Decentralized non-convex optimization via bi-level SQP and ADMM

Alternating Direction Method of Multipliers Session, IEEE CDC 2022

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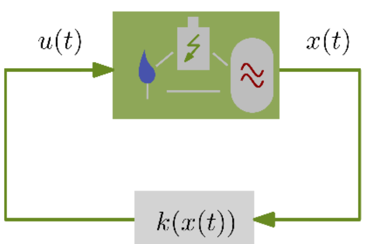
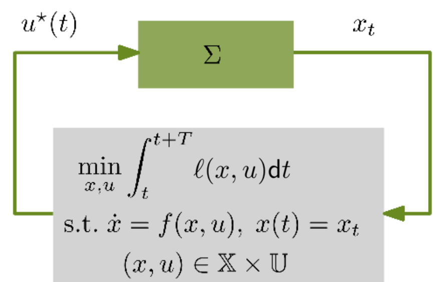
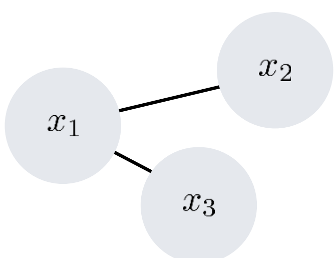




Problem Statement

Partially separable NLP

$$
\begin{array}{llll}
\min\limits_{x_1,\dots,x_S} & \sum\limits_{i\in\mathcal{S}} f_i(x_i) & & f_i \in \mathcal{C}^2 : \mathbb{R}^{n_i} \to \mathbb{R} \\
\text{s.t.} & \cancel{g_i(x_i) = 0 \quad | \, \nu_i \quad \forall i \in \mathcal{S} \quad g_i \in \mathcal{C}^2 : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{gi}}} & & g_i(x_i) \text{ in paper} \\
 & h_i(x_i) \le 0 \quad | \, \mu_i & \forall i \in \mathcal{S} & h_i \in \mathcal{C}^2 : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{hi}} \\
 & \sum\limits_{i\in\mathcal{S}} E_i x_i = c \quad | \, \lambda & & E_i \in \mathbb{R}^{n_c \times n_i}
\end{array}
$$



Desirable algorithmic properties

- Communication only between neighbors (decentralized method)
- Convergence guarantees for non-convex problems
- Low-complexity computations per subsystem

How to design a suitable method?

Bertsekas, D. and Tsitsiklis, J. “Parallel and distributed computation: numerical methods.” Athena Scientific, 2015

Sequential Quadratic Programming (SQP)

Centralized NLP

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & h(x) \leq 0 \mid \mu \\ & Ex = c \mid \lambda \end{aligned}$$

$$p^k = \begin{bmatrix} x^k \\ \mu^k \\ \lambda^k \end{bmatrix}$$

QP approximation (convex)

$$H^k \approx \nabla^2_{xx} L(p^k)$$

$$\begin{aligned} \min_{s} \quad & \frac{1}{2} s^\top H^k s + \nabla f^{k\top} s \\ \text{s.t.} \quad & h^k + \nabla h^{k\top} s \leq 0 \mid \mu^{QP} \\ & E(x^k + s) = c \mid \lambda^{QP} \end{aligned}$$

$$d^k = \begin{bmatrix} s^k \\ \mu^{QP,k} - \mu^k \\ \lambda^{QP,k} - \lambda^k \end{bmatrix}$$

SQP

- Evaluate sensitivites
- Solve QP for d^k
- $p^{k+1} = p^k + d^k$

Local convergence via Newton's method

Boggs, P. and Tolle, J. “Sequential quadratic programming.” *Acta numerica,* 1995
Nocedal, J. and Wright, S. “Numerical Optimization.” *Springer Science & Business Media*, 2006

Sequential Quadratic Programming (SQP)

KKT conditions

$$\nabla_x L(p) = 0$$
$$h(x) \leq 0,\ \mu \geq 0,\ \mu^\top h(x) = 0$$
$$Ex - c = 0$$

$$F(p) \doteq \begin{bmatrix} \nabla_x L(p) \\ \mathrm{min}(-h(x), \mu) \\ Ex - c \end{bmatrix} = 0$$

Newton: $p^{k+1} = p^k - \nabla(F^k)^{-1} F^k$

Assumption 1 *(KKT point)*
The KKT point $p^\star$ satisfies

i) Strict complementarity

ii) SOSC

iii) LICQ

Local convergence of SQP

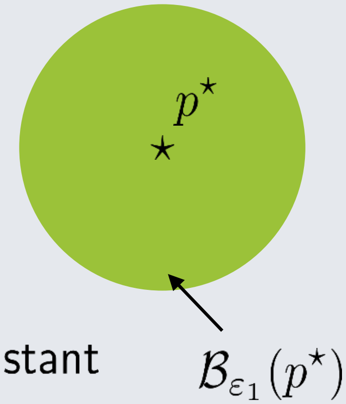
If $p^0 \in \mathcal{B}_{\varepsilon_1}(p^\star)$, then:

i) $\nabla F^{k^{-1}}$ exists

ii) $p^{k+1} \in \mathcal{B}_{\varepsilon_1}(p^\star)$

iii) Active set $\mathcal{A}^k$ stays constant

iv) $\{p^k\} \to p^\star$ q-superlinearly



Apply SQP to partially separable NLP?

Boggs, P. and Tolle, J. “Sequential quadratic programming.” *Acta numerica,* 1995
Nocedal, J. and Wright, S. “Numerical Optimization.” *Springer Science & Business Media*, 2006

Decentralized SQP

Partially separable NLP

$$\min_{x_1,\dots,x_S} \sum_{i\in\mathcal{S}} f_i(x_i)$$

$$\text{s.t.} \quad h_i(x_i) \le 0 \mid \mu_i$$

$$\sum_{i\in\mathcal{S}} E_i x_i = c \mid \lambda$$

$$p^k = \begin{bmatrix} x^k \\ \mu^k \\ \lambda^k \end{bmatrix}$$

QP approximation (convex)

$$H_i^k \approx \nabla^2_{x_i x_i} L(p^k)$$

$$\min_{s_1,\dots,s_S} \sum_{i\in\mathcal{S}} \frac{1}{2} s_i^\top H_i^k s_i + \nabla f_i^{k\top} s_i$$

$$\text{s.t.} \quad h_i^k + \nabla h_i^{k\top} s_i \le 0 \mid \mu_i^{QP}$$

$$\sum_{i\in\mathcal{S}} E_i(x_i^k + s_i) = c \mid \lambda^{QP}$$

Decentralized SQP (d-SQP)

Evaluate sensitivities per subsystem

Solve partially separable convex QP

$$x_i^{k+1} = x_i^k + s_i \qquad \mu_i^{k+1} = \mu_i^{QP} \qquad \lambda^{k+1} = \lambda^{QP}$$

Solve in decentralized fashion?

Solve the QP with ADMM to obtain a decentralized SQP scheme

Alternating Direction Method of Multipliers (ADMM)

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Inner QP

$$\min_{s_1,\dots,s_S} \sum_{i\in\mathcal{S}} \frac{1}{2}{s_i}^\top H_i^k s_i + \nabla f_i^{k\top} s_i$$

$$\text{s.t.} \quad \left. h_i^k + \nabla h_i^{k\top} s_i \le 0 \right\} \mathbb{S}_i$$

$$\left. \sum_{i\in\mathcal{S}} E_i(x_i^k + s_i) = c \right\} \mathbb{E}$$

$$\bar{s}_i \in \mathbb{R}^{ni}$$

$$\min_{s_i,\bar{s}_i,i\in\mathcal{S}} \sum_{i\in\mathcal{S}} \frac{1}{2}{s_i}^\top H_i^k s_i + \nabla f_i^{k\top} s_i$$

$$\text{s.t.} \quad s_i \in \mathbb{S}_i$$

$$s_i - \bar{s}_i = 0 \mid \gamma_i$$

$$\sum_{i\in\mathcal{S}} E_i(x_i^k + \bar{s}_i) = c$$

$$L_\rho(s,\bar{s},\nu) = \sum_{i\in\mathcal{S}} L_{\rho,i}(s_i,\bar{s}_i,\nu_i) = \sum_{i\in\mathcal{S}} \frac{1}{2}{s_i}^\top H_i^k s_i + \nabla f_i^{k\top} s_i + \gamma_i^\top(s_i - \bar{s}_i) + \frac{\rho}{2}\|s_i - \bar{s}_i\|_2^2$$

ADMM

$$s_i^{l+1} = \operatorname*{argmin}_{s_i\in\mathbb{S}_i} L_{\rho,i}(s_i, \bar{s}_i^l, \gamma_i^l)$$

$$\bar{s}^{l+1} = \operatorname*{argmin}_{\bar{s}\in\mathbb{E}} \sum_{i\in\mathcal{S}} L_{\rho,i}(s_i^{l+1}, \bar{s}_i, \gamma_i^l)$$

$$\gamma_i^{l+1} = \gamma_i^l + \rho(s_i^{l+1} - \bar{s}_i^{l+1})$$

Decentralized averaging

Convex inner QP $\implies$ convergence

Boyd, S., Parikh, N., Chu, E., Peleato, B. and Eckstein, J. “Distributed optimization and statistical learning via the alternating direction method of multipliers.” *Found. Trends Mach. Learn.,* 2011

Decentralized SQP

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Partially separable NLP

$$\min_{x_1,\dots,x_S} \sum_{i\in\mathcal{S}} f_i(x_i)$$
$$\text{s.t.} \quad h_i(x_i) \le 0 \mid \mu_i$$
$$\sum_{i\in\mathcal{S}} E_i x_i = c \mid \lambda$$

$$p^k = \begin{bmatrix} x^k \\ \mu^k \\ \lambda^k \end{bmatrix}$$

QP approximation (convex)

$$H_i^k \approx \nabla^2_{x_i x_i} L(p^k)$$

$$\min_{s_1,\dots,s_S} \sum_{i\in\mathcal{S}} \frac{1}{2} {s_i}^\top H_i^k s_i + \nabla f_i^{k\top} s_i$$
$$\text{s.t.} \quad h_i^k + \nabla h_i^{k\top} s_i \le 0 \mid \mu_i^{QP}$$
$$\sum_{i\in\mathcal{S}} E_i(x_i^k + s_i) = c \mid \lambda^{QP}$$

Decentralized SQP (d-SQP)

Evaluate sensitivities per subsystem

Solve partially separable convex QP with ADMM

Terminate early?

$s_i, \bar{s}_i, \lambda$

$$x_i^{k+1} = x_i^k + \bar{s}_i \qquad \mu_i^{k+1} = \mu_i^{QP} \qquad \lambda^{k+1} = \lambda^{QP}$$

Inexact decentralized SQP steps! Convergence?

Inexact SQP

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KKT

$$F(p) = \begin{bmatrix} \nabla_x L(p) \\ \min(-h(x), \mu) \\ Ex - c \end{bmatrix}$$

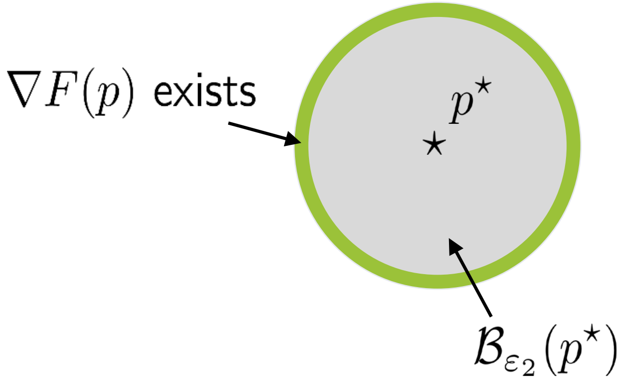
Newton: $F^k + \nabla F^k d^k = 0$

Inexact Newton: $\|F^k + \nabla F^k d^k\| \leq \eta^k \|F^k\|$ (IN)

Lemma *(Inexact SQP convergence)*
Let Assumption 1 hold and let d^k satisfy (IN). If $p^0 \in \mathcal{B}_{\varepsilon_2}(p^\star)$, then:

i) $\{p^k\} \to p^\star$ *q-linearly, if* $\eta^k \leq \eta$

ii) $\{p^k\} \to p^\star$ *q-superlinearly, if* $\eta^k \to 0$



How to evaluate (IN) outside $\mathcal{B}_{\varepsilon_1}$?

Dembo, R. S., Eisenstat, S. C. and Steihaug, T. “Inexact newton methods" *SIAM J. Nume Anal.,* 1982
Morini, B. “Convergence behavior of inexact newton methods.” *Math. Comput.,* 1999

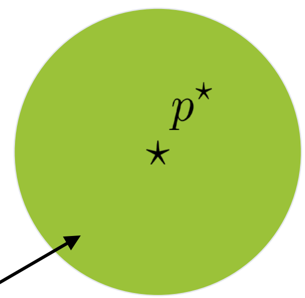
Evaluating the Stopping Criterion

KKT

$$F(p) = \begin{bmatrix} \nabla_x L(p) \\ \min(-h(x), \mu) \\ Ex - c \end{bmatrix}$$

Newton: $F^k + \nabla F^k d^k = 0$

Inexact Newton: $\|F^k + \nabla F^k d^k\| \leq \eta^k \|F^k\|$ (IN)



$\nabla_p \min(-h(x), \mu) = ?$

Optimal active set $\mathcal{A}^\star \implies \min(-h(x), \mu) = 0$

Modified stopping criterion (decentralized)

$$\tilde{F}(p) \doteq \begin{bmatrix} \nabla_x L(p) \\ Ex - c \end{bmatrix}$$

$\|\tilde{F}^k + \nabla \tilde{F}^k d^k\| \leq \eta^k \|\tilde{F}^k\|$ (SC)

If ADMM terminates at $\mathcal{A}^\star$, then (SC) $\implies$ (IN)

Decentralized SQP

Decentralized SQP (d-SQP)

Evaluate sensitivities per subsystem

Solve partially separable convex QP with ADMM until $\|\tilde{F}^k + \nabla \tilde{F}^k d^k\| \leq \eta^k \|\tilde{F}^k\|$

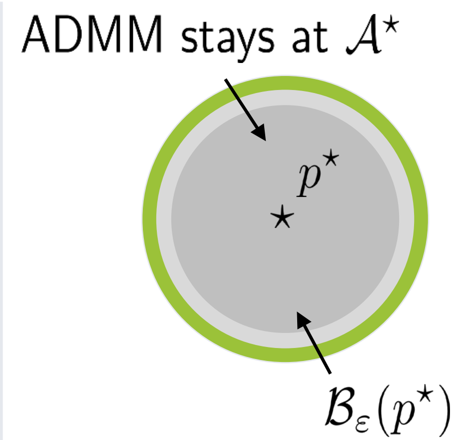
$s_i, \bar{s}_i, \lambda$

$x_i^{k+1} = x_i^k + \bar{s}_i \quad \mu_i^{k+1} = \mu_i^{QP} \quad \lambda^{k+1} = \lambda^{QP}$

Theorem *(d-SQP convergence)*
Let Assumption 1 hold and initialize ADMM with $s_i = 0$ *and* $\gamma_i = E_i^\top \lambda^k$*. If* $p^0 \in \mathcal{B}_\varepsilon(p^\star)$*, then:*

i) $\{p^k\} \to p^\star$ *q-linearly in the outer iterations, if* $\eta^k \leq \eta$

ii) $\{p^k\} \to p^\star$ *q-superlinearly in the outer iterations, if* $\eta^k \to 0$



d-SQP is guaranteed to converge locally for non-convex problems!

Numerical Example: AC-OPF

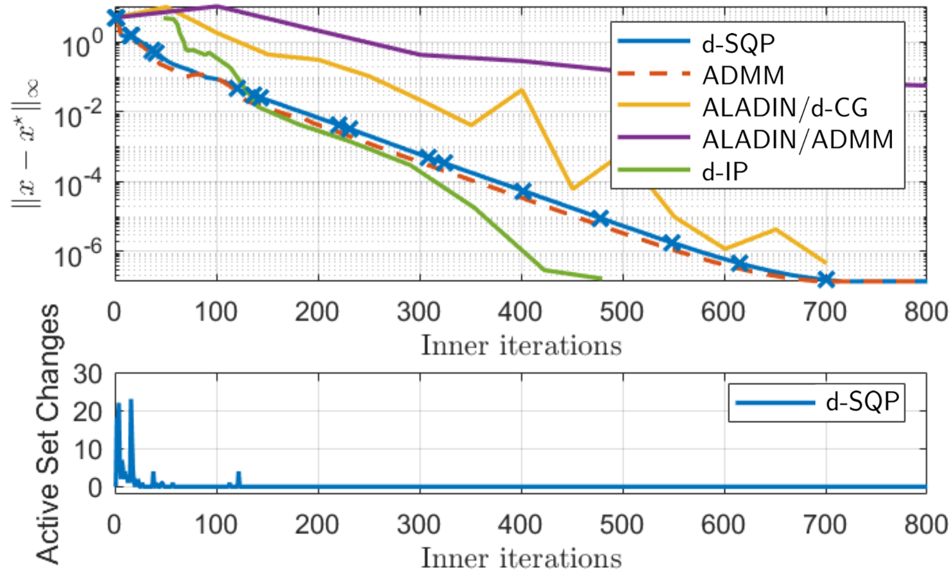
Partially separable NLP

$$\min_{x_1,\dots,x_S} \sum_{i\in\mathcal{S}} f_i(x_i) \qquad x \in \mathbb{R}^{576}$$

$$\text{s.t.} \quad g_i(x_i) = 0 \qquad g(x) \in \mathbb{R}^{470}$$

$$h_i(x_i) \le 0 \qquad h(x) \in \mathbb{R}^{792}$$

$$\sum_{i\in\mathcal{S}} E_i x_i = 0 \qquad E \in \mathbb{R}^{52\times 576}$$



Method	Only neighbor-to-neighbor communication	Proven convergence
d-SQP	yes	yes
ADMM	yes	no
ALADIN/d-CG	no	yes
ALADIN/ADMM	yes	yes
d-IP	no	yes

Computation time until $\|x - x^\star\| < 10^{-6}$
Matlab, one computer for all subsystems

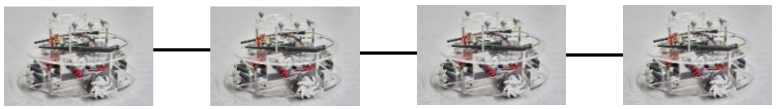
d-SQP	30 s
ADMM	67 s

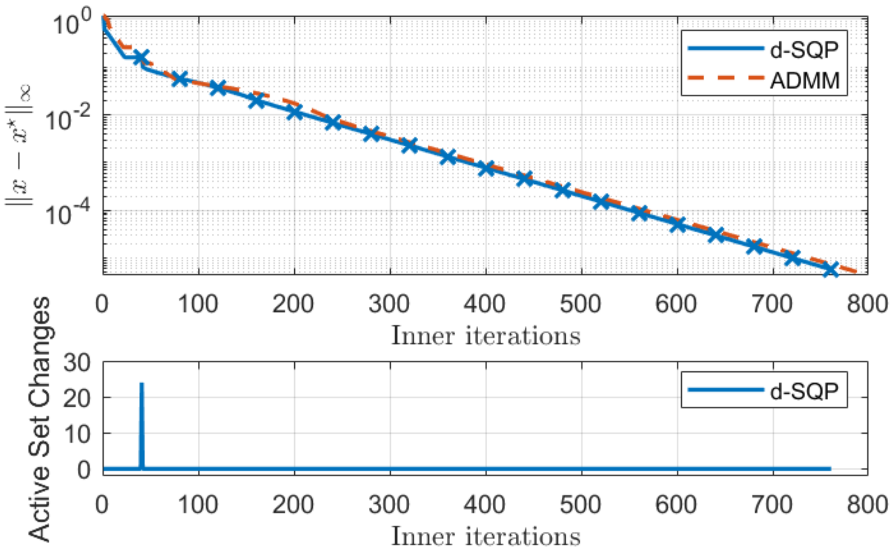
d-SQP shows competitive performance while only solving QPs

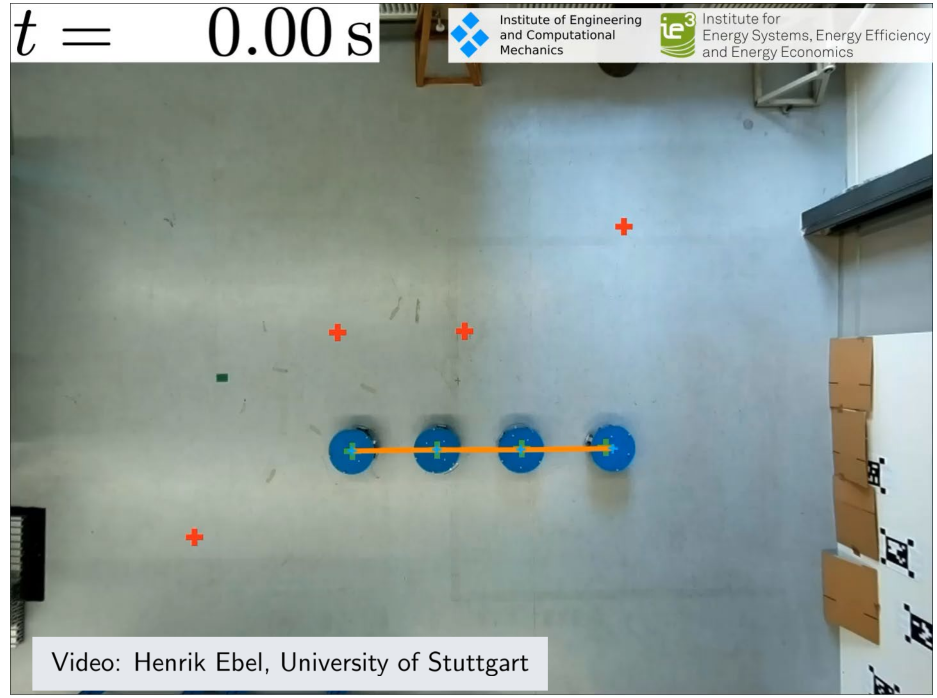
Erseghe, T. "Distributed optimal power flow using ADMM." IEEE Trans. Power Syst., 2014
Frank, S and Rebennack, S. "An introduction to optimal power flow: theory, formulation, and examples." *IIE transactions*, 2016
Engelmann, A. et al. "Decomposition of nonconvex optimization via bi-level distributed ALADIN." *IEEE Trans. Control Netw. Syst.*, 2020

Application Example: Distributed MPC

Mobile robots (4 subsystems)







Computation time to solve optimal control problem
C++, one computer per robot

median	33.81 ms
maximum	53.06 ms

$< \Delta t = 200\,\mathrm{ms}!$

Promising results for distributed NMPC

Stomberg, G., Ebel, H., Faulwasser, T. and Eberhard, P. "Distributed Model Predictive Formation Control in Real-Time", arXiv, 2022

Summary

Decentralized SQP (d-SQP)

Evaluate sensitivities per subsystem

Solve partially separable convex QP with ADMM

$s_i, \bar{s}_i, \lambda$

$x_i^{k+1} = x_i^k + \bar{s}_i \quad \mu_i^{k+1} = \mu_i^{QP} \quad \lambda^{k+1} = \lambda^{QP}$

Key features and outlook

- Local convergence gurantees for non-convex NLPs
- Communication only between coupled subsystems (decentralized method)
- Solves convex QPs on a subsystem level
- Open problem: globalization

Thank you

Stomberg, G., Engelmann, A. and Faulwasser T. “Decentralized non-convex optimization via bi-level SQP and ADMM”, 61st CDC, 2022
Stomberg, G., Ebel, H., Faulwasser, T. and Eberhard, P. “Distributed Model Predictive Formation Control in Real-Time”, arXiv, 2022

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Erseghe, T. “Distributed optimal power flow using ADMM.” IEEE Trans. Power Syst., vol. 29, no. 5, pp. 2370-2380, 2014

Frank, S and Rebennack, S. "An introduction to optimal power flow: Theory, formulation, and examples." *IIE transactions* 48.12: 1172-1197, 2016

Engelmann, A., Jiang, Y., Houska, B. and Faulwasser, T. “Decomposition of nonconvex optimization via bi-level distributed ALADIN.” *IEEE Trans. Control Netw. Syst.,* vol. 7, no. 4, pp. 1848–1858, 2020

Stomberg, G., Ebel, H., Faulwasser, T. and Eberhard, P. “Distributed Model Predictive Formation Control in Real-Time”, arXiv, 2022