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A bundle–Newton method for nonsmooth unconstrained minimization

Ladislav Lukšan *, Jan Vlček [1]

Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic

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Abstract

An algorithm based on a combination of the polyhedral and quadratic approximation is given for finding stationary points for unconstrained minimization problems with locally Lipschitz problem functions that are not necessarily convex or differentiable. Global convergence of the algorithm is established. Under additional assumptions, it is shown that the algorithm generates Newton iterations and that the convergence is superlinear. Some encouraging numerical experience is reported. © 1998 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

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1. Introduction

This paper describes a bundle-type method for minimizing a locally Lipschitz continuous function $f : \mathbb{R}^N \to \mathbb{R}$. We assume that for each $y \in \mathbb{R}^N$ we can compute $f(y)$, an arbitrary subgradient $g(y)$, i.e. one element of the subdifferential $\partial f(y)$ (called generalized gradient in Clarke, 1983) and an $N \times N$ symmetric matrix $G(y)$ as a substitute for the Hessian matrix. The function f is often (as in all problems that we have solved) continuous piecewise-C^2, i.e. $\mathbb{R}^N$ is composed of regions inside which both the gradient and the Hessian matrix exist and are continuous. In that case, if f is not twice differentiable at y, we can take the gradient and the Hessian matrix at some point "infinitely close" to y as $g(y)$ and $G(y)$, respectively. If f is convex, then for all y except in a set of zero (Lebesgue) measure, f is differentiable at y and has second-order approximation around y (see Hiriart-Urruty and Lemarechal, 1993).

* Corresponding author.

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Our method is based on the following model, which generalizes a long-known *cutting plane* model due to Kelley (1960) and Cheney and Goldstein (1959). At step k, let $x_1,\dots,x_k$ be the iterates and $y_1,\dots,y_k$ be the trial points that have been generated, together with the corresponding function values $f(y_1),\dots,f(y_k)$, subgradients $g_1 \in \partial f(y_1),\dots,g_k \in \partial f(y_k)$, matrices $G_1 = G(y_1),\dots,G_k = G(y_k)$ and damping parameters $\varrho_j \in [0,1], j = 1,\dots,k$. We define the quadratic approximation of f around y_j by

$$f_j^{\#}(x) = f(y_j) + g_j^{\mathrm{T}}(x-y_j) + \tfrac{1}{2}\varrho_j(x-y_j)^{\mathrm{T}}G_j(x-y_j), \quad j = 1,\dots,k, \tag{1}$$

choose some index set $J_k \subset \{1,\dots,k\}$ and define the piecewise quadratic function

$$f_k^{\square}(x) = \max\{f_j^{\#}(x) \mid j \in J_k\}. \tag{2}$$

Minimizing this model is equivalent to the nonlinear programming problem

$$\text{(A)} \qquad \underset{(x,z)\in\mathbb{R}^{N+1}}{\text{minimize}} \quad z$$
$$\text{subject to} \quad f_j^{\#}(x) \leqslant z, \;\; j \in J_k,$$

which can be solved by the sequential quadratic programming (SQP) method, whose rate of convergence is of a second order (see Fletcher, 1987). The iteration step of the SQP method can be written as a quadratic programming (QP) problem

$$\text{(B)} \qquad \underset{(x,z)\in\mathbb{R}^{N+1}}{\text{minimize}} \quad z + \tfrac{1}{2}(x-x_k)^{\mathrm{T}}W_k(x-x_k)$$
$$\text{subject to} \quad f_j^{\#}(x_k) + g_j^{\#}(x_j)^{\mathrm{T}}(x-x_k) \leqslant z, \;\; j \in J_k,$$

where if we denote by λ_j^k, $j \in J_k$, the Lagrange multipliers at step k,

$$W_k = \sum_{j\in J_{k-1}} \lambda_j^{k-1}\varrho_j G_j, \tag{3}$$

$$g_j^{\#}(x) = \nabla f_j^{\#}(x) = g_j + \varrho_j G_j(x-y_j), \quad j = 1,\dots,k. \tag{4}$$

The idea of using a quadratic model is not new. Lemarechal (1978) in his pioneering work proposed an algorithm where the following QP problem was solved in each step

$$\text{(C)} \qquad \underset{(x,z)\in\mathbb{R}^{N+1}}{\text{minimize}} \quad z + \tfrac{1}{2}(x-x_k)^{\mathrm{T}}A_k(x-x_k)$$
$$\text{subject to} \quad [f(y_j) + g_j^{\mathrm{T}}(x_k-y_j)] + g_j^{\mathrm{T}}(x-x_k) \leqslant z, \;\; j = 1,\dots,k,$$

where A_k was some symmetric positive definite $N \times N$ matrix, which was intended to accumulate information about the curvature of f around x_k. Mifflin (1982) slightly modified the algorithm and showed that if f is inf-compact and the matrices A_k stay uniformly bounded and positive definite then at least one cluster point of $\{x_k\}$ is stationary. Later (in Mifflin, 1984), he considered the problem of minimizing $f_2^{\square}(x)$ to motivate algorithms having a subproblem, which was similar to (B), and investigated conditions for obtaining better than linear convergence. Other ideas for developing a rapidly convergent algorithm, based on QP subproblem (B), can be found in Mifflin

(1992). Kiwiel (1989) presented the algorithm, where subproblem (C) was solved, in which he combined features of the ellipsoid and bundle methods and reduced the number of stored subgradients using two strategies: subgradient selection and aggregation.

Because our model uses more second-order information and follows closely the minimax analogy, we expect faster convergence. Note that it is not necessary to evaluate the matrices G_j analytically – when we used finite difference approximation (without respecting discontinuities), the number of iterations was practically the same.

Our algorithm is based on a line search concept. We also mention two significant first order methods of Schramm and Zowe (1992) and Kiwiel (1996), based on a restricted step (trust region) approach.

The paper is organized as follows. The algorithm is derived in Section 2, its global convergence is proved in Section 3 and its superlinear convergence is studied in Section 4. Particularly, under additional assumptions, we show a self-cleaning property of the algorithm and its reduction to the Newton method. In Section 5 some numerical experience is reported, which demonstrates faster convergence in comparison with first-order bundle methods.

Throughout the paper, we use $\|.\|$ to denote the spectral matrix norm.

2. Derivation of the method

The algorithm given below generates a sequence $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^N$ that should converge to a minimizer of f, search directions $\{d_k\} \subset \mathbb{R}^N$ and stepsizes $\{t_L^k\} \subset [0,1]$, related by $x_{k+1} = x_k + t_L^k d_k$, $k \geqslant 1$. The method also calculates trial points $y_{k+1} = x_k + t_R^k d_k \in \mathbb{R}^N$ for $k \geqslant 1$ with $y_1 = x_1$, subgradients $g_k \in \partial f(y_k)$, symmetric matrices G_k and damping parameters $\varrho_k \in [0,1]$ for $k \geqslant 1$, where $t_R^k \in (0,1]$ are the auxiliary trial stepsizes.

We take a *serious step* from x_k to x_{k+1}, and $y_{k+1} = x_{k+1}$ if we find t_L^k satisfying $t_L^k \geqslant t_0$ and

$$f(x_{k+1}) \leqslant f(x_k) + m_L t_L^k v_k, \tag{5}$$

where $m_L \in (0, \frac{1}{2})$, $t_0 \in (0,1)$ are parameters and $v_k < 0$ is the predicted amount of descent (if $v_k = 0$ the algorithm will stop with x_k; see below). Otherwise a *short step* if (5) holds, but $t_L^k \in (0, t_0)$ or a *null step* $x_{k+1} = x_k$ will improve the quadratic approximation $f_{k+1}^{\square}$. Letting

$$f_j^k = f_j^{\#}(x_k), \quad g_j^k = g_j^{\#}(x_k) = g_j + \varrho_j G_j(x_k - y_j), \qquad j = 1,\ldots,k, \;\; k \geqslant 1, \tag{6}$$

we can write (B) equivalently in the form (here z is not the same as in (B))

$$\text{(D)} \qquad \underset{(x,z)\in\mathbb{R}^{N+1}}{\text{minimize}} \quad z + \tfrac{1}{2}(x - x_k)^{\mathrm{T}} W_k (x - x_k)$$

$$\text{subject to} \quad -\beta_j^k + (x - x_k)^{\mathrm{T}} g_j^k \leqslant z, \;\; j \in J_k,$$

where $\beta_j^k = f(x_k) - f_j^k$. To guarantee the property that $\min_x f_k^{\square}(x) \leqslant f(x_k)$ in our model, it would be useful to have $0 \leqslant \beta_j^k = f(x_k) - f_j^{\#}(x_k), j \in J_k$, because then it would be the case that $\min_x f_k^{\square}(x) \leqslant f_k^{\square}(x_k) \leqslant f(x_k)$ by Eq. (2). Note that it can happen that $\beta_j^k < 0$ even when f is convex. Furthermore, $f_k^{\square}$ closely approximates f only when trial points $y_j, j \in J_k$ are in the neighbourhood of x_k. Thus we generalize the locality measures introduced by Kiwiel (1985) and replace β_j^k by $\alpha_j^k = \max[|f_j^k - f(x_k)|, \gamma(s_j^k)^{\omega}]$ for $j \in J_k$ (the absolute value is not necessary, but significantly improves numerical results), where

$$s_j^k = |y_j - x_j| + \sum_{i=j}^{k-1} |x_{i+1} - x_i| \geqslant |y_j - x_k|, \quad k = 1, \ldots, k, \ k \geqslant 1, \tag{7}$$

and $\gamma > 0$, $\omega \geqslant 1$ are parameters (Kiwiel, 1985, uses $\omega = 2$).

Because the method needs a positive definite matrix in problem (D), we replace W_k by its positive definite modification $\bar{G}_p^k$. To reduce the bundle size, we use the subgradient aggregation strategy of Kiwiel (1985).

We shall now state the method in detail.

Algorithm 2.1

Step 0 (Initialization). Select the starting point $x_1 \in \mathbb{R}^N$, a final accuracy tolerance $\varepsilon \geqslant 0$, a bundle dimension $M \geqslant 2$, a distance measure parameter $\gamma > 0$, line search parameters $m_L \in (0, \frac{1}{2}), m_R \in (m_L, 1)$, a lower bound for long serious steps $t_0 \in (0,1)$, an upper bound $C_S > 0$ for distance between x_k and y_k, an upper bound for damped matrices $C_G > 0$, a matrix selection parameter $i_{\mathrm{m}} \geqslant 0$, a bundle reset parameter $i_{\mathrm{r}} \geqslant 0$ and a locality measure parameter $\omega \geqslant 1$. Set $y_1 = x_1$ and compute $f(y_1), g_1 \in \partial f(y_1)$ and a symmetric matrix G_1. Initialize the iteration counter $k = 1$, the number of consecutive null and short steps $i_{\mathrm{n}} = 0$, the number of serious steps from the last bundle reset $i_{\mathrm{s}} = 0$, $J_1 = \{1\}, \varrho_1 = 1, s_p^1 = s_1^1 = 0$, $f_p^1 = f_1^1 = f(y_1)$, $g_p^1 = g_1$ and $G_p^1 = G_1$.

Step 1 (Direction finding). If both of the steps $k-1$ and $k-2$ were serious and $\lambda_{k-1}^{k-1} = 1$ or if $i_{\mathrm{s}} > i_{\mathrm{r}}$, then set $G = G_k$, otherwise set $G = G_p^k$. If $i_{\mathrm{n}} \leqslant i_{\mathrm{m}}$, modify G to obtain a positive definite matrix $\bar{G}_p^k$, otherwise set $\bar{G}_p^k = \bar{G}_p^{k-1}$. Find the solution $(d_k, \hat{v}_k)$ to the kth QP subproblem

$$(\mathscr{P}) \qquad \begin{aligned} &\text{minimize} \quad \hat{v} + \tfrac{1}{2} d^{\mathrm{T}} \bar{G}_p^k d \quad \text{over all } (d, \hat{v}) \in \mathbb{R}^N \times \mathbb{R} \\ &\text{subject to} \quad -\alpha_j^k + d^T g_j^k \leqslant \hat{v} \ \text{ for } j \in J_k, \\ &\qquad\qquad\quad\ -\alpha_p^k + d^T g_p^k \leqslant \hat{v} \ \text{ if } i_s \leqslant i_r, \end{aligned}$$

where

$$\alpha_j^k = \max[|f_j^k - f(x_k)|, \gamma(s_j^k)^{\omega}] \quad \text{for } j \in J_k, \tag{8a}$$

$$\alpha_p^k = \max[|f_p^k - f(x_k)|, \gamma(s_p^k)^{\omega}], \tag{8b}$$

which can be obtained by solving the kth subproblem dual (see Lemarechal, 1978): Find values of the multipliers λ_j^k, $j \in J_k$, and λ_p^k to

$$(\mathscr{P}') \qquad \text{minimize} \quad \frac{1}{2}\left|H_k\left(\sum_{j\in J_k}\lambda_j g_j^k + \lambda_p g_p^k\right)\right|^2 + \sum_{j\in J_k}\lambda_j \alpha_j^k + \lambda_p \alpha_p^k$$

$$\text{subject to} \quad \lambda_j \geqslant 0, \;\; j \in J_k, \;\; \lambda_p \geqslant 0, \;\; \sum_{j\in J_k}\lambda_j + \lambda_p = 1,$$

$$\lambda_p = 0 \;\text{ if }\; i_s > i_r,$$

with

$$d_k = -H_k^2\left(\sum_{j\in J_k}\lambda_j^k g_j^k + \lambda_p^k g_p^k\right), \tag{9}$$

$$\hat{v}_k = -d_k^{\mathrm{T}}\bar{G}_p^k d_k - \sum_{j\in J_k}\lambda_j^k \alpha_j^k - \lambda_p^k \alpha_p^k, \tag{10}$$

where $H_k = (\bar{G}_p^k)^{-1/2}$. If $i_s > i_r$ set $i_s = 0$. Set

$$(\tilde{g}_p^k, \tilde{f}_p^k, G_p^{k+1}, \tilde{s}_p^k) = \sum_{j\in J_k}\lambda_j^k(g_j^k, f_j^k, \varrho_j G_j, s_j^k) + \lambda_p^k(g_p^k, f_p^k, G_p^k, s_p^k), \tag{11}$$

$$\tilde{\alpha}_p^k = \max[|\tilde{f}_p^k - f(x_k)|, \gamma(\tilde{s}_p^k)^\omega], \tag{12}$$

$$v_k = -|H_k\tilde{g}_p^k|^2 - \tilde{\alpha}_p^k, \tag{13}$$

$$w_k = \tfrac{1}{2}|H_k\tilde{g}_p^k|^2 + \tilde{\alpha}_p^k. \tag{14}$$

Step 2 (Stopping criterion). If $w_k \leqslant \varepsilon$, then stop.

Step 3 (Line search). By a line search procedure as given below find step sizes t_L^k, t_R^k such that $0 \leqslant t_L^k \leqslant t_R^k \leqslant 1$ and such that the corresponding points $x_{k+1} = x_k + t_L^k d_k, y_{k+1} = x_k + t_R^k d_k$ satisfy the serious descent criterion (5) and either a serious step $t_L^k = t_R^k \geqslant t_0$ is taken, or a *short step* $0 < t_L^k < t_0$, $t_L^k \leqslant t_R^k$ or a null step $0 = t_L^k < t_R^k$ occur. Calculate $f_{k+1} = f(y_{k+1}), g_{k+1} \in \partial f(y_{k+1})$ and a symmetric matrix G_{k+1}. If $t_L^k < t_0$ set $i_n = i_n + 1$, otherwise set $i_n = 0$ and $i_s = i_s + 1$.

Step 4 (Updating). If $i_n \leqslant 3$, set $\varrho_{k+1} = \min[1, C_G/\|G_{k+1}\|]$, otherwise set $\varrho_{k+1} = 0$. Calculate the values

$$s_j^{k+1} = s_j^k + |x_{k+1} - x_k|, \quad j \in J_k, \tag{15a}$$

$$s_{k+1}^{k+1} = |x_{k+1} - y_{k+1}|, \tag{15b}$$

$$s_p^{k+1} = \tilde{s}_p^k + |x_{k+1} - x_k|, \tag{15c}$$

$$f_j^{k+1} = f_j^k + (x_{k+1} - x_k)^{\mathrm{T}} g_j^k + \tfrac{1}{2}\varrho_j(x_{k+1} - x_k)^{\mathrm{T}} G_j (x_{k+1} - x_k), \quad j \in J_k, \tag{16a}$$

$$f_{k+1}^{k+1} = f_{k+1} + (x_{k+1} - y_{k+1})^{\mathrm{T}} g_{k+1} + \tfrac{1}{2}\varrho_{k+1}(x_{k+1} - y_{k+1})^{\mathrm{T}} G_{k+1}(x_{k+1} - y_{k+1}), \tag{16b}$$

$$f_p^{k+1} = \tilde{f}_p^k + (x_{k+1} - x_k)^{\mathrm{T}}\tilde{g}_p^k + \tfrac{1}{2}(x_{k+1} - x_k)^{\mathrm{T}} G_p^{k+1}(x_{k+1} - x_k), \tag{16c}$$

$$g_j^{k+1} = g_j^k + \varrho_j G_j (x_{k+1} - x_k), \quad j \in J_k, \tag{17a}$$

$$g_{k+1}^{k+1} = g_{k+1} + \varrho_{k+1} G_{k+1}(x_{k+1} - y_{k+1}), \tag{17b}$$

$$g_p^{k+1} = \tilde{g}_p^k + G_p^{k+1}(x_{k+1} - x_k). \tag{17c}$$

Select a set J_{k+1} satisfying $J_{k+1} \subset \{k - M + 2, \dots, k+1\} \cap \{1, 2, \dots\}$ and $k + 1 \in J_{k+1}$.

Step 5. Increase k by 1 and go to Step 1.

A few comments on the algorithm are in order.

The situation when $i_s > i_r$ and thus $\lambda_p = 0$ will be called the bundle reset, significant only for the theory contained in Section 4.

Note that one of the j constraints in $(\mathscr{P})$ may be the same as the p constraint, e.g. when $k = 1$; it must be respected when solving $(\mathscr{P})$.

It follows from Eqs. (13) and (14) that $v_k < 0$ when the stopping criterion is not satisfied. This criterion is presented in the form usual in bundle methods, but in practice it can be advantageous to modify it, e.g. to the form

$$\text{if } |H_k \tilde{g}_p^k|^2 + c \cdot \tilde{\alpha}_p^k / (|f(x_k)| + \delta) \leqslant 2\varepsilon \text{ then stop,}$$

Where c, δ are suitable positive constants (e.g. $c = 100, \delta = 0.001$).

The condition $i_n \leqslant 3$ in step 4 was established empirically. The choice $\varrho_k \leqslant \min[1, C_G/\|G_k\|], k \geqslant 1$, guarantees the boundedness of $\{\varrho_k G_k\}$, because we always have

$$\varrho_k \|G_k\| \leqslant C_G. \tag{18}$$

The updating rules for s_j^{k+1}, f_j^{k+1} and $g_j^{k+1}, j \in J_{k+1}$ follow from Eqs. (7), (1) and (6), respectively. Following the Kiwiel (1985) aggregation strategy we obtain the updating rules for s_p^{k+1}, f_p^{k+1} and g_p^{k+1}.

The parameters i_m, i_r are not meant to improve the efficiency of the method. We need them for convergence proofs.

We shall now present a line search algorithm and subsequent lemma given in similar form by Kiwiel (1985). The choice $\vartheta > 1$ is intended not to prevent rapid convergence of some interpolation procedures at step (vi) (Kiwiel, 1985, uses $\vartheta = 1$). Note that the termination conditions for short and null steps (which occur when $t_L^k < t_0$) in step (v) of the following procedure correspond to

$$-\alpha_{k+1}^{k+1} + d_k^{\mathrm{T}} g_{k+1}^{k+1} \geqslant m_R v_k, \quad |x_{k+1} - y_{k+1}| \leqslant C_S. \tag{19}$$

Line Search Procedure 2.2

(i) Set $t_L = 0$ and $t = t_U = 1$. Choose $\zeta \in (0, \frac{1}{2}), \vartheta \geqslant 1$.

(ii) If $f(x_k + td_k) \leqslant f(x_k) + m_L t v_k$ set $t_L = t$, otherwise set $t_U = t$.

(iii) If $t_L \geqslant t_0$ set $t_R = t_L$ and return.

(iv) Calculate $g \in \partial f(x_k + td_k)$, a symmetric matrix G and

$$\varrho = \begin{cases} \min[1, C_G/\|G\|] & \text{if } i_n \leqslant 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$f = f(x_k + td_k) + (t_L - t)g^{\mathrm{T}}d_k + \tfrac{1}{2}\varrho(t_L - t)^2 d_k^{\mathrm{T}} G d_k,$$
$$\beta = \max[|f - f(x_k + t_L d_k)|, \gamma|t_L - t|^{\omega}|d_k|^{\omega}]$$

(at termination $x_k + t_L d_k$ and $x_k + td_k$ correspond to x_{k+1} and y_{k+1}, respectively).

(v) If $-\beta + d_k^{\mathrm{T}}(g + \varrho(t_L - t)Gd_k) \geqslant m_R v_k$ and $(t - t_L)|d_k| \leqslant C_S$, then set $t_R = t$ and return.

(vi) Choose $t \in [t_L + \zeta(t_U - t_L)^{\vartheta}, t_U - \zeta(t_U - t_L)^{\vartheta}]$ by some interpolation procedure and go to (ii).

Lemma 2.3. *Let f satisfy the following "semismoothness" hypothesis (see Lemma* 3.3.3 *and Remark* 3.3.4 *in Kiwiel,* 1985):
for any $x \in \mathbb{R}^N, d \in \mathbb{R}^N$ and sequences $\{\bar{g}_i\} \subset \mathbb{R}^N$ and $\{t_i\} \subset \mathbb{R}_+$ satisfying $\bar{g}_i \in \partial f(x + t_i d)$ and $t_i \downarrow 0$, one has

$$\lim_{i\to\infty} \sup \bar{g}_i^{\mathrm{T}} d \geqslant \lim_{i\to\infty} \inf [f(x + t_i d) - f(x)]/t_i.$$

Then Line Search Procedure 2.2 *terminates with $t_L^k = t_L$ and $t_R^k = t$ satisfying* (5).

Proof. Assume, for contradiction purposes, that the search does not terminate. Let $t^i, t_L^i, t_U^i, g^i, \varrho^i, G^i$ and β^i denote the values taken on by $t, t_L, t_U, g, \varrho, G$ and β, respectively at the ith iteration of the procedure, hence $t^i \in \{t_L^i, t_U^i\}$ for all i. Since $\zeta \in (0, \frac{1}{2}), (t_U^i - t_L^i)^{(\vartheta-1)} \leqslant 1, t_L^i \leqslant t_L^{i+1} \leqslant t_U^{i+1} \leqslant t_U^i$ and $t_U^{i+1} - t_L^{i+1} \leqslant t_U^i - t_L^i - \zeta(t_U^i - t_L^i)^{\vartheta}$ for all i, there exists $\tilde{t} \geqslant 0$ satisfying $t_L^i \uparrow \tilde{t}, t_U^i \downarrow \tilde{t}$. Let

$$S = \{t \geqslant 0 \mid f(x_k + td_k) \leqslant f(x_k) + m_L t v_k\}.$$

Since $\{t_L^i\} \subset S, t_L^i \uparrow \tilde{t}$ and f is continuous, we have

$$f(x_k + \tilde{t}d_k) - f(x_k) \leqslant m_L \tilde{t} v_k, \tag{20}$$

i.e. $\tilde{t} \in S$. Let $I = \{i \mid t^i \notin S\}$. We prove first, that the set I is infinite. If there existed $i_0 \in I$ satisfying $t^i \in S$ for all $i > i_0$, it would be $t_U^{i_0} = t_U^i \downarrow \tilde{t}$ for all $i > i_0$, which implies $\tilde{t} = t_U^{i_0} \notin S$, which is a contradiction. Thus I is infinite and we have

$$f(x_k + t^i d_k) - f(x_k) > m_L t^i v_k \quad \text{for all } i \in I.$$

By (20), we obtain

$$[f(x_k + t^i d_k) - f(x_k + \tilde{t}d_k)]/(t^i - \tilde{t}) > m_L v_k \quad \text{for all } i \in I,$$

hence

$$\begin{aligned} m_L v_k &\leqslant \liminf_{i\to\infty,\ i\in I} \frac{f(x_k + \tilde{t}d_k + (t^i - \tilde{t})d_k) - f(x_k + \tilde{t}d_k)}{t^i - \tilde{t}} \\ &\leqslant \limsup_{i\to\infty,\ i\in I}\ d_k^{\mathrm{T}} g^i, \end{aligned} \tag{21}$$

where $g^i \in \partial f(x_k + t^i d_k)$. For sufficiently large i we have $(t^i - t_L^i)|d_k| \leqslant C_S$ and by step (v) of the procedure

$$-\beta^i + d_k^{\mathrm{T}}(g^i + \varrho^i(t_L^i - t^i)G^i d_k) < m_R v_k \quad \text{for all large } i.$$

But $\beta^i \to 0, (t_L^i - t^i)\varrho^i d_k^T G^i d_k \to 0$ as $i \to \infty$, since $t_L^i \uparrow \tilde{t}$, $t^i \to \tilde{t}, f$ is continuous, subgradient mapping $\partial f(.)$ is locally bounded (see Kiwiel, 1985) and $\{\varrho^i \|G^i\|\}$ is bounded by (18). Thus lim $\sup_{i\to\infty} d_k^T g^i \leqslant m_R v_k$ and by (21) we obtain $m_L v_k \leqslant m_R v_k$, which contradicts $0 < m_L < m_R < 1$ and $v_k < 0$. Therefore the search terminates and obviously (5) holds at termination. □

3. Global convergence

In this section we will establish the global convergence of the method, generalizing and modifying the nonconvex approach of Kiwiel (1985). We suppose that each execution of Line Search Procedure 2.2 is finite and that the values s_j^{k+1} and g_j^{k+1} are defined by the updating rules Eqs. (15a) and (17a), respectively, also for $j \notin J_k$, i.e. for all $j = 1, \dots, k$, $k \geqslant 1$, and define additional multipliers $\lambda_j^k = 0$ for $j \in \{1, \dots, k\} \setminus J_k$, $k \geqslant 1$. Convergence results assume that the final accuracy tolerance ε is set to zero.

Lemma 3.1. *Suppose that $k \geqslant 1$ is such that Algorithm* 2.1 *did not stop before the kth iteration. Then there exist numbers $\hat{\lambda}_j^k$, $j = 1, \dots, k$, satisfying*

$$(G_p^{k+1}, \tilde{g}_p^k, \tilde{s}_p^k) = \sum_{j=1}^k \hat{\lambda}_j^k (\varrho_j G_j, g_j^k, s_j^k), \quad \hat{\lambda}_j^k \geqslant 0, \; j = 1, \dots, k, \; \sum_{j=1}^k \hat{\lambda}_j^k = 1. \tag{22}$$

Proof. The proof will proceed by induction. If $k = 1$ then we can set $\hat{\lambda}_1^k = 1$. Suppose that Eq. (22) holds for some $k \geqslant 1$. Let

$$\hat{\lambda}_j^{k+1} = \lambda_j^{k+1} + \lambda_p^{k+1}\hat{\lambda}_j^k \quad \text{for } j \leqslant k, \qquad \hat{\lambda}_{k+1}^{k+1} = \lambda_{k+1}^{k+1}.$$

Then $\hat{\lambda}_j^{k+1} \geqslant 0$ for all $j \leqslant k+1$ and $\sum_{j=1}^{k+1} \hat{\lambda}_j^{k+1} = \sum_{j=1}^{k+1} \lambda_j^{k+1} + \lambda_p^{k+1}(\sum_{j=1}^k \hat{\lambda}_j^k) = 1$. From Eqs. (11) and (22) we obtain

$$\begin{aligned} G_p^{k+2} &= \sum_{j=1}^{k+1} \lambda_j^{k+1} \varrho_j G_j + \lambda_p^{k+1}\left(\sum_{j=1}^{k} \hat{\lambda}_j^k \varrho_j G_j\right) \\ &= \lambda_{k+1}^{k+1} \varrho_{k+1} G_{k+1} + \sum_{j=1}^{k} \varrho_j (\lambda_j^{k+1} + \lambda_p^{k+1}\hat{\lambda}_j^k) G_j \\ &= \sum_{j=1}^{k+1} \hat{\lambda}_j^{k+1} \varrho_j G_j, \end{aligned}$$

and, letting $\delta_k = x_{k+1} - x_k$,

$$\begin{aligned} (\tilde{g}_p^{k+1}, \tilde{s}_p^{k+1}) &= \sum_{j=1}^{k+1} \lambda_j^{k+1} (g_j^{k+1}, s_j^{k+1}) + \lambda_p^{k+1}(\tilde{g}_p^k + G_p^{k+1}\delta_k, \tilde{s}_p^k + |\delta_k|) \\ &= \sum_{j=1}^{k+1} \lambda_j^{k+1} (g_j^{k+1}, s_j^{k+1}) + \sum_{j=1}^{k} \lambda_p^{k+1}\hat{\lambda}_j^k (g_j^k + \varrho_j G_j \delta_k, s_j^k + |\delta_k|) \end{aligned}$$

$$= \lambda_{k+1}^{k+1}(g_{k+1}^{k+1}, s_{k+1}^{k+1}) + \sum_{j=1}^{k}[\lambda_j^{k+1} + \lambda_p^{k+1}\hat{\lambda}_j^k](g_j^{k+1}, s_j^{k+1})$$
$$= \sum_{j=1}^{k+1}\hat{\lambda}_j^{k+1}(g_j^{k+1}, s_j^{k+1})$$

from Eqs. (17) and (15). The induction is then established with $k+1$ replacing k. □

Lemma 3.2. *Let $\bar{x} \in \mathbb{R}^N$ be given and suppose that there exist matrices $\bar{G}_j$, vectors $\bar{q}, \bar{y}_j, \bar{g}_j$ and numbers $\bar{s}_j, \bar{\lambda}_j$* for $j = 1, \ldots, L$, $L \geqslant 1$, *satisfying*

$$(\bar{q}, 0) = \sum_{j=1}^{L}\bar{\lambda}_j(\bar{g}_j + \bar{G}_j(\bar{x} - \bar{y}_j), \bar{s}_j), \quad \bar{\lambda}_j \geqslant 0, \ j = 1, \ldots, L, \ \sum_{j=1}^{L}\bar{\lambda}_j = 1, \tag{23}$$

$$|\bar{y}_j - \bar{x}| \leqslant \bar{s}_j, \quad \bar{g}_j \in \partial f(\bar{y}_j), \qquad j = 1, \ldots, L. \tag{24}$$

Then $\bar{q} \in \partial f(\bar{x})$.

Proof. Let $J = \{j \mid \bar{\lambda}_j > 0\}$. By Eq. (23), $\bar{s}_j = 0$ for all $j \in J$, hence (24) implies $\bar{y}_j = \bar{x}$, $j \in J$, so $\bar{g}_j \in \partial f(\bar{x})$ for all $j \in J$. Thus we have $\bar{q} = \sum_{j\in J}\bar{\lambda}_j\bar{g}_j, \bar{\lambda}_j > 0$ for $j \in J, \sum_{j\in J}\bar{\lambda}_j = 1$, so $\bar{q} \in \partial f(\bar{x})$ by the convexity of $\partial f(\bar{x})$. □

Lemma 3.3. *If Algorithm* 2.1 *terminates at the kth iteration, then the point $\bar{x} = x_k$ is stationary for f.*

Proof. If the algorithm terminates at step 2 due to $w_k = 0$, then, since $\varepsilon = 0$ and $\tilde{\alpha}_p^k \geqslant 0$, we have $\tilde{g}_p^k = 0, \tilde{\alpha}_p^k = \tilde{s}_p^k = 0$ by Eq. (14) and nonsingularity of H_k. From Eq. (7) we obtain $|y_j - \bar{x}| \leqslant s_j^k$ for $j \leqslant k$. Using Lemma 3.1, Eq. (6) and Lemma 3.2 with $L = k$, $\bar{G}_j = \varrho_j G_j$, $\bar{q} = \tilde{g}_p^k$, $\bar{y}_j = y_j$, $\bar{g}_j = g_j$, $\bar{s}_j = s_j^k$, $\bar{\lambda}_j = \hat{\lambda}_j^k$ for $j \leqslant k$ we have $0 = \bar{q} \in \partial f(\bar{x})$. □

From now on we suppose that the algorithm does not terminate, i.e. $w_k > 0$ for all k.

Lemma 3.4. *Suppose that N-vectors p, g, Δ and numbers $c, v, w, \beta, m \in (0,1), \alpha \geqslant 0$ satisfy*

$$w = \tfrac{1}{2}|p|^2 + \alpha, \qquad v = -(|p|^2 + \alpha), \qquad -\beta - g^{\mathrm{T}}p \geqslant mv,$$
$$c = \max[|g|, |p|, \sqrt{\alpha}]. \tag{25}$$

Let

$$Q(\nu) = \tfrac{1}{2}|\nu g + (1-\nu)(p + \Delta)|^2 + \nu\beta + (1-\nu)\alpha \quad \textit{for } \nu \in \mathbb{R}. \tag{26}$$

Then

$$\min\{Q(\nu) \mid \nu \in [0,1]\} \leqslant w - w^2\frac{(1-m)^2}{8c^2} + 4c|\Delta| + \tfrac{1}{2}|\Delta|^2.$$

Proof. Simple calculations yield

$$Q(\nu) = Q_1(\nu) + Q_2(\nu),$$

where

$$Q_1(v) \triangleq \tfrac{1}{2}|p|^2 + \alpha + v(-|p|^2 - \alpha + \beta + p^{\mathrm{T}}g) = w + v(v + \beta + p^{\mathrm{T}}g),$$
$$Q_2(v) \triangleq \tfrac{1}{2}v^2|p + \Delta - g|^2 + \Delta^{\mathrm{T}}((p + \Delta/2)(1 - 2v) + vg).$$

From Eq. (25) we have for $v \in [0, 1]$

$$\begin{aligned} Q_1(v) &\leqslant w + v(1-m)v \leqslant w - v(1-m)w, \\ Q_2(v) &\leqslant \tfrac{1}{2}v^2(2c + |\Delta|)^2 + (\tfrac{1}{2} - v)|\Delta|^2 + 2c|\Delta| \\ &= 2c^2v^2 + 2cv^2|\Delta| + \tfrac{1}{2}(1-v)^2|\Delta|^2 + 2c|\Delta| \\ &\leqslant 2c^2v^2 + 4c|\Delta| + \tfrac{1}{2}|\Delta|^2. \end{aligned}$$

Denoting $\tilde{Q}(v) = 2c^2v^2 - v(1-m)w$, we check that $\tilde{Q}$ is minimized by $\bar{v} = (1-m)w/(4c^2) \leqslant 1 \cdot (3/2)c^2/(4c^2) < 1$, yielding $\tilde{Q}(\bar{v}) = -(1-m)^2w^2/(8c^2)$, $\bar{v} \in [0, 1]$, which completes the proof. □

We define ($\hat{w}_k$ is the optimal value of the kth QP subproblem($\mathscr{P}'$))

$$\sigma(x) = \liminf_{k\to\infty} \max[w_k, |x_k - x|] \quad \text{for } x \in \mathbb{R}^N, \tag{27}$$

$$\hat{\alpha}_p^k = \sum_{j\in J_k} \lambda_j^k \alpha_j^k + \lambda_p^k \alpha_p^k, \qquad \hat{w}_k = \tfrac{1}{2}|H_k \tilde{g}_p^k| + \hat{\alpha}_p^k. \tag{28}$$

Lemma 3.5. (i) *At the kth iteration of Algorithm* 2.1, *one has*

$$\tilde{\alpha}_p^k \leqslant \hat{\alpha}_p^k, \qquad w_k \leqslant \hat{w}_k. \tag{29}$$

(ii) *Suppose that there exist* $\bar{x} \in \mathbb{R}^N$ *and an infinite set* $K \subset \{1, 2, \ldots\}$ *satisfying* $x_k \xrightarrow{K} \bar{x}$. *Then* $f(x_k) \downarrow f(\bar{x})$ *and* $t_L^k v_k \to 0$.

Proof. (i) By Eqs. (12), (11), (8), and (28) and, since the functions $\xi \to \gamma|\xi|^\omega$ for $\gamma > 0, \omega \geqslant 1$ and $(\xi, \eta) \to \max[\xi, \eta]$ are convex,

$$\tilde{\alpha}_p^k \leqslant \max\left[\sum_{j\in J_k}\lambda_j^k|f_j^k - f(x_k)| + \lambda_p^k|f_p^k - f(x_k)|, \sum_{j\in J_k}\lambda_j^k\gamma(s_j^k)^\omega + \lambda_p^k\gamma(s_p^k)^\omega\right]$$

$$\leqslant \sum_{j\in J_k}\lambda_j^k \max\left[|f_j^k - f(x_k)|, \gamma(s_j^k)^\omega\right] + \lambda_p^k \max\left[|f_p^k - f(x_k)|, \gamma(s_p^k)^\omega\right] = \hat{\alpha}_p^k,$$

which yields (29).

(ii) Let $x_k \xrightarrow{K} \bar{x}$. Continuity of f implies $f(x_k) \xrightarrow{K} f(\bar{x})$, so $f(x_k) \downarrow f(\bar{x})$ follows from the monotonicity of $\{f(x_k)\}$ due to (5). Since $m_L \in (0, \frac{1}{2})$, $t_L^k \geqslant 0$, $v_k < 0$ and (5) is always fulfilled, we have $0 \leqslant -t_L^k v_k \leqslant [f(x_k) - f(x_{k+1})]/m_L \to 0$, which implies $t_L^k v_k \to 0$ and completes the proof. □

Lemma 3.6. *Suppose that* $\{x_k\}$ *is bounded (e.g. when the level set* $\{x \in \mathbb{R}^N \mid f(x) \leqslant f(x_l)\}$ *is bounded for some* $l \geqslant 1$) *and* $\sigma(\bar{x}) = 0$ *for some point* $\bar{x} \in \mathbb{R}^N$. *Then* $0 \in \partial f(\bar{x})$.

Proof. By Eq. (27) there exists an infinite set $K \subset \{1, 2, \dots\}$ such that $x_k \xrightarrow{K} \bar{x}, w_k \xrightarrow{K} 0$. Let $I = \{1, \dots, N+2\}$. From Lemma 3.1 and the Caratheodory theorem (see Hiriart-Urruty and Lemarechal, 1993) we deduce the existence of vectors $g^{k,i}$ and numbers $\lambda^{k,i}, s^{k,i}$ for $i \in I, k \geqslant 1$, satisfying

$$(\tilde{g}_p^k, \tilde{s}_p^k) = \sum_{i \in I} \lambda^{k,i}(g^{k,i}, s^{k,i}), \quad \lambda^{k,i} \geqslant 0, \ i \in I, \ \sum_{i \in I} \lambda^{k,i} = 1, \tag{30}$$

with $(g^{k,i}, s^{k,i}) \in \{(g_j^k, s_j^k) \mid j = 1, \dots, k\} \subset \mathbb{R}^N \times \mathbb{R}, i \in I, k \geqslant 1$. In view of Eq. (6) we can assign to every $k \geqslant 1$ and ever $i \in I$ an index $j = j(k, i), 1 \leqslant j \leqslant k$, satisfying

$$g^{k,i} = g_j^k = g_j + \varrho_j G_j(x_k - y_j), \qquad s^{k,i} = s_j^k, \tag{31}$$

with $g_j \in \partial f(y_j), \varrho_j \in [0, 1]$. By (19) and the fact that $x_j = y_j$ for serious steps, we always have $|x_j - y_j| \leqslant C_S$. Thus $\{y_j\}$ is bounded and there exist points $\bar{y}_i, i \in I$, and an infinite set $K_1 \subset K$ satisfying $y_{j(k,i)} \to \bar{y}_i$ as $k \xrightarrow{K_1} \infty$ for $i \in I$. By the local boundedness and the upper semicontinuity of ∂f (see Kiwiel, 1985), there exist vectors $\bar{g}_i \in \partial f(\bar{y}_i), i \in I$, and an infinite set $K_2 \subset K_1$ satisfying $g_{j(k,i)} \xrightarrow{K_2} \bar{g}_i$ for $i \in I$. Since $\{\varrho_j G_j\}, \{\lambda^{k,i}\}$ are bounded by (18), there exist matrices $\bar{G}_i$, numbers $\bar{\lambda}_i, i \in I$, and an infinite set $\bar{K} \subset K_2$ satisfying $\varrho_{j(k,i)} G_{j(k,i)} \xrightarrow{\bar{K}} \bar{G}_i, \lambda^{k,i} \xrightarrow{\bar{K}} \bar{\lambda}_i$ for $i \in I$.

Letting $k \in \bar{K}$ approach infinity in Eqs. (30) and (31) we obtain $\tilde{g}_p^k \xrightarrow{\bar{K}} \sum_{i \in I} \bar{\lambda}_i(\bar{g}_i + \bar{G}_i(\bar{x} - \bar{y}_i)) \triangleq \bar{q}$. From $w_k \xrightarrow{K} 0$, (14), (12) and combining (18) with Lemma 3.1 we have $\tilde{g}_p^k \xrightarrow{K} 0 = \bar{q}$ and $\tilde{\alpha}_p^k \xrightarrow{K} 0$, which yields $\tilde{s}_p^k \xrightarrow{K} 0$, hence $\lambda^{k,i} s^{k,i} \xrightarrow{K} 0$ for $i \in I$ by Eq. (30) and nonnegativity of all $\lambda^{k,i} s^{k,i}$. Therefore from $\lambda^{k,i} \xrightarrow{\bar{K}} \bar{\lambda}_i, i \in I$ and Eq. (7) we obtain $s^{k,i} \xrightarrow{\bar{K}} s_i \geqslant |\bar{x} - \bar{y}_i|$, setting $\bar{s}_i = 0$ for $\bar{\lambda}_i \neq 0$. If $\bar{\lambda}_i = 0$ we set $\bar{s}_i = |\bar{x} - \bar{y}_i|$. Obviously $\bar{\lambda}_i \geqslant 0, i \in I, \sum_{i \in I} \bar{\lambda}_i = 1$, so $0 = \bar{q} \in \partial f(\bar{x})$ by Lemma 3.2. □

Lemma 3.7. *Let $\bar{x} \in \mathbb{R}^N$ be given and suppose that $\{H_k\}$ is bounded and there exists an infinite set $K \subset \{1, 2, \dots\}$ such that $x_k \xrightarrow{K} x, \sigma(\bar{x}) > 0$. Then for any $i \geqslant 0$ $x_{k+i} \to \bar{x}$ and $t_L^{k+i} \to 0$ as $k \xrightarrow{K} \infty$. Moreover, for any fixed $r \geqslant 0$ there exists $\bar{k} \geqslant 0$ such that $w_{k+i} \geqslant \sigma(\bar{x})/2$ and $t_L^{k+i} < t_0$ for all $k > \bar{k}, k \in K$ and $0 \leqslant i \leqslant r$.*

Proof. (i) We shall first establish $x_{k+i} \xrightarrow{K} \bar{x}$ for any $i \geqslant 0$. For $i = 0$ it is true by assumption. By induction, let it be true for any fixed $i \geqslant 0$. Since $\{H_k\}, \{t_L^k\}$ are bounded, we have

$$|x_{k+i+1} - x_{k+i}| = t_L^{k+i} |H_{k+i}^2 \tilde{g}_p^{k+i}| \leqslant \|H_{k+i}\| \sqrt{t_L^{k+i}} \sqrt{-t_L^{k+i} v_{k+i}} \to 0$$

by Eqs. (9), (11)–(13) and Lemma 3.5(ii), which implies $x_{k+i+1} \xrightarrow{K} \bar{x}$ and completes the induction.

(ii) Next we show that $t_L^{k+i} \xrightarrow{K} 0$ for any fixed $i \geqslant 0$. We assume that it is not true, i.e. that there exist $\hat{t} > 0$ and an infinite set $\bar{K} \subset K$, satisfying $t_L^{k+i} \geqslant \hat{t}$ for all $k \in \bar{K}$. By Eqs. (13) and (14) and Lemma 3.5(ii) we get $0 \leqslant \hat{t} w_{k+i} \leqslant -t_L^{k+i} v_{k+i} \to 0$ for $k \in \bar{K}$, which yields $w_{k+i} \xrightarrow{\bar{K}} 0$, so $\sigma(\bar{x}) = 0$, since $x_{k+i} \xrightarrow{\bar{K}} \bar{x}$. It is a contradiction, yielding the desired assertion.

(iii) Let $r \geqslant 0$ be fixed. For any $i \geqslant 0$, since $x_{k+i} \xrightarrow{K} \bar{x}$ together with $\sigma(\bar{x}) > 0$ and, since $t_L^{k+i} \xrightarrow{K} 0$, there exist $k_i \geqslant 0$, satisfying $w_{k+i} \geqslant \sigma(\bar{x})/2$ and $t_L^{k+i} < t_0$ for all $k > k_i, k \in K$. Setting $\tilde{k} = \max\{k_i \mid 0 \leqslant i \leqslant r\}$ completes the proof. □

Note that the boundedness of $\{H_k\}$ can be provided numerically. If we modify the matrix G_p^k in Step 1 using a factorization method by Gill and Murray (1974), then there exists a constant $c > 0$ satisfying $\|(\bar{G}_p^k)^{-1}\| \leqslant c$ for all $k \geqslant 1$. This follows easily from the fact that $\bar{G}_p^k = L_k D_k L_k^{\mathrm{T}}$, where D_k is a diagonal matrix with elements greater than some positive constant and L_k is a unit lower-triangular matrix with bounded off-diagonal elements.

Theorem 3.8. *Suppose $\{x_k\}$ and $\{H_k\}$ are bounded. Then every accumulation point of $\{x_k\}$ is stationary for f.*

Proof. Suppose $x_k \xrightarrow{K} \bar{x}$. In view of Lemma 3.6, it suffices to show that $\sigma(\bar{x}) = 0$. For contradiction purposes, let $\sigma(\bar{x}) > 0$ or $\sigma(\bar{x}) = +\infty$.

As in the proof of Lemma 3.6 we establish boundedness of $\{y_k\}, \{\varrho_k G_k\}, \{g_k\}$ and also of $\{g_k^k\}, \{H_k g_k^k\}$ and $\{\alpha_k^k\}$ by Eqs. (17b), (15b) and (16b), and continuity of f. Since the multipliers $\lambda_k = 1, \lambda_j = 0$ for $j \in J_k \setminus \{k\}$ and $\lambda_p = 0$ are feasible for the kth dual subproblem $(\mathscr{P}')$ for all $k \geqslant 1$, it holds $\hat{w}_k \leqslant \frac{1}{2}|H_k g_k^k|^2 + \alpha_k^k, k \geqslant 1$, and using (29) and (14) we deduce that $\{w_k\}, \{H_k \tilde{g}_p^k\}, \{\tilde{g}_p^k\}$ and $\{\tilde{\alpha}_p^k\}$ are bounded and $\sigma(\bar{x})$ is finite. Denote

$$c = \sup\left\{|H_k g_k^k|, |H_k \tilde{g}_k^k|, \sqrt{\tilde{\alpha}_p^k} | k \geqslant 1\right\}, \quad \Delta_k = H_{k+1}(g_p^{k+1} - \tilde{g}_p^k), \qquad k \geqslant 1,$$
$$\delta = \sigma(\bar{x})/2, \qquad \bar{c} = \delta(1 - m_R)/(4c), \qquad r = (\tfrac{3}{2})c^2/\bar{c}^2 + i_{\mathrm{m}}. \tag{32}$$

Arguing as in the first part of the proof of Lemma 3.7 and from Eq. (15c) and Lemma 3.5(ii) we obtain $x_{k+1} - x_k \to 0, s_p^{k+1} - \tilde{s}_p^k \to 0, f(x_{k+1}) - f(x_k) \to 0$. Combining (18) with Lemma 3.1 and using Eqs. (16c) and (17c) we get $f_p^{k+1} - \tilde{f}_p^k \to 0$ and $\Delta_k \to 0$. Since for $\omega \geqslant 1$ the function $\xi \to \xi^\omega$ is Lipschitz continuous on any bounded subset of $\mathbb{R}_+, \tilde{s}_p^k \leqslant (\tilde{\alpha}_p^k/\gamma)^{1/\omega}$ for $k \geqslant 1$ and $\{\tilde{\alpha}_p^k\}$ is bounded, there is a constant $c_L > 0$ such that $|(s_p^{k+1})^\omega - (\tilde{s}_p^k)^\omega| \leqslant c_L |s_p^{k+1} - \tilde{s}_p^k|$ for $k \geqslant 1$. Using Eqs. (8b) and (12), and relation $|\max[a,b] - \max[c,d]| \leqslant |a-c| + |b-d|$, holding for $a,b,c,d \in \mathbb{R}$, we have for $k \geqslant 1$

$$\begin{aligned} |\alpha_p^{k+1} - \tilde{\alpha}_p^k| &= |\max[|f_p^{k+1} - f(x_{k+1})|, \gamma(s_p^{k+1})^\omega] - \max[|\tilde{f}_p^k - f(x_k)|, \gamma(\tilde{s}_p^k)^\omega]| \\ &\leqslant |f_p^{k+1} - \tilde{f}_p^k| + |f(x_{k+1}) - f(x_k)| + \gamma c_L |s_p^{k+1} - \tilde{s}_p^k| \to 0 \end{aligned}$$

and thus there exists a number $\bar{k} \geqslant 0$ satisfying

$$4c|\Delta_k| + |\Delta_k|^2/2 + |\alpha_p^{k+1} - \tilde{\alpha}_p^k| < \bar{c}^2 \quad \text{for all } k > \bar{k}. \tag{33}$$

Let $\tilde{k}$ be the number defined in Lemma 3.7. Choose $k_0 \in K$ satisfying $k_0 > \max[\tilde{k}, \bar{k}]$, any integer $i \in [i_{\mathrm{m}}, r]$ and set $k = k_0 + i$. It follows from Lemma 3.7 that $w_k \geqslant \delta$, $t_L^k < t_0$ and $i_{\mathrm{n}} > i_{\mathrm{m}}$ after Step 3 of Algorithm 2.1. Thus $\bar{G}_p^{k+1} = \bar{G}_p^k$ in the next Step 1 and $H_{k+1} = H_k$. Since no bundle resetting occurs (i.e $i_s \leqslant i_r$) for short and null steps, the multipliers $\lambda_{k+1} = v, \lambda_j = 0$ for $j \in J_{k+1} \setminus \{k+1\}$, $\lambda_p = 1 - v$, $v \in [0, 1]$ are feasible for the $(k+1)$th dual subproblem $(\mathscr{P}')$ and we get by Eqs. (28) and (29)

$$w_{k+1} \leqslant \tfrac{1}{2}|vH_{k+1}g_{k+1}^{k+1} + (1-v)H_{k+1}g_p^{k+1}|^2 + v\alpha_{k+1}^{k+1} + (1-v)[\tilde{\alpha}_p^k + (\alpha_p^{k+1} - \tilde{\alpha}_p^k)]. \quad (34)$$

In view of Eq. (19) we can apply Lemma 3.4 with $p = H_k\tilde{g}_p^k = -H_{k+1}^{-1}d_k$, $g = H_{k+1}g_{k+1}^{k+1}, \Delta = \Delta_k, v = v_k, w = w_k, \beta = \alpha_{k+1}^{k+1}, \alpha = \tilde{\alpha}_p^k$ and $m = m_R$, to obtain

$$w_{k+1} \leqslant w_k - w_k^2\frac{(1-m_R)^2}{8c^2} + 4c|\Delta_k| + \frac{1}{2}|\Delta_k|^2 + |\alpha_p^{k+1} - \tilde{\alpha}_p^k| < w_k - \bar{c}^2, \quad (35)$$

where the first inequality follows from Lemma 3.4 and the second from the definition of $\bar{c}$ in Eq. (32), the fact that $w_k \geqslant \delta$ and (33). For the largest $n \leqslant r$ it follows from (35), (14) and definition of c and r in Eq. (32) that

$$w_{k_0+n+1} < w_{k_0+i_m} - \bar{c}^2(n+1-i_m) < c^2/2 + c^2 - \bar{c}^2(r - i_m) = 0,$$

which is impossible. Therefore $\sigma(\bar{x}) = 0$, yielding the desired result. □

4. Superlinear convergence

In this section we show that the convergence rate of Algorithm 2.1 is superlinear and from some index on we have Newton iterations under the following assumptions: the trial points sequence $\{y_k\}$ converges to $\bar{x}$, the problem function f is strongly convex with modulus $C_F > 0$ (i.e. $f(x) - (C_F/2)|x|^2$ is convex) and has continuous second order derivatives in some neighbourhood $B(\bar{x})$ of $\bar{x}$, the number of serious steps is infinite, the locality measure parameter $\omega = 1$ and G_k are the Hessian matrices $\nabla^2 f(y_k)$.

We suppose that the final accuracy tolerance $\varepsilon = 0$ and C_F is large enough to ensure that in Step 1 of Algorithm 2.1 the matrices G_k are not modified for all $y_k \in B(\bar{x})$.

Lemma 4.1. *Let the number of serious steps generated by Algorithm 2.1 be infinite. Then for each $k_1 \geqslant 1$, there is a number $k_2 > k_1$ such that $J_k \subset \{k_1, k_1+1, \ldots\}$ and*

$$\left(G_p^{k+1}, g_p^{k+1}, s_p^{k+1}\right) = \sum_{j=k_1}^{k} \hat{\lambda}_j^k \left(\varrho_j G_j, g_j^{k+1}, s_j^{k+1}\right), \quad \hat{\lambda}_j^k \geqslant 0, \ k_1 \leqslant j \leqslant k, \ \sum_{j=k_1}^{k} \hat{\lambda}_j^k = 1 \quad (36)$$

for all $k \geqslant k_2$.

Proof. Choose $k_2 \geqslant k_1 + M - 1$ ($M \geqslant 2$ is the bundle dimension) such that in the k_2th step the bundle resetting was performed, i.e. $\lambda_p^{k_2} = 0$. Let $k \geqslant k_2$.

The bundle definition yields $J_k \subset \{k - M + 1, \ldots, k\} \subset \{k_1, k_1+1, \ldots\}$, which implies $\lambda_j^k = 0$ for $j < k_1$. Thus, letting $\hat{\lambda}_j^k$ be the same as in Lemma 3.1, we have $\hat{\lambda}_j^k = \lambda_p^k \hat{\lambda}_j^{k-1}$ for $j < k_1$ from the proof of Lemma 3.1. Since $\lambda_p^{k_2} = 0$, we obtain by induction for $k = k_2, k_2+1, \ldots,$ that $\hat{\lambda}_j^k = 0$ in Eq. (22) for $j < k_1$. Using Eqs. (15), (17), and (22) we get

$$\begin{aligned}\left(g_p^{k+1}, s_p^{k+1}\right) &= \left(\tilde{g}_p^k + G_p^{k+1}(x_{k+1} - x_k), \tilde{s}_p^k + |x_{k+1} - x_k|\right) \\ &= \sum_{j=k_1}^{k} \hat{\lambda}_j^k \left(g_j^k + \varrho_j G_j(x_{k+1} - x_k), s_j^k + |x_{k+1} - x_k|\right) \\ &= \sum_{j=k_1}^{k} \hat{\lambda}_j^k \left(g_j^{k+1}, s_j^{k+1}\right),\end{aligned}$$

which together with Eq. (22) completes the proof. □

Lemma 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied. Suppose that* $\{x_k\}, \{y_k\}$ *are sequences generated by Algorithm* 2.1, $y_k \to \bar{x}$, *the function* f *has locally Lipschitz continuous first derivatives at* $\bar{x}$, $\{H_k\}$ *is bounded and* $\omega = 1$. *Then* $\nabla f(\bar{x}) = 0$ *and there exists a number* $\tilde{k}$ *such that QP subproblem* ($\mathscr{P}$) *has only one active constraint with the index* k *whenever* $k \geqslant \tilde{k}$ *and* $y_k = x_k$.

Proof. By assumption there exists a neighbourbood $B(\bar{x})$ of $\bar{x}$ and a constant C_L satisfying

$$|g_i - g_j| \leqslant C_L |y_i - y_j| \quad \text{for all} \quad y_i, y_j \in B(\bar{x}). \tag{37}$$

By (19) and in virtue of $x_j = y_j$ for serious steps, we always have $|x_k - y_k| \leqslant C_S$. Therefore $\{x_k\}$ is bounded and since the set $\{k \mid x_k = y_k\}$ is infinite by assumption, we can apply Theorem 3.8, obtaining $0 \in \partial f(\bar{x}) = \{\nabla f(\bar{x})\}$ by continuity of ∇f at $\bar{x}$ (see Clarke, 1983). Thus $g_k \to 0$ and we can choose a number k_1 such that $y_k \in B(\bar{x})$ and

$$(C_L + C_G) C_H^2 |g_k| < \gamma \tag{38}$$

for all $k \geqslant k_1$, where $C_H = \sup\{\|H_k\| \mid k \geqslant 1\}$ and γ is the distance measure parameter.

Let $\tilde{k}$ be a number k_2 determined by Lemma 4.1 and suppose that $k > \tilde{k}$ and $y_k = x_k$. Then $\alpha_k^k = 0$ and $g_k^k = g_k$ by Eqs. (8a), (15b), (16b) and (17b), and the reduced QP subproblem

$$(\mathscr{R}) \qquad \underset{(u,z) \in \mathbb{R}^{N+1}}{\text{minimize}} \quad z + \tfrac{1}{2} u^{\mathrm{T}} \bar{G}_p^k u \qquad \text{subject to} \quad -\alpha_k^k + u^{\mathrm{T}} g_k^k \leqslant z$$

(similar to the QP subproblem ($\mathscr{P}$)) has the solution

$$u_k = -H_k^2 g_k, \qquad z_k = -u_k^{\mathrm{T}} \bar{G}_p^k u_k = g_k^{\mathrm{T}} u_k. \tag{39}$$

Since $k \geqslant \tilde{k}$, we deduce from Lemma 4.1 that $j \geqslant k_1$ for any $j \in J_k$ and, hence, $y_j \in B(\bar{x})$. By Eqs. (6), (39) and (18)

$$(g_j^k - g_k)^{\mathrm{T}} u_k \leqslant |g_j - g_k - \varrho_j G_j(y_j - x_k)|\ |u_k| \leqslant (C_L + C_G) C_H^2 |g_k| |y_j - x_k|$$

for all $j \in J_k$. Observe that the assumption $x_k = y_k$ can be fulfilled only for serious or short steps ($x_k \neq x_{k-1}$), hence $s_j^k > 0$ for $j < k$ by Eq. (15a). Thus, since $\omega = 1$ is assumed, one always has

$$(g_j^k - g_k)^{\mathrm{T}} u_k < \gamma s_j^k \leqslant \alpha_j^k \tag{40}$$

for all $j \in J_k \setminus \{k\}$ by Eqs. (38), (7) and (8a). Similarly Eqs. (36), (40) and (8b) imply

$$(g_p^k - g_k)^{\mathrm{T}} u_k = \sum_{j=k_1}^{k-1} \hat{\lambda}_j^{k-1} (g_j^k - g_k)^{\mathrm{T}} u_k < \gamma \sum_{j=k_1}^{k-1} \hat{\lambda}_j^{k-1} s_j^k = \gamma s_p^k \leqslant \alpha_p^k. \tag{41}$$

From Eqs. (39)–(41) we get $-\alpha_j^k + u_k^{\mathrm{T}} g_j^k < z_k$ for $j \in J_k \setminus \{k\}$ and $-\alpha_p^k + u_k^{\mathrm{T}} g_p^k < z_k$, hence (u_k, z_k) also solves the QP subproblem $(\mathscr{P})$, which completes the proof. □

Lemma 4.3. *Let the assumptions of Lemma 4.2 be satisfied. Suppose that the function f is strongly convex with modulus $C_F > 0$ and has continuous second-order derivatives in some neighbourhood of $\bar{x}$. Then there exists a number $\bar{k}$ such that $y_{k+1} = x_{k+1} = x_k - G_k^{-1} g_k$ (Newton step) for all $k \geqslant \bar{k}$.*

Proof. Let $K = \{k \mid x_k = y_k \text{ and } \bar{G}_p^k = G_k\}$.

(i) At first we establish the existence of a number k_0 such that $y_{k+1} = x_{k+1} = x_k + d_k$ for all $k \in K, k \geqslant k_0$. Suppose that $k \in K, k \geqslant \tilde{k}$, where $\tilde{k}$ is defined in Lemma 4.2. Then, by Lemma 4.2, $\lambda_k^k = 1$ and $\lambda_j^k = 0$ for all $j \neq k$. Hence, we have $\tilde{\alpha}_p^k = \alpha_k^k = 0$ and $\tilde{g}_p^k = g_k^k = g_k$ by Eqs. (11), (12), (15b), (16b) and (17b), which gives

$$d_k = -G_k^{-1} g_k, \qquad v_k = g_k^{\mathrm{T}} d_k = -d_k^{\mathrm{T}} G_k d_k \tag{42}$$

by Eqs. (9) and (13). Reasoning as in the proof of Lemma 4.2, we obtain $g_k \to 0$, hence $d_k \to 0$ by the boundedness of $\{H_k\}$. A Taylor series about x_k and Eq. (42) yield

$$f(x_k + d_k) - f(x_k) = d_k^{\mathrm{T}} g_k + \tfrac{1}{2} d_k^{\mathrm{T}} G_k d_k + \Delta_k = \tfrac{1}{2} v_k + \Delta_k, \tag{43}$$

where $\Delta_k = o(d_k^{\mathrm{T}} d_k)$ by continuity of $\nabla^2 f$. It follows from the strong convexity of f with modulus C_F that the smallest eigenvalue of $\nabla^2 f$ is minorized by C_F (see Hiriart-Urruty and Lemarechal, 1993). Thus there exists $k_0 \geqslant \tilde{k}$ such that

$$\Delta_k \leqslant (\tfrac{1}{2} - m_L) C_F |d_k|^2, \; d^{\mathrm{T}} G_k d \geqslant C_F |d|^2 \;\text{ for all } d \in \mathbb{R}^N, \; k \in K, \; k \geqslant k_0. \tag{44}$$

From Eqs. (42)–(44) we obtain

$$f(x_k + d_k) - f(x_k) \leqslant \tfrac{1}{2} v_k + (\tfrac{1}{2} - m_L) d_k^{\mathrm{T}} G_k d_k = m_L v_k,$$

hence Eq. (5) with $t_L^k = 1$ holds for $k \in K, \; k \geqslant k_0$.

(ii) Choose $\bar{k} \geqslant k_0 \geqslant \tilde{k}$ such taht in the $\bar{k}$th step the bundle resetting was performed. Then $\bar{k} \in K$ and thus the $\bar{k}$th step is serious by the part (i) of the proof. Since the $(\bar{k} - 1)$th step was serious, it follows from Lemma 4.2, the positive definiteness of $G_{\bar{k}+1}$ and Algorithm 2.1 that $\bar{k} + 1 \in K$. Now we can complete the proof by induction. □

In view of (44) the strong convexity and second-order differentiability assumptions of Lemma 4.3 imply the boundedness of $\{G_k^{-1}\}$ and, hence, the boundedness of $\{H_k\}$ which is assumed in Lemma 4.2.

Theorem 4.4. *Let the assumptions of Lemma* 4.3 *be satisfied. Then, after a sufficient number of steps, Algorithm* 2.1 *generates Newton iterations purely and* $\{x_k\}$ *converges to* $\bar{x}$ *superlinearly.*

Proof. Suppose that $k > \bar{k}$, where $\bar{k}$ is defined by Lemma 4.3. Write $e_k = x_k - \bar{x}$. Since $\nabla f(\bar{x}) = 0$ by, e.g., Lemma 4.2, we obtain from $y_k = x_k$ and $y_{k+1} = x_{k+1} = x_k - G_k^{-1}g_k$

$$e_{k+1} = -G_k^{-1}(g_k - G_k e_k) = -G_k^{-1}[\nabla f(\bar{x} + e_k) - \nabla f(\bar{x}) - \nabla^2 f(x_k)e_k].$$

By continuity of $\nabla^2 f$ and in view of the boundedness of $\{G_k^{-1}\}$, easy calculations give

$$|e_{k+1}|/|e_k| \leqslant \|G_k^{-1}\| \left\| \int_0^1 [\nabla^2 f(\bar{x} + \xi e_k) - \nabla^2 f(\bar{x} + e_k)\mathrm{d}\xi \right\| \to 0. \qquad \square$$

5. Numerical examples

The above concept was implemented in FORTRAN 77 as BNL. In this section we compare our results for 18 standard examples from literature with those obtained by the ellipsoid bundle method (EB) of Kiwiel (1989), by the BT algorithm (trust region concept) of Schramm and Zowe (1992) and by our implementation of the proximal bundle method (PBL, line search concept). Problems 1–14 are described in Mäkelä and Neittaanmäki (1992), problems 15–18 and also 10–12 in Kiwiel (1989). In Table 1 we give optimal values of tested functions.

The parameters of the algorithm had the values $M = N + 3$, $\zeta = m_L = 0.01$, $\vartheta = 1, m_R = 0.5, t_0 = 0.001, C_S = C_G = 10^{50}, i_\mathrm{m} = i_\mathrm{r} = 100$. The algorithm of Lukšan

Table 1
Test problems

Nr.	N	Problem	Minimum	Nr.	N	Problem	Minimum
1	2	Rosenbrock	0.0	10	4	Rosen	−44.0
2	2	Crescent	0.0	11	5	Shor	22.600162
3	2	CB2	1.9522245	12	10	Maxquadl	−0.84140833
4	2	CB3	2.0	13	20	Maxq	0.0
5	2	DEM	−3.0	14	20	Maxl	0.0
6	2	QL	7.20	15	5	Colville	−32.348679
7	2	LQ	−1.4142136	16	15	SHELL DUAL	32.348679
8	2	Mifflin 1	−1.0	17	30	MXHILB	0.0
9	2	Mifflin 2	−1.0	18	30	L1HILB	0.0

Table 2
Our test results

Nr.	BNL $\omega = 1$				BNL $\omega = 2$				PBL		
	N_i	N_f	F	γ	N_i	N_f	F	γ	N_i	N_f	F
1	51	52	0.12×10^{-18}	0.5	59	60	0.367×10^{-15}	1.3	42	45	0.381×10^{-6}
2	7	8	0.168×10^{-10}	10^{-4}	7	8	0.168×10^{-10}	0.001	18	20	0.679×10^{-16}
3	9	10	1.9522245	0.25	8	9	1.9522245	1.0	32	34	1.9522245
4	14	15	2.0000000	0.01	13	14	2.0000000	0.1	14	16	2.0000000
5	15	16	−3.0000000	0.1	14	15	−3.0000000	0.25	17	19	−3.0000000
6	4	6	7.2000000	10^{-10}	4	6	7.2000000	10^{-10}	13	15	7.2000015
7	16	17	−1.4142136	10^{-10}	16	17	−1.4142136	10^{-10}	11	12	−1.4142136
8	11	13	−1.0000000	0.1	12	14	−1.0000000	0.08	66	68	−0.99999941
9	10	11	−1.0000000	10^{-10}	10	11	−1.0000000	10^{-10}	13	15	−1.0000000
10	13	15	−44.000000	10^{-10}	13	15	−44.000000	10^{-10}	43	45	−43.999999
11	7	8	22.600173	10^{-10}	7	8	22.600173	10^{-10}	27	29	22.600162
12	12	14	−0.84140833	10^{-4}	12	14	−0.84140833	0.01	80	81	−0.84140833
13	38	39	0.33×10^{-8}	10^{-10}	38	39	0.33×10^{-8}	10^{-10}	161	162	0.166×10^{-7}
14	24	25	0.453×10^{-8}	10^{-10}	24	25	0.453×10^{-8}	10^{-10}	39	40	0.242×10^{-12}
15	18	20	−32.348679	0.08	18	20	−32.348679	0.25	62	64	−32.348679
16	247	258	32.348679	10^{-3}	423	482	32.348680	0.06	1410	1501	32.349129
17	14	15	0.5×10^{-8}	10^{-10}	14	15	0.480×10^{-8}	10^{-10}	19	20	0.424×10^{-8}
18	14	15	0.141×10^{-8}	10^{-10}	13	14	0.110×10^{-8}	10^{-10}	19	20	0.99×10^{-9}
$\sum$	524	557			705	786			2086	2206	
	Time = 9.17 s				Time = 12.80 s				Time = 16.04 s		

Table 3
Test results for EB and BT methods

Nr.	BT			Nr.	BT			EB		
	N_i	N_f	F		N_i	N_f	F	N_i	N_f	F
1	79	88	0.130E-11	10	22	32	−43.99998	20	20	−43.9998
2	24	27	0.944E-06	11	29	30	−22.60016	45	45	22.60017
3	13	16	1.952225	12	45	56	−0.8414083	58	58	−0.84135
4	13	21	2.000000	13	125	128	0.0	–	–	–
5	9	13	−3.000000	14	74	84	0.0	–	–	–
6	12	17	−7.200009	15	–	–	–	45	45	−32.3486
7	10	11	−1.414214	16	–	–	–	191	600	32.3538
8	49	74	−1.000000	17	–	–	–	15	15	0.13×10^{-7}
9	6	13	−1.000000	18	–	–	–	16	16	0.77×10^{-8}

(1984) was employed for solving the QP subproblem. To cut off useless iterations, the algorithm stopped:

if $|H_k \tilde{g}_p^k|^2 + 100\tilde{\alpha}_p^k/(|f(x_k)| + 0.001) \leqslant 2 \cdot 10^{-6}$

or $(|f(y_k) - f(x_{k-1})|)/\max[1, |f(y_k)|] \leqslant 10^{-8}$ in two consecutive iterations.

Our results are summarized in Table 2, in which the following notation is used. N_i is the number of iterations, N_f is the number of objective function (and also subgradient and matrix G_k) evaluations, F is the objective function value at termination and γ is the distance measure parameter value (values of γ were chosen experimentally). In Table 3 we compare our results with those obtained by the EB and BT methods.

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