Gauss Approximation

- Problem Formulation
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- Solution of Gauss' Approximation Problem

Contents

Problem Formulation

Gauss' approximation problem is to construct a polynomial p of degree $\leq n$ which solves

$$\min_{p \in P_n} \|f - p\| \quad \text{with} \quad \|g\| = \sqrt{\int_a^b g(x)^2 \, \mathrm{d}x} \; .$$

denoting the L_2-norm. Here P_n denotes the set of polynomials $p : \mathbb{R} \to \mathbb{R}$ with degree $\leq n$ and $f : \mathbb{R} \to \mathbb{R}$ is a given function.

L_2-Scalar Products

Recall that the L_2-scalar product of two functions $f, g : [a, b] \to \mathbb{R}$ on an interval $[a, b]$ is given by

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, \mathrm{d}x \ .$$

In this notation, the L_2-norm can be written in the form

$$\|f\| = \sqrt{\langle f, f \rangle}$$

In particular, the Cauchy-Schwartz inequality can be written in the form

$$\langle f, g \rangle \leq \|f\| \cdot \|g\| \ .$$

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Optimality Conditions

Theorem The polynomial p is a solution of the minimization problem

$$\min_{p \in P_n} \|f - p\| \ .$$

if and only if we have $\langle f - p, q \rangle = 0$ for all $q \in P_n$.

Proof:

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if and only if we have $\langle f - p, q \rangle = 0$ for all $q \in P_n$.

Proof:

Step 1: If $p \in P_n$ is an optimal approximation, the function

$F(t) := \|f - p - tq\|^2$ must have a minimizer at $t = 0$ for all $q \in P_n$.

Thus, we must have

$$0 = \frac{\partial}{\partial t} \|f - p - tq\|^2 \Big|_{t=0} = \langle f - p, q \rangle \; .$$

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Theorem The polynomial p is a solution of the minimization problem

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Proof:

Step 2: The other way around, if p satisfies $\langle f - p, q \rangle = 0$ for all $q \in P_n$, we have

$$\|f - p\|^2 \ = \ \langle f - p, f - q \rangle + \underbrace{\langle f - p, q - p \rangle}_{=0} \leq \|f - p\| \, \|f - q\|$$

and thus $\|f - p\| \leq \min_{q \in P_n} \|f - q\|$, i.e., p is a minimizer. $\square$

Uniqueness of Solutions

In the following, we check that the Gauss problem has at most one solution:

If two functions $p_1, p_2 \in P_n$ satisfy the optimality condition

$$\langle f - p_1, q \rangle = \langle f - p_2, q \rangle = 0 \quad \text{for all} \quad q \in P_n \ ,$$

we also have $\langle p_1 - p_2, q \rangle = 0$. Thus, for $q = p_1 - p_2$, we find

$$\|p_1 - p_2\| = 0 \ ,$$

which implies $p_1 = p_2$.

Proving existence is a bit more difficult; we will come back to it later...

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Gram-Schmidt Algorithm

Let's recall some basic linear algebra:

Assume we have k vectors $a_1, \ldots, a_k \in \mathbb{R}^n$. Gram-Schmidt's algorithm can be used to check for linear independence:

Gram-Schmidt Algorithm:

For $i = 1, \ldots, k$:

- Orthogonalization. $\overline{q}_i = a_i - \langle q_1, a_i \rangle q_1 - \ldots - \langle q_{i-1}, a_i \rangle q_{i-1}$.
- Test for dependence. If $\overline{q}_i = 0$, quit.
- Normalization. $q_i = \frac{\overline{q}_i}{\|\overline{q}_i\|}$.

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Gram-Schmidt Algorithm

The Gram-Schmidt Algorithm computes the vectors $q_1, \ldots, q_k$. These vectors are orthonormal. This can be proven by induction:

- The vector $q_1 = \frac{a_1}{\|a_1\|}$ is normalized.
- Assume the vectors $q_1, \ldots, q_{i-1}$ are already orthonormal. Then, the vector $\overline{q}_i$ satisfies

$$\begin{aligned} \langle \overline{q}_i, q_j \rangle &= \langle a_i, q_j \rangle - \textstyle\sum_{k=1}^{i-1} \langle q_k, a_i \rangle \langle q_k, q_j \rangle \\ &= \langle a_i, q_j \rangle - \textstyle\sum_{k=1}^{i-1} \langle q_k, a_i \rangle \delta_{k,j} = 0 \end{aligned}$$

for all $j \in \{1, \ldots, i-1\}$, i.e., the vectors $q_1, \ldots, q_i$ are orthonormal.

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Gram-Schmidt Algorithm for Functions

The Gram-Schmidt Algorithm can be generalized for any Hilbert space.

In particular, we can apply it to functions, where $\langle \cdot, \cdot \rangle$ denotes the L_2 scalar product on the interval $[-1, 1]$.

Example: Start with $a_0(x) = 1$, $a_1(x) = x$, $a_2(x) = x^2$, ..., $a_n = x^n$:

- $q_0(x) = \sqrt{\frac{1}{2}}$.
- $q_1(x) = \sqrt{\frac{3}{2}}x$.
- $q_2(x) = \sqrt{\frac{5}{8}}(3x^2 - 1)$.
- ...
- $q_n(x) = \sqrt{\frac{2n+1}{2}} \frac{1}{2^n n!} \frac{\partial^n}{\partial x^n} \left(x^2 - 1\right)^n$. (Exercise)

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Legendre Polynomials

The orthogonal polynomials

$$L_n(x) = \frac{1}{2^n n!} \frac{\partial^n}{\partial x^n} \left(x^2 - 1\right)^n .$$

are called Legendre polynomials. They satisfy

$$\langle L_i, L_j \rangle = \frac{2}{2i+1} \delta_{i,j}$$

by construction.

Contents

Solution of Gauss' Approximation Problem

We represent the polynomial p with respect to orthonal basis functions $q_0, \ldots, q_n$,

$$p(x) = \sum_{i=0}^{n} c_i q_i(x) \; .$$

The coefficients $c_0, \ldots, c_n$ can be found by substituting the orthogonal polynomials in the optimality condition

$$\forall q \in P_n, \qquad \langle f - p, q \rangle = 0 \; .$$

This yields

$$c_i = \langle p, q_i \rangle = \langle f, q_i \rangle$$

for all $i \in \{1, \ldots, n\}$.

Summary

- Gauss' approximation problem is to find polynomials $p \in P_n$, which solve

$$\min_{p \in P_n} \ \|f - p\|$$

for a given (L_2-integrable) function f.

- Gram Schmidt algorithm can be used to construct orthogonal polynomials $q_0, \ldots, q_n \in P_n$, which satisfy

$$\langle p_i, p_j \rangle = \delta_{i,j} \ .$$

- The solution polynomial p is unique and can be written in the form $p(x) = \sum_{i=0}^{n} c_i q_i(x)$. Here, the coefficients $c_0, \ldots, c_n$ are given by

$$\forall i \in \{0, \ldots, n\}, \qquad c_i = \langle p, q_i \rangle = \langle f, q_i \rangle \ .$$