Partial Differential Equations

- Introduction
- Dirichlet's Problem for Laplacian Operator
- Linear Second Order Elliptic PDEs
- Regularity of Solutions
- Second Order Parabolic PDEs

Contents

Introduction to PDE

- General second order partial differential equation (PDE) have the form

$$F(x, u, \nabla u, \nabla^2 u) = 0 \; .$$

- We solve PDEs on a (usually bounded) open set $\Omega \subseteq \mathbb{R}^n$.
- Additionally, there may be conditions on u at the boundary $\partial\Omega$.
- Contrast to ODEs: there is no general theory for nonlinear PDE!!!
- PDE theory is much more difficult than ODE theory.
- Long list of open research problems.
- Later in this course, we will focus on a special class of nonlinear PDE, named, Hamilton Jacobi Bellman equations.

Introduction to PDE

- In order to get started we first focus on linear second order PDEs,

$$Lu = -\sum_{i,j} \partial_i (a_{i,j} \partial_j u) + \sum_i \partial_i (b_i u) + cu = f \ .$$

- We assume that $a_{i,j} = a_{j,i}, b_i, c \in L^\infty(\Omega)$ and $f \in H^{-1}(\Omega)$.
- If there exists a $\theta > 0$ such that

$$\text{a.a. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n, \qquad \sum_{i,j} a_{i,j}(x) \xi_i \xi_j \geq \theta \|x\|_2^2 \ ,$$

  we say that the linear PDE is uniformly elliptic.

- Linear PDEs are generally relatively well-understood.
- Uniformly elliptic linear PDEs are particularly easy to analyze (at least compared to more general PDEs...).

Introduction to PDE

Physical Motivation

- In applications, u is often the density of a quantity (e.g. a chemical concrentration at equilibrium).
- The second order term "$\sum_{i,j} a_{i,j}\partial_i\partial_j u$" often models diffusion.
- The first order term "$\sum_i b_i\partial_i u$ typically models transport.
- And, the term cu models an increase or depletion (e.g. if a chemical reaction is going on...)
- Another example: in electrostatics Poisson's equation, "$\Delta u = f$" describes the potential of an electric field "u" in the dependence on the (scaled) volume charge density "f".

Contents

Dirichlet Problem

- In order to get started, it is helpful to study Dirichlet's PDE

$$-\Delta u = f \qquad \text{with} \qquad u|_{\partial\Omega} = 0$$

  on an open set $\Omega \subseteq \mathbb{R}^n$ that is bounded in some direction.

- Assume for a moment that u, f, and the boundary of Ω are smooth.

- Under this assumption, we can multiply with a test function $\phi \in C_0^\infty(\Omega)$ and integrate to find

$$\int_\Omega \nabla u^\intercal \nabla \phi \, \mathrm{d}x \; = \; \int_\Omega f \phi \, \mathrm{d}x \; .$$

- The boundary term, $\int_{\partial\Omega} \phi \nabla u^\intercal n \, \mathrm{d}S = 0$, vanishes as $\phi = 0$ on $\partial\Omega$.

Dirichlet Problem

- The above condition can also be written in the form

$$\langle u, \phi \rangle_{H_0^1} \stackrel{\text{def}}{=} \int_\Omega Du \cdot D\phi \, \mathrm{d}x \;=\; \int_\Omega f\phi \, \mathrm{d}x \;=\; \Lambda_f(\phi) \,, \qquad (1)$$

  where D denotes the weak gradient operator.

- As $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we may enforce (1) for all $\phi \in H_0^1(\Omega)$.
- The right hand expression is well-defined for any $\Lambda_f \in H^{-1}(\Omega)$.
- Sloppy notation: just write f instead of Λ_f.

Definition

- Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in H^{-1}(\Omega)$. We call $u \in H_0^1(\Omega)$ a weak solution of Dirichlet's PDE if (1) holds for all $\phi \in H_0^1(\Omega)$.

Dirichlet Problem

Theorem

- Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded in at least one direction and let $\Lambda_f \in H^{-1}(\Omega)$ be given. Then there exists a unique weak solution $u \in H_0^1(\Omega)$ to Dirichlet's PDE.

Proof

- Recall that $H_0^1(\Omega)$ with inner product $\langle u, \phi \rangle_{H_0^1}$ is a Hilbert space, which follows from Poincare's inequality.
- The theorem follows now directly from Frechét Riesz' theorem.

Contents

Linear Elliptic PDE

- Introduce the linear second order operator

$$Lu \quad \stackrel{\text{def}}{=} \quad -\sum_{i,j} \partial_i(a_{i,j}\partial_j u) + \sum_i \partial_i(b_i u) + cu$$

- Let $\mu \in \mathbb{R}$ be a parameter. Consider the PDE

$$Lu + \mu u = f \qquad \text{with} \qquad u|_{\partial\Omega} \quad = \quad 0 \; . \tag{2}$$

- We assume $a_{i,j} = a_{j,i}, b_i, c \in L^\infty(\Omega)$ and $f \in H^{-1}(\Omega)$ and

$$\exists \theta > 0 : \quad \text{a.a. } x \in \Omega, \; \forall \xi \in \mathbb{R}^n, \qquad \sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq \theta \|x\|_2^2 \; .$$

Linear Elliptic PDE

- We define the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ as

$$a(u,v) \quad \stackrel{\text{def}}{=} \quad \int_\Omega \left(\sum_{i,j} a_{i,j} \partial_i u \partial_j v - \sum_i b_i u \partial_i v + cuv \right) \, \mathrm{d}x \; .$$

Definition

- Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in H^{-1}(\Omega)$. We call $u \in H_0^1(\Omega)$ a weak solution of (2) if

$$\forall \phi \in H_0^1(\Omega), \qquad a(u,\phi) + \mu \langle u, \phi \rangle_{L^2} = \langle f, \phi \rangle_{L^2} \; .$$

Linear Elliptic PDE

Lemma

- Under the above assumption, we can find constants $C_1, C_2 < \infty$ and $\gamma \in \mathbb{R}$ such that for all $u, v \in H_0^1(\Omega)$

$$\begin{array}{rrcl} & C_1 \|u\|_{H_0^1}^2 & \leq & a(u,u) + \gamma \|u\|_{L^2} \\ \text{and} & |a(u,v)| & \leq & C_2 \|u\|_{H_0^1} \|v\|_{H_0^1} \ . \end{array}$$

Proof

- The triangle inequality + Cauchy Schwarz directly yields

$$a(u,v) \leq \left(\sum_{i,j} \|a_{i,j}\|_{L^\infty} + \sum_i \|b_i\|_{L^\infty} + \|c\|_{L^\infty} \right) \|u\|_{H_0^1} \|v\|_{H_0^1} \ .$$

Linear Elliptic PDE

Proof (continued)

- Next, we derive the estimate

$$\begin{aligned}
\theta \|Du\|_{L^2}^2 &= \theta \int_\Omega |Du|^2 \, \mathrm{d}x \\
&\leq \sum_{i,j} \int_\Omega a_{i,j} \delta_i u \delta_j u \, \mathrm{d}x \\
&\leq a(u,u) + \sum_i \int_\Omega b_i u \partial_i u \, \mathrm{d}x - \int_\Omega c u^2 \, \mathrm{d}x \\
&\leq a(u,u) + \sum_i \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i u\|_{L^2} - c_0 \|u\|_{L^2}^2 \\
&\leq a(u,u) + \beta \|u\|_{L^2} \|Du\|_{L^2} - c_0 \|u\|_{L^2}^2
\end{aligned}$$

with $c_0 \overset{\text{def}}{=} \inf_\Omega c$ and $\beta \overset{\text{def}}{=} \sqrt{\sum_i \|b_i\|_{L^\infty}^2}$.

Linear Elliptic PDE

Proof (continued)

- The above estimate implies

$$\begin{aligned}\frac{\theta}{2}\|u\|_{H_0^1}^2 &\leq a(u,u) + \beta\|u\|_{L^2}\|Du\|_{L^2} - \frac{\theta}{2}\|Du\|_{L_2}^2 + \left(\frac{\theta}{2} - c_0\right)\|u\|_{L^2}^2 \\ &\leq a(u,u) + \left(\frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0\right)\|u\|_{L^2}^2 \; .\end{aligned}$$

- Thus, the statement of the lemma holds with

$$C_1 = \frac{\theta}{2} \qquad \text{and} \qquad \gamma = \frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0 \; .$$

Remark

- Moving the term $\frac{\theta}{2}\|u\|_{L^2}^2$ to the right side in the above estimation feels a bit arbitrary (this is not sharp). On bounded domains, we can also use Poincare's inequality to get smaller values for γ.

Linear Elliptic PDE

Theorem

- Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in H^{-1}(\Omega)$, and let the above assumptions be satisfied. Then the PDE

$$Lu + \mu u = f \qquad \text{with} \qquad u|_{\partial\Omega} \;=\; 0$$

  has a unique solution for all $\mu \geq \gamma$.

Proof

- Let us introduce the bilinear form

$$a_\mu(u,v) \;\stackrel{\text{def}}{=}\; a(u,v) + \mu\langle u,v\rangle_{L^2} \; .$$

Linear Elliptic PDE

Proof (continued)

- The conditions of the Lax Milgram lemma for a_μ are satisfied, as

$$\begin{aligned} |a_\mu(u,v)| &\leq (C_2 + |\mu|)\|u\|_{H_0^1}\|v\|_{H_0^1} \\ a_\mu(u,u) &\geq a(u,u) + \gamma\|u\|_{L^2} \ \geq \ C_1\|u\|^2_{H_0^1} \ . \end{aligned}$$

- Thus, there exists a unique $u \in H_0^1(\Omega)$ such that

$$\forall \phi \in H_0^1(\Omega), \qquad \langle f, \phi \rangle = a_\mu(u,\phi) \ .$$

This is equivalent to the statement of the theorem.

Linear Elliptic PDE

Remark

- The statement of the above theorem does not hold in general without conditions on μ.
- For instance, the one dimensional PDE

$$-u''(x) - u(x) = 0 \qquad \text{with} \qquad u(-\pi) = u(\pi) = 0$$

  has infinitely many solutions, $u(x) = a\sin(x)$, with arbitrary $a \in \mathbb{R}$.
- More generally, the Dirichlet Laplacian has infinitely many eigenvalues.

Advanced Topics

Formal Adjoint

- The bilinear form a is in general not symmetric.
- The adjoint bilinear form is denoted by $a^*(u,v) \stackrel{\text{def}}{=} a(v,u)$.
- It can formally be associated with the adjoint PDE

$$L^* v = f \quad \text{with} \quad v\big|_{\partial\Omega} = 0,$$

  whose weak solutions $v \in H_0^1(\Omega)$ satisfy

$$\forall u \in H_0^1(\Omega), \qquad a^*(v,u) = \langle f, u \rangle_{L^2} \ .$$

- Here, we formally have introduced the adjoint linear operator

$$L^* v \ \stackrel{\text{def}}{=} \ -\sum_{i,j} \partial_i (a_{ij} \partial_j v) - \sum_i b_i \partial_i v + \left(c - \sum_i \partial_i b_i \right) v \ .$$

Advanced Topics

Fredholm Alternative

- The Lax-Milgram theorem only yields a sufficient condition under which we can ensure the existence of a unique solution.
- The analysis of more general elliptic PDEs requires more advanced tools from functional analysis that we did not discuss in this course.
- Nevertheless, we state without proof that either
  1. $Lu = f$ has a unique weak solution $u \in H_0^1(\Omega)$ for all $f \in L^2(\Omega)$, or
  2. there exists a weak solution $0 \neq u \in H_0^1(\Omega)$ of $Lu = 0$.
- This is called *Fredholm Alternative*.

Advanced Topics

Fredholm Alternative

- In the second case, if $Lu = 0$ has a nontrivial solution in $H_0^1(\Omega)$, then the dimension of the subspace $N \subseteq H_0^1(\Omega)$ of solutions is finite and equals the dimension of the subspace $N^* \subseteq H_0^1(\Omega)$ of solutions $v \in H_0^1(\Omega)$ of the adjoint problem $L^* v = 0$.
- Additionally, it turns out that $Lu = f$ has a solution $u \in H_0^1(\Omega)$ if and only if $\langle f, v \rangle_{L^2} = 0$ for all $v \in N^*$.
- The above results can also be used as a starting point for analyzing eigenvalue problems of the form $Lu = \lambda u$ (spectral theory).

Contents

Interior Regularity

Basic Intuition

- "Solutions of elliptic PDEs are as smooth as their data allows"

Motivation

- Consider the following a-priori estimate for a function $u \in C_0^\infty(\Omega)$:

$$
\begin{aligned}
\int_\Omega (\Delta u)^2 \,\mathrm{d}x &= \sum_{i,j=1}^{n} \int_\Omega (\partial_{i,i}^2 u)(\partial_{j,j}^2 u)\,\mathrm{d}x \\
&= -\sum_{i,j=1}^{n} \int_\Omega (\partial_{i,ij}^3 u)(\partial_j^2 u)\,\mathrm{d}x \\
&= \sum_{i,j=1}^{n} \int_\Omega (\partial_{i,j}^2 u)(\partial_{ij}^2 u)\,\mathrm{d}x \;=\; \int_\Omega |D^2 u|^2 \,\mathrm{d}x \,.
\end{aligned}
$$

Interior Regularity

- If our smooth function $u \in C_0^\infty(\Omega)$ satisfies $\Delta u = f$, then

$$\|D^2 u\|_{L^2} = \|f\|_{L^2} \ .$$

- The above equation implies that the second derivative of a smooth solution to $\Delta u = f$ is bounded by the L_2-norm of f.
- **Conjecture:** if $f \in L^2(\Omega)$, then weak solutions of $\Delta u = f$ satisfy $u \in H^2_{\mathrm{loc}}(\Omega)$.
- Our goal is to prove that this conjecture is true.

Interior Regularity

General Problem Formulation

- Focus on uniformly elliptic operators of the form

$$Lu = -\sum_{i,j} \partial_i (a_{i,j} \partial_j u) \quad \text{and define} \quad a(u,v) \stackrel{\text{def}}{=} \sum_{i,j} \int_\Omega a_{i,j} \partial_i u \partial_j v \; .$$

- The techniques below easily generalize for lower order terms, too.
- Recall that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ if

$$\forall v \in H_0^1(\Omega), \qquad a(u,v) = \langle f, v \rangle_{L^2}$$

- Keep things general: no boundary condition on u.

Interior Regularity

Outline of Main Idea (assume for a moment that u is smooth...)

- Choose $\eta \in C_0^\infty(\Omega)$, $0 \le \eta \le 1$ with $\eta = 1$ on $\Omega' \subset\subset \Omega$.
- Introduce the test function $v = -\partial_k(\eta^2 \partial_k u)$.
- By partial integration, we find

$$\begin{aligned}
\langle Lu, v\rangle_{L^2} &= \int_\Omega \sum_{i,j} \partial_i(a_{i,j}\partial_j u)\partial_k(\eta^2\partial_k u))\,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j} \partial_k(a_{i,j}\partial_j u)\partial_i(\eta^2\partial_k u))\,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j} \eta^2 a_{i,j}(\partial_j\partial_k u)(\partial_i\partial_k u)\,\mathrm{d}x + F \\
\text{with}\quad F &= \int_\Omega \sum_{i,j} \big\{\eta^2(\partial_k a_{i,j})(\partial_i u)(\partial_j\partial_k u) \\
&\qquad +2\eta\partial_j\eta\,[a_{i,j}(\partial_i\partial_k u)(\partial_k u) + (\partial_k a_{i,j})(\partial_i u)(\partial_k u)]\big\}
\end{aligned}$$

Interior Regularity

Outline of Main Idea (continued)

- We can use the uniform ellipticity to derive the bound

$$\theta \int_{\Omega'} |D\partial_k u|^2 \, \mathrm{d}x \leq \int_{\Omega} \sum_{i,j} \eta^2 a_{i,j} (\partial_i \partial_k u)(\partial_j \partial_k u) = \langle f, v \rangle_{L^2} - F$$

- Plan: F has terms that are linear terms in second derivatives. They can be “absorbed” by the quadratic terms on the left finding that

$$\|D\partial_k u\|^2_{L^2(\Omega')} \;\leq\; C(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)})$$

  for a constant $C < \infty$.

- BUT: we need to replace derivatives by difference quotients.

Difference Quotients

Motivation

- Assume $\Omega' \subset\subset \Omega$ are open sets.
- We are not so sure whether weak derivatives of $u \in L^2(\Omega')$ exists.
- But we can always choose a h (with $|h|$ sufficiently small) and define

$$D_i^h u \stackrel{\text{def}}{=} \frac{u(x + he_i) - u(x)}{h} \quad \text{and} \quad D^h u \stackrel{\text{def}}{=} (D_1^h u, \ldots, D_n^h u) .$$

- Assume we can find a sufficiently small $\overline{h} > 0$ and $C < \infty$ such that

$$\forall h \in \left[-\overline{h}, \overline{h} \right], \qquad \| D^h u \|_{L^2(\Omega')} \le C .$$

- Recall from Lecture 2: every bounded sequence in $L^2(\Omega')$ has a weakly convergent subsequence.
- By using this result, we can show that $u \in W^{1,2}(\Omega')$ (see next slides).

Difference Quotients

Theorem

- Assume that $u \in L^2(\Omega')$ satisfies $\|D^h u\|_{L^2} \leq C$ for a constant $C < \infty$. Then $u \in W^{1,2}(\Omega')$ and $\|Du\|_{L^2} \leq C$.

Proof

- For any given test function $\phi \in C_0^\infty(\Omega')$, we have

$$\int_{\Omega'} u(x) \left[\frac{\phi(x + he_i) - \phi(x)}{h} \right] \mathrm{d}x = - \int_{\Omega'} \left[\frac{u(x) - u(x - he_i)}{h} \right] \phi(x) \, \mathrm{d}x$$

- The above equation can be also be written in the form

$$\int_{\Omega'} u D_i^h \phi \, \mathrm{d}x \; = \; - \int_{\Omega'} (D_i^{-h} u) \phi \, \mathrm{d}x$$

Difference Quotients

Proof (continued)

- Due to our assumption that $\|D^h u\|_{L^2} \leq C$, there exists a $v_i \in L^2(\Omega')$ and a sequence $h_k \to 0$ with $D_i^{h_k} u \rightharpoonup v_i$, where the symbol "$\rightharpoonup$" indicates weak convergence in $L^2(\Omega')$.
- Consequently, it follows that

$$
\begin{aligned}
\int_{\Omega'} u\phi_{x_i}\,\mathrm{d}x &= \int_{\Omega} u\phi_{x_i}\,\mathrm{d}x \;=\; \lim_{k\to\infty}\int_{\Omega} uD_i^{h_k}\phi\,\mathrm{d}x \\
&= -\lim_{k\to\infty}\int_{\Omega} D_i^{-h_k}u\phi\,\mathrm{d}x \;=\; -\lim_{k\to\infty}\int_{\Omega} v_i\phi\,\mathrm{d}x\ .
\end{aligned}
$$

  Thus, $v_i = D_i u$ is a weak derivative of u and $Du \in L^2(\Omega')$.
- Since $u \in L^2(\Omega')$, we have $u \in W^{1,2}(\Omega')$ and also $\|Du\|_{L^2} \leq C$.

Interior Regularity

- We can use difference quotients to establish an interior regularity result for uniformly elliptic PDE $Lu = f$.

Theorem

- Recall that $\Omega' \subset\subset \Omega$ are open. If $a_{i,j} \in C^1(\Omega)$ and $f \in L^2(\Omega)$, then any weak solution $u \in H^1(\Omega)$ of $Lu = f$ satisfies $u \in H^2(\Omega')$. Moreover, there exists a $C < \infty$ (depending only on n, Ω, Ω' and the coefficients $a_{i,j}$) such that

$$\|u\|_{H^2(\Omega')} \;\leq\; C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \;.$$

Interior Regularity

Proof

- Choose $\eta \in C_0^\infty(\Omega)$, $0 \leq \eta \leq 1$ with $\eta = 1$ on $\Omega' \subset\subset \Omega$.
- Introduce the test function $v = -D_k^{-h}(\eta^2 D_k^h u) \in H_0^1(\Omega)$.
- Take above derivation, but replace derivatives by difference quotients:

$$
\begin{aligned}
\langle Lu, v\rangle_{L^2} &= \int_\Omega \sum_{i,j} \partial_i(a_{i,j}\partial_j u) D_k^{-h}(\eta^2 D_k^h u))\,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j} \eta^2 a_{i,j}(D_k^h \partial_j u)(D_k^h \partial_i u)\,\mathrm{d}x + F \\
\text{with}\quad F &\stackrel{\text{def}}{=} \int_\Omega \sum_{i,j} \Big\{ \eta^2 (D_k^h a_{i,j})(\partial_i u)(\partial_j D_k^h u) \\
&\qquad + 2\eta\partial_j\eta \left[a_{i,j}^{k,h}(D_k^h \partial_i u)(D_k^h u) + (D_k^h a_{i,j})(\partial_i u)(D_k^h u) \right] \Big\} \\
\text{and}\quad a_{i,j}^{k,h} &\stackrel{\text{def}}{=} a_{i,j}(x + he_k)
\end{aligned}
$$

Interior Regularity

Proof (continued)

- Uniform ellipticity yields

$$\theta \int_\Omega \eta^2 |D_k^h Du|^2 \mathrm{d}x \leq \int_\Omega \sum_{i,j} \eta^2 a_{i,j} (D_k^h \partial_i u)(D_k^h \partial_j u)\, \mathrm{d}x \; .$$

- Thus, we find

$$\theta \int_\Omega \eta^2 |D_k^h Du|^2 \mathrm{d}x \;\leq\; -\int_\Omega f D_k^{-h}(\eta^2 D_k^h u)\, \mathrm{d}x - F \; .$$

- Next, we use Cauchy-Schwarz inequality to find

$$\left| \int_\Omega f D_k^{-h}(\eta^2 D_k^h u)\, \mathrm{d}x \right| \leq \|f\|_{L^2(\Omega)} \left\| D_k^{-h}(\eta^2 D_k^h u) \right\|_{L^2(\Omega)} \; .$$

Interior Regularity

Proof (continued)

- We proceed term-by-term. First, since $\mathrm{Supp}(\eta) \subset \Omega$,

$$
\begin{aligned}
\left\| D_k^{-h}(\eta^2 D_k^h u) \right\|_{L^2(\Omega)} &\leq \left\| \partial_k(\eta^2 D_k^h u) \right\|_{L^2(\Omega)} \\
&\leq \left\| (\eta^2 \partial_k D_k^h u) \right\|_{L^2(\Omega)} + \| 2\eta(\partial_k \eta) D_k^h u \|_{L^2(\Omega)} \\
&\leq \| \eta \partial_k D_k^h u \|_{L^2(\Omega)} + \tilde{C} \| Du \|_{L^2(\Omega)} \; .
\end{aligned}
$$

- And, similarly,

$$
|F| \; \leq \; \tilde{C} \left(\| Du \|_{L^2(\Omega)} \left\| \eta D_k^h Du \right\|_{L^2(\Omega)} + \| Du \|^2_{L^2(\Omega)} \right) \; .
$$

as long as the constant $\tilde{C} < \infty$ is sufficiently large.

Interior Regularity

Proof (continued)

- If we substitute the previous bounds, we find

$$\begin{aligned}\theta\|\eta D_k^h Du\|_{L^2(\Omega)} \;\leq\; \tilde{C}\Big(&\|f\|_{L^2(\Omega)}\|\eta D_k^h Du\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)} \\ &\|Du\|_{L^2(\Omega)}\|\eta D_k^h Du\|_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\Big)\end{aligned}$$

- Additionally, we have the upper bounds

$$\begin{aligned}\|f\|_{L^2(\Omega)}\|\eta D_k^h Du\|_{L^2(\Omega)} &\leq \epsilon\|\eta D_k^h Du\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon}\|f\|^2_{L^2(\Omega)} \\ \|Du\|_{L^2(\Omega)}\|\eta D_k^h Du\|_{L^2(\Omega)} &\leq \epsilon\|\eta D_k^h Du\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon}\|Du\|^2_{L^2(\Omega)}\end{aligned}$$

- Choose ϵ such that $4\tilde{C}\epsilon = \theta$ and substitute the above estimates to find (see next slide)

Interior Regularity

Proof (continued)

- ... (after substituting)

$$\frac{\theta}{4}\|\eta D_k^h Du\|^2_{L^2(\Omega)} \leq \hat{C}\left(\|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\right)$$

  for a constant $\hat{C} < \infty$.

- and since $\eta = 1$ on Ω'

$$\|D_k^h Du\|^2_{L^2(\Omega')} \;\leq\; \overline{C}\left(\|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\right)$$

  for some $\overline{C} < \infty$ that does not depend on h.

- Now, our theorem for difference quotients yields $u \in W^{1,2}(\Omega')$.

Proof (continued)

- The above estimate can further be refined by using that

$$
\begin{aligned}
\theta \int_\Omega |Du|^2 \, \mathrm{d}x &\leq \int_\Omega \sum_{i,j} a_{i,j} \partial_i u \partial_j u \, \mathrm{d}x \\
&= \int_\Omega f u \, \mathrm{d}x \\
&\leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\
&\leq \frac{1}{2} \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right).
\end{aligned}
$$

  This implies $\|D^2 u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$ for a $C < \infty$, as claimed by the theorem.

Interior Regularity

Remarks

- If $u \in H^2_{\text{loc}}(\Omega)$ and $f \in L^2(\Omega)$ is a weak solution of $Lu = f$, then this equation relates the weak derivatives a.e.. Such solutions are sometimes called "strong solutions".
- Repeated application of the above argument yields that if $a_{i,j} \in C^{k+1}(\Omega)$ and $f \in H^k(\Omega)$, then every weak solution $u \in H^1(\Omega)$ satisfies $u \in H^{k+2}(\Omega')$.
- If $a_{i,j}, f \in C^\infty(\Omega)$, we even have that any weak solution $u \in H^1(\Omega)$ satisfies $u \in C^\infty(\Omega)$ (due to Sobolev's embedding theorem).

From Interior to Boundary Regularity

More Remarks

- If the boundary Ω is "sufficiently regular" one can extend the above regularity results to the boundary.
- As we did not cover "trace operators" in this lecture, we will state the following theorem without proof.

Theorem

- If Ω is a bounded open set with C^{k+2}-boundary, $a_{i,j} \in C^{k+1}(\overline{\Omega})$, and $f \in H^k(\Omega)$, then every weak solution $u \in H_0^1(\Omega)$ of $Lu = f$ satisfies $u \in H^{k+2}(\Omega)$.

Maximum Principle

- If the solution u of a PDE is sufficiently regular, e.g. $u \in C^2(\Omega)$, and attains a maximum at a point $x \in \Omega$ inside its open domain Ω, then

$$\nabla u(x) = 0 \qquad \text{and} \qquad \nabla^2 u(x) \preceq 0 \, .$$

- Now, let L be an elliptic operator in non-divergence form

$$Lu = -\sum_{i,j} a_{i,j} \partial_i \partial_j u + \sum_i b_i \partial_i u = -\mathrm{Tr}(A \nabla^2 u) + b^\mathsf{T} \nabla u \, .$$

- Obviously, if $u \in C^2(\Omega)$ has a maximum at $x \in \Omega$, then

$$Lu(x) = -\underbrace{\mathrm{Tr}(A \nabla^2 u(x))}_{\leq 0} + \underbrace{b^\mathsf{T} \nabla u(x)}_{=0} \geq 0 \, .$$

Maximum Principle

- The above consideration can be refined by analyzing the function

$$u^\epsilon(x) = u(x) + \epsilon e^{\lambda x_1}$$

- If we assume that $Lu \leq 0$ and $\lambda > \frac{|b_1|}{a_{11}}$ then

$$\begin{aligned} Lu^\epsilon(x) &\leq L(\epsilon e^{\lambda x_1}) \\ &= \epsilon e^{\lambda x_1}\left[-a_{11}\lambda^2 + b_1\lambda\right] \;<\; 0\,. \end{aligned} \tag{3}$$

- Next, let Ω be open and bounded. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, the above consideration yields for all $\epsilon > 0$ the implication

$$Lu \leq 0 \qquad \Longrightarrow \qquad \max_{x \in \overline{\Omega}} u^\epsilon(x) \;=\; \max_{x \in \partial\Omega} u^\epsilon(x)\,.$$

Maximum Principle

- By taking the limit $\epsilon \to 0$, we arrive at the so-called weak maximum principle, which shows that sub-solutions of elliptic operators always take their maximum at the boundary.

Theorem

- If $\Omega \subseteq \mathbb{R}^n$ is open and bounded and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \leq 0$ in Ω, then we have

$$\max_{x \in \overline{\Omega}} u(x) \;=\; \max_{x \in \partial\Omega} u(x) \; .$$

Maximum Principle

Remarks

- The above maximum principle has many variants.
- For instance, if L has a zeroths order coefficient $c \geq 0$,

$$Lu \;=\; -\mathrm{Tr}(A\nabla^2 u) + b^\mathsf{T}\nabla u + cu,$$

  we can show that $Lu \leq 0$ in Ω implies

$$\max_{x\in\overline{\Omega}} u(x) \;\leq\; \max_{x\in\partial\Omega} \max\{0, u(x)\}\;.$$

  Similarly, if $Lu = 0$ in Ω, then $|u|$ takes its maximum on $\partial\Omega$.
- Also for $c = 0$, $Lu \leq 0$, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ one can show

$$\forall x \in \Omega, \qquad u(x) < \max_{x\in\partial\Omega}\; u(x)\;.$$

Contents

Parabolic Equations

- Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $\Omega_T = \Omega \times (0, T]$.
- We consider the boundary value problem

$$\begin{array}{rcll} \partial_t u + Lu & = & f & \text{in} \quad \Omega_T \\ u & = & 0 & \text{on} \quad \partial\Omega \times [0, T] \\ u & = & g & \text{on} \quad \Omega \times \{t = 0\} \end{array}$$

with

$$Lu = -\sum_{i,j} \partial_j (a_{i,j} \partial_i u) + \sum_i b_i \partial_i u + cu \ .$$

- If L is uniformly elliptic, $\sum_{i,j} a_{i,j} \xi_i \xi_j \geq \theta |\xi|^2$, then $\partial_t + L$ is called a parabolic differential operator.

Parabolic Equations

Protoype Example

- The most basic example for a parabolic equation is the heat equation

$$u_t - \Delta u = 0 \; .$$

  Often, the initial temperature distribution and the temperature at the boundary is given—or can even be controlled.

- More generally, parabolic PDEs often model the time evolution of the density of some quantity—for example, a chemical concrentration.

Weak Solutions

- We'll assume $a_{i,j} = a_{j,i}, b_i, c \in L^\infty(\Omega_T)$ and, usually,

$$f \in L^2(\Omega_T) \qquad \text{as well as} \qquad g \in L^2(\Omega)$$

- The bilinear form of the parabolic PDE is given by

$$B(t,u,v) \stackrel{\text{def}}{=} \int_\Omega \left[\sum_{i,j} a_{i,j}(\partial_i u)(\partial_j v) + \sum_i b_i(\partial_i u)v + cuv \right] \mathrm{d}x$$

- In some applications, B depends on t (if the coefficients depend on t).
- The function u is sometimes regarded as a map $u : [0,T] \to H_0^1(\Omega)$. It is defined by $[u(t)](x) \stackrel{\text{def}}{=} u(t,x)$.

Notation

- A general map $u : [0,T] \to X$ into a Banach space $(X, \|\cdot\|)$ is called Bochner p-integrable if it is (strongly) measurable and

$$\int_0^T \|u(t)\|^p \,\mathrm{d}t \;<\; \infty$$

  The set of such maps u is denoted by $L^p(0,T;X)$,

- $C(0,T;X)$ denotes the set of continuous functions $u : [0,T] \to X$,
- and $W^{1,p}(0,T;X)$ denotes the set of functions $u \in L^p(0,T;X)$ that have a weak time derivative $u' \in L^p(0,T;X)$.
- Sometimes u and u' are different spaces. In the context of parabolic PDEs we are interested in functions $u \in L^2(0,T;H_0^1(\Omega))$ with $u' \in L^2(0,T;H^{-1}(\Omega))$.

Weak Solutions

Definition

- A function $u \in L^2(0,T;H_0^1(\Omega))$ with $u' \in L^2(0,T;H^{-1}(\Omega))$ is called a weak solution of the parabolic PDE if

$$\langle u', v\rangle + B(t,u,v) \ = \ \langle f, v\rangle_{L^2} \quad \text{and} \quad u(0) = g$$

  for all $v \in H_0^1(\Omega)$ and almost all $t \in [0,T]$.

- Here, $\langle u', v\rangle$ refers to the duality pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
- It follows from Sobolev's embedding theorem that $u \in C(0,T,L^2(\Omega))$ (possibly after modifying u on a Lebesgue null set). As such, the condition $u(0) = g$ makes sense.

Existence of Solutions

- There are two key strategies for analyzing solutions of the PDE

$$\begin{aligned} \partial_t u + Lu &= f \quad \text{in} \quad \Omega_T \\ u &= 0 \quad \text{on} \quad \partial\Omega \times [0,T] \\ u &= g \quad \text{on} \quad \Omega \times \{t=0\}\,. \end{aligned}$$

- The first strategy is to discretize in time, solve a sequence of elliptic PDEs, derive energy estimates, and then take the limit.
- And the second strategy proceeds by discretizing in space (Galerkin approximation) and use Gronwall's inequality to pass to the limit.
- Both strategies work, but let's focus on the Galerkin method.

Galerkin Method

- Let $\varphi_1, \varphi_2, \varphi_3, \ldots C^\infty(\Omega)$ be a sequence of functions such that

  1. $\varphi_1, \varphi_2, \varphi_3, \ldots$ is an orthogonal basis of $H_0^1(\Omega)$, and
  2. $\varphi_1, \varphi_2, \varphi_3, \ldots$ is an orthonormal basis of $L^2(\Omega)$.

- The main idea is to first construct approximate solutions

$$u_m(t) \stackrel{\text{def}}{=} \sum_{k=1}^{m} d_k(t)\varphi_k$$

  with time-varying coefficients $d_k(t)$, such that

$$\langle u_m'(t), \varphi_k\rangle + B(t, u_m, \varphi_k) = \langle f(t), \varphi_k\rangle \qquad \text{and} \qquad d_k(0) = \langle g, \varphi_k\rangle$$

- The coefficient functions d_k can be found by solving a linear ODE.

Galerkin Method

- In detail, we work out the terms

$$
\begin{aligned}
\dot{d}_k(t) &= \langle u_m'(t), \varphi_k \rangle \\
E_{k,l}(t) &\stackrel{\text{def}}{=} -B(t, \varphi_l, \varphi_k) \\
e_k(t) &\stackrel{\text{def}}{=} \langle f(t), \varphi_k \rangle \\
d_k^0 &\stackrel{\text{def}}{=} \langle g, \varphi_k \rangle
\end{aligned} \tag{4}
$$

  such that the linear ODE takes the form

$$
\dot{d}(t) \;=\; E(t)d(t) + e(t) \quad \text{with} \quad d(0) = d^0 \; .
$$

- Recall from Lecture 3: linear ordinary differential equations are Lipschitz continuous and, consequently, have a unique solution.

Energy Estimates

- Our next goal is to analyze convergence of the Galerkin approximation u_m to a solution of the parabolic ODE.
- For this aim, we first need to establish an "energy estimate".

Theorem

- There exists a constant C (depending only on Ω and T) such that

$$\max_{t \in [0,T]} \|u_m(t)\|_{L^2(\Omega)} + \|u_m\|_{L^2(0,T;H_0^1(\Omega))} + \|u_m'\|_{L^2(0,T;H^{-1}(\Omega))}$$

$$\leq \; C \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(\Omega)} \right) \; .$$

Energy Estimates

Proof.

- Let us start with the equations

$$\langle u_m', \varphi_k \rangle + B(t, u_m, \varphi_k) = \langle f, \varphi_k \rangle$$

multiply with $d_k(t)$ and sum over k to find

$$\langle u_m'(t), u_m(t) \rangle + B(t, u_m(t), u_m(t)) = \langle f(t), u_m(t) \rangle \ .$$

- Analogous to the elliptic PDE, we can find $\beta > 0$ and $\gamma \geq 0$ such that

$$\beta \|u_m(t)\|_{H_0^1}^2 \ \leq \ B(t, u_m(t), u_m(t)) + \gamma \|u_m(t)\|_{L^2(\Omega)}^2 \ .$$

Energy Estimates

Proof (continued...)

- Next, we bound the term

$$|\langle f(t), u_m(t)\rangle| \quad \leq \quad \frac{1}{2}\|f(t)\|_{L^2(\Omega)} + \frac{1}{2}\|u_m(t)\|_{L^2(\Omega)}$$

- Also notice that

$$\frac{\partial}{\partial t}\frac{1}{2}\|u_m(t)\|^2_{L^2(\Omega)} \;=\; \langle u_m'(t), u_m(t)\rangle \; .$$

- Thus, in summary, we have

$$\frac{\partial}{\partial t}\|u_m(t)\|^2_{L^2(\Omega)} + 2\beta\|u_m(t)\|^2_{H_0^1} \;\leq\; C_1\|u_m(t)\|^2_{L^2(\Omega)} + C_2\|f(t)\|_{L^2(\Omega)}$$

  for suitable constants C_1, C_2.

Energy Estimates

Proof (continued...)

- Now, on the one hand Gronwall's inequality yields

$$\max_{t\in[0,T]} \|u_m(t)\|_{L^2(\Omega)} \ \le\ C\left(\|g\|_{L^2}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}\right) \ .$$

- Similarly, integrating the inequality from the previous slide yields

$$\|u_m\|_{L^2(0,T;H_0^1(\Omega))}^2 \ \le\ C\left(\|g\|_{L^2}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}\right)^2 \ .$$

- It remains to bound the norm of u_m'. For this aim, we first note that for any function $v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1} \le 1$, we have

$$\begin{aligned} |\langle u_m'(t), v\rangle| \ &=\ |\langle u_m', v^\perp\rangle| \ =\ |\langle f(t), v^\perp\rangle - B(t, u_m, v^\perp)| \\ &\le\ C\left(\|f(t)\|_{L^2(\Omega)}^2 + \|u_m(t)\|_{H_0^1(\Omega)}^2\right) \ , \end{aligned}$$

  where $v^\perp$ denotes the projection of v onto $\operatorname{span}(\varphi_1, \varphi_2, \ldots, \varphi_m)$.

Energy Estimates

Proof (continued...)

- Thus, we find that

$$\begin{aligned} \|u_m'(t)\|_{H^{-1}(\Omega)} &= \sup_{v \in H_0^1(\Omega)} \frac{|\langle u_m'(t), v\rangle|}{\|v\|_{H_0^1}} \\ &\leq C\left(\|f(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_{H_0^1(\Omega)}\right) . \end{aligned}$$

- Taking squares + integration over t and collecting the previous inequalities yields the desired energy estimate.

Existence of Solutions

Theorem

- There exists a weak solution to the parabolic PDE on Ω_T under the above assumptions.

Proof

- Due to the energy estimate, the Galerkin approximation sequences u_m and u'_m have weakly convergent subsequences; that is
  1. $u_{m_l} \rightharpoonup u$ weakly in $L^2(0, T; H_0^1(\Omega))$, and
  2. $u'_{m_l} \rightharpoonup u'$ weakly in $L^2(0, T; H^{-1}(\Omega))$.
- Our goal is to show that u is a weak solution.

Existence of Solutions

Proof (continued...)

- For any $v \in C^1([0,T]; H_0^1(U))$ of the form

$$v(t) \;=\; \sum_{k=1}^{N} \tilde{d}_k(t)\varphi_k$$

  with $N \leq m$, we have that

$$\int_0^T \langle u_m'(t), v(t)\rangle + B(t, u_m(t), v(t))\,\mathrm{d}t \;=\; \langle f(t), v(t)\rangle$$

- Thus, if we set $m = m_l$ and pass to weak limits,

$$\int_0^T \langle u'(t), v(t)\rangle + B(t, u(t), v(t))\,\mathrm{d}t \;=\; \int_0^T \langle f(t), v(t)\rangle\,\mathrm{d}t$$

  for all $v \in L^2(0,T; H_0^1(\Omega))$.

Existence of Solutions

Proof (continued...)

- Consequently, we must have

$$\langle u'(t), v(t)\rangle + B(t, u(t), v(t)) \;=\; \langle f(t), v(t)\rangle$$

  for all $v \in H_0^1(\Omega)$ and almost all $t \in [0, T]$.

- Moreover, from the above integral equation and partial integration,

$$\int_0^T \langle v'(t), u(t)\rangle + B(t, u(t), v(t))\,\mathrm{d}t \;=\; \int_0^T \langle f(t), v(t)\rangle\,\mathrm{d}t + \langle v(0), u(0)\rangle$$

  for all $v \in C^1(0, T; H_0^1(\Omega))$ with $v(T) = 0$.

Existence of Solutions

Proof (continued...)

- Similarly,

$$\int_0^T \langle u_m(t), v'(t)\rangle + B(t, u_m(t), v(t))\,\mathrm{d}t \;=\; \langle f(t), v(t)\rangle + \langle v(0), g\rangle$$

  implies (by passing to weak limits)

$$\int_0^T \langle u(t), v'(t)\rangle + B(t, u(t), v(t))\,\mathrm{d}t \;=\; \langle f(t), v(t)\rangle + \langle v(0), g\rangle$$

- By comparing the above relations and using that $v(0)$ is arbitrary, it follows that $u(0) = g$. This completes our proof.

Uniqueness of Solutions

Theorem

- The above constructed weak solution of the parabolic PDE is unique.

Proof

- It is sufficient to check this for $f = 0$ and $g = 0$ (Why?).
- By substituting the test function $v = u$, we have

$$\frac{\partial}{\partial t} \frac{\|u(t)\|_{L^2}^2}{2} \;=\; -B(t, u(t), u(t)) \;\leq\; \gamma \|u(t)\|_{L^2}^2$$

- Since $u(0) = 0$, Gronwall's lemma yields $\|u(t)\|_{L^2}^2 \leq 0$.
- This completes the proof.