Hamilton-Jacobi-Bellman Equations

- Introduction
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- Hamilton-Jacobi-Bellman Equation

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Optimal Control

Problem Formulation

- Solve the continuous-time optimal control problem

$$V(0,y) \;=\; \min_{x,u} \int_0^T \ell(x(\tau),u(\tau))\,\mathrm{d}\tau + m(x(T))$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) &=& f(x(t),u(t))\,, \quad \text{a.a. } t \in [0,T] \\ x(0) &=& y \end{cases}$$

- Functions f, ℓ, and m are Lipschitz continuous (often even smooth).
- BUT: the control u is merely Lebesgue measurable / integrable.
- In practice, we often have additional control constraints, $u(t) \in \mathbb{U}$, and state constraints, $x(t) \in \mathbb{X}$, where $\mathbb{U}$ and $\mathbb{X}$ are given closed sets.

Optimal Value Function

- For general $t \in [0, T]$, we define the optimal value function by

$$V(t, y) \stackrel{\text{def}}{=} \min_{x,u} \int_t^T \ell(x(\tau), u(\tau))\, \mathrm{d}\tau + m(x(T))$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) &= f(x(\tau), u(\tau))\,, \quad \text{a.a. } \tau \in [t, T] \\ x(t) &= y \end{cases}$$

- Sometimes V is also called the "cost-to-go" function.
- Notice that the terminal value function, $V(T, y) = m(y)$, is given.
- Recall that our goal is to find $V(0, y)$. This means that we wish to propagate V "backwards"; that is, t runs from T to 0.

Bellman's Principle of Optimality

- The value function V satisfies Bellman's functional equation,

$$V(t,y) \;=\; \min_{x,u} \int_t^{t+h} \ell(x(\tau),u(\tau))\,\mathrm{d}\tau + V(t+h, x(t+h))$$

$$\text{s.t.} \quad \left\{ \begin{array}{rcl} \dot{x}(\tau) & = & f(x(\tau),u(\tau))\,, \quad \text{a.a. } \tau \in [t,t+h] \\ x(t) & = & y \end{array} \right.$$

  for all $t \in [0,T]$ and all $h \in [t,T]$.

- This means that if we have the function $V(t+h,\cdot)$ at a later time, $t+h$, we can find $V(t,\cdot)$ by solving an optimal control problem on the time horizon $[t,t+h]$.

- Question: what happens for infinitesimal small $h \to 0^+$?

Hamilton-Jacobi Equations

Completely unreasonable assumption:

- The functions x, u, ℓ, m and V are all smooth.

Intuitive but sloppy derivation of HJB:

- If we temporarily accept the above assumption, we have

$$\begin{aligned}
V(t^+, y) &\doteq V(t, y) + \dot{V}(t, y)h \\
x(t^+) &\doteq y + hf(y, u(t)) \\
V(t^+, x(t^+)) &\doteq V(t^+, y) + h\nabla V(t^+, y)^\mathsf{T} f(y, u(t)) \\
\int_t^{t^+} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau &\doteq h\, \ell(y, u(t))
\end{aligned}$$

with $t^+ = t + h$ and "$\doteq$" means "equal up to terms of order $O(h^2)$".

Hamilton-Jacobi Equations

- By substituting the previous relations we find

$$
\begin{array}{lrcl}
 & V & = & \min_u \left\{ h\,\ell + V + h\dot{V} + h\nabla V^{\mathsf{T}} f \right\} \\
\Longleftrightarrow & 0 & = & \dot{V} + H(y, \nabla V) \\
\text{with} & H(y,\lambda) & \overset{\text{def}}{=} & \min_u \left\{ \ell(y,u) + \lambda^{\mathsf{T}} f(y,u) \right\} \ .
\end{array}
$$

- The function H is called the Hamiltonian function.
- The partial differential equation

$$
\begin{array}{rcll}
-\dot{V} & = & H(y, \nabla V) & \text{on } [0,T] \times \mathbb{R}^n \\
V & = & m & \text{on } \{t = T\} \times \mathbb{R}^n
\end{array}
$$

is called the Hamilton-Jacobi-Bellman (HJB) equation.

Hamilton-Jacobi Equations

Remarks

- The above derivation was based on unreasonable assumptions. In general, neither x, nor u, nor V are smooth. It is, however, the goal of this lecture to make sense of the formal HJB-PDE

$$\begin{aligned} -\dot{V} &= H(y, \nabla V) && \text{on } [0,T] \times \mathbb{R}^n \\ V &= m && \text{on } \{t = T\} \times \mathbb{R}^n \end{aligned}$$

  in a more general setting and to understand the relation of this PDE to its underlying optimal control problem.

Hamilton-Jacobi Equations

More Remarks

- The Hamiltonian function

$$H(y,\lambda) \;=\; \min_u \,\{\ell(y,u) + \lambda^\intercal f(y,u)\}$$

  is—obviously—only well-defined if the minimum over u exists.

- Many but not all practical optimal control problems are formulated in such a way that the function

$$\widetilde{H}(y,\lambda,u) \;\overset{\text{def}}{=}\; \ell(y,u) + \lambda^\intercal f(y,u)$$

  is smooth and strongly convex in u.

- The condition, $\nabla^2_{uu}\widetilde{H} \;\succ\; 0$, is called the *strong Legendre condition*.

Hamilton-Jacobi Equations

More Remarks

- For the case that (closed) control constraints are present, the Hamiltonian function is defined by

$$H(y,\lambda) \;=\; \min_{u\in\mathbb{U}} \left\{\ell(y,u)+\lambda^{\mathsf{T}} f(y,u)\right\} \;.$$

- In practice, $\mathbb{U}$ is often a simple interval box (control bounds).
- The corresponding parametric minimizer is denoted by

$$u^{\star}(y,\lambda) \;\stackrel{\text{def}}{=}\; \underset{u\in\mathbb{U}}{\operatorname{argmin}} \left\{\ell(y,u)+\lambda^{\mathsf{T}} f(y,u)\right\}$$

- In general, $u^{\star}(y,\lambda)$ may be multi-valued but at if $\mathbb{U}$ is compact and convex and if the strong Legendre condition is satisfied, it is unique.

Hamilton-Jacobi Equations

Control Regularization

- In order to simplify our analysis a bit, we will sometimes assume the function $u^\star$ is not too irregular.
- In practice, one often adds control penalty terms of the form "$\|u\|_R^2$" (with weight $R \succ 0$) to the Lagrange term in order to regularize.

Auxiliary Theorem from Convex Optimization

- Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a compact and convex set with non-empty interior and let $\Phi \in C^\infty(\mathbb{X} \times \mathbb{R}^m)$ be such there exists a $\theta > 0$ with $\nabla_x^2 \Phi(x,p) \succ \theta \|x\|_2^2$ for all $x \in \mathrm{int}(X)$ and all $p \in \mathbb{R}^m$, then

$$\forall p \in \mathbb{R}^m, \qquad x^\star(p) \stackrel{\mathrm{def}}{=} \underset{x \in \mathbb{X}}{\mathrm{argmin}}\, \Phi(x,p)$$

is a locally Lipschitz continuous function on $\mathbb{R}^m$.

Hamilton-Jacobi Equations

Proof (rough sketch only)

- Since Φ is strongly convex in x and X satisfies Slater's constraint qualification, the minimizer $x^\star(p)$ is unique.
- Let $N(x)$ denote the normal cone of X at x, given by

$$N(x) \overset{\text{def}}{=} \{ \, y \in \mathbb{R}^n \mid \forall z \in X, \; y^\mathsf{T}(z - x) \leq 0 \, \} \; .$$

- The first order necessarily optimality condition takes the form

$$-\Phi_x(x, p) \; \in \; N(x) \qquad \text{with} \qquad \Phi_x \overset{\text{def}}{=} \nabla_x \Phi \; .$$

- Let us set $a = x^\star(p)$ and $b = x^\star(q)$ for two parameters $p, q \in \mathbb{R}^m$.

Hamilton-Jacobi Equations

Proof (rough sketch continued...)

- Then there exist vectors $n_a \in N(a)$ and $n_b \in N(b)$ with

$$-\Phi_x(a,p) = n_a \qquad -\Phi_x(b,q) = n_b$$

- Note that the definition of the normal cone implies that

$$(b-a)^\mathsf{T} n_a \ \leq \ 0 \qquad \text{and} \qquad (a-b)^\mathsf{T} n_b \ \leq \ 0 \ .$$

- Thus, on the one hand, we have

$$\begin{aligned}
(b-a)^\mathsf{T}\Phi_x(b,p) &= (b-a)^\mathsf{T}\left[\Phi_x(b,p)-\Phi_x(a,p)\right] - (b-a)^\mathsf{T} n_a \\
&\geq \int_0^1 (b-a)^\mathsf{T}\Phi_{xx}(a+s(b-a),p)(b-a)\,\mathrm{d}s \\
&\geq \theta\|b-a\|^2 \ .
\end{aligned}$$

Hamilton-Jacobi Equations

Proof (rough sketch continued...)

- ... and, on the other hand, we have

$$
\begin{aligned}
(b-a)^{\mathsf{T}}\Phi_x(b,p) &= (b-a)^{\mathsf{T}}\left[\Phi_x(b,p)-\Phi_x(b,q)\right]-(b-a)^{\mathsf{T}}n_b \\
&\leq (b-a)^{\mathsf{T}}\left[\Phi_x(b,p)-\Phi_x(b,q)\right] \\
&\leq L\|b-a\|\|p-q\| \, ,
\end{aligned}
$$

  where $L < \infty$ denotes a suitable local bound on the derivative of Φ_x with respect to the parameter.

- In summary, we find that $x^\star$ is locally Lipschitz continuous, because

$$
\theta\|b-a\|^2 \;\leq\; L\|b-a\|\|p-q\| \quad \Longrightarrow \quad \|b-a\| \leq \frac{L}{\theta}\|p-q\| \, .
$$

Hamilton-Jacobi Equations

Remark

- The assumption that Φ is smooth can, of course, be replaced by weaker (but more technical) assumptions. For instance, the above proof formally only uses that Φ_x is locally Lipschitz continuous in p.

Back to HJB Equations:

- If $\widetilde{H}$ is sufficiently smooth and satisfied the strong Legendre condition, and if the set $\mathbb{U}$ is compact, convex, with non-empty interior, the above theorm can be applied to show that $u^{\star}$ is Lipschitz continuous.

Hamilton-Jacobi Equations

Rough plan for the main steps:

1. Analyze / solve the HJB equation

$$
\begin{aligned}
-\dot{V} &= H(y, \nabla V) \qquad && \text{on } \; [0,T] \times \mathbb{R}^n \\
V &= m && \text{on } \; \{t = T\} \times \mathbb{R}^n
\end{aligned}
$$

   in order to find the value function V (if it exists).

2. The optimal feedback law is given by

$$\mu(t,x) \; \stackrel{\text{def}}{=} \; u^{\star}(x, \nabla V(t,x)) \; .$$

3. The optimal state x can eventually be found by forward simulation

$$\dot{x} \; = \; f(x, \mu(t,x)) \qquad \text{with} \qquad x(0) = x_0 \; .$$

Contents

Nonlinear First Order PDEs

Goal

- We are interested in analyzing and solving first order PDEs

$$F(x, V, \nabla V) = 0 \qquad u|_{\partial\Omega} = g$$

on a suitable open domain Ω for given functions F and g.

Major Problems

- In the context of linear elliptic and parabolic PDE analysis, the second order term has helped us a lot with our analysis—but here we have no such second order term.
- There is—to date—no uniform theory for nonlinear PDEs.

Nonlinear First Order PDEs

Example

- Let us consider the scalar but nonlinear PDE

$$|\nabla V| - 1 = 0 \quad \text{with} \quad V(-1) = V(1) = 0$$

  on the interval $\Omega = (0, 1)$.

- There is obviously no classical (smooth) solution.
- The function $V(x) = 1 - |x|$ is a weak solution (solves the PDE a.e.)
- The function $V(x) = |x| - 1$ is another weak solution.
- More generally, all piecewise linear functions with slopes ± 1 that satisfy the boundary conditions are weak solutions ($V \in H_0^1(\Omega)$).

Nonlinear First Order PDEs

Motivation

- Let us modify the PDE slightly and set

$$|\nabla V^\epsilon| - 1 = \epsilon \Delta V^\epsilon \quad \text{with} \quad V^\epsilon(-1) = V^\epsilon(1) = 0$$

  on the interval $\Omega = (0, 1)$ for a small $\epsilon > 0$.

- What happens if we take the limit $\epsilon \to 0$?
- Let us assume that $V_\epsilon \to V$ uniformly on $[-1, 1]$.
- If V would have a strict local minimizer $x \in (-1, 1)$, then V^ϵ would have a minimizer at a nearby point $x_\epsilon \in (-1, 1)$. But then

$$|\nabla V^\epsilon(x^\epsilon)| - 1 = -1 = \epsilon \Delta V^\epsilon > 0,$$

  which is a contradiction.

Nonlinear First Order PDEs

Motivation

- In summary, the only possible piecewise linear solution V of the PDE that may be obtained as a limit of the form $V_\epsilon \to V$ with

$$|\nabla V^\epsilon| - 1 = \epsilon \Delta V^\epsilon \quad \text{with} \quad V^\epsilon(-1) = V^\epsilon(1) = 0$$

  is the function $V(x) = 1 - |x|$, since this is the only function that has no local minimizers in $(-1, 1)$.

- In the following, we will see that $V(x) = 1 - |x|$ can indeed be obtained by taking such a limit. It is the unique viscosity solution of

$$|\nabla V| - 1 = 0 \quad \text{with} \quad V^\epsilon(-1) = V^\epsilon(1) = 0 \; .$$

One-Sided Differentials

Definition

- For an open set $\Omega \subseteq \mathbb{R}^n$ and a function $V : \Omega \to \mathbb{R}$, we define the super- and sub-differentials of V at $x \in \Omega$ by

$$D^+V(x) \overset{\text{def}}{=} \left\{ \lambda \in \mathbb{R}^n \;\middle|\; \limsup_{y \to x} \frac{V(y) - V(x) - \lambda^\mathsf{T}(y-x)}{\|x-y\|} \leq 0 \right\}$$

$$D^-V(x) \overset{\text{def}}{=} \left\{ \lambda \in \mathbb{R}^n \;\middle|\; \liminf_{y \to x} \frac{V(y) - V(x) - \lambda^\mathsf{T}(y-x)}{\|x-y\|} \geq 0 \right\}$$

- The term $V(x) + \lambda^\mathsf{T}(y-x)$ can be interpreted as tangent that touches V at x from above/below.

One-Sided Differentials

Theorem

- Let $V \in C(\Omega)$ be a continuous function. Then we have $\lambda \in D^+V(x)$ if and only if there exists a $\varphi \in C^1(\Omega)$ with $\lambda = \nabla\phi(x)$ and such that $V - \varphi$ has a local maximum at x.
- Similarly, $\lambda \in D^-V(x)$ if and only if there exists a $\varphi \in C^1(\Omega)$ with $\lambda = \nabla\phi(x)$ and such that $V - \varphi$ has a local minimum at x.

Proof.

- Assume $\lambda \in D^+V(x)$ and choose any monotonically increasing function $\sigma \in C(\mathbb{R})$ with $\sigma(0) = 0$ such that

$$V(y) \ \leq \ V(x) + \lambda^{\mathsf{T}}(y - x) + \sigma(\|y - x\|)\|y - x\|$$

for all y in a sufficiently small neighborhood of x.

One-Sided Differentials

Proof (continued...)

- Define $\rho(r) \overset{\text{def}}{=} \int_0^r \sigma(s)\,\mathrm{d}s$ such that

$$\rho(0) = \rho'(0) = 0 \quad \text{and} \quad \rho(2r) \geq \int_r^{2r} \sigma(s)\,\mathrm{d}s \geq r\sigma(r)\ .$$

- This construction is such that

$$\varphi(y) \ \overset{\text{def}}{=}\ V(x) + \lambda^{\mathsf{T}}(y-x) + \rho(2\|y-x\|)$$

  is a C^1-function with $\varphi(x) = V(x)$ and $\nabla\varphi(x) = \lambda$.

- Moreover for all y in a small neighborhood of x it holds that

$$V(y) - \varphi(y) \ \leq\ \sigma(\|y-x\|)\|y-x\| - \rho(2\|y-x\|) \ \leq\ 0\ .$$

  That is, $V - \varphi$ has a local maximum at x.

One-Sided Differentials

Proof (continued...)

- The other way around if $\nabla\varphi(x) = \lambda$ and $V - \varphi$ has a local maximum at x, then

$$\limsup_{y \to x} \frac{V(y) - V(x) - \lambda^\mathsf{T}(y-x)}{\|y-x\|}$$

$$\leq \quad \limsup_{y \to x} \frac{\varphi(y) - \varphi(x) - \lambda^\mathsf{T}(y-x)}{\|y-x\|} \;=\; 0\,.$$

  Thus, $\lambda \in D^+V(x)$ as claimed.

- The statement about the subdifferential is, of course, analogous.

One-Sided Differentials

Remarks:

- Since one can always add constants, we may additionally assume $\varphi(x) = V(x)$. Thus, we have $D^+V(x) \neq \varnothing$ iff there exists a smooth function $\varphi \geq V$ that "touches" the graph of V at x from above.
- In the above proof, we may replace $\varphi(y)$ by $\varphi(y) + \|y - x\|^2$. Thus, it is not restrictive to additionally assume that $V - \varphi$ has a strict local maximum at x.
- Analogous remarks apply to subdifferentials.

One-Sided Differentials

Theorem:

Assume that $V \in C(\Omega)$. Then:

1. If V is differentiable at $x \in \Omega$, then $D^+V(x) = D^-V(x) = \{\nabla V(x)\}$.
2. If $D^+V(x) \neq \varnothing$ and $D^-V(x) \neq \varnothing$, then V is differentiable at x.
3. The sets

$$\Omega^+ \stackrel{\text{def}}{=} \left\{ x \in \Omega \;\middle|\; D^+V(x) \neq \varnothing \right\}$$

and $$\Omega^- \stackrel{\text{def}}{=} \left\{ x \in \Omega \;\middle|\; D^-V(x) \neq \varnothing \right\}$$

are dense in Ω.

One-Sided Differentials

Proof.

1. If V is differentiable at x, then $\nabla V(x) \in D^+V(x)$. Moreover, if $\varphi \in C^1(\Omega)$ is such that $V - \varphi$ has a local maximum at x, then we necessarily have $\nabla\varphi(x)) = \nabla V(x)$. That is, $D^+V(x)$ contains only the vector $\nabla V(x)$.

2. If $D^+V(x) \neq \varnothing$ and $D^-V(x) \neq \varnothing$, then there exists a $\delta > 0$ and $\varphi_1, \varphi_2 \in C^1(\Omega)$ with

$$\varphi_1(x) = u(x) = \varphi_2(x) \quad \text{and} \quad \varphi_1(y) \leq u(y) \leq \varphi_2(y)$$

for all y with $\|x - y\| < \delta$. This is sufficient to conclude that V is differentiable at x, where $\nabla u(x) = \nabla\varphi_1(x) = \nabla\varphi_2(x)$.

One-Sided Differentials

Proof (continued).

3. Let $x_0 \in \Omega$ and $\epsilon > 0$ be given. We consider the function

$$\varphi(x) \stackrel{\text{def}}{=} \frac{1}{\epsilon^2 - \|x - x_0\|^2}$$

on the open ball $B_\epsilon(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \epsilon\}$. Because $\varphi(x) \to \infty$ for $\|x - x_0\| \to \epsilon$, the function $V(x) - \varphi(x)$ has a local maximum at a point $y \in B_\epsilon(x_0) \subseteq \Omega$. Thus, $D^+V(y) \neq \varnothing$. This implies that Ω^+ is dense in Ω and an analogous statement holds for the set Ω^-.

Viscosity Solutions

Definitions

- We say that $V \in C(\Omega)$ is a viscosity subsolution of the PDE

$$F(x, V, \nabla V) = 0$$

  on Ω if $F(x, V(x), \lambda) \leq 0$ for all $x \in \Omega^+$ and all $\lambda \in D^+V(x)$.

- Similarly, $V \in C(\Omega)$ is a viscosity supersolution if $F(x, V(x), \lambda) \leq 0$ for all $x \in \Omega^-$ and all $\lambda \in D^-V(x)$.

- Finally, V is both a viscosity sub- and a viscosity super solution, then we say that V is a viscosity solution.

Viscosity Solutions

Remarks

- Unfortunately, the sign of F matters, which can easily lead to confusion: a viscosity solution of "$F(x, V, \nabla V) = 0$" is not necessarily a viscosity solution of "$-F(x, V, \nabla V) = 0$" and vice versa. Example: $V(x) = 1 - |x|$ is a viscosity solution of $|\nabla V| - 1 = 0$, but it is not a viscosity solution of $1 - |\nabla V| = 0$.
- The concept of "viscosity solutions" of PDE is generally different from the concept of "weak solutions". Nevertheless, if V is a Lipschitz continuous viscosity solution of $F(x, V, \nabla V) = 0$, then V satisfies this equation almost everywhere in Ω.

Viscosity Solutions of HJB Equations

- The previous definition formally applies to HJB equations as well, as long as we define

$$F(x, V(x), \nabla V(x)) \;\stackrel{\text{def}}{=}\; \partial_t V(t,y) + H(x, \nabla V(t,y))$$

  with $x \stackrel{\text{def}}{=} (t, y^\intercal)^\intercal$.

- This means that V is a viscosity subsolution of the HJB if for all C^1-functions φ such that $V - \varphi$ has a local maximum at (t, y)

$$\dot{\varphi}(t,y) + H(y, \nabla \varphi(t,y)) \leq 0 \; .$$

  Viscosity supersolutions satisfy a similar, reversed inequality.

Stability of Viscosity Solutions

- The set of solutions to nonlinear first order PDEs is not necessarily closed with repsect to uniform convergence.
- For instance, the sequence (with index $m = 1, 2, 3, \ldots$)

$$V_m(x) \stackrel{\text{def}}{=} \min_{k \in \mathbb{N}} \left| x - \frac{k}{m} \right|$$

  satisfies the PDE $|\nabla V_m| - 1 = 0$ almost everywhere. We also have uniform convergence,

$$V(x) \stackrel{\text{def}}{=} \lim_{m \to \infty} V_m(x) = 0,$$

  but the limit function V does not satisfy $|\nabla V| - 1 = 0$.
- Clearly, the problem is that the derivatives do not converge.

Stability of Viscosity Solutions

- The goal of the following analysis is to show that for viscosity solutions a stability result holds without requiring the convergence of derivatives.
- For this aim, we first prove the following auxiliary result.

Lemma

- Assume $V \in C(\Omega)$ and $\varphi \in C^1(\Omega)$ such that $V - \varphi$ has a strict local minimum at $x \in \Omega$. If $V_m \to V$ uniformly, there exists a sequence $x_m \to x$ with $V_m(x_m) \to V(x)$ such that $V_m - \varphi$ has a local minimum at x_m.

Stability of Viscosity Solutions

Proof.

- If $V - \varphi$ has a strict local minimum at $x \in \Omega$, there there exists for every sufficiently small $\delta > 0$ an $\epsilon > 0$ such that

$$\forall y \in B_\delta(x), \quad V(y) - \varphi(x) > V(x) - \varphi(x) + \epsilon \; .$$

- If $V_m \to V$ uniformly, we can find a sufficiently large N such that

$$\forall y \in B_\delta(x), \quad V_m(y) - \varphi(x) > V_m(x) - \varphi(x) + \frac{\epsilon}{2} \; .$$

  Thus, $V_m - \varphi$ has a local minimum at a point $x_m \in B_\delta(x)$.

- The desired sequence x_m can be constructed by sending $\epsilon, \delta \to 0$.

Stability of Viscosity Solutions

Theorem.

- Let $V_m \in C(\Omega)$ be viscosity subsolutions of

$$F_m(x, V_m, \nabla V_m) = 0$$

on Ω and $F_m \to F$ uniformly on compact substets of $\Omega \times \mathbb{R} \times \mathbb{R}^n$. If $V_m \to V \in C(\Omega)$, then V is a subsolution of

$$F_m(x, V, \nabla V) = 0$$

Of course, the same statement also holds for viscosity supersolutions.

Stability of Viscosity Solutions

Proof.

- Let $\varphi \in C^1(\Omega)$ be such that $V - \varphi$ has a strict local maximum at a point $x \in \Omega$. By the above auxiliary result, there exists a sequence $x_m \to x$ of local maxima of $V_m - \varphi$, and $u_m(x_m) \to u(x)$. Since V_m is a viscosity subsolution, we have

$$F_m(x, V_m(x_m), \nabla\varphi(x_m)) \ \leq \ 0 \ .$$

Thus, by continuity, we can take the limit $m \to \infty$ and obtain

$$F(x, V(x), \nabla\varphi(x)) \ \leq \ 0 \ .$$

Thus, V is a subsolution as claimed.

Vanishing Viscosity Limits

Basic Intuition:

- The concept of viscosity solutions for HJBs was introduced by Crandall and Lions in the 1970s.
- Among their main sources of inspiration was the viscous equations

$$F(x, V_\epsilon, \nabla V_\epsilon) \;=\; \epsilon \Delta V_\epsilon$$

  with viscosity (or "stochastic diffusion") parameter $\epsilon > 0$.
- During our analysis of linear elliptic PDEs, we have seen that the second order terms of the form "$\epsilon \Delta V_\epsilon$" often lead to a regularization of the solutions of PDE as long as $\epsilon > 0$.
- But here we want to analyze the "vanishing viscosity limit": $\epsilon \to 0$.

Vanishing Viscosity Limits

Theorem

- If we assume that $V_\epsilon \in C^\infty(\Omega)$ is a family of smooth solutions of

$$F(x, V_\epsilon(x), \nabla V_\epsilon(x)) \;=\; \epsilon\, \Delta V_\epsilon(x)$$

  and that $V_\epsilon \to V$ converges uniformly on an open set Ω, then V is a viscosity solution of the first order PDE $\quad F(x, V(x), \nabla V_\epsilon) = 0$.

Proof.

- Assume $x \in \Omega$ and $\lambda \in D^+V(x)$. This means that we can find $\varphi \in C^1(\Omega)$ with $\nabla\varphi(x) = \lambda$ and such that $V - \varphi$ has a strict local maximum at x.

Vanishing Viscosity Limits

Proof (continued...)

- For any $\delta > 0$ we can further find $0 < \hat{\delta} \leq \delta$ and $\psi \in C^2(\Omega)$ such that

  1. $|\nabla \varphi(y) - \nabla \varphi(x)| \leq \delta$ for all $y \in B_{\hat{\delta}}(x)$,
  2. $\|\psi - \varphi\|_{C^1} \leq \delta$, and
  3. $V_\epsilon - \psi$ has a local maximum in $B_{\hat{\delta}}(x)$ for all sufficiently small $\epsilon > 0$.

- Let x_ϵ denote such a local maximum of $V_\epsilon - \psi$. As V_ϵ is smooth,

$$\nabla u_\epsilon(x_\epsilon) = \nabla \psi(x_\epsilon) \quad \text{and} \quad \Delta u_\epsilon(x_\epsilon) = \Delta \psi(x_\epsilon) \; .$$

  Thus, we also have $F(x_\epsilon, V_\epsilon, \nabla \psi(x_\epsilon)) \; \leq \; \epsilon \, \Delta \psi(x_\epsilon)$.

Vanishing Viscosity Limits

Proof (continued...)

- Clearly, we can pick a sequence $\epsilon_k \to 0$ such that

$$\lim_{k\to\infty} x_{\epsilon_k} \;=\; \widetilde{x} \;\in\; B_{\hat{\delta}}(x)\;.$$

As $\psi \in C^2(\Omega)$ we can take the limit for $k \to \infty$ to find

$$F(x, V(\widetilde{x}), \nabla\psi(\widetilde{x})) \;\leq\; 0\;.$$

- By construction of ψ, we also have

$$\|\nabla\psi(\widetilde{x}) - \lambda\| \;\leq\; \|\nabla\psi(\widetilde{x}) - \nabla\varphi(\widetilde{x})\| + \|\nabla\varphi(\widetilde{x}) - \nabla\varphi(x)\| \leq 2\delta\;.$$

Thus, by sending $\delta \to 0$ we find $F(x, V(x), \lambda) \leq 0$ and V is a subsolution as claimed.

Comparison Principles

Motivation

- Recall that in the context of analyzing linear elliptic/parabolic equation, we have been establishing a "maximum principles" for sufficiently smooth solutions of certain PDEs.
- It turns out that similar conceptual ideas are applicable to analyzing viscosity solutions of HJB equations.
- A surprinsing result is that this is possible under a rather general continuity assumption on the viscosity solution—without needing any other regularity assumption.

Comparison Principles

Theorem

- Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, let $V_1 \in C(\overline{\Omega})$ be a viscosity sub- and $V_2 \in C(\overline{\Omega})$ a viscosity supersolution of

  $$V + H(x, \nabla V) = 0$$

  in Ω with $V_1 \leq V_2$ on $\partial\Omega$. If H satisfies

  $$|H(x,\lambda) - H(y,\lambda)| \leq \omega(\|x-y\|(1+\|\lambda\|))$$

  for a continuous non-decreasing function $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\omega(0) = 0$, then $V_1(x) \leq V_2(x)$ for all $x \in \overline{\Omega}$.

Comparison Principles

Main Idea of the Proof

- Let us assume for a moment that V_1 and V_2 are smooth. If $V_1 - V_2$ has a positive maximum at a point $x \in \Omega$, then $\lambda \stackrel{\mathrm{def}}{=} \nabla V_1(x) = \nabla V_2(x)$ satisfies

$$
\begin{aligned}
& V_1(x) + H(x,\lambda) && \leq && 0 \\
\text{and} \qquad & V_2(x) + H(x,\lambda) && \geq && 0\,.
\end{aligned}
$$

  If we substract the second from the first inequality, we find $V_1(x) - V_2(x) \leq 0$, which contradicts our assumption that $V_1 - V_2$ has a positive maximum at x.

Comparison Principles

Proof.

- If V_1 and/or V_2 are non-smooth the above argument fails to work, since $D^+V_1(x)$ and/or $D^-V_2(x)$ could be empty at x.
- Instead, we attempt to find nearby points x_ϵ, y_ϵ with $V_1(x_\epsilon) > V_2(y_\epsilon)$ such that there exists $\lambda \in D^+V_1(x_\epsilon)$ and $\lambda \in D^-V_2(y_\epsilon)$.
- Here, the key idea is analyze the augmented difference function

$$\Phi_\epsilon(x,y) \stackrel{\text{def}}{=} V_1(x) - V_2(y) - \frac{\|x-y\|}{2\epsilon}$$

on the domain $\overline{\Omega} \times \overline{\Omega}$.

Comparison Principles

Proof (continued...)

- If $V_1 > V_2$ at some point inside Ω and if $\epsilon > 0$ is sufficiently small, the function

$$\Phi_\epsilon(x,y) \stackrel{\text{def}}{=} V_1(x) - V_2(y) - \frac{\|x-y\|^2}{2\epsilon}$$

  has a positive maximum at a point $(x_\epsilon, y_\epsilon) \in \Omega \times \Omega$.

- Define $\varphi_1(x) \stackrel{\text{def}}{=} V_2(y_\epsilon) + \frac{1}{2\epsilon}\|x - y_\epsilon\|^2$. Then $\Phi(x, y_\epsilon) = V_1(x) - \varphi_1(x)$ has a maximum at x_ϵ and

$$\lambda \stackrel{\text{def}}{=} \frac{x_\epsilon - y_\epsilon}{\epsilon} = \nabla\varphi_1(x_\epsilon) \in D^+V_1(x_\epsilon) \, .$$

- Analogously, we conclude $\lambda \in D^-V_2(y_\epsilon)$.

Comparison Principles

Proof (continued...)

- If a positive global maximizer $x_0 \in \Omega$ of $V_1 - V_2$ exists, we have

$$\Phi_\epsilon(x_\epsilon, y_\epsilon) \geq \delta > 0 \quad \text{with} \quad 0 < \delta \overset{\text{def}}{=} \max_{x \in \overline{\Omega}} V_1(x) - V_2(x) \; .$$

- Since $|V_1| \leq M$ and $|V_2| \leq M$ on $\overline{\Omega}$ for sufficiently large $M < \infty$,

$$\Phi_\epsilon(x, y) \leq 2M - \frac{\|x - y\|^2}{2\epsilon}$$

and $\Phi_\epsilon(x, y) \leq 0$ if $\|x - y\|^2 \geq 4M\epsilon$. Thus, we must have

$$\|x_\epsilon - y_\epsilon\| \leq 2\sqrt{M\epsilon} \; .$$

Comparison Principles

Proof (continued...)

- Recall that (by continuity of V_1 and V_2) we have $x_\epsilon, y_\epsilon \in \Omega \times \Omega$ and

$$\lambda = \frac{x_\epsilon - y_\epsilon}{\epsilon} \in D^+ V_1(x_\epsilon) \cap D^- V_2(y_\epsilon)$$

  as well as

$$V_1(x_\epsilon) + H(x_\epsilon, \lambda) \leq 0 \quad \text{and} \quad V_2(y_\epsilon) + H(y_\epsilon, \lambda) \geq 0 \; .$$

- Moreover,

$$\begin{aligned} V_1(x_\epsilon) - V_2(x_\epsilon) &= \Phi_\epsilon(x_\epsilon, x_\epsilon) \; \leq \; \Phi_\epsilon(x_\epsilon, y_\epsilon) \\ &\leq V_1(x_\epsilon) - V_2(x_\epsilon) + |V_2(x_\epsilon) - V_2(y_\epsilon)| - \frac{\|x_\epsilon - y_\epsilon\|^2}{2\epsilon} \end{aligned}$$

Comparison Principles

Proof (continued...)

- Consequently, we have

$$\frac{\|x_\epsilon - y_\epsilon\|^2}{2\epsilon} \leq |V_2(x_\epsilon) - V_2(y_\epsilon)| \quad \Longrightarrow \quad \limsup_{\epsilon \to 0^+} \frac{\|x_\epsilon - y_\epsilon\|^2}{2\epsilon} \;\leq\; 0 \; .$$

- In summary,

$$\begin{aligned} 0 \;\; < \;\; \delta \; \leq \; \Phi_\epsilon(x_\epsilon, y_\epsilon) \; \leq \; V_1(x_\epsilon) - V_2(y_\epsilon) \\ \leq \;\; |H(x_\epsilon, \lambda) - H(y_\epsilon, \lambda)| \; \leq \; \omega\left(\|x_\epsilon - y_\epsilon\| \left(1 + \frac{\|x_\epsilon - y_\epsilon\|}{\epsilon} \right) \right) \end{aligned}$$

As the latter term converges to 0 for $\epsilon \to 0$, we obtain the desired contradiction. This concludes the proof.

Uniqueness of Viscosity Solutions

- An immediate consequence of the above comparison principle is that the nonlinear first order PDE

$$V + H(x, \nabla V) = 0 \qquad \text{with} \qquad V|_{\partial\Omega} = V_{\text{boundary}}$$

  has—under the above equicontinuity assumption on H—at most one viscosity solution.

- This is because if V_1 and V_2 are two viscosity solutions we have $V_1 = V_2$ on the boundary, and, consequently, $V_1 \leq V_2$. Similarly, by exchanging the roles of V_1 and V_2, we also have $V_1 \geq V_2$.

Uniqueness of Viscosity Solutions

- The above ideas can be extended to analyze PDEs of the form

$$\dot{V} + H(x, \nabla V) = 0 \quad \text{with} \quad V(0, x) = V_0(x)$$

- Motivation: for $t \leftarrow T - t$, $H \leftarrow -H$ this is a HJB equation.
- We'll assume that H satisfies

$$\begin{aligned} |H(x, \lambda) - H(y, \lambda)| &\leq C\|x - y\|(1 + \|\lambda\|) \\ |H(x, \lambda) - H(x, \kappa)| &\leq C\|\lambda - \kappa\| \end{aligned}$$

for a Lipschitz constant $C < \infty$.

Uniqueness of Viscosity Solutions

Theorem

- Let H satisfy the Lipschitz continuity assumption from the previous slide. If $V_1 \in C^0(\Omega_T)$ is a bounded viscosity subsolution of

$$\dot{V} + H(x, \nabla V) = 0$$

on $\Omega_T \stackrel{\text{def}}{=} [0,T] \times \mathbb{R}^n$ and $V_2 \in C^0(\Omega_T)$ a bounded viscosity supersolution, with $V_1(0,x) \leq V_2(0,x)$ for all $x \in \mathbb{R}^n$, then we have

$$V_1(t,x) \ \leq \ V_2(t,x)$$

for all $(t,x) \in \Omega_T$.

Uniqueness of Viscosity Solutions

Proof.

- The following proof is indirect and in many ways similar to our previous theorem: if we don't have $V_1 \leq V_2$ on Ω_T, we can find $\sigma, \omega > 0$ such that

$$\sup_{(t,x)\in\Omega_T} \{V_1(t,x) - V_2(t,x) - 2\omega t\} \;\geq\; \sigma \; .$$

- We'll again work with an augmented difference function, namely,

$$\begin{aligned} \Phi_\epsilon(t,x;s,y) \quad &\stackrel{\text{def}}{=} \quad V_1(t,x) - V_2(s,y) - \omega(t+s) \\ &\quad\; -\epsilon(\|x\|^2 + \|y\|^2) - \frac{\|t-s\|^2 + \|x-y\|^2}{\epsilon^2} \; . \end{aligned}$$

Uniqueness of Viscosity Solutions

Proof (continued...)

- The global maximum $(t_\epsilon, x_\epsilon, s_\epsilon, y_\epsilon) \in ((0,T] \times \mathbb{R}^n)^2$ of Φ_ϵ for sufficiently small $\epsilon > 0$ satisfies

$$\Phi_\epsilon(t_\epsilon, x_\epsilon, s_\epsilon, y_\epsilon) \;\geq\; \max_{(t,x)\in\Omega_T} \Phi_\epsilon(t,x,t,x) \geq \frac{\sigma}{2} \; .$$

- Again very similar to the proof of the previous theorem, we define

$$\begin{aligned}\varphi_1(t,x) \quad &\overset{\text{def}}{=} \quad V_2(s_\epsilon, y_\epsilon) + \omega(t + s_\epsilon) \\ &\qquad + \epsilon(\|x\|^2 + \|y_\epsilon\|^2) + \frac{|t - s_\epsilon|^2 + \|x - y_\epsilon\|^2}{\epsilon^2}\end{aligned}$$

such that $V_1 - \varphi_1$ has a maximum at (t_ϵ, x_ϵ).

Uniqueness of Viscosity Solutions

Proof (continued...)

- Since V_1 is a viscosity subsolution, we find

$$\begin{aligned}&\dot{\varphi}_1(t_\epsilon, x_\epsilon) + H(x_\epsilon, \nabla \varphi_1(t_\epsilon, x_\epsilon))\\ &= \omega + \tfrac{2(t_\epsilon - s_\epsilon)}{\epsilon^2} + H\left(x_\epsilon, \tfrac{2(x_\epsilon - y_\epsilon)}{\epsilon^2} + 2\epsilon x_\epsilon\right) \ \le \ 0 \ .\end{aligned}$$

- An analogous argument for the supersolution yields

$$\begin{aligned}&\dot{\varphi}_2(s_\epsilon, y_\epsilon) + H(y_\epsilon, \nabla \varphi_2(s_\epsilon, y_\epsilon))\\ &= -\omega + \tfrac{2(t_\epsilon - s_\epsilon)}{\epsilon^2} + H\left(y_\epsilon, \tfrac{2(x_\epsilon - y_\epsilon)}{\epsilon^2} - 2\epsilon y_\epsilon\right) \ \ge \ 0\end{aligned}$$

with

$$\begin{aligned}\varphi_2(t,x) \ \overset{\text{def}}{=} \ & V_1(t_\epsilon, x_\epsilon) - \omega(t_\epsilon + s)\\ & - \epsilon(\|x_\epsilon\|^2 - \|y\|^2) + \frac{|t_\epsilon - s|^2 + \|x_\epsilon - y\|^2}{\epsilon^2} \ .\end{aligned}$$

Uniqueness of Viscosity Solutions

Proof (continued...)

- By subtracting the latter inequalities and using Lipschitz continuity:

$$
\begin{aligned}
0 &< \omega \\
&= \frac{1}{2}H\left(y_\epsilon, \frac{2(x_\epsilon - y_\epsilon)}{\epsilon^2} - 2\epsilon y_\epsilon\right) - \frac{1}{2}H\left(x_\epsilon, \frac{2(x_\epsilon - y_\epsilon)}{\epsilon^2} + 2\epsilon x_\epsilon\right) \\
&\leq C\epsilon(\|x_\epsilon\| + \|y_\epsilon\|) + C\|x_\epsilon - y_\epsilon\|\left(1 + \frac{\|x_\epsilon - y_\epsilon\|}{\epsilon^2} + \epsilon(\|x_\epsilon\| + \|y_\epsilon\|)\right).
\end{aligned}
$$

- If we show that the latter term converges to 0 for $\epsilon \to 0$, we obtain the desired contradiction. Here, we need the boundedness assumption on V_1 and V_2: there exists $M < \infty$ such that

$$
\|x_\epsilon\| \leq \frac{M}{\sqrt{\epsilon}}, \quad \|y_\epsilon\| \leq \frac{M}{\sqrt{\epsilon}}, \quad \|t_\epsilon - s_\epsilon\| \leq M\epsilon, \quad \text{and} \quad \|x_\epsilon - y_\epsilon\| \leq M\epsilon \; .
$$

Uniqueness of Viscosity Solutions

Proof (continued...)

- In summary, the term $\epsilon(\|x_\epsilon\| + \|y_\epsilon\|) \leq 2M\sqrt{\epsilon}$ goes to zero for $\epsilon \to 0$.
- Moreover, since $\Phi_\epsilon(t_\epsilon, x_\epsilon; s_\epsilon, y_\epsilon) \geq \Phi_\epsilon(t_\epsilon, x_\epsilon; t_\epsilon, x_\epsilon)$, we have

$$
\begin{aligned}
& V_1(t_\epsilon, x_\epsilon) - V_2(s_\epsilon, y_\epsilon) - \omega(t_\epsilon + s_\epsilon) - \epsilon(\|x_\epsilon\|^2 + \|y_\epsilon\|^2) - \tfrac{|t_\epsilon - s_\epsilon|^2 + \|x_\epsilon - y_\epsilon\|^2}{\epsilon^2} \\
\geq \;& V_1(t_\epsilon, x_\epsilon) - V_2(t_\epsilon, x_\epsilon) - 2\omega t_\epsilon - 2\epsilon\|x_\epsilon\|^2 \\
\Longrightarrow \quad & \tfrac{|t_\epsilon - s_\epsilon|^2 + \|x_\epsilon - y_\epsilon\|^2}{\epsilon^2} \leq V_2(t_\epsilon, x_\epsilon) - V_2(s_\epsilon, y_\epsilon) \\
& \qquad\qquad + \omega(t_\epsilon - s_\epsilon) + \epsilon(\|x_\epsilon\|^2 - \|y_\epsilon\|^2) \; .
\end{aligned}
$$

  Due to continuity, the right hand goes to 0 for $\epsilon \to 0$. By collecting all the above estimates, we complete our indirect proof.

Contents

End-Cost Problems

- Before we use the above viscosity solution analysis for optimal control, let us first attempt to understand the simplified optimal control problem

$$\min_{x,u} \quad m(x(T))$$

$$\text{s.t.} \quad \left\{ \begin{array}{rcll} \dot{x}(t) & = & f(x(t), u(t))\,, & \text{a.a. } t \in [0,T], \\ u(t) & \in & \mathbb{U} & \text{f.a. } t \in [0,T], \\ x(0) & = & x_0 & \end{array} \right.$$

which has "only" a Mayer term $m \in C^1(\mathbb{R}^n)$ while $x_0 \in \mathbb{R}^n$ is given.

- For simplicity, we additionally assume that $f \in C^1(\mathbb{R}^n \times \mathbb{U})$, while $U \subseteq \mathbb{R}^{n_u}$ is a closed set.

Needle Variations

- Let us assume that u is an optimal control problem input that happens to be continuous at a time $s \in (0,T]$.
- This means that u yields an objective value that is at least as small as the one that we get with the input

$$u_\epsilon(t,v) \overset{\text{def}}{=} \begin{cases} v & \text{if } \ t \in [s-\epsilon, s] \\ u(t) & \text{otherwise .} \end{cases}$$

- The function u_ϵ is called a "needle variation" of the input u.
- Let $\xi_\epsilon(t,v)$ denote the parametric solution of the ODE

$$\dot{\xi}_\epsilon(t,v) \ = \ f(\xi_\epsilon(t,v), u_\epsilon(t,v)) \qquad \text{with} \qquad \xi_\epsilon(0,v) = x_0 \ .$$

First Order Optimality Conditions

- Next, we define the first order variation

$$a(t) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \frac{\xi_\epsilon(t,v) - x(t)}{\epsilon}$$

- As long as u is continuous at $t = s$, we have

$$\begin{array}{rrcl} & a(s) & = & f(x(s),v) - f(x(s),u(s)) \\ \text{and} \quad \text{a.a.}\, t \in [s,T], & \dot{a}(t) & = & A(t)a(t)\ , \end{array}$$

  where $A(t) \stackrel{\text{def}}{=} D_x f(x(t),u(t))$ denotes the Jacobian of f.

- And, since we assume that u is optimal, $m(\xi_\epsilon(T,v)) \leq m(x(T))$,

$$\nabla m(x(T))^\mathsf{T} a(T) \ \geq \ 0\ .$$

Forward versus Backward Differentiation

Language and Notation

- The vector-valued function a satisfies $a(t) = X(t,s)a(s)$, where X is the solution of the matrix differential equation

$$\text{a.a. } t \in [0,T], \quad \frac{\mathrm{d}}{\mathrm{d}t} X(t,s) \; = \; A(t)X(t,s) \qquad \text{with} \qquad X(s,s) = I \; .$$

- The matrix $X(t,s) = \frac{\partial x(t)}{\partial x(s)}$ is often called the first order variation of the given ODE (sometimes also called “Wronski matrix”).
- Since a is propagated forward in time (from s to T), it can be interpreted as a directional forward derivative.
- In this context, the vector $a(s)$ is called the “seed” vector.

Forward versus Backward Differentiation

Language and Notation

- Notice that the optimality condition has the form

$$0 \;\leq\; \nabla m(x(T))^\intercal a(T) \;=\; \nabla m(x(T))^\intercal X(T,s) a(s) \;=\; b(s)^\intercal a(s),$$

  where we define

$$b(s) \;\overset{\text{def}}{=}\; X(t,s)^\intercal b(T) \qquad \text{and} \qquad b(T) \;=\; \nabla m(x(T))$$

- The vector-valued function b can be interpreted as a directional backward derivative, because it satisfies

$$\forall t \in [s,T], \qquad -\dot{b}(t) \;=\; A(t)^\intercal b(t) \; .$$

- In this context, $b(T)$ is called the backward seed.

Forward versus Backward Differentiation

- The equivalence of backward and forward first order optimality conditions alternatively follows by verifying $b(t)^\intercal a(t)$ is invariant:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} b(t)^\intercal a(t) &= b(t)^\intercal \dot{a}(t) + a(t)^\intercal \dot{b}(t) \\ &= b(t)^\intercal \left[A(t) - A(t) \right] a(t) \; = \; 0 \,. \end{aligned} \tag{1}$$

- The first order optimality condition then takes the form

$$b(s)^\intercal a(s) \; = \; b(T)^\intercal a(T) \; \geq \; 0 \,.$$

- Key point: one "backward sweep" is enough to collect the optimality conditions for all $s \in (0, T]$. This is a big advantage compared to the forward differentiation viewpoint.

Forward versus Backward Differentiation

- In summary, if (x, u) is optimal and u continuous at s, then we must have

$$b(s)^\mathsf{T} a(s) \; = \; b(s)^\mathsf{T} (f(x(s), v) - f(x(s), u(s))) \; \leq \; 0$$

for all admissible variations $v \in \mathbb{R}^n$, or, equivalently,

$$b(s)^\mathsf{T} \dot{x}(s) \; = \; b(s)^\mathsf{T} f(x(s), u(s)) \; = \; \min_{v \in \mathbb{U}} \; b(s)^\mathsf{T} f(x(s), v) \; .$$

- It turns out that if there exists a Lebesgue integrable optimal control input, then the above condition holds at all Lebesgue points of u and, consequently, almost everywhere.

Pontryagin's Principle

- Let us introduce a shorthand for the parametric minimizer,

$$u^\star(x,\lambda) \overset{\text{def}}{=} \underset{u \in \mathbb{U}}{\operatorname{argmin}} \; \lambda^\mathsf{T} f(x,u) \; .$$

- Pontryagin's first order optimality conditions for a Lebesgue integrable optimal control input can then be summarized in the form of a boundary value problem (after renaming $\lambda \leftarrow b$),

$$\begin{aligned} \dot{x} &= f(x, u^\star(x,\lambda)) \qquad &, \qquad x(0) = x_0 \; , \\ -\dot{\lambda} &= -D_x f(x, u^\star(x,\lambda))^\mathsf{T} \lambda \qquad &, \qquad \lambda(T) = \nabla m(x(T)) \; . \end{aligned}$$

- This is called Pontryagin's Minimum Principle (or Maximum Principle if we replace $m \leftarrow -m$ as Pontryagin did...)

Pontryagin's Principle

Remarks

- Pontryagin's boundary value problem can be used as a starting point for developing numerical algorithms: if we discretize the differential equation, we need to solve nonlinear equations (e.g., with a Newton or Newton-type method). This is called an "Indirect Method" for optimal control, which is, however, often numerically ill-conditioned. (We'll discuss later how to fix this.)
- Pontryagin's boundary value problem corresponds to a necessary but not to a sufficient condition for optimality (see next slide).

Pontryagin's Principle

Example 1

- Consider the non-convex optimal control problem

$$\min_{x,u} \; x_2(T) \quad \text{s.t.} \quad \begin{cases} \forall t \in [0,T], \\ \dot{x}_1(t) = u(t), \\ \dot{x}_2(t) = -x_1(t)^2, \\ x_1(0) = x_2(0) = 0, \\ u(t) \in [-1,1] \, . \end{cases}$$

The functions $x_1(t) = x_2(t) = u(t) = 0$ and $\lambda(t) = (0,1)^\mathsf{T}$ satisfy Pontryagin's boundary value problem, but correspond to a local maximum. (Exercise: use the PMP to work out all optimal solutions!)

Pontryagin's Principle

Lagrange Terms (= "Running Cost")

- If we have an optimal control problem with Lagrange term,

$$\min_{x,u} \int_0^T \ell(x(t),u(t))\,\mathrm{d}t + m(x(T)) \quad \text{s.t.} \quad \begin{cases} \forall t \in [0,T], \\ \dot{x}(t) = f(x(t),u(t)), \\ x(0) = x_0, \\ u(t) \in \mathbb{U}\ , \end{cases}$$

we can reformulate the problem by introducing an auxiliary state,

$$\widetilde{x}(t) \overset{\text{def}}{=} \begin{pmatrix} x(t) \\ x_{n+1}(t) \end{pmatrix}, \qquad \begin{cases} \dot{x}_{n+1}(t) & = & \ell(x(t),u(t)) \\ x_{n+1}(0) & = & 0\ . \end{cases}$$

Pontryagin's Principle

Lagrange Terms (continued)

- The corresponding auxiliary multiplier $\dot{\lambda}_{n+1} = 0$ and $\lambda_{n+1}(T) = 1$.
- Thus, the parametric minimizer $u^\star$ is given by

$$u^\star(x,\lambda) \overset{\text{def}}{=} \underset{u \in \mathbb{U}}{\operatorname{argmin}}\ \ell(x,u) + \lambda^\mathsf{T} f(x,u) \ = \ \underset{u \in \mathbb{U}}{\operatorname{argmin}}\ \widetilde{H}(x,u,\lambda) \ .$$

- We also recall our definition $H(x,\lambda) \overset{\text{def}}{=} \min_{u \in \mathbb{U}} \widetilde{H}(x,u,\lambda)$.
- Pontryagin's boundary value problem then has the form

$$\begin{aligned} \dot{x} &= \nabla_\lambda H(x,\lambda) \quad , \quad & x(0) &= x_0 \ , \\ \dot{\lambda} &= -\nabla_x H(x,\lambda) \quad , \quad & \lambda(T) &= \nabla m(x(T)) \ . \end{aligned}$$

Pontryagin's Principle

Advanced Topic: Hamiltonian Systems

- The Pontryagin differential equation can also be written in the form

$$\dot{z} \;=\; F(z) \;\stackrel{\mathrm{def}}{=}\; \Omega\, \nabla H(z) \qquad \text{with} \qquad \Omega \;\stackrel{\mathrm{def}}{=}\; \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

  with the stacked state $z \stackrel{\mathrm{def}}{=} (x^\intercal, \lambda^\intercal)^\intercal$.

- Differential equations of this form are called Hamiltonian systems.

- The Jacobian F' of F has a very special structure, because

$$\Omega F' \;=\; -(F')^\intercal \Omega \qquad \text{and, in particular,} \qquad \mathrm{div}(F) = 0\;.$$

- The state z evolves on a so-called symplectic manifold.

Pontryagin's Principle

Example 2

- Consider the Linear Quadratic Regulator (LQR) problem

$$\min_{x,u} \int_0^T \frac{1}{2}(x^\intercal Q x + u^\intercal R u)\,\mathrm{d}t \qquad \text{s.t.} \qquad \left\{ \begin{array}{l} \dot{x} = Ax + Bu, \\ x(0) = x_0 \,. \end{array} \right.$$

- The Hamiltonian function is given by

$$\begin{aligned} H(x,\lambda) &= \min_u \ \frac{1}{2}x^\intercal Q x + \frac{1}{2}u^\intercal R u + \lambda^\intercal A x + \lambda^\intercal B u \\ &= \frac{1}{2}x^\intercal Q x + \lambda^\intercal A x - \frac{1}{2}\lambda^\intercal B R^{-1} B^\intercal \lambda \end{aligned}$$

- Thus, the Pontryagin differential equation becomes

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^\intercal \\ -Q & -A^\intercal \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} \quad \text{with} \quad \left\{ \begin{array}{l} x(0) = x_0, \\ \lambda(T) = 0 \,. \end{array} \right.$$

Pontryagin's Principle

Example 2 (continued)

- The LQR problem can be solved by working out the matrix exponential

$$X(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{pmatrix} \stackrel{\mathrm{def}}{=} \exp\left(\begin{pmatrix} A & -BR^{-1}B^{\mathsf{T}} \\ -Q & -A^{\mathsf{T}} \end{pmatrix} t\right)$$

  and compute $\lambda_0 = -X_{22}(T)^{-1}X_{21}(T)x_0$.

- The optimal state x, co-state λ, and control u are given by

$$\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = X(t)\begin{pmatrix} x_0 \\ \lambda_0 \end{pmatrix} \qquad \text{and} \qquad u(t) = -R^{-1}B^{\mathsf{T}}\lambda(t)\,.$$

Pontryagin's Principle

State Constraints

- Pontryagin's maximum principle can be extended for systems with state constraints.
- In order to understand the main idea, it is helpful to first study the case that we have terminal equality constraints,

$$\min_{x,u} \; m(x(T)) \quad \text{s.t.} \quad \begin{cases} \text{a.a. } t \in [0,T], \\ \forall i \in \{1,\ldots,n_h\}, \\ \dot{x}(t) = f(x(t),u(t)), \\ x(0) = x_0, \; u(t) \in \mathbb{U}, \\ h_i(x(T)) = 0\,. \end{cases}$$

Pontryagin's Principle

Terminal Set

- The associated terminal set,

$$S \stackrel{\mathrm{def}}{=} \{ \, x \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, n_h\}, \quad h_i(x) = 0 \, \}$$

  can be interpreted as a manifold in $\mathbb{R}^n$.
- In general, even if the h_is are smooth, S may be non-smooth.
- Nevertheless, in many practical problems the gradients

$$\nabla m(x_T^\star), \ \nabla h_1(x_T^\star), \ \nabla h_2(x_T^\star), \ \ldots, \ \nabla h_{n_h}(x_T^\star),$$

  turn out to be linearly independent at the optimal terminal point $x_T^\star$.
- If so, we say that the (generalized) linear independence constraint qualification (LICQ) is satisfied at the optimal solution.

Pontryagin's Principle

Terminal Set

- Let's assume that m and the h_is are smooth and that the (generalized) LICQ conditions holds at a given point $x^\star$.
- In this case, the tangent space of S at $x^\star$ is given by

$$\mathcal{T}_S \;=\; \left\{\, a \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, n_h\}, \quad \nabla h_i(x^\star)^\mathsf{T} a = 0 \,\right\} \;.$$

- Moreover, the tangent cone of the set

$$S^- \;=\; \left\{\, x \in S \mid m(x) \leq m(x^\star) \,\right\}$$

is given by

$$\mathcal{T}_{S^-} \;=\; \left\{ a \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \forall i \in \{1, \ldots, n_h\}, \quad \nabla h_i(x^\star)^\mathsf{T} a \;=\; 0 \\ \text{and} \quad \nabla m(x^\star)^\mathsf{T} a \;\leq\; 0 \end{array} \right\} \;.$$

Pontryagin's Principle

Lemma

- Under the above mentioned assumptions, a vector $\lambda \in \mathbb{R}^n$ satisfies $\lambda^\mathsf{T} a \geq 0$ for all $a \in \mathcal{T}_{S^-}$ if and only if there exist $\kappa_0 \leq 0$, $\kappa_i \in \mathbb{R}$, with

$$\lambda \; = \; \kappa_0 \nabla m(x^\star) + \sum_{i=1}^{n_h} \kappa_i \nabla h_i(x^\star) \; .$$

Remark

- In convex optimization, the above lemma is known as Farkas' lemma. It also holds in general (without needing LICQ conditions), but on the next slide we only quickly discuss a simplified proof under the assumption that LICQ holds.

Pontryagin's Principle

Proof

- Introduce the shorthands $w_0 \stackrel{\text{def}}{=} \nabla m(x^\star)$ and $w_i \stackrel{\text{def}}{=} \nabla h_i(x^\star)$.
- If LICQ holds, we can augment these vectors to find a basis

$$w_0, w_1, \ldots, w_{n_h}, w_{n_h+1}, \ldots, w_{n-1}$$

  of $\mathbb{R}^n$. Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}^n$ be an associated dual basis with

$$a_i^\mathsf{T} w_j = \delta_{i,j} \ .$$

- Next, we have $a \in \mathcal{T}_{S^-}$ if and only if

$$a = c_0 a_0 + \sum_{i=n_h+1}^{n-1} c_i a_i$$

  for some coefficients $c_0 \leq 0$, $c_i \in \mathbb{R}$.

Pontryagin's Principle

Proof (continued...)

- By construction, any vector $\lambda \in \mathbb{R}^n$ can be written as

$$\lambda = \kappa_0 w_0 + \sum_{i=1}^{n_h} \kappa_i w_i + \sum_{i=n_h+1}^{n-1} \kappa_i w_i \ .$$

- Thus, for $v \in \mathcal{T}_{S^-}$, we find

$$\lambda^\mathsf{T} a \ = \ \kappa_0 c_0 + \sum_{i=n_h+1}^{n-1} \kappa_i a_i \ .$$

- Thus, $\lambda^\mathsf{T} a \geq 0$ for all $a \in \mathcal{T}_{S^-}$ if and only if $\kappa_0 \leq 0$ and $\kappa_i = 0$ for all $i \geq n_h + 1$. This yields the statement of the lemma.

Pontryagin's Principle

- Let $(x^\star, u^\star)$ be an optimal solution for the state and control of the optimal control problem with terminal equality.
- Assume that LICQ holds at $x^\star(T)$.
- Next, let $a^{s,v}$ be a tangent vector that is obtained by a needle variation of $u^\star$ at time τ with pertubation $v \in \mathbb{U}$ recalling that

$$\dot{a}^{s,v}(t) = A(t) a^{s,v}(t) \; , \quad a^{s,v}(s) = f(x^\star(s), v) - f(x^\star(s), u^\star(s)) \; .$$

- The corresponding cone Γ of feasible pertubation directions is then given by

$$\Gamma \overset{\text{def}}{=} \{ \, a^{s,v}(t) \mid v \in \mathbb{U}, \; s \in (0, T] \, \} \; .$$

Pontryagin's Principle

- Since we assume that $(x^\star, u^\star)$ is optimal the cones Γ and $\mathcal{T}_{S^-}$ must be weakly separated. This means, that there exists a non-zero vector $\lambda(T)$ with

$$\begin{aligned} \lambda(T)^\intercal a &\leq 0 \quad \text{for all} \quad a \in \mathcal{T}_{S^-} \\ \lambda(T)^\intercal a &\geq 0 \quad \text{for all} \quad a \in \Gamma \end{aligned}$$

- The above condition on the existence of a "separating hyperplane" is sometimes called the *geometric version* of Pontryagin's principle.
- An associated analytic version is obtained by using again that

$$\lambda(T)^\intercal a^{s,v}(T) \geq 0 \qquad \Longleftrightarrow \qquad \lambda(s)^\intercal a^{s,v}(s) \geq 0$$

as long as λ satisfies its associated adjoint variational equation.

Pontryagin's Principle

- In detail, we need to enforce

$$0 \;\leq\; \lambda(s)^\intercal a^{s,v}(s) \;=\; \lambda(s)^\intercal \left(f(x^\star(s), v) - f(x^\star(s), u^\star(s))\right)$$

  for all $v \in \mathbb{U}$ and all $s \in (0, T]$.

- Consequently, a necessary condition for optimality is that there exists a non-zero $\lambda : [0, T] \to \mathbb{R}^n$ and coefficients $\kappa_0 \leq 0$, $\kappa_i \in \mathbb{R}$ such that

$$\begin{aligned}
\dot{x}^\star &= \nabla_\lambda H(x^\star, \lambda^\star), \quad x(0) = x_0 \\
\dot{\lambda} &= -\nabla_x H(x^\star, \lambda^\star), \quad \lambda(T) = \kappa_0 \nabla m(x^\star(T)) + \sum_{i=1}^{n_h} \kappa_i \nabla h_i(x^\star(T))
\end{aligned}$$

  with $H(x, \lambda) = \min_{u \in \mathbb{U}} \lambda^\intercal f(x, u)$. This is the analytic version of Pontryagin's principle.

Pontryagin's Principle

Remarks

- Pontryagin's principle has many variants.
- For instance, the above derivation can be generalized easily for terminal inequality constraints of the form $h_i(x(T)) \leq 0$. (Exercise)
- The above derivation assumes that the vectors $\nabla m(x^\star(T))$ and $\nabla h_i(x^\star(T))$ are all linearly independent. This assumption may seem odd on the first view: if we have $n_h = n$ terminal constraints, we would need $n+1$ linearly independent vectors in $\mathbb{R}^n$, which is impossible. However, this paradox is easy to resolve: we may assume that the last state merely integrates the Lagrange term is, consequently, unconstrained. Thus, we may indeed assume $n_h \leq n-1$ without loss of generality.

Pontryagin's Principle

Euler-Lagrange Equation

- An important special case of Pontryagin's principle is obtained for the optimal control problem

$$\min_{x,u} \int_0^T \ell(x,u)\,\mathrm{d}t \qquad \text{s.t.} \qquad \left\{ \begin{array}{l} \dot{x} = u, \\ x(0) = x_0, \quad x(T) = x_T \ . \end{array} \right.$$

- In this case, the Hamiltonian function has the form

$$H(x,u) \;=\; \min_u \; \ell(x,u) + \lambda u \ .$$

- If ℓ is smooth, Pontryagin's differential equation for λ yields

$$\begin{array}{rrcl} & -\dot{\lambda} & = & \nabla_x H(x,u) \;=\; \dfrac{\partial}{\partial x}\ell(x,\dot{x}), \\ \text{where} & \lambda & = & -\nabla_u \, \ell(x,u^\star(x,\lambda)) \;=\; -\dfrac{\partial}{\partial \dot{x}}\,\ell(x,\dot{x}) \ . \end{array}$$

Pontryagin's Principle

Euler-Lagrange Equation

- By substituting these equations, Pontryagin's differential equation can be simplified a bit, finding

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\,\ell(x,\dot{x}) = \frac{\partial}{\partial x}\ell(x,\dot{x}) \; .$$

- This equation is called the Euler-Lagrange equation. In out case, it must be solved together with the boundary conditions

$$x(0) \;=\; x_0 \quad \text{and} \quad x(T) = x_T \; .$$

- The Euler-Lagrange equation has the advantage that we managed to eliminate the co-state λ explicitly. But it has the disadvantage that it is, in general, a second order differential equation.

Pontryagin's Principle

Historical Example (Bernoulli 1696)

- Find a path from a given point A to a given point B such that an object slides (without friction) along this path in minimum time.
- This problem is called the *Brachistochrone Problem*.
- If we denote the path by $y(z)$, then the velocity $\dot{s}(t)$ of the object (with mass m) satisfies the kinetic = potential energy balance,

$$\frac{1}{2} m \dot{s}(t)^2 = m g y(z(t)) \ .$$

- The points $y(0) = 0$ and $y(a) = b$ are given.
- The total length of the path from $(0,0)$ to $(z, y(z))$ is given by

$$s(z) \ = \ \int_0^z \sqrt{1 + y'(z')} \, \mathrm{d}z' \ .$$

Pontryagin's Principle

Historical Example (continued...)

- We change variables from t to s to z to work out the total time,

$$T \;=\; \int_0^T \mathrm{d}t \;=\; \int_0^{s(a)} \frac{1}{v(s)} \mathrm{d}s \;=\; \int_0^a \frac{\sqrt{1+y'(z)}}{\sqrt{2gy(z)}}\, \mathrm{d}z$$

- By switching back to optimal control notation, this means that we want to minimize the stage cost

$$\ell(x,\dot{x}) = \sqrt{\frac{1+\dot{x}^2}{2gx}} \qquad \text{s.t.} \qquad x(0)=0 \quad \text{and} \quad x(T)=b\;,$$

where have matched the variable names $x \leftarrow y$, $t \leftarrow z$, and $T \leftarrow a$.

Pontryagin's Principle

Historical Example (continued...)

- The corresponding Euler-Lagrange equation has the form

$$-\frac{1}{2}\sqrt{\frac{1+\dot{x}^2}{x^3}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{x}}{\sqrt{x(1+\dot{x}^2)}} \quad \text{with} \quad \left\{ \begin{array}{l} x(0) = 0 \\ x(T) = b \, . \end{array} \right.$$

- After some lengthly computations this differential equation can be solved explicitly (Exercise).
- In practice, such explicit solutions are, however, often useless, as we also want to model other effects like friction or other forces that act on the object. In this case, one needs to solve the problem numerically (we'll discuss this later on in the course).

Contents

HJB Equation: Back to Square One

- In the introduction of this lecture, we have derived the Hamilton-Jacobi Bellman (HJB) equation in a sloppy (but hopefully intuitive) way— under rather questionable assumptions.
- The goal of the following slides is to make this derivation mathematically precise, without making dubious assumptions.
- In particular, we want to understand why the concept of viscosity solution is so very important in the context of HJBs.
- Moreover, we also wish to understand in detail what the relation between the HJB equation and Pontryagin's principle are.

Lipschitz-Continuous Value Functions

- The following slides concern the optimal control problem

$$V(t_0, y) \;=\; \inf_{x,u} \int_{t_0}^{T} \ell(x(\tau), u(\tau))\,\mathrm{d}\tau + m(x(T))$$

$$\text{s.t.} \quad \left\{ \begin{array}{rcll} \dot{x}(t) & = & f(x(t), u(t)) & \text{a.a. } t \in [t_0, T], \\ x(t_0) & = & y, & \\ u(t) & \in & \mathbb{U} & \text{a.a. } t \in [t_0, T] \,. \end{array} \right.$$

- We assume that f, ℓ, and m are bounded and Lipschitz continuous,
- that the set $\mathbb{U} \subset\subset \mathbb{R}^{n_u}$ is non-empty and compact, and that
- the controls $u : [0, T] \to \mathbb{R}^{n_u}$ are Lebesgue measurable.
- Note: we are writing "$\inf$" as a minimizer might not exist.

Lipschitz-Continuous Value Functions

Theorem

- If the assumptions from the previous slide are satisfied, then V is bounded and Lipschitz-continuous. That is, there exists a constant $C < \infty$ such that

$$\begin{array}{rrcl} & V(s,y) & \leq & C \\ \text{and} & |V(s,y) - V(s',y')| & \leq & C\left(|s-s'| + \|y-y'\|\right) \end{array}$$

for all $s, s' \in [0,T]$ and all $y, y' \in \mathbb{R}^n$.

Proof

- Let $(s', y') \in [0,T] \times \mathbb{R}^n$ and $\epsilon > 0$ be given. (see next slide...)

Lipschitz-Continuous Value Functions

Proof (continued...)

- We use the notation $\xi(t;s,y,u)$ to denote the parametric solution of the ODE

$$\dot{\xi} = f(\xi,u) \quad \text{with} \quad \xi(s;s,y,u) = y \; .$$

Moreover, we define

$$J(s,y,u) \;\overset{\text{def}}{=}\; \int_{t_0}^{T} \ell(\xi(\tau;s,y,u),u(\tau))\,\mathrm{d}\tau + m(\xi(T;s,y,u)) \; .$$

- Next, we choose a measurable control $u_\epsilon : [0,T] \to \mathbb{U}$ such that

$$J(s',y',u_\epsilon) \;\leq\; V(s',y') + \epsilon \; .$$

Lipschitz-Continuous Value Functions

Proof (continued...)

- Due to Gronwall's lemma, there exists a constant $C_1 < \infty$ such that

$$\|\xi(t; s, y, u_\epsilon) - \xi(t; s', y', u_\epsilon)\| \;\leq\; C_1 \left(|s - s'| + \|y - y'\|\right)$$

  for all $t \in [0, T]$ and all $(s, y) \in [0, T] \times \mathbb{R}^n$.

- Thus, by exploiting the Lipschitz continuity of ℓ and m, we also have

$$J(s, y, u_\epsilon) \;\leq\; J(s', y', u_\epsilon) + C_2 \left(|s - s'| + \|y - y'\|\right)$$

  for some constant $C_2 < \infty$.

- Consequently, in summary, we find

$$V(s, y) \;\leq\; J(s, y, u_\epsilon) \;\leq\; V(s', y') + \epsilon + C_2 \left(|s - s'| + \|y - y'\|\right) \; .$$

Lipschitz-Continuous Value Functions

Proof (continued...)

- By taking the limit for $\epsilon \to 0$, we find that

$$V(s,y) \;\leq\; V(s',y') + C_2\left(|s-s'| + \|y-y'\|\right) \;.$$

- By interchanging the roles of (s,y) and (s',v'), we find that V is indeed Lipschitz continuous,

$$|V(s,y) - V(s',y')| \;\leq\; C_2\left(|s-s'| + \|y-y'\|\right) \;.$$

- Similarly, boundedness of V follows from boundedness of ℓ and m.

Bellman's Principle of Optimality (Revisited)

Theorem

- The function V satisfies the dynamic programming equation

$$V(s,y) \; = \; \inf_u \int_s^\tau \ell(\xi(t;s,y,u),u(t))\,\mathrm{d}t + V(\tau,\xi(\tau;s,y,u))$$

  for all intermediate times $\tau \in [s,T]$ and all $(s,y) \in [0,T) \times \mathbb{R}^n$.

Proof (sketch only)

- The argument is the same as for the case that a minimizer exist, but here we need to concatenate an ϵ-suboptimal control $u_{\epsilon,1}$ on the interval $[s,\tau]$ with an ϵ-suboptimal control $u_{\epsilon,2}$ on the interval $(\tau,T]$ to compose an 2ϵ-suboptimal input on the interval $[s,T]$. The proof follows then after passing to the limit $\epsilon \to 0$.

Optimal Control Inputs

- Let us introduce the auxiliary function

$$\Phi(t; s, y, u) \stackrel{\text{def}}{=} \inf_u \int_s^t \ell(\xi(\tau; s, y, u), u(\tau)) \, \mathrm{d}\tau + V(t, \xi(t; s, y, u)) \; .$$

- Due the principle of optimality, we have

$$\Phi(t_1, s, y, u) \; \leq \; \Phi(t_2, s, y, u)$$

  for all $s \leq t_1 \leq t_2 \leq T$ and any given $y \in \mathbb{R}^n$ and $u \in \mathcal{U}$.

- In particular, u is optimal for the given initial value y at time s if and only if $\Phi(\cdot, s, u, y)$ is constant on the interval $[s, T]$.

Optimal Control Inputs

- If we assume for a moment that $u \in \mathcal{U}$ is an optimal control input for a given state $y \in \mathbb{R}^n$ at time s, if u is continuous at s and V differentiable at (s, y), then we must have

$$\begin{aligned} 0 &= \left. \frac{\mathrm{d}}{\mathrm{d}t} \Phi(t; s, y, u) \right|_{t=s} \\ &= \ell(y, u(s)) + \dot{V}(s, y) + \nabla V(s, y)^{\mathsf{T}} f(y, u(s)) \; . \end{aligned}$$

- Similarly, since Φ is non-decreasing for any constant input $u(t) = v$,

$$0 \leq \ell(y, v) + \dot{V}(s, y) + \nabla V(s, y)^{\mathsf{T}} f(y, v) \; .$$

for all $v \in \mathbb{U}$.

Optimal Control Inputs

- The two relations from the previous slide imply that

$$\begin{array}{lrcl} & 0 & = & \min_{v \in \mathbb{U}} \left\{ \ell(y,v) + \dot{V}(s,y) + \nabla V(s,y)^\intercal f(y,v) \right\} \\ \text{and} & u(s) & \in & \operatorname*{argmin}_{v \in \mathbb{U}} \left\{ \ell(y,v) + \dot{V}(s,y) + \nabla V(s,y)^\intercal f(y,v) \right\} \ . \end{array}$$

- Recalling our definition

$$H(y,\lambda) \stackrel{\text{def}}{=} \min_{u \in \mathbb{U}} \left\{ \ell(y,u) + \lambda^\intercal f(y,u) \right\}$$

we find that V satisfies the PDE

$$\dot{V}(s,y) + H(y, \nabla V(s,y)) \ = \ 0$$

at all points (s,y) at which V is differentiable.

Viscosity Solution of HJB

- After the above preliminary training of our intuition based on auxiliary assumptions on the differentiability on V and continuity of u, we are ready to prove a theorem, which removes these auxiliary assumptions:

Theorem

- The stage cost V of the above optimal control problem, where f, ℓ, m are Lipschitz continuous and ℓ and m bounded, is the unique viscosity solution of the HJB

$$-\left[\dot{V} + H(y, \nabla V)\right] = 0 \quad \text{with} \quad V(T, y) = m(y) \; .$$

Viscosity Solution of HJB

Proof

- First recall that V is Lipschitz continuous (see previous slides)
- Also, it's clear that we must have $V(T, y) = m(y)$.
- But we need to show that for $\varphi \in C^1(\Omega_T)$ with $\Omega_T \stackrel{\text{def}}{=} (0, T) \times \mathbb{R}^n$:

1. If $V - \varphi$ has a local maximum at $(t_0, x_0) \in \Omega_T$, then

$$\varphi_t(t_0, x_0) + \min_{v \in \mathbb{U}} \{\ell(x_0, v) + \nabla\varphi(t_0, x_0)^\intercal f(x_0, v)\} \ \geq \ 0 \ .$$

2. If $V - \varphi$ has a local minimum at $(t_0, x_0) \in \Omega_T$, then

$$\varphi_t(t_0, x_0) + \min_{v \in \mathbb{U}} \{\ell(x_0, v) + \nabla\varphi(t_0, x_0)^\intercal f(x_0, v)\} \ \leq \ 0 \ .$$

Viscosity Solution of HJB

Proof (continued)

- In order to verify the first inequality, we may assume (for all t, x)

$$V(t_0, x_0) = \varphi(t_0, x_0) \quad \text{and} \quad V(t, x) \le \varphi(t, x)$$

- Next, assume (indirect proof) that there exists $v \in \mathbb{U}$ and $\theta > 0$ with

$$\varphi_t(t_0, x_0) + \ell(x_0, v) + \nabla\varphi(t_0, x_0)^\mathsf{T} f(x_0, v) \; < \; -\theta \; .$$

  In this case, we can a choose u with $u(t) = v$ for $t \in [t_0, t_0 + \delta]$ and $\delta > 0$ sufficiently small such that

$$\varphi_t(t_0, x_0) + \nabla\varphi(t_0, x_0)^\mathsf{T} f(x_0, v) \; < \; -\ell(x_0, v) - \theta \; .$$

  for all t, x with $|t - t_0| < \delta$ and $|x - x_0| < \delta$.

Viscosity Solution of HJB

Proof (continued)

- Let us set $x(t) \stackrel{\text{def}}{=} \xi(t; t_0, x_0, u)$ (at $u(t) = v$ on $[t_0, t_0 + \delta]$). Then,

$$
\begin{aligned}
& V(t_0 + \delta, x(t_0 + \delta)) - V(t_0, x_0) \\
\leq \; & \varphi(t_0 + \delta, x(t_0 + \delta)) - \varphi(t_0, x_0) \\
= \; & \int_{t_0}^{t_0+\delta} \frac{\mathrm{d}}{\mathrm{d}t} \varphi(t, x(t)) \, \mathrm{d}t \\
= \; & \int_{t_0}^{t_0+\delta} \left(\dot{\varphi}(t, x(t)) + \nabla \varphi(t, x(t))^\intercal f(x(t), v) \right) \, \mathrm{d}t \\
\leq \; & - \int_{t_0}^{t_0+\delta} \ell(x(t), v) \, \mathrm{d}t - \delta \theta \; ,
\end{aligned}
$$

  but this contradicts Bellman's principle of optimality! Thus, the first inequality holds.

Viscosity Solution of HJB

Proof (continued)

- In order to verify the second inequality, we may assume (for all t, x)

$$V(t_0, x_0) = \varphi(t_0, x_0) \quad \text{and} \quad V(t, x) \geq \varphi(t, x)$$

- Next, assume that there exists $\theta > 0$ with

$$\varphi_t(t_0, x_0) + \ell(x_0, v) + \nabla\varphi(t_0, x_0)^\mathsf{T} f(x_0, v) \;>\; -\theta \;.$$

  for all $v \in \mathbb{U}$.

- Thus, this way around, we have an inequality of the form

$$\varphi_t(t, x) + \nabla\varphi(t, x)^\mathsf{T} f(x, v) \;>\; \theta - \ell(x, v) \;.$$

  for all t, x with $|t - t_0| < \delta$ and $|x - x_0| < \delta$.

Viscosity Solution of HJB

Proof (continued)

- Thus $x(t) \stackrel{\mathrm{def}}{=} \xi(t; t_0, x_0, u)$ for an arbitrary control $u \in \mathcal{U}$ satisfies

$$
\begin{aligned}
& V(t_0+\delta, x(t_0+\delta)) - V(t_0, x_0) \\
\geq \; & \varphi(t_0+\delta, x(t_0+\delta)) - \varphi(t_0, x_0) \\
= \; & \int_{t_0}^{t_0+\delta} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, x(t))\,\mathrm{d}t \\
= \; & \int_{t_0}^{t_0+\delta} \left(\dot{\varphi}(t, x(t)) + \nabla\varphi(t, x(t))^{\mathsf{T}} f(x(t), v)\right)\,\mathrm{d}t \\
\geq \; & -\int_{t_0}^{t_0+\delta} \ell(x(t), v)\,\mathrm{d}t + \delta\theta \;,
\end{aligned}
$$

but, since this holds for all feasible controls, this yields also a contradiction to Bellman's princple of optimality. QED.

State-Of-The-Art & Open Problems in HJB Theory

- As mentioned earlier, there is—to date—no complete theory for nonlinear PDE; and the same holds true for HJB equations.
- The concept of viscosity solutions seems to be a promising concept: the above theorem looks alright for finite horizon problems.
- There are, however, open problems regarding the existence of optimal controls (partly solved by Filipov under certain convexity assumptions)
- Extensions for the case of HJBs for systems with state constraints became more mature during last two decades (based on work by H. Frankowska and co-workers), but regarding the general nonlinear case many open issues remain.

State-Of-The-Art & Open Problems in HJB Theory

- The infinite horizon case is—to date—not-well understood. Some researchers have attempted to introduce exponential discounts of the stage costs. For this case, one can establish some kind of infinite horizon HJB theory, but passing to the ergodic limit for vanishing discounts seems to fail (at least, based on all that I have seen in the literature so far...).
- There are many extensions of the HJB, for example, for stochastic control, where the viscosity term $\epsilon \Delta V$ models stochastic pertubation of the nonlinear differential equation (a so-called McKean-Vlasov SDE). Modern extensions of HJB theory also include differential game theory and applications in mean field optimal control.