Scalar Linear Control Systems

- Introduction
- Explicit Solution
- Control Parameterizations

Contents

Scalar Linear Control System

We use the notation

$$\dot{x}(t) = ax(t) + bu(t)$$

to denote scalar linear control systems in standard form.

- Function $x : \mathbb{R} \to \mathbb{R}$ is called state trajecory;
- Function $u : \mathbb{R} \to \mathbb{R}$ is called control input;
- Constants a and b are given.

Elimination of offsets

If we have an affine control system of the form

$$\dot{y}(t) = ay(t) + bv(t) + c \tag{1}$$

we eliminate the offset. There are two ways to do this:

1. If $a \neq 0$, subtract $-\frac{c}{a}$, i.e., set $x(t) = y(t) + \frac{c}{a}$, such that

$$\dot{x}(t) = \dot{y}(t) = a\left(x(t) - \frac{c}{a}\right) + bv(t) + c = ax(t) + bv(t)\ .$$

2. If $b \neq 0$, we can, altenatively, shift the input, $u(t) = v(t) + \frac{c}{b}$, then

$$\dot{y}(t) = ay(t) + b\left(u(t) - \frac{c}{b}\right) + c = ay(t) + bu(t)\ .$$

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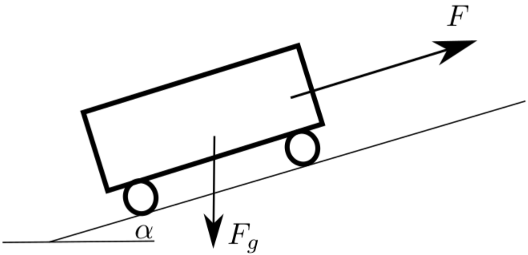
Example 1: car on a hill



- Newton's equation of motion

$$\dot{v}(t) = \frac{F(t)}{m} - g\sin(\alpha) \; ,$$

- Set $x = v$, $a = 0$, $b = \frac{1}{m}$:

$$\dot{x}(t) = ax(t) + bu(t) \qquad \text{with} \qquad u(t) = F(t) - mg\sin(\alpha) \; .$$

Integeral Form

By integrating a linear control system on both sides:

$$x(t) = x_0 + \int_0^t \left(ax(\tau) + bu(\tau)\right) \mathrm{d}\tau \ .$$

This is called the integral form of the linear control system ($x(0) = x_0$).

Main advantage of integral forms:

1. We don't need derivatives! For example, if the function u has a jump.

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Discontinuous control inputs

Example:

$$\dot{x}(t) = u(t) \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad u(t) = \begin{cases} 0 & \text{if } t < 1 \\ +1 & \text{otherwise} \end{cases}$$

Soution for the state trajectory,

$$x(t) = \begin{cases} 0 & \text{if } t < 1 \\ t - 1 & \text{otherwise} \end{cases},$$

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Explicit solution

The explicit solution of the linear control system is given by

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau$$

as long as u is an integrable function.

Proof

$$\begin{aligned}
\dot{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau\right) \\
&= ae^{at}x_0 + e^{a(t-t)}bu(t) + \int_0^t ae^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau \\
&= ax(t) + bu(t)\,.
\end{aligned}$$

and

$$x(0) = e^0 * x_0 + \int_0^0 e^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau = x_0\,.$$

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Parameterization of control functions

General affine parameterizations of the control function can be written in the form

$$u(t) = \sum_{i=0}^{N} v_i \varphi_i(t) \ .$$

- The functions $\varphi_0, \varphi_1, \ldots \varphi_N : \mathbb{R} \to \mathbb{R}$ are given basis functions
- The scalars $v_0, v_1, \ldots, v_N \in \mathbb{R}$ are the control parameterization coefficients

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Response functions

If we send the i-th basis function to the real system, i.e., $u(t) = \varphi_i(t)$, the associated solution of the differential equation

$$\dot{x}(t) = ax(t) + bu(t) \quad \text{with} \quad x(0) = 0$$

is given by

$$x(t) = \Phi_i(t) \qquad \text{with} \qquad \Phi_i(t) = \int_0^t e^{a(t-\tau)} b\varphi_i(\tau)\,\mathrm{d}\tau$$

We call the functions Φ_i response functions.

Linear Superposition Principle

If we send the input

$$u(t) = \sum_{i=0}^{N} v_i \varphi_i(t) \; .$$

to the real system, the response is given by

$$x(t) = \Psi_0(t) x_0 + \sum_{i=0}^{N} v_i \Phi_i(t) \quad \text{with} \quad \Psi_0(t) = e^{at} \; .$$

Proof:

$$\begin{aligned}
x(t) &= e^{at} x_0 + \int_0^t e^{a(t-\tau)} b u(\tau) \, \mathrm{d}\tau \\
&= e^{at} x_0 + \int_0^t e^{a(t-\tau)} b \sum_{i=0}^{N} v_i \varphi_i(\tau) \, \mathrm{d}\tau \\
&= e^{at} x_0 + \sum_{i=0}^{N} \left[\left(\int_0^t e^{a(t-\tau)} b \varphi_i(\tau) \, \mathrm{d}\tau \right) v_i \right] = \Psi_0(t) x_0 + \sum_{i=0}^{N} \Phi_i(t) v_i \; ,
\end{aligned}$$

Linear Superposition Principle

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$$\begin{aligned} x(t) &= e^{at} x_0 + \int_0^t e^{a(t-\tau)} b u(\tau)\, \mathrm{d}\tau \\ &= e^{at} x_0 + \int_0^t e^{a(t-\tau)} b \sum_{i=0}^{N} v_i \varphi_i(\tau)\, \mathrm{d}\tau \\ &= e^{at} x_0 + \sum_{i=0}^{N} \left[\left(\int_0^t e^{a(t-\tau)} b \varphi_i(\tau)\, \mathrm{d}\tau \right) v_i \right] = \Psi_0(t) x_0 + \sum_{i=0}^{N} \Phi_i(t) v_i \ , \end{aligned}$$

Linear Superposition Principle

If we have $x_0 = 0$, linear superposition principle can be summarized as

Input: **Output:**

$$u(t) = \sum_{i=0}^{N} v_i \varphi_i(t) \qquad \Longrightarrow \qquad x(t) = \sum_{i=0}^{N} v_i \Phi_i(t) \ .$$

Example 1: piecewise constant controls

Main idea: divide a given time interval, $[t_0, t_N]$, into N sub-intervals,

$$t_0 < t_1 < t_2 < \ldots < t_N \ .$$

Piecewise constant control function u given by

$$u(t) = \begin{cases} v_0 & \text{if } t \in [t_0, t_1] \\ v_1 & \text{if } t \in [t_1, t_2] \\ v_2 & \text{if } t \in [t_2, t_3] \\ \vdots & \\ v_{N-1} & \text{if } t \in [t_{N-1}, t_N] \ , \end{cases} \tag{2}$$

with parameters $v_0, v_1, \ldots, v_{N-1} \in \mathbb{R}$.

Example 1: piecewise constant controls

Basis functions:

$$\varphi_i(t) = \begin{cases} 1 & \text{if } t \in [t_i, t_{i+1}] \\ 0 & \text{otherwise.if} \end{cases}$$

Response functions:

$$\begin{aligned} \Phi_i(t) &= \int_0^t e^{a(t-\tau)} b \varphi_i(\tau)\, \mathrm{d}\tau \\ &= \begin{cases} 0 & \text{if } t < t_i \\ \frac{e^{a(t-t_i)}-1}{a} b & \text{if } t_i \le t \le t_{i+1} \\ \frac{e^{a(t-t_i)}-e^{a(t-t_{i+1})}}{a} b & \text{if } t > t_{i+1} \end{cases} \end{aligned}$$

(if $a \neq 0$; otherwise take the limit $\lim_{a\to 0}$)

Example 2: Frequency response

Consider periodic input functions with a given frequency ω,

$$u(t) = v_0 \cos(\omega t) + v_1 \sin(\omega t) \; .$$

These functions can be represented by choosing the basis functions

$$\varphi_0(t) = \cos(\omega t) = \mathsf{Re}\left(e^{i\omega t}\right) \quad \text{and} \quad \varphi_1(t) = \sin(\omega t) = \mathsf{Im}\left(e^{i\omega t}\right)$$

with $i = \sqrt{-1}$.

Example 2: Frequency response

The response function Φ_0 and Φ_1 are the real- and imaginary part of the function

$$
\begin{aligned}
\Phi(t) &= \Phi_0(t) + i\Phi_1(t) \\
&= \int_0^t e^{a(t-\tau)} b \cos(\omega\tau)\,\mathrm{d}\tau + i\left[\int_0^t e^{a(t-\tau)} b \sin(\omega\tau)\,\mathrm{d}\tau\right] \\
&= \int_0^t e^{a(t-\tau)} b \left(\cos(\omega\tau) + i\sin(\omega\tau)\right)\,\mathrm{d}\tau \\
&= \int_0^t e^{a(t-\tau)} b e^{i\omega\tau}\,\mathrm{d}\tau = \frac{e^{at} - e^{i\omega t}}{a - i\omega} b
\end{aligned}
$$

Example 2: Frequency response

Thus, we find

$$\Phi_0(t) \quad = \quad \mathsf{Re}\left(\frac{e^{at} - e^{i\omega t}}{a - i\omega} b\right) = \frac{abe^{at}}{a^2 + \omega^2} - \frac{ab\cos(\omega t)}{a^2 + \omega^2} + \frac{\omega b \sin(\omega t)}{a^2 + \omega^2}$$

and

$$\Phi_1(t) \quad = \quad \mathsf{Im}\left(\frac{e^{at} - e^{i\omega t}}{a - i\omega} b\right) = \frac{\omega b e^{at}}{a^2 + \omega^2} - \frac{\omega b\cos(\omega t)}{a^2 + \omega^2} - \frac{ab \sin(\omega t)}{a^2 + \omega^2}$$

Example 2: Frequency response

If we introduce the phase shift

$$\theta = \arccos\left(\frac{a}{\sqrt{a^2+\omega^2}}\right)$$

we can use the addition theorems for the sine and cosine function to write

$$\begin{aligned}
\frac{ab\cos(\omega t)}{a^2+\omega^2} - \frac{\omega b\sin(\omega t)}{a^2+\omega^2} &= \frac{b}{\sqrt{a^2+\omega^2}}\cos(\omega t+\theta) \qquad (3)\\
\frac{\omega b\cos(\omega t)}{a^2+\omega^2} + \frac{ab\sin(\omega t)}{a^2+\omega^2} &= \frac{b}{\sqrt{a^2+\omega^2}}\sin(\omega t+\theta) \qquad (4)
\end{aligned}$$

Example 2: Frequency response

Thus, we have

$$\Phi_0(t) = \frac{abe^{at}}{a^2+\omega^2} - \frac{b}{\sqrt{a^2+\omega^2}}\cos(\omega t+\theta)$$

and

$$\Phi_1(t) = \frac{\omega be^{at}}{a^2+\omega^2} - \frac{b}{\sqrt{a^2+\omega^2}}\sin(\omega t+\theta)$$

Interpretation:

1. The term $\frac{\omega be^{at}}{a^2+\omega^2}$ vanishes for $t \to \infty$ if $a < 0$ (transient term).
2. The constant θ may be interpreted as a phase shift.
3. The term $\frac{b}{\sqrt{a^2+\omega^2}}$ may be interpreted as a signal amplification factor.

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