b|}{|j\omega_0 + a|} = \frac{|b|}{\sqrt{\omega_0^2 + a^2}}, \quad \varphi = (\operatorname{sgn}(b) - 1)90^\circ - \tan^{-1}\frac{\omega_0}{a}$$

It is clear that measurements of A and φ uniquely determine a, b.

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Sufficiently Rich

Consider the second order plant

$$y = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} u = G(s) u$$

where $a_1, a_0 > 0$. We can show that $u(t) = \sin \omega_0 t + \sin \omega_1 t$

where $\omega_0 \neq \omega_1$ leads to the steady-state response

$$y(t) = A_0 \sin(\omega_0 t + \varphi_0) + A_1 \sin(\omega_1 t + \varphi_1)$$

where

$A_0 = |G(j\omega_0)|, \varphi_0 = \angle G(j\omega_0), A_1 = |G(j\omega_1)|, \varphi_1 = \angle G(j\omega_1)$.

By measuring $A_0, A_1, \varphi_0, \varphi_1$ we can determine uniquely

a_1, a_0, b_1, b_0 by solving four algebraic equations.

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Definition: A signal $u : \mathbb{R}_+ \to \mathbb{R}$ is called *sufficiently rich* of order n if it consists of at least $\frac{n}{2}$ distinct nonzero frequencies.

Example: $u = \sum_{i=1}^{10} \sin \omega_i t$,with $\omega_i \neq \omega_j$ for all $i \neq j$, is sufficiently rich of order 20.

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From SR to PE

Consider the signal vector generated as

$$\phi = H(s)u \tag{1}$$

where $u \in \mathbb{R}$ and $H(s)$ is a vector whose elements are transfer functions that are proper with stable poles. The PE property of ϕ is related to the sufficient richness of u by the following theorem:

Theorem:[PE condition] Consider the signal vector ϕ in (1) and assume the complex vectors $H(j\omega_1), \ldots, H(j\omega_n)$ are linearly independent on the complex space $\mathcal{C}^n$ for all distinctive $\omega_i \in \mathcal{R}$. **Then, ϕ is P.E. if and only if u is sufficiently rich of order n.**

The proof of Theorem is given in Ioannou's book in Section 5.6.

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From SR to PE(Supplementary)

Lemma A: If the autocovariance of a function $\phi : \mathcal{R}^+ \mapsto \mathcal{R}^n$ defined as

$$R_\phi(t) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \phi(\tau)\phi^\top(t+\tau)d\tau \geq 0$$

exists and is uniform with respect to t_0 (that is *stationary*), then u is PE if and only if $R_\phi(0)$ is positive definite.

Remark: Theorem can be proved if we establish that $R_\phi(0)$ is positive definite if and only if u is sufficiently rich of order n.

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From SR to PE(Supplementary)

Definition: the Fourier transform of $R_u(t)$ is given by

$$S_u(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} R_u(\tau) d\tau$$

is referred to as the spectral measure of u.

Lemma B: Given $S_u(\omega), R_u(t)$ can be calculated using the inverse Fourier transform, i.e.,

$$R_u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} S_u(\omega) d\omega$$

Furthermore, we have

$$\int_{-\infty}^{\infty} S_u(\omega) d\omega = 2\pi R_u(0)$$

From SR to PE(Supplementary)

Lemma C Consider the system

$$y = H(s)u$$

where $H(s)$ is a strictly proper transfer function matrix of dimension $m \times n$ with stable poles and real impulse response $h(t)$. If u is stationary, with autocovariance $R_u(t)$, then y is stationary, with autocovariance

$$R_y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) R_u(t + \tau_1 - \tau_2) h^{\top}(\tau_2) \, d\tau_1 d\tau_2$$

and spectral distribution

$$S_y(\omega) = H(-j\omega) S_u(\omega) H^{\top}(j\omega)$$

From SR to PE(Supplementary)

Brief proof :

If: We will show the result by contradiction. Because u is stationary and $R_\phi(0)$ is uniform with respect to t, we take $t = 0$ and obtain

$$R_\phi(0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(\tau)\phi^\top(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_\phi(\omega)d\omega$$

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If u has a sinusoidal component at frequency ω_0 then $S_u(\omega)$ has a point mass (a delta function) at ω_0 and $-\omega_0$. Using the condition that u is sufficiently rich of order n, i.e., u has spectral lines at n points, we can express $S_u(\omega)$ as

$$S_u(\omega) = \sum_{i=1}^{n} f_u(\omega_i)\, \delta(\omega - \omega_i)$$

where $f_u(\omega_i) > 0$. Now we obtain

$$R_\phi(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\phi(\omega) d\omega = \frac{1}{2\pi} \sum_{i=1}^{n} f_u(\omega_i)\, H(-j\omega_i)\, H^\top(j\omega_i)$$

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Suppose that $R_\phi(0)$ is not positive definite, then there exists $x \in \mathcal{R}^n$ with $x \neq 0$ such that

$$x^\top R_\phi(0) x = \sum_{i=1}^{n} f_u\left(\omega_i\right) x^\top H\left(-j\omega_i\right) H^\top\left(j\omega_i\right) x = 0$$

Because $f_u\left(\omega_i\right) > 0$ and each term under the summation is nonnegative, above equation can be true only if:

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$$x^\top H\left(-j\omega_i\right) = 0, \quad i = 1, 2, \ldots, n$$

However, this contractions with the condition that $\left\{H\left(j\omega_1\right), H\left(j\omega_2\right), \ldots, H\left(j\omega_n\right)\right\}$ are linearly dependent. Hence, $R_\phi(0)$ is positive definite.

From SR to PE(Supplementary)

Brief proof :

Only if: We also prove this by contradiction. Assume that $R_\phi(0)$ is positive definite but u is sufficiently rich of order $r < n$, then we can express $R_\phi(0)$ as

$$R_\phi(0) = \frac{1}{2\pi} \sum_{i=1}^{r} f_u\left(\omega_i\right) H\left(-j\omega_i\right) H^\top\left(-j\omega_i\right)$$

where $f_u\left(\omega_i\right) > 0$. Note that the right hand side indicates the rank of $R_\phi(0)$ can be at most $r < n$, which contradicts with the assumption that $R_\phi(0)$ is positive definite.

From SR to PE

Example: Again, recall the regressor signal of our two parameter case where $\phi = [u, -y]^\top$ and it can be regarded as the output of

$$\phi = H(s)u, \quad H(s) = \begin{bmatrix} 1 \\ -\frac{b}{s+a} \end{bmatrix}$$

In this case $n = 2$, $H(s)$ is stable and proper and

$$\bar{H} := [H(j\omega_1), \quad H(j\omega_2)] = \begin{bmatrix} 1 & 1 \\ -\frac{b}{j\omega_1 + a} & -\frac{b}{j\omega_2 + a} \end{bmatrix}$$

is non-singular, which implies the independence of $H(j\omega_i)$ for all distinctive $\omega_i \in \mathcal{R}$.

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Example Continue: Next, choose

$$u = \sin \omega_0 t$$

for any $\omega_0 \neq 0$. According to the PE condition Theorem, we have $\phi(t)$ is PE.

Check the steady state:

$$\phi = \begin{bmatrix} \sin \omega_0 t \\ c_0 \sin(\omega_0 t + \varphi_0) \end{bmatrix}$$

where

$$c_0 = \frac{|b|}{\sqrt{\omega_0^2 + a^2}}, \quad \varphi_0 = \arg\left(\frac{-b}{j\omega_0 + a}\right)$$

From SR to PE

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From SR to PE

Example Continue: Now

$$\phi\phi^T = \begin{bmatrix} \sin^2 \omega_0 t & c_0 \sin \omega_0 t \sin(\omega_0 t + \varphi_0) \\ c_0 \sin \omega_0 t \sin(\omega_0 t + \varphi_0) & c_0^2 \sin^2(\omega_0 t + \varphi) \end{bmatrix}$$

Calculate the PE integral and choose $T_0 = \frac{\pi}{\omega_0}$, we have

$$\int_t^{t+T_0} \phi(\tau)\phi^T(\tau) d\tau = \frac{T_0}{2} \begin{bmatrix} 1 & c_0 \cos \varphi_0 \\ c_0 \cos \varphi_0 & c_0^2 \end{bmatrix}$$

which is a positive definite matrix. We can verify that for

$$\alpha_0 = \frac{1}{2} \frac{(1 - \cos^2 \varphi_0) c_0^2}{1 + c_0^2} > 0$$

From SR to PE

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From SR to PE

Exercise: Consider the plant model:

$$y = \frac{b\left(s^2+1\right)}{(s+5)^3}u$$

where b is the only unknown parameter and only y and u are available signals. Design a parameter estimation algorithm such that the estimates converges to the true value exponentially fast.