Lyapunov Stability

Contents

Non-autonomous system

Recall the non-autonomous system described by differential equations of form

$$\dot{x}(t) = f(t, x), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

where $t_0 \geq 0$, $f : [t_0, \infty) \times \mathcal{D} \mapsto \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x, assume the domain $\mathcal{D}$ contains the equilibrium at origin, that is

$$f(t, 0) = 0 \quad , \forall t \geq 0$$

Let $V : [t_0, \infty) \times \mathcal{D} \mapsto \mathbb{R}$ be a $\mathcal{C}^1$ function with derivative is

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t)$$

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Extension of Lyapunov stability of Autonomous system

Theorem: Let $x = 0$ be an equilibrium point for the non-autonomous system and let $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function, then origin is

- stable, if V is p.d. and $\dot{V}$ is n.s.d.
- asymptotically stable, if V is p.d. and $\dot{V}$ is n.d.
- g.a.s., if $\mathcal{D} = \mathbb{R}^n$, V is p.d. and radially unbounded, $\dot{V}$ is n.d.
- e.s. if exist positive constant k_1, k_2, k_3, a such that

$$k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a, \text{ and, } \dot{V} \leq -k_3 \|x\|^a$$

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Example: Consider the system

$$\dot{x}_1 = -x_1 - g(t)x_2$$

$$\dot{x}_2 = x_1 - x_2$$

where $g(t)$ is a $\mathcal{C}^1$ function and satisfies

$$0 \leq g(t) \leq k, \quad \dot{g}(t) \leq g(t), \forall t \geq 0$$

study the stability property of the equilibrium $x = (0, 0)$ with the Lyapunov candidate function

$$V(t, x) = x_1^2 + [1 + g(t)]x_2^2$$

Comparison Functions

Definition: A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Moreover, it is said to belong to class $\mathcal{K}_\infty$ (or sometimes $\mathcal{KR}$), if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition: A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$, if

- for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ w.r.t r
- for each fixed r, the mapping $\beta(r, s)$ is decreasing w.r.t s, and $\beta(r, s) \to 0$ as $s \to \infty$.

Example: i) r^2 ii) arctan(r) iii) $\frac{r}{sr+1}$ iv) $|x_0|\mathrm{e}^{-t}$

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Rephrase of $\delta - \epsilon$ requirement

Lemma: The equilibrium $x = 0$ of the non-autonomous system is

- *uniformly stable*, if there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant c independent of t_0 such that

$$||x(t)|| \leq \alpha(||x(t_0)||), \quad \forall t \geq t_0 \geq 0, \forall ||x(t_0)|| < c$$

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Decrescent and Radially Unbounded

Definition: A $V : [t_0, \infty)\mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ function is said to be *decrescent* if there exists a class $\mathcal{K}$ function $\phi(\cdot)$ such that

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Consider a linear time-varying system

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with the assumption that $A(t)$ is piecewise continuous and nonsingular for all $t \geq t_0$. From linear system theory, we know its solution is given by

$$x(t; t_0, x_0) = \Phi(t, t_0)x_0$$

where $\Phi(t, t_0)$ is called state transition matrix satisfies

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From the perspective of Lyapunov definition:

Theorem: The equilibrium of LTV System (1) is

- stable iff $c(t_0) \triangleq \sup_{t \geq t_0} \|\Phi(t, t_0)\| < \infty$ and u.s. iff there exists $c_0 = \sup_{t_0 \geq 0} \|c(t_0)\| < \infty$
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Thereom: Assume that the elements of $A(t)$ are *uniformly bounded*, the equilibrium of System (1) is g.u.a.s(g.e.s) iff, for any given symmetric p.d. *continuous and bounded* matrix $Q(t)$, there is a continuously differentiable bounded p.d. symmetric matrix P satisfies the Lyapunov equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

$V(t,x) = x^\top P(t)x$ is the Lyapunov function verifying the condition for g.e.s.

The complete proof can be found in Khali's nonlinear system Section 4.6.

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The complete proof can be found in Khali's nonlinear system Section 4.6.

From the perspective of Lyapunov equation:

Thereom: Assume that the elements of $A(t)$ are *uniformly bounded*, the equilibrium of System (1) is g.u.a.s(g.e.s) iff, for any given symmetric p.d. *continuous and bounded* matrix $Q(t)$, there is a continuously differentiable bounded p.d. symmetric matrix P satisfies the Lyapunov equation

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From the perspective of $A(t)$:Hurwitz of $A(t)$ indicates stable?

Counterexample:

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5\sin t \cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

The eigenvalues of $A(t)$ for each fixed t,

$$\lambda(A(t)) = -0.25 \pm j0.5\sqrt{1.75}$$

have negative real parts and are also independent of t. Despite this the equilibrium at origin is unstable because

$$\Phi(t, 0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

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Theorem: Let element of $A(t)$ of the LTV System (1) be differentiable and bounded functions of time, and there is a positive constant σ such that

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Then the equilibrium at origin is said to be g.u.a.s (g.e.s), if

$$||\dot{A}(t)|| \in \mathcal{L}_2.$$

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$$\dot{x}_1 = x_2$$

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with $u = -x_1$

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Consider the non-autonomous system

$$\dot{x} = f(x,t)$$

assume f is $\mathcal{C}^1$ w.r.t x. Then, in the neighborhood of equilibrium, f has a Taylor expansion that can be write as

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Theorem: Consider the equilibrium $x_e = 0$ of the nonlinear system $\dot{x} = f(t, x)$ and the equilibrium $z_e = 0$ of the LTV system $\dot{z} = A(t)z$, where $A(t)$ represents the Jacobian matrix of $f(t, x)$ evaluated at origin. Assume $A(t)$ is uniformly bounded and the remaining term $g(t, x)$ satisfies

$$\lim_{|x| \to 0} \sup_{t \geq 0} \frac{|g(t, x)|}{|x|} = 0$$

Then the following statements are true:

- z_e is u.a.s (e.s), then x_e is l.u.a.s (l.e.s)
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This is also known as *First Method of Lyapunov.*

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$$A' = \left.\frac{\partial f}{\partial x}\right|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

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Unfortunately, if z_e is stable or uniformly stable, NO conclusion can be drawn about x_e.

Example: Consider the scalar system

$$\dot{x} = ax^3$$

Linearizing the system around origin yields

$$A = \frac{\partial f}{\partial x}\bigg|_{x=0} = 3ax^2\Big|_{x=0} = 0$$

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Contents

Converse Theorem

Theorem: Consider the non-autonomous system

$$\dot{x} = f(t, x), \quad x(0) = x_0 \in \mathbb{R}^n$$

with $x_e = 0$ and f is Lipschitz continuous in x and piecewise continuous in t for all $x \in \mathcal{D} := \{x \in \mathbb{R}^n | \|x\| < r\}$ and $t \geq 0$.

If origin is u.a.s, then there exist a $\mathcal{C}^1$ function $V(t, x)$ and class $\mathcal{K}$ function $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ defined on $[0, r)$ such that

$$\alpha_1(\|x\|^2) \leq V(t, x) \leq \alpha_2(\|x\|^2)$$
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Example: Consider a system

$$\dot{z} = f(t,z), \quad z(0) = z_0 \in \mathbb{R}^m$$

$$\dot{x} = Ax + Bz, \quad x(0) = x_0 \in \mathbb{R}^n$$

with $z = 0$ is an exponentially stable equilibrium of $z-$subsystem, A is Hurwitz and B is a finite constant matrix. Analyze the stability property of the closed-loop system.

Summary

Four types of system:

$$\dot{x} = f(t,x), \quad \dot{x} = f(x), \quad \dot{x} = A(t)x, \quad \dot{x} = Ax$$

Three stability properties:

i) stable (u.s)

ii) a.s (u.a.s, g.u.a.s,)

iii) e.s. (g.e.s)

Two Lyapunov function criteria:

a) p.d(p.s.d.)

b) n.d.(n.s.d)

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Exercise 1:

$$\dot{x} = -\frac{1}{1+t}x$$

Exercise 2:

$$\ddot{x} + 2\dot{x}^3 + 2x = 0$$

Contents

Input-Output stability

Consider an LTI system described by the convolution of two functions $u, h : \mathcal{R}^+ \to \mathcal{R}$ defined as*

$$y(t) = u * h \triangleq \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t u(t-\tau)h(\tau)d\tau$$

We say above system is $\mathcal{L}_p$ stable if $u \in \mathcal{L}_p \Rightarrow y \in \mathcal{L}_p$ and $\|y\|_p \leq c\|u\|_p$ for some constant $c \geq 0$ and any $u \in \mathcal{L}_p$. When $p = \infty, \mathcal{L}_p$ stability, i.e., $\mathcal{L}_\infty$ stability, is also referred to as bounded-input bounded-output (BIBO) stability.

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*Let $H(s)$ be the Laplace transform of the I/O operator $h(\cdot).H(s)$ is called the transfer function and $h(t)$ is the impulse response of the system.

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Consider the system

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Definition :The system above is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

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Suppose the unforced system

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has a globally uniformly asymptotically stable equilibrium point at the origin $x = 0$. What can we say about the behavior of the system in the presence of a bounded input $u(t)$?

Lemma : Suppose $f(t, x, u)$ is continuously differentiable and globally Lipschitz in (x, u), uniformly in t. If the unforced system (has a globally exponentially stable equilibrium point at the origin $x = 0$, then the forced system is input-to-state stable.

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