STOCHASTIC STABILITY AND CONTROL

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Harold J. Kushner

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Preface

In recent years there has been a great deal of activity in the study of qualitative properties of solutions of differential equations using the Liapunov function approach. Some effort has also been devoted to the application of the "Liapunov function method" to the design of controls or to obtain sufficient conditions for the optimality of a given control (say, via the Hamilton–Jacobi equation, or dynamic programming approach to optimal control).

In this monograph we develop the stochastic Liapunov function approach, which plays the same role in the study of qualitative properties of Markov processes or controlled Markov processes as Liapunov functions do for analogous deterministic problems. Roughly speaking, a stochastic Liapunov function is a suitable function of the state of a process, which, considered as a random process, possesses the supermartingale property in a neighborhood. From the existence of such functions, many properties of the random trajectories, both asymptotic and finite time, can be inferred. The motivation for the work was the author's interest in stochastic problems in control, and the methods to be discussed enlighten many such problems; nevertheless, much of the material seems to have an independent probabilistic interest.

Although some discrete time results are given, we have emphasized processes with a continuous time parameter, which require somewhat more elaborate methods. Actually, most of the proofs are not difficult in that they do not involve highly detailed or subtle arguments; some of the proofs are difficult in the sense that we have found it necessary to refer, in their proof, to theorems which are subtle. Some of these are discussed in the background material of Chapter I, and others in the referenced works of E. B. Dynkin.

The analysis requires the introduction of the weak infinitesimal operators of (strong) Markov processes, either as a general abstract object, or in the forms in which it appears for special cases. Instead of doing the analysis for various special processes, we chose the economical alternative of treating general cases, and listing special cases in remarks or corollaries.

Chapter I is devoted to a discussion of many of the concepts from probability theory which are used in the sequel. Chapter II, on stability, is the longest and probably the most basic part; the material underlies most of the results of the other chapters. The chapter is by no means exhaustive. We have concentrated on several results which we feel to be important (and can prove); there are certainly many other cases of potential interest, both obvious and not. The chapter contains a number of nonlinear and linear examples. Nevertheless, a quick survey of the examples reveals a shortcoming that we share with the deterministic method; namely, the difficulties in finding suitable Liapunov functions. In particular, we have not been able to obtain a family of Liapunov functions which can be used to completely characterize the asymptotic stability properties of the solutions of linear differential equations with homogeneous Markov process coefficients (i.e., obtain necessary and sufficient conditions in terms of (say) the transition functions of the coefficient processes, for certain types of statistical asymptotic stability).

Chapter III is devoted to the study of first exit times or, equivalently, to the problem of obtaining useful upper bounds to the probability that the state of the process will leave some given set at least once by a given time. The approach is expected to be useful, in view of the importance of such problems in control and the difficulty of obtaining useful estimates.

Chapter IV is devoted to problems in optimal control; in particular, to the determination of sufficient conditions for the optimality of a control. Some of the material provides a stochastic analog to the

Hamilton–Jacobi equation, or dynamic programming, approach to sufficient conditions for optimality in the deterministic case, and provides a justification for some well-known formal results obtained by dynamic programming.

In Chapter V, we discuss several uses of stochastic Liapunov functions in the design of controls. The method may be applied to the computation of a control which reduces some "cost" or ensures that some stability property obtains.

It is a pleasure to express my appreciation to P. L. Falb and W. Fleming for their helpful criticisms on parts of the manuscript, and to Mrs. KSue Brinson for the excellent typing of several drafts.

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April 1967 HAROLD J. KUSHNER

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Contents

I / INTRODUCTION

In this chapter, we list and discuss some results and definitions in various branches of the theory of Markov processes which are to be used in the sequel. The chapter is for introductory purposes only, and all the ideas which are discussed are treated in greater detail in the listed references. Occasionally, to simplify the introduction, the list of properties which characterize a definition will be shortened, and reference made to the precise definition. The sample space is denoted by Ω, with generic point ω. The range of the process is in a Euclidean space E, and $\mathscr{E}$ is a Borel field of sets in E.

1. Markov Processes

Discrete time Markov process

Let $x_1, \ldots$ be a discrete time parameter stochastic process with the associated transition function $P(s, x; n+s, \Gamma)$. Then the function $P(s, x; n+s, \Gamma)$ is to be interpreted as the probability that x_{n+s} is in $\Gamma \in \mathscr{E}$, given that $x_s = x$. Suppose that the conditional distribution function satisfies

$$P\{x_{n+s} \in \Gamma | x_1, \ldots, x_s\} = P\{x_{n+s} \in \Gamma | x_s\} \tag{1-1}$$

for all Γ in $\mathscr{E}$ and nonnegative n and s, with probability one. Then the process is termed a Markov process and, in addition, the Chapman-

Kolmogorov equation (1-2) holds:

$$P(s, x; n + m + s, \Gamma) = \int_E P(s, x; s + m, dy)\, P(s + m, y; n + m + s, \Gamma). \quad (1\text{-}2)$$

The following alternative characterization of a Markov process will be useful. To each point x in E and nonnegative integer $s < \infty$, a probability measure $P_{x,s}\{A\}$ is associated. The measure, on an appropriate σ-algebra of sets in Ω is defined by* its values on the Borel sets

$$P_{x,s}\{x_{n+s} \in \Gamma\} = P(s, x; s + n, \Gamma)$$

and satisfies with probability one

$$P_{x,m}\{x_{m+n+s} \in \Gamma | x_1, \ldots, x_{m+s}\} = P(m + s, x_{m+s}; s + n + m, \Gamma)$$

for each Γ in $\mathscr{E}$. $P_{x,m}\{A|b\}$ is interpreted as the probability of the event A, with respect to the measure $P_{x,m}$ conditioned on the minimum σ-algebra on Ω over which b is measurable.† Thus $P_{x,s}\{x_{n+s} \in \Gamma\}$ is the probability that x_{n+s} is in Γ, given that the process starts at s with $x_s = x$, a constant. If $P_{x,s}\{x_{n+s} \in \Gamma\}$ does not depend on s, we write $P_x\{x_n \in \Gamma\}$ for the probability that $x_{n+s} \in \Gamma$, given that the process starts at x with $x_s = x$, for any initial time s.

The sample paths of stochastic process are not always defined for all time (for either continuous or discrete parameter processes). There may be a nonzero probability that the process will escape to infinity in a finite time, or at least leave the domain on which it is defined as a Markov process at some finite time. Consider the trivial Markov process whose paths are the solutions of the differential equation $\dot{x} = x^2$ where the initial value x_0 (taken at $t = 0$) is a random variable. Then, for any $T < \infty$, it is easily seen that $x_t \to \infty$ as $t \to \xi(x_0) \leqq T$, with the probability that $x_0 \geqq 1/T$.

In general to each initial condition x in E there is associated a

* $\{x_{n+s} \in \Gamma\}$ is defined as the ω set $\{\omega\colon x_{n+s} \in \Gamma\}$.

† Sometimes the appropriate σ-algebra is written in lieu of b.

random variable ξ which takes values in $[0, \infty]$. ξ is termed the "killing time" and the process x_n is defined for $n < \xi$ only. If $\xi < \infty$ with a nonzero probability, then $P_{x,m}\{x_n \in E\}$ will be less than 1 for large n. If the sample paths are defined for all time with probability one, then $\xi = \infty$ with probability one.

It is often convenient to suppose that $x_n = \infty$ for $n \geqq \xi$. However, for most purposes of the sequel, the killing time will be unimportant, and unless mentioned otherwise, we assume that it is equal to ∞ with probability one. Generally we will be concerned with the behavior of a process up to the first instant of time that the process exists from a given set. It will be required that the process be defined up to this time, and this will be true either by assumption or as a known property of the specific case treated. An exception to this is Theorem 8, Chapter II, in which it is proved that $\xi = \infty$ with probability one (a type of Lagrange stability) under stated conditions on the process.

Continuous time Markov processes

If the time indices are allowed to take any values in $[0, \infty)$, then the discrete case definitions carry over to the continuous parameter case. Precise definitions and results on continuous parameter processes are given in Dynkin [1, 2]. Other useful sources on discrete or continuous parameter processes are Feller [1], Chung [1], Doob [1], Loeve [1], and Bharucha-Reid [1].

Occasionally, to assist in the simplification of notation, it will be helpful to consider time as a component of the Markov process. According to the definitions above, if x_t is a Markov process, then so is the pair* (x_t, t). We will occasionally use the convention of writing the state (x, s) (or (x_s, s)) simply as x (or x_s). The new state space $E \times [0, \infty)$ (which contains the (x, t) sets Γ) is then written simply as E.

* The pair (x_t, t) does not satisfy the precise definition of a Markov process as given by Dynkin [2], p. 78, condition 3.1G. This deficiency is easily overcome by a standard enlargement of the space Ω which leaves all other properties intact. See footnote on p. 79 of Dynkin [2].

This will allow several results which are stated more succinctly in the homogeneous case terminology, to extend to the nonhomogeneous case.

Notation

The following terminology will be used. Suppose that time is not a component of the state. Let the process x_s be homogeneous. Then $P_x\{x_t \in \Gamma\} = P(s, x; s+t, \Gamma) \equiv P(x, t, \Gamma)$ with probability one for any $s \geqq 0$. Also, $E_x f(x_t) = \int f(y)\, P(x, t, dy)$. If the process is not homogeneous, let $P_{x,s}\{x_{t+s} \in \Gamma\} = P(s, x; s+t, \Gamma)$ and $E_{x,s} f(x_{t+s}) = \int f(y)$ $P(s, x; s+t, dy)$. In interpreting $P_{x,s}\{x_{t+s} \in \Gamma\}$ it is to be understood that x is the initial value of the process *at initial time s*. $P_x(d\omega)$ and $P_{x,s}(d\omega)$ are the probability measures corresponding to the homogeneous and nonhomogeneous cases, respectively.

If time is considered as a component of the state, then either the homogeneous case or the nonhomogeneous case terminology may be applied.

Continuity

The process x_t is said to be *stochastically continuous at the point x* if

$$P_x\{\|x_\delta - x\| \geqq \varepsilon\} \to 0, \qquad \|x\|^2 = \sum x_i^2$$

as $\delta \to 0$, for any $\varepsilon > 0$. If

$$P_x\{\sup_{\delta \geqq \Delta \geqq 0} \|x_\Delta - x\| \geqq \varepsilon\} \to 0$$

uniformly for x in a set M, as $\delta \to 0$, for any $\varepsilon > 0$, then the process is *uniformly stochastically continuous* in the set M.

2. Strong Markov Processes

In this section, it is assumed that the killing time ξ equals infinity with probability one. This assumption will not restrict the results of the following chapters, but will allow a discussion of a number of concepts and results of Dynkin [2] (which we will use later) with a reasonably unburdened notation.

Markov time

Let $\mathscr{F}_t$ be the minimum σ-algebra on Ω determined by conditions on x_s, $0 \leqq s \leqq t$, $x_0 = x$ (and completed with respect to the measure $P_x(d\omega)$). A random variable τ taking values in $[0, \infty]$ and depending also on x is called a Markov time if the event $\{\tau \leqq t\}$ is contained in $\mathscr{F}_t$ for each $t < \infty$ and fixed x. τ is also called a "time" or "optional time." More intuitively, τ is a functional of the sample paths and, whether or not the time τ has "arrived" by the time t, can be determined by observing the sample paths only up to and including t.

Define the first exit time of x_t from an open set Q by $\tau = \inf\{t: x_t \text{ not in } Q\}$. It is intuitively clear that if x_t is continuous from the right with probability one, then τ is a Markov time (since whether or not the process has left Q at least once by time t can be determined (with probability one) by observing x_s, $s \leqq t$). More precisely (see Loeve [1], p. 580), let R_n be the set of x points whose distance (Euclidean) from $E - Q$ is less than $1/n$ and let $\{r_i\}$ be the rationals. Then

$$\{\tau \leqq t\} = \{x_t \notin Q\} \cup \bigcap_n \bigcup_{r_i \leqq t} \{x_{r_i} \in R_n\} + N \in \mathscr{F}_t,$$

where N is a null set.

A much more thorough characterization of the first entrance and exit times which are also Markov times is given in Dynkin [2, Chapter 4, Section 1]. See also Itô [2].

If a Markov time is undefined for some ω, we set it equal to infinity.

Strong Markov process

Markov times play an important role in the analysis of a Markov process, and will play a crucial role in the sequel. The events of greatest interest to us will occur at Markov times; for example, the first time that a controlled process enters a target set; the first time that a trajectory leaves a given neighborhood of the origin (which is a property of interest in stability), or the first time that two processes are within ε of one another. In addition, much of the analysis concerns the behavior of sequences of random variables which are the values of a Markov process at a sequence of Markov times.

It is useful to further restrict the class of Markov process with which we deal so that the analysis with the Markov times will be feasible. The definitions which follow are according to Dynkin [2]. First, let τ be a Markov time and define the σ-algebra $\mathscr{F}_\tau$ as follows: let the set A be in $\mathscr{F}_\tau$ if

$$A \cap \{\tau \leqq t\} \in \mathscr{F}_t \tag{2-1}$$

for each $t \geqq 0$. The verification that $\mathscr{F}_\tau$ is a σ-field is straightforward; for example, let A_n, $n = 1, \ldots$ be in $\mathscr{F}_\tau$. Then since $A_n \cap \{\tau \leqq t\}$ is in $\mathscr{F}_t$ for each t and n, $(\bigcup_n A_n) \cap \{\tau \leqq t\}$ is also in $\mathscr{F}_t$ for each $t \geqq 0$; hence, $\bigcup_n A_n$ is in $\mathscr{F}_\tau$, and so forth.

$\mathscr{F}_\tau$ includes those events which can be determined by observations on the process up to and including, but no later than, τ. Two examples will clarify this. Let $\tau = s$, a constant. τ is obviously a Markov time. For $t < s$, $\{\tau \leqq t\}$ is empty, and all sets A satisfy (2-1). For $t \geqq s$, $\{\tau \leqq t\} = \Omega$ and only the sets A in $\mathscr{F}_s$ satisfy $\Omega \cup A \in \mathscr{F}_t$, all $t \geqq s$. Thus $\mathscr{F}_\tau = \mathscr{F}_s$.

Consider also the simple example where Ω contains all possible outcomes of two successive coin tosses; $\Omega = \{HH, HT, TH, TT, \phi\}$. Let $\tau = 1$ if the first toss is heads and $\tau = 2$ otherwise. τ is a random time. Also, $\mathscr{F}_1 = \{\Omega, \phi, HH \cup HT, TT \cup TH\}$ and $\mathscr{F}_2$ contains all possible combinations of points in Ω. It is easily verified that $\mathscr{F}_\tau$ is the field over $\{\Omega, \phi, HH \cup HT, TH \cup TT, TH, TT\}$, that is, the collection of events which can be observed at $n = 1$, if the first toss is a head, or at $n = 2$ otherwise.

In what follows we always assume that the process x_t is continuous on the right with probability one; *no other processes will be considered* *in the monograph.* The process x_t is termed* a *strong Markov process* if, for any Markov time τ and any $t \geqq 0$, Γ in $\mathscr{E}$, and x in E, the conditional probability satisfies

$$P_x\{x_{t+\tau} \in \Gamma | \mathscr{F}_\tau\} = P(t, x_\tau, \Gamma) \tag{2-2}$$

with probability one. Equation (2-2) has the interpretation that the probability of $\{x_{t+\tau} \in \Gamma\}$, conditioned upon the history up to τ, equals the probability of $\{x_{t+\tau} \in \Gamma\}$, conditioned upon x_τ only. For example, let x_t be a continuous process, and let τ be the first exit time from a bounded open set Q. Then $x_\tau \in \partial Q$, and (2-2) means that the *conditional* probability that the path will be in Γ, t units of time after contacting ∂Q, is the same whether only the position on ∂Q at τ is given, or whether the method of attaining the position on ∂Q at τ is given.

Since (2-2) holds for τ equal to any finite constant, any strong Markov process is also a Markov process. The reverse is not true. See the counterexample in Loeve [1], pp. 577–578. Nevertheless, to the author's knowledge, all Markov processes which have been studied as models of physical processes are strongly Markovian.

Discrete parameter processes

All discrete parameter Markov processes are also strong Markov processes. This follows directly from Loeve [1, p. 581]. The basic distinction between discrete and continuous parameter processes which yields the result is that Markov times for discrete parameter processes can take only countably many values. That this is sufficient may be seen from the following argument. Suppose that τ is a Markov time for the process $x_1, \ldots$. Assume, for simplicity, that the process is homogeneous. To prove the result we must show that

$$P_x\{x_{\tau+n} \in \Gamma | x_r, r \leqq \tau\} = P(n, x_\tau, \Gamma) \tag{2-3}$$

* If x_t is not right continuous, then Dynkin [2], p. 99, requires that it be measurable.

with probability one. By the left side of (2-3), we mean $P_x\{x_{\tau+n}\in\Gamma\,|\mathscr{F}_\tau\}$. Equation (2-3) is equivalent to the statement that

$$P_x\{A\cap\{x_{n+\tau(\omega)}\in\Gamma\}\}=\int_A P_x(d\omega)\,P(n,x_{\tau(\omega)},\Gamma) \tag{2-4}$$

for any set A in $\mathscr{F}_\tau$. Define the event $B_j=\{\tau=j\}$. The B_j are disjoint sets and their union (including the event $B_\infty=\{\tau=\infty\}$) is Ω. Thus (2-4) may be written as

$$\begin{aligned}\sum_j P_x\{A\cap B_j\cap\{x_{n+j}\in\Gamma\}\} &= \int_{A\cap(\cup B_j)} P_x(d\omega)\,P(n,x_{\tau(\omega)},\Gamma)\\ &=\sum_j\int_{A\cap B_j} P_x(d\omega)\,P(n,x_j,\Gamma).\end{aligned} \tag{2-5}$$

However, the defining relation (2-6) for discrete parameter Markov processes

$$P_x\{x_{n+j}\in\Gamma|x_r,\,r\leqq j\}=P(n,x_j,\Gamma)\qquad\text{(with probability one)} \tag{2-6}$$

implies (2-5) and concludes the demonstration.

Feller processes

Suppose that the function $f(x)$ is bounded, continuous, and has compact support. If the function

$$E_x f(x_t)=F(x)$$

is continuous in x, for each $t\geqq 0$, the process is termed a *Feller process*. Also, *every right continuous Feller Markov process is a strong Markov process* (Dynkin [2, Theorem 5.10]). Indeed, this criterion provides a quite useful computational check for the strong Markov property; see for example, Sections 4 and 5. Other, more abstract, criteria for the strong Markov property are given in Dynkin [2], Chapter 3, Section 3.

WEAK INFINITESIMAL OPERATOR OF A STRONG MARKOV PROCESS

The function $f(x)$ is said to be in the domain of the weak infinitesimal operator $\tilde{A}$ of the process x_s, and we write $\tilde{A}f(x) = h(x)$, if the limit

$$\lim_{\delta \to 0} \frac{E_x f(x_\delta) - f(x)}{\delta} = h(x) \tag{2-7}$$

exists pointwise in E, and satisfies*

$$\lim_{\delta \to 0} E_x h(x_\delta) = h(x). \tag{2-8}$$

Clearly, $\tilde{A}$ is linear. It will be calculated for Itô and Poisson processes in Sections 4 and 5, respectively.

Suppose that x_t is the solution to an ordinary deterministic differential equation $\dot{x} = g(x)$. Then "$f(x)$ is in the domain of $\tilde{A}$" implies that $f(x)$ has continuous first partial derivatives and that $\dot{f}(x_t)$ is continuous in t. Under these conditions $\tilde{A}f(x) = f_x'(x)\,\dot{x} = \dot{f}(x)$. In general, $\tilde{A}f(x)$ is interpreted as the average time rate of change of the process $f(x_s)$ at time s, given that $x_s = x$. Note that $\tilde{A}$ is not necessarily a local (in the ordinary Euclidean topology) operator. The value of the function $\tilde{A}f(x) = h(x)$ at some point x_0 will depend on the values of $f(x_s)$ (with initial condition x_0) which are attainable for small s. For example, if x_t is a Poisson process, then $h(x_0)$ will depend on the values of $f(x)$ which can be reached by a single jump from x_0. For a detailed development of the properties and uses of $\tilde{A}$ and related operators, the reader is referred to the monograph of Dynkin [2]. Its application in the sequel is due to formula (2-9).

Consider the form and domain of the weak infinitesimal operator when the process is not homogeneous or when $f(x, t)$ depends explicitly on time. Suppose that for each constant b in some interval the function of x given by $f(x, b)$ is in the domain of $\tilde{A}$ and $\tilde{A}f(x, b) = h(x, b)$. Suppose also that $f(x, t)$ has a *continuous and bounded deriva-*

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* If the limits (2-7) and (2-8) are uniform in x in E, then $f(x)$ is in the domain of the strong infinitesimal operator of the process.

tive, $f_t(x, t)$, with respect to t, for each x in E and $t \leqq T$. Let

$$E_{x,t}\, h(x_{t+\delta}, t+\delta) \to h(x, t)$$
$$E_{x,t} f_t(x_{t+\delta}, t+\alpha) \to f_t(x, t)$$

as $\delta \to 0$ and $\delta \geqq \alpha \to 0$. Then (2-7a) and (2-8a) hold and $f(x, t)$ is in the domain of $\tilde{A}$. (In (2-7a) and (2-8a) the symbol $\tilde{A}'$ is used for the weak infinitesimal operator of x_s operating on $f(x, t)$, where t is fixed.)

$$\begin{aligned}
\lim_{\delta \to 0} & \frac{E_{x,t} f(x_{t+\delta}, t+\delta) - f(x, t)}{\delta} \\
&= \lim_{\delta \to 0} \frac{E_{x,t} f(x_{t+\delta}, t+\delta) - E_{x,t} f(x_{t+\delta}, t)}{\delta} \\
&\quad + \lim_{\delta \to 0} \frac{E_{x,t} f(x_{t+\delta}, t) - f(x, t)}{\delta} \\
&= \lim_{\substack{\delta \to 0 \\ \delta \geqq \alpha \to 0}} E_{x,t} f_t(x_{t+\delta}, t+\alpha) + \tilde{A}' f(x, t) \\
&= f_t(x, t) + h(x, t)
\end{aligned} \tag{2-7a}$$

$$\lim_{\delta \to 0} E_{x,t}[f_t(x_{t+\delta}, t+\delta) + h(x_{t+\delta}, t+\delta)] = f_t(x, t) + h(x, t). \tag{2-8a}$$

Dynkin's formula

Suppose that x_t is a right continuous strong Markov process and τ is a random time with $E_x \tau < \infty$. Let $f(x)$ be in the domain of $\tilde{A}$, with $\tilde{A} f(x) = h(x)$. Then (Dynkin [2], p. 133)

$$E_x f(x_\tau) - f(x) = E_x \int_0^\tau h(x_s)\, ds = E_x \int_0^\tau \tilde{A} f(x_s)\, ds. \tag{2-9}$$

Equation (2-9) will be termed *Dynkin's formula*. The deterministic counterpart is the basic formula of the calculus:

$$f(x_t) - f(x) = \int_0^t \dot{f}(x_s)\, ds.$$

3. Stopped Processes

Define $t \cap s = \min(t, s)$. Suppose that τ is the first time of exit of x_s from an open set Q. The process $\tilde{x}_t \equiv x_{t \cap \tau}$ is a *stopped process*:

$$\tilde{x}_t = x_t \qquad (t < \tau),$$
$$\tilde{x}_t = x_\tau \qquad (t \geqq \tau).$$

We use the notation P_x^Q and E_x^Q (in lieu of P_x and E_x) for the stopped process only when confusion may otherwise arise. If x_t is right continuous, then so is $\tilde{x}_t$. The process $\tilde{x}_t$ is strongly Markovian (Dynkin [2], Theorem 10.2). Denote the corresponding weak infinitesimal operator by $\tilde{A}_Q$.

If the limit (3-2) exists for the process $\tilde{x}_t$ and $\lim_{\delta \to 0} E_x^Q h(\tilde{x}_\delta) = h(x)$, then $f(x)$ is in the domain of $\tilde{A}_Q$:

$$\begin{aligned} E_x^Q f(\tilde{x}_\delta) &= E_x[\chi_{\tau \geqq \delta}] f(x_\delta) + E_x[\chi_{\tau < \delta}] f(x_\tau) \\ \tilde{A}_Q f(x) &= \lim_{\delta \to 0} \frac{E_x[\chi_{\tau \geqq \delta}] f(x_\delta) + E_x[\chi_{\tau < \delta}] f(x_\tau) - f(x)}{\delta} = h(x). \end{aligned} \tag{3-1}$$

χ_A is the characteristic function of the set A.

Suppose that x is not in Q. Then $\tilde{A}_Q f(x) = 0$. If x is in Q, then $\tau > 0$ with probability one by right continuity. If $f(x)$ is in the domain of $\tilde{A}$ and

$$\lim_{\delta \to 0} \frac{(E_x^Q f(\tilde{x}_\delta) - f(x)) + (f(x) - E_x f(x_\delta))}{\delta} \to 0 \tag{3-2}$$

and

$$E_x^Q h(\tilde{x}_\delta) \to h(x) \qquad \text{as} \qquad \delta \to 0 \tag{3-3a}$$
$$E_x h(x_\delta) \to h(x) \tag{3-3b}$$

for all $x \in Q$, then $f(x)$ is also in the domain of $\tilde{A}_Q$ and $\tilde{A}_Q f(x) = \tilde{A} f(x)$ for x in Q. Equation (3-2) equals

$$\frac{E_x \chi_{\delta > \tau}[f(x_{\delta \cap \tau}) - f(x_\delta)]}{\delta}.$$

If $V(x)$ is bounded and continuous, then the limit (3-2) is zero if

$$\limsup_{\delta \to 0} \frac{P_x\{\tau < \delta\}}{\delta} < \infty. \tag{3-4}$$

Equation (3-4) is true for the Itô and Poisson processes to be discussed, and essentially implies that the transition density $P(t, x, \Gamma)$ is differentiable with respect to t (from above) at $t = 0$.

If $f(x)$ is in the domain of $\tilde{A}$ and (3-2) holds, then $f(x)$ is in the domain of $\tilde{A}_Q$ if $h(x)$ is continuous and bounded in Q. This follows since the difference between the left-hand sides of (3-3a) and (3-3b) is

$$\lim_{\delta \to 0} E_x \chi_{\delta > \tau} [h(x_{\delta \cap \tau}) - h(x_\delta)]$$

which equals zero under the above condition.

It is implicitly assumed in much of the text that if $Q \supset P$, then $V(x)$ in the domain of $\tilde{A}_Q$ implies $V(x)$ is in the domain of $\tilde{A}_P$ and $\tilde{A}_Q V(x) = \tilde{A}_P V(x)$ in P.

4. Itô Processes

A class of continuous time Markov processes whose members are often used as models of stochastic control systems are the solution processes of the stochastic differential (Itô) equation

$$dx = f(x, t)\, dt + \sigma(x, t)\, dz. \tag{4-1}$$

z_t is a normalized vector Wiener process with $E(z_t - z_s)(z_t - z_s)' = I|t - s|$, where I is the identity matrix. Define the matrix

$$\{S_{ij}(x, t)\} = S(x, t) = \sigma'(x, t)\, \sigma(x, t).$$

In some engineering literature, (4-1) appears as

$$\dot{x} = f(x, t) + \sigma(x, t)\psi, \tag{4-2}$$

where ψ is called "white Gaussian noise." The resemblence of (4-2) to an ordinary differential equation is misleading, since ψ is not a function. Some relations between the solutions of (4-1) (or (4-2)) and the solution of ordinary differential equations are discussed in Example 2 of Chapter II. See also Wong and Zakai [1, 2]. If $\sigma(x, s) \equiv 0$, then (4-1) is interpreted to be an ordinary differential equation. Equation (4-1) is given a precise interpretation by Itô [1], who writes it as an integral equation:

$$x_t = x_0 + \int_0^t f(x_s, s)\, ds + \int_0^t \sigma(x_s, s)\, dz_s .$$

The second integral, called a stochastic integral, is defined roughly as follows. A random function V_s and variable τ, respectively, are said to be nonanticipative if V_s and the event $\{\tau \leqq s\}$, respectively, are independent of $z_t - z_u$ for all triples $t \geqq u \geqq s$. Suppose that $g(\omega, s)$ is a scalar-valued nonanticipative random function which satisfies

$$\int_0^T Eg^2(\omega, s)\, ds < \infty . \tag{4-3}$$

Let $\{t_i\}$ be an increasing sequence of numbers tending to T. Now, let $\hat{g}(\omega, s)$ satisfy (4-3) and take the value (independent of s) $\hat{g}_i(\omega)$ in the interval $[t_i, t_{i+1})$. Define the stochastic integral $\hat{g}(\omega, s)$ by

$$\int_0^t \hat{g}(\omega, s)\, dz_s = \sum_i \hat{g}_i(\omega)\,(z_{t_{i+1}} - z_{t_i}) . \tag{4-4}$$

Approximate $g(\omega, s)$ to be a mean fundamental sequence of such simple functions. The integral $\int_0^t g(\omega, s)\, dz_s$ is defined as a (mean square or probability one) limit of the corresponding sequence (4-4). With this definition, and the conditions (4-5), Itô uses a Picard iteration technique to construct a sequence of processes which converge with probability one to a process which is defined as the solution to (4-1):

$$\begin{aligned} &f(x, t), \qquad \sigma(x, t) \qquad \text{are continuous functions} \\ &\|f(x, t)\|^2 = \sum_i |f_i(x, t)|^2 \leqq K(1 + \|x\|^2) \\ &\|\sigma(x, t)\|^2 = \sum_{i,j} |\sigma_{ij}(x, t)|^2 \leqq K(1 + \|x\|^2) \\ &\|f(x + \alpha, t) - f(x, t)\| \leqq K\,\|\alpha\| \\ &\|\sigma(x + \alpha, t) - \sigma(x, t)\| \leqq K\,\|\alpha\| \end{aligned} \tag{4-5}$$

where K *is a nonnegative real number.*

The scalar case of (4-1) is discussed by Doob [1], and the vector case by Dynkin [2] and Skorokhod [1]. Under (4-5), the solution of (4-1) is a Markov process with killing time equal to infinity. The process is continuous with probability one and, for any $0 \leqq a < b < \infty$ and initial condition x,

$$E_{x,a} \max_{a \leqq t \leqq b} \|x_t\|^2 < \infty.$$

x_t is independent of $z_s - z_u$, all $s > u \geqq t$. Also

$$P_{x,t}\{\max_{\delta \geqq \Delta \geqq 0} \|x_{t+\Delta} - x\| \geqq \varepsilon > 0\} \leqq (1 + \|x\|^2)^{3/2}\, O(\delta^{3/2}). \quad (4\text{-}6)$$

$O(\cdot)$ is uniform in t and x, but depends on ε. The scalar version of (4-6) is given in Doob [1], p. 285, and the vector version is derived in an identical manner. By (4-6), the process is uniformly stochastically continuous in any compact set.

Skorokhod [1] shows that, for fixed t, x_t is continuous in probability with respect to the initial condition. In other words, let the solutions x_t and y_t of (4-1) correspond to initial values x and y, respectively. Then

$$P_{x,y,t}\{\|x_{t+\delta} - y_{t+\delta}\| \geqq \varepsilon\} \to 0 \quad (4\text{-}7)$$

as $\|x - y\| \to 0$, for any $\varepsilon > 0$, $\delta > 0$. Since $E_{x,a} \max_{a \leqq t \leqq b} \|x_t\|^2 < \infty$, for each fixed a, b, and x, Chebychev's inequality gives

$$P_{x,t}\{\|x_T - x\| > N\} \leqq \frac{K_1}{N^2}, \quad (4\text{-}8)$$

where K_1 is some real number. See Doob [1], p. 285, for the derivations of the scalar versions of (4-9) and (4-10):

$$E_{x,t}(x_{t+\delta} - x) = \int_t^{t+\delta} f(s, x)\, ds + (1 + \|x\|^2)^{1/2}\, O(\delta^{3/2}), \quad (4\text{-}9)$$

$$E_{x,t}(x_{t+\delta} - x)(x_{t+\delta} - x)'$$

$$= \int_t^{t+\delta} S(s, x)\, ds + (1 + \|x\|^2)^{1/2}\, O(\delta^{3/2}). \quad (4\text{-}10)$$

STRONG MARKOV PROPERTY

Next, we show that the solution (in the sense of Itô) of (4-1) is a Feller process and, hence, since the paths x_t are continuous, x_t is a strong Markov process. Let $g(x)$ be a bounded continuous real-valued function and consider t as a component of the state; then we must show that $E_x g(x_t)$ is continuous in x for each $t > 0$. Suppose that $|g(x)| \leqq B < \infty$ and x_t and y_t are solutions of (4-1) with initial conditions x and y, respectively. Then we must show that $|E_x g(x_t) - E_y g(y_t)| \to 0$ as $\|x - y\| \to 0$. We now divide the possible situations into the following three cases: either $\|x_t - y_t\| \geqq \rho$, or else $\|x_t - y_t\| < \rho$ occurs together with either $\|x_t\| \geqq N$ or $\|x_t\| < N$. Thus

$$|E_x g(x_t) - E_y g(y_t)| \leqq P_{x,y}\{\|x_t - y_t\| \geqq \rho\}\, 2B + P_x\{\|x_t\| \geqq N\}\, 2B + \sup_{\substack{\|u\| \leqq \rho \\ \|w\| \leqq N}} |g(w+u) - g(w)|. \tag{4-11}$$

Given any $\delta > 0$, we will find an $\varepsilon > 0$ so that if $\|x - y\| \leqq \varepsilon$, then each term on the right of (4-11) is less than $\delta/3$. Choose $N < \infty$ so that $P_x\{\|x_t\| > N\} \leqq \delta/6B$. Then choose $\rho > 0$ so that the last term of (4-11) is less than $\delta/3$. Finally choose $\varepsilon > 0$ so that $P_{x,y}\{\|x_t - y_t\| \geqq \rho\} < \delta/6B$, if $\|x - y\| < \varepsilon$. The demonstration is complete.

DIFFERENTIAL GENERATOR

The operator

$$\mathscr{L} = \sum_i f_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij}(x, t) \frac{\partial^2}{\partial x_i \, \partial x_j} + \frac{\partial}{\partial t} \tag{4-12}$$

is known as the differential generator of the process x_t.

Suppose that the function $g(x, t)$ is bounded and has bounded and continuous first and second partial derivatives with respect to the x_i, and a bounded and continuous derivative with respect to t. By using the evaluations (4-9) and (4-10) it is not hard to show that for each fixed t and x

$$\lim_{\delta\to 0}\left\|\frac{E_{x,t}\,g(x_{t+\delta},t+\delta)-g(x,t)}{\delta}-\mathscr{L}g(x,t)\right\|=0. \tag{4-13}$$

Let $\mathscr{L}g(x,t)=h(x,t)$. The verification of

$$\lim_{\delta\to 0} E_{x,t}\,h(x_{t+\delta},t+\delta)=h(x,t) \tag{4-14}$$

is straightforward, and will be demonstrated for the term $g_{x_ix_j}(x,t)$ $S_{ij}(x,t)$ only. By our assumptions on $g(x,t)$ and $S(x)$, $|g_{x_ix_j}(x,t)\,S_{ij}(x,t)|\leqq K_2(1+\|x\|^2)$, for some real number K_2. Also, $E_x\max_{\delta\geqq u\geqq 0}\|x_u\|^2<\infty$ for any $\delta>0$. Then it follows directly from the dominated convergence theorem, and the continuity of both the process x_t and the functions $S_{ij}(\cdot,\cdot)$ and $g_{x_ix_j}(\cdot,\cdot)$, that

$$\lim_{\delta\to 0} E_{x,t}\,g_{x_ix_j}(x_{t+\delta},t+\delta)\,S_{ij}(x_{t+\delta},t+\delta)=g_{x_ix_j}(x,t)\,S_{ij}(x,t).$$

Equation (4-13) together with (4-14) imply that, on the class of functions described by the first sentence of the previous paragraph,

$$\tilde{A}=\mathscr{L}.$$

ITÔ'S LEMMA

Let x_t be an Itô process, and let $F(x,t)$ have continuous derivatives $F_t(x,t)$, $F_{x_i}(x,t)$, and $F_{x_ix_j}(x,t)$ for $0\leqq t\leqq T$ and $\|x\|<\infty$, and suppose that $\sup_{0\leqq t\leqq T}|F(x_t,t)|<\infty$ with probability one. Then, for $0\leqq s\leqq t\leqq T$, (see Itô [1], Dynkin [2], Theorem 7.2, Skorokhod [1], Theorem 2.5),

$$F(x_t,t)-F(x_s,s)=\int_s^t \mathscr{L}F(x_u,u)\,du+\int_s^t F_x'(x_u,u)\,\sigma(x_u,u)\,dz_u. \tag{4-15}$$

with probability one.

Now, consider a more general Itô process. Assume that $f(x,t,\omega)$

and $\sigma(x, t, \omega)$ satisfy (4-5) uniformly in ω and are nonanticipative random functions if $x_s^{\bullet}$ is nonanticipative. Then the equation

$$dx = f(x, t, \omega)\, dt + \sigma(x, t, \omega)\, dz \tag{4-16}$$

may be interpreted in very much the same way as (4-1). The solutions are continuous with probability one, $E \max_{\infty > b > s > a \geqq 0} x_s' x_s < \infty$, etc. Of course, x_s is not necessarily a Markov process. Furthermore, (4-15) holds where

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_i f_i(x, t, \omega) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij}(x, t, \omega) \frac{\partial^2}{\partial x_i\, \partial x_j}.$$

Equation (4-15) also holds if t and s are nonanticipative random variables. Write I_τ as the indicator of the (ω, t) set where $\tau \leqq t$, and suppose that τ and $q(\omega, t)$ are nonanticipative, and τ is uniformly bounded. Then

$$\int_0^\tau q(\omega, t)\, dz_t = \int_0^\infty I_\tau q(\omega, t)\, dz_t.$$

If $\int_0^\infty E \| I_\tau q(\omega, t) \|^2\, dt < \infty$, then

$$E \int_0^\infty I_\tau q(\omega, t)\, dz_t = 0.$$

Stopped processes

The processes obtained upon stopping the solutions of (4-1) at random times (of the process z_t) are discussed in Dynkin [2], Chapter 11, Section 3. Suppose that the stopping time τ is the first moment of exit of x_t from an open set Q. By continuity, x_τ is on ∂Q with probability one. By (4-6), the limit (3-4) is zero. Let $g(x, t)$ be the restriction to $Q + \partial Q$ of a function which, together with its derivatives, satisfies the boundedness and continuity conditions following equation (4-12). Then $g(x, t)$ is in the domain of $\tilde{A}_Q$ and $\tilde{A} g(x, t) = \tilde{A}_Q g(x, t)$ in Q.

WEAKENING OF CONDITION (4-5)

Suppose that there is an increasing sequence of open sets Q_m, tending to E, so that (4-5) is satisfied in each $Q_m + \partial Q_m$ (with finite Lipschitz constant and bound K_m'). Construct a sequence of functions $f^m(x, t)$ and $\sigma^m(x, t)$, agreeing with $f(x, t)$ and $\sigma(x, t)$, respectively, on $Q_m + \partial Q_m$, and satisfying (4-5) for some finite K_m. Corresponding to each of these functions there is a well-defined unique solution to (4-1), which we denote by x_t^m. The processes x_t^m, $m \geqq n$, take identical values until the first exit time from Q_n, and this first exit time from Q_n is identical for all x_t^m, $m \geqq n$. (See, for example, Dynkin [2], Chapter 11, Section 3, or Doob [2].) The sequence of first exit times, τ_m, of x_t^m from Q_m, increase and tend to a finite or infinite limit ζ. By this procedure we may define a unique solution x_t, to (4-1), for $t < \zeta$, with (4-5) being satisfied only locally. If $\zeta(\omega) < \infty$, then we say that $x_t(\omega)$ has a finite escape time.

In the sequel, we will generally be concerned with processes which are either stopped at or defined until the first moment of exit from open sets. It will then be required that (4-5) hold in these sets only.

5. Poisson Differential Equations

Let q_t be a vector-valued process whose components are independent Poisson step processes with $a_i\Delta + o(\Delta)$ the probability that the ith component will experience a jump in $[t, t + \Delta)$. Given that a jump occurs, let $P_i(dy)$ be the corresponding probability measure on the jump amplitude. Let $P_i(dy)$ have compact support. Let*

$$\int y P_i(dy) = 0$$

for each i. Write†

$$dx = f(x, t)\, dt + \sigma(x, t)\, dq\,. \tag{5-1}$$

* $Ey_i = 0$ is a helpful assumption, since then the q_t in (5-1) is a martingale, and a very close analog with the results of the previous section is available.

† Actually, (5-1) was also studied first by Itô [1].

The theory of equations of the form (5-1) is precisely that of equations (4-1). The definitions and existence and uniqueness proofs are identical. This derives from the fact that both the Wiener process z_t and the Poisson process q_t are martingales and have independent infinitely divisible increments. Both (5-1) and (4-1) are special cases of the equations discussed by Itô [1] and Skorokhod [1].

Let (4-5) hold. Then

$$E_{x,a} \max_{b \geqq t \geqq a} \|x_t\|^2 < \infty \tag{5-2}$$

$$P_{x,y,t}\{\|x_{t+\delta} - y_{t+\delta}\| \geqq \rho\} \to 0 \tag{5-3}$$

as $\|x-y\| \to 0$. Equation (5-3) is the analog of (4-7):

$$P_{x,t}\{\max_{\delta \geqq \Delta \geqq 0} \|x_{t+\Delta} - x\| > \varepsilon\} = O(\delta). \tag{5-4}$$

The derivation of (5-4) is almost exactly that of (4-6). (The scalar case derivation of (4-6) appears in Doob [1], p. 385; note that the last term of Doob's equation (3.19), p. 285, is now $O(\delta)$, rather than $O(\delta^{3/2})$, since our q_t is a Poisson process. All other terms are similar in both cases.) The analog of (4-8) is derived by using (5-2). Equation (4-9) also holds here. Also, for fixed x and t,

$$P_{x,t}\{\|x_{t+\delta} - x - \int_t^{t+\delta} \sigma(x,s)\, dq_s - \int_t^{t+\delta} f(x,s)\, ds\| > \varepsilon\} = O(\delta^{3/2}) \tag{5-5}$$

The normed term in (5-5) equals

$$\int_t^{t+\delta} [f(x_s,s) - f(x,s)]\, ds + \int_t^{t+\delta} [\sigma(x_s,s) - \sigma(x,s)]\, dq_s$$

whose mean square value is of the order of δ. (The integrals are evaluated exactly as in the scalar case in Doob [1], p. 284, (3.16) and (3.17).)

It is readily verified that x_t is a Feller process and, hence, since the paths are continuous from the right, it is a strong Markov process.

Equation (5-4) implies that, at least on bounded continuous functions, if $g(x, t)$ is in the domain of $\tilde{A}$, then it is in the domain of $\tilde{A}_Q$ and $\tilde{A}g(x, t) = \tilde{A}_Q g(x, t)$ in Q (see (3-1)).

Define the operator $\mathscr{D}$. Let y^i be a vector with y in the ith component and zeros elsewhere:

$$\begin{aligned}\mathscr{D}g(x, t) &= \frac{\partial g(x, t)}{\partial t} + \sum_i f_i(x, t) \frac{\partial g(x, t)}{\partial x_i} \\ &\quad + \sum_i \int [g(x + \sigma(x, t)\, y^i, t) - g(x, t)]\, a_i P_i(dy). \\ &\equiv h(x, t). \end{aligned} \tag{5-6}$$

Suppose that $g(x, t)$, $g_t(x, t)$, $g_{x_i}(x, t)$ are bounded and continuous, and the support of $P_i(dy)$ is compact. The evaluation (5-5) then yields

$$\lim_{\delta \to 0} \left\| \frac{E_{x,t}\, g(x_{t+\delta}, t + \delta) - g(x, t)}{\delta} - \mathscr{D}g(x, t) \right\| = 0 \tag{5-7}$$

and

$$\lim_{\delta \to 0} E_{x,t}\, h(x_{t+\delta}, t + \delta) = h(x, t).$$

Thus

$$\mathscr{D} = \tilde{A}$$

on the class of functions described.

Second form of the Poisson equation

Let us consider another form of a "Poisson driven" process. We use the homogeneous case notation. Let $\dot{x} = f(x, y)$, where $f(x, y)$ is bounded and satisfies a uniform Lipschitz condition in x, and y_t is a Poisson step process, taking values $y^1, \ldots, y^m$. Let $\alpha_{ij}\delta + o(\delta) = P\{y_{t+\delta} = y^j \mid y_t = y^i\}$. The pair (x_t, y_t) is clearly a right continuous strong Markov process with no finite escape time. In fact, x_t is uniformly continuous.

Suppose that $g(x, y)$ is bounded and has continuous and bounded derivatives $g_{x_i}(x, y)$, for each y^j. Then we define the operator $\mathscr{A}$:

$$\begin{aligned}\mathscr{A}g(x, y^i) &= \sum_j g_{x_j}(x, y^i) f_j(x, y^i) + \sum_j \alpha_{ij} [g(x, y^j) - g(x, y^i)] \\ &= h(x, y^i). \qquad (5\text{-}8)\end{aligned}$$

We will show that on the described class of functions, $\mathscr{A}$ is the weak infinitesimal operator of the process (x_t, y_t). The fact that $E_{x,y^i}\, h(x_t, y_t) \to h(x, y^i)$ is obvious. This property holds for any function $h(x, y)$ which is bounded and continuous in x for each value of y, $y = y^1, \ldots, y^m$. Now, supposing that $g(x, y)$ and its derivatives are bounded in absolute value by G,

$$\begin{aligned}[E_{x,y^i}\, g(x_\delta, y_\delta) &= g[x + \int_0^\delta f(x_s, y^i)\, ds, y^i]\, (1 - \sum_j \alpha_{ij}\delta + o(\delta)) \\ &+ E_{x,y^i} \sum_j \alpha_{ij}\delta\, g[x + \int_0^\Delta f(x_s, y^j)\, ds + \int_\Delta^\delta f(x_s, y^j)\, ds, y^j] + G\, o(\delta), \qquad (5\text{-}9)\end{aligned}$$

where Δ is the (random) time of a transition in y_t (supposing that there is one transition), and the expectation on the right of (5-9) is over $\Delta(\omega)$. Continuing the evaluation of (5-9), we have

$$\begin{aligned}E_{x,y^i}\, g(x_\delta, y_\delta) &= \delta \sum_j g_{x_i}(x, y^i) f_j(x, y^i) + \delta \sum_j g(x, y^j)\, \alpha_{ij} \\ &\quad - \delta \sum_j g(x, y^i)\, \alpha_{ij} + G\, o(\delta) + g(x, y^i),\end{aligned}$$

which yields the desired result that (5-8) equals $\tilde{A}$ on $g(x, y)$.

Let $Q_1, \ldots, Q_m$ be open x sets. Let Q have the form $Q = Q_1 \times \{y^1\} + \cdots + Q_m \times \{y^m\}$. Write $\hat{Q} = \bigcup_{i=1}^m Q_i$. Let τ be the first exit time of (x_t, y_t) from the open set Q. Since for (x, y) in Q, $P_{x,y}\{\tau < t\} = O(t)$, we have $\tilde{A} = \mathscr{A} = \tilde{A}_Q$ on the described class of functions. If $g(x, y)$ and

$f(x, y)$ satisfy the imposed uniform boundedness, differentiability, and uniform Lipschitz conditions only on $\hat{Q}$ for each y^i, then we still have $\tilde{A}_Q = \mathscr{A}$, and the process $(x_{t \cap \tau}, y_{t \cap \tau})$ is a right continuous strong Markov process.

6. Strong Diffusion Processes

Markov processes with continuous paths and for which there are Hölder continuous functions $a_{ij}(x, t)$, $b_i(x, t)$, and $c(x, t)$ given by (6-1) are called diffusion processes. Let ε be any positive real number:

$$\lim_{s \to 0} \frac{1}{s} \int_{\{\|x_{t+s} - x\| < \varepsilon\}} (x_{t+s} - x)\, P_{x,t}(d\omega) = \{b_i(x, t)\} = b(x, t)$$

$$\lim_{s \to 0} \frac{1}{s} \int_{\{\|x_{t+s} - x\| < \varepsilon\}} (x_{t+s} - x)(x_{t+s} - x)'\, P_{x,t}(d\omega) = \{a_{ij}(x, t)\} \qquad (6\text{-}1)$$

$$\lim_{s \to 0} \frac{1}{s}\Big[1 - \int_{\{x_{t+s} \in E\}} P_{x,t}(d\omega)\Big] = c(x, t) \geqq 0 .$$

Itô processes (Section 4) constitute one example (where $c = 0$). ($c(x, t)\,\Delta$ is essentially the probability of escape to infinity from x in the interval $[t, t + \Delta)$.) Under conditions (6-2) and (6-3), there are some rather strong properties associated with the process. We now suppose that (6-2) and (6-3) hold in a bounded open region G in E.

For some real positive μ and any vector ξ,

$$\sum_{i,j} a_{ij}(x, t)\, \xi_i \xi_j \geqq \mu \|\xi\|^2, \qquad (6\text{-}2)$$

$a_{ij}(x, t)$, $b_i(x, t)$, $c(x, t)$ are continuous and bounded and satisfy a uniform Hölder condition in x, (6-3)

$$\partial G \quad \text{is of class}^* \quad H_{2+\alpha}. \qquad (6\text{-}4)$$

* The local representation of the boundary ∂G has Hölder continuous second derivatives.

Let a_{ij}, b_i, and c depend only on x, and not on t. The following references are all to Dynkin [2]. Under (6-2) and (6-3), there is a neighborhood U of each x in G with $E_x\tau < \infty$, where τ is the first exit time from U (Theorem 5.8). Since G has a compact closure, (6-2) and (6-3) imply the following (Theorems 5.10 and 13.18) for x in G and on the class of bounded functions of x with continuous first and second derivatives*:

$$\tilde{A} = \tilde{A}_Q = L - c(x).$$

$$L - c(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i} - c(x).$$

The process is stochastically continuous in G, and the killing time is greater than zero with probability one (Lemma 5.11). In fact, if $c(x) = 0$, the process is continuous and defined with probability one up to at least the first moment of contact with ∂G. All processes with the same differential generator $L - c(x)$ in G are equivalent in G (p. 160) (that is, they have the same probability transition function).

Furthermore (Theorems 5.8, 5.11, 13.11, 13.12, and 13.16), to each operator $L - c(x)$ satisfying (6-2) and (6-3) on G, there is a unique process† on G (terminated upon first exit from G) with differential generator $L - c(x)$. The transition density of this process is the unique fundamental solution $p(t, x, y)$ of

$$\frac{\partial p}{\partial t} = (L - c(x))\,p$$

with boundary condition $p(t, x, y) \to 0$ as $x \to \partial G$ and $t > 0$. Let G be bounded and ∂G satisfy (6-4). Let $g(x)$ satisfy a uniform Hölder condition and be bounded in G, and let $\varphi(x)$, defined on ∂G, be continuous. Then (6-5) has a unique solution which is (6-6) (Theorem 13.16):

* We write L in lieu of $\mathscr{L}$, since $\mathscr{L}$ includes a term $\partial/\partial t$.

† The process is unique if the initial distribution is fixed.

$$Lf - c(x)f = -g(x) \tag{6-5}$$

$$f(x) \to \varphi(x) \quad \text{as} \quad x \to \partial G;$$

$$f(x) = E_x \int_0^\tau \exp\left[-\int_0^t c(x_s)\,ds\right] g(x_t)\,dt$$

$$+ E_x \exp\left[-\int_0^t c(x_s)\,ds\right] \varphi(x_\tau); \tag{6-6}$$

τ is the first instant of contact of x_t with ∂G.

In particular, let $\varphi(x) = 0$ on ∂G and $g(x) = 1, c = 0$. Then the unique solution of $Lf = -1$ is $f(x) = E_x\tau$, which must thus be finite. Suppose that G is of the form of Figure 1, with smooth exterior

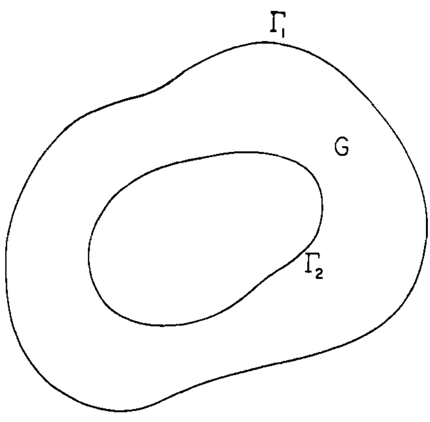

Figure 1

boundary Γ_1 and smooth interior boundary Γ_2. Let $\varphi(x) = 1$ on Γ_1 and $\varphi(x) = 0$ on Γ_2 and let $c = g = 0$. Then $Lf = 0$ has the unique solution

$$f(x) = E_x\varphi(x_\tau) = P_x\{x_t \text{ touches } \Gamma_1 \text{ before } \Gamma_2\}.$$

7. Martingales

The references of this section are specifically to Doob [1]. (See also Loeve [1].) Let y_n, $n = 1, \ldots$ be a sequence of random variables and B_n, $n = 1, \ldots$ a nondecreasing sequence of σ-algebras. If y_n is measurable with respect to B_n and satisfies

$$E[y_{n+1}|B_n] = y_n$$

with probability one, then the sequence $\{y_n, B_n\}$ is a martingale. If

$$E[y_{n+1}|B_n] \leqq y_n \tag{7-1}$$

with probability one, then the sequence $\{y_n, B_n\}$ is a supermartingale. Equation (7-1) implies, for each set A in B_n, and any $n \geqq 1$, $m \geqq 0$,

$$\int_A y_{n+m} P(d\omega) \leqq \int_A y_n P(d\omega).$$

Let x_n, $n = 1, \ldots$, be a Markov process, and B_n the minimum σ-algebra over which x_i, $i \leqq n$, is measurable. If $E|V(x_n)| < \infty$ for each $n < \infty$ and

$$E_x[V(x_{n+1})|B_n] \leqq V(x_n) \tag{7-2}$$

with probability one, then $\{V(x_n), B_n\}$, $n < \infty$ is a supermartingale. The left-hand side of equation (7-2) equals $E_{x_n} V(x_{n+1})$ with probability one. (This holds also for continuous parameter processes.)

Let τ be an (integral-valued) Markov time. If $\{y_n, B_n\}$ is a supermartingale or martingale, then so is $\{y_{n \cap \tau}, B_n\}$ (Chapter 7, Theorem 2.1).

Let $y_i \geqq 0$ and suppose that $\{y_n, B_n\}$ is a supermartingale. The arguments yielding equation (3.2) of Chapter 7 of Doob [1] also give, for each $\lambda \geqq 0$,

$$P\{\max_{n \geqq i \geqq 1} y_i \geqq \lambda\} \leqq E\frac{y_1}{\lambda}. \tag{7-3}$$

(Theorem 4.1s (Doob)). Let $\{y_n, B_n\}$ be a submartingale. Let l.u.b. $E|y_n| = M < \infty$. Then $\lim_n y_n = y_\infty$ exists with probability one and

$E|y_\infty| \leqq M$. (If $y_n \geqq 0$ is a supermartingale, the theorem applies to $\{-y_n, B_n\}$. Then the hypothesis is satisfied if $E|y_1| < \infty$.) If the y_n are uniformly integrable, then $E|y_\infty - y_n| \to 0$.

The continuous parameter results are essentially identical to the discrete parameter results. Separability of the process is always assumed. If (7-1) holds and $B_t \supset B_s$, $t \geqq s$, and y_t is measurable over B_t and $E|y_t| < \infty$, then $\{y_t, B_t\}$, $t < \infty$, is a supermartingale.

For a nonnegative supermartingale

$$P\{\sup_{T \geqq t \geqq 0} y_t \geqq \lambda\} \leqq E\frac{y_0}{\lambda}. \tag{7-4}$$

Let $\{y_t, B_t\}$ be a submartingale. Let l.u.b. $E|y_t| < \infty$. Then there exists a random variable y_∞ such that $y_t \to y_\infty$ with probability one. If the y_t are uniformly integrable, then $E|y_\infty - y_n| \to 0$.

(Theorem 11.6 (Doob); see also remark on p. 379). Let τ be a non-anticipative time for the process $y_t, 0 \leqq t < \infty$, and let $\{y_t, B_t\}$ be a submartingale, where y_t is continuous from the right with probability one. Then $\{y_{t \cap \tau}, B_t\}$ is a submartingale. (y_τ is, of course, evaluated as the limit of $y_{\tau+t}$ as $t \to 0$ from above.)

II / STOCHASTIC STABILITY

1. Introduction

Deterministic stability is a branch of the qualitative theory of dynamical systems. In particular, the majority of presently available results which are termed stability results pertain to certain qualitative and quantitative (which do not involve the actual computation of a solution) properties of differential equations. Consider the differential equation $\dot{x} = f(x, t)$ with initial condition x_0 belonging to a set R. In what follows, R may vary but will always be a nonempty bounded open set containing the origin $x = \{0\}$. Let P be a set containing R. Some typical problems which may be grouped under the title "stability problems" are:

(P1) Let P be given. Is there an R such that if $x_0 \in R$, then $x_t \in P$ for all finite t?

(P2) In reference to (P1), is there some R corresponding to each open set P containing the origin?

(P3) In reference to (P1), estimate the largest set R. (The estimate is a quantitative property.)

(P4) For a given set of initial values R, is the set P, containing the range of the trajectory for all $t < \infty$, bounded?

(P5) What is the smallest set containing the asymptotic values of the solution for all x_0 in R?

(P6) For fixed P and x_0 in R, estimate the first time of exit of x_t from P.

(P7) Will the trajectory intersect a given set S in finite time, for any x_0 in R?

(P8) Let $\dot{x} = f(x, t) + p$. Will $\|x_t\|$ be uniformly bounded if $\|p_t\|$ is?

The reader is referred to the monographs of LaSalle and Lefschetz [1], Hahn [1], Krasovskii [1], Aiserman and Gantmacher [1], or Lefschetz [1] for more specific details and references on the deterministic "stability" problems.

Related questions arise in the analysis of stochastic processes; in particular, in the analysis of stochastic processes that are models of automatic control systems. The processes with which we will deal are right continuous strong Markov processes. Not all Markov processes are strongly Markovian (see counterexample in Loeve [1], p. 578); however, to the author's knowledge, all Markov processes which have been studied as models of physical problems have the strong Markov property.

Some typical problems subsumed under the title of finite time stochastic stability are mentioned in the introduction to Chapter III. Stochastic analogies to the listed deterministic problems can easily be drawn. Let x_t be a right continuous strong Markov process, whose initial value, a nonrandom constant x_0, lies in a nonempty open set R containing the origin. P is a set which contains R.

(P1)′ Is there an R so that $P_x\{x_t \notin P, \text{ some } t < \infty\} \leqq \rho < 1$ for some given ρ, P and any x in R?

(P2)′ In reference to (P1)′, is there some R corresponding to each given P and $\rho < 1$?

(P3)′ For fixed P and ρ in (P1)′, estimate the largest set R.

(P4)′ What is the value of $\min_{x_0 \in R} P_{x_0}\{x_t \in P, \text{ all } t < \infty\}$, when P is a given bounded set?

(P5)′ What is the smallest set to which x_t tends with probability one, as $t \to \infty$?

(P6)′ Estimate the probability $P_x\{x_t \notin P, \text{ some } t \leqq T\}$, $x \in R$.

(P7)′ Is there a finite-valued Markov time τ such that $x_t \to S$, as $t \to \tau$?

The obvious analogs of (P8) are of no probabilistic interest. Two related problems of interest are:

(P8)′ Let $\dot{x} = f(x) + p$, where p_t is a Markov process. Is the pair (x_t, p_t) a positive recurrent process? Is $E\,\|x_t\|^2$ uniformly bounded if $E\,\|p_t\|^2$ is?

(P6)′ is a type of finite time stability or first exit time result. (See Chapter III.) For a generic source of such problems in control theory, consider the system $\dot{x} = f(x) + p_t$. Suppose first that $p_t \equiv 0$. Under this condition, suppose that if $x = x_0$ is in R, then x_t is in R, for all $t < \infty$, and that if x is in $E - R$, then x_t will be in $E - R$, all $t < \infty$, R is the "desired" region of operation. Let the forcing term p_t be a stochastic process. Then p_t may eventually drive x_t into $E - R$ with probability one; however, if the first time of entrance into $E - R$ is "large," the fact that $x_t \to E - R$ may be unimportant for practical purposes. In any case, estimates of the probabilities of entrance times are of interest.

The methods of this chapter are applicable to the determination ((P7)′) of whether or not x_t tends to a given set S as $t \to \infty$, or intersects S at some finite-valued Markov time. The results are pertinent to the problem of choosing a control which will transfer the initial condition x to a given target set S with probability one (or, perhaps, with at least probability ρ, where ρ is given).

Uniform boundedness of the sample paths ((P4)′) is also of some interest. Consider the case where a dynamical system is removed from a stable equilibrium state by a random disturbance whose "intensity" or "magnitude" eventually decreases to zero as $t \to \infty$; for example, an earthquake, or an echoing series of shocks. Suppose that one of the states of the dynamical system is the stress associated with a particular member. Then the probability that this stress is uniformly bounded by some given number (perhaps the average breaking stress minus a safety factor) is of interest. This problem is distinguished from the finite time stability problem in that, here, the shocks decrease with time, and the state may indeed be uniformly bounded (for $t < \infty$) in a prescribed way with a nonzero probability.

Problems on an infinite time interval, where the noise intensity is proportional to a function of x, but does not depend on time, also occur. Consider the scalar problem

$$\dot{x} = f(x) + u(x) + u(x)\,\xi,$$

where ξ_t is the solution of an Itô equation

$$d\xi = -\,a\xi\,dt + \sigma\,dz \qquad (a > 0).$$

The "control" $u(x)$ is to be selected to ensure that $x_t \to 0$ with probability one. Since the noise effects are proportional to $u(x)$, it is not immediately obvious that there is a $u(x)$ which will accomplish the desired task (unless $xf(x) < 0$, in which case $u(x) = 0$ will do). Suppose that, if $\xi_t \equiv 0$, then a suitable $u(x)$ has the property: $|u(x)| \to 0$ as $|x| \to 0$. Then, with the same $u(x)$ and the true ξ_t, it may be expected that, at least for small σ, $x_t \to 0$ with probability one. Some such situations are discussed in the examples. For any particular $u(x)$, an estimate of $P_x\{\sup_{\infty > t \geqq 0} |x_t| \geqq \varepsilon\}$, as well as the asymptotic behavior of x_t, is of interest.

Definitions

For further use, and to fix ideas on the properties with which the chapter will deal, we list some definitions and some properties which will be discussed in the sequel. The reader should be cautioned not to take the assigned terminology too seriously. The terminology is largely motivated by the terminology used for related deterministic properties. There is some discrepancy in usage among authors in the field and, in any case, the most appropriate usage remains to be determined, as the mathematical results and practical applications develop further and become codified. There are stochastic Liapunov theorems corresponding to all the properties which are defined by (D1) to (D9).

(D1) *The origin is stable with probability one* if and only if, for any

$\rho > 0$ and $\varepsilon > 0$ there is a $\delta(\rho, \varepsilon) > 0$ such that, if $\|x_0\| < \delta(\rho, \varepsilon)$, then, with $x_0 = x$,

$$P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq \varepsilon\} \leqq \rho .$$

The "with probability one" clause is motivated by the arbitrariness of ρ. An alternative definition which is less ambiguous is:

(D2) The system is stable with respect to the triple (Q, P, ρ) if and only if x in Q implies that

$$P_x\{x_t \in P, \quad \text{all} \quad t < \infty\} \leqq \rho .$$

More important than mere knowledge of the satisfaction of the conditions of the definitions are the upper bounds of

$$P_x\{\sup_{\infty > t \geqq 0} V(x_t) \geqq \varepsilon\},$$

which are provided by the theorems. $V(x)$ is a stochastic Liapunov function.

(D3) *The origin is asymptotically stable with probability one*, if and only if it is stable with probability one and $x_t \to 0$ with probability one, for all x_0 in some neighborhood of the origin R. If R is the whole space, we add "*in the large.*"

(D4) For a given initial condition x, x_t *is uniformly bounded by* ε *with probability* ρ if and only if

$$P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq \varepsilon\} \leqq 1 - \rho .$$

The stability properties of some components of the Markov process may not be of interest. For example, let y_t be a Markov process and let $\dot{w} = f(w, y)$. Then, under suitable conditions on $f(w, y)$, (w_t, y_t) is a Markov process, but it is possible that only w_t is of interest in some problems. Then definition (D2) is still applicable, and the probability bounds still yield useful information. (D1) is irrelevent, but may be modified as: Let (w_t, y_t) be a Markov process. Let w take values in the

Euclidean space E_w. The origin of E_w is said to be stable with probability one if and only if, for each $y_0 = y$, $\rho > 0$, and $\varepsilon > 0$, there is a $\delta(\rho, \varepsilon, y) > 0$ so that, if $\|w_0\| < \delta(\rho, \varepsilon, y)$, we have, with $w_0 = w$,

$$P_{y,w}\{\sup_{\infty > t \geqq 0} \|w_t\| \geqq \varepsilon\} \leqq \rho\,.$$

In fact, the case where some components of the Markov process are of no interest is probably the general case. Some of the states of a Markov model of a control process will be the observed states, or the quantities of interest in the control process. However, in order to "Markovianize" a control process, many "states" may have to be added.

(D5) *The origin is exponentially stable with probability one* if and only if it is stable with probability one and, for all $T < \infty$,

$$P_x\{\sup_{\infty > t \geqq T} \|x_t\| \geqq \varepsilon\} \leqq K\, e^{-\alpha T},$$

where $K < \infty$ and $\alpha > 0$.

(D6) The origin is unstable with probability ρ if and only if

$$\lim_{\varepsilon \to 0} \lim_{x \to 0} P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq \varepsilon\} = \rho\,.$$

(D7) The process is *ultimately bounded with probability one and with bound m* if and only if, for each x in E, there is a finite-valued (with probability one) random time $\tau(x)$ such that

$$P_x\{\sup_{\infty > t > \tau(x)} \|x_t\| \leqq m\} = 1\,,$$

or, alternatively,

$$\lim_{T \to \infty} P_x\{\sup_{\infty > t \geqq T} \|x_t\| > m\} = 0\,.$$

If the time $\tau(x)$ does not depend on x (and is finite with probability one), then the bound is said to be uniform.

(D8) Let x_t and $\hat{x}_t$ be processes corresponding to the initial conditions x and $\hat{x}$, respectively. The process is said to be *equi-*

distance bounded with probability one* if and only if for fixed $\|x - \hat{x}\|$,

$$P_{x,\hat{x}}\{\sup_{\infty > t \geqq 0} \|x_t - \hat{x}_t\| \geqq \varepsilon\} \to 0$$

uniformly in x and $\hat{x}$, as $\varepsilon \to \infty$.

(D9) The process is said to be *equistable with probability one* if and only if it satisfies (D8) and, for any fixed $\varepsilon > 0$,

$$P_{x,\hat{x}}\{\sup_{\infty > t \geqq 0} \|x_t - \hat{x}_t\| \geqq \varepsilon\} \to 0$$

as $\|x - \hat{x}\| \to 0$, uniformly in x and $\hat{x}$.

The properties (D1) and (D9) depend on the behavior of the sample paths over a time interval of positive length (as opposed to moments, for example, which depend only on the distribution of x_t at fixed instants of t, and only indirectly on the path characteristics). Most of the sequel is devoted to this type of property. An exception is Theorem 6 that allows the estimation of $\lim_{t \to \infty} E_x \tilde{A} V(x_t)$ (if such exists and is independent of x) and of $\lim \int_0^T E_x \tilde{A} V(x_t)\, dt/T$ for suitable functions $V(x)$.

Finite time or first exit time results appear in Chapter III.

The idea of a Liapunov function

The Liapunov function approach to the study of the stability of deterministic systems can be briefly introduced as follows. Suppose that the nonnegative continuous function $V(x)$, satisfying $V(0) = 0$, $V(x) > 0$, $x \neq 0$, has continuous first partial derivatives in the bounded set $Q_m = \{x\colon V(x) < m\}$, $m < \infty$. Let $\dot{x} = f(x)$ have a unique solution in Q_m. Now the sets Q_r, each containing the origin, decrease monotonically to the origin, as $r \to 0$. Suppose that $\dot{V}(x_t) = V_x'(x_t) f(x_t) \equiv -k(x_t) \leqq 0$ in Q_m, where $k(x)$ is continuous. The fact that $\dot{V}(x_t) \leqq 0$ in Q_m together with the definition of Q_m implies that, if $V(x_0) < m$, then

* Following the deterministic definition of Yoshizawa [1].

$V(x_t) < m$; hence, x_t is in Q_m for all $t < \infty$, provided only that x_0 is in Q_m. This is a stability result. The function $V(x)$ is termed a Liapunov function.

We may proceed further. Write

$$V(x_0) - V(x_t) = \int_0^t k(x_s)\, ds = - \int_0^t \dot{V}(x_s)\, ds. \tag{1-1}$$

Since $k(x)$ is nonnegative and continuous in the bounded set Q_m, and x_t is in Q_m, $t < \infty$, and since the left side of (1-1) is bounded, it is obvious that $x_t \to \{x: k(x) = 0\}$ as $t \to \infty$. Also, it is clear that $V(x_t)$ decreases monotonically to some nonnegative constant v. If, for example, $k(x) = 0$ in Q_m implies $x = 0$, then $x_t \to 0$ as $t \to \infty$.

These results are among many which may be obtained by the use of Liapunov functions. (See the references previously cited.) The "Liapunov function method" is a method of obtaining information on a family of solutions of $\dot{x} = f(x)$ without actually computing the numerical values of each solution. In lieu of the problem of numerical solution, there is the problem of obtaining useful Liapunov functions.

The stochastic Liapunov function approach

In studying the qualitative properties of a random process, we study the qualitative properties of a family of functions (the sample paths) simultaneously. If $\dot{V}(x_t) \leqq 0$ for each sample path, then the deterministic theorems may be used, and the problem may not have much probabilistic interest. A basic property of deterministic Liapunov functions, obtained in the previous subsection, is $V(x_t) \downarrow v \geqq 0$. While we do not ordinarily expect $\dot{V}(x_t) \leqq 0$ in the stochastic case, it is reasonable to expect that some properties of a stability nature could be inferred from the form of $\tilde{A}V(x)$, the natural stochastic analog of the deterministic derivative. Further, since we are interested in making inferences concerning the asymptotic value of $V(x_t)$ from local properties, the possible applicability of the martingale theorems is suggested.

If $V(x) \geqq 0$ and, for any $\Delta > 0$,

$$E_{x_s} V(x_{s+\Delta}) - V(x_s) \leqq 0 \tag{1-2}$$

with probability one, then the supermartingale theorems apply and one may infer $V(x_t) \to v \geqq 0$ with probability one. Just as the form of the functions $k(x)$ was helpful in obtaining information on the asymptotic values of x_s in the deterministic case, we expect that the form of (1-2) would yield similar information in the stochastic case. In fact, Dynkin's formula provides the needed connection between $V(x)$, $\tilde{A}V(x)$, and the martingale theorems, and plays the same role in the stochastic proofs as (1-1) plays in the deterministic proofs. Let $V(x) \geqq 0$ be in the domain of $\tilde{A}$ and let τ be a Markov time satisfying $E_x \tau < \infty$. Then, supposing that $\tilde{A}V(x) = -k(x) \leqq 0$,

$$V(x) - E_x V(x_\tau) = E_x \int_0^\tau k(x_s)\, ds = -E_x \int_0^\tau \tilde{A}V(x_s)\, ds \geqq 0 . \tag{1-3}$$

It is reasonable to expect that $V(x_t)$ is a supermartingale, and also that $x_t \to \{x: k(x) = 0\}$ with probability one (these facts will be proved in Section 2). A number of difficulties attend the immediate application of (1-3). First, the domain of $\tilde{A}$ is usually too restricted to include many functions $V(x)$ of interest. Even if $V(x)$ is in the domain of $\tilde{A}$, $k(x)$ may be nonnegative in a subset of E only and, furthermore, to investigate asymptotic properties, we must let $\tau \to \infty$. These difficulties may be circumvented by first applying (1-3) to an appropriate stopped process derived from x_t, and then taking limits.

History

The probability literature contains much material concerned with criteria under which random processes have some given qualitative property. However, the first suggestions that there may be a stochastic Liapunov method analogous to the deterministic Liapunov method seems to have appeared in the papers of Bertram and Sarachik [1] and Kats and Krasovskii [1]. Both papers are concerned with moments only. Bucy [1] recognized that stochastic Liapunov functions should have the supermartingale property and proved a theorem on "with probability one" convergence for discrete parameter processes. Bucy's work is probably the first to treat a nonlinear stochastic stability problem in any generality. Some results, of the Liapunov form, were given by Khas'minskii [3] for strong diffusion processes. The martingale theorems have had, of course, wide application in providing probability bounds on stochastic processes, although their applicability to the problems of the references had not been exploited very much prior to these works. Other works are those of the author included here and in Kushner [3–6], Wonham [2, 3] (for strong diffusion processes), Khas'minskii [2], Kats [1], and Rabotnikov [1]. There is certainly much other material on "stochastic stability," concerned with more direct methods. See Kozin [1–3] and Caughy and Gray [1].

2. Theorems. Continuous Parameter

ASSUMPTIONS

The following assumptions are collected here for use in the following theorems.

For some fixed m,

(A1) $V(x)$ is nonnegative and is continuous in the open set $Q_m \equiv \{x\colon V(x) < m\}$.

(A2) x_t is a right continuous strong Markov process defined until at least some $\tau' > \tau_m = \inf\{t\colon x_t \notin Q_m\}$ with probability one (or, for all $t < \infty$, if $x_t \in Q$, all $t < \infty$).
If $x_t(\omega)$ is in Q_m, for all $t < \infty$, define $\tau_m(\omega) = \infty$.

(A3) Write $\tilde{A}_m$ for $\tilde{A}_{Q_m}$, the weak infinitesimal operator of $x_{t \cap \tau_m}$.

(A4) $V(x)$ is in the domain of $\tilde{A}_m$.

(A5) $P_x\{\sup_{t \geq s \geq 0} \|x_s - x\| > \varepsilon\} \to 0$ as $t \to 0$ for $x \in Q_m$ and any $\varepsilon > 0$.

Denote the ω set $\{\omega\colon x_t \in Q_m, \text{ all } t < \infty\}$ by B_m.

Lemma 1 establishes a result which is to be used repeatedly. It also illustrates the basic use to which Dynkin's formula will be put in the sequel.

Lemma 1. Assume (A1) to (A4). Let $\tilde{A}_m V(x) \leqq 0$.* Then† $V(x_{t \cap \tau_m})$ is a nonnegative supermartingale of the stopped process $x_{t \cap \tau_m}$ and, for $\lambda \leqq m$, and initial condition $x_0 = x$ in Q_m,

$$P_x\{\sup_{\infty > t \geqq 0} V(x_{t \cap \tau_m}) \geqq \lambda\} \leqq \frac{V(x)}{\lambda}. \tag{2-1}$$

Also there is a random variable $c(\omega)$, $0 \leqq c(\omega) < m$, such that, with probability one relative to B_m, $V(x_t) \to c(\omega)$, as $t \to \infty$. $P_x\{B_m\} \geqq 1 - V(x)/m$.

Proof. By Dynkin's formula

$$E_x V(x_{t \cap \tau_m}) - V(x) = E_x \int_0^{t \cap \tau_m} \tilde{A}_m V(x_s)\, ds \leqq 0. \tag{2-2}$$

Thus $E_x V(x_{t \cap \tau_m}) \leqq V(x)$. Also, since $V(x)$ is in the domain of $\tilde{A}_m$, $E_x V(x_{t \cap \tau_m}) \to V(x)$, as $t \to 0$. These two facts imply the supermartingale property (Dynkin [2], Theorem 12.6). Equation (2-1) is the supermartingale probability inequality, and can also be derived from (2-2). The existence of a $c(\omega)$ follows from the supermartingale convergence theorem and the last statement of the hypothesis follows from (2-1).

Remark. Suppose the state is (x, t). Define Q_m as in (A1). If $\tilde{A}_m V(x, t) \leqq 0$ in $Q_m(\tilde{A}_m$ is the weak infinitesimal operator of the process $(x_{t \cap \tau_m}, t \cap \tau_m))$, then the conclusions of Lemma 1 hold.

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* For greater clarity, we could write the redundant statement $\tilde{A}_m V(x) \leqq 0$ in Q_m. $\tilde{A}_m V(x)$ is, of course, defined only in Q_m.

† An obvious set of increasing σ-fields can be adjoined to $V(x_{t \cap \tau_m})$ to complete the description of the supermartingale.

Equation (2-1) is written as

$$P_{x,T}\{\sup_{\infty > t \geqq T} V(x_t, t) \geqq \lambda\} \leqq \frac{V(x, T)}{\lambda}, \tag{2-1$'$}$$

where (x, T) is in Q_m and $\lambda \leqq m$.

Theorem 1 is the basic result on stochastic stability, and is the stochastic analog of Liapunov's theorem on stability. The probability estimate (or (D2)) is more important than the mere fact that (D1) is satisfied.

Theorem 1. (*Stability*) Assume (A1) to (A4) for some $m > 0$. Let $V(0) = 0$, $x \in Q_m$. Then the system is stable relative to $(Q_r, Q_m, 1 - r/m)$, for any $r = V(x_0) = V(x) \leqq m$ (see (D2) in Section 1). Also, for almost all ω in B_m, $V(x_{t \cap \tau m}) \to c(\omega) \leqq m$. If $V(x) > 0$ for $x \neq 0$ and $x \in Q_m$, then the origin is stable with probability one.

Proof. The proof is a direct consequence of Lemma 1, the definitions of stability, and the properties of the function $V(x)$.

Lemma 2 establishes an upper bound on the average time that the process x_t spends in a set. It will later be used to prove, among other things, that certain sets are reached in finite average time.

Lemma 2. Let $V(x)$ be nonnegative in an open region Q (Q is not necessarily bounded), x_t a right continuous strong Markov process in Q, and τ the random first exit time from the open set $P \subset Q$. Let $V(x)$ be in the domain of $\tilde{A}_Q$. Let $\tilde{A}_Q V(x) \leqq -b < 0$ in P. Then the average time, $E_x\tau$, spent interior to P up to the first exit time is no greater than $V(x)/b$.

Proof. By Dynkin's formula

$$V(x) - E_x V(x_{t \cap \tau}) = -E_x \int_0^{t \cap \tau} \tilde{A}_Q V(x_s)\, ds \geqq bE_x \int_0^{t \cap \tau} I_x(s, \omega)\, bs$$
$$= bE_x(t \cap \tau), \tag{2-3}$$

where $I_x(s, \omega)$ is the indicator function of the (s, ω) set where

$\tilde{A}_Q V(x) \leqq -b$ (it depends on $x_0 = x$). The nonnegativity of $V(x)$ and (2-3) clearly imply the theorem.

DEFINITION

An ε-neighborhood of a set M relative to an open set Q is an open set $N_\varepsilon(M) \subset Q$, such that

$$N_\varepsilon(M) = \{x \in Q \quad \text{such that} \quad \|x - y\| < \varepsilon \quad \text{for some} \quad y \in M\}.$$

$\bar{N}_\varepsilon(M) = N_\varepsilon(M) + \partial N_\varepsilon(M)$. Note that it is not necessary that $N_\varepsilon(M)$ contain M (although $N_\varepsilon(M)$ will contain M in Theorem 2).

Theorems 2 and 3 are stochastic analogs of some Liapunov theorems on asymptotic stability.

Theorem 2. (*Asymptotic Stability*) Assume (A1) to (A5) and also $\tilde{A}_m V(x) = -k(x) \leqq 0$. Let Q_m be bounded and define $P_m = Q_m \cap \{x\colon k(x) = 0\}$. For each d less than some $d_0 > 0$, let there exist an $\varepsilon_d > 0$ such that, for the ε_d-neighborhood $N_{\varepsilon_d}(P_m)$ of P_m relative to Q_m, we have $k(x) \geqq d > 0$ on $Q_m - N_{\varepsilon_d}(P_m)$. (A condition implying this is uniform continuity of $k(x)$ on P_m and $k(x) > 0$ for some $x \in Q_m$.) Then

$$x_t \to P_m$$

with a probability no less than $1 - V(x)/m$.

Suppose that the hypotheses hold for all $m > 0$. Define $P = \bigcup_1^\infty P_m$. and let $N_\varepsilon(P)$ be the ε-neighborhood of P relative to $Q = \bigcup_1^\infty Q_m$. If, for each d, $0 < d \leqq d_0$, there is an $\varepsilon_d > 0$ such that $k(x) \geqq d$ on $Q - N_{\varepsilon_d}(P)$, then $x_t \to P$ with probability one.

Proof. Note that the statement of the last paragraph follows from the statement of the first paragraph. Note also that if $k(x) \equiv 0$ in Q_m, the theorem is trivially true as a consequence of Theorem 1. By Lemma 1, x_t remains strictly interior to Q_m with a probability $\geqq [1 - V(x)/m]$.

Fix d_1 and d_2 such that $d_0 > d_1 > d_2 > 0$. Let ε_i correspond to d_i

(that is, in $Q_m - N_{\varepsilon_i}(P_m)$, we have $k(x) \geqq d_i$). It is no loss of generality to suppose that $N_{\varepsilon_2}(P_m)$ is properly contained in $N_{\varepsilon_1}(P_m)$. Define $I_x(s, \omega, \varepsilon_i)$ as the indicator of the (s, ω) set where $x_s(\omega)$ is in $Q_m - N_{\varepsilon_i}(P_m)$ and write

$$T_x(t, \varepsilon_i) = \int_{t \cap \tau_m}^{\tau_m} I_x(s, \omega, \varepsilon_i)\, ds .$$

$T_x(t, \varepsilon_i)$ is the total time spent in $Q_m - N_{\varepsilon_i}(P_m)$ after time t and before either $t = \infty$ or the first exit time from Q_m (with $x_0 = x$). $T_x(t, \varepsilon_i) = 0$ if $\tau_m \leqq t$. By Theorem 1,

$$P_x\{x_t \quad \text{leaves } Q_m \text{ at least once before} \quad t = \infty\} \equiv 1 - P_x\{B_m\} \leqq \frac{V(x)}{m}$$

By Lemma 2 we conclude that $T_x(t, \varepsilon_i) < \infty$ with probability one and, hence, $T_x(t, \varepsilon_i) \to 0$ as $t \to \infty$.

Now we distinguish two possibilities: (a) there is a random variable $\tau(\varepsilon_1) < \infty$ with probability one such that $x_t \in N_{\varepsilon_1}(P_m)$ with probability one (relative to B_m) for all $t > \tau(\varepsilon_1)$; or (b) for $\omega \in B_m$, x_t moves from $N_{\varepsilon_2}(P_m)$ to $Q_m - N_{\varepsilon_1}(P_m)$ and back to $N_{\varepsilon_2}(P_m)$ infinitely often in any interval $[t, \infty)$. ((a) and (b) are all possibilities, since the path $x_t(\omega \in B_m)$ can stay exterior to $N_{\varepsilon_2}(P_m)$ for only a finite total time, since $T_x(t, \varepsilon_2) \to 0$ as $t \to \infty$.) Consider case (b). Since $T_x(t, \varepsilon_i) \to 0$ with probability one as $t \to \infty$ there are infinitely many movements from $N_{\varepsilon_2}(P_m)$ to $Q_m - N_{\varepsilon_1}(P_m)$ and back to $N_{\varepsilon_2}(P_m)$ in a *total* time which (in the integral sense) is arbitrarily small. We now show that the probability of case (b) is zero.

Fix $\delta_1 > 0$. Choose $h > 0$ so that

$$\sup_{x \in Q_m - N_{\varepsilon_2}(P_m)} P_x\{\sup_{h \geqq s \geqq 0} \|x_s - x\| \geqq \varepsilon_1 - \varepsilon_2\} < \delta_1 . \tag{2-4}$$

By stochastic continuity and compactness of $Q_m + \partial Q_m$, for each $\delta_1 > 0$ there is some $h > 0$ for which (2-4) holds. For each $\delta_2 > 0$, and the fixed initial condition x, there is a $t < \infty$ so that

$$P_x\{T_x(t, \varepsilon_2) > h\} < \delta_2 . \tag{2-5}$$

Under case (b), (2-4) and (2-5) are contradictory. Thus, we conclude that

$$P_x\{x_s \in Q_m - N_{\varepsilon_1}(P_m) \quad \text{for some} \quad s \in (t, \infty)\} \to 0$$

as $t \to \infty$, for $x \in Q_m$. Since ε_1 and ε_2 are arbitrary, the theorem is proved.

REMARK ON TIME DEPENDENCE

Suppose that either the process x_t is not homogeneous or that the Liapunov function $V(x, t)$ depends on time, so that as a consequence either k or V are time dependent. Then Theorem 2 may still be used as stated, although it is the custom in the deterministic theory to reword the hypothesis so that certain trivial possibilities are eliminated. For example, suppose that $k(x, t) = g(x)/(1 + t)$, where $g(x) > 0$ for $x \neq 0$. Then, considering t as a state, all that we may conclude (from Theorem 3, since Q_m is not compact in this case) is that either $t \to \infty$ or $g(x) \to 0$ with some probability.

We may reword the hypothesis in the following way. Let $V(x, t)$ be nonnegative, continuous, and in the domain of $\tilde{A}_m$ (corresponding to the process $(x_{t \cap \tau_m}, t \cap \tau_m))$ with $\tilde{A}_m V(x, t) = -k(x, t) \leqq 0$ in Q_m. (Note that $Q_m = \{x, t\colon V(x, t) < m\}$.) Suppose that $k(x, t) \geqq k_1(x)$ in Q_m, where $k_1(x)$ satisfies the conditions on $k(x)$ in Theorem 2. Let $x_T = x$ be the initial condition. Then $x_t \to \{x\colon k_1(x) = 0\} \cap \{x\colon V(x, t) < m$ for some $t \geqq T\}$ with a probability no less than $1 - V(x, T)/m$.

Many special results, analogous to deterministic results, may be developed for time-dependent processes or functions, but we will not pursue the matter here.

Theorem 3 will be useful in examples where the "stability" of only a few components of x is of interest. (For example, where some components are of the nature of "parameters.") In such cases the sets Q_m are not always bounded. Define the event $D_{\varepsilon_i} \equiv \{\omega\colon \omega \in B_m$, and $\int_0^\infty I_x(s, \omega, \varepsilon_i)\, ds < \infty\}$. D_{ε_i} includes those ω such that $x_t(\omega) \in Q_m$, all $t < \infty$, and $x_t(\omega) \in Q_m - N_{\varepsilon_i}(P_m)$ for only a finite total time.

Theorem 3. (*Asymptotic Stability*) Assume the conditions of Theorem 2, except that boundedness of Q_m and P_m is not required.

Let, for any $\varepsilon_i > 0$, $P_x\{\|x_t\| \to \infty, x_t \in Q_m, D_{\varepsilon_i}\} = 0$. Then the conclusions of Theorem 2 hold. They also hold if (A5) holds uniformly in x in $Q_m - N_\varepsilon(P_m)$, for each $\varepsilon > 0$.

Proof. If Q_m and P_m are not bounded, then the proof of Theorem 2 implies that either: (c) there is a random variable $\tau_1(\omega) < \infty$ with probability one such that $x_t \in N_{\varepsilon_1}(P_m)$, all $t > \tau_1(\omega)$, for $\omega \in B_m$; or (d) for *any* compact set* $A \subset Q_m$, there is a random variable $\tau(A) < \infty$ with probability one such that x_t is not A for $t > \tau(A)$ and $\omega \in B_m$. The event D_{ε_i}, $\varepsilon_i > 0$, occurs in any case. (Equations (2-4) and (2-5) are valid when Q_m is replaced by compact A.) Under condition (d), there is a monotone increasing sequence of sets A_n, $A_n \uparrow Q_m$, and random variables $\tau(A_n) < \infty$ with probability one for each n, which satisfy (d). Then, under condition (d), $\|x_t\| \to \infty$, and $x_t \in Q_m$ for all t. Since $P_x\{\|x_t\| \to \infty, x_t \in Q_m, D_{\varepsilon i}\} = 0$, case (c) must hold. Since ε_1 is arbitrary, $x_t \to P_m$ with probability one, relative to B_m. The last statement follows from the proof of Theorem 2, and the observation that, under the uniformity hypothesis, there can be no escape to infinity in finite (with probability one) time, unless the tails of the trajectories are entirel- contained in $N_\varepsilon(P_m)$.

The following corollary will be useful in the chapters on optimal cony trol and design of controls. The set S will correspond to a "target" set.

Corollary 3-1. Assume (A1) to (A5) with the following exception: $V(x)$ is not necessarily nonnegative. Let $S = \{x : V(x) \leqq 0\}$, and define $Q'_m = \{x : m > V(x) > 0\}$. Define τ_m as $\inf\{t : x_t \notin Q'_m\}$. Let $\tilde{A}_m$ be the weak infinitesimal operator corresponding to the stopped process $x_{t \cap \tau_m}$. Let $V(x)$ be in the domain of $\tilde{A}_m$ with $\tilde{A}_m V(x) = -k(x) \leqq 0$ in Q'_m.

Suppose that $k(x)$ is uniformly continuous** in the set $P'_m \equiv$

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* The words "any compact set A" could be replaced by the words "any set A for which (A5) holds uniformly, bounded or not."

** More generally let $N_\varepsilon(P'_m)$ be an ε-neighborhood of P'_m relative to Q'_m. Let $k(x)$ satisfy, on $Q'_m - N_{\varepsilon_i}(P'_m)$ the conditions on $Q'_m - N_{\varepsilon_i}(P_m)$ of Theorem 2. Then Corollary 3–1 remains true.

$\{x: k(x) = 0\} \cap \{Q'_m + \partial S\}$ and $k(x) > 0$ at some x in Q'_m. Then there is a random variable $\tau \leqq \infty$ such that $x_t \to \{x: k(x) = 0\} \cup \{Q'_m + S\}$ as $t \to \tau$, with a probability no less than $1 - V(x)/m$. Also

$$P_x\{ \sup_{\tau \geqq t \geqq 0} V(x_t) \geqq m\} \leqq \frac{V(x)}{m}.$$

Proof. The proof follows closely the proofs of Theorem 2 and Lemma 1 and will be omitted.

This situation envisioned in Corollary 3-1 is depicted in Figure 1. The behavior of the process after τ is not of interest. Note that if m is arbitrary in Corollary 3-1, and $k(x) > 0$ in each Q'_m, then there is a random variable $\tau' \leqq \infty$ such that $x_t \to S$ with probability one as $t \to \tau'$.

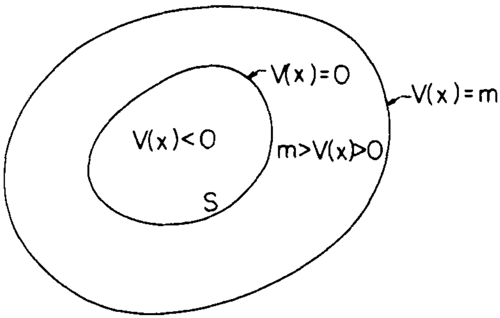


Figure 1

The result for Itô and Poisson processes are collected in Corollaries 3-2 and 3-3. The proofs involve merely the substitution of the properties of these processes into the hypotheses of Theorem 2 or 3.

Corollary 3-2. Let x_t be an Itô process, the coefficients of whose differential generator satisfy the boundedness and Lipschitz conditions (of Chapter I, Section 4) in Q_m. Let $V(x)$ be a continuous nonnegative function which is bounded and has bounded continuous first and second derivatives in Q_m. (If $S_{ij}(x) \equiv 0$ in Q_m, then $\partial^2 V(x)/\partial x_i \, \partial x_j$ need not be continuous.) Then $V(x)$ is in the domain of $\tilde{A}_m$. With the homogeneous case notation,

$$\tilde{A}_m V(x) = \mathscr{L} V(x) = \sum_i f_i(x) \frac{\partial V(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij}(x) \frac{\partial^2 V(x)}{\partial x_i \, \partial x_j} = -k(x).$$

$$S(x) = \{S_{ij}(x)\} = \sigma'(x)\,\sigma(x).$$

Let $k(x) \geqq 0$ in Q_m, and if Q_m is unbounded, let $P_x\{\|x_t\| \to \infty, x_t \in Q_m, D_\varepsilon\} = 0$, for each $\varepsilon > 0$. Then $x_t \to \{x: k(x) = 0\} \cap Q_m$ with at least the probability $1 - V(x)/m$.

Remark on strong diffusion processes

Define $G_r = \{x: \|x\| \leqq r\}$ and $R > r > 0$. Suppose that x_t is a strong diffusion process in $G_R - G_r$, for some fixed $R > 0$ and each $r > 0$ (that is, for each $r > 0$, there is an $m_r > 0$ so that $\xi' S(x)\,\xi \geqq m_r \|\xi\|^2$, for any vector ξ). Note that the process is not necessarily a strong diffusion process in the set G_R, since m_r may tend to zero as $r \to 0$. In fact, if the origin is stable with probability one ((2-6)) this must be the case, since, otherwise, the escape time from G_R is finite with probability one, for any initial value x, including $x = 0$. Let

$$P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq R\} \to 0, \qquad \text{as} \qquad x \to 0. \tag{2-6}$$

Then there is a nonnegative continuous function $V(x)$ satisfying $V(0) = 0$, $V(x) = 1$, $x \in \partial G_R$, $V(x) > 0$, $x \neq 0$, and $x \in G_R$, and $V(x)$ is in the domain of the weak infinitesimal operator of each process $x_{t \cap \tau_r}$, where τ_r is the first exit time from $G_R - G_r - \partial G_R = D_r$. Also, $\tilde{A}_{D_r} V(x) = LV(x) = 0$ in D_r for each $r > 0$ and

$$P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq R\} \leqq P_x\{\sup_{\infty > t \geqq 0} V(x_t) \geqq 1\} \leqq V(x). \tag{2-7}$$

In other words, the existence of a stochastic Liapunov function is also a necessary condition for stability of the origin with probability one (2-6). The sufficiency follows from Theorem 5 (we say nothing there about the properties of $V(x)$ near $x = 0$). Also $x_t \to 0$ with a probability at least $1 - P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq R\}$. Thus, stability of the origin with probability one implies asymptotic stability with probability one, in this case. The result is due to Khas'minskii [3], and a proof follows.

Define the function $\varphi(x)$ on the boundary of $G_R - G_r$. Let $\varphi(x) = 1$ on ∂G_R and $\varphi(x) = 0$ on ∂G_r. Then, since x_t is a strong diffusion process in D_r, the equation (see Chapter I, Section 6) $LV_r(x) = 0$ with boundary condition $V_r(x) = \varphi(x)$, for $x \in \partial D_r$ has a unique continuous solution which is given by

$$V_r(x) = E_x \varphi(x_{\tau_r}) \equiv P_x\{x_t \quad \text{reaches} \quad \partial G_R \quad \text{before} \quad x_t \quad \text{reaches} \quad \partial G_r\},$$

where τ_r is the first exit time from D_r. Also $E_x \tau_r < \infty$, $x \in D_r$, and $0 \leqq V_r(x) \leqq 1$.

Let $\{r_i\}$ be a sequence of values of r tending to zero. It is easy to show, using the strong maximum principle for elliptic operators, that $V_{r_n}(x)$ is a nondecreasing sequence in each $G_R - G_r$ (for $r_n \leqq r$). Define $V(x)$:

$$\begin{aligned} V(x) = \lim_{r \to 0} V_r(x) &= \lim_{r \to 0} P_x\{x_t \quad \text{reaches} \quad \partial G_R \quad \text{before} \quad \partial G_r\} \\ &= P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq R\}. \end{aligned} \tag{2-8}$$

The last term on the right of (2-8) should be $P_x\{\sup_{\tau > t \geqq 0} \|x_t\| \geqq R\}$, where $\tau = \lim_{r \to 0} \tau_r$. If $\tau < \infty$ with a probability which is greater than zero, then either $\|x_t\| = R$ for some $t < \infty$, or $\|x_t\| \to 0$, $t \to \tau$. In the latter case $\sup_{\infty > t \geqq 0} \|x_t\| < R$ with probability one (relative to $\{\|x_t\| \to 0 \text{ as } t \to \tau\}$). This may be proved by use of the hypothesis (2-6). Thus equation (2-8) is valid.

By hypothesis the right side of (2-8) goes to zero as $x \to 0$; thus $V(0) = 0$. Also, by the method of constructing the $V_r(x)$, $0 \leqq V(x) \leqq 1$ and $V(x) = 1$ on ∂G_R. $V(x)$ is the limit of a nondecreasing uniformly bounded sequence of harmonic functions; hence, $LV(x) = LV_r(x) = 0$ in each D_r, $r > 0$.

Now an argument similar to that used in Theorem 5 may be applied to complete the proof. $V(x)$ is in the domain of the weak infinitesimal operator of the process in each D_r, and the operator, acting on $V(x)$ in this region, is L. In Theorem 5, we require $\tilde{A}_{m,\varepsilon} V(x) \leqq -\delta_\varepsilon$ in

$Q_R - Q_\varepsilon$ (or, equivalently, $LV(x) \leqq -\delta_r < 0$ for each $r > 0$ in D_r), in order to assure that the first exit time from the set $Q_R - Q_\varepsilon$ would be finite with probability one. The latter property holds here for each D_r. Thus, pursuing the argument of Theorem 5, we conclude that $x_t \to 0$ with at least the probability $1 - P_x\{\sup_{\infty > t \geqq 0} \|x_t\| \geqq R\}$. Equation (2-7) is obvious from (2-8).

Corollary 3-3. Let the nonnegative function $V(x)$ have bounded and continuous first partial derivatives in Q_m and let

$$dx = f(x)\,dt + \sigma(x)\,dz$$

be the Poisson equation of Section 5, Chapter I. Let $f(x)$ and $\sigma(x)$ satisfy a uniform Lipschitz condition and be bounded in Q_m. Let $a_i\Delta + o(\Delta)$ be the probability that z_{it} has a jump in $[t, t + \Delta)$ and $P_i(dy)$ the density of the jump in z_{it}. Let each $P_i(dy)$ have compact support. Suppose that $V(x + \sigma(x)\,y^i)$ is bounded* and continuous for x in Q_m and for each y^i for which $P_i(dy)$ is not zero. Then $V(x)$ is in the domain of $\tilde{A}_m$ and

$$\begin{aligned}\tilde{A}_m V(x) = \mathscr{D}V(x) &= \sum_i f_i(x)\, V_{x_i}(x) \\ &\quad + \sum_i \int [V(x + \sigma(x)\,y^i) - V(x)]\, a_i P_i(dy) \\ &= -k(x).\end{aligned}$$

Suppose that $k(x)$ satisfies the conditions of either Theorem 2 or 3. (Note that $k(x)$ is continuous in Q_m.) Then

$$x_t \to \{x : k(x) = 0\} \cap Q_m$$

with probability at least $1 - V(x)/m$.

Proof. The theorem follows from Theorem 3 and Section 5, Chapter I.

If $\tilde{A}_m V(x) = -k(x) \leqq 0$ in Q_m and $k(x)$ is proportional to $V(x)$ in Q_m, then a type of exponential convergence occurs. In particular:

* y^i is a vector with y in the ith component and zero elsewhere.

Theorem 4. (*Exponential Asymptotic Stability*) Assume (A1) to (A5), $V(0)=0$, $\tilde{A}_m V(x) \leqq -\alpha V(x)$ in Q_m for some $\alpha > 0$. Then, with $x_0 = x$,

$$P_x\{\sup_{\infty > t \geqq T} V(x_t) \geqq \lambda\} \leqq \frac{V(x)}{m} + \frac{V(x)\, e^{-\alpha T}}{\lambda}.$$

If the hypothesis holds for arbitrary m, then

$$P_x\{\sup_{\infty > t \geqq T} V(x_t) \geqq \lambda\} \leqq \frac{V(x)\, e^{-\alpha T}}{\lambda}.$$

Proof. Let $\tau_m = \inf\{t: x_t \notin Q_m\}$. Define the process $\tilde{x}_t$

$$\begin{aligned} \tilde{x}_t &= x_t \qquad (t < \tau_m), \\ \tilde{x}_t &= 0 \qquad (t \geqq \tau_m). \end{aligned}$$

Let $\tilde{E}_x$ correspond to the process $\tilde{x}_t$. The process $\tilde{x}_t$, $t < \infty$, is a right continuous strong Markov process (although τ_m is not necessarily a Markov time of the process $\tilde{x}_t$). By Dynkin's formula,

$$\begin{aligned} \tilde{E}_x V(\tilde{x}_t) - V(x) &\leqq E_x V(x_{t \cap \tau_m}) - V(x) = E_x \int_0^{\tau_m \cap t} \tilde{A}_m V(x_s)\, ds \\ &\leqq -\alpha E_x \int_0^{\tau_m \cap t} V(x_s)\, ds = -\alpha \tilde{E}_x \int_0^t V(\tilde{x}_s)\, ds \\ &= -\alpha \int_0^t \tilde{E}_x V(\tilde{x}_s)\, ds, \end{aligned}$$

which implies, by the Gronwall-Bellman lemma, that

$$\tilde{E}_x V(\tilde{x}_T) \leqq V(x)\, e^{-\alpha T}.$$

Now, the facts

$$\begin{aligned} &\tilde{E}_x V(\tilde{x}_t) \leqq V(x) \\ &\tilde{E}_x V(\tilde{x}_t) \to V(x), \qquad \text{as} \qquad t \to 0 \end{aligned}$$

imply that $V(\tilde{x}_t)$ is a nonnegative supermartingale (Dynkin [2], Theorem 12.6). The supermartingale inequality yields

$$P_x\{\sup_{\infty > t \geqq T} V(\tilde{x}_t) \geqq \lambda\} \leqq \frac{\tilde{E}_x V(\tilde{x}_T)}{\lambda} \leqq \frac{V(x)\, e^{-\alpha T}}{\lambda}$$

which, together with

$$P_x\{\sup_{\infty > t \geqq 0} |V(\tilde{x}_t) - V(x_t)| > 0\} = P_x\{\sup_{\infty > t \geqq 0} V(x_t) \geqq m\} \leqq \frac{V(x)}{m}$$

implies the theorem.

An alternative proof can be based on the observation that $\tilde{A}_m e^{\alpha t} V(x) \leqq 0$ in $\{x\colon V(x) < m\} \equiv Q_m$. Then $e^{\alpha t \cap \tau_m} V(x_{t \cap \tau_m})$ is a non-negative supermartingale. Thus

$$\begin{aligned} P_x\{\sup_{\infty > t \geqq 0} V(x_t) \exp \alpha t \geqq m\} &= P_x\{V(x_t) \geqq m \exp - \alpha t, \text{ some } t < \infty\} \\ &\leqq \frac{V(x)}{m}, \end{aligned}$$

with $\lambda < m$ and $T > 0$; we have

$$\begin{aligned} &P_x\{\sup_{\infty > t \geqq T} V(x_t) \geqq \lambda\} \\ &\quad \leqq P_x\{\sup_{\infty > t \geqq T} e^{\alpha t} V(x_t) \geqq \lambda\, e^{\alpha T}\} \\ &\quad \leqq P_x\{\tau_m < T\} + P_x\{\sup_{\infty > t \geqq T} e^{\alpha t \cap \tau_m} V(x_{t \cap \tau_m}) \geqq \lambda\, e^{\alpha T}\} \\ &\quad \leqq P_x\{\tau_m < T\} + \frac{E_x V(x_{T \cap \tau_m})\, e^{\alpha T \cap \tau_m}}{\lambda\, e^{\alpha T}}. \end{aligned}$$

Since $P_x\{\tau_m < T\} \leqq V(x)/m$ and $E_x V(x_{T \cap \tau_m}) \exp \alpha\, T \cap \tau_m \leqq V(x)$, Theorem 4 is again proved.

It is not usually necessary that $V(x)$ be in the domain $\tilde{A}_m$ in all of Q_m. In particular, if we are concerned only with the proof that $x_t \to 0$ as $t \to \infty$, then the properties of $V(x)$ at $x = 0$ may not be important. It may only be required that $V(x)$ be in the domain of the weak infinitesimal operator in the complement (with respect to Q_m) of each neighborhood of $\{x = 0\}$. Theorem 5, giving just such an extension of Theorem 3, will be useful to improve the result in one of the examples where the process is an Itô process and the Liapunov function has a cusp at the origin. The theorem may be extended to include non-compact Q_m.

Note that stochastic continuity is *not explicitly used* in the proof. The supermartingale inequalities (2-9) and (2-10) are sufficient to give the result in the case of Theorem 5, since $\{x: k(x)=0\}$ must be included in $\{x: V(x)=0\}$. Define, for $\varepsilon < m, \tau_m(\varepsilon) = \inf\{t: x_t \notin Q_m - Q_\varepsilon - \partial Q_\varepsilon\}$

Theorem 5. Assume (A1) to (A3), $V(0)=0$, and that the set Q_m is bounded. Denote the first exit time of x_t from $Q_m - Q_\varepsilon - \partial Q_\varepsilon$ by $\tau_m(\varepsilon)$. Let $\tilde{A}_{m,\varepsilon}$ denote the weak infinitesimal operator of the process $x_{t\cap\tau_m}(\varepsilon)$. Let $V(x)$ be in the domain of $\tilde{A}_{m,\varepsilon}$ for $\varepsilon > 0$ arbitrarily small, and let $\tilde{A}_{m,\varepsilon} V(x) = -k(x) \leqq 0$ in $Q_m - Q_\varepsilon - \partial Q_\varepsilon$. For each small $\varepsilon > 0$, let there be a $\delta > 0$ so that $k(x) \geqq \delta$ in $Q_m - Q_\varepsilon - \partial Q_\varepsilon$. Suppose that $\{x: V(x)=0\}$ is an absorbing set; that is, exit is impossible.* Then $x_t \to \{x: V(x)=0\}$ with a probability at least $1 - V(x)/m$, as $t \to \infty$.

Proof. Define the sequence ε_i, $\varepsilon_i = \varepsilon^i \varepsilon_{i-1}$, $\varepsilon_1 = \varepsilon$, where $0 < \varepsilon < m$, and $\varepsilon < 1$. The Q_{ε_i} decrease to $\{x: V(x)=0\}$, and $V(x)$ is in the domain of $\tilde{A}_{m,\varepsilon_i}$, for each $i < \infty$. Let δ_i correspond to ε_i. Then, by Lemma 2, $E_x \tau_m(\varepsilon_i) < \infty$ and the $\tau_m(\varepsilon_i)$, $i < \infty$, are all finite with probability one. Denote by τ the limit $\tau = \lim_n \tau_m(\varepsilon_n)$. τ is a Markov time of x_t, since the nondecreasing $\tau_m(\varepsilon_i)$ are. Let $x = x_0 \neq 0$ and let y be any point in $Q_{\varepsilon_n} + \partial Q_{\varepsilon_n}$. Note that

$$P_x\{\sup_{\infty > t \geqq 0} V(x_{t\cap\tau_m(\varepsilon_n)}) \geqq m\} \leqq \frac{V(x)}{m} \tag{2-9}$$

$$P_y\{V(x_t) \geqq \varepsilon_{n-1} \text{ before } V(x_t) \leqq \varepsilon_{n+1}\} \leqq \frac{V(y)}{\varepsilon_{n-1}} \leqq \frac{\varepsilon_n}{\varepsilon_{n-1}} = \varepsilon^n. \tag{2-10}$$

The probability of the set of trajectories $(\Omega - B_\varepsilon)$ which leave Q_m before entering any arbitrary Q_ε, $\varepsilon > 0$, is less than $V(x)/m$. Then

* The assumption that $\{x: V(x) = 0\}$ is an absorbing set is convenient if $\tau < \infty$ (see proof) with a nonzero probability. It eliminates the need for introducing assumptions which guarantee it, as well as the attendent arguments. In any case, either $V(x_t) \to 0$ as $t \to \infty$ or $V(x_s) = 0$ for some $s < \infty$, with probability $\geqq [1 - V(x)/m]$.

$x_t(\omega) \to Q_{\varepsilon_n} + \partial Q_{\varepsilon_n}$, as $t \to \tau_m(\varepsilon_n)$ with at least the probability $1 - V(x)/m$. Once in $Q_{\varepsilon_n} + \partial Q_{\varepsilon_n}$, the probability that a trajectory will leave $Q_{\varepsilon_{n-1}}$ before entering $Q_{\varepsilon_{n+1}} + \partial Q_{\varepsilon_{n+1}}$ is bounded by ε^n by (2-10). Thus, for $\omega \in B_{\varepsilon_n}$, $V(x_t)$ is bounded by ε_{n-1} in the time interval $[\tau_m(\varepsilon_n), \tau_m(\varepsilon_{n+1}))$ with a probability no less than $(1 - \varepsilon_n/\varepsilon_{n-1}) = (1 - \varepsilon^n)$.

Thus, x_t converges uniformly to zero, as $t \to \tau$, with a probability no less than

$$\left(1 - \frac{V(x)}{m}\right)\prod_2^\infty (1 - \varepsilon^n). \tag{2-11}$$

Since ε is arbitrary, $V(x_t) \to 0$ with probability $\geqq [1 - V(x)/m]$, as $t \to \tau$, and the theorem is proved.

The Liapunov function method may be used to estimate the asymptotic values of moments, providing that they exist. In any case, we have:

Theorem 6. Assume (A1) to (A4). Let $V(x)$ be in the domain of $\tilde{A}_m$ for each $m > 0$. Suppose that $V(x)$ is bounded for finite x, and that x_t does not have a finite escape time (with probability one) to ∞. Let

$$\tilde{A}_m V(x) \leqq -k(x) + c^2 \tag{2-12}$$

$$k(x) \geqq 0 \quad \text{in each} \quad Q_m, \qquad \infty > c > 0.$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{E_x k(x_s)\, ds}{c^2} \leqq 1. \tag{2-13}$$

Strict equality in (2-12) implies strict equality in (2-13). If $E_x k(x_s)$ has a limit, then it is at most c^2.

Proof. By Dynkin's formula and (2-12)

$$V(x) - E_x V(x_{t\cap\tau_m}) = -E_x \int_0^{t\cap\tau_m} \tilde{A}_m V(x_s)\, ds$$

$$\geqq E_x \int_0^{t\cap\tau_m} [k(x_s) - c^2]\, ds, \tag{2-14}$$

where τ_m is the first exit time from Q_m. Since $t<\infty$, $\infty>c>0$, and $k(x)\geqq 0$, and also since, by hypothesis, $\tau_m \to \infty$ with probability one, we may replace the limit of the right-hand integral of (2-14) (as $m\to\infty$) by the integral from 0 to t (dominated convergence theorem). Also, by Fubini's theorem, the order of integration may be changed. Thus

$$V(x) - \lim_{m\to\infty} E_x V(x_{t\cap\tau_m}) \geqq \int_0^t E_x k(x_s)\,ds - c^2 t.$$

Now, (2-13) follows by letting $t\to\infty$ in

$$\frac{V(x)}{c^2 t} \geqq \frac{V(x) - \lim_{m\to\infty} E_x V(x_{t\cap\tau_m})}{c^2 t} \geqq \frac{1}{t}\int_0^t \frac{E_x k(x_s)\,ds}{c^2} - 1.$$

Remark. Generally, in applications of Theorem 6, when $V(x)$ is large, $\tilde{A}_m V(x)$ is negative ($V(x)$ decereases "on the average" along trajectories of the process) and $\tilde{A}_m V(x)$ is positive when $V(x)$ is small ($V(x)$ then increases along trajectories of the process "on the average"). x_t "oscillates" in some random fashion about the set $\bigcup_{m=1}^{\infty}(\{x\colon\ \tilde{A}_m V(x)=0\}\cap Q_m)$. Since, in applications generally $\tilde{A}_m V(x)=\tilde{A}_n V(x)$, $n>m$, in Q_m, the sets $\{x\colon \tilde{A}_m V(x)=0\}\cap Q_m$ will generally be increasing.

Extension. Let $\tilde{A}_m V(x) = -k(x)+g(x)$, $k(x)\geqq 0$, $g(x)\geqq 0$. Then Theorem 6 may be extended to read

$$\lim_{t\to\infty} \frac{\int_0^t E_x k(x_s)\,ds}{\int_0^t E_x g(x_s)\,ds} = 1$$

provided only that some condition is given which guarantees that

$$E_x \int_0^{\tau_m\cap t} (-k(x_s)+g(x_s))\,ds = -E_x \int_0^{\tau_m\cap t} k(x_s)\,ds + E_x \int_0^{\tau_m\cap t} g(x_s)\,ds$$

and

$$\lim_{t\to\infty} \frac{V(x)}{\int_0^t E_x g(x_s)\,ds} = 0.$$

While the problem of existence of stochastic Liapunov functions has not been solved,* some existence results are possible. Theorem 7 is applicable to the case where, for example, x_t is an Itô process, $h(x_s) = \|x_s\|$, and $E_x\|x_s\|$ tends to zero exponentially. See Kats and Krasovskii [1] and also Hahn [1], Theorem 24.5.

Theorem 7. Let x_t be a right continuous strong Markov process. Let $h(x)$ satisfy $h(0) = 0$, $h(x) > 0$ for $x \neq 0$, and be continuous at 0. Suppose that $x_0 = x$ and

$$F(x) = E_x \int_0^\infty h(x_s)\, ds < \infty, \qquad \text{all } x \text{ in } E$$

$$\frac{E_x \int_0^{\delta \cap \tau_Q} h(x_s)\, ds}{\delta} \to h(x) \qquad \text{as} \qquad \delta \to 0$$

$$E_x h(x_{s \cap \tau_Q}) \to h(x) \qquad \text{as} \qquad s \to 0$$

where τ_Q is the first exit time from an arbitrary neighborhood, Q, of the origin. Suppose that

$$F(x) \to 0 \qquad \text{as} \qquad x \to 0,$$

$$F(x) \to \infty \qquad \text{as} \qquad \|x\| \to \infty.$$

Then $F(x)$ is a stochastic Liapunov function satisfying the conditions of Theorem 2, and $x_s \to 0$ with probability one, as $s \to \infty$, for any x_0.

Proof. First, we show that $V(x)$ is in the domain of $\tilde{A}_Q$, for each neighborhood Q of the origin. Let x be in Q. By Dynkin [2], Theorem 3.11,

$$E_x F(x_{\delta \cap \tau_Q}) = E_x E_{x_{\delta \cap \tau_Q}} \int_0^\infty h(x_{s+\delta \cap \tau_Q})\, ds = E_x \int_{\delta \cap \tau_Q}^\infty h(x_s)\, ds.$$

* Stochastic counterparts of the deterministic existence theorems will appear in H. J. Kushner, Converse theorems for stochastic Liapunov functions, *SIAM J. Control* **5**, No. 2 (1967).

Then, under the hypothesis,

$$\frac{E_x F(x_{\delta\cap\tau_Q}) - F(x)}{\delta} = \frac{-E_x \int_0^{\delta\cap\tau_Q} h(x_s)\,ds}{\delta} \to -h(x)$$

$$E_x h(x_s) \to h(x)$$

and, hence,

$$\tilde{A}_Q F(x) = -h(x), \quad \text{each} \quad Q \subset E \quad \text{and} \quad x \in Q.$$

By hypothesis, $Q_m = \{x\colon F(x) < m\}$ is bounded and $F(x)$ is continuous. Thus $F(x)$ satisfies the hypothesis on $V(x)$ of Theorem 2. Q.E.D.

Theorem 8. Assume (A1) to (A4), except that the killing time ζ may be less than infinity. Let $0 \leqq V(x) \to \infty$ as $\|x\| \to \infty$ and, for each $m > 0$, $\tilde{A}_m V(x) = g(x)$, where $g(x) \leqq 0$ for large $\|x\|$. Let there be an infinite sequence of compact sets $G_i \uparrow E$, such that, if $\|x_t\| \to \infty$ as $t \to \zeta < \infty$, then (with probability one) $x_{\tau(i)} \in G_i$ for some sequence of random times $\tau(i) \uparrow \zeta$. Then there is no finite escape time ($\zeta = \infty$ with probability one). If $\tilde{A}_m V(x) \leqq -\varepsilon < 0$ in $Q_m - G$, for all large m and where G is compact, then x_t always returns to G with probability one.

Remark. The purpose of the condition on the G_i and $\tau(i)$ is to eliminate the possibility that x_t either disappears or jumps directly to infinity from some finite point.* A simple example will illustrate the problem. Let x_t be a constant, $t < \zeta$. Let the killing time ζ have the distribution $P\{\zeta \leqq t + \delta | \zeta > t\} = c\delta + o(\delta)$. Then $E_x V(x_{t+\Delta}) = (1 - c\delta + o(\delta))\, V(x)$ and

$$\tilde{A}_m V(x) = -cV(x)$$

for all $V(x)$. But we may infer neither stability nor even boundedness of the x_t process (in any time interval).

Proof. For large i each G_i is contained in some bounded Q_{α_i} and contains some Q_{β_i}. For all large α, $g(x) \leqq 0$ in $E - Q_\alpha$. Let $Q_m \supset Q_r \supset Q_p$

* An equivalent condition is that for each bounded open set Q with first exit time τ_Q, there is an $\varepsilon_Q > 0$ with probability one and $\zeta \geqq \tau_Q + \varepsilon_Q$.

with $x = x_0$ in $Q_r - Q_p - \partial Q_p$ and $g(x) \leqq 0$ in $Q_m - Q_p - \partial Q_p$. Define $\tau_m = \inf\{t: x_t \notin Q_m - Q_p - \partial Q_p\}$. Then $\tau_m < \zeta$ with probability one, for each m, and

$$E_x V(x_{\tau_m \cap t}) - V(x) = E_x \int_0^{\tau_m \cap t} \tilde{A}_m V(x_s)\, ds = E_x \int_0^{\tau_m \cap t} g(x_s)\, ds \leqq 0.$$

Since $\tau_m < \zeta$ we have $mP_x\{\sup_{\tau_m \cap t \geqq s \geqq 0} V(x_s) \geqq m\} \leqq E_x V(x_{\tau_m \cap t})$ for all $t < \infty$ and, hence,

$$P_x\{\sup_{\tau_m \geqq s \geqq 0} V(x_s) \geqq m\} \leqq \frac{V(x)}{m}. \tag{2-15}$$

Since (2-15) holds for arbitrary $m < \infty$, x_t must always enter $Q_p + \partial Q_p$ before going to infinity, with probability one. This implies that there is no finite escape time. The last statement of the theorem follows from the above reasoning and Lemma 2.

Many other stochastic Liapunov theorems are possible. The proof of Theorem 9 is a slight variation on the proof of Theorem 2 and will not be given.

Theorem 9. Assume (A1) to (A5) with $\tilde{A}_m V(x) \leqq -k(x) + \varphi_t$, for each m, where $k(x) \geqq 0$ and $\varphi_t \to 0$ as $t \to \infty$. Let $k(x) \to \infty$ and $V(x) \to \infty$ as $\|x\| \to \infty$. Let $k(x)$ be continuous on the compact set $\{x: k(x) = 0\} = P$. Then, for each neighborhood $N(P)$, there is a finite-valued random time τ_N such that $x_t \in N(P)$, all $t > \tau_N$, with probability one. Suppose that $k(x) \leqq 0$ in $Q_m \supset P$ only. Then $x_t \to P$ with the probability that x_t does not escape from Q_m.

The following theorem is proved by use of the supermartingale $V(x) + \int_t^T \varphi_s\, ds$; the proof is omitted.

Theorem 10. Assume (A1) to (A4) and let $V(x) \to \infty$ as $\|x\| \to \infty$. Let $\tilde{A}_m V(x) \leqq \varphi_t$ where $\varphi_t \geqq 0$ is integrable over $[0, \infty)$. Then there is a finite-valued random variable v such that $V(x_t) \to v$ with probability one. Also

$$P_x\{\sup_{\infty > t \geqq 0} V(x_s) \geqq m\} \leqq \frac{V(x) + \int_0^\infty \varphi_s\, ds}{m}.$$

There is a simple stochastic analog of a theorem of Yoshizawa [1] concerning a bound on the "sensitivity" of the trajectory to the initial condition. The proof is similar to that of Theorem 1 or 2 and is omitted. The processes x_s, y_s have initial conditions x, y, respectively. If x_s and y_s are right continuous strong Markov processes, then so is (x_s, y_s).

Theorem 11. Let (A1) to (A5) hold for the process (x_s, y_s) and function $V(x, y, t)$ in the bounded open set $Q = \{x, y \colon W(\|x - y\|) < m\}$, where $W(0) = 0$ and $W(r)$ is continuous and strictly monotone increasing for $r \leqq m$. Let

$$W(\|x - y\|) \leqq V(x, y, t).$$

Let

$$\tilde{A}_Q V(x, y, t) \leqq -k(x, y, t) \leqq -k_1(\|x - y\|) \leqq 0.$$

Then

$$P_{x,y}\{\sup_{\infty > t \geqq 0} \|x_t - y_t\| \geqq W^{-1}(m)\} \leqq P_{x,y}\{\sup_{\infty > t \geqq 0} V(x_t, y_t, t) \geqq m\} \leqq \frac{V(x, y, 0)}{m}.$$

If $k_1(r)$ is continuous on $P = \{x, y \colon k_1(\|x - y\|) = 0\} \cap Q$, then we have $\|x_t - y_t\| \equiv r_t \to \{r : k_1(r) = 0\}$ with a probability at least $1 - V(x, y, 0)/m$.

3. Examples

First, several simple scalar Itô processes will be investigated.

Example 1. Let x_t be the solution of the scalar Itô equation

$$dx = ax\,dt + \sigma x\,dz. \tag{3-1}$$

The function $V(x) = x^2$ satisfies the conditions of Corollary 3-2 in each set $Q_m = \{x \colon x^2 < m^2\}$, and

$$\tilde{A}_m V(x) = \mathscr{L} V(x) = x^2(2a + \sigma^2).$$

If $2a + \sigma^2 \leqq 0$, Corollary 3-2 yields the estimate

$$P_x\{\sup_{\infty > t \geqq 0} x_t^2 \geqq m^2\} \leqq \frac{x^2}{m^2}.$$

Let $2a + \sigma^2 < 0$; then $x_t \to 0$ with at least the probability $1 - x^2/m^2$. Since m is arbitrary here, $x_t \to 0$ with probability one.

Since $\mathscr{L}V(x) = -\alpha V(x)$, $\alpha = |2a + \sigma^2|$, and m is arbitrary, Theorem 4 yields, for any $\lambda < \infty$,

$$P_x\{\sup_{\infty > t \geqq T} V(x_t) \geqq \lambda\} \leqq \frac{V(x)(\exp - \alpha T)}{\lambda}.$$

Example 2. Let x_t be the solution of (3-1). An improved result will be obtained with the use of the Liapunov function $V(x) = |x|^s$, $\infty > s > 0$. If $s < 2$, the second derivative of $V(x)$ does not exist at $x = 0$. Nevertheless, Theorem 5 is still applicable. Then, appealing to Corollary 3-2 and Theorem 5, we have in each $Q_m - Q_\varepsilon - \partial Q_\varepsilon$

$$\tilde{A}_{m,\varepsilon} V(x) = \mathscr{L}V(x) = bx^2,$$

where we suppose that

$$b = a + \frac{(s-1)\sigma^2}{2} < 0.$$

Hence,

$$P_x\{\sup_{\infty > t \geqq 0} |x_t| \geqq m\} = P_x\{\sup_{\infty > t \geqq 0} |x_t|^s \geqq m^s\}$$

$$\leqq \frac{V(x)}{m^s} = \frac{|x|^s}{m^s}. \tag{3-2}$$

Clearly the larger is s, the smaller is (3-2). In general, Liapunov functions of higher algebraic order are preferable, if they exist, since they yield better estimates of the probabilities, as in the case (3-2). If $a < \sigma^2/2$, there is some $\infty > s > 0$ such that $b < 0$. Then $x_t \to 0$ with probability one. Thus, even if $a > 0$, there may still be asymptotic stability. Except for some specially constructed examples, there does not seem to be a vector analog of this result. In our case, it is verifiable directly by noting that the exact solution to (3-1) is (with probability one)

$$x_t = x_0 \exp\left[\left(a - \frac{\sigma^2}{2}\right)t + z_t\right]. \tag{3-3}$$

If $a - \sigma^2 = -c < 0$, then

$$x_t = x_0 \exp\left[-c + \frac{z_t}{t}\right]t$$

which tends to zero with probability one, as $t \to \infty$.

In deterministic stability studies, if a function $V(x)$ is a Liapunov function in a region Q then so is $V_\alpha(x) = V^\alpha(x)$, $\alpha > 1$, in the same region. There is not necessarily a stochastic analog of this property. (If there were, then all estimates of the form (3-2) could be written with $s = \infty$.) That $V(x)$ is a stochastic Liapunov function in Q implies

$$E_x \int_0^\infty \tilde{A}_Q V(x_s)\, ds = \int_0^\infty E_x \tilde{A}_Q V(x_s)\, ds < \infty$$

or that $E_x \tilde{A}_Q V(x_s)$ decreases sufficiently rapidly as $s \to \infty$. But this does not imply that $E_x \tilde{A}_Q V^\alpha(x_s)$ will decrease sufficiently rapidly to ensure that, for any $\alpha > 1$,

$$\int_0^\infty E_x \tilde{A}_Q V^\alpha(x_s)\, ds < \infty .$$

It is these "integral moment" properties which effectively determine the probability bounds.

Remarks on the model (3-1). Astrom [1] gives a thorough analysis, together with numerical results on the sample path behavior, of the scalar equation $dx = \sigma_1 x\, dz_1 + \sigma_2\, dz_2$, where z_{1t} and z_{2t} are (possibly correlated) Wiener processes.

Let

$$\dot{x} = \sigma x \xi$$

where ξ_s is integrable on each $[0, T]$. Then

$$x_t = x_0 \exp \sigma \int_0^t \xi_s \, ds .$$

The solution still holds (with probability one) if ξ_s is the solution of the Itô equation

$$d\xi = -\alpha\xi \, dt + \alpha \, dz \qquad (\alpha > 0) .$$

As $\alpha \to \infty$, $\int_0^t \xi_s \, ds \to z_t$ and

$$x_t \to x_0 \exp \sigma z_t$$

in mean square. The last equation is the solution to the Itô equation

$$dx = \frac{\sigma^2 x}{2} dt + \sigma x \, dz .$$

The conclusion to the discussion of this paragraph is the following. Suppose that ξ_s is a random function with a "large" bandwidth, which acts on the system whose output is governed by $\dot{x} = x\sigma\xi$, and which we wish to model by an Itô equation. Then the model $dx = \sigma^2 x/2dt + \sigma x \, dz$ may be preferable to the model $\dot{x} = \sigma x \, dz$. This modelling question, which is concerned with the relation of the solution of ordinary and stochastic differential equations, is discussed by Wong and Zakai [1, 2].

Example 3. Consider the scalar problem

$$dx = -ax \, dt + \sigma(x) \, dz \qquad (a > 0) .$$

The exponential form

$$V(x) = \exp \lambda |x|^\alpha$$

is useful in the study of a variety of scalar situations. The constants α and λ are to be selected. Here, and in Example 4, we set $\alpha = 1$. Since $V(x)$ does not have a second derivative at $x = 0$, but is in the domain of the weak infinitesimal operator of x_t in all $Q_m - Q_\varepsilon - \partial Q_\varepsilon$, $\varepsilon > 0$, Theorem 5, rather than Theorem 2, must be applied. Suppose that

$$\sigma^2(x) = |x|\,\sigma^2.$$

(Note that the corresponding $\sigma(x)$ does not satisfy a Lipschitz condition in any Q_m, owing to its behavior at $x = 0$. This is unimportant. We define the origin as an absorbing point, and the solution may be defined up to the time of contact with the origin by a method analogous to that of the last subsection of Section 4, Chapter I. Furthermore, the results below and the argument used in the proof of Theorem 5 show that either $x_t \to 0$ as $t \to \infty$, or $x_s = 0$ for some finite-valued Markov time s.)

For $x \neq 0$,

$$\mathscr{L}V(x) = \lambda V(x)\left[-a|x| + \frac{\lambda\sigma^2(x)}{2}\right]. \tag{3-4}$$

Setting $\lambda = 2a/\sigma^2$ yields $\mathscr{L}V(x) = 0$. Then, by an appeal to Theorem 5,

$$P_x\{\sup_{\infty > t \geqq 0} |x_t| \geqq \rho\} = P_x\{\sup_{\infty > t \geqq 0} e^{\lambda V(x_t)} \geqq e^{\lambda\rho}\} \leqq \frac{e^{\lambda V(x)}}{e^{\lambda\rho}}$$
$$= \exp\left[\frac{2a(|x| - \rho)}{\sigma^2}\right].$$

Example 4. Take the model of Example 3 but set

$$\sigma^2(x) = (1 - e^{-|x|})\,\sigma^2.$$

The remarks in Example 3 regarding the properties of the process in the neighborhoods Q_ε hold here also. For $x \neq 0$,

$$\mathscr{L}V(x) = \lambda V(x)\left[-a|x| + \frac{\lambda\sigma^2(1 - e^{-|x|})}{2}\right]. \tag{3-5}$$

Set $\lambda = 2a/\sigma^2$; this yields $\mathscr{L}V(x) \leqq 0$, for $x \neq 0$. By Theorem 5, we conclude, as in Example 3, that

$$P_x\{\sup_{\infty > t \geqq 0} |x_t| \geqq \rho\} = P_x\{\sup_{\infty > t \geqq 0} e^{\lambda|x_t|} \geqq e^{\lambda\rho}\} \leqq \exp\left[\frac{2a(|x| - \rho)}{\sigma^2}\right].$$

The bound is identical to that obtained in Example 3, even though the variance $\sigma^2(x)$ is uniformly bounded here. The probability bounds are determined by the choice of λ, whose value in both cases depends upon the properties of $\sigma^2(x)$ at $x=0$ (or, alternatively, on the requirement that the bracketed terms of (3-4) and (3-5) be nonpositive).

Example 5. A method of computing Liapunov functions for scalar Itô processes will be given. (See also Khas'minskii [1], where a similar method is used in the study of processes which are "dominated" in a certain sense by a one-dimensional process and Feller [2].) Let the model be

$$dx = f(x)\,dt + \sigma(x)\,dz \qquad (3\text{-}6)$$
$$f(0) = \sigma(0) = 0.$$

In order to obtain a tentative Liapunov function, we proceed formally at first. Assume that $V(0)=0$, $V(x)>0$ for $x \neq 0$. Let $V(x)$ be in the domain of $\tilde{A}_{m,\varepsilon}$ in $Q_m - Q_\varepsilon - \partial Q_\varepsilon$ and let $\tilde{A}_{m,\varepsilon}V(x) \leqq 0$ in $Q_m - Q_\varepsilon - \partial Q_\varepsilon$, where each Q_m is bounded. Then, in $Q_m - Q_\varepsilon - \partial Q_\varepsilon$,

$$\tilde{A}_{m,\varepsilon}V(x) = \mathscr{L}V(x) = V_x(x) f(x) + \frac{V_{xx}(x)\,\sigma^2(x)}{2} \leqq 0$$

or, equivalently,

$$\frac{V_{xx}(x)}{V_x(x)} \leqq -\frac{2f(x)}{\sigma^2(x)}. \qquad (3\text{-}7)$$

The function defined by (3-8) satisfies the relation (3-7), provided that the right-hand integral exists:

$$V_1(x) = \int_0^x ds \left[\exp - \int^s \frac{2f(u)\,du}{\sigma^2(u)}\right]. \qquad (3\text{-}8)$$

(The exponent of the exponential is written as an indefinite integral since, in many applications, the integral from 0 to s may not exist.) Since $x=0$ is an absorbing point, *the rest of the discussion may be specialized to* $x \geqq 0$. It is readily verified that $\mathscr{L}V_1(x) = 0$ on $Q_m - Q_\varepsilon - \partial Q_\varepsilon$ for any $0<\varepsilon<m<\infty$. From the nonformal point of view, the existence of the integrals in (3-8) must be verified.

Some special cases where the preceeding method is applicable are of interest. Suppose that

$$\begin{aligned} \sigma^2(x) &= c_1 x^2 + c_2|x| \qquad && (c_1 > 0, \quad c_2 > 0), \\ f(x) &= -ax && (a > 0). \end{aligned}$$

Then, computation of the integral (3-8), and omission of the constants of integration, yields the tentative Liapunov function

$$V_1(x) = \left(x + \frac{c_2}{c_1}\right)^{2a/c_1 + 1}.$$

The function $V_1(x)$ is in the domain of $\tilde{A}_{m,\varepsilon}$ for each $\varepsilon > 0$. By the method of construction, $\mathscr{L}V_1(x) = 0$. In order to infer asymptotic stability of the origin from Theorem 2 or 5, we require a Liapunov function $V(x)$ with $\mathscr{L}V(x) = -k(x) \leqq 0$, where $k(x) = 0$ implies $x = 0$. To achieve this, we take

$$V(x) = \left(x + \frac{c_2}{c_1}\right)^p - \left(\frac{c_2}{c_1}\right)^p. \tag{3-9}$$

where $p = 2a/c_1 + 1 - \delta$, for some arbitrarily small positive δ. Now $V(0) = 0$ and $V(x) > 0$, $x > 0$, and

$$\mathscr{L}V(x) = -\left(\frac{\delta p c_1}{2}\right)\left(x + \frac{c_2}{c_1}\right)^{p-1} x.$$

Thus by Theorem 5, Corollary 3-2, and the arbitrariness of m, $x_t \to 0$ with probability one, and for $x = x_0 > 0$,

$$\begin{aligned} P_x\{\sup_{\infty > t \geqq 0} x_t \geqq \lambda\} &= P_x\left\{\sup_{\infty > t \geqq 0}\left[\left(x_t + \frac{c_2}{c_1}\right)^p - \left(\frac{c_2}{c_1}\right)^p\right]\right. \\ &\qquad \left. \geqq \left(\lambda + \frac{c_2}{c_1}\right)^p - \left(\frac{c_2}{c_1}\right)^p\right\} \\ &\leqq \frac{\left(x + \dfrac{c_2}{c_1}\right)^p - \left(\dfrac{c_2}{c_1}\right)^p}{\left(\lambda + \dfrac{c_2}{c_1}\right)^p - \left(\dfrac{c_2}{c_1}\right)^p}. \end{aligned}$$

Example 6. Another application of the method of construction given in Example 5 will be considered. We suppose again that the origin is an absorbing point and that $x > 0$. Let $f(x)$ be differentiable and less than $-ax$ in the region $0 \leqq x \leqq \bar{x}$ and also $a > 0$. Let $\sigma^2(x) = c_1 x^2 + c_2 |x|$. Now, we replace the right side of (3-7) by its *minorant* $2ax/\sigma^2(x)$ and compute a tentative Liapunov function, $V_2(x)$, using the minorant.

Thus

$$V_2(x) = \int_0^x ds \left[\exp \int^s \left(\frac{2au}{\sigma^2(u)} \right) du \right]$$

where

$$V_2(x) = V_1(x),$$
$$\mathscr{L} V_2(x) \leqq 0 \qquad (x \neq 0 \quad \text{and} \quad x \leqq \bar{x}).$$

Now proceeding as in Example 5, and using the function $V(x)$ defined by (3-9) (which is in the domain of $\tilde{A}_{m,\varepsilon}$ for each $\varepsilon > 0$ and $m \leqq V(\bar{x})$), we may apply Theorem 5 and Corollary 3-2, which yield

$$P_x\{ \sup_{\infty > t \geqq 0} V(x_t) \geqq m \} \leqq \frac{V(x)}{m}$$

and $x_t \to 0$ with at least the probability $1 - V(x)/V(\bar{x})$.

Example 7. Consider the two-dimensional nonlinear Itô equation, where $g(x_1)$ satisfies a local Lipschitz condition

$$dx_1 = x_2 \, dt$$
$$dx_2 = -g(x_1)\, dt - ax_2 \, dt - x_2 c \, dz$$
$$\int_0^t g(s)\, ds \to \infty \qquad \text{as} \qquad t \to \infty \tag{3-10}$$
$$sg(s) > 0, \qquad s \neq 0$$
$$g(0) = 0.$$

The function

$$V(x) = x_2^2 + 2\int_0^{x_1} g(s)\,ds$$

is a Liapunov function for the deterministic problem where $c = 0$ (see LaSalle and Lefschetz [1], p. 59 ff). Each Q_m is bounded and $V(x)$ is in the domain of each $\tilde{A}_m$. (Since $S_{11}(x) = S_{12}(x) = 0$, it is not required that $V_{x_1 x_i}(x)$ exist, $i = 1, 2$.) We have

$$\mathscr{L}V(x) = x_2^2(-2a + c^2). \tag{3-11}$$

Suppose that $a > c^2/2$. Then (3-11) (by appeal to Corollary 3-2, and the arbitrariness of m) implies $x_{2t} \to 0$ with probability one.

It will now be proved that $x_{1t} \to 0$ with probability one. By Lemma 1, and the fact that, for any $m > 0$,

$$P_x\{\sup_{\infty > t \geqq 0} V(x_t) \geqq m\} \leqq \frac{V(x)}{m}, \tag{3-12}$$

there is a random variable $b(\omega) < \infty$ with probability one, such that $V(x_t) \to b(\omega)$ with probability one, as $t \to \infty$. Since $x_{2t} \to 0$ with probability one, this implies that

$$G(x_{1t}) \equiv 2\int_0^{x_{1t}} g(s)\,ds \to b(\omega) < \infty$$

as $t \to \infty$. Since $G(0) = 0$ and $G(s)$ is strictly increasing as $|s|$ increases on either side of $s = 0$, for any ω there are at the most two values, $\alpha_1(\omega)$ and $\alpha_2(\omega)$, such that $G(\alpha_1(\omega)) = G(\alpha_2(\omega)) = b(\omega)$; $\alpha_1(\omega) \geqq 0$, $\alpha_2(\omega) \leqq 0$. Thus $x_{1t} \to \{\alpha_1(\omega), \alpha_2(\omega)\}$. In fact, in the limit, each path x_{1t} tends to only one of $\alpha_1(\omega)$, $\alpha_2(\omega)$, since it cannot (with probability one) jump from $\alpha_1(\omega)$ to $\alpha_2(\omega)$ and back to $\alpha_1(\omega)$. (Since $x_{1t} \to \{\alpha_1(\omega), \alpha_2(\omega)\}$, for any $\varepsilon > 0$ there is a random variable $\tau_\varepsilon(\omega) < \infty$ with probability one such that $\min\,[(x_{1t} - \alpha_1(\omega))^2 + x_{2t}^2, (x_{1t} - \alpha_2(\omega))^2 + x_{2t}^2] < \varepsilon$ with probability one for $t \geqq \tau_\varepsilon(\omega)$. x_t cannot move from a neighborhood of $\{\alpha_1(\omega), 0\}$ to a neighborhood of $\{\alpha_2(\omega), 0\}$ without traversing the path between the points. By stochastic continuity, the probability of such a jump is zero.) Thus $x_{1t} \to \alpha(\omega), |\alpha(\omega)| < \infty$ with probability one, where $\alpha(\omega)$ is either $\alpha_1(\omega)$ or $\alpha_2(\omega)$.

Now, for any $1 > \rho > 0$, there is a compact set M so that x_t, $t < \infty$, is confined to M with probability at least $1 - \rho$ (by (3-12)). Thus, by the local Lipschitz condition on $g(s)$, for each $\rho > 0$ there is a real-valued $K_\rho < \infty$, so that, uniformly in t,

$$|g(x_{1t}) - g(\alpha(\omega))| \leqq K_\rho |x_{1t} - \alpha(\omega)|$$

with a probability at least $1 - \rho$. This and the convergence of x_{1t} imply that $g(x_{1t}) \to g(\alpha(\omega))$ with probability one.

The final evaluation of $\alpha(\omega)$ is obtained by integrating the second equation of (3-10) between $T_n - \Delta$ and T_n, where Δ is positive but arbitrary. Let $T_n \to \infty$ and suppose that the intervals $[T_n, T_n - \Delta)$ are disjoint:

$$x_{2T_n} - x_{2T_n - \Delta} = -\int_{T_n - \Delta}^{T_n} g(x_{1t})\, dt - a \int_{T_n - \Delta}^{T_n} x_{2t}\, dt - c \int_{T_n - \Delta}^{T_n} x_{2t}\, dz_t. \quad (3\text{-}13)$$

Now, all with probability one as $n \to \infty$, the left side of (3-13) goes to zero, the first term on the right tends to $-\Delta g(\alpha(\omega))$, and the second term on the right tends to zero. Thus

$$-g(\alpha(\omega))\,\Delta = \lim_{n \to \infty} c \int_{T_n - \Delta}^{T_n} x_{2t}\, dz_t. \quad (3\text{-}14)$$

The last term on the right side of (3-13) represents a sequence of orthogonal random variables with uniformly bounded variance (since $E_x x_{2t}^2 \leqq E_x V(x_t) \leqq V(x)$). Thus, the sequence, since it converges, can converge only to zero (with probability one). Thus, by (3-14), $g(\alpha(\omega)) = 0$, and, hence, $\alpha(\omega) = 0$ with probability one. In conclusion, $x_t \to 0$ with probability one under the condition $-2a + c^2 < 0$. Also, for any $m > 0$,

$$P_x\{\sup_{\infty > t \geqq 0} x_{2t}^2 + 2\int_0^{x_{1t}} g(s)\, ds \geqq m\} \leqq \frac{V(x)}{m}. \quad (3\text{-}15)$$

Let us try to improve the probability estimate (3-15) for small c^2. Let $V_n(x) = V^n(x)$, $n \geqq 1$. For each $m > 0$, $V_n(x)$ is in the domain of $\tilde{A}_m$ (where for this example we let Q_m still denote the set $\{x: V(x) < m\}$):

$$\begin{aligned}\mathscr{L}V^n(x) &= nV^{n-1}(x)\,\mathscr{L}V(x) + \frac{n(n-1)}{2}V^{n-2}(x)\left(\frac{\partial V(x)}{\partial x_2}\right)^2 c^2x_2^2 \\ &\leqq nV^{n-1}(x)\,x_2^2\left[-2a + c^2 + \frac{(n-1)\,c^2(4x_2^2)}{2(x_2^2 + 2\int_0^{x_1} g(s)\,ds)}\right] \\ &\leqq 2nV^{n-1}(x)\,x_2^2\,[-a + c^2(n-1))]. \qquad (3\text{-}16)\end{aligned}$$

If $a \geqq (n-1)\,c^2$, then $\mathscr{L}V^n(x) \leqq 0$ and we have

$$P_x\{\sup_{\infty > t \geqq 0} V(x) \geqq m\} = P_x\{\sup_{\infty > t \geqq 0} V^n(x) \geqq m^n\} \leqq \frac{V^n(x)}{m^n}, \qquad (3\text{-}17)$$

which is better bound than (3-15).

Example 8. Suppose that $\sigma^2(x) = \sigma^2 x_2^2/(1 + x_2^2)$ in Example 7. The function

$$V_1(x) = \exp \lambda\, V(x),$$

where $V(x)$ is the Liapunov function of Example 7, is in the domain of $\tilde{A}_m$ for each $m > 0$, and

$$\begin{aligned}\mathscr{L}V_1(x) = \tilde{A}_m V_1(x) &= 2\lambda V_1(x)\left[-ax_2^2 + (1 + 2\lambda x_2^2)\frac{\sigma^2(x)}{2}\right] \\ &\leqq 2\lambda V_1(x)\,x_2^2\left[-a + \frac{\sigma^2}{2(1 + x_2^2)} + \frac{\lambda x_2^2\sigma^2}{(1 + x_2^2)}\right] \\ &\leqq 2\lambda V_1(x)\,x_2^2\,[-a + \sigma^2(\lambda + \tfrac{1}{2})].\end{aligned}$$

If $a > \sigma^2/2$, then letting $\lambda = (a - \sigma^2/2)/\sigma^2$ yields $\mathscr{L}V_1(x) \leqq 0$ and the bound

$$\begin{aligned}P_x\{\sup_{\infty > t \geqq 0} V(x_t) \geqq \rho\} &= P_x\{\sup_{\infty > t \geqq 0} V_1(x_t) \geqq \exp \lambda\rho\} \\ &\leqq \exp \lambda\big(V(x) - \rho\big).\end{aligned}$$

Example 9. A Stochastic van der Pol Equation. Consider the form of the van der Pol equation

$$d(\dot{x}) + \varepsilon(1 - x^2)\,\dot{x}\,dt + bx\,dt = cx\,dz \qquad (\varepsilon > 0), \tag{3-18}$$

or

$$\begin{aligned} dx_1 &= x_2\,dt \\ dx_2 &= -\,bx_1\,dt - \varepsilon(1 - x_1^2)\,x_2\,dt + cx_1\,dz. \end{aligned} \tag{3-19}$$

With $c = 0$, the origin is asymptotically stable, and there is an unstable limit cycle. Equation (3-18) is also equivalent to the system

$$\begin{aligned} dy_1 &= y_2\,dt - \varepsilon\left(y_1 - \frac{y_1^3}{3}\right)dt \\ dy_2 &= -\,by_1\,dt + cy_1\,dz. \end{aligned} \tag{3-20}$$

Equation (3-20) is obtained from (3-18) by integrating (3-18) and letting $y_1 = x$ and $y_{2t} = \int_0^t [-\,by_{1s}\,ds + cy_{1s}\,dz_s]$. The identity between (3-18) and (3-20) can also be seen by noting that (at least until the first exit time from an arbitrary compact set) y_1 is differentiable and $dy_{1t}^i/dt = (i-1)\,y_{1t}^{i-1}\,\dot{y}_{1t}$. See also the discussion of the deterministic problem in LaSalle and Lefschetz [1], pp. 59–62, where the following Liapunov function is used:

$$V(y) = \frac{y_2^2}{2} + \frac{by_1^2}{2}.$$

Let $\tilde{A}_m$ be the weak infinitesimal operator of the process (3-20) in Q_m. Q_m is bounded, and the terms of (3-20) satisfy a uniform Lipschitz condition in each Q_m. $V(y)$ is in the domain of $\tilde{A}_m$ for each $m > 0$:

$$\mathscr{L}V(y) = y_1^2\left[\left(\frac{c^2}{2} - b\varepsilon\right) + \left(\frac{b\varepsilon y_1^2}{3}\right)\right]$$

Suppose that $c^2/2 - b\varepsilon < 0$. Then $\mathscr{L}V(y) < 0$ on the set

$$P = \{y : y_1^2 < p\}$$

$$p = \frac{3(b\varepsilon - c^2/2)}{b\varepsilon}.$$

Define

$$m = \frac{bp}{2} = \frac{3(b\varepsilon - c^2/2)}{2\varepsilon}.$$

Then, in the open set Q_m, $\mathscr{L}V(y) < 0$, provided that $y_1 \neq 0$. By Corollary 3-2,

$$P_y\{\sup_{\infty > t \geqq 0} V(y_t) \geqq m\} \leqq \frac{V(y)}{m} \tag{3-21}$$

and $x_{1t} = y_{1t} \to 0$ with a probability at least $1 - V(y)/m$.

By a method similar to that used in Example 7, it can be shown that $y_t \to 0$ for ω in $\{\omega\colon V(x_t) < m, \text{ all } t < \infty\} \equiv B_m$. Let $\Delta > 0$ and define $t_n = t_{n-1} + \Delta$, $t_0 = 0$. Write t' for $t \cap \tau_m$. Then, from (3-20),

$$y_{1t'_n} - y_{1t'_{n-1}} = \int_{t'_{n-1}}^{t'_n} y_{2t}\, dt - \varepsilon \int_{t'_{n-1}}^{t'_n} \left(y_{1t} - \frac{y_{1t}^3}{3}\right) dt. \tag{3-22}$$

There is a random variable $c(\omega)$, $0 \leqq c(\omega) < m$, such that $V(x_t) \to c(\omega)$ (with probability one, relative to B_m) for ω in B_m. *Let ω be in B_m.* Since $y_{1t} \to 0$, we have $y_{2t}^2 \to 2c(\omega)$. As in Example 7, either $y_{2t} \to \sqrt{2c(\omega)}$ or $y_{2t} \to -\sqrt{2c(\omega)}$. Since the term on the left and the term on the far right of (3-22) tend to zero as $n \to \infty$, the first term on the right also tends to zero as $n \to \infty$. This implies that $c(\omega) = 0$. Note that $P_y\{B_m\}$ is given by (3-21).

If c^2 is small, there is a simple device which will improve the estimate (3-21). Let $V_n(y) = V^n(y)$. $V_n(y)$ is in the domain of $\tilde{A}_m$ for each $m > 0$:

$$\begin{aligned}
\mathscr{L}V_n(y) &= nV^{n-1}(y)\, y_1^2 \left[\left(\frac{c^2}{2} - b\varepsilon\right) + \frac{b\varepsilon y_1^2}{3} + \frac{(n-1)\, y_2^2 c^2}{(y_2^2 + b y_1^2)}\right] \\
&\leqq nV^{n-1}(y)\, y_1^2 \left[\left(\frac{c^2}{2} - b\varepsilon\right) + \frac{b\varepsilon y_1^2}{3} + (n-1)\, c^2\right].
\end{aligned}$$

Let c^2 be sufficiently small so that

$$b\varepsilon > c^2(n - \tfrac{1}{2}). \tag{3-23}$$

Then $\mathscr{L}V_n(y) < 0$ (if $y_1 \neq 0$) provided that

$$y_1^2 < p_n = \frac{3(b\varepsilon - c^2(n - \frac{1}{2}))}{b\varepsilon}.$$

Define m_n,

$$m_n = \frac{bp}{2} = \frac{3(b\varepsilon - c^2(n - \frac{1}{2}))}{2\varepsilon}.$$

Then, $\mathscr{L}V_n(y) < 0$ $(y_1 \neq 0)$ in the set $Q_{m_n} \equiv \{y\colon V(y) < m_n\} = \{y\colon V_n(y) < m_n^n\}$, and

$$\begin{aligned} P_y\{\sup_{\infty > t \geqq 0} V(y_t) \geqq m\} &\leqq P_y\{\sup_{\infty > t \geqq 0} V(y_t) \geqq m_n\} \\ &= P_y\{\sup_{\infty > t \geqq 0} V^n(y_t) \geqq m_n^n\} \leqq \frac{V^n(y)}{m_n^n}. \end{aligned}$$

There is some n for which $V^n(y)/m_n^n$ is minimum, within the constraint (3-23). Also, $x_{1t} = y_{1t} \to 0$ with a probability at least $1 - V^n(y)/m_n^n$.

An application of Theorem 6 and its extension is given in:

Example 10. Let $\varepsilon < 0$ in (3-20). Then the deterministic problem $(c = 0)$ has an unstable origin and a stable limit cycle. With

$$V(y) = \frac{y_2^2}{2} + \frac{by_1^2}{2},$$

we have, for each $m > 0$,

$$\mathscr{L}V(y) = \tilde{A}_m V(y) = \left(\frac{c^2}{2} + b\varepsilon\right)y_1^2 - \frac{b\varepsilon}{3}y_1^4.$$

Also there is no finite escape time. Suppose that (as is true)

$$E_x \int_0^\infty y_{1s}^2\, ds = \infty.$$

Then

$$\lim_{T \to \infty} \frac{\int_0^T E_x y_{1s}^4\, ds}{\int_0^T E_x y_{1s}^2\, ds} = \frac{c^2/2 + b\varepsilon}{b\varepsilon/3}.$$

If $E_x y_{1s}^4$ and $E_x y_{1s}^2$ both converge to nonzero limits, then

$$\frac{\lim E_x y_{1s}^4}{\lim E_x y_{1s}^2} = \frac{c^2/2 + b\varepsilon}{b\varepsilon/3}.$$

Moment results on another type of stochastic van der Pol equation appear in Zakai [1, 2].

Example 11. A Two-Dimensional Poisson Parameter Problem. Let

$$\dot{x} = (A + Y)\, x,$$

where Y_t is a matrix-valued Poisson process with two possible values, $Y_t = D$ or $Y_t = -D$. Let the probabilities of transition be $P\{Y_{t+\Delta} = D \mid y_t = -D\} = P\{Y_{t+\Delta} = -D \mid y_t = D\} = p\Delta + o(\Delta)$. Suppose that both $M \pm D$ are positive definite. The function

$$V(x, Y) = x'(M + Y)\, x$$

is bounded and has bounded derivatives in every bounded set of x points and tends to infinity as $\|x\| \to \infty$. For the process stopped on exit from each region $Q_m = \{x, y\colon V(x, y) < m\}$, $V(x, y)$ is in the domain of the weak infinitesimal operator (which, on $V(x)$, is $\mathscr{A}$).

To continue, let the real parts of the eigenvalues of $(A + D)$ be negative. We write

$$\mathscr{A} V(x, Y) = -\, x' C_+ x,$$

when the initial condition is $Y = +D$, and

$$\mathscr{A} V(x, Y) = -\, x' C_- x$$

when the initial condition is $Y = -D$:

$$-\, C_+ = (A + D)'(M + D) + (M + D)'(A + D) - 2pD \tag{3-24a}$$

$$-\, C_- = (A - D)'(M - D) + (M - D)'(A - D) + 2pD \tag{3-24b}$$

If $(M \pm D)$ and both C_+ and C_- were positive definite, then Theorem 2 would apply. To be specific, suppose that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix} \qquad (d > b > 0)$$

($A - D$ is an unstable matrix, so that the problem is not trivial.) If F is a stable matrix and G positive definite and symmetric, then there exists a positive definite symmetric matrix solution to $-G = F'B + BF$ (Bellman [1]). Thus, by choosing a diagonal $C_+ = C$, with positive entries c_1, c_2 chosen so that $C_+ - 2pD$ is positive definite and symmetric, there is a positive definite symmetric solution $(M + D)$ to (3-24a):

$$M + D = \begin{bmatrix} \dfrac{(d+b)\,c_1}{2} + \dfrac{1}{2(b+d)}(c_2 + 2p\,d + c_1) & \dfrac{c_1}{2} \\ \\ \dfrac{c_1}{2} & \dfrac{1}{(d+b)}(c_2 + 2p\,d + c_1) \end{bmatrix}$$

Also $M - D$ is positive definite. Substitute $(M - D)$ into (3-24b) and compute $-C_-$; this yields that C_- is positive definite provided that

$$c_1\left[-c_1 - \frac{(d-b)}{(d+b)}(c_2 + c_1 + 2pd) - 4\,(d-b)\,d + 2pd\right] - d^2\,[2 - c_1]^2 > 0. \qquad (3\text{-}25)$$

If p is sufficiently large, then the inequality is satisfied. As p increases, so does the probability of a transition in any interval. We will not pursue the matter further. Certainly, stability is implied by weaker conditions than (3-25). If (3-25) holds, then, for any $m > 0$,

$$P_{x,Y}\{\sup_{\infty > t \geqq 0} x_t'(M + Y_t)\,x_t \geqq m\} \leqq \frac{x'(M + Y)\,x}{m}.$$

The form of $V(x, Y)$ was suggested by the following observation. If $Y = -D$, then, although the derivative of $x_t'(M - D)\,x_t$ may be positive, the average rate of decrease in $x'(M + Y_t)\,x$ (initial condition $Y = -D$) for fixed x may be sufficiently negative for $\mathscr{A}\,V(x, Y) \leqq 0$.

4. Discrete Parameter Stability

The statements of the stability theorems for the discrete parameter process are similar to the continuous parameter theorems. The proofs are generally simpler, since the technicalities centering about the weak infinitesimal operator are not required. To illustrate the method, we state and prove only one theorem.

Theorem 12. Let $x_1, \ldots$ be a discrete parameter Markov process, $V(x) \geqq 0$, and $Q_m = \{x: V(x) < m\}$. In Q_m, let

$$E_{x,n} V(x_{n+1}) - V(x) = -k(x) \leqq 0.$$

Then

$$P_{x,0}\{\sup_{\infty > n \geqq 0} V(x_n) \geqq m\} \leqq \frac{V(x)}{m}. \tag{4-1}$$

There is a random variable v, $0 \leqq v \leqq m$, such that $V(x_n) \to v$ with probability $\geqq [1 - V(x)/m]$. Also $k(x_n) \to 0$ in Q_m with at least the probability that x_n is in Q_m, all $n < \infty$.

Proof. Let τ be the first exit time of x_n from Q_m. Define $\tilde{x}_n = x_{n \cap \tau}$; $\tilde{x}_n$ is a Markov process. Also, $V(\tilde{x}_n)$ is a nonnegative supermartingale since $\tilde{E}_{x,n} V(\tilde{x}_{n+1}) - V(\tilde{x}) \leqq 0$ for $\tilde{x}$ either in Q_m or not in Q_m. Equation (4-1) and the sentence following it follow from the supermartingale properties and the fact that $V(\tilde{x}_n) \geqq m$ if $V(x_i) \geqq m$ for some $i \leqq n$.

Define $\tilde{k}(y)$ by

$$\tilde{E}_{x,n} V(\tilde{x}_{n+1}) - V(\tilde{x}) = -\tilde{k}(\tilde{x}).$$

If $\tilde{x}$ is in Q_m, then $\tilde{k}(\tilde{x}) = k(\tilde{x})$, and if $\tilde{x}$ is not in Q_m, then $\tilde{x}_{n+1} = \tilde{x}_n = \tilde{x}$ and $\tilde{k}(\tilde{x}) = 0$; thus $\tilde{k}(\tilde{x}) \geqq 0$. Also,

$$\tilde{E}_{x,0} V(x_n) - V(\tilde{x}) = -\sum_{i=1}^{n-1} \tilde{E}_{x,0} \tilde{k}(\tilde{x}_i). \tag{4-2}$$

Equation (4-2) stays bounded as $n \to \infty$ since $V(x) \geqq 0$ and $\tilde{k}(x) \geqq 0$. Thus $\rho_i \equiv P_{x,0}\{\tilde{k}(x_i) \geqq \varepsilon\}$ is a summable sequence for each $\varepsilon > 0$. By the Borel-Cantelli lemma, $\tilde{k}(\tilde{x}_n) \to 0$ with probability one. Since $\tilde{x}_n = x_n$, all $n < \infty$, with probability $\geqq [1 - V(x)/m]$, the theorem follows.

5. On the Construction of Stochastic Liapunov Functions

Generally, in the application of the Liapunov function method, a Liapunov function $V(x)$ is given and $\tilde{A}_Q V(x) = -k(x)$ computed. The usefulness of the function for stability inferences depends heavily on the form of $k(x)$, and much effort is usually expended in search for functions $V(x)$ yielding useful $k(x)$. In the deterministic theory, a great deal of effort has also been devoted to the reverse problem: Choose a $k(x) \geqq 0$ of the desired form and seek the function $V(x)$ giving $\dot{V}(x) = -k(x)$ in some region Q. If such a $V(x)$ can be found, and if it has the appropriate properties, then stability inferences can be drawn. This reverse approach is intriguing, but it has not, to date, produced many useful Liapunov functions, except for the case of time varying linear systems. See Zubov [1], Rekasius [1], Geiss [1], and Infante [1].

We now give the stochastic analog of the method of partial integration, an approach which attempts to obtain $V(x)$ by integrating a given $k(x_s)$, with the use of the system equations. In the stochastic case, the average value of the integral is the quantity of interest. Only diffusion processes will be considered. Our starting point will be Dynkin's formula, instead of the (deterministic tool) integral-derivative relation of the ordinary calculus. Suppose $\mathscr{L}V(x)$ is given, then under suitable conditions on $V(x)$ and τ,

$$V(x) - E_x V(x_\tau) = -E_x \int_0^\tau \mathscr{L}V(x_s)\, ds = E_x \int_0^\tau k(x_s)\, ds. \tag{5-1}$$

In order to reduce the number of technicalities in the discussion, all operations in the development will be formal. τ (in (5-1)) will not be specified; we assume that, on all functions considered, $\mathscr{L}V(x) = \tilde{A}_Q V(x)$, and Q will not be specified. Further, $E_x V(x_\tau)$ is assumed equal to zero. These assumptions provide for convenience in the derivations, and are not seriously restrictive from the point of view of obtaining a result, since, in any case, any computed Liapunov function must be checked against the theorems of Section 2. (Roughly, if τ is the first exit time of x_t from an open set Q, and $\mathscr{L}V(x) = \tilde{A}_Q V(x) = -k(x)$ in Q, then $\mathscr{L}E_x V(x_\tau) = 0$ and $E_x V(x_\tau)$ contributes nothing of value to either $V(x)$ or $\tilde{A}_Q V(x)$.) The method is illustrated by two examples.

Example 1. Suppose that the process is the solution to the Itô equation

$$dx_1 = x_2\,dt$$
$$dx_2 = -(bx_2 + 2cx_1 + 3x_1^2)\,dt + \sigma x_2\,dz \qquad (b > 0, \quad c > 0).$$

Suppose that we seek a function $V(x)$ which is a Liapunov function in a neighborhood of the origin and which satisfies

$$\mathscr{L}V(x) = -x_2^2.$$

Define

$$I_1 = E_x \int_0^\tau x_{1t} x_{2t}\,dt,$$
$$I_2 = E_x \int_0^\tau x_{2t} x_{1t}^2\,dt, \tag{5-2}$$
$$I_3 = E_x \int_0^\tau x_{2t}^2\,dt.$$

Note that, formally, I_3 satisfies $\mathscr{L}I_3 = -x_2^2$, and we will solve for I_3.

Upon applying Dynkin's formula (and omitting the $E_x g(x_\tau)$ terms), and substituting the values of the $\mathscr{L}x_i^n$, we have

$$x_2^2 = E_x \int_0^\tau (-\mathscr{L}x_{2t}^2)\,dt = 4cI_1 + 6I_2 + (2b - \sigma^2)\,I_3, \tag{5-3a}$$

$$x_1^2 = E_x \int_0^\tau (-\mathscr{L}x_1^2)\,dt = -2I_1, \tag{5-3b}$$

$$x_1^3 = E_x \int_0^\tau (-\mathscr{L}x_1^3)\,dt = -3I_2. \tag{5-3c}$$

If $\sigma^2 < 2b$, the solution of the linear set (5-3) for the function I_3, which we now denote by $V(x)$, is

$$V(x) = \frac{x_2^2 + 2cx_1^2 + 2x_1^3}{2b - \sigma^2}. \tag{5-4}$$

It is indeed true that $\mathscr{L}V(x) = -x_2^2$ and, further, it is readily verified that Theorem 2 may be applied to $V(x)$ in a neighborhood of the origin.

Equation (5-4) was not hard to calculate. The answer (5-4) was known: the application of $\mathscr{L}$ to each term of (5-4) individually yields only terms which are integrands in (5-2); hence, the I_i could be calculated by use of the integral formula. In general, we pursue the following procedure. Choose a collection of functions $g_i(x)$, $i = 1, \ldots, n$; these correspond to the left sides of (5-3). Compute each $\mathscr{L}g_i(x) = \sum_j \alpha_{ij} h_{ij}(x)$. Hopefully, the set $\{h_{ij}\}$ will contain only n distinct functions and one of these will (hopefully) be the desired function $k(x)$, the desired value of $-\mathscr{L}V(x)$. Finally, by writing

$$g_i(x) = E_x \int_0^\tau -\mathscr{L}g_i(x_s)\,ds = -\sum_j \alpha_{ij} E_x \int_0^\tau h_{ij}(x_s)\,ds$$

we may solve the set of linear equations for the desired Liapunov function

$$V(x) = E_x \int_0^\tau k(x_s)\,ds.$$

Of course, it is usually no easy matter (if not impossible) to select a proper set $\{g_i(x)\}$ (so that there are only n distinct terms in $\{h_{ij}(x)\}$). As in the deterministic case, the choice of $\{g_i(x)\}$ seems to be a matter of trial and error. The computed $V(x)$ must be tested according to the appropriate theorem.

If the number of terms in $\{h_{ij}(x)\}$ cannot be reduced to n ($k(x)$ must be a distinct term in $\{h_{ij}(x)\}$ in any case), it may still be possible to compute a Liapunov function which gives some information. An example follows.

Example 2. Let

$$dx = (-a + y)x\,dt \qquad (a > 0) \tag{5-5}$$

where the time varying coefficient y_t is the solution of

$$dy = -by\,dt + \sigma(y)\,dz \qquad (b > 0). \tag{5-6}$$

Our aim is to obtain a function $V(x, y)$ which is uniformly (in y) positive definite in a neighborhood of the origin (of the x space) and which satisfies

$$\mathscr{L}V(x, y) = -x^2.$$

We restrict our search to functions of the form $V(x, y) = x^2 P(y)$, where $P(y)$ is a polynomial.

Define the sequence $I_1, \ldots$:

$$I_{-1} = I_{-2} = 0, \qquad I_n = E_{x,y} \int_0^\tau x_t^2 y_t^n\,dt \qquad (n > 0).$$

The desired $V(x, y)$ equals I_0. Then, a formal application of Dynkin's formula gives

$$\begin{aligned} x^2 y^n &= E_{x,y} \int_0^\tau (-\mathscr{L}x_t^2 y_t^n)\,dt \\ &= (2a + nb)\,I_n - 2I_{n+1} - \frac{n(n-1)\,J_{n-2}}{2} \end{aligned} \tag{5-7}$$

where $J_{n-2} = E_{x,y} \int_0^\tau \sigma^2(y_t)\,x_t^2 y_t^{n-2}\,dt$. We now make the additional assumption, which will be helpful in the following calculations, that $J_{n-2} = \sigma^2(y)\,I_{n-2}$. Let us solve (5-7) under the supposition that $I_n = 0$, $n \geqq 3$. Then, providing that

$$\Delta = \inf_y \{2a(2a + 2b)(2a + b) - 4\sigma^2(y)\} > 0,$$

we have

$$I_0 = V(x, y) = \frac{x^2[(2a+b)(2a+2b) + 2(2a+2b)y + 4y^2]}{\Delta} \tag{5-8}$$

$$\begin{aligned}\mathscr{L}V(x, y) &= \frac{2x^2[-a(2a+b)(2a+2b) + 2\sigma^2(y) + 4y^3]}{\Delta} \\ &= -x^2\left(1 - \frac{8y^3}{\Delta}\right).\end{aligned} \tag{5-9}$$

Suppose that $\sigma^2(y)$ and y_0 are such that $y_t^3 < \Delta/8$ for all $t < \infty$ and the bracketed term in (5-8) is always positive. Then (5-8) is a stochastic Liapunov function. The set of such $\sigma^2(y)$ and initial conditions y_0 is neither empty nor trivial. Let $a = b = 1$, and max $\sigma^2(y) = 1$. Then $\Delta = 20$, and the conditions are satisfied if $\sigma^2(y) = 0$, for $|y| > (20/8)^{1/3} > |y_0|$. By truncating the set of linear equations at larger n, and solving the set for a new approximation to I_0, a larger allowable range of variation in y_t is obtained.

III / FINITE TIME STABILITY AND FIRST EXIT TIMES

1. Introduction

The behavior of many tracking and control systems is of interest for a finite time only, and their study requires estimates of the quantities

$$P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq m\}, \qquad P_x\{\sup_{N \geqq n \geqq 0} V(x_n) \geqq m\} \tag{1-1}$$

for some suitable function $V(x)$. First exit time probabilities have a traditional importance in communication theory (see, for example, Rice [1]) where they are used, for example, to study the statistical properties of local extrema of random functions, or of clipped random functions; the study of control systems via a study of appropriate first exit times is certainly not new (see, for example, Ruina and Van Valkenburg [1]). Nevertheless, most of the available work that is pertinent to control systems seems to bear mainly on asymptotic properties or to involve numerical solutions of partial differential equations which are associated with certain types of first exit time problems. In this chapter, the stochastic Liapunov function method of obtaining upper bounds to (1-1) is discussed, and several theorems and examples are given.

Most of the published work on stochastic control and tracking systems seems to be concerned with the calculation of moments, especially with second moments. Moments have generally been easier to calculate or to estimate than the values of (1-1). However, for many problems, knowledge of (1-1) has greater pertinence than knowledge of moments.

Many tracking or breakdown problems must be considered only as finite time problems, since either a stationary distribution of error may not exist, owing to eventual breakdown or loss of track, or else the "transient" period is the time of greatest interest. Estimates of (1-1) are also of importance when the system, even in the absence of the disturbing noise, is unstable outside of a given region. The noise may eventually drive the system out of the stable region with probability one, and then estimates of the probabilities of the times of these occurrences are of interest.

Tracking systems occur in many forms. For example, an aircraft may be tracked through the use of the noise corrupted signals which are returned to a tracking radar system, or a parameter of a control system may be tracked either directly by using noise disturbed observations of the value of the parameter, or indirectly by noise disturbed observations on the values of the states of the system which is parametrized by the parameter. In such cases, bounds on the probability that the maximum tracking error (within some given time interval) will be greater than some given quantity are of interest. This is especially true in systems which lose track at the moment that the error exceeds some given quantity. The orbiting telescope is a problem of some current interest, where first exit times are important. To allow successful photography of astronomical objects, the direction of point (attitude) of the telescope must remain within a given region for at least the time required to take the photograph. However, the attitude is influenced by the random forces acting on the satellite, and in its control system.

Problems in hill climbing are often of the same type. Given some maximum seeking method of the gradient estimation type, one would like a reasonably good upper bound of the probability that at least one member of some finite sequence of estimates of the peak deviates by more than a given amount from the true location of the peak. Such estimates provide a useful evaluation of the hill climbing procedure,*

* Two opposing demands on the design of maximum seeking methods (which are based on the use of noise corrupted observations) are (1), to have the probability

and are of particular use in cases where there are many relative maxima, or where large errors cannot be tolerated; for example, where a parameter of an operating plant is being adjusted. Similarly, the use of first exit probabilities provide one natural way of evaluating other "so-called" adaptive control systems.

The reader is also referred to the work on deterministic finite time stability in Infante and Weiss [1, 2]. The finite time results differ from the infinite time results in that $\tilde{A}_m V(x)$ may be positive for some or all values of x. Nontrivial statements regarding the values of the probabilities (1-1) may, nevertheless, be made.

If the system has some "finite time stability" properties, but is not stable in the sense of Chapter II, then a positive definite Liapunov function cannot satisfy $\tilde{A}_m V(x) \leqq 0$ in the set of interest, since this would imply a result of the type of Chapter II which gives (stability) information on the trajectories over the infinite interval $[0, \infty)$. Generally, in this chapter, $\tilde{A}_m V(x)$ will be positive for all or some values of x in the set of interest. Nevertheless, the form of $\tilde{A}_m V(x)$ will suggest time dependent nonnegative supermartingales which, in turn, will be useful to obtain upper bounds to the probabilities (1-1).

An analog to the deterministic case is helpful. Let $V(x)$ be differentiable in the set $Q_m = \{x : V(x) < m\}$, $m > 0$. Let $\dot{x} = f(x)$ and assume that in the set Q_m, $\dot{V}(x) = (\partial V(x)/\partial x)' f(x) \leqq \varphi$, where φ is a nonnegative constant. Then elementary considerations show that x_t is in the set Q_m for at least the time $T = (m - V(x))/\varphi$, where $x = x_0$ is the initial condition. This is a result in deterministic finite time stability.

2. Theorems

Theorem 1. Assume (A1) to (A5) of Chapter II. In Q_m, let

$$\tilde{A}_m V(x) \leqq -\mu V(x) + \varphi_t \qquad (\mu > 0), \tag{2-1}$$

of "bad" estimates as low as possible (for example, keep the probability of at least one "bad" estimate below a given quantity), and (2), to have the largest possible "rate of improvement" in the estimate of the maximum. The greater "caution" demanded by (1) slows down (2); see Kushner [9].

where φ_t is nonnegative and continuous in $[0, T]$. Define in $[0, T]$

$$\Phi_t = \int_0^t \varphi_s \, ds$$

$$W(x, t) = e^{c\Phi_t} V(x) + \left(\frac{e^{c\Phi_T} - e^{c\Phi_t}}{c} \right),$$

where c is selected so that

$$\mu \geqq \max_{t \leqq T} c\varphi_t .$$

Let λ be such that if $V(x_t) \geqq m$, then $W(x, t) \geqq \lambda$. Then, with $x_0 = x$,

$$P_x\{ \sup_{T \geqq t \geqq 0} W(x_t, t) \geqq \lambda\} \leqq \frac{V(x) + (e^{c\Phi_T} - 1)/c}{\lambda}. \qquad (2\text{-}2)$$

With suitable choices of λ and c, the appropriate equation, (2-3) or (2-4), holds, where $\Phi'_T = \Phi_T \mu / \max \varphi_t$:

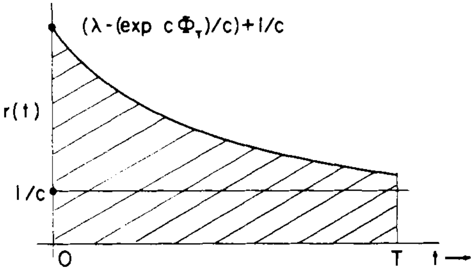
$$P_x\{ \sup_{T \geqq t \geqq 0} V(x_t) \geqq m\} \leqq 1 - \left(1 - \frac{V(x)}{m}\right) e^{-\Phi_T/m} \qquad (2\text{-}3)$$

$$\left(m \geqq \max \frac{\varphi_t}{\mu} \right),$$

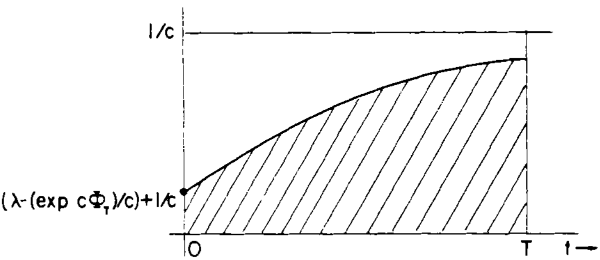
$$P_x\{ \sup_{T \geqq t \geqq 0} V(x_t) \geqq m\} \leqq \frac{V(x) + (e^{\Phi'_T} - 1)\,(\max_{t \leqq T} \varphi_t/\mu)}{m e^{\Phi'_T}} \qquad (2\text{-}4)$$

$$\left(m \leqq \max \frac{\varphi_t}{\mu} \right).$$

Remark. Let $r(t) = e^{-c\Phi_t}(\lambda - e^{c\Phi_T}/c) + 1/c$. The definition of $W(x, t)$ together with equation (2-2) imply that (x_t, t) stays within the hatched area of Figure 1 with at least the probability given by (2-2). To obtain (2-3) or (2-4), λ is chosen so that $\max_{T \geqq t \geqq 0} r(t) = m$.



Maximum of r(t) in [0,T] occurs at t=0



Maximum of r(t) in [0,T] occurs at t=T

Figure 1

Remark. Note that, for $V(x_0) = V(x) \neq 0$ and $T = 0$, the right sides of (2-3) and (2-4) do not reduce to zero. This behavior is due to the method of proof which does not distinguish between initial conditions (of $V(x_t)$) which are fixed constants of value $V(x)$ and initial conditions which are random variables whose expectation is $V(x)$. The results remain valid (as they do in Chapter II) if the initial condition is a random variable (nonanticipative) with mean value $V(x)$.

The fact that $W(x_t, t)$ (stopped on exit from some region) may be a nonnegative supermartingale is suggested by the form of $\tilde{A}_m V(x)$ in (2-1).

Proof. Define $\hat{Q}_\lambda = \{x, t: W(x, t) < \lambda, t < T\}$ and let τ be the first time of exit of (x_t, t) form $\hat{Q}_\lambda$. (Labeling the vertical axis in Figure 1 by $V(x)$, the scaled $\hat{Q}_\lambda$ corresponds to the interior of the shaded areas.) Let $s_t = t \cap \tau$. Both $\tilde{x}_t = x_{t\cap\tau}$ and $\hat{x}_t = (\tilde{x}_t, s_t)$ are right continuous strong Markov processes in $[0, T]$. In reference to the $\hat{x}_t$ process, let $\hat{P}_{x,t}\{B\}$ be the probability of the event B given the initial condition $\hat{x}_t = x$, $s_t = t$ where $(x, t) \in \hat{Q}_\lambda$. Now, let $(x, t) \in \hat{Q}_\lambda$. Then the stochastic continuity of the process $\tilde{x}_t$ at t implies that $\hat{P}_{x,t}(\tau > t + h) \to 1$ as h decreases to zero. Consequently $\hat{E}_{x,t}(\exp[c\Phi(t+h)\cap\tau] - e^{c\Phi_t})/h \to c\varphi_t e^{c\Phi_t}$ as $h \to 0$. Let $\hat{A}_\lambda$ be the weak infinitesimal operator of $\hat{x}_t$. The operation of $\hat{A}_\lambda$ on $W(x, t)$, for (x, t) in $\hat{Q}_\lambda$, is easily calculated to be (note that if (x_t, s_t) is in $\hat{Q}_\lambda$, then $t = s_t$)

$$\hat{A}_\lambda W(x, s) = e^{c\Phi_s}[c\varphi_s V(x) + \tilde{A}_m V(x)] - \varphi_s e^{c\Phi_s} \leqq 0. \tag{2-5}$$

Note that $W(x, s) \geqq 0$ for (x, s) in $\hat{Q}_\lambda$. Now an application of Dynkin's formula yields

$$P_x\{\sup_{T \geqq t \geqq 0} W(x_t, t) \geqq \lambda\} = \hat{P}_{x,0}\{\sup_{\infty > t \geqq 0} W(\tilde{x}_t, s_t) \geqq \lambda\} \leqq \frac{W(x, 0)}{\lambda}$$

which is (2-2).

Inequalities (2-3) and (2-4) are obtained by a weakening of the bound (2-2). If $W(x, t) \geqq \lambda$ for some $t \leqq T$, then

$$V(x_t) \geqq r(t) = e^{-c\Phi_t}\left[\lambda - \left(\frac{e^{c\Phi_T}}{c}\right)\right] + \frac{1}{c}$$

for some $t \leqq T$ (and vice versa). (See the remark following the theorem statement.) Also, $P_x\{\sup_{T\geqq t\geqq 0} V(x_t) \geqq \sup_{T\geqq t\geqq 0} r(t)\}$ is no greater than the bound given by (2-2). $r(t)$ is either monotonic nonincreasing or monotonic nondecreasing, as t increases to T, depending on whether or not λ is greater than or less than $e^{c\Phi_T}/c$. Each case will be considered separately.

Let the maximum of $r(t)$ (*in* $[0, T]$) *occur at* $t = 0$. Then, choosing λ so that the maximum of $r(t)$ equals m, we have $\lambda = m + (e^{c\Phi_T} - 1)/c$. Note that, for consistency, we require $\lambda \geqq e^{c\Phi_T}/c$; thus $c \geqq 1/m$ is required in this case, and the right side of (2-2) equals

$$B(c) = \frac{V(x) + (e^{c\Phi_T} - 1)/c}{m + (e^{c\Phi_T} - 1)/c}.$$

A further constraint on the value of c is $c \leqq \mu/\max\varphi_t$, which assures that (2-5) is nonpositive in $\hat{Q}_\lambda$. Suppose that the interval $A = [1/m, \mu/\max\varphi_t]$ is not empty (a necessary and sufficient condition for the maximum of $r(t)$, in $[0, T]$, to occur at $t = 0$). The value of c in A which minimizes $B(c)$ is $c = 1/m$, the smallest number in A. Equation (2-3) follows immediately.

If A is empty, then the maximum of $r(t)$ in $[0, T]$ occurs at $t = T$. To satisfy $c \leqq \mu/\max\varphi_t$, we set $c = \mu/\max\varphi_t$. Also, choose λ so that $m = \max_{t\leqq T} r(t) = r(T) = \lambda e^{-c\Phi_T}$. The right side of (2-2) now equals

$$B(c) = \frac{V(x) + (e^{c\Phi_T} - 1)/c}{me^{c\Phi_T}}$$

and (2-4) follows immediately upon substitution of the chosen c. (It can also be verified directly that $\lambda \leqq e^{-c\Phi_T}/c$ in this case.)

Remark. In the examples, we use the special case $\varphi_t = \varphi$, a constant. Then $\Phi_T = \varphi T$ and $\max_{t<T}\varphi_t/\mu = \varphi/\mu$.

Corollary 1-1. Assume the conditions of Theorem 1, except that $\mu \geqq 0$. Then

$$P_x\{\sup_{T\geqq t\geqq 0} V(x_t) \geqq m\} \leqq \frac{V(x) + \varphi T}{m}. \tag{2-6}$$

Proof. The proof is the same as that of Theorem 1, except that $W(x, t) = V(x) + \Phi_T - \Phi_t$ is used. (Alternatively, let $c \to 0$ in Theorem 1.) For $m \leqq \Phi_T$, the bound (2-6) is trivial.

Remark. The theorems of this chapter are only particular cases of the general idea that if $W(x, t)$, $t \leqq T$, is a nonnegative supermartingale in a suitable region, then

$$P_x\{\sup_{T\geqq t\geqq 0} W(x_t, t) \geqq \lambda\} \leqq \frac{W(x, 0)}{\lambda}.$$

Theorem 1 is given in the form in which it appears, since form (2-1) arose in a number of examples, with the use of some simple $V(x)$. It is likely that other cases, not covered in the theorems or by the examples, will be of greater use for some problems, either in that the corresponding Liapunov functions will be easier to find or that they will yield better bounds. One such special case is given by Corollary 1-2, and the Wiener process example following it.

Remark. The forms of $\tilde{A}_m$ on given domains for Itô and Poisson processes are given in Chapter 1, Sections 4 and 5.

Remark. The bound (2-6) may seem poor, and it probably is for most problems that one is likely to encounter. Yet there are simple systems for which the probability in (2-6) does increase linearly with time. Let $\dot{x} = 1$, and let the initial condition be a random variable which is uniformly distributed in the interval [0, 1]. Then $P_x\{\sup_{T \geqq t \geqq 0} x_t \geqq 1\} = T$ for $T \leqq 1$.

The Liapunov function approach is often a crude approach. Given a tentative Liapunov function, its application to systems of rather diverse natures may yield very similar forms for $\tilde{A}_m V(x)$. If there is to be no further analysis or search for other possible Liapunov functions, then we must be content with the result given by a function which does not distinguish between quite different systems.

Corollary 1-2. Assume (A1) to (A5) and let $\tilde{A}_m V(x) \leqq \mu V(x)$. Let $m \geqq \lambda e^{\mu_1 T}$, $\mu_1 \geqq \mu > 0$. Then

$$P_x\{\sup_{T \geqq t \geqq 0} e^{-\mu_1 t} V(x_t) \geqq \lambda\} \leqq \frac{V(x)}{\lambda}. \tag{2-7}$$

Proof. The proof is very similar to the first part of the proof of Theorem 1 and will be omitted. The following example illustrates a case where the value of μ_1 may be chosen as a function of T, to yield a useful estimate.

Example. Let $dx = \sigma\, dz$, where z_t is a Wiener process. Let $V(x) = \cosh \beta x = (e^{\beta x} + e^{-\beta x})/2$. Then, for each $m < \infty$, $\tilde{A}_m$ acting on $V(x)$

(or on a function which is equal to $V(x)$ on Q_m and which is bounded and has bounded and continuous second derivatives) is $(\sigma^2/2)\, d^2/dx^2$. Thus

$$\tilde{A}_m V(x) = \frac{\sigma^2 \beta^2}{2} V(x).$$

By reasoning as in Theorem 1, for $\mu_1 = \beta^2\sigma^2/2 = \mu$, we have

$$\hat{A}_\lambda e^{-\mu_1 t} V(x) = 0.$$

Define m by

$$\cosh \beta m = (\cosh \beta\varepsilon) \exp(\mu T). \tag{2-8}$$

Then

$$\begin{aligned} P_x\{\sup_{T \geqq t \geqq 0} |x_t| \geqq m\} &\leqq P_x\{\sup_{T \geqq t \geqq 0} e^{-\mu t} \cosh \beta x_t \geqq \cosh \beta\varepsilon\} \\ &\leqq \frac{\cosh \beta x}{\cosh \beta\varepsilon} = \frac{\cosh \beta x}{\cosh \beta m} (\exp \mu T) \\ &\leqq 2 \exp[\mu T + \beta(x - m)]. \end{aligned} \tag{2-9}$$

The last term on the right of (2-9) is minimized by $\beta = (m - x)/\sigma^2 T$ yielding a bound

$$2 \exp - \frac{(m - x)^2}{2\sigma^2 T}.$$

The success of the method for more difficult problems depends, of course, on the availability of functions satisfying $\tilde{A}_m V(x) = \mu V(x)$, for an appropriate range of μ.

Suppose that x_s and y_s are two processes (not necessarily with the same transition function) whose initial conditions are close. Theorem 2 gives a "Liapunov function" estimate for the probability that the processes have separated by at least some fixed distance at least once in a given time interval. The proof is essentially that of Theorem 1 and will be omitted. Such estimates are particularly important in control applications (as, for example, in the study of the deviations of a randomly perturbed trajectory from the "nominal" unperturbed trajectory).

Theorem 2. Let (A1) to (A5) hold for processes x_t, y_t (with initial conditions x, y, respectively) and function $V(x, y, t) \geqq 0$ in the bounded set $Q = \{x, y: W_1(\|x-y\|) < m\}$, where $W_1(r) \geqq 0$ is continuous and increasing and $W_1(0) = 0$, and $V(x, y, t) \geqq W_1(\|x-y\|)$, for $t \leqq T$. Let

$$\tilde{A}_m V(x, y, t) \leqq -\mu V(x, y, t) + \varphi_t$$

where $\mu > 0$ and φ_t is nonnegative and continuous in $[0, T]$. Define $W(x, y, t)$ as in Theorem 1, with $V(x, y, t)$ replacing $V(x)$. Then, (2-2)–(2-4) hold, with $V(x_t, y_t, t)$ replacing $V(x_t)$ and $V(x, y, 0)$ replacing $V(x)$. Also, with these substitutions the left sides of (2-3) and (2-4) majorize the quantity

$$P_{x,y}\{\sup_{T \geqq t \geqq 0} W_1(\|x_t - y_t\|) \geqq m\}.$$

Theorem 3 is the discrete time parameter version of Theorem 1.

Theorem 3. Let x_n, $n = 0, \ldots, N$ be a Markov process and $V(x)$ a continuous nonnegative function with

$$E_{x_n} V(x_{n+1}) \leqq \frac{V(x_n)}{\beta} + \varphi_n \tag{2-10}$$

in $Q_m = \{x: V(x) < m\}$, where $\beta > 1$, $\varphi_n \geqq 0$. Define

$$W(x, n) = F_n(K)\, V(x) + KF_N(K) - KF_n(K),$$

where $F_n(K) = \prod_0^{n-1} [K/(K - \varphi_i)]$, $F_0(k) = 1$, and $\beta > K/(K - \varphi_i)$. Let λ be such that if $V(x) \geqq m$, then $W(x, n) \geqq \lambda$. Then with $x_0 = x$,

$$P_x\{\sup_{N \geqq n \geqq 0} W(x_n, n) \geqq \lambda\} \leqq \frac{V(x) + KF_N(K) - K}{\lambda} \equiv B(K). \tag{2-11}$$

With appropriate choices of λ and K, and letting $m > \max \varphi_i = \varphi'$, we obtain

$$P_x\{\sup_{N\geqq n\geqq 0} V(x_n)\geqq m\}\leqq 1-\left(1-\frac{V(x)}{m}\right)\prod_0^{N-1}\left(1-\frac{\varphi_i}{m}\right), \tag{2-12}$$

$$m\geqq\frac{\varphi'\beta}{\beta-1}$$

$$P_x\{\sup_{N\geqq n\geqq 0} V(x_n)\geqq m\}\leqq\frac{V(x)}{m}\prod_0^{N-1}\left(1-\frac{\varphi_i}{m'}\right)+\frac{m'}{m}\left[1-\prod_0^{N-1}\left(1-\frac{\varphi_i}{m'}\right)\right],$$

$$m'=\frac{\varphi'\beta}{\beta-1}>m \tag{2-13}$$

Remark. In the special case where $\varphi_i=\varphi$, the right side of (2-13) reduces to

$$\frac{V(x)\,\beta^{-N}}{m}+\frac{(1-\beta^{-N})\,\varphi\beta}{(\beta-1)\,m}. \tag{2-13a}$$

Proof. Define τ as the first Markov time (integral valued) for which $W(x_n,n)\geqq\lambda$, $n\leqq N$, and let $\hat{Q}_\lambda$ be as in the proof of Theorem 1. Except for the fact that Dynkin's formula is not needed, the proof is similar to that of Theorem 1. The stopped process $\tilde{x}_n=x_{n\cap\tau}$ is a Markov process, and so is $\hat{x}_n=(\tilde{x}_n,\tau\cap n)$. A straightforward calculation shows that $W(x_n,n)$ is a nonnegative supermartingale if $\beta>K/(K-\varphi_i)$ (for each i) or $K>\varphi'\beta/(\beta-1)$ (with adjunction of the appropriate family of σ-fields), for $n\leqq N$. (For (x_n,n) in Q_λ, $Ex_n W(x_{n+1},n+1)-W(x_n,n)\leqq 0$.) Inequality (2-11) is the nonnegative supermartingale probability inequality. Define

$$\begin{aligned} r(n)&=F_n^{-1}[\lambda-KF_N(K)+KF_N(K)]\\ &=\prod_0^{n-1}\left(\frac{K-\varphi_i}{K}\right)\left[\lambda-K\prod_0^{N-1}\left(\frac{K}{K-\varphi_i}\right)\right]+K. \end{aligned}$$

The maximum of $r(n)$, in the interval $[0,N]$, occurs at either $n=0$ or $n=N$, depending on whether λ is greater than or less than

$K\prod_0^{N-1}(K/(K-\varphi_i)) = KF_N(K)$. If λ is chosen so that $\max_{0\leqq n\leqq N} r(n) = m$, then

$$P_x\{\sup_{N\geqq n\geqq 0} V(x_n) \geqq m\} \leqq P_x\{\sup_{N\geqq n\geqq 0} W(x_n, n) \geqq \lambda\} \leqq B(K).$$

Assume $\lambda \geqq KF_N(K)$ and let $\max_{N\geqq n\geqq 0} r(n) = m$. Then $\lambda = m + KF_N(K) - K$. The last two sentences imply that $m \geqq K$. Thus we require

$$m \geqq K \geqq \frac{\varphi'\beta}{\beta - 1}.$$

Now

$$B(K) = \frac{V(x) + K(F_N(K) - 1)}{m + K(F_N(K) - 1)}$$

which is minimized (in the alloted interval $m \geqq K \geqq \varphi'\beta/(\beta - 1)$) by the maximum value $K = m$, and (2-12) follows immediately upon making this substitution.

Now, let $\lambda \leqq KF_N(K)$; then the maximum of $r(n)$ occurs at $n = N$, and, setting the maximum equal to m yields $\lambda = mF_N(K)$. For consistency, we now require that $K > m$. Since $K \geqq \varphi'\beta/(\beta - 1)$ is also required, and (in this case) $m < \varphi'\beta/(\beta - 1)$, and

$$B(K) = \frac{V(x) + K(F_N(K) - 1)}{mF_N(K)}$$

decreases as K decreases, we set $K = \varphi'\beta/(\beta - 1) = m' > m$, the minimum allowed value. Equation (2-13) follows immediately upon this substitution.

Corollary 2-1. Assume the conditions of Theorem 2, except that $\beta \geqq 1$. Then

$$P_x\{\sup_{N\geqq n\geqq 0} V(x_n) \geqq m\} \leqq \frac{V(x) + \sum_0^{N-1}\varphi_i}{m}. \tag{2-14}$$

Proof. The proof is similar to that of Theorem 2, using $W(x, n) = V(x) + \sum_n^{N-1}\varphi_i$.

STRONG DIFFUSION PROCESSES

It is instructive to consider a derivation of a result of the type of Theorem 1 from another point of view. The derivation is for the special case of the strong diffusion process in a compact region and makes use of the relationship between such processes and parabolic differential equations.

Let G be an open region with boundary ∂G and compact closure. In order to apply the result for parabolic equations it is required that ∂G satisfy some smoothness condition. We suppose that ∂G is of class $H_{2+\alpha}$ (see Chapter I, Section 6). If g is given by $Q_m = \{x: V(x) < m\}$, where Q_m is bounded and $V(x)$ has Hölder continuous second derivatives, the smoothness condition on ∂G is satisfied.

Theorem 4. Let x_t be a strong diffusion process in G, where G and ∂G satisfy the conditions in the previous paragraph. Let $P(x, t)$ be a continuous nonnegative function with continuous second derivatives in the components of x and a continuous first derivative in t, in $\hat{G} \equiv [G + \partial G] \times [0, T]$. If

$$\left(L - \frac{\partial}{\partial t}\right) P(x, t) \leqq 0 \tag{2-15}$$

and

$$\begin{aligned} &P(x, 0) \geqq 0, \\ &P(x, t) \geqq 1, \qquad T \geqq t > 0, \qquad x \in \partial G. \end{aligned} \tag{2-16}$$

Then, for $t \leqq T$,

$$P_x\{x_s \notin G, \quad \text{some} \quad s \leqq t\} \leqq P(x, t). \tag{2-17}$$

Remark. Although Theorem 4 is applicable only to strong diffusion processes, it is suggestive as a criterion for evaluating stochastic Liapunov functions to which Theorems 1 and 4 are to be applied. Let $P_1(x, t)$ and $P_2(x, t)$ satisfy the conditions on $P(x, t)$ of Theorem 4. Let $P_1(x, t) \geqq P_2(x, t)$ on $\partial G \times [0, T] + G \times \{0\}$,

$$\left(L - \frac{\partial}{\partial t}\right) P_1 \leqq \left(L - \frac{\partial}{\partial t}\right) P_2 \leqq 0$$

in G. Then Theorem 4 and the strong maximum principle for parabolic operators (Friedman [1], Chapter 2) yields

$$P_1(x, t) \geqq P_2(x, t) \geqq P_x\{x_s \notin G, \quad \text{some} \quad s \leqq t\}.$$

Thus $P_2(x, t)$ is no worse an estimate than $P_1(x, t)$, and will be better if the inequalities are strict at some point.

Let x_t be a strong diffusion process and suppose that $V(x)$ has continuous second derivatives. Let $P_1(x, t)$ be the right side of (2-3) of Theorem 1 (where $\varphi_t = \varphi$ and $\mu \geqq \varphi/m$) and $L = \tilde{A}_m$, and let $G = \{x: V(x) < m\}$ be bounded. On ∂G, $V(x) = m$ and $P_1(x, t) = 1$. At $t = 0$, $P_1(x, 0) = V(x)/m \geqq 0$. Also

$$\left(L - \frac{\partial}{\partial t}\right) P_1(x, t) = e^{-\varphi T/m}\left[-\frac{\varphi}{m}\left(1 - \frac{V(x)}{m}\right) + \frac{LV(x)}{m}\right].$$

Since $LV(x) \leqq -\mu V(x) + \varphi$ and $\mu \geqq \varphi/m$ by assumption,

$$\left(L - \frac{\partial}{\partial t}\right) P_1(x, t) \leqq 0,$$

and we conclude that the right side of (2-3) satisfies the conditions on $P(x, t)$ of Theorem 4, under the appropriate conditions on the process x_t. A similar statement can be made for the right side of (2-4).

Proof of Theorem 4. Under the conditions on $S_{ij}(x), f_i(x)$, and on ∂G, Theorem 13.18 of Dynkin [2] (see Section 6, Chapter I) yields that $p(t, x, y)$, the probability transition density of the process x_t in G (and stopped on ∂G), satisfies $(L - \partial/\partial t)u = 0$. (Here $P_x(x_t \in \Gamma \subset G) = \int_\Gamma (p(t, x, y)\, dy)$. The function $p(t, x, y)$ is continuous and has continuous second derivatives for $t > 0$. Also $p(t, x, y) \to 0$, $x \to \partial G$, $t > 0$, and $p(t, x, y) \leqq Kt^{-1/2} \exp - \|y - x\|^2/2th$, for some positive real numbers K and h.

So, the function $1 - \int_G P(t, x, y)\, dy = Q(x, t)$ is the probability that the first time that x_s exists from G is no larger than t. $Q(x, t)$ satisfies $(L - \partial/\partial t)\, Q(x, t) = 0$ with boundary condition $Q(x, t) \to 1$ as $x \to \partial G$, $t > 0$ and $Q(x, t) \to 0$ as $t \to 0$, $x \in G$. Noting that $(L - \partial/\partial t)\, P(x, t) \leqq 0$ and that $P(x, t) \geqq Q(x, t)$ on $G \times \{0\} + \partial G \times [0, T]$, an application of the strong maximum principle for parabolic operators yields that

$$P(x,t) \geqq Q(x,t) \qquad \text{in} \qquad [G + \partial G] \times [0, T].$$

Thus, for $t \leqq T$,

$$Q(x,t) = P_x\{x_s \notin G, \quad \text{some} \quad s \leqq t\} \leqq P(x,t). \qquad \text{Q.E.D.}$$

3. Examples

Example 1. A Hill Climbing Problem. In this example, bounds on the errors associated with a standard simple gradient hill climbing method are developed. Let $f(x)$ be a smooth scalar-valued function with a unique maximum θ_n, at time n, which varies in time according to the rule $\theta_{n+1} = \theta_n + \psi_n$, where the ψ_n are assumed to be independent random variables. The quantities x and θ_n are scalar valued, and x_n is the nth sequential estimate of θ_n. For the most general problem, it would be assumed that not much a priori information concerning $f(x)$ is available, but that observations on $f(x)$, corrupted by additive noise, can be taken.

To facilitate a simple development we suppose that $f(x)$ takes the simple form $f(x) = -k(x - \theta_n)/2$ and let two observations (at $x_n + c$ and at $x_n - c$) be taken simultaneously. The sequence x_n is given by the gradient procedure

$$\begin{aligned} x_{n+1} &= x_n + \frac{a\,[\text{observation at } (x_n + c) - \text{observation at } (x_n - c)]}{2c} \\ &= x_n + \frac{a[f(x_n + c) - f(x_n - c) + \xi_n]}{2c}. \end{aligned} \tag{3-1}$$

The random variable ξ_n is the total observation noise and the members of $\{\xi_n\}$ are assumed to be independent. a and c are suitable positive constants. Equation (3-1) can be written as

$$x_{n+1} - \theta_{n+1} = (1 - ak)(x_n - \theta_n) + v_n$$

$$v_n = \frac{a\xi_n}{2c} - \psi_n.$$

Define $Ev_n^2 = m_2$ and $Ev_n^4 = m_4$ and let $Ev_n = Ev_n^3 = 0$. In order to apply Theorem 3, the assumption that $|1 - ak| < 1$ will also be needed. Let $e_n = x_n - \theta_n$; then

$$e_{n+1} = (1 - ak)e_n + v_n. \tag{3-2}$$

The estimates of $P_x\{\sup_{N \geqq n \geqq 0} |e_n| \geqq \varepsilon\}$ given by two different Liapunov functions will be compared. First, define $V_1(e) = e^2$. Then (3-2), applied to $V_1(e)$, gives

$$E_{e_n} e_{n+1}^2 = (1 - ak)^2 e_n^2 + m_2,$$
$$\beta = (1 - ak)^{-2}, \qquad \varphi_n = m_2.$$

Thus Theorem 3 is applicable and, with the use of $m = \varepsilon^2$ and $e_0^2 = e^2$, gives the bounds

$$P_x\{\sup_{N \geqq n \geqq 0} |e_n| \geqq \varepsilon\} = P_x\{\sup_{N \geqq n \geqq 0} e_n^2 \geqq \varepsilon^2\}$$
$$\leqq 1 - \left(1 - \frac{e^2}{\varepsilon^2}\right)\left(1 - \frac{m_2}{\varepsilon^2}\right)^N \tag{3-3a}$$

if

$$\varepsilon^2 \geqq \frac{\varphi\beta}{\beta - 1} = \frac{m_2}{[1 - (1 - ak)^2]}$$

and

$$P_x\{\sup_{N \geqq n \geqq 0} |e_n| \geqq \varepsilon\} \leqq \frac{e^2(1 - ak)^{2N}}{\varepsilon^2} + \frac{m_2[1 - (1 - ak)^{2N}]}{\varepsilon^2[1 - (1 - ak)^2]} \tag{3-3b}$$

if

$$\varepsilon^2 \leqq \frac{m_2}{[1 - (1 - ak)^2]}.$$

The bound in (3-3b) is separated into two terms, the first depending on the initial error, and the second depending on the noise variance. As N increases, the contribution of the initial error decreases exponentially, while the noise contribution increases as a constant minus a decreasing exponential.

Now let us try $V_2(e) = Be^4 + e^2$ as a Liapunov function.

By (3-2)

$$
\begin{aligned}
E_{e_n}(Be_{n+1}^4 + e_{n+1}^2) &= B(1-ak)^4 e_n^4 \\
&\quad + \left[B\binom{4}{2}(1-ak)^2 m_2 + (1-ak)^2\right] e_n^2 \\
&\quad + (Bm_4 + m_2) \\
&\leqq \frac{Be_n^4 + e_n^2}{\beta} + \varphi,
\end{aligned}
$$

$$
\beta = \left[B\binom{4}{2}(1-ak)^2 m_2 + (1-ak)^2\right]^{-1} \geqq 1,
$$

$$
\varphi = Bm_4 + m_2. \tag{3-4}
$$

Apply Theorem 3 to $V_2(e)$ and let $m = Be^4 + \varepsilon^2$; this gives the bounds

$$
\begin{aligned}
P_x\{\sup_{N \geqq n \geqq 0} |e_n| \geqq \varepsilon\} &= P_x\{\sup_{N \geqq n \geqq 0} V_2(e_n) \geqq V_2(\varepsilon)\} \\
&\leqq 1 - \left(1 - \frac{Be^4 + e^2}{B\varepsilon^4 + \varepsilon^2}\right)\left(1 - \frac{Bm_4 + m_2}{B\varepsilon^4 + \varepsilon^2}\right)^N
\end{aligned} \tag{3-5a}
$$

when

$$
m = B\varepsilon^4 + \varepsilon^2 \geqq \frac{Bm_4 + m_2}{1 - 1/\beta},
$$

and

$$
\begin{aligned}
&P_x\{\sup_{N \geqq n \geqq 0} |e_n| \geqq \varepsilon\} \\
&\leqq \frac{(Be^4 + e^2)\beta^{-N} + (1 - \beta^{-N})(Bm_4 + m_2)/(1 - \beta^{-1})}{B\varepsilon^4 + \varepsilon^2}
\end{aligned} \tag{3-5b}
$$

when

$$
B\varepsilon^4 + \varepsilon^2 < \frac{Bm_4 + m_2}{1 - 1/\beta}.
$$

Let us compare the two forms $V_1(e)$ and $V_2(e)$. To simplify the comparison, let (3-3a) and (3-5a) be applicable, and let $e = 0$. The bounds (3-3a) and (3-5a) reduce to $1 - (m_2/\varepsilon^2)^N$ and $1 - (Bm_4 + m_2)^N/(B\varepsilon^4 + \varepsilon^2)^N$, respectively. The ratio of the powered quantities is (the second to the first)

$$\frac{1 + Bm_4/m_2}{1 + B\varepsilon^2}.$$

It is now readily seen that $V_1(e)$ is preferable to $V_2(e)$ if and only if $m_4 \geqq \varepsilon^2 m_2$. If $V_2(e)$ is preferable to $V_1(e)$ there remains the problem of choosing B, but we will not pursue it.

Example 2. A Second Order Nonlinear Itô Equation. We consider the particular stochastic form of Lienard's equation

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= -(2x_1 + 3x_1^2 + x_2)\, dt + \sigma\, dz. \end{aligned} \tag{3-6}$$

The function

$$V(x) = \frac{x_2^2}{2} + \frac{(x_1 + x_2)^2}{2} + 2x_1^2(1 + x_1)$$

is a Liapunov function for the deterministic problem.* The deterministic system has a saddle point at $x = (-2/3, 0)$ (see LaSalle and Lefschetz [1], p. 64) and the origin is asymptotically stable in the sense of Liapunov. At $x = (-2/3, 0)$, $V(x) = 14/27 \equiv \hat{m}$. Thus, the probability that $\sup_{T \geqq t \geqq 0} V(x_t) \geqq \hat{m}$ is essentially the probability that the object will be lost in the time interval $[0, T]$. $Q_{\hat{m}}$ is within the domain of attraction of the origin for the deterministic problem. $V(x)$ is in the domain of $\tilde{A}_m$ for $m \leqq \hat{m}$. Suppose that $m \leqq \hat{m}$:

$$\tilde{A}_m V(x) = \mathscr{L} V(x) = -x_2^2 - 3x_1^2(\tfrac{4}{3} + x_1) + \sigma^2.$$

A majorization of $V(x)$ yields

$$V(x) \leqq 3\left(\frac{x_2^2}{2} + x_1^2\right) + 2x_1^3$$

* This Liapunov function is a special case of one constructed by E. F. Infante (personal communication) for equation (3-6) with arbitrary coefficients.

and

$$\frac{\mathscr{L}V(x) - \sigma^2}{V(x)} \geqq \frac{2}{3}\left[1 + \frac{5x_1^3/6}{x_2^2/2 + x_1^2 + 2x_1^3/3}\right]. \tag{3-7}$$

In Q_m, the least value of the right side of (3-7) occurs at $(x_1 = -q_m, x_2 = 0)$ where $q_m < 2/3$. Thus

$$\frac{-[\mathscr{L}V(x) - \sigma^2]}{V(x)} \geqq \frac{2}{3}\left[1 - \frac{5q_m/6}{1 - 2q_m^2/3}\right] \equiv \mu_m \tag{3-8}$$

and

$$\mathscr{L}V(x) \leqq -\mu_m V(x) + \sigma^2 .$$

Now, we seek a good estimate of

$$P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq \hat{m}\}, \tag{3-9}$$

and, to obtain it, we would like to use the stronger part of Theorem 1, namely (2-3), if applicable. This form, whose application to the region Q_m requires $\hat{m} \geqq \varphi/\mu_{\hat{m}} = \sigma^2/\mu_{\hat{m}}$, is not directly applicable, however, since $\mu_{\hat{m}} = 0$. Suppose that there is some $m_0 < \hat{m}$ such that $m_0 \geqq \sigma^2/\mu_{m_0}$ and $V(x) < m_0$. Then the form (2-3) may be used to estimate $P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq m_0\}$, and a bound on (3-9) obtained by the inequality

$$P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq \hat{m}\} \leqq P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq m_0\}. \tag{3-10}$$

We now outline a procedure for optimizing m_0. (Note first, however, that for some (small) values of T and $V(x)$, the form (2-4) applied to $Q_{\hat{m}}$ may be preferable to the following procedure.) Since the exponent in (2-3) is σ^2/m, provided $m \geqq \sigma^2/\mu_m$, the remarks above suggest that we seek the largest value of $m < \hat{m}$ for which $m \geqq \sigma^2/\mu_m$. This is the largest m (or, equivalently, the largest $q_m < \frac{2}{3}$) for which

$$V(-q_m, 0) = m = q_m^2(\tfrac{5}{2} - 2q_m) \geqq \frac{\sigma^2}{\frac{2}{3}[1 - (5q_m/6)/(1 - 2q_m/3)]} = \frac{\sigma^2}{\mu_m}$$
$$= \frac{3\sigma^2(1 - 2q_m/3)}{2(1 - 3q_m/2)} .$$

Denoting the maximum by $\hat{m}$, we have

$$P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq \hat{m}\} \leqq P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq \tilde{m}\}$$
$$\leqq 1 - \left(1 - \frac{V(x)}{\tilde{m}}\right) \exp - \frac{\sigma^2 T}{\tilde{m}}.$$

For small σ^2, the estimate can be improved by the use of either $V^n(x)$ or $\exp \psi V(x)$, although, since the computations are somewhat more complex, the problem will not be pursued here.

Example 3. In this example, a simple first order linear Itô equation is considered. This is the simplest model of a continuous time tracker.

Let x_t be the position of the object to be tracked, and let us suppose that the system $dx = \sigma_1\, dw$, where w_t is a Wiener process, models x_t. Let m_t and $e_t = m_t - x_t$ denote the estimate of x_t, and the tracking error, respectively. A fairly common form for the observations on x_t is

$$g(e_t) + \psi_t,$$

where ψ_t is a noise term and $g(e)e \geqq 0$. The observation may be a function of e_t, rather than of x_t, since the physical observer or tracker may center his "sighting" at m_t, the estimate of x_t, and the observation would then be a function, $g(e_t)$, of the difference between the center of the sighting and the true location.

Let the ψ_t be "white" Gaussian noise and represent the m_t process by

$$dm = -g(e)\, dt + \sigma_2\, du,$$

where u_t is a Wiener process (to account for the supposed white Gaussian observation noise). Then

$$de = -g(e)\, dt + \sigma\, dz,$$

where z_t is a Wiener process and $\sigma\, dz = -\sigma_1\, dw + \sigma_2\, du$. If $g(e) = 0$ for $|e| \geqq \varepsilon$, then the only "restoring force," when $|e| \geqq \varepsilon$, is the noise, and track may be considered to be lost if $|e| \geqq \varepsilon$.

In the remainder of this example, we consider the process in $\{e: |e| < \varepsilon\}$ and the linear case $g(e) = ae$. Let $V(e) = |e|^n \; n \geqq 2$. Then

$$\tilde{A}_m V(e) = \mathscr{L} V(e) = -na\,|e|^n + \sigma^2 \frac{n(n-1)}{2} |e|^{n-2} \tag{3-11}$$

which takes the form

$$\mathscr{L} V(e) \leqq -\mu V(e) + \varphi \tag{3-12}$$

$$\mu = na, \qquad \varphi = \frac{\sigma^2 n(n-1)\varepsilon^{n-2}}{2}.$$

Let $m = \varepsilon^n \geqq \varphi/\mu = \sigma^2 \varepsilon^{n-2}(n-1)/2a$. Then (2-3) is applicable and gives the bound

$$P_x\{\sup_{T \geqq t \geqq 0} |e_t| \geqq \varepsilon\} = P_x\{\sup_{T \geqq t \geqq 0} |e_t|^n \geqq \varepsilon^n\}$$

$$\leqq 1 - \left(1 - \frac{|e|^n}{\varepsilon^n}\right) \exp\left[-\frac{n(n-1)}{2\varepsilon^2}\sigma^2 T\right]. \tag{3-13}$$

When putting (3-11) into the form (3-12), a helpful tradeoff between the first and second forms of terms of (3-14) is possible. Write (3-11) as

$$\mathscr{L} V(e) = -\gamma na\,|e|^n + \left[\frac{\sigma^2 n(n-1)}{2} |e|^{n-2} - (1-\gamma)\, na\,|e|^n\right]$$

$$\leqq -\mu_\gamma |e|^n + \varphi_\gamma, \qquad 1 \geqq \gamma \geqq 0. \tag{3-14}$$

A judicious choice of γ can reduce the bound (3-13). To facilitate a simple illustration, let

$$m \geqq \frac{\sigma^2(n-1)}{2}\left[\frac{(n-1)\sigma^2}{2a}\right]^{(n-2)/2} \equiv D.$$

The reason for this assumption will appear shortly. It implies that form (2-3) is applicable for some $0 < \gamma < 1$; φ_γ is the maximum of the second term of (3-11) and equals

$$\varphi_\gamma = \sigma^2(n-1)\left[\frac{(n-1)(n-2)\sigma^2}{2(1-\gamma)\,an}\right]^{(n-2)/2}$$

and occurs at $e^2 = (n-1)(n-2)\,\sigma^2/(2an(1-\gamma)) = e_\gamma^2$; *if* $e_\gamma > \varepsilon$, *then the maximum value of the second term of* (3-11) *occurs at* $\varepsilon = e$. *Let* $\varepsilon_\gamma \leqq \varepsilon$. To make the most effective use of Theorem 1 (assuming (2-3) is applicable) we want to select γ so that φ_γ/m is a minimum subject to the constraint $\mu_\gamma \geqq \varphi_\gamma/m$ (which assures us that (2-3) applies); i.e., if m is sufficiently large, we would like to select $\gamma = \gamma^*$ so that $m = \varphi_{\gamma^*}/\mu_{\gamma^*}$. (See Figure 2.) The value of γ which minimizes $\varphi_\gamma/\mu_\gamma$ is $\gamma^* = 2/n$ and

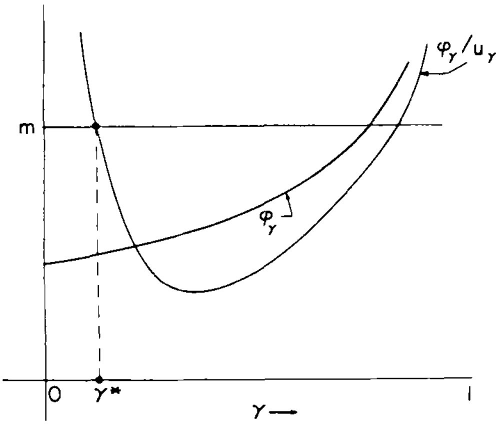


Figure 2

yields $\varphi_{\gamma^*}/\mu_{\gamma^*} = D$. Thus, we require

$$m \geqq \varphi_{\gamma^*}/\mu_{\gamma^*} \geqq D\,.$$

The inequalities are satisfied by our assumption on m. Thus, corresponding to some $0 \leqq \gamma \leqq 1$, the equality $m = \varphi_\gamma/\mu_\gamma$ holds. With this γ, (2-3) yields the bound

$$P_x\{\sup_{T \geqq t \geqq 0} |e_t| \geqq \varepsilon\} \leqq 1 - \left(1 - \frac{e^n}{\varepsilon^n}\right) e^{-MT} \tag{3-15}$$

$$M = \mu_\gamma = \frac{\varphi_\gamma}{m_\gamma} = \gamma n a = \frac{\varphi_\gamma}{\varepsilon^n}\,.$$

If the hypotheses of the construction are satisfied, namely $m \geqq D$, then (3-15) is preferable to (3-13), provided that the same value of n is used in each. However, the best value of n in (3-13) is $n = 2$ (at least for $e = 0$; $n \geqq 2$ by assumption), and the computations which have been made have not proved that (3-15) is preferable to (3-13) when each is minimized seperately with respect to n. Nevertheless, the method does seem to have a general interest.

Example 4. A Two-Dimensional Linear Itô Equation. Let the system be modeled by

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= (-x_1 - x_2)\, dt + \sigma\, dz\,. \end{aligned} \tag{3-16}$$

We will use the stochastic Liapunov function

$$V(x) = \tfrac{3}{2}x_1^2 + x_1 x_2 + x_2^2 = x'Bx$$

with

$$\tilde{A}_m V(x) = \mathscr{L}V(x) = -x_1^2 - x_2^2 + \sigma^2 = -x'Ax + \sigma^2\,. \tag{3-17}$$

Equation (3-17) can be put into the form

$$\tilde{A}_m V(x) \leqq -\mu V(x) + \varphi$$

$$\varphi = \sigma^2\,, \qquad \mu = \min_x (x_1^2 + x_2^2)/(\tfrac{3}{2}x_1^2 + x_1 x_2 + x_2^2)\,.$$

The minimum of the ratio is easily shown to be the least value of $|\lambda|$ for which

$$(A + \lambda B)\, x = 0$$

has a nontrivial solution, and it equals $\mu = 1 - 0.4\sqrt{1.25} \geqq 0.552$. Thus

$$\tilde{A}_m V(x) \leqq -0.552 V(x) + \sigma^2\,.$$

Suppose that $m \geqq \sigma^2/(0.552)$. Then equation (2-3) gives the bound

$$P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq m\} \leqq 1 - \left(1 - \frac{V(x)}{m}\right) \exp\left(-\frac{\sigma^2 T}{m}\right). \tag{3-18}$$

IMPROVING THE BOUND

We will give a qualitative description of the situation when the Liapunov function $V^n(x)$ is used:

$$\tilde{A}_m V^n(x) = \mathscr{L}V^n(x)$$
$$= nV^{n-1}(x)\left[-x_1^2 - x_2^2 + \sigma^2 + \frac{(2x_2 + x_1)^2 \sigma^2 (n-1)}{2V(x)}\right]. \tag{3-19}$$

There are many ways of factoring (3-19) into the form $\mathscr{L}V^n(x) \leqq -\mu V^n(x) + \varphi$. Let $\bar{\mu} = 0.552$ and write

$$\mathscr{L}V^n(x) \leqq -\mu_n V^n(x) + \varphi_n$$

where

$$\mu_n = n\bar{\mu}\alpha_n, \qquad 1 > \alpha_n > 0$$

$$\varphi_n = \max_x \left[n\bar{\mu}(\alpha_n - 1) V^n(x) + nV^{n-1}(x)\sigma^2 + \frac{(2x_2 + x_1)^2 V^{n-2}(x)\sigma^2(n-1)n}{2}\right]. \tag{3-20}$$

The first term in the bracket of (3-20) is negative and dominates the other terms for large $\|x\|$. Thus, for any $1 > \alpha_n > 0$, the term being maximized will have a finite maximum which will occur at a finite point. φ_n increases, as α_n increases to one. φ_n also increases with n roughly as n^n. Choose $1 > \alpha_n > 0$ so that φ_n is as small as possible subject to the constraint $m^n \geqq \varphi_n/\mu_n$. (To simplify the discussion, we assume that $m^n \geqq \varphi_n/\mu_n$ for the range of n of interest. For any fixed m, this will hold for at most finitely many n. If $m^n \geqq \varphi_n/\mu_n$ cannot be achieved, then the argument must be based on (2-4) rather than on (2-3).)

For any n, if m is *sufficiently large*, then the exponent in (2-3) or (3-21) satisfies $\varphi_n/m^n < \varphi_r/m^r$ for all $r < n$. Also, $V^n(x)/m^n < V^r(x)/m^r$, $r < n$. We may conclude that, for each value of m, there is an optimum n (n is not required to be integral valued, provided $n \geqq 1$); the optimum value of n is an increasing function of m. While no computations have been made, it seems reasonable to expect that, if the optimum $n(m)$ were substituted into (3-21), then, for fixed T, the graph of the right side of (3-21), considered as a function of m, would have a bell shape, rather than the shape of a simple exponential:

$$P_x\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq m\} = P_x\{\sup_{T \geqq t \geqq 0} V^n(x_t) \geqq m^n\}$$
$$\leqq 1 - \left(1 - \frac{V^n(x)}{m^n}\right) \exp\left(-\frac{\varphi_n T}{m^n}\right). \tag{3-21}$$

A related approach involves the use of the Liapunov function $\exp \psi V(x)$ in lieu of $V^n(x)$. The choice of ψ replaces the choice of n. We have

$$\exp \psi V(x) = \psi \exp \psi V(x)\left[-x_1^2 - x_2^2 + \sigma^2 + \frac{\psi\sigma^2}{2}(2x_2 + x_1)^2\right], \tag{3-22}$$

which may be treated similarly to $\mathscr{L}V^n(x)$. For more details, see Example 2 of Chapter V which concerns the design of a control for the system (3-16).

IV / OPTIMAL STOCHASTIC CONTROL

1. Introduction

This section motivates the more general discussion of the succeeding sections. Let x_t^u be a family of strong Markov processes. The parameter u associated with each member is termed a control. The control determines the probability transition function $p^u(t, x; s, y)$ of the process x_t^u. Usually each u may be identified with a specific member of a given family of functions which takes values $u(x_t^u, t)$ depending only on x_t^u and t, at time t. The object of control theory is to select the control so that the corresponding process possesses some desired property. One object may be to transfer an initial state x to some target set S with probability one, that is, choose a u so that $x_t^u \to S$ with probability one); another object could be for x_t^u to follow as closely as possible (in a suitable statistical sense) some preassigned path.

Write $\tilde{A}^u$ and $\tilde{A}_m^u$ for the weak infinitesimal operators of the process x_t^u and the process stopped on exit from an open set Q_m, respectively, and write E_x^u for the expectation given that the initial state is x. Henceforth x_t^u will be written as x_t, and $u(x_t^u, t)$ may be written u_t. The specific control associated with x_t will be clear from the context. The problem may be developed further by associating to each control a cost

$$C^u(x) = E_x^u\, b(x_{\tau_u}) + E_x^u \int_0^{\tau_u} k(x_s, u_s)\, ds\,. \tag{1-1}$$

If we wish to attain a target set S, then τ_u is the random time of arrival at S. A problem of optimal control is to select u so that $x_t \to S$ with probability one as t approaches some random time τ_u (possibly infinite valued) and which minimizes the cost $C^u(x)$ with respect to other controls of a specified class of "admissible comparison controls."

A DYNAMIC PROGRAMMING ALGORITHM

Let us first consider the control of processes governed by Itô equations, and, for the moment, proceed in a formal way. Suppose that the control process is modeled by the homogeneous process

$$dx = f(x, u)\, dt + \sigma(x, u)\, dz$$
$$\mathscr{L}^u = \sum_j f_i(x, u) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij}(x, u) \frac{\partial^2}{\partial x_i\, \partial x_j}, \tag{1-2}$$

and assume $\tilde{A}^u = \mathscr{L}^u$ on all functions to which the operator is to be applied. The function u takes values $u(x_t)$ at time t. The system (1-2) has a well-defined solution in the sense of Itô for any sufficiently smooth f, u, and σ. Thus, the control u indeed determines the process.

Let a target set S be given and let (1-1) represent the cost given that $x_0 = x$ is not in S. Denote the minimum cost by $V(x)$; in other words, $V(x) \leqq C^u(x)$ for all controls u lying in a specified comparison class. If $V(x)$ is indeed the minimum cost, then the principle of dynamic programming (Bellman [2]) implies that $V(x)$ satisfies the functional equation

$$V(x) = \min_u E_x^u \left[V(x_{\Delta \cap \tau_u}) + \int_0^{\Delta \cap \tau_u} k(x_s, u_s)\, ds \right]. \tag{1-3}$$

The time index $\Delta \cap \tau_u$ appears since control is terminated at τ_u. The initial time in (1-3) is set equal to zero for convenience. Since u depends only on x, the process is homogeneous in time and the convention is inconsequential. If $V(x)$ is given, then, by the algorithm of dynamic programming, the control u minimizing the right side of (1-3) is the optimal control.

Continuing formally, let us put (1-3) into a more convenient form. Divide (1-3) by Δ and let

$$\lim_{\Delta\to 0}\frac{E_x^u V(x_{\Delta\cap\tau_u}) - V(x)}{\psi} = \mathscr{L}^u V(x) = \tilde{A}^u V(x)$$

$$\lim_{\Delta\to 0} E_x^u \frac{1}{\Delta}\int_0^{\Delta\cap\tau_u} k(x_s, u_s) = k(x, u(x)). \tag{1-4}$$

This yields

$$0 = \min_u \left[\mathscr{L}^u V(x) + k(x, u)\right], \tag{1-5}$$

which is a nonlinear partial differential equation for $V(x)$, and also gives the optimum control $u(x)$ in terms of the derivatives of $V(x)$. In other words, if u is the function minimizing (1-5) and w is some other control, then

$$0 = \mathscr{L}^u V(x) + k(x, u) \leqq \mathscr{L}^w V(x) + k(x, w). \tag{1-6}$$

$V(x)$ is, of course, subject to the boundary condition $V(x) = b(x)$ for x on ∂S. In a sense, for stochastic control, (1-5) replaces the Hamilton–Jacobi equation corresponding to the deterministic optimal control problem, where the Itô equation is replaced by an ordinary differential equation. See, for example, Kalman [1], Dreyfus [1], and Athans and Falb [1].

A slightly different control problem occurs if we add the requirement that the process is to be terminated upon the first time x_t exits from an open set G containing S, provided that this occurs before x_t reaches S. Then by letting τ_u be the least time to either ∂G or ∂S, whichever is contacted first, and assigning the penalty $b(x_{\tau_u})$ to the stopping position, the problem is unchanged except that the solution of (1-5) is subject to the boundary condition $b(x)$ on $\partial S \cup \partial G$.

A seemingly different type of control problem appears when no target set S is specified, and the control is to be terminated at a given finite time T. Now, let u take values $u(x_t, t)$ depending on both x_t and t. By letting τ_u equal the minimum of T and the first time of contact with ∂G, and letting $x_t = x \in G$ be the initial condition, the cost is written as

$$C^u(x, t) = E_{x,t}^u\, b(x_{\tau_u}, \tau_u) + E_{x,t}^u \int_t^{\tau_u} k(x_s, u_s, s)\, ds, \qquad (1\text{-}7)$$

where $E_{x,t}^u$ is the expectation given $x_t = x \in G$ and $t \leqq T$.

The associated dynamic programming equation is (we now allow all functions to depend explicitly on t)

$$0 = \min_u \left[\frac{\partial V(x, t)}{\partial t} + \mathscr{L}^u V(x, t) + k(x, u, t) \right], \qquad (1\text{-}8)$$

where $V(x, t)$ is the minimum cost function (the minimum of $C^u(x, t)$ over u).

Actually, from the point of view of formal development, the problem leading to (1-7) and (1-8) is a special case of the preceeding problem. It is not necessary to introduce time explicitly since some component of x_t can be considered to be time; that is, $dx_{n+1} = dt$ or $x_{n+1,t} = t$. In fact, unless otherwise mentioned, we will omit the t argument, and assume that some state is time.

The control forms which have been discussed up to now are functions of only the present value x_t at time t. Their values at t have not been allowed to depend on the values of x_s, $s < t$. Let P be the class of controls whose values depend only on the present state x_t. Let P_1 be the class of controls whose values have an arbitrary dependence upon the present and also upon the past history of the state. Is the class P no worse than the class P_1? In other words, for each control w in P_1 is there a control $u(w)$ in P such that $C^{u(w)}(x) \leqq C^w(x)$? The question as formulated is still a little vague. Nevertheless, it is intuitively reasonable that, if the uncontrolled process has the Markov property, that the class P is "just as good" as the class P_1. Derman [1] gives some results on this problem for the discrete parameter control problem and Fleming [2] for the continuous parameter diffusion process; see also Theorem 8 and Example 1 of this chapter.

DISCUSSION

A number of difficulties with the development leading to (1-5) and (1-6) are readily apparent. Let Q be the family of controls (functions of x) such that, if $u \in Q$, then (1-2) has a unique solution (in the sense of Itô) which is a well-defined strong Markov process. The class Q is essentially limited to functions satisfying at least a local Lipschitz condition in x. There may be no u in Q such that $C^u(x) \leqq C^w(x)$ for every other control w in Q.* Even if an optimum control u in Q did exist, there is no guarantee that the cost $C^u(x)$ is in the domain of $\tilde{A}^u$ and that $\tilde{A}^u = \mathscr{L}^u$ when acting on $C^u(x)$; that is, the limits (1-4) may be meaningless. Third, if a smooth solution to (1-5) did exist, there is no guarantee that the control which minimizes (1-5) will be in Q; in particular, it may not satisfy even a local Lipschitz condition in x. (If u does not satisfy this condition, then the corresponding Itô process has not been defined, although there are diffusion processes whose differential generators have coefficients that are less smooth than that required (at present) for the existence and uniqueness of solutions to the Itô equation.)

The family of comparison controls Q is not clear from the derivation of (1-5) and (1-6). Also, without further analysis, we cannot assume as is done in (1-4) that $P_x^u\{\Delta < \tau_u\} \to 1$ as $\Delta \to 0$. (This property holds for all the right continuous processes of concern here.) Lastly, the solution to (1-5), subject to the appropriate boundary conditions, may not be unique.

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* Consider a control problem where a target set S is to be attained. Typically, the question of the existence of an optimal control is broken into two questions. The first is a question of attainability and the second of optimality. First: Is there a control with which the set S will be attained with probability one? It is possible to treat this with the methods developed here for stochastic stability. See Chapter V. The second question is: Given that there is one control accomplishing the desired task, then is there an optimal control? The latter question has been well treated for the deterministic problem. See, for example, Lee and Marcus [1], Roxin [1], and Fillipov [1]. For the stochastic problem, the possibilities are great and the results more fragmentary; see Kushner [7] and Fleming and Nisio [1]. See the survey papers Kushner [2, 10] for more references to the literature on stochastic control.

The purpose of the next section is to resolve these and related questions.

A DETERMINISTIC RESULT

To motivate the theorems of the sequel, we first consider the control of the deterministic system $\dot{x} = f(x, u)$. Let Q be a class of controls each member of which is a function whose values depend on time, and impose conditions on Q and f so that $\dot{x} = f(x, u)$ has a unique continuous solution and $\|f(x, u)\|$ is bounded in the time interval $[0, \infty)$, for each u in Q. Let S be a compact target set and define the cost as

$$C^u(x) = b(x_{\tau_u}) + \int_0^{\tau_u} k(x_s, u_s)\, ds$$

where τ_u is the time of arrival at ∂S. Let $b(x)$ be continuous and let $k(x, u) > 0$ outside of S. Now, let $V(x)$ be a nonnegative function with continuous derivatives, and which tends to infinity as $\|x\| \to \infty$. Define the operator $H^u = \sum_i f_i(x, u)\, \partial/\partial x_i$. Let

$$H^u V(x) = \dot{V}(x) = -\, k(x, u).$$

Then $V(x)$ is a Liapunov function for $\dot{x} = f(x, u)$, and any trajectory of the system is uniformly bounded in time. Also, with $x_0 = x$ and $t \leqq \tau_u$,

$$V(x_t) - V(x) = \int_0^t H^u V(x_s)\, ds = -\int_0^t k(x_s, u_s)\, ds. \qquad \text{(1-9)}$$

Owing to the boundedness of the solutions and of $\|\dot{x}\|$ and to the properties of $V(x)$ and $\dot{V}(x)$ outside S, we see that, as t increases to some finite or infinite value τ_u, x_t must approach some point x_{τ_u} on ∂S, and $V(x_t)$ must approach $b(x_{\tau_u})$. Hence $V(x) = C^u(x)$.

Now, let w be any other control in Q which transfers x to ∂S and let $H^w V(x) + k(x, w) \geqq H^u V(x) + k(x, u)$. Then, again $V(x_t) \to b(x_{\tau_w})$, and a simple calculation yields

$$C^u(x) = \int_0^{\tau_u} k(x_s, u_s)\, ds + b(x_{\tau_u}) \leqq C^w(x) = \int_0^{\tau_w} k(x_s, w_s)\, ds + b(x_{\tau_w}).$$

In other words, if $0 = H^u V(x) + k(x, u) \leqq H^w V(x) + k(x, w)$ for all w in Q, then u is an optimal control relative to the set of controls Q. Also, the minimum cost is a solution to $H^u V(x) + k(x, u) = 0$ with boundary data $b(x)$ and the optimum u minimizes $H^u V(x) + k(x, u)$. ($H^u V(x) + k(x, u) = 0$ is an equation of the Hamilton–Jacobi type.)

Except for a few results concerning existence and uniqueness for some problems in the control of strong diffusion processes, the approach of the sequel is similar to the deterministic approach just described. Given candidates for the minimum cost and optimum control, we discuss criteria which may be used to test for optimality. The processes, target sets, and conditions will vary from theorem to theorem. The stochastic situation is much more difficult than the deterministic counterpart just described.

The stochastic analog

Stochastic analogs of the deterministic result are the burden of part of Section 2. In order to base a stochastic proof upon the deterministic model of the previous subsection, an analog of the integral formula (1-9) is needed. Dynkin's formula

$$E_x^u V(x_t) - V(x) = E_x^u \int_0^{\tau} \tilde{A}^u V(x_s)\, ds \tag{1-10}$$

will provide this. τ is a Markov time with finite average value, and $V(x)$ is in the domain of $\tilde{A}^u$. The use of (1-10) would appear to be hindered by these conditions. Nevertheless, by first applying (1-10) under these conditions, we may subsequently extend its domain of validity in such a way as to provide a useful tool. The extension is accomplished by truncating either $V(x)$ or the process x_t and then taking limits, and is developed in the proofs which follow.

2. Theorems

TERMINOLOGY

In Theorem 1, the following conventions and assumptions are used for the target set S. Let $x = (\bar{x}, \bar{\bar{x}})$, where $\bar{x}_t$ is continuous with probability one, for $t \leqq \tau_u$, if $\tau_u < \infty$, and for $t < \infty$ otherwise. The component $\bar{\bar{x}}_t$ is only right continuous on the same time interval. Let $\bar{x}$ be in the Euclidean space E and $\bar{\bar{x}}$ in $\bar{\bar{E}}$, $E = \bar{E} \times \bar{\bar{E}}$. The object of the control is to transfer the initial value of $\bar{x}_t$, denoted by $\bar{x}$, to $\bar{S}$ in $\bar{E}$. These conventions will simplify the proof. They do not seem to be a serious compromise of generality, since discontinuous processes will probably enter the problem as parameters or, at least, not as variables whose values are to be transferred to some set.

For each control under consideration, the corresponding process is assumed to be a right continuous strong Markov process. The previous usage $Q_m = \{x: V(x) < m\}$, where $V(x)$ is a given function, will be retained. Note also that $\tilde{A}_m^u$ refers to the weak infinitesimal operator of the process which corresponds to u and which is stopped on exit from a given set Q_m. The set Q_m (and function $V(x)$) will be clear from the context. Except where explicitly noted, u will always denote a real- or vector-valued function whose values will depend either only on x, or on x and t, as indicated. We define $\tilde{A}_{m,b}^u$ as the weak infinitesimal operator of the process (with control u) stopped on exit from $Q_m - S = \{x: V(x) \leqq m\} - S$. τ_u is always the first entrance time into the set S, or the time of termination of control if this occurs first.

Theorem 1 gives conditions on a function $V(x)$ and control u so that $V(x) = C^u(x)$. It is assumed that $b(x) \equiv 0$ and $\bar{x}_t \to \partial S$ with probability one as $t \to \tau_u$. Theorem 2 gives conditions on a function $V(x)$ so that $V(x) = C^u(x)$ and $b(x)$ may take nonzero values if $x_t \to S$ as $t \to \tau_u$, where $\tau_u < \infty$ with probability one. Theorem 3 gives a readily check-able condition under which a condition (uniform integrability) required in Theorems 1 and 2 is satisfied. Theorem 4 is an optimality theorem; conditions on two controls u and v are given so that $C^u(x) \leqq C^v(x)$. The analog of Theorem 2 for fixed time of control appears in Theorem 5. Theorems 6 and 7 give special forms (stronger results under stronger conditions) of Theorems 2 and 5, respectively, for the special case of strong diffusion processes. Finally, there are examples and a discrete parameter theorem.

Theorem 1. Assume the target set conventions of the first paragraph of Section 2. Let $b(x) \equiv 0$. Let $V(x) \geqq 0$ be a continuous function defined on E and which takes the value zero on S. For each $m > 0$, let $V(x)$ be in the domain of $\tilde{A}^u_{m,b}$ and

$$\tilde{A}^u_{m,b} V(x) = - k(x, u) \leqq 0 .$$

Suppose* that $\bar{x}_t \to \partial \bar{S}$ as $t \to \tau_u$. There is a nondecreasing sequence of Markov times τ_i satisfying $E^u_x \tau_i < \infty$, $\tau_i \to \tau_u$ with probability one and $\tau_i \leqq \inf\{t: V(x_t) \geqq i\}$. Let x be in $E - S$. Assume that either (2-1) holds, or that (2-2) together† with either (2-3) or (2-4) holds:

$$E^u_x V(x_{\tau_i}) \to 0 \qquad \text{as} \qquad i \to \infty ; \tag{2-1}$$

$$\text{the } V(x_{\tau_i}) \text{ are uniformly integrable}‡; \tag{2-2}$$

$$V(x) \text{ is uniformly continuous on } \partial S ; \tag{2-3}$$

$$P^u_x\{\sup_{\tau_u > t \geqq 0} \|x_t\| \geqq N\} \to 0 \qquad \text{as} \qquad N \to \infty . \tag{2-4}$$

* This is a question in stochastic stability, and can be verified immediately if $k(x, u)$ has the appropriate form; see for example Lemma 2 or Corollary 3-1 of Chapter II.

† If $V(x) \to \infty$ as $\|x\| \to \infty$, then the nonpositivity of $\tilde{A}_{m,b}{}^u V(x)$ implies (2-4). See Chapter II.

‡ A sequence f_i is uniformly integrable if, for every $\varepsilon > 0$, there is an $N < \infty$ independent of i such that $E_x{}^u\, |f_i|\, \chi_{(|f_i| \geqq N)} \leqq \varepsilon$, where χ_A is the characteristic function of the set A. If $f_i \to f$ with probability one, then uniform integrability implies $E_x{}^u f_i \to E_x{}^u f$.

Then

$$V(x) = C^u(x) = E_x^u \int_0^{\tau_u} k(x_s, u_s)\, ds. \tag{2-5}$$

Proof. Define $\tau_i = \min(i, \tau_u, \inf\{t\colon V(x_t) \geqq i\})$. Since $P_x\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq i\} \leqq V(x)/i$ implies that $\inf\{t\colon V(x_t) \geqq i\} \to \infty$ as $i \to \infty$ (with probability one), we have $\tau_u \cap \inf\{t\colon V(x_t) \geqq i\} \to \tau_u$ (with probability one) as $i \to \infty$. Thus $\{\tau_i\}$ satisfies the hypothesis. First, we prove that (2-2) and (2-4) imply (2-1). If the sample function x_t, $t < \tau_u$, is *uniformly bounded*, then, by the continuity of $V(x)$, the sample values $V(x_{\tau_i}) \to 0$ as $\tau_i \to \tau_u$. Equation (2-4) implies that the sample functions x_t are uniformly bounded with a probability arbitrarily close to one. Thus $V(x_{\tau_i}) \to 0$ with probability one. (This is true even though the x_t may not converge to a unique limit on ∂S. By hypothesis $x_t \to \partial S = \partial \bar{S} \times \bar{E}$ with probability one. Thus, for each $\varepsilon > 0$, there is a finite-valued random variable ϕ_ε such that the distance between x_t and ∂S is less than ε for $t > \phi_\varepsilon$, with probability one. $V(x_t) \to 0$ as $t \to \tau_u$, since $V(x)$ is uniformly continuous on the range of x_t, $t < \tau_u$, with a probability arbitrarily close to one.) Finally, this fact together with (2-2) implies (2-1).

Similarly, it is proved that (2-2) and (2-3) imply (2-1). (Note that the fact that $V(x_{\tau_i}) \to 0$ with probability one follows immediately from the uniform continuity of $V(x)$ on ∂S together with the fact that $x_t \to \partial S$ with probability one as $t \to \tau_u$.)

By Dynkin's formula

$$\begin{aligned} V(x) &= E_x^u V(x_{\tau_i}) - E_x^u \int_0^{\tau_i} \tilde{A}_{i,b}^u V(x)\, ds \\ &= E_x^u V(x_{\tau_i}) + E_x^u \int_0^{\tau_i} k(x_s, u_s)\, ds. \end{aligned}$$

Since $k(x) \geqq 0$, the integral tends to $E_x^u \int_0^{\tau_u} k(x_s, u_s)\, ds = C^u(x)$. Since the first term on the right tends to zero as $i \to \infty$, the proof is complete.

Theorem 2. Let the nonnegative function $V(x)$ be defined and continuous on E and take the nonnegative value $b(x)$ on the target set

S. Assume that $V(x)$ is in the domain of $\tilde{A}^u_{m,b}$, for each $m > 0$, and that

$$\tilde{A}^u_{m,b} V(x) = -k(x, u) \leqq 0$$

in $E - S + \partial S$, and that the control u transfers the initial condition to S at τ_u, where $\tau_u < \infty$ with probability one. There is a nondecreasing sequence of Markov times τ_i such that $\tau_i \to \tau_u$ with probability one, $E^u_x \tau_i < \infty$ and $\tau_i \leqq \inf\{t : V(x_t) \geqq i\}$. For some such sequence, let either (2-1a) or (2-2a) hold:

$$E^u_x V(x_{\tau_i}) \to E^u_x V(x_{\tau_u}) = E^u_x b(x_{\tau_u}); \tag{2-1a}$$

$$\text{the } V(x_{\tau_i}) \text{ are uniformly integrable}. \tag{2-2a}$$

Then

$$V(x) = C^u(x) = E^u_x b(x_{\tau_u}) + E^u_x \int_0^{\tau_u} k(x_s, u_s)\, ds. \tag{2-5a}$$

Remark. The primary difference between the problems stated in Theorems 1 and 2 is that here the limit $\lim_t x_t = x_{\tau_u}$ must exist with probability one. Otherwise, the "terminal cost" term $E^u_x b(x_{\tau_u})$ is meaningless. This is the reason for the requirement that τ_u be essentially finite valued.

Proof. Since $\tilde{A}^u_{m,b} V(x) \leqq 0$, in $E - S + \partial S$ we have, for $x \in E - S$,

$$P_x\{\sup_{\tau_u > s \geqq 0} V(x_s) \geqq m\} \leqq \frac{V(x)}{m}$$

which goes to zero as $m \to \infty$. Define $\tau_i = \min(i, \tau_u, \inf\{t : V(x_t) \geqq i\})$. As in Theorem 1, $\{\tau_i\}$ satisfies the hypothesis. Also, since $\tau_u < \infty$ with probability one, there is an $i(\omega)$ which is finite valued with probability one, such that $x_{\tau_i} = x_{\tau_u}$ for all $i > i(\omega)$. Hence $\lim_{i\to\infty} V(x_{\tau_i}) = V(x_{\tau_u})$ with probability one, whether or not the process $V(x_t)$ is continuous. Now (2-2a) implies (2-1a). Applying Dynkin's formula to τ_i and $V(x)$

we have

$$V(x) = E_x^u V(x_{\tau_i}) + E_x^u \int_0^{\tau_i} k(x_s, u_s)\, ds.$$

Using (2-1a) and taking limits we obtain (2-5a).

Remark on the significance of (2-2) *or* (2-2a). *Whether or not* any of (2-1)–(2-4), (2-1a), or (2-2a) holds, the nonnegativity of $k(x, u)$ and the monotone convergence theorem imply that, for any sequence of Markov times t,

$$V(x) = E_x^u \int_0^{\tau_u} k(x_s, u_s)\, ds + \lim_{t \to \tau_u} E_x^u V(x_t).$$

By Fatou's lemma (if $b(x)$ is identically zero set $b(x_{\tau_u}) = V(x_{\tau_u}) = 0$ in what follows),

$$\lim_{t \to \tau_u} E_x^u V(x_t) \geqq E_x^u V(x_{\tau_u}) = E_x^u b(x_{\tau_u}).$$

Hence

$$V(x) \geqq C^u(x) \geqq 0$$

and, for x on S,

$$V(x) = b(x) = C^u(x).$$

Also, since $V(x)$ is in the domain of each $\tilde{A}_{m,b}^u$, the function defined by

$$C_1^u(x) \equiv E_x^u \int_0^{\tau_u} k(x_s, u_s)\, ds + E_x^u b(x_{\tau_u})$$

may be assumed* to be in the domain of each $\tilde{A}_{m,b}^u$ with $\tilde{A}_{m,b}^u C_1^u(x) =$

* It is not always true that integrals of the form $E_x{}^u \int_0^{\tau_u} g(x_s)\, ds = G(x)$ are in the domain of $\tilde{A}_{m,b}{}^u$ for any $m > 0$. If this is so and $g(x)$ is continuous, then $\tilde{A}_{m,b}{}^u\, G(x) = -g(x)$. The assumption is made for the sake of argument, but it will not be pursued here. See Dynkin [2], Chapter 5, for a discussion of such questions.

$-k(x, u)$. Thus

$$\tilde{A}^u_{m,b}(V(x) - C^u_1(x)) = 0. \tag{2-6}$$

Define

$$\begin{aligned} q(x) &= V(x) - C^u_1(x) \\ &= E^u_x V(x_{t\cap\tau_u}) - E^u_x b(x_{\tau_u}) - E^u_x \int_{\tau_u \cap t}^{\tau_u} k(x_s, u_s)\, ds . \end{aligned}$$

Then (2-6) implies that the process $q(x_{s\cap\tau_u})$ is a nonnegative martingale with value $q(x) = 0$ on S. Also, for any $s \leqq t < \infty$,

$$\begin{aligned} E^u_x q(x_{s\cap\tau_u}) &= E^u_x V(x_{t\cap\tau_u}) - E^u_x b(x_{\tau_u}) - E^u_x \int_{\tau_u \cap t}^{\tau_u} k(x_s, u_s)\, ds \\ &= \lim_{t\to\infty} E^u_x V(x_{t\cap\tau_u}) - E^u_x b(x_{\tau_u}) . \end{aligned}$$

If the only function of the process $x_{t\cap\tau_u}$ which is a martingale and which satisfies the above boundary and expectation conditions is the trivial function $q(x) \equiv 0$, then condition (2-2) is satisfied and (2-5) holds.

To see that there can be such functions $q(x)$ which are not identically zero, consider the scalar uncontrolled Itô process whose stopped subprocesses have the weak infinitesimal operators $\tilde{A}^0_{m,b}$ and with the origin as the target set S:

$$dx = -x\, dt + \sqrt{\tfrac{2}{3}}x\, dz$$

Suppose that

$$k(x, u) = k(x) = \frac{4x^2}{3}, \qquad b(0) = 0 .$$

Let $V(x) = x^2$. Then $V(x)$ is in the domain of $\tilde{A}^0_{m,b}$, and on $V(x)$ in the set Q_m, $\mathscr{L}^0 = \tilde{A}^0_{m,b}$ for each m, and

$$\mathscr{L}^0 V(x) = -\frac{4x^2}{3} = -k(x) .$$

Now let $V_1(x) = x^2 + x^4$; $V_1(x)$ is also in the domain of $\tilde{A}^0_{m,b}$ for each m, and $V_1(0) = 0$ (note that Q_m is now relative to $V_1(x)$) and $\tilde{A}^0_{m,b} = \mathscr{L}^0$ on $V_1(x)$:

$$\mathscr{L}^0 V_1(x) = -\frac{4x^2}{3} = -k(x) = \mathscr{L}^0 V(x).$$

In this case it can be shown that $\tau_0 = \infty$ with probability one (see the proof of Khas'minskii [3]); there are analogous examples, however, when τ_0 is finite with probability one.

The process x_t may be essentially written as

$$x_t = x_0 \exp\left(\tfrac{2}{3} z_t - \tfrac{4}{9} t\right).$$

A direct computation yields

$$E_{x_s} x^4_{t+s} = x^4_s$$

with probability one, verifying that the difference $V_1(x_t) - V(x_t) = x_t^4$ is a martingale. Hence, the stopped process $x^4_{t \cap \tau_m}$ is also a martingale. Write $V_i = V(x_{\tau_i})$. By a supermartingale convergence theorem of Doob [1], the supermartingale sequence V_i converges to a finite valued random variable v with probability one if $E|V_i| \leqq M < \infty$. Also, if the sequence V_i is uniformly integrable, then $E_x V_i \to E_x v$. Thus, uniform integrability (condition (2-2)) together with the convergence of the V_i to the appropriate boundary values is equivalent to (2-1) or (2-1a). Theorem 3 gives a general and usable criterion for the uniform integrability of the sequence $V(x_{\tau_i})$; hence, under the conditions of Theorem 3, if $V(x_{\tau_i})$ converges to the desired boundary value $b(x_{\tau_u})$ with probability one, then $E^u_x\, V(x_{\tau_i}) \to E^u_x b(x_{\tau_u})$.

Theorem 3. Let the function $V(x)$ and the sequence τ_i satisfy the conditions of Theorem 1 or 2. Let there exist a nonnegative function $\hat{F}(x)$ in the domain of $\tilde{A}^u_{m,b}$ with $\tilde{A}^u_{m,n}\, \hat{F}(x) \leqq 0$ for each m. Let

$$\inf_{\{x : V(x) = \lambda\}} \frac{\hat{F}(x)}{V(x)} = g(\lambda) \to \infty$$

as $\lambda \to \infty$. Then the $V(x_{\tau_i})$ are uniformly integrable. In particular

$$E_x^u V(x_{\tau_i}) \to E_x^u V(x_{\tau_u}).$$

Proof. Define the random sequences $V_i = V(x_{\tau_i})$ and $F_i = \hat{F}(x_{\tau_i})$. For each ω in the set $B_{mi} \equiv \{\omega : V_i \geqq m\}$, $V_i / F_i \leqq 1/g(m)$.

Thus,

$$\int_{B_{mi}} V_i P_x^u(d\omega) = \int_{B_{mi}} F_i \frac{V_i}{F_i} P_x^u(d\omega) \leqq \frac{E_x^u F_i}{g(m)}. \tag{2-7}$$

Since $\tilde{A}_{m,b}^u \hat{F}(x) \leqq 0$, the process $\hat{F}(x_{t \cap \tau_u})$ is a nonnegative supermartingale. The discrete parameter process obtained by sampling a supermartingale at a sequence of Markov times is a discrete parameter supermartingale (Doob [1], Chapter VIII, Theorems 11.6 and 11.8). Thus the process F_i is a discrete parameter nonnegative supermartingale. By the martingale property,

$$E_x^u F_i \leqq \hat{F}.$$

Substituting this into the right side of (2-7) one obtains a bound which is independent of i and tends to zero as $m \to \infty$. Hence the sequence V_i is uniformly integrable. Q.E.D

Remark on the selection of $\hat{F}(x)$ for Itô processes. In Example 3, $\hat{F}(x)$ will be taken in the special form $\hat{F}(x) = F(V(x))$, where $F(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$. Let $F(\lambda)$ have continuous second derivatives and let $V(x)$ be in the domain of $\tilde{A}_{m,b}^u$ for each $m > 0$. Then $F(V(x))$ is in the domain of $\tilde{A}_{m,b}^u$ for each $m > 0$, and

$$\begin{aligned} \tilde{A}_{m,b}^u F(V(x)) &= \mathscr{L}^u F(V(x)) = F_V(V(x))\, \mathscr{L}^u V(x) \\ &\quad + \frac{1}{2} F_{VV}(V(x)) \sum_{i,j} \left(\frac{\partial V(x)}{\partial x_i}\right) \left(\frac{\partial V(x)}{\partial x_j}\right) S_{ij}(x) \\ &= -F_V(V(x))\, k(x,u) + F_{VV}(V(x))\, s(x), \end{aligned} \tag{2-8}$$

$$s(x) = \frac{V_x(x)'\, S(x)\, V_x(x)}{2}$$

where $V_x(x)$ is the gradient of $V(x)$ and

$$k(x, u) = -\mathscr{L}^u V(x).$$

Let $m(s)$ be any continuous function which satisfies

$$\frac{k(x, u)}{s(x)} \geqq m(V(x)) \geqq 0 \tag{2-9}$$

in $E - S$. If $F(V(x))$ is to satisfy the conditions of Theorem 2 then (2-8) must be nonpositive in $E - S$, or, equivalently,

$$\frac{F_{VV}(V(x))}{F_V(V(x))} \leqq \frac{k(x, u)}{s(x)}$$

in $E - S$. The function $F(\lambda)$ defined by

$$F(\lambda) = \int^{\lambda} \exp\left\{\int^{s} m(y)\, dy\right\} ds \tag{2-10}$$

satisfies $\mathscr{L}^u F(V(x)) \leqq 0$ in $E - S$. If the condition

$$\exp \int^{s} m(y)\, dy \to \infty \qquad \text{as} \qquad s \to \infty \tag{2-11}$$

is also satisfied, then $F(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$, and since $F(\lambda)$ has continuous second derivatives, it will satisfy the hypotheses of Theorem 3. If $m(s)$ is the trivial function $m(s) \equiv 0$, then $F(\lambda)/\lambda \equiv 1$.

Thus, *if the bound* (2-9) *satisfies* (2-11), *the* $V(x_{\tau_i})$ *are uniformly integrable.*

If the process x_s is confined to a set $Q_r \supset S$ with probability one, then the V_i are uniformly integrable, since they are uniformly bounded. This boundedness is guaranteed if $S(x) \to 0$ uniformly as $x \to \partial Q_r$ from the interior and $k(x, u) > \varepsilon > 0$ in some neighborhood of ∂Q_r (relative to Q_r) for some $\varepsilon > 0$. To show this we need only observe that the given conditions imply the existence of a function $m(\lambda)$ which is continuous in $Q_r - S$, satisfies (2-9), and tends to infinity as $\lambda \uparrow r$.

Thus $F(\lambda)/\lambda$ (defined by (2-10)) tends to infinity as $\lambda \to r$, and finally, this implies that

$$P_x\{\sup_{\tau_u > t \geqq 0} V(x_\tau) > r\} = 0.$$

A PARTICULAR FORM FOR $F(V(x))$

For Itô processes, it is convenient to use the particular form

$$F(\lambda) = \lambda \log(A + \lambda) \tag{2-12}$$

for a suitably large real number A. Then

$$\begin{aligned}\mathscr{L}^u F(V(x)) = &-\left[\log(A + V(x)) + \frac{V(x)}{V(x) + A}\right] k(x, u) \\ &+ \frac{[2 - V(x)/(A + V(x)]}{2(A + V(x))} [V_x'(x)\, S(x)\, V_x(x)].\end{aligned} \tag{2-13}$$

AN OPTIMALITY THEOREM

The cost corresponding to control u (of the statement of Theorem 2) may be compared to the costs corresponding to other controls by the following considerations. Let w be a control which takes the initial condition x to S with probability one, for each $x \in E - S$. Let $V(x)$ satisfy the conditions of Theorem 2 and suppose that $V(x)$ is also in the domain of $\tilde{A}^w_{m,b}$ for each $m > 0$ and that

$$\tilde{A}^w_{m,b} V(x) \geqq -k(x, w). \tag{2-14}$$

Suppose that $\tau_w < \infty$ with probability one, and that, with control w, there is a sequence τ_i of Markov times satisfying the conditions of Theorem 2 (but converging to τ_w). Then, by Dynkin's formula, for each i,

$$V(x) \leqq E^w_x V(x_{\tau_i}) + E^w_x \int_0^{\tau_i} k(x_s, w_s)\, ds.$$

By taking limits, we obtain

$$V(x) \leqq C^w(x) + \delta(x),$$

where

$$C^w(x) = E_x^w b(x_{\tau_w}) + E_x^w \int_0^{\tau_w} k(x_s, w_s)\, ds$$

$$\delta^w(x) = \lim_{i\to\infty} E_x^w V(x_{\tau_i}) - E_x^w V(x_{\tau_w}) \geqq 0.$$

In order to compare the controls u and w, we need to evaluate $\delta^w(x)$. If $\delta^w(x) = 0$, then $C^u(x) = V(x) \leqq C^w(x)$. In general, u is optimal (in the sense of minimizing the cost) with respect to at least all controls w for which $\tilde{A}^w_{m,b} V(x) \geqq -k(x, w)$, and $\delta^w(x) = 0$.

In the deterministic problem, if $V(x)$ is continuous and if a control w transfers the initial point x to a finite point on ∂S, then $V(x_t) \to V(x_{\tau_w}) = b(x_{\tau_w})$. In the stochastic problem, if w transfers the initial point x to S in finite time and $V(x)$ is continuous and $E - S$ is *bounded*, then $V(x_t) \to V(x_{\tau_w}) = b(x_{\tau_w})$ with probability one, and $E_x^w V(x_{\tau_i}) \to E_x^w b(x_{\tau_w})$. Some of the most common models taken for continuous time control processes do not have a bounded state space. Then, although we may have $x_t \to S$ and $V(x_t) \to V(x_{\tau_w})$, both with probability one as $t \to \tau_w < \infty$, we may not have $E_x^w V(x_{\tau_i}) \to E_x^w V(x_{\tau_w})$, as the example following Theorem 2 showed. Thus, in order to compare $C^u(x)$ and $C^w(x)$ in this case, some further constraints on the effects of the comparison control w are required. Theorem 4 sums up this discussion.

Theorem 4. Let both the controls u and w transfer the initial condition x to the set S with $\tau_u < \infty$ and $\tau_w < \infty$ with probability one. Let the nonnegative continuous function $V(x)$ take the value $b(x)$ in S, and be in the domain of both $\tilde{A}^u_{m,b}$ and $\tilde{A}^w_{m,b}$ for each $m > 0$. Suppose that

$$\tilde{A}^u_{m,b} V(x) = -k(x, u) \tag{2-14a}$$

$$\tilde{A}^w_{m,b} V(x) = -k_w(x, w) \geqq -k(x, w). \tag{2-14b}$$

Let either $k_w \leqq 0$ or

$$P_x^w\{\sup_{\tau_w > t \geqq 0} V(x_t) \geqq m\} \to 0 \qquad \text{as} \qquad m \to \infty.$$

For the process corresponding to each control, there exists a sequence of Markov times satisfying the condition of Theorem 2. For any such pair of sequences, let either ($v = u$ or w)

$$E_x^v V(x_{\tau_i}) \to E_x^v V(x_{\tau_v}) \tag{2-15}$$

or let the $V(x_{\tau_i})$ be uniformly integrable. Then

$$C^u(x) \leqq C^w(x).$$

Remark. Theorem 4 is a precise statement of the dynamic programming result of Section 1 for a more general process. In Section 1, $\tilde{A}^v_{m,b} = \mathscr{L}^v$ was used. Conditions on $k(x, v)$ which imply that $x_s \to S$ with probability one are discussed as stability results in Chapter II.

Proof. The conditions imply that the appropriate sequences $\{\tau_i\}$ exist. The rest of the proof follows from Theorem 2, and the discussion preceeding the statement of Theorem 4.

Corollary 4-1 follows immediately from Theorem 4, and applies to the case where $C^u(x) = P_x^u\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq \lambda\}$ and the target set S is contained strictly interior to Q_λ; that is, the cost is the probability that x_t will leave Q_λ at least once before absorbtion on S at τ_u.

Corollary 4-1. Let $k(x, u) = 0$ with $V(x) = 0$ on S (and $V(x) = \lambda$ on ∂Q_λ) where S is strictly interior to Q_λ. Let $V(x)$ and the controls u and w satisfy the other conditions of Theorem 4. Then

$$P_x^u\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq \lambda\} \leqq P_x^w\{\sup_{\tau_w > t \geqq 0} V(x_t) \geqq \lambda\}.$$

For future reference in Example 3 we write:

Corollary 4-2. Let u and w be controls and let the corresponding processes be Itô processes, for $t < \tau_u$ and $t < \tau_w$, respectively. Let (2-14) and the conditions preceding it in Theorem 4 be satisfied. Suppose that $A > 0$ and

$$\mathscr{L}^u V(x) \log (A + V(x)) \leqq 0$$
$$\mathscr{L}^w V(x) \log (A + V(x)) \leqq 0 .$$

Then

$$C^u(x) \leqq C^w(x).$$

Proof. The proof follows from Theorems 1 to 4.

Corollary 4-3. Let the set $E - S + \partial S$ be bounded. Let the continuous nonnegative function $V(x)$ satisfy the boundary condition $b(x)$ on S and let $V(x)$ be in the domains of both $\tilde{A}^u_{m,b}$ and $\tilde{A}^w_{m,b}$. Let the controls u and w transfer the initial condition x to S with probability one as $t \to \tau_u$ or τ_w, respectively. Then, if

$$\tilde{A}^u_{m,b} V(x) = -k(x, u) \leqq 0,$$
$$\tilde{A}^w_{m,b} V(x) \geqq -k(x, w),$$

we have

$$C^u(x) \leqq C^w(x).$$

Proof. $V(x)$ is continuous and bounded in the bounded set $E - S + \partial S$. Thus, for both controls u and w, the corresponding processes $V(x_t)$ converge to the appropriate boundary condition ($V(x_{\tau_u})$ or $V(x_{\tau_w})$). Also, for the associated sequences $\{\tau_i\}$, we have both $E^w_x V(x_{\tau_i}) \to E^w_x V(x_{\tau_w})$ and $E^u_x V(x_{\tau_i}) \to E^u_x V(x_{\tau_u})$. The rest of the proof follows from Theorem 1 or 2. Note that, if $b(x) \equiv 0$, the arrival times τ_u and τ_w may take infinite values.

Control over a fixed time interval

If the control is to be exercised over a fixed time interval $(t, T]$ only, the result is essentially a special case of Theorem 2. We now introduce the time variable t explicitly, and list some conventions which provide for the immediate application of Theorem 2. The pair (x_t, t) is considered as a right continuous Markov process, for $t \leqq T$, and the conventions of Chapter I, Section 3, are used to define the weak infinitesimal operator $\tilde{A}_m^u$ acting on the function $V(x, t)$ in the set $Q_m = \{x, t: V(x, t) < m\}$. Define the target set $S_T = S \times [0, T] + \{T\} \times E$. We allow all components of x_t to be only right continuous. Define $\tau_u = T \cap \inf\{t: x_s \in S\}$. The terminal cost is the nonnegative continuous function $b(x, t)$ defined on the target set S_T. Let the initial value $x_t = x \in E - S + \partial S$ be given at time $t < T$. Then, we define the cost corresponding to control u and the interval $(t, T]$ by

$$C^u(x, t) = E_{x,t}^u \int_t^{\tau_u} k(x_s, u_s, s)\, ds + E_{x,t}^u\, b(x_{\tau_u}, \tau_u),$$

where $k(x, u, s) \geqq 0$. In Theorem 5, *we use $\tilde{A}_{m,b}^u$ as the weak infinitesimal operator of the nonhomogeneous process x_s* (with control u) *stopped at τ_u, and acting on functions which may depend on time.*

Theorem 5. Let the killing times for the processes corresponding to controls u and w be no less than T with probability one. Let the nonnegative function $V(x, t)$ be continuous in all its arguments and suppose that it satisfies the boundary condition $V(x, t) = b(x, t) \geqq 0$ on S_T. Assume that, in $(E - S) \times [0, T]$

$$\tilde{A}_{m,b}^u V(x, t) = -k(x, u, t) \leqq 0 \tag{2-16a}$$

for each m and

$$\tilde{A}_{m,b}^w V(x, t) = -k_w(x, w, t) \geqq -k(x, w, t), \tag{2-16b}$$

where either $k_w(x, w, t) \geqq 0$ or

$$P_x^w\{\sup_{\tau_w > t \geqq 0} V(x_t, t) \geqq m\} \to 0 \quad \text{as} \quad m \to \infty. \tag{2-17}$$

To each control u and w, there is a corresponding sequence of nondecreasing Markov times with the properties; $\tau_i \to \tau_v$ with probability one; $E_{x,t}^v \tau_i < \infty$ (where v is either u or w); $\tau_i \leqq \inf\{t: V(x_s, s) \geqq i\}$;

for each ω, there is an integer $i(\omega) < \infty$ so that $\tau_i = \tau_v$ for all $i > i(\omega)$. Finally, suppose that either (2-18) or (2-19) hold:

$$E_{x,t}^{v} V(x_{\tau_i}, \tau_i) \to E_{x,t}^{v} V(x_{\tau_v}, \tau_v) \tag{2-18}$$

the $V(x_{\tau_i}, \tau_i)$ are uniformly integrable for both u and w. (2-19)

Then

$$V(x, t) = C^u(x, t) \leqq C^w(x, t). \tag{2-20}$$

Proof. Let the control be u. By (2-16a), the process $V(x_t, t)$, stopped at τ_u, is bounded with probability one, for $t \leqq \tau_u$; that is,

$$P_x^u\{\sup_{T \geqq \tau_u \geqq s \geqq t} V(x_s, s) \geqq m\} \leqq \frac{V(x, t)}{m},$$

which goes to zero, as $m \to \infty$. Thus there is a sequence $\{\tau_i\}$ satisfying the conditions of the hypothesis (for example, $\tau_i = \min(i, \tau_u, \inf\{t: V(x_t) \geqq i\})$). Let x be in $E - S$ and $t < T$. Dynkin's formula may be applied to τ_i and $V(x, t)$ and yields

$$\begin{aligned} V(x, t) &= E_{x,t}^u V(x_{\tau_i}, \tau_i) - E_{x,t}^u \int_t^{\tau_i} \tilde{A}_i^u V(x_s, s)\, ds \\ &= E_{x,t}^u V(x_{\tau_i}, \tau_i) + E_{x,t}^u \int_t^{\tau_i} k(x_s, u_s, s)\, ds. \end{aligned}$$

The last term on the right converges to

$$E_{x,t}^u \int_t^{\tau_u} k(x_s, u_s, s)\, ds$$

as $i \to \infty$. By (2-18), $V(x, t) = C^u(x, t)$. In any case, since $x_{\tau_i} = x_{\tau_u}$ for some $i < \infty$ with probability one (since $\tau_i = \tau_u$ for some $i < \infty$ with probability one), we have $\lim_{i \to \infty} V(x_{\tau_i}, \tau_i) = V(x_{\tau_u}, \tau_u)$, with probability one. Now (2-19) implies $E_{x,t}^u V(x_{\tau_i}, \tau_i) \to E_{x,t}^u V(x_{\tau_u}, \tau_u)$, and the left side of (2-20) follows.

If the control w satisfies either $k_w(x, w, s) \geqq 0$ in $(E - S) \times [0, T]$ or condition (2-17), then we also infer that $V(x_{\tau_i}, \tau_i) \to V(x_{\tau_w}, \tau_w)$ with probability one. Then (2-19) implies (2-18). Combining (2-18) with Dynkin's formula, we obtain

$$V(x) \leqq \lim_{i\to\infty} E^w_{x,t} V(x_{\tau_i}, \tau_i) + \lim_{i\to\infty} \int_t^{\tau_i} k(x_s, w_s, s)\, ds$$
$$= C^w(x, t)$$

and (2-20) follows.

Strong diffusion process

A stronger result is available when x_t is a strong diffusion process and $E - S$ is bounded. In Theorem 6 the control depends on x only, and time is not one of the components of x.

Theorem 6. Let Q be a bounded open region of class* $H_{2+\alpha}$, and let the process x_t, corresponding to controls $v = u$ or w, be a strong diffusion process in $Q + \partial Q$, with differential generator

$$L^v = \mathscr{L}^v = \sum_i f_i(x, v) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij}(x, v) \frac{\partial^2}{\partial x_i\, \partial x_j}$$

where the control $v(x)$, and the functions $f_i(x, v)$ and $S_{ij}(x, v)$ satisfy a uniform Hölder condition in $Q + \partial Q$. Then each v transfers the initial condition $x \in Q$ to ∂Q in time τ_v with $E^v_x \tau_v < \infty$.

Let the function $V(x)$ have continuous second derivatives at each x in Q, let $V(x) = b(x)$ on ∂Q, where $b(x)$ is continuous, and suppose that

$$\mathscr{L}^u V(x) = -\, k(x, u) \leqq 0$$
$$\mathscr{L}^w V(x) \geqq -\, k(x, w)$$

* See Chapter I, Section 6, for the definition.

at each x in Q, where $k(x, u)$ satisfies a uniform Hölder condition in its arguments in $Q + \partial Q$. Then

$$V(x) = C^u(x) \leqq C^w(x).$$

For any such $k(x, v)$, the equation $\mathscr{L}^v A(x) = -k(x, v)$ has a unique solution satisfying $A(x) = b(x)$ on ∂Q and $A(x)$ has Hölder continuous second derivatives in compact subsets of Q.

Proof. The proof is almost an immediate consequence of the remarks in Chapter I, Section 6. Under the hypothesis on $\mathscr{L}^v$, it is the differential generator of a strong diffusion process and, hence, $E_x^v \tau_v < \infty$. Since, in addition, the process is continuous with probability one, and Q is bounded, the cost $C^u(x)$ is well defined (either by the arguments of Theorem 2 or Section 6 of Chapter I). Also, there is a unique solution to $\mathscr{L}^u A(x) = -k(x, u)$ with $A(x) = b(x)$ on ∂Q. By Section 6 of Chapter I, this solution must be $C^u(x)$; also, by the uniqueness and the hypothesis on $V(x)$, $V(x) = C^u(x)$.

The same reasoning yields a unique solution to $\mathscr{L}^w W(x) = -k(x, w)$, with boundary condition $W(x) = b(x)$ on ∂Q. Also, $W(x) = C^w(x)$. By the hypothesis,

$$\mathscr{L}^w U(x) \leqq 0$$

where we define

$$U(x) = C^w(x) - C^u(x)$$

in Q and $U(x) = 0$ on ∂Q. Finally, the strong maximum principle yields $C^w(x) \geqq C^u(x)$.

Theorem 7 is the analog of Theorem 5 for strong diffusion process.

Theorem 7. Let Q be a bounded open set of class $H_{2+\alpha}$ and x_t a strong diffusion process in Q, for $0 \leqq t \leqq T$, with differential generator

$$\mathscr{L}^v = \sum_i f_i(x, v, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij}(x, v, t) \frac{\partial^2}{\partial x_i \, \partial x_j} + \frac{\partial}{\partial t}.$$

Let each considered control v be of the form $v(x, t)$. For any vector ξ let

$$\sum_{i,j} S_{ij}(x, v, t)\, \xi_i \xi_j \geqq \mu \|\xi\|^2 \qquad (\mu > 0)$$

in $D + \partial D$, where $D = Q \times [0, T]$. In $D + \partial D$, let $f_i(x, v, t)$, $v_i(x, t)$, $S_{ij}(x, v, t)$, and $k(x, v, t)$ be continuous and satisfy uniform Hölder conditions in x and v. Suppose that the function $V(x, t)$ has continuous second partial derivatives in the components of x, and a continuous derivative with respect to t, and $V(x, t)$ equals the continuous function $b(x, t)$ on $Q \times \{T\} + \partial Q \times [0, T]$, and that

$$\mathscr{L}^u V(x, t) = -\, k(x, u, t)$$
$$\mathscr{L}^w V(x, t) \geqq -\, k(x, w, t),$$

where both u and w satisfy the conditions on v. Then

$$C^w(x, t) \geqq C^u(x, t).$$

Proof. Except that the corresponding theorems for parabolic equations are required, the proof is essentially the same as that of Theorem 6, and will not be given.

General nonanticipative comparison controls

The control u of Theorem 4 or 5, whose values at t depend on either x_t or x_t and t, is optimal with respect to a class of comparison controls whose values at t depend only on either x_t or x_t and t. When the controlled processes are the solutions of stochastic differential equations of the Itô type, it is possible to widen the class of comparison controls to include a family of nonanticipative functions.* We suppose that

$$dy = f(y, w, t)\, dt + \sigma(y, w, t)\, dz$$

* The possibility of the extension was pointed out to the author by W. H. Fleming. See also Fleming [2].

where the control w takes values in a set U and is nonanticipative, and also that the possible w are further restricted so that the solution process x_t is defined in the time interval of interest.

Let $\mathscr{B}_t$ be the least σ-algebra over which z_s, $s \leqq t$, is measurable. Then w_t is measurable over $\mathscr{B}_t$. Let y_t be the process corresponding to control w_t, and x_t to u_t (whose values depend on x_t and t only). y may not be a Markov process. The cost corresponding to control w in the interval $[t, \tau_w]$ is the random variable

$$C^w(\mathscr{B}_t, t) = E^{\mathscr{B}_t} b(y_{\tau_w}, \tau_w) + E^{\mathscr{B}_t} \int_t^{\tau_w} k(y_s, w_s, s)\, ds\,,$$

where τ_w is a nonanticipative random variable which is the terminal time of control (as defined in Theorem 4 or 5). The cost is a random variable simply because the value of the control at $s \geqq t$ may depend on values of y_s for $s < t$.

Theorem 8. Suppose that the Itô process corresponding to the continuous and locally Lipschitz control $u(x)$ or $u(x, t)$ satisfies the conditions of Theorem 4 or 5 (depending on whether or not the terminal time T is fixed) and $\mathscr{L}^u V(x, t) = -k(x, u, t) \leqq 0$ in $E - S$ and $V(x, t) = C^u(x, t)$. For any other number v in U and any x in $E - S$, let $\mathscr{L}^v V(x, t) \geqq -k(x, v, t)$. Let there be a sequence of nonanticipative random variables $\{\tau_i\}$ tending to τ_w with probability one and satisfying

$$E^{\mathscr{B}_t} V(y_{\tau_i}, \tau_i) \to E^{\mathscr{B}_t} V(y_{\tau_w}, \tau_w) \tag{2-21}$$

with probability one and

$$E\left|\int_t^{\tau_i} V_y'(y_\rho, \rho)\, \sigma(y_\rho, w_\rho, \rho)\, dz_\rho\right|^2 < \infty\,. \tag{2-22}$$

Let $\sup_{\tau_i \geqq t \geqq 0} V(y_t, t) < \infty$ with probability one. Then

$$V(x, t) = C^u(x, t) \leqq C^w(\mathscr{B}_t, t)$$

with probability one.

Proof. The inequalities and equalities are understood to hold only with probability one. Itô's lemma (Chapter I, Section 4) is applicable to $V(y, t)$ and τ_i and

$$V(y_{\tau_i}, \tau_i) - V(y_t, t) = \int_t^{\tau_i} \mathscr{L}^w V(y_\rho, \rho)\, d\rho + \int_t^{\tau_i} V_y'(y_\rho, \rho)\, \sigma(y_\rho, w_\rho, \rho)\, dz_\rho. \tag{2-23}$$

Since $\mathscr{L}^v V(x, t) \geqq -k(x, v, t)$ for all $v \in U$ and x in $E - S$, (2-23) and (2-22) yield

$$E^{\mathscr{B}_t} V(y_t, t) \leqq E^{\mathscr{B}_t} V(y_{\tau_i}, \tau_i) + E^{\mathscr{B}_t} \int_t^{\tau_i} k(y_\rho, w_\rho, \rho)\, d\rho.$$

By (2-21),

$$V(y_t, t) = E^{\mathscr{B}_t} V(y_t, t) \leqq C^w(\mathscr{B}_t, t).$$

Since $C^u(x, t) = V(x, t)$ by the hypothesis, we have for any random variable y_t,

$$C^u(y_t, t) \leqq C^w(\mathscr{B}_t, t)$$

and the theorem is proved.

"Practical optimality": Compactifying the state space

Commonly, the descriptions of the process x_s at large $\|x\|$ are idealizations which provide for a relatively simple or tractable mathematical model. Often, the behavior of the process model at large $\|x\|$ will be grossly different from the behavior of the "physical" process at large $\|x\|$. For example, the paths of the "physical" process may be uniformly bounded, whereas those of the model may not be (for example, if the model is "Gaussian"). In such cases, the model (which is equivalent to $\tilde{A}^u$) and the assigned cost $k(x, u)$ cannot be taken too seriously for large $\|x\|$. Recall also that the possible behavior of $V(x_t)$ for large $\|x\|$ demanded, in the theorems, the introduction of technical conditions which would ensure that $E_x^u V(x_{\tau_i}) \to E_x^u V(x_{\tau_u})$. These considerations suggest that a modification of the problem statement allowing for some freedom in the choice of the process for large $\|x\|$ would be useful to simplify the statements of the theorems, and to simplify the checking of the "optimality" of any given control.

Let us modify the problem in the following way. Define the rate of cost to be $\bar{k}(x, u)$. Let u be a fixed control transferring x to S at $\tau_u < \infty$ with probability one. Assume that there is a finite region of the state space, R, which incorporates part of S and such that $P_x^u\{x_s \notin R$, some $s < \tau_u\} < \delta$, where $x = x_0$ in R and δ is given and is small. (Naturally $R - S$ is not to be empty.) The discussion of the preceding paragraph suggests that if $x_0 = x$ is in R, then the behavior, outside R, of the process x_t corresponding to the model, should not be important and should be changeable if a change would be helpful in studying the "optimality" properties of a given control.

Denote the weak infinitesimal operator of x_s stopped on first entrance into S by $\tilde{A}_b^u$. *Let* $k(x, u) = \bar{k}(x, u)$ in R; then $\bar{k}(x, u)$ in $E - R$ may be altered in any way consistent with $\bar{k}(x, u) \leqq 0$ (which the theorems require*). Let $R_1 \supset R$ be a compact set, and modify $\tilde{A}_b^u$ in $R_1 - R$ so that the process x_s (with control u) never leaves R_1. (For example, for Itô processes, let the $\sigma_{ij} \to 0$ as $x \to \partial R_1$ and alter the $f_i(x, u)$ in $R_1 - R$ so that the paths of the resulting system never leave R_1 and return to R with probability one, if they ever leave R. In, general, let us alter the process so that, with an arbitrary control, it is deterministic in $E - R_1$.

Now consider the altered control problem. Denote the alteration of $\tilde{A}_b^u$ by $\bar{A}_b^u$. Let $V(x)$ be in the domain of $\bar{A}_b^u$. We still have $\bar{A}^u V(x) = -k(x, u)$ in R. Now *define* the function $k(x, u) \geqq 0$ *in* R_1 *by the expression* $\bar{A}_b^u V(x) = -k(x, u)$. Then $E_x^u V(x_{\tau_i}) \to E_x^u b(x_{\tau_u})$ and

$$C^u(x) = V(x) = E_x^u b(x_{\tau_u}) + E_x^u \int_0^{\tau_u} k(x_s, u_s)\, ds.$$

* Theorems 6 and 7 do not require $k \leqq 0$, but there the space was bounded for all mathematical purposes.

Let w be any control transferring x to S with probability one, let $V(x)$ be in the domain of $\bar{A}_b^w$, and let

$$\bar{A}_b^w V(x) \geqq \bar{A}_b^u V(x).$$

Then, by Theorem 2, and noting that the paths of x_s, corresponding to control w, are bounded (since $x_s \to S$ with probability one and x_s is deterministic and bounded in $E - R_1$) we have

$$C^u(x) = V(x) \leqq C^w(x).$$

For the altered process, the control u is optimal with respect to w. If an arbitrary nonzero δ yields a compact R, and if there is a process alteration so that (for the controls u, w) the paths do not leave R_1 and satisfy the other conditions above, then we say that u is "practically optimum" with respect to w.

Let $V(x) \to \infty$ and $\|x\| \to \infty$, and suppose that $V(x)$ is in the domain of $\tilde{A}_{m,b}^u$ and $\tilde{A}_{m,b}^u V(x) = -k(x,u) \leqq 0$, all $m > 0$. Suppose that the initial value x_0 is such that $V(x_0) \leqq V(\hat{x})$. Then, since

$$P_x^u \{ \sup_{\tau_u \geqq t \geqq 0} V(x_t) \geqq m \} \leqq \frac{V(x)}{m},$$

a candidate for the region R is

$$R = \left\{ y : V(y) \leqq \frac{V(\hat{x})}{\delta} \right\}.$$

Note that R is compact, since $V(x) \to \infty$ as $\|x\| \to \infty$.

3. Examples

Example 1. In the first example, we prove that the usual solution (as, for example, given in Florentin [1]) of the fixed time of control, linear system, and quadratic loss problem for the Itô equation is indeed an optimal control, and we exhibit a class of controls with respect to which it is optimal. Let the loss and system be represented by

$$
\begin{aligned}
k(x, u, t) &= x'C(t)\, x + u'D(t)\, u \\
b(x) &= x'Mx \\
dx &= A(t)\, x\, dt + B(t)\, u\, dt + \sigma(t)\, dz
\end{aligned} \tag{3-1}
$$

where $D(t)$ satisfies $\xi' D(t)\xi \geqq \beta \|\xi\|^2$ for all ξ and all t in $[0, T]$ where $\beta > 0$. The control interval is $[0, T]$. $C(t)$ and M are symmetric and nonnegative definite. The target set is the entire space. First, we obtain the usual result formally by the use of dynamic programming.

A standard application of the dynamic programming result of Section 1 yields that the optimal control is the control which produces the minimum in (3-2):

$$
\begin{aligned}
0 = \min_v \{ & V_t(x, t) + V_x'(x, t)\,(Ax + Bv) + x'Cx + v'Dv \\
& + \tfrac{1}{2} \sum_{i,j} V_{x_i x_j}(x, t)\, S_{ij} \},
\end{aligned} \tag{3-2}
$$

where $V(x, t)$ is the minimum cost. Denote the optimal control by u. Then, formally,

$$
\begin{aligned}
V(x, t) &= C^u(x, t) \\
&= \min_v E_{x,t} \{ b(x_T) + \int_0^T k(x_t, v_t, t)\, dt \}.
\end{aligned}
$$

From (3-2),

$$
u(x, t) = - \frac{D^{-1}(t)\, B'(t)\, V_x(x, t)}{2}. \tag{3-3}
$$

Equation (3-2) is to be solved by assuming a solution of the form

$$
V(x, t) = x'P(t)\, x + Q(t),
$$

where $P(t)$ is nonnegative definite and symmetric and $Q(t) \geqq 0$:

$$
\begin{aligned}
V_x(x, t) &= 2P(t)\, x \\
u(x, t) &= - D^{-1}(t)\, B(t)\, P(t)\, x.
\end{aligned}
$$

Upon substituting $V(x, t)$ and $u(x, t)$ into (3-2) and collecting coefficients of like powers of x, we obtain that the satisfaction of ($p_{ij}(t)$ are the elements of $P(t)$)

$$\dot{P} + A'P + PA + C - P(BD^{-1}B')P = 0, \qquad P(T) = M \tag{3-4}$$

$$\dot{Q} + \sum_{i,j} p_{ij}S_{ij} = 0, \qquad Q(T) = 0. \tag{3-5}$$

is (at least formally) a sufficient condition for the optimality of (3-3) and minimality of $V(x, t)$. Equation (3-5) is not important since the control depends only on P. Note that if P is nonnegative, then $\Sigma p_{ij}S_{ij} = \text{trace } PS \geqq 0$ and, hence $Q(t) \geqq 0$ for $0 \leqq t \leqq T (Q(T) = 0)$. Equation (3-4) is known as the matrix Ricatti equation. Since $D(t)$ is positive definite and $C(t)$ and M semidefinite, there is a nonnegative definite solution to the Ricatti equation in the interval $[0, T]$ which satisfies $P(t) \to M$ as $t \to T$. Since $u(x, t)$ is continuous in t and satisfies a uniform Lipschitz condition in x, the process (3-1) corresponding to $u(x, t)$ is defined and finite with probability one in the time interval $[0, \infty)$ and enjoys the property

$$E_{x,0}^{u} \sup_{0 \leqq s \leqq T} x_s' x_s < \infty . \tag{3-6}$$

By construction $\mathscr{L}^u V(x, t) = -k(x, u, t)$, and $V(x, t)$ is in the domain of $\tilde{A}_m^u = \mathscr{L}^u$ for each $m > 0$. It will be verified that

$$E_{x,t}^{u}(x_s' P(s)\, x_s) \to E_{x,t}^{u}(x_T' M x_T)$$

for any sequence of *Markov times* s which tend monotonically to T. But, by the property (3-6) and the dominated convergence theorem, $E_{x,t}^{u}(x_s'(P(s) - M)\, x_s) \to 0$ as $t \leqq s \to T$; by (3-6) and the fact that $P_{x,t}^{u}\{\sup_{0 \geqq \delta \geqq \tau} \| x_{r+\delta} - x_r \| \geqq \varepsilon > 0\} \to 0$ as $\tau \to 0$ $(r \geqq t)$, we have $E_{x,t}^{u}(x_s - x_T)'\, M(x_s - x_T) \to 0$ as the Markov times $s \to T$. Thus, by Theorem 5, $C^u(x, t) = V(x, t)$.

Let w be another control, which is continuous in t and uniformly Lipschitz continuous in x. Then (3-6) holds for w, and $V(x, t)$ is in the domain of $\tilde{A}_m^w = \mathscr{L}^w$. The proof that $E_{x,t}^{w}\, V(x_s, s) \to E_{x,t}^{w}\, V(x_T, T)$ for any sequence of nondecreasing Markov times s tending to T is similar to the proof in the case of u. By virtue of the minimization in

(3-2),

$$\mathscr{L}^{w}V(x, t) \geqq \mathscr{L}^{u}V(x, t).$$

Thus, by Theorem 5, we conclude that

$$C^{u}(x, t) = V(x, t) \leqq C^{w}(x, t)$$

for any such control w, and u is optimal.

The case where $\sigma(t)$ is a linear function of x may be treated in exactly the same way, and the dynamic programming solution is indeed an optimal control with respect to the same class of comparison controls w. Finally, by appealing to Theorem 8, we note that $u(x, t)$ is also optimal with respect to any bounded nonanticipative control. Also, the result of Example 1 is slightly easier to prove by a direct application of Itô's lemma, as in Theorem 8.

Example 2. A short treatment of the combined linear filtering and control problem with a quadratic rate of loss will be given. Example 2 is more difficult than Example 1. The added difficulty is due to the fact that the control driving (3-7) is a functional of the observations on (3-7). It must be proved that the control at time t is actually a function of the expectation of x_t conditioned upon the observations up to t. The analysis is possible owing to the linearity of the system.

The system to be controlled is represented by

$$dx = A(t)\,x\,dt + B(t)\,u\,dt + W(t)\,dz, \tag{3-7}$$

where the initial condition $x_0 = x$ is a normally distributed random variable. We suppose that the available observations can be represented by the function y_t:

$$dy = H(t)\,x\,dt + W_1(t)\,dz_1, \tag{3-8a}$$

or

$$y_t = \int^{t} \left(H(\tau)\,d\tau + W_1(\tau)\,dz_{1\tau}\right), \tag{3-8b}$$

where $W_1' W_1$ is a positive definite matrix for each t in the interval of interest. In more intuitive terms equations (3-8a) and (3-8b) imply that the "function" $q(t) = dy/dt = H(t)\,x_t + W_1(t)\,\xi_t$ is observed, where ξ_t is white Gaussian noise. Mathematically, it is only the integrated "observation," namely (3-8b), which can be treated.

Let $W\,dz = W_2\,dz_1 + W_3\,dz_2$, where z_{1t} and z_{2t} are independent Wiener processes. Define $m_t \equiv E[x_t \mid y_s, s \leqq t]$ (the expectation conditioned upon the minimum σ-algebra over which y_s, $s \leqq t$, is measureable). The development requires that, until further notice, we set the control u equal to zero in (3-7). When $u = 0$, we denote the corresponding processes by $\hat{x}_t$, $\hat{y}_t$, and $\hat{m}_t$.

It is known (Kalman and Bucy [1]) that a version of the conditional expectation $\hat{m}_t$ satisfies the equation

$$\begin{aligned} d\hat{m} &= A\hat{m}\,dt + K\,dl \\ dl &\equiv d\hat{y} - H\hat{m}\,dt = H(\hat{x} - \hat{m})\,dt + W_1\,dz_1 \\ K &= [\Sigma H' + W_2 W_1 W_2'][W_1 W_1']^{-1} \\ \dot{\Sigma} &= A\Sigma + \Sigma A' + [\Sigma H' + W_2 W_1 W_2'](W_1 W_1')^{-1}[H\Sigma + W_2 W_1' W_2'] \\ &\quad + WW'. \end{aligned} \tag{3-9}$$

$\Sigma(t)$ is, of course, the covariance of $(m_t - x_t)$, and its initial value $\Sigma(t_0)$ is a given positive definite matrix. Actually, the derivation in Kalman and Bucy [1] is still formal. The form (3-9) can be established as a special case of a result which describes the differential equations satisfied by a version of the conditional expectation of functions of the solution of Itô equations. Let x_t be the solution to an Itô equation and let the observations be $y_s = \int^s g(x_\tau, \tau)\,d\tau + \int^s W_1\,dz_{1\tau}$; then the results of Kushner [11] give conditions on x_s and on the possibly nonlinear functions $g(x, t)$ and $h(x)$, under which a version of $E[h(x_t) \mid y_s, s \leqq t]$ (considered as a stochastic process) satisfies a stochastic differential equation of the Itô type. From the equations for the case where $g(x, t)$ is linear in x, and $h(x)$ is linear and quadratic in the components of x, the form (3-9) can be derived. See also Kushner [1].

Let $\hat{B}_t$ and B_t be the minimal σ-algebras over which $\hat{y}_s$, and $\hat{y}_s - \int^s H(\tau)\,\hat{m}_\tau\,d\tau \equiv l_s$, $s \leqq t$, are measurable, respectively. Since $\hat{m}_\tau$ is measurable over $\hat{B}_t$, $\tau \leqq t$, B_t is contained in $\hat{B}_t$ and any function which is measurable over B_t is also measurable over $\hat{B}_t$. The process l_s (that is, the solution of the stochastic differential equation $dl = d\hat{y} - H\hat{m}\,dt = H(\hat{x} - \hat{m})\,dt + W_1\,dz_1$) is a vector-valued Wiener process (the process l_s is unnormalized; its covariance is not t times the identity). This is easily verified by noting that it is a Gaussian process with orthogonal (hence independent) and infinitely divisible increments and $E(l_t - l_s)(l_t - l_s)' = \int_s^t W_1'(\tau)\,W_1(\tau)\,d\tau$. The details are left to the reader. (See remark at the end of the example.) From this, it follows that $\hat{m}_t$ is actually the solution to a linear Itô equation (3-9).

Now define $\bar{m}_t$ by the "feedback" system (3-10) or (3-11). We suppose that the control is of the form $u_t = u(\bar{m}_t, t)$, where $u(\bar{m}, t)$ is continuous and satisfies a uniform Lipschitz condition in $\bar{m}$. Let $\Phi(t, s)$ be the fundamental matrix of $\dot{x} = Ax$ and let the initial condition for (3-11) be given at time s and suppose that $\bar{m}_s = \hat{m}_s$. (In other words, the initial condition is the initial value of the conditional expectation of x_s.) Thus,

$$\bar{m}_t = \hat{m}_t + U_{s,t}, \qquad U_{s,t} = \int_s^t \Phi(t, \tau)\,B(\tau)\,u(m_\tau, \tau)\,d\tau \tag{3-10}$$

or

$$\bar{m}_t = \Phi(t, s)\,\hat{m}_s + \int_s^t \Phi(t, \tau)\,K(\tau)\,dl_\tau + \int_s^t \Phi(t, \tau)\,B(\tau)\,u(\bar{m}_\tau, \tau)\,d\tau.$$

Equation (3-10) is equivalent to the Itô equation

$$d\bar{m} = A\bar{m}\,dt + Bu(\bar{m}, t)\,dt + K\,dl \tag{3-11}$$

and has a unique solution with probability one in the interval $[s, \infty)$, with $\bar{m}_s = \hat{m}_s$ given.

Let us add the same control $u(\bar{m}, t)$ to (3-7). Then

$$x_t = \hat{x}_t + U_{s,t}$$

or

$$dx_t = Ax\,dt + Bu(\bar{m}, t)\,dt + W\,dz. \tag{3-12}$$

We will show first that $\bar{m}_t = m_t = E[x_t \mid y_s, s \leqq t]$ with probability one.

First, we show $\bar{m}_t = E[x_t \mid \hat{B}_t]$. Let $P(d\omega)$ be the probability measure.

Since $\bar{m}_t$ is measurable over B_t (it is a functional of $l_s, s \leqq t$), it is measurable over $\hat{B}_t$. Also, let Λ be any set in $\hat{B}_t$. Then the definition $\hat{m}_t = E[\hat{x}_t | \hat{B}_t]$ and evaluation

$$\int_\Lambda \bar{m}_t P(d\omega) = \int_\Lambda \hat{m}_t P(d\omega) + \int_\Lambda U_{s,t} P(d\omega) = \int_\Lambda \hat{x}_t P(d\omega) + \int_\Lambda U_{s,t} P(d\omega)$$
$$= \int_\Lambda x_t P(d\omega)$$

yields $\bar{m}_t = E[x_t | \hat{B}_t]$ with probability one, that is, $\bar{m}_t$ is a version of the conditional expectation of x_t given the observations on the *uncontrolled process* $\hat{x}_s, s \leqq t$.

Although the control is a random process, the actual value that a particular realization assumes at time t is known. Since $d\hat{y} - H\hat{m}\, dt = dy - H\bar{m}\, dt$,

$$\begin{aligned} d\bar{m} &= [A\bar{m} + Bu(\bar{m}, t)]\, dt + K(d\hat{y} - H\hat{m}\, dt) \\ &= [A\bar{m} + Bu(\bar{m}, t)]\, dt + K(dy - H\bar{m}\, dt). \end{aligned} \tag{3-13}$$

Thus $\bar{m}_t$ and, hence, the control, is computed from the observations on $x_s, s \leqq t$. The observations on the controlled and uncontrolled process differ by

$$y_t - \hat{y}_t = \int_0^t H(\tau) \int_0^\tau \Phi(\tau, s)\, B(s)\, u(\bar{m}_s, s)\, ds\, d\tau. \tag{3-14}$$

Equations (3-13) and (3-14) imply that the observations on the controlled and uncontrolled process are nonanticipative functionals of one another. Hence $\bar{m}_t = E[x_t | \hat{y}_s, s \leqq t] = E[x_t | y_s, s \leqq t]$.

Let the realization of the conditional mean at time t be given as the number $m_t = m$. Then the cost, defined by

$$C^u(m, t) = E^u_{m,t} x'_T M x_T + E^u_{m,t} \int_t^T (x'_s\, C(s)\, x_s + u'_s D(s)\, u_s)\, ds$$

is truly a function of m. The terms of the integrand which contain x_s

can be replaced by terms containing only m_s. We will show this to be true for the right-hand term only. We have

$$x_s'C(s)\,x_s = m_s'C(s)\,m_s + (x_s - m_s)'\,C(s)\,(x_s - m_s) + 2(x_s - m_s)\,C(s)\,m_s$$

and

$$E_{m,t}^u \int_t^T x_s'\,C(s)\,x_s\,ds = \int_t^T E_{m,t}^u x_s'\,C(s)\,x_s\,ds\,. \tag{3-15}$$

Noting that $(x_s - m_s)$ and m_s are independent Gaussian random variables, and using the above equalities, we obtain that (3-15) is equal to

$$E_{m,t}^u \int_t^T (m_s'\,C(s)\,m_s\,ds + q(s))\,ds$$

where $q(s)$ is a function of s only; it depends on the covariance of $(x_s - m_s)$ but does not depend on either m_s or u_s. We may write $C^u(m,t)$ as (modulo some known function of time)

$$C_1^u(m,t) = E_{m,t}^u \int_t^T (m_s'\,C(s)\,m_s + u_s'D(s)\,u_s)\,ds + E_{m,t}^u m_T' M m_T\,. \tag{3-16}$$

We may write (3-13) as

$$\begin{aligned} dm &= [Am + Bu(m,t)]\,dt + K\,dl \\ dm &= [Am + Bu(m,t)]\,dt + KW_1\,db \end{aligned} \tag{3-17}$$

where b_s is a Wiener process. Finally, the optimization problem is equivalent to the minimization of (3-16) subject to the constraint (3-17). The remainder of the procedure is exactly as for Example 1. The optimal control is (3-3) with m replacing x. Again, a family of comparison controls is the family of continuous $u(m,t)$ satisfying a uniform Lipschitz condition in m. The family may be extended to include a class of local Lipschitz controls and bounded nonanticipative controls.

Remark. In order to prove that $v_t - v_0 \equiv \int_0^t K_s\, dl_s$ is a (unnormalized) Wiener process, we need only show that it is a (vector-valued) Gaussian martingale. The Gaussian property is obvious. Since

$$v_t - v_0 = \hat{m}_t - \hat{m}_0 - \int_0^t A\hat{m}_\tau\, d\tau,$$

$v_t - v_0$ is measurable over the minimum σ-algebra determined by conditions on the $\hat{m}_\tau$, $0 \leqq \tau \leqq t$, the martingale property then follows from the evaluation

$$\begin{aligned} &E\{v_t - v_s | \hat{m}_r, r \leqq s\} \\ &\quad = E\{\int_s^t [KH(\hat{x}_\tau - \hat{m}_\tau)\, d\tau + KW_1\, dz_\tau] | \hat{m}_r, r \leqq s\} = 0 \end{aligned}$$

with probability one. The last equality follows from the independence of the Gaussian variables $\hat{x}_\tau - \hat{m}_\tau$ and $\hat{m}_r$, $r \leqq \tau$.

The process v_t has orthogonal, hence independent, increments, and is therefore a Wiener process. Furthermore, the evaluation

$$E(\int_t^{t+h} K_s\, dl_s)(\int_t^{t+h} K_s\, dl_s)' = \int_t^{t+h} K_s W_{1s} W_{1s}' K_s'\, ds + o(h)$$

implies that there is a normalized Brownian motion b_t such that

$$\int_0^t K_s\, dl_s = \int_0^t K_s W_{1s}\, db_s.$$

Example 3. We consider a stochastic version of the so-called "norm invariant" problem (Athans and Falb [1]). Let the target set be $S = \{x : \|x\| \leqq r\}$ and let the process be

$$dx = f(x)\, dt + u\, dt + \sigma I\, dz \tag{3-18}$$

where σ is a constant and I is the identity matrix. It is supposed that $x'f(x) = 0$ (which implies that the uncontrolled system is conservative if $\sigma = 0$). The control values are confined to a sphere of radius ρ; that is, $u'u \leqq \rho^2$. We seek the control which minimizes the *average* time required to transfer $x_0 = x$ to ∂S; thus $k(x, u) = 1$ and $b(x) = 0$.

It will be shown that the uniformly Lipschitz control (3-19) is an optimal control:

$$u = -\frac{\rho x}{\|x\|}. \tag{3-19}$$

Equation (3-19) is also the optimal control for the corresponding deterministic problem (when $\sigma = 0$). Owing to the symmetry of the system (3-18), this result is not surprising.

First we will use (3-19) and find a function $V(x)$ which satisfies $V(x) > 0$ and $\mathscr{L}^u V(x) = -1$ for $\|x\| > r$, and $V(x) = 0$ for $\|x\| = r$. Then the optimality will be proved.

The symmetry of (3-18) suggests that $C^u(x)$ is of the form $C^u(x) = g(\|x\|)$ for some monotonically increasing function $g(w)$.

Let $w = \|x\|$. Then, supposing that the candidate $V(x)$ is of the form $g(\|x\|)$, the relation

$$\frac{\partial^2 V(x)}{\partial x_i^2} = \frac{\partial g(w)}{\partial w}\left(\frac{1}{w} - \frac{x_i^2}{w^3}\right) + \frac{\partial g(w)}{\partial w^2}\left(\frac{x_i^2}{w^2}\right)$$

implies that

$$\begin{aligned}
\mathscr{L}^u V(x) &= \frac{\partial g(w)}{\partial w}\left(\frac{x'f(x)}{w} + \frac{u'x}{w}\right) \\
&\quad + \frac{\sigma^2}{2}\sum_i\left[\frac{\partial g(w)}{\partial w}\left(\frac{1}{w} - \frac{x_i^2}{w^3}\right) + \frac{\partial^2 g(w)}{\partial w^2}\frac{x_i^2}{w^2}\right] \\
&= \frac{\partial g(w)}{\partial w}\left(-\rho + \frac{(n-1)\sigma^2}{2w}\right) + \frac{\partial^2 g(w)}{\partial w^2}\frac{\sigma^2}{2} = -1,
\end{aligned} \tag{3-20}$$

where n is the dimension of the vector x. Equation (3-20) admits of a solution of the form

$$g(w) = A_0 + A_1 w + A_2 \log w + \sum_1^\infty \frac{B_i}{w^i}. \tag{3-21}$$

Substituting (3-21) into (3-20), and equating coefficients of like terms in w on each side of (3-20) we obtain the coefficients A_i and B_i:

$$A_1 = \frac{1}{\rho}, \qquad A_2 = (n-1)\frac{\sigma^2}{2\rho^2},$$
$$B_1 = \frac{-(n-2)(n-1)}{\rho}\left(\frac{\sigma^2}{2\rho}\right)^2 \leqq 0, \tag{3-22}$$
$$B_{d+1} = \frac{\sigma^2 n B_n (n-d-2)}{2\rho(d+1)} \leqq 0.$$

Since the set (3-22) contains only n nonzero terms,

$$V(x) = g(\|x\|) = A_0 + A_1\|x\| + A_2 \log\|x\| + \sum_1^{n-2} \frac{B_i}{\|x\|^i} \tag{3-23}$$

where the empty sum defined by $\sum_1^0 B_i/w^i = 0$. The value of A_0 is obtained from the boundary condition

$$g(r) = 0 = A_0 + A_1 r + A_2 \log r + \sum_1^{n-2} \frac{B_i}{r^i}.$$

Since $A_i \geqq 0$ and $B_i \leqq 0$, we have $g(w) > 0$, for $w > r$.

The optimality of (3-19) and minimality of (3-23) will now be verified by the use of Theorems 1 and 3. By construction, $V(x) > 0$ and $\mathscr{L}^u V(x) = -1$ in $E - S$, and $V(x) = 0$ on ∂S. We may define $V(x) = 0$ on S. $V(x)$ is in the domain of $\tilde{A}_{m,b}$, for each $m > 0$ and $\tilde{A}_{m,b} V(x) = \mathscr{L}^u V(x)$. $\mathscr{L}^u V(x) = -1$ implies that $x_s \to \partial S$ with probability one in a finite average time for x in $E - S$. Finally, by the use of Theorem 3 and $F(V(x)) = V(x)\log(A + V(x))$ and

$$\mathscr{L}^u F(V(x)) \leqq 0$$

for large A, and x in $E - S$, the uniform integrability condition of Theorem 1 is satisfied. Thus $V(x) = C^u(x)$.

Now, noting that the control (3-19) absolutely minimizes $\mathscr{L}^w V(x) + 1$, where w is subject to the constraint $w'w \leqq \rho^2$, Theorems 3 and 4 imply that (3-19) is optimal with respect to at least the family of uniform Lipschitz controls w such that

$$\mathscr{L}^w V(x) \log(A + V(x)) \leqq 0$$

in $E - S$.

4. A Discrete Parameter Theorem

Theorem 9 is the discrete parameter version of Theorem 2. The discrete parameter case has been more extensively studied than the continuous parameter case. See Howard [1], Blackwell [1], Derman [1], [2], and Bellman and Dreyfus [1] for some representative results. We denote the target set by S and suppose that the transition probability of the homogeneous* Markov process $x_1, \ldots$ depends on the control. The values of the control u_n depend only on x_n. N_u is the first Markov time of arrival at S, and we suppose that the process is stopped at N_u. The cost is

$$C^u(x) = E_x^u b(x_{N_u}) + E_x^u \sum_{1}^{N_u - 1} k_1(x_{i+1}, u_i)$$

$$b(x) \geqq 0, \qquad k_1(x, u) \geqq 0,$$

conditioned on the initial condition $x_1 = x$. Write

$$E_{x,n}^u k_1(x_{n+1}, u_n) = k(x, u_n).$$

Let $k(x, u) = 0$ for x in S.

By the algorithm of dynamic programming, the minimum cost, $V(x)$, satisfies the functional equation

$$V(x) = \min_u E_{x,n}[V(x_{n+1}) + k_1(x_{n+1}, u_n)]. \tag{4-1}$$

Theorem 9. Let u transfer $x_1 = x$ to S with probability one. (With $N_u < \infty$ with probability one if $b(x) \not\equiv 0$.) Let $V(x)$ be a continuous nonnegative function satisfying $b(x) = V(x)$ for $x \in S$ and

$$E_{x,n}^u V(x_{n+1}) - V(x) = -k(x, u) \leqq 0. \tag{4-2}$$

in $E - S$. Let either (4-3) or (4-4) together with either (4-5) or (4-6) hold:

$$E_x^u V(x_{n \cap N_u}) \to E_x^u b(x_{N_u}) \quad \text{as} \quad n \to \infty, \tag{4-3}$$

$$V(x_{n \cap N_u}) \text{ are uniformly integrable}, \tag{4-4}$$

$$V(x) \text{ uniformly continuous on } S, \tag{4-5}$$

* The nonhomogeneous case result is the same, except for notation.

$$P_x^u\{\sup_{N_u > i \geqq 1} \|x_i\| \geqq N\} \to 0 \qquad \text{as} \qquad N \to \infty. \tag{4-6}$$

Then

$$C^u(x) = V(x).$$

Let w be a control such that the above conditions are satisfied except that

$$E_{x,n}^w V(x_{n+1}) - V(x) \geqq -k(x, w).$$

Then

$$C^u(x) = V(x) \leqq C^w(x).$$

Proof. If $b(x) \equiv 0$, then $x_n \to S$ and either (4-5) or (4-6) imply $V(x_n) \to 0$. If $b(x) \not\equiv 0$, then $x_n \to S$, $N_u < \infty$, and either (4-5) or (4-6) imply $V(x_n) \to 0$ (all with probability one). Then (4-4) implies (4-3) whether or not $b(x) \equiv 0$. Next, we note that the application of (4-3) to

$$V(x) = E_x^u \sum_1^{n \cap N_u} k(x_i, u_i) + E_x^u V(x_{n+1 \cap N_u})$$

implies that $V(x) = C^u(x)$. Similarly, it can be proved that $V(x) \leqq C^w(x)$.

V / THE DESIGN OF CONTROLS

1. Introduction

In this chapter we discuss some methods of designing controls which are suggested by the results of Chapters II to IV. If a control problem is posed as a well-formulated mathematical optimization problem, as in the introduction to Chapter IV, then it is natural at least to attempt to compute the optimizing control. Owing to the difficulty of the computational problem (even if existence and other theoretical questions were settled), this is not always possible. In addition, the practical control problem is not usually posed as a well-formulated mathematical optimization problem.

The goal which the control is designed to accomplish may be phrased somewhat loosely. We may desire a control which will guarantee that a given target set is attained with probability one at some random time which is specified only by a bound on its average value or on the probability of large values. Or we may require that the system satisfy some particular statistical stability property, either in some asymptotic sense, or with bounded paths, or with a condition on the average rate of decrease of some error for large values of the error. Any member of some family of loss functions may be satisfactory. It may only be desired that the control, which accomplishes a given task, not take "large" values with a high probability. For the deterministic problem there is interest in the use of Liapunov function methods to design controls which will satisfy such qualitative requirements; for example, Kalman and Bertram [1], LaSalle [1], Geiss [1], Nahi [1], Rekasius [1], and Johnson [1]; see Kushner [5] for the stochastic case.

In the next section, we give two examples of the use of the stability results to design controls which will assure that the resulting process has some specified stability property. The design of controls to improve the cost is discussed in Section 3.

It is doubtful that the design of a control can be based solely on the Liapunov function method; however, it should provide some helpful assistance.

2. The Calculation of Controls Which Assure a Given Stability Property

Example 1. We will compute a control which will assure that the stability properties of (2-1) satisfy a given specification:

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= (-x_1 - x_2 + u)\, dt + \sigma\, dz\,. \end{aligned} \tag{2-1}$$

The function

$$V(x) = \tfrac{3}{2}x_1^2 + x_1 x_2 + x_2^2 \tag{2-2}$$

is in the domain of $\tilde{A}_m^u$ for each m, and $\tilde{A}_m^u = \mathscr{L}^u$ for any uniformly Lipschitz control:

$$\mathscr{L}^u V(x) = -x_1^2 - x_2^2 + u(2x_2 + x_1) + \sigma^2\,.$$

Suppose that $u(x)$ is uniformly Lipschitz. Define $S = \{x : x'x \leqq R^2\}$ where $R^2 \geqq \sigma^2$. Let $u \equiv 0$. Outside of $S, \mathscr{L}^0 V(x) < 0$ and tends to $-\infty$ as $\|x\|$ tends to infinity. By Chapter II, Corollary 2-1, there is a time τ_0 (possibly taking infinite values) so that $x_t \to \partial S$ with probability one as $t \to \tau_0$. Similarly, if the control $u(x)$ satisfies *sign* $u(x) = -$*sign* $(2x_2 + x_1)$, there is a τ_u such that $x_t \to \partial S$ with probability one as $t \to \tau_u$. We will select a control u so that

$$P_x^u\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq \varepsilon\} \leqq \rho\,, \tag{2-3}$$

where ρ and ε are fixed; that is, the control will assure that the probability that x_t leaves Q_ε before touching ∂S is no greater than ρ.

Let $c > 0$. Then, for any stochastic Liapunov function $V(x)$, $V(x) + c$ gives a worse estimate than $V(x)$. This is easily seen by noting that

$$\tilde{A}_m(V(x) + c) = \tilde{A}_m V(x)$$

(provided only that the probability that x_t is killed in $[t, t + \Delta)$, given $x_t = x \in Q_m$, is $o(\Delta)$) and

$$\begin{aligned} P_x\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq \varepsilon\} &= P_x\{\sup_{\tau_u > t \geqq 0} V(x_t) + c \geqq \varepsilon + c\} \\ &\leqq \frac{V(x) + c}{\varepsilon + c} > \frac{V(x)}{\varepsilon} \\ &\geqq P_x\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq \varepsilon\}. \end{aligned}$$

Therefore, to improve the estimate in our problem, we use $\bar{V}(x) = V(x) - v \geqq 0$ in $E - S$, where $v = \min_{x \in \partial S} V(x)$.

Define $\bar{W}(x) = \exp \psi \bar{V}(x) - 1$, $m = \exp \psi \bar{\varepsilon}$, and $\bar{\varepsilon} = \varepsilon - v$. $\bar{W}(x)$ is in the domain of $\tilde{A}_m^u$ and $\tilde{A}_m^u = \mathscr{L}^u$. Note that $\bar{W}(x) \geqq 0$, where $\bar{V}(x) \geqq 0$.

$$\begin{aligned} \mathscr{L}^u \bar{W}(x) &= \psi(\bar{W}(x) + 1) \\ &\cdot \left[-x_1^2 - x_2^2 + \sigma^2 + u(2x_2 + x_1) + \frac{\psi\sigma^2}{2}(2x_2 + x_1)^2 \right]. \end{aligned} \tag{2-4}$$

If $u(x)$ is such that $\mathscr{L}^u \bar{W}(x) \leqq 0$ in $E - S$, then by Chapter II, Corollary 2-1,

$$\begin{aligned} P_x^u\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq \varepsilon\} &= P_x^u\{\sup_{\tau_u > t \geqq 0} \bar{W}(x_t) \geqq \exp \psi\bar{\varepsilon} - 1\} \\ &\leqq \frac{\exp \psi \bar{V}(x) - 1}{\exp \psi \bar{\varepsilon} - 1} = B(\psi). \end{aligned}$$

To complete the choice of $u(x)$, fix the initial condition $V(x)$, choose ψ so that $B(\psi) = \rho$ and, finally, choose $u(x)$ so that $u(2x_2 + x_1)$ cancels the noise contribution $(\psi\sigma^2/2)(2x_2 + x_1)^2$; one choice, which is not necessarily the "smallest," is $u = \psi\sigma^2(2x_2 + x_1)/2$.

It cannot be claimed that the chosen control is best in the sense of minimizing the left side of (2-2), or even that it improves the stability in any absolute sense. All that can be claimed is that (2-3) is satisfied. Nevertheless, if this cannot be mathematically ascertained by some other technique, and some other control, then the method employed does have value for the problem.

It should be mentioned that, in a realistic control problem, the specification ((2-3)) would rarely be given so precisely; a fair amount of freedom may be available in the choice of both S and "Q_ε," and other Liapunov functions, giving information on different regions, may also be useful. In fact, it would be desirable to compare several Liapunov functions, bounds, and regions.

Example 2. We consider now a finite time problem for the system (2-1). Let $V(x)$ be defined as in Example 1 and let $\hat{W}(x) = \exp \psi V(x) - 1$, $W(x) = \exp \psi V(x)$. Determine a control which assures that

$$P_x^u\{\sup_{T \geqq t \geqq 0} V(x_t) \geqq \varepsilon\} \leqq \rho. \tag{2-5}$$

Recall that (Chapter III, Example 3), $x_1^2 + x_2^2 \geqq \beta V(x)$, where $\beta = 0.552$. Let $u = -(2x_2 + x_1)\sigma^2\psi/2$ and write (analogous to (2-4))

$$\mathscr{L}^u \hat{W}(x) \leqq -\hat{\mu}\psi \hat{W}(x) + \psi W(x)[\hat{\mu} + \sigma^2 - \beta V(x)]. \tag{2-6}$$

The second term of (2-6) has a maximum, denoted by φ, at any point x such that

$$V(x) = \max\left[0, \frac{\psi(\hat{\mu} + \sigma^2) - \beta}{\psi\beta}\right].$$

If the maximum occurs at $V(x) = 0$, then $\varphi = \psi(\hat{\mu} + \sigma^2)$; otherwise, $\varphi = \beta \exp[\psi(\hat{\mu} + \sigma^2)/\beta - 1]$. In what follows, we suppose that the latter case holds; we also suppose that ε is large enough so that, for all ψ of interest,

$$m \equiv e^{\psi\varepsilon} - 1 \geqq \frac{\varphi}{\mu} = \frac{\beta \exp[\psi(\hat{\mu} + \sigma^2)/\beta - 1]}{\hat{\mu}\psi}, \quad \varepsilon > \frac{\hat{\mu} + \sigma^2}{\beta}. \tag{2-7}$$

We allow these assumptions merely for ease of presentation.

Under condition (2-7), Theorem 1 of Chapter III gives

$$P_x^u\{\sup_{T\geqq t\geqq 0} V(x_t)\geqq\varepsilon\} = P_x^u\{\sup_{T\geqq t\geqq 0} \hat{W}(x_t)\geqq \exp\psi\varepsilon - 1 = m\}$$

$$\leqq 1-\left(1-\frac{\exp\psi V(x)-1}{\exp\psi\varepsilon - 1}\right)e^{-TB(\psi)} \tag{2-8}$$

$$B(\psi)=\frac{\beta\exp[\psi(\hat{\mu}+\sigma^2)/\beta - 1]}{\exp\psi\varepsilon}.$$

ψ can be selected so that the right-hand side of (2-8) equals ρ, as required by (2-5). Note that, as $\psi\to\infty$, the right side of (2-8) tends to zero for all fixed $T<\infty$, provided (2-7) holds. Of course, as ψ increases, so does the magnitude of the control.

3. Design of Controls to Decrease the Cost

Theorem 1. Suppose that the pair $(V(x), u)$ satisfies the conditions of Theorem 1 (or Theorem 2 if $b(x)\not\equiv 0$), Chapter IV (so that $V(x)=C^u(x)$). Let w be any control which transfers the initial condition x to S with probability one and for which $V(x)$ is in the domain of $\tilde{A}_m^w$, for each $m>0$. If $b(x)\not\equiv 0$, then let $\tau_w<\infty$ with probability one. Let

$$\tilde{A}_m^w V(x)\leqq -k(x,w)\leqq 0. \tag{3-1}$$

Then

$$C^w(x)\leqq C^u(x). \tag{3-2}$$

If (3-1) is strict for some x, then so is (3-2) for some x.

Proof. By Theorem 1 (or Theorem 2, if applicable) of Chapter IV, $V(x)=C^u(x)$. Let τ_i be a sequence of Markov times (for the process with control w) tending to τ_w with probability one and satisfying $E_x^w\tau_i<\infty$, $\tau_i\leqq\inf\{t\colon V(x_t)\geqq i$, with control $w\}$. Then, by Dynkin's formula and (3-1),

$$V(x) \geqq E_x^w V(x_{\tau_i}) + E_x^w \int_0^{\tau_i} k(x_s, w_s)\, ds. \tag{3-3}$$

Now applying Fatou's lemma and using the fact that $V(x_{\tau_i}) \to V(x_{\tau_w})$ with probability one, we obtain

$$\begin{aligned} V(x) &\geqq \liminf E_x^w V(x_{\tau_i}) + \liminf E_x^w \int_0^{\tau_i} k(x_s, w_s)\, ds \\ &\geqq E_x^w b(x_{\tau_w}) + E_x^w \int_0^{\tau_w} k(x_s, w_s)\, ds = C^w(x), \end{aligned} \tag{3-4}$$

whether or not $b(x) \equiv 0$. Q.E.D.

Remark. Note that it is not required that $E_x^w V(x_{\tau_i}) \to E_x^w b(x_{\tau_w})$. If this is not the case, then the difference is in favor of the control w.

If $E - S + \partial S$ is bounded, then it is again only required that the processes x_t converge either to S if $b(x)$ is identically zero or to a specific point on S in finite time if $b(x)$ is not identically zero.

Remark. Suppose that $V(x)$ is know to equal the cost $C^u(x) = E_x^u b(x_{\tau_u}) + E_x^u \int_0^{\tau_u} k(x_s, u_s)\, ds$, and is in the domain of $\tilde{A}_m^u$, for each $m > 0$, as well. Suppose also that $k(x, u)$ is continuous at each x and u and that u_s is also right continuous with probability one for $t < \tau_u$. Then $\tilde{A}_m^u V(x) = -k(x, u)$. (This can be demonstrated by an application of Theorem 5.2 and Lemma 5.6 of Dynkin [2].) Then any control w for which $V(x)$ is in the domain of $\tilde{A}_m^w$ and $\tilde{A}_m^w V(x) \leqq -k(x, w)$ yields a cost that is no larger than $C^u(x)$, provided $V(x_t) \to V(x_{\tau_w})$ as $t \to \tau_w$.

The theorem provides a procedure which, at least in principle, may be used to compute a sequence of controls, each member of the sequence yielding a smaller cost than the previous member. Suppose that the cost, corresponding to u, can be computed, and suppose that the pair $(C^u(x), u)$ satisfies the conditions on $(V(x), u)$ of the theorem. Suppose also that a control w can be found which satisfies the conditions of the theorem. Then $C^w(x) \leqq C^u(x)$. If $C^w(x)$ can be computed, and $(C^w(x), w)$ has the properties of $(V(x), u)$ of the theorem, then the procedure may be repeated, etc.

For Itô processes, the computations involve, under suitable conditions, the solution of a partial differential equation. See, for example, Chapter IV. If a solution to the appropriate equation is available, it must still be checked against the conditions of the theorems of Chapter IV. The limiting properties of at least the sequence of solutions to this partial differential equation, for strong diffusion processes in bounded domains, is discussed by Fleming [1]. See also Fleming [2]. We will not pursue these interesting results here.

Remark. The following form for Itô processes will be useful in the example. Let

$$dx = f(x, u)\,dt + \sigma(x)\,dz$$

where σ does not depend on the control. Suppose that the triple $V(x)$, $u = 0$ and w, satisfies the conditions of Theorem 1, and that, on $V(x)$, $\tilde{A}_m^0 = \mathscr{L}^0$ and $\tilde{A}_m^w = \mathscr{L}^w$. Let

$$C^u(x) = E_x^u \int_0^{\tau_u} [k(x_s) + l(x_s, u_s)]\,ds$$

where

$$k(x) \geqq 0, \qquad l(x, u) \geqq 0, \qquad l(x, 0) = 0.$$

Let

$$\mathscr{L}^0 V(x) + k(x) = 0.$$

By the theorem, $C^0(x) \geqq C^w(x)$. The inequality is strict for some x, if

$$\mathscr{L}^w V(x) - \mathscr{L}^0 V(x) + l(x, w) < 0,$$

or, equivalently, if

$$'V_x(x)\,[f(x, w) - f(x, 0)] + l(x, w) < 0.$$

The Liapunov function approach to design

Suppose that some stochastic Liapunov function $V(x) \geqq 0$ is given. Then the control problem may be studied in several ways. The common factor underlying the several approaches is that $V(x)$ is assumed to equal the cost associated with some control, and some loss function $k(x, u)$. The control may be given a priori, but the loss function is determined by $V(x)$ and u.

First let $b(x) \equiv 0$ and define the loss to be $k(x, u) = -\tilde{A}_m^u V(x) \geqq 0$ and suppose that the value of $\tilde{A}_m V(x)$ in any fixed set does not depend on m. Define $R^u = \{x : k(x, u) \leqq 0\}$. Suppose that there is some $\gamma > 0$ so that $Q_\gamma + \partial Q_\gamma = \{x : V(x) \leqq \gamma\} \supset R^u$. Then, for the loss $k(x, u)$ and target set $Q_\gamma + \partial Q_\gamma = S$, the cost corresponding to control u is $C^u(x) = V(x) - \gamma$. (We suppose, of course, that $(V(x), u)$, statisfies the conditions of Theorem 1.) Let w be any control satisfying the conditions of Theorem 1, and suppose also that $S \supset R^w$. Then the Theorem says that $C^w(x) \leqq C^u(x)$. Thus, given some suitable $V(x)$, we constructed a control problem and then improved the control. Obviously, the usefulness of the procedure is dependent upon whether the chosen $V(x)$ yields a $k(x, u)$ and an S of suitable forms. In any case, with the given $V(x)$ and computed S and any $k(x, u) \leqq 0$ in $E - S$, some type of stochastic stability is guaranteed.

Now, choose a set S which has a smooth boundary and which contains R^u, and define the function $V(x) = b(x)$ on S. The procedure outlined above may be repeated. Now, $C^u(x) = V(x)$ and, if there is a w satisfying Theorem 1, then w is at least as useful as u.

The method is applied to a control process in Example 3. More examples are given in Kushner [5].

Example 3. Let the control system be defined by

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= (-x_1 - x_2 + u)\, dt + \sigma\, dz . \end{aligned}$$

Define

$$V(x) = \tfrac{3}{2}x_1^2 + x_1 x_2 + x_2^2 .$$

$V(x)$ is in the domain of $\tilde{A}_m^u$ for each $m > 0$ and each locally Lipschitz control $u(x)$. On $V(x)$, $\mathscr{L}^u = \tilde{A}_m^u$. In particular,

$$\mathscr{L}^0 V(x) = - x'x + \sigma^2 .$$

We will now construct a control problem based on $V(x)$, and compare the control $u = 0$ to another control. Define $R^0 = \{x : x'x \leqq \sigma^2\}$ and let $S = Q_\gamma + \partial Q_\gamma = \{x : V(x) \leqq \gamma\} \supset R^0$. By Corollary 2-1, Chapter II, there is some $\tau_0 \leqq \infty$ such that $x_s \to \partial Q_\gamma$ as $s \to \tau_0$, with probability one. In addition, it is not hard to verify (although we omit the proof)* that $E_x^0 V(x_{\tau_n}) \to \gamma$ for any sequence of random times tending to τ_0. Thus, by Theorem 1, Chapter IV,

$$V(x) - \gamma = C^0(x) = E_x^0 \int_0^{\tau_0} (x_s' x_s - \sigma^2)\, ds .$$

We conclude that $V(x) - \gamma$ is the cost corresponding to the control problem with target set S, loss $x'x - \sigma^2$, and control $u = 0$. Suppose now that we wish to find a control $u = w$ for which there is a random time τ_w such that $x_t \to \partial S$ as $t \to \tau_w$, and, in addition, for which $(x_0 = x \notin \partial S)$

$$C^0(x) > C^w(x) = E_x^w \int_0^{\tau_w} (x_s' x_s - \sigma^2 + w_s^2)\, ds .$$

The procedure to be followed is outlined in the last remark following Theorem 1.

Let w be a uniformly Lipschitz control. Define the loss

$$\begin{aligned} k(x, w) &= k(x) + l(w) \\ k(x) &= x'x - \sigma^2 , \qquad l(w) = w^2 \\ \mathscr{L}^w V(x) &= - x_1^2 - x_2^2 + \sigma^2 + w(x_1 + 2x_2) . \end{aligned}$$

* Suppose that the distance between ∂S and ∂R^u is greater than zero. Then $- x'x + \sigma^2 < - \varepsilon < 0$ in $E - S$, and it is easy to show that $E_x{}^0 V(x_{\tau_i}) \to \gamma$. Let $\alpha > 0$. Then $\mathscr{L}^0 V^{1+\alpha}(x) = (1 + \alpha) V^\alpha(x) [- x'x + \sigma^2 + \sigma^2 \alpha (2x_2 + x_1)^2/2]$ which, for small α, is nonpositive in $E - S$. An appeal to Theorem 3, Chapter IV, completes the demonstration for this case.

If $\operatorname{sign} w = -\operatorname{sign}(x_1 + 2x_2)$, then $\mathscr{L}^w V(x) \leqq \mathscr{L}^0 V(x)$ and, hence, $x_t \to \partial S$ as $t \to \tau_w$ (that is, τ_w is defined). Suppose that w satisfies, in addition,

$$\mathscr{L}^w V(x) \leqq -k(x, w)$$

or, equivalently, if

$$V_x'(x)\left(f(x, w) - f(x, 0)\right) + w^2 = (x_1 + 2x_2)\, w + w^2 < 0,$$

then the control $u = w$ is better than the control $u = 0$. The control minimizing the left side of the last expression (and which also satisfies the other requirements) is

$$w = -\frac{x_1 + 2x_2}{2}.$$

Other forms, such as $w_1 = w$, if $|w| \leqq 1$, and $w_1 = \operatorname{sign} w$ otherwise, are also satisfactory.

It does not appear easy to compute explicitly the improvement in the cost obtained by the use of w, $C^0(x) - C^w(x)$. Nevertheless, the use of control w allows an improved estimate of $P_x^u\{\sup_{\tau_u > t \geqq 0} V(x_t) \geqq m\}$. In fact, this estimate is essentially provided by Example 1.

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