Scalar Linear Systems

- Introduction
- Background: Differentiable Functions
- Scalar Linear Time-Invariant Differential Equation
- Modeling examples

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Model Based Control

Overview

- This lecture introduces scalar linear differential equations
- we will learn how to develop mathematical models for dynamic systems
- in the following weeks we will learn about model-based control

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Derivatives

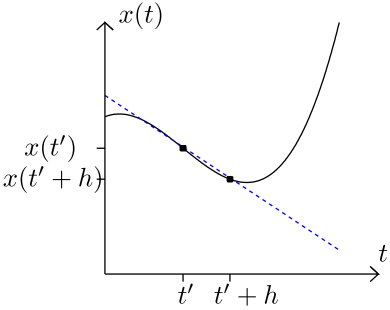
A function $x : D \to \mathbb{R}^n$ is called differentiable on an open set $D \subseteq \mathbb{R}$ if its derivative

$$\dot{x}(t) \;=\; \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

exists for all $t \in D$.

Relation to Interpolation

The term $\frac{x(t'+h)-x(t')}{h}$ can be interpreted as the slope of a line that passes through the points $p_1 = (t', x(t'))$ and $p_2 = (t'+h, x(t'+h))$.



Simple differentiable functions

- The function $x(t) = t$ is differentiable on $D = \mathbb{R}$.

  Its derivative function satisfies $\dot{x}(t) = 1$.

- The function $x(t) = \exp(t)$ is differentiable on $D = \mathbb{R}$.

  Its derivative function satisfies $\dot{x}(t) = \exp(t) = x(t)$.

- The function $x(t) = \sin(t)$ is differentiable on $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

  Its derivative function satisfies $\dot{x}(t) = \cos(t) = \sqrt{1 - x(t)^2}$.

- The function $x(t) = \sqrt{2t}$ is differentiable on $D = \mathbb{R}_{++}$.

  Its derivative function satisfies $\dot{x}(t) = \frac{1}{\sqrt{2t}} = x(t)^{-1}$.

- The function $x(t) = t^{-1}$ is differentiable on $\mathbb{R} \setminus \{0\}$.

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Higher order derivatives

Higher order derivatives can be defined recursively. For example

$$\ddot{x}(t) \;=\; \lim_{h\to 0} \frac{\dot{x}(t+h) - \dot{x}(t)}{h}$$

General recursion:

$$\frac{\partial^{m+1}}{\partial t^{m+1}} x(t) = \lim_{h\to 0} \frac{\frac{\partial^m}{\partial t^m} x(t+h) - \frac{\partial^m}{\partial t^m} x(t)}{h} \; .$$

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Examples for higher order derivatives

The function $x(t) = t^k$, $k \in \mathbb{N}$, is smooth on $D = \mathbb{R}$. Its m-th derivative function satisfies

$$\frac{\partial^m}{\partial t^m} x(t) = \frac{k!}{(k-m)!} t^{k-m} = \frac{k!}{(k-m)!} x(t)^{\frac{k-m}{k}}$$

if $m \leq k$ and $\frac{\partial^m}{\partial t^m} x(t) = 0$ otherwise.

Examples for higher order derivatives

The function $x(t) = \exp(t)$ is smooth on $D = \mathbb{R}$. Its m-th derivative function satisfies

$$\frac{\partial^m}{\partial t^m} x(t) = \exp(t) = x(t) \ .$$

Examples for higher order derivatives

The function $x(t) = \sin(t)$ is smooth on $D = \mathbb{R}$. Its even derivative functions satisfy

$$\frac{\partial^{2m}}{\partial t^{2m}} x(t) = (-1)^m \sin(t) = (-1)^m x(t) \; .$$

Fundamental theorem of calculus

The fundamental theorem of calculus can be summarized as

$$x(0) + \int_0^t \dot{x}(\tau)\,\mathrm{d}\tau \;=\; x(t)\;.$$

Important consequences:

- Integration by parts

$$\int_0^t x(\tau)\dot{y}(\tau)\,\mathrm{d}\tau \;=\; x(t)y(t) - x(0)y(0) - \int_0^t \dot{x}(\tau)y(\tau)\,\mathrm{d}\tau\;.$$

- General Leibniz integral rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} g(t,\tau)\,\mathrm{d}\tau \;=\; g(t,b(t))\,\dot{b}(t) - g(t,a(t))\,\dot{a}(t) + \int_{a(t)}^{b(t)} g_t(t,\tau)\,\mathrm{d}\tau\;.$$

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Dirac distribution

Parametric functions of the form

$$\delta_h(t) := \begin{cases} \frac{1}{h} & \text{if } t \in \left[-\frac{h}{2}, \frac{h}{2}\right] \\ 0 & \text{otherwise} , \end{cases}$$

satisfy

$$\lim_{h \to 0^+} \int_{-\infty}^{\infty} x(\tau)\delta_h(\tau - t)\, \mathrm{d}\tau \ = \ x(t) \ .$$

In practice, we write

$$\int_{-\infty}^{\infty} x(\tau)\delta(\tau - t)\, \mathrm{d}\tau \ = \ x(t) \ ,$$

as if we could swap the limit for $h \to 0$ and the integration over τ. The limit expression "$\delta = \lim_{h \to 0^+} \ \delta_h$" is called Dirac distribution.

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Piecewise differentiable functions

If $x : D \to \mathbb{R}^{n_x}$ is differentiable at all $t' \in D \setminus \{t_0, t_1, \ldots, t_N\}$, we define the "weak derivative" function

$$\dot{x}(t) = \left\{ \begin{array}{ll} \lim_{h \to 0} \frac{x(t+h)-x(t)}{h} & \text{if } t \notin \{t_0, t_1, \ldots, t_N\} \\ d_i & \text{if } \ t = t_i \ \text{ for an index } i \in \{0, \ldots, N\} \end{array} \right\} .$$

for arbitrary constants $d_0, d_1, \ldots, d_N \in \mathbb{R}^{n_x}$.

- Important property: for all differentiable functions $y : [\underline{t}, \overline{t}] \to \mathbb{R}^{n_x}$ with $y(\underline{t}) = y(\overline{t}) = 0$ and $[\underline{t}, \overline{t}] \subseteq D$ we have

$$\int_{\underline{t}}^{\overline{t}} \dot{x}(t)^\intercal y(t) \, \mathrm{d}t \ = \ - \int_{\underline{t}}^{\overline{t}} x(t)^\intercal \dot{y}(t) \, \mathrm{d}t \ .$$

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Example: absolute value function

The function $x(t) = |t|$ is non-differentiable at $t = 0$,

but a weak derivative is given by

$$\dot{x}(t) = \operatorname{sgn}(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases} .$$

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Scalar Linear Time-Invariant System

Let $a, b \in \mathbb{R}$ be scalar constants. The differential equation

$$\dot{x}(t) \;=\; ax(t) + b \quad \text{with} \quad x(0) = x_0$$

is called a scalar linear time-invariant system.

- $x : \mathbb{R} \to \mathbb{R}$ is called the *state trajectory*
- $x_0 \in \mathbb{R}$ is called the *initial value*
- t is called the *free variable*
- in practice, t often (but not always) denotes time.

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Shifting the free variable

If we start at time $t_0 \neq 0$, we write

$$\dot{y}(t) = ay(t) + b \quad \text{with} \quad y(t_0) = x_0 \; .$$

If we set $y(t) = x(t - t_0)$ this is equivalent to the differential equation

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- **We may assume $t_0 = 0$, as we can always shift t otherwise.**

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The homogeneous case

For the special case $x_0 = 0$ and $b = 0$, we obtain

$$\dot{x}(t) \ = \ ax(t) \quad \text{with} \quad x(0) = 0 \ .$$

This differential equation only admits the trivial solution $x(t) = 0$.

Proof: The auxiliary function $v(t) = x(t)e^{-at}$ satisfies

$$\dot{v}(t) = \dot{x}(t)e^{-at} - ax(t)e^{-at} = (ax(t) - ax(t))e^{-at} = 0 \ ,$$

i.e., v must be a constant function. Now, $v(0) = x(0) = 0$ implies $v(t) = 0$ for all $t \in \mathbb{R}$. Consequently, $x(t) = v(t)e^{at} = 0$ is the only possible solution.

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Steady states

For $a \neq 0$, $x_{\mathsf{s}} = -\frac{b}{a}$ is called the steady state of the differential equation.

- Motivation: the differential equation

$$\dot{x}(t) = ax(t) + b \quad \text{with} \quad x(0) = x_{\mathsf{s}}$$

  admits only the constant solution $x(t) = x_{\mathsf{s}}$ for all $t \in \mathbb{R}$.

Proof: The shifted state trajectory $y(t) = x(t) - x_{\mathsf{s}}$ satisfies the homogeneous differential equation

$$\dot{y}(t) = \dot{x}(t) = ax(t) + b = a(y(t) + x_{\mathsf{s}}) + b = ay(t) \quad \text{with} \quad y(0) = 0\ ,$$

which implies $y(t) = 0$ and thus $x(t) = x_{\mathsf{s}}$ for all $t \in \mathbb{R}$.

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Uniqueness of solutions

Assume x_1 and x_2 both satisfy the linear differential equation $\dot{x}(t) = ax(t) + b$ and $x(0) = x_0$. Then the difference function

$$y(t) = x_1(t) - x_2(t)$$

satisfies the homogeneous differential equation

$$\dot{y}(t) = \dot{x}_1(t) - \dot{x}_2(t) = ax_1(t) - ax_2(t) = ay(t) \quad \text{and} \quad y(0) = 0\ .$$

Consequently, $y(t) = 0$, which is equivalent to $x_1(t) = x_2(t)$.

Construction of solutions

Which functions x satisfy the differential equation

$$\dot{x}(t) \ = \ ax(t) + b \quad \text{with} \quad x(0) = x_0 \ ?$$

Let us try functions of the form

$$x(t) = ce^{at} + d \quad \text{such that} \quad \dot{x}(t) = cae^{at} \ ,$$

where c and d are real valued coefficients. By comparing coefficients:

$$\begin{aligned} ad + b &= 0 \\ c + d &= x_0 \ . \end{aligned}$$

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Construction of solutions

If we have $a \neq 0$, this equation system can be solved:

$$c = x_0 + \frac{b}{a} \quad \text{and} \quad d = -\frac{b}{a} \ ,$$

which yields the solution (for $a \neq 0$)

$$x(t) \ = \ x_0 e^{at} + \frac{e^{at} - 1}{a} b \ = \ e^{at}(x_0 - x_{\mathsf{s}}) + x_{\mathsf{s}} \ .$$

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Construction of solutions

For the special case $a = 0$, we compute the limit

$$\begin{aligned} x(t) &= \lim_{a \to 0} \left\{ x_0 e^{at} + \frac{e^{at} - 1}{a} b \right\} \\ &= x_0 + bt \, , \end{aligned}$$

which satisfies $\dot{x}(t) = b$ and $x(0) = x_0$ as expected.

Limit behavior for $t \to \infty$

- For $a > 0$: $\lim_{t\to\infty} x(t) \;=\; \left\{ \begin{array}{rl} +\infty & \text{if } x_0 > x_\mathrm{s} \\ x_\mathrm{s} & \text{if } x_0 = x_\mathrm{s} \\ -\infty & \text{if } x_0 < x_\mathrm{s} \end{array} \right\}.$

- For $a = 0$: $\lim_{t\to\infty} x(t) \;=\; \left\{ \begin{array}{rl} +\infty & \text{if } b > 0 \\ x_0 & \text{if } b = 0 \\ -\infty & \text{if } b < 0 \end{array} \right\}.$

- For $a < 0$: $\lim_{t\to\infty} x(t) \;=\; -\frac{b}{a} \;=\; x_\mathsf{s}$; convergence to steady state.

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Reversing the free variable

If $x(t)$ satisfies $\dot{x}(t) = ax(t) + b$, then $y(t) = x(-t)$ satisfies the reverse differential equation

$$\dot{y}(t) = -\dot{x}(-t) = -ay(t) - b \quad \text{with} \quad y(0) = x(-0) = x_0 \ .$$

Relations between forward and reverse differential equation for $a \neq 0$:

- the steady states coincide:

$$y_{\mathsf{s}} = -\frac{-b}{-a} = -\frac{b}{a} = x_{\mathsf{s}} \ ,$$

- we have

$$(y(t) - x_{\mathsf{s}})(x(t) - x_{\mathsf{s}}) = (x_0 - x_{\mathsf{s}})^2 = \text{const.} \ .$$

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Contents

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- Scalar Linear Time-Invariant Differential Equation
- Modeling examples

Application example: charging of a capacitor

Consider simple electrical circuit consisting of

- an (initially uncharged) capacitor with capacitance C,
- a resistor with resistant R, and
- a battery with constant voltage V_0

The current $I(t)$ in the circuit satisfies

$$\dot{I}(t) = -\frac{1}{RC}I(t) \qquad \text{with} \qquad I(0) = \frac{V_0}{R}$$

The explicit solution is given by

$$I(t) = \frac{V_0}{R}e^{-\frac{t}{RC}} \ .$$

The current decreases over time as the capacitor is charged.

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Application example: intensity of light in water

If we send light through a thin layer of water its irradiance, i.e., the light's power per unit area is reduced.

- Power per unit area that is absorbed by a thin layer of height $\Delta z \ll 1$m is approximately $P = c\Delta z I$.
- c is a constant that depends on the clarity of the water
- I denotes the irradiance of the incoming light.
- If $I(z)$ denotes the irradiance of sun light at depth z under surface of the sea, the irradiance at depth $z + \Delta z$ is approximately

$$I(z+\Delta z) = I(z) - c\Delta z I(z) \quad \Longleftrightarrow \quad \frac{I(z+\Delta z) - I(z)}{\Delta z} = -cI(z)\ .$$

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$$\frac{\partial}{\partial z} I(z) = -cI(z) \qquad \text{with} \qquad I(0) = I_0 \ ,$$

Here, the free variable is not time, but the depth under water!

The solution function is given by $I(z) = I_0 \exp(-cz)$.

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