Pontryagin-Koopman Operators—Open Problems

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Internal communication: this document summarizes a couple of research direction in the context of Pontryagin Koopman operators, which could, in my opinion, lead to a sequence of interesting publications / PhD theses if we solve them.

Introduction

Let $x = f(x, u)$ be a control system with stage cost $l(x, u)$ and Hamiltonian

$$H(x, y) = \min_u y^\intercal f(x, u) + l(x, u) \; .$$

The associated right-hand side of the Pontryagin differential equation is given by

$$F = \begin{bmatrix} \nabla_x H \\ -\nabla_y H \end{bmatrix}$$

and the associated Koopman-Pontryagin operator U_t satisfies

$$U_t \Phi = \Phi \circ \Gamma_t \; ,$$

where Γ_t denotes the flow field of Pontryagin's differential equation. In the following we are mostly interested in the differential Pontryagin Koopman operator $L = \dot{U}_t$, which satisfies

$$L\Phi = F^\intercal \nabla \Psi \; .$$

1 Open Problem 1 [Existence of eigenfunctions in certain Sobolev spaces]

One of the most important questions that I currently have is about the existence of eigenfunctions Ψ with square-integrable derivative, which are such that

$$L\Psi = \kappa \Psi \qquad \text{and} \qquad \int_{\mathbb{R}^{2n}} \|\nabla \Psi(x)\|_2^2 \, \mathrm{d}x < \infty \; .$$

Under which conditions on f and l can we ensure that such functions exist? This problem has not been addressed in our paper by Colin yet, which is, however, important to establish the generality of these results. In order to understand why this question about existence may be non-trivial, first notice that for the LQR-case, where F is linear, the main eigenfunctions of L are linear functions, too. This means that the derivative of Ψ is constant and the integral does not exist (except for the uninteresting case that Ψ is constant such that $\nabla \Psi = 0$, which is not admissible as eigenfunction). However, the main conjecture is that—after adding suitable barrier functions to F—the existence of eigenfunctions with L_2-integrable derivative can be established.

1.1 Main intuition from 1D PDEs

In order to outline a bit where my intuition for the above conjecture comes from, consider 1D PDEs of the form

$$(x + x^{2n+1})\nabla\Psi_n(x) = \Psi_n(x)$$

in dependence on the integer $n \in \mathbb{N}$. Here, we have a linear term "x", which can be interpreted as the dominant term of a dynamic system close to steady-state and a term, x^{2n+1}, which can be interpreted as an indefinite barrier that dominates for $|x| > 1$ (and potentially large n). Now, a very interesting observation is that one can work out eigenfunctions explcitly, namely,

$$\Psi_n(x) = \frac{x}{\sqrt[2n]{1 + x^{2n}}} \qquad \text{with} \qquad \nabla\Psi_n(x) = \frac{1}{(1 + x^2)^{\frac{2n+1}{2n}}} \; .$$

Just plot these functions for different n to see what's going on! As you will see an interesting aspect is that the function Ψ_n converges to the unit saturation function for $n \to \infty$ (BTW: the derivatives converges to something entirely different—we see a so-called non-uniform convergence effect here !!!) In my opinion, the behavior of the above functions Ψ_n looks promising: if we make the barrier more extreme (i.e. if we make n bigger), the tails of $\nabla\Psi$ converge to zero quickly so that the integral over the square of these derivatives exists for sufficiently large n. Now, the main question is whether we can establish such a convergence results for more general ODEs and barrier terms. Is it possible to construct (symplectic!) barriers that reproduce this behavior in the multivariate case ???

2 Open Problem 2 [Symplectic Basis]

If we introduce the bilinear form

$$\omega(\phi, \Phi) = \int_{\mathbb{R}^{2n}} \nabla\phi(z)^\intercal \Omega \nabla\Phi(z) \, \mathrm{d}z$$

with $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and the set W of functions with square integrable weak derivative, then (W, ω) is a symplectic space (which is important in our context as U_t is symplectic in this space). It's not difficult to see that the kernel

$$\ker(\omega) = \{(\phi, \Phi) \in W \mid \omega(\phi, \Phi) = 0 \; \}$$

of ω in W is infinite dimensional. But what can we say about the dimension of its complement,

$$W \setminus \ker(W) \; ?$$

Is this set finite dimensional? Can we find a symplectic basis of this set? Or, alternatively, can we find another useful characterization of this set?

3 Open Problem 3 [Very important conjecture]

Let L be a differential Pontryagin Koopman operator for a stabilizable system with regular steady state. Actually, I have the conjecture that (possibly under additional regularity assumptions) that there exist precisely $2n$ independent eigenfunctions of L that are elements of $W \setminus \ker(W)$. (and I have absolutely no idea how to prove it...)

4 Open Problem 4 (Algebraic properties of the kernel of L)

Let L be the differential Pontryagin Koopman operator for a reasonably regular optimal control problem. If Ψ^+ is an eigenfunction to the eigenvalue κ and Ψ^+ is an eigenfunction to a mirrored eigenvalue, $-\kappa^+$, then the function

$$\Psi = \Psi^+\Psi^-$$

is in the kernel of L, which we define as

$$\ker(L) = \{ \ \phi \in W \mid L\phi = 0 \ \} \ .$$

The proof follows directly from the product rule of differentiation: since $L\Psi^+ = \kappa\Psi^+$ as well as $L\Psi^- = -\kappa\Psi^-$ we have

$$L\Psi = F^\intercal \nabla(\Psi^+\Psi^-) = F^\intercal \nabla \Psi^+ \Psi^- + F^\intercal \nabla \Psi^- \Psi^+ = \kappa\Psi^+\Psi^- - \kappa\Psi^+\Psi^- = 0 \ .$$

Now, a very important observation is that the kernel of Pontryagin Koopman operator is extremely rich. For example, the function Ψ inherits the roots of Ψ^+ and Ψ^-, which means that elements of this kernel can be used to simultaneously characterize stable and unstable manifolds of the Pontryagin ODE. Moreover, one can show that the Hamiltonian H is contained in this kernel,

$$H \in \ker(L) \ .$$

(I leave the proof as a little exercise to you). This result alone is a very important side remark as it means that the (nonlinear !) Hamilton Jacobi Bellman PDE can be analyzed by analyzing the kernel of a ("simple") linear operator. However, my open problem goes much further: if $f, g \in \ker(L)$, we also have $f + g \in \ker(L)$ as well as $f * g \in \ker(L)$. This is an almost trivial observation, as these inclusions are simple consequences of the definition of L and sum / product rule of differentiation. However, the implication of this is that $(\ker(L), +, *)$ is a commutative ring—and all out of the sudden a completely new world is opening up thinking of the fact that commutative algebra is rather well-establised field of research that we can now rely on. In particular, my question is whether we can characterize the algebraic properties of this ring. For example: what are the prime ideals of this ring and how can we compute them? Can we develop some kind of "Euclid" algorithm to write the Hamiltionian H as a product of prime ideals in this ring? Are there cases in which H is irreducible and what does this mean physically? [May be we can at some point also talk to Manolis about this, but let's first try and see how far we get by ourselves; I am currently mostly playing with a polynomial basis, more later...]

5 Open Problem 5 (Practical Solver)

I think we have all made the experience that the standard Galerkin projection method is very unstable and certainly not the way to go for analyzing Pontryagin-Koopman operators. Thus, from a practical perspective, the most important open problem is to develop a tailored numerical solver, which can find the stable manifold such that we can actually construct globally optimal control laws. One of my main motivations to analyze the kernel of L is actually to develop a numerical stable solver (as soon as we understand that L is degenerate == has a nontrivial kernel), we are a big step further. In particular, because this kernel is non-trivial, standard Galerkin methods cannot possibly work well—and, as far as I can see, it should help us a lot if we could understand the properties of this kernel to be able to move on.

6 Open Problem 6 (Stability Boundaries)

My sixth open problem is a problem that I discussed briefly with Milan Korda when I met him in Nice in December. Basically, my question is what happens to the Koopman eigenfunctions if we cross the stability domain of an ODE—or, in our context, if we leave the domain in which we can stabilize a nonlinear control system. Milan gave a partial answer to that mentioning that there are non-continuities at the crossing facets (just dig out his paper, you'll see). However, my stomach feeling tells me that there is potentially much more that we can infer from the behavior of the Koopman eigenmodes at these critical facets. Unfortunately, I am not sure yet how to translate this feeling into formulas yet...

7 Open Problem 7 (Optimal Periodic Orbits)

Can we use the properties of the Pontryagin Koopman operator to "count" the number of optimal periodic orbits or, perhaps, even answer questions about their existence? This question may be extremely difficult to address; I have at the moment no really good ideas yet (apart from a dumb feeling that there may be something possible).

8 Open Problem 8 (Explicit Case Studies)

Can we find classes of optimal control problems (apart from linear-quadratic problems) for which we can construct all eigenfunctions of the associated Pontryagin Koopman operators explicitly? I think Junyan had some good ideas in this direction already, but there may be much more possible. I think that this is vert interesting, as these explicit cases might help us to better understand the properties of Pontryagin-Koopman operators or formulate / prove / disprove the above or many other possible conjectures.