INTRODUCTION TO CALCULUS

Continuity

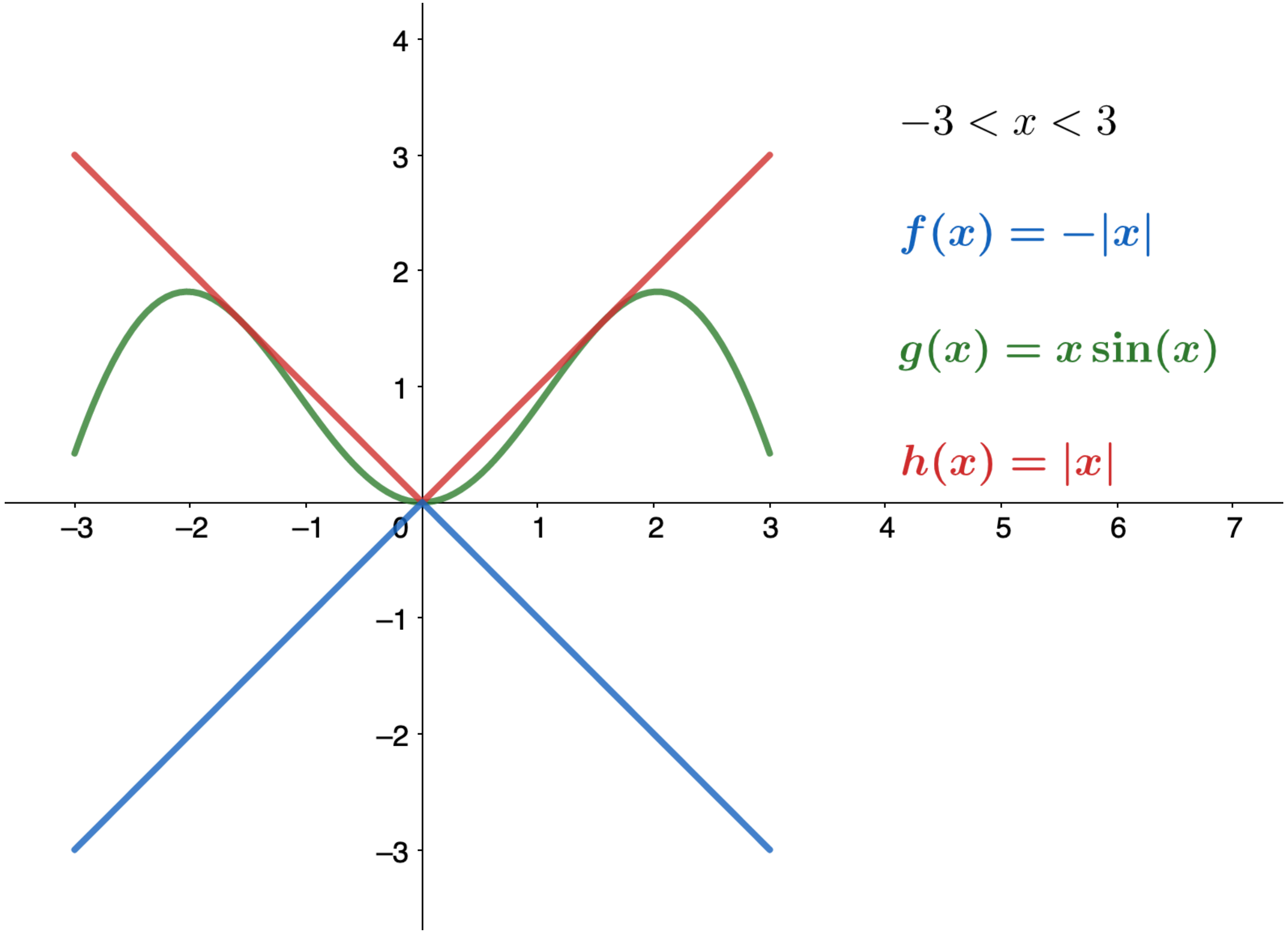
Intermediate Value Theorem

Squeeze Theorem

HOW TO EVALUATE LIMITS OF TRIGONOMETRIC FUNCTIONS?

- $\sin x$
- $\cos x$
- $\tan x$
- $\ldots$

CALCULATE LIMITS BY "SQUEEZING" A FUNCTION



$$-3 < x < 3$$

$$f(x) = -|x|$$

$$g(x) = x\sin(x)$$

$$h(x) = |x|$$

THE SQUEEZE THEOREM

THEOREM 2.7

defined for
$x \neq a$

The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing *a*. If

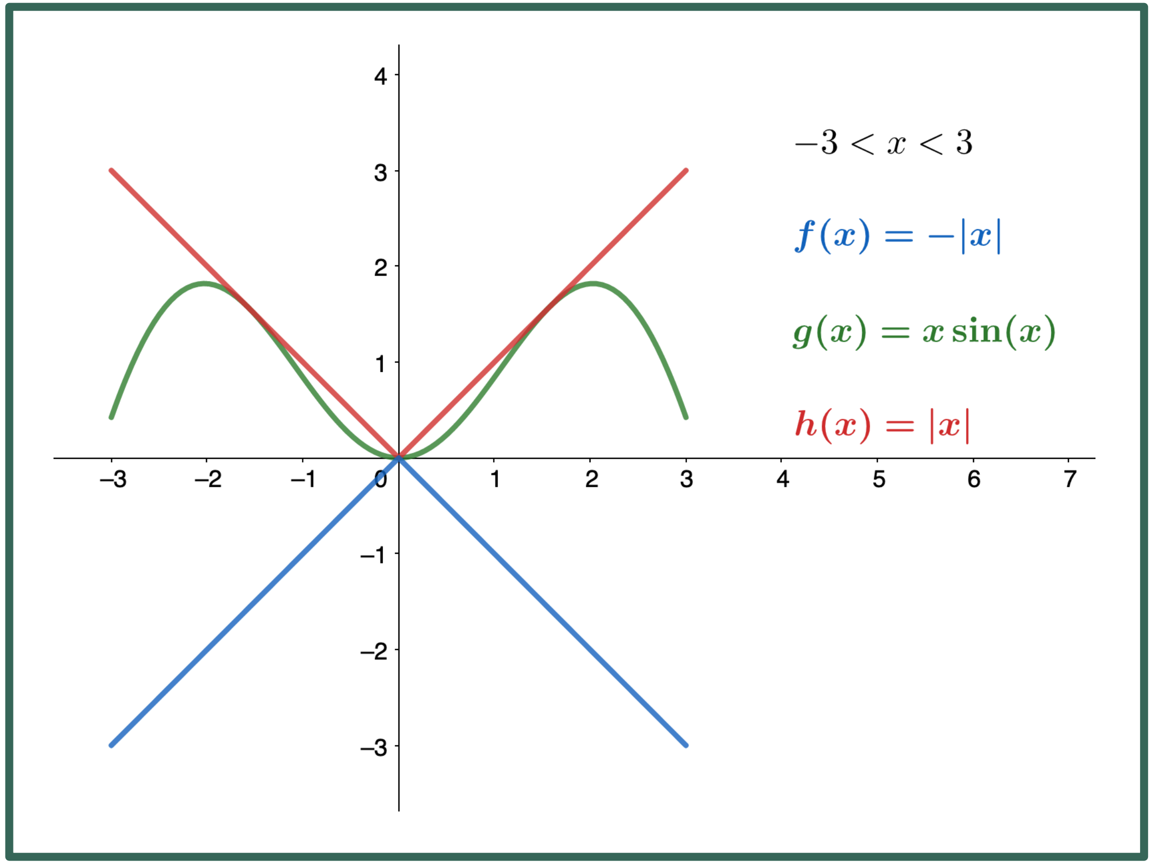
$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing *a* and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where *L* is a real number, then $\lim_{x \to a} g(x) = L$.

EXAMPLE ONE

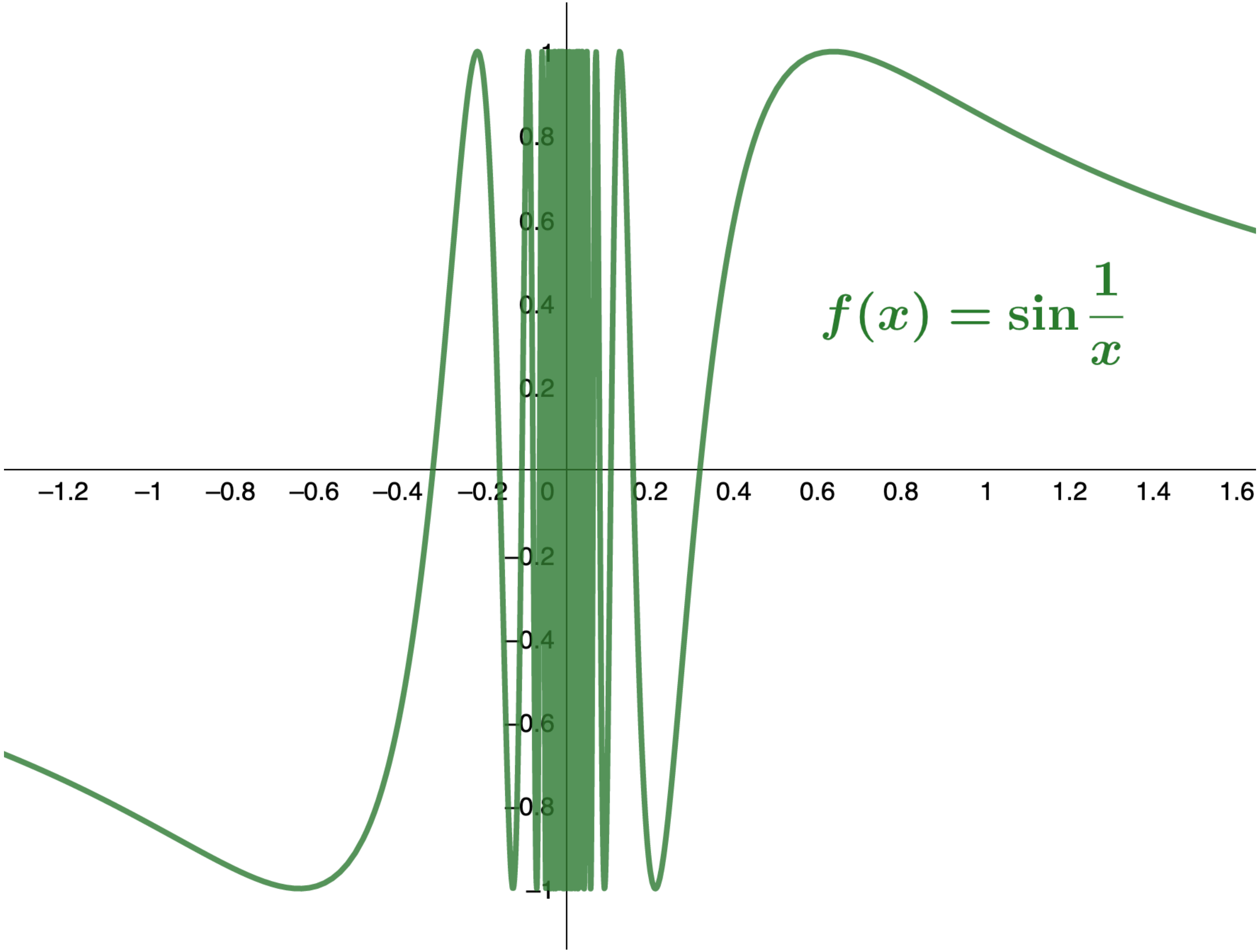


$$\lim_{x\to 0} g(x) = ?$$

- $-1 \leq \sin x \leq 1$
- $f(x) \leq g(x) \leq h(x)$

$$\lim_{x\to 0} f(x) = 0$$

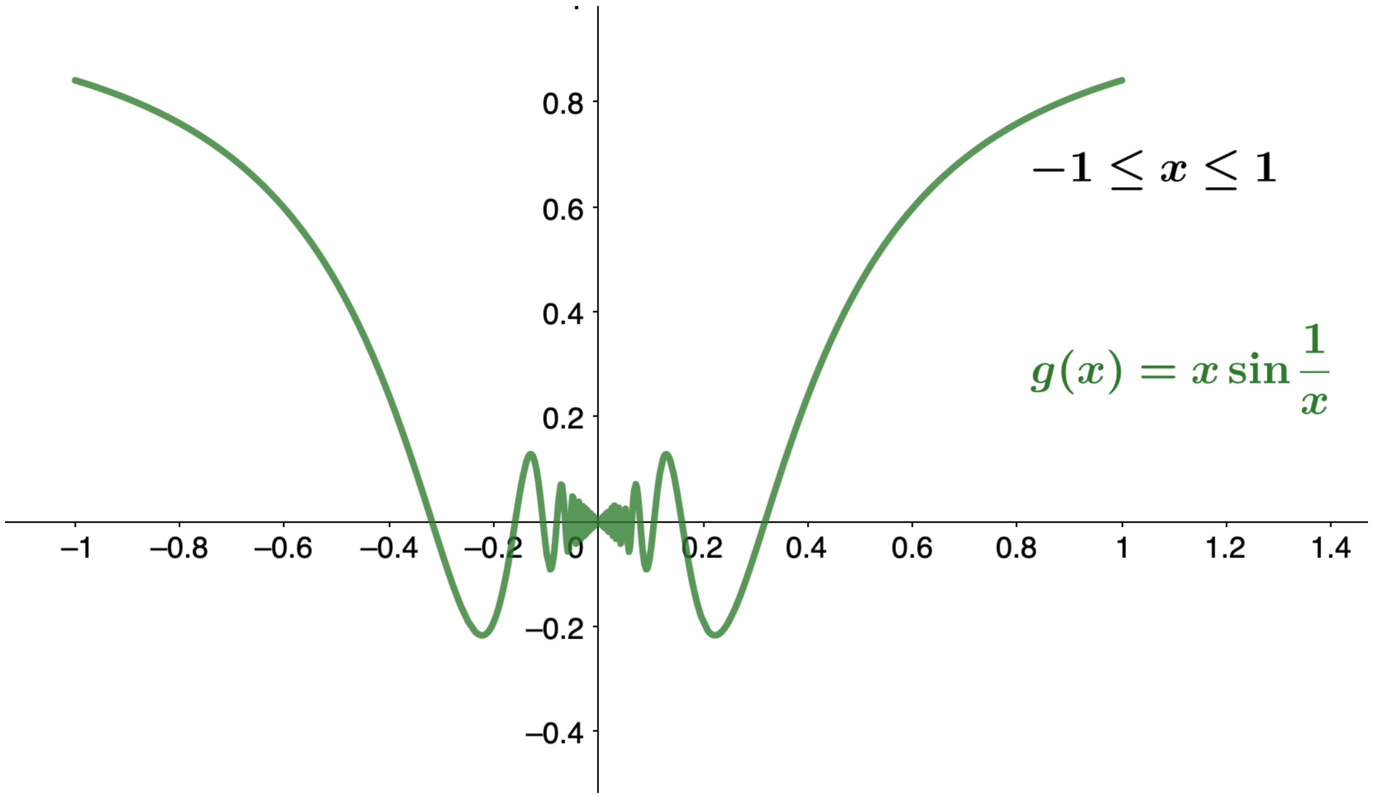
$$\lim_{x\to 0} h(x) = 0$$

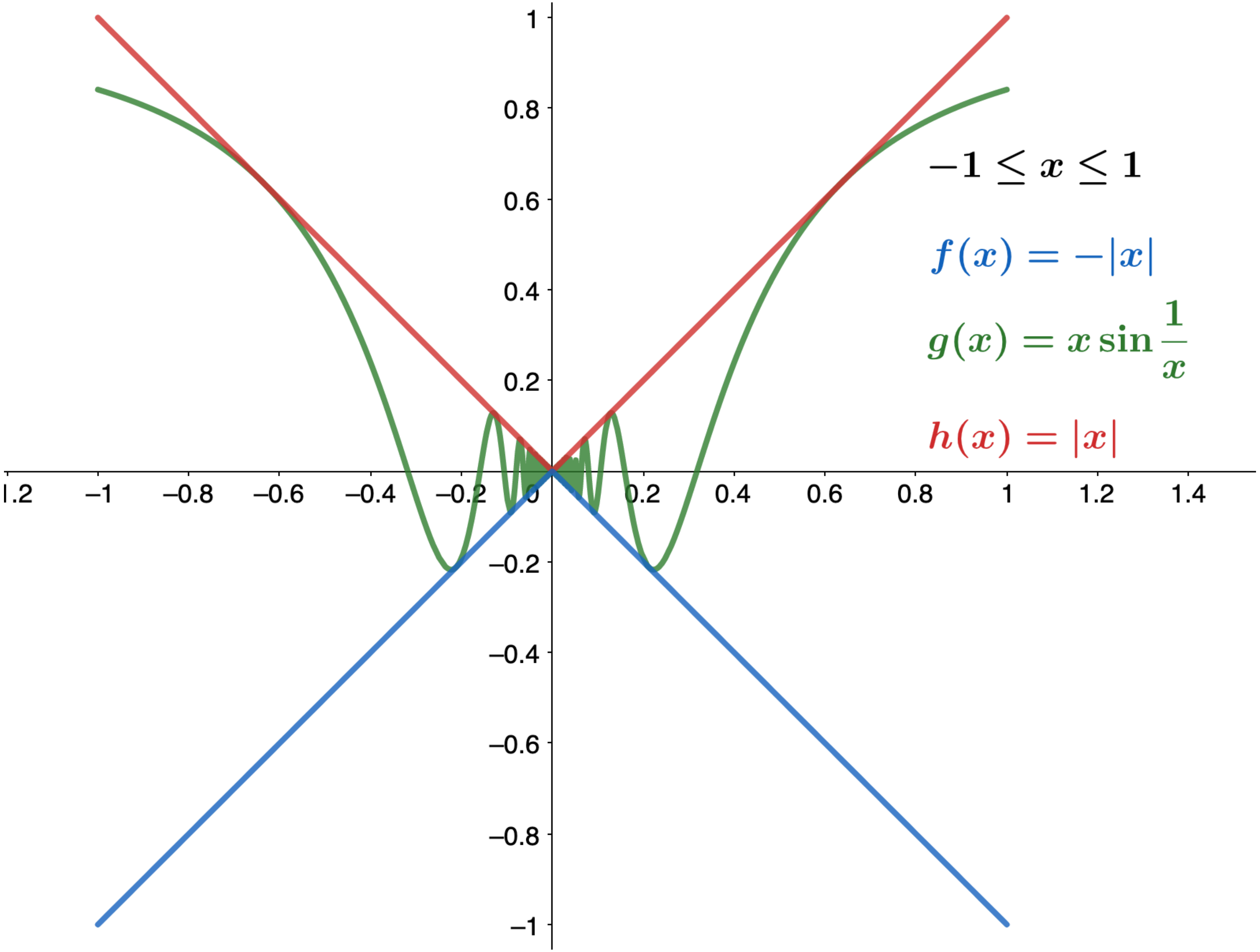

$f(x) = \sin\frac{1}{x}$
−1.2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1

EXAMPLE TWO

EXAMPLE TWO

- $\lim_{x \to 0} g(x) = ?$




$-1 \leq x \leq 1$
$f(x) = -|x|$
$g(x) = x \sin \frac{1}{x}$
$h(x) = |x|$
1
0.8
0.6
0.4
0.2
–0.2
–0.4
–0.6
–0.8
–1
–1.2
–1
–0.8
–0.6
–0.4
–0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4

EXAMPLE TWO

Evaluate the limit using the Squeeze Theorem

- $\lim\limits_{x\to 0} x \cos\frac{1}{x}.$

EXERCISE ONE A

EXERCISE ONE B

Evaluate the limit using the Squeeze Theorem

- $\lim_{x \to 0} x^2 \cos \frac{1}{x}$.

EXERCISE TWO

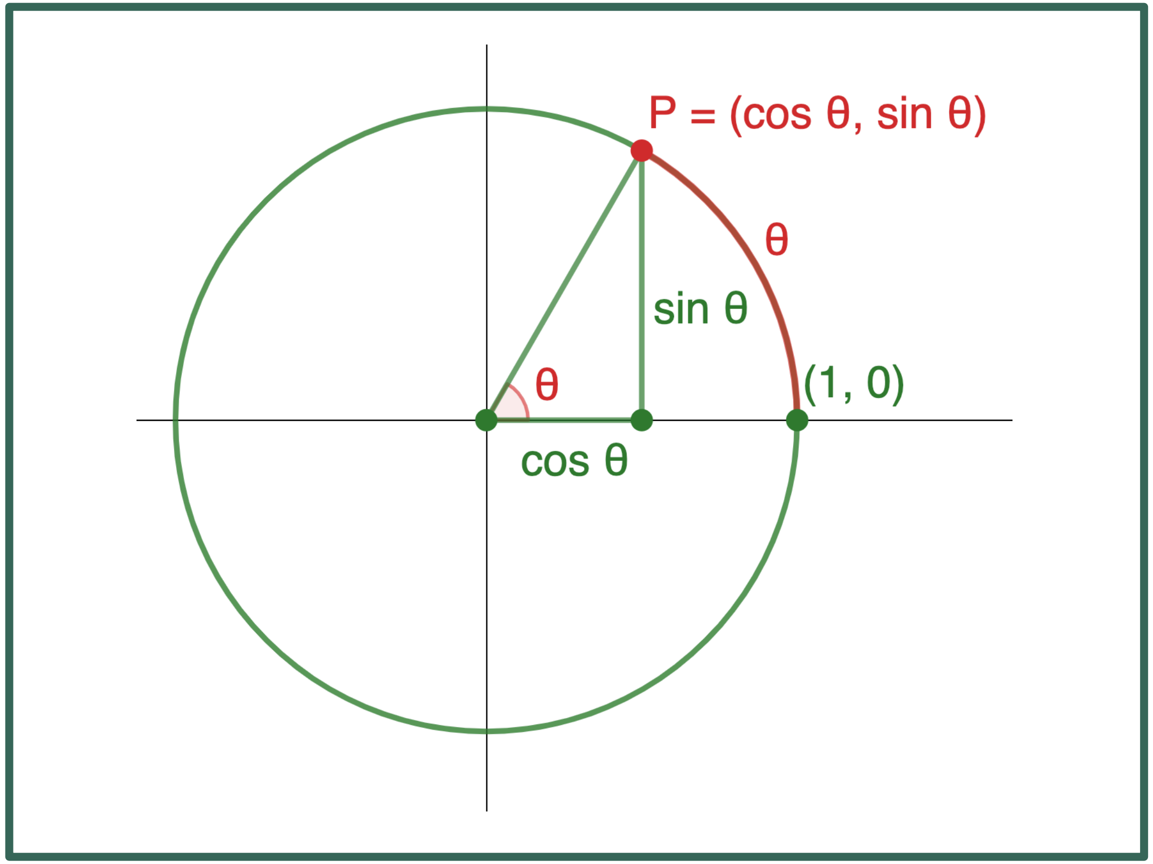
Evaluate the limit using the Squeeze Theorem

- $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$.

THANKS TO THE SQUEEZE THEOREM, NOW WE CAN…

- $\lim_{x \to 0} \sin x$
- $\lim_{x \to 0} \cos x$
- $\lim_{x \to 0} \tan x$
- …

EVALUATE $\lim_{x \to 0} \sin x$

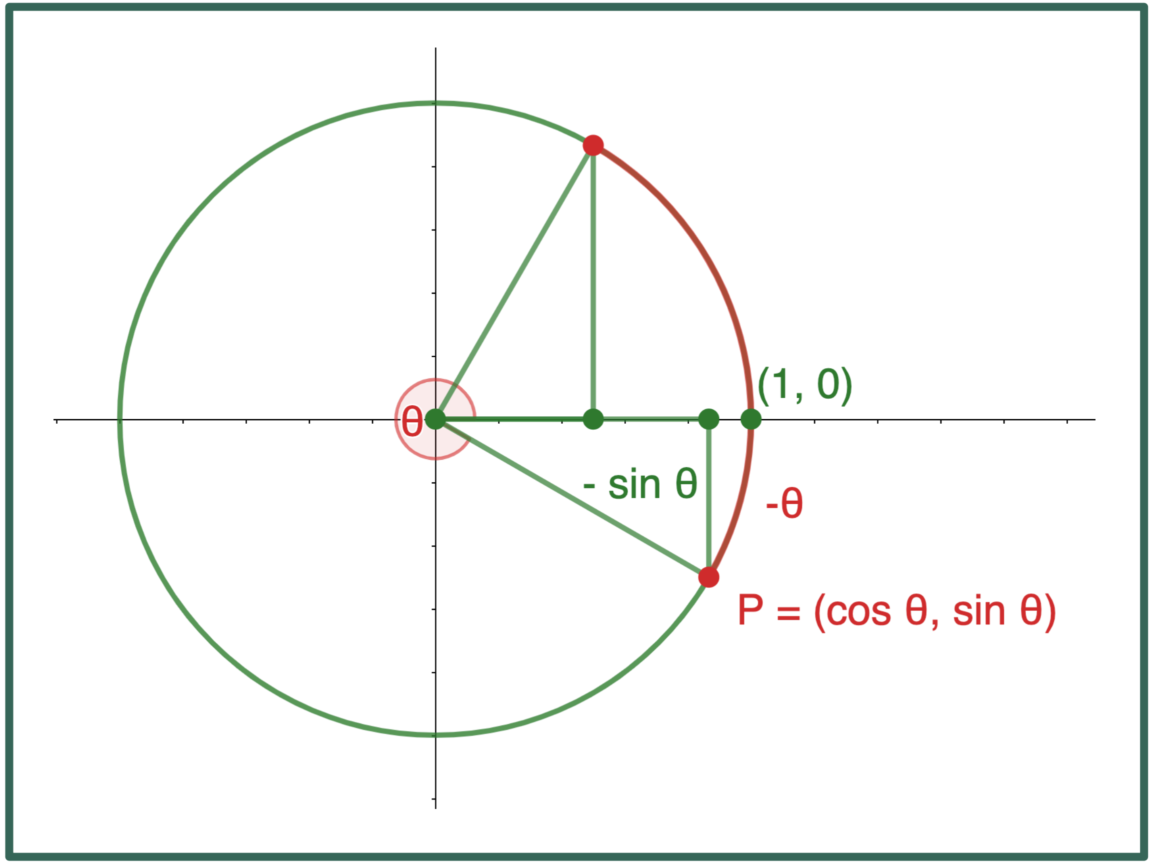


- For $0 < \theta < \frac{\pi}{2}$, $0 < \sin\theta < \theta$.

Use the squeeze theorem!

- $\lim_{\theta \to 0^+} 0 = \lim_{\theta \to 0^+} \theta = 0$
- $\lim_{\theta \to 0^+} \sin\theta = 0$

EVALUATE $\lim_{x \to 0} \sin x$



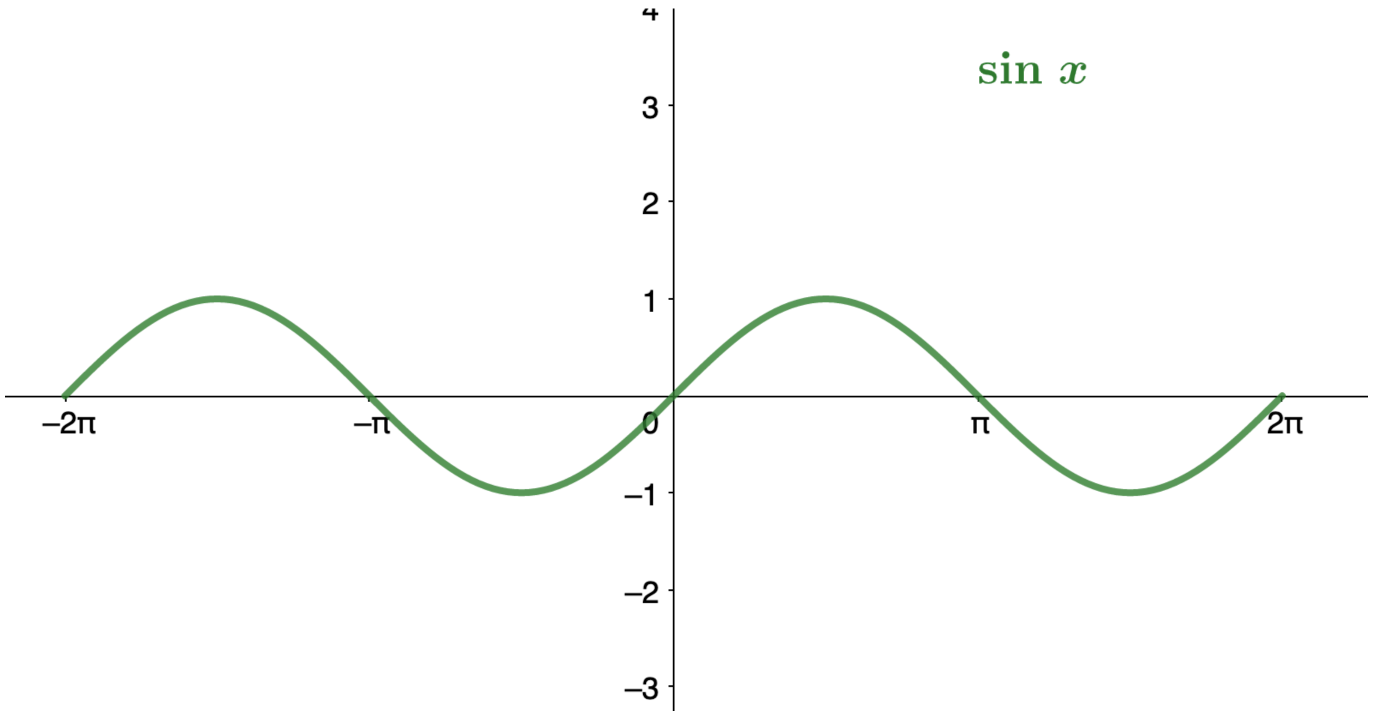
- For $-\frac{\pi}{2} < \theta < 0, 0 < -\sin\theta < -\theta.$

Use the squeeze theorem!

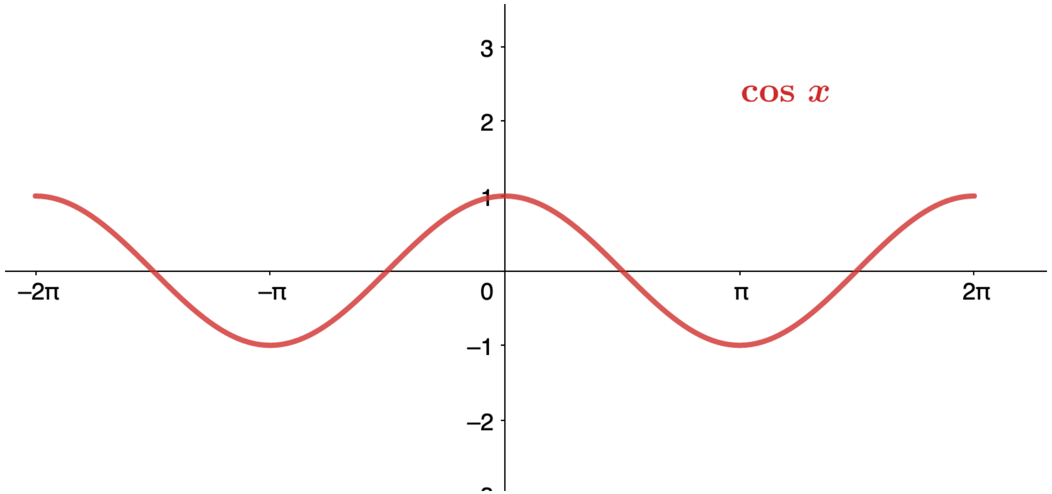
- $\lim_{\theta \to 0^-} 0 = \lim_{\theta \to 0^-} -\theta = 0$
- $\lim_{\theta \to 0^-} -\sin\theta = 0$
- $\lim_{\theta \to 0^-} \sin\theta = 0$

EVALUATE $\lim_{x \to 0} \sin x$

- $\lim_{\theta \to 0^+} \sin\theta = 0$
- $\lim_{\theta \to 0^-} \sin\theta = 0$
- $\lim_{\theta \to 0} \sin\theta = 0$

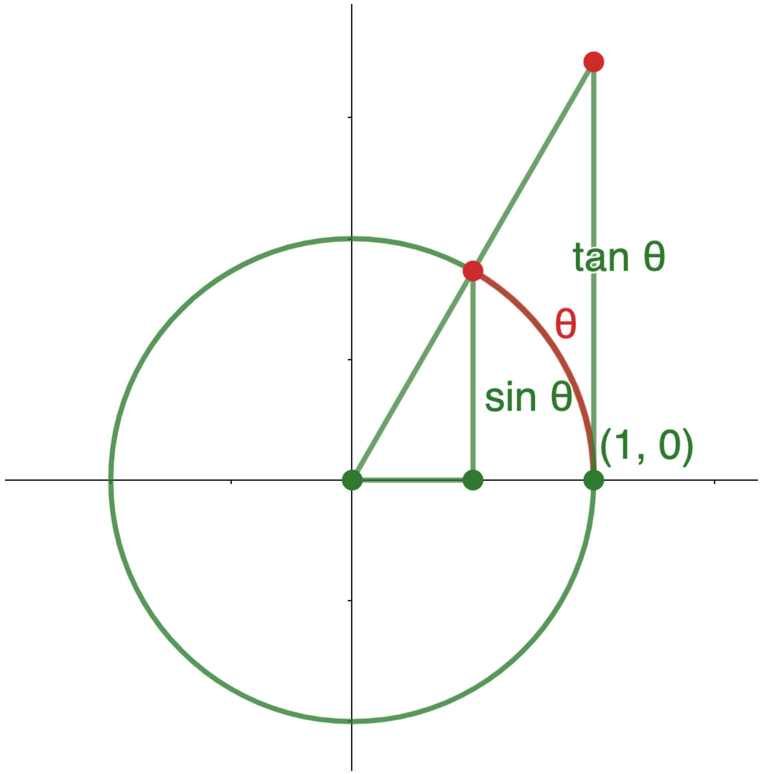


EVALUATE $\lim_{x \to 0} \cos x$



- For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos\theta = \sqrt{1 - \sin^2\theta}$.
- $\lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2\theta} = 1$

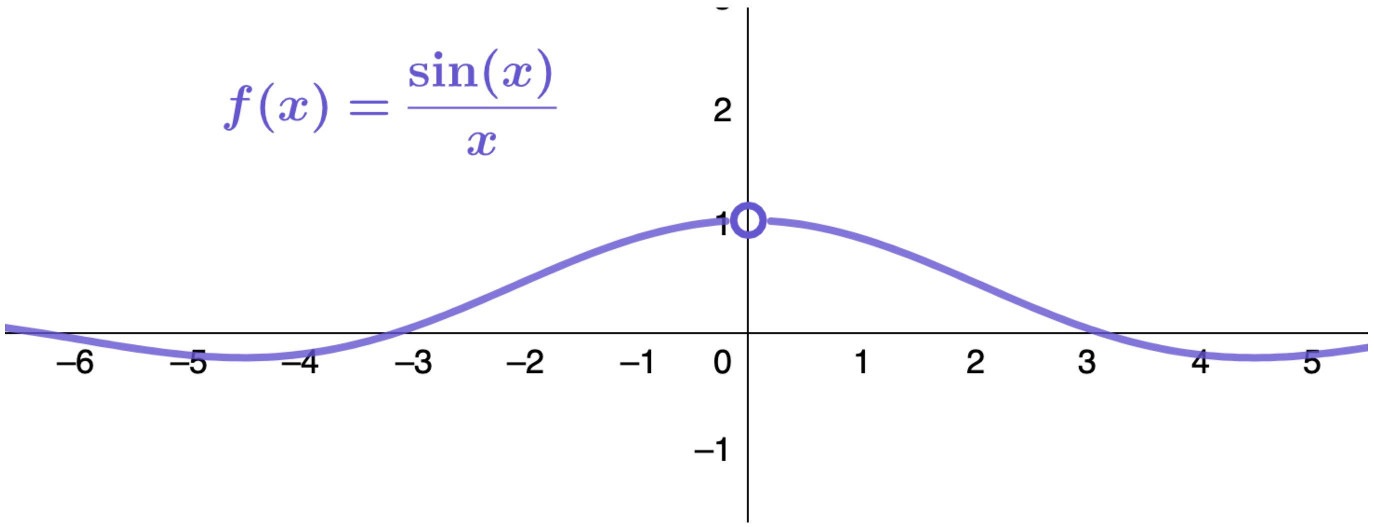
EVALUATE $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$



- For $0 < \theta < \frac{\pi}{2}$, $0 < \sin\theta < \theta < \tan\theta$.
- Hence, $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$.
- That is, $\cos\theta < \frac{\sin\theta}{\theta} < 1$.
- $\lim_{\theta \to 0^+} \cos\theta = \lim_{\theta \to 0^+} 1 = 1$
- $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

EVALUATE $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$

- $\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$
- $\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1$
- $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

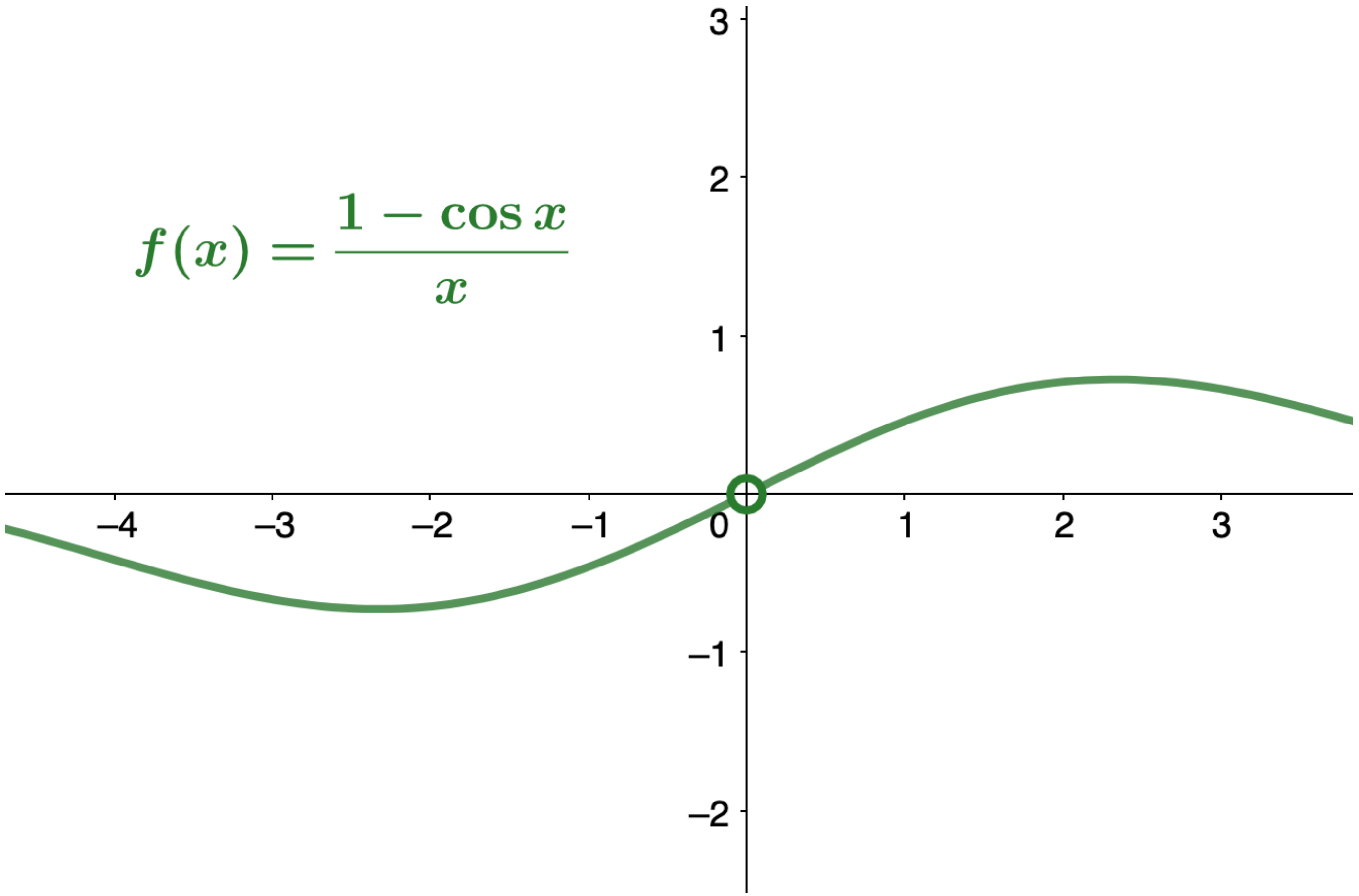


EXERCISE ONE

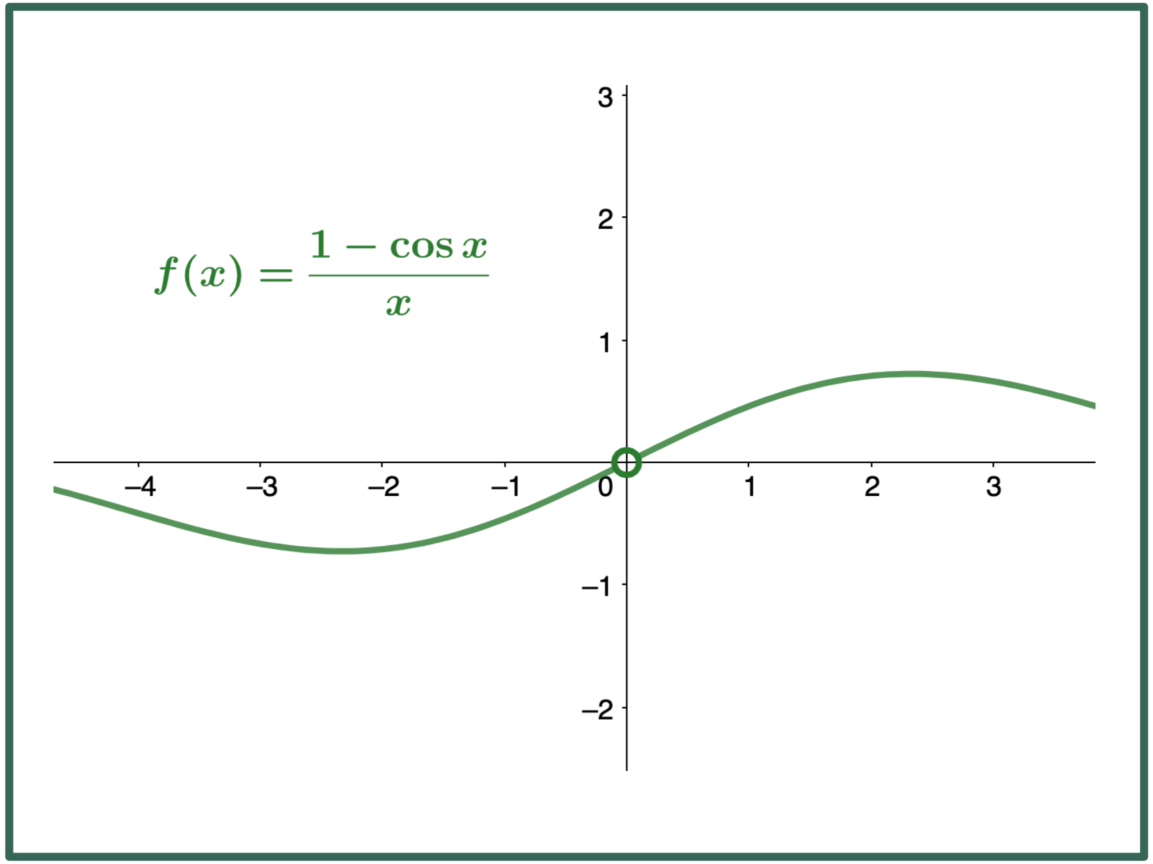
Evaluate $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta}$.

Hint

- $\sin^2 \theta + \cos^2 \theta = 1$



EXERCISE ONE

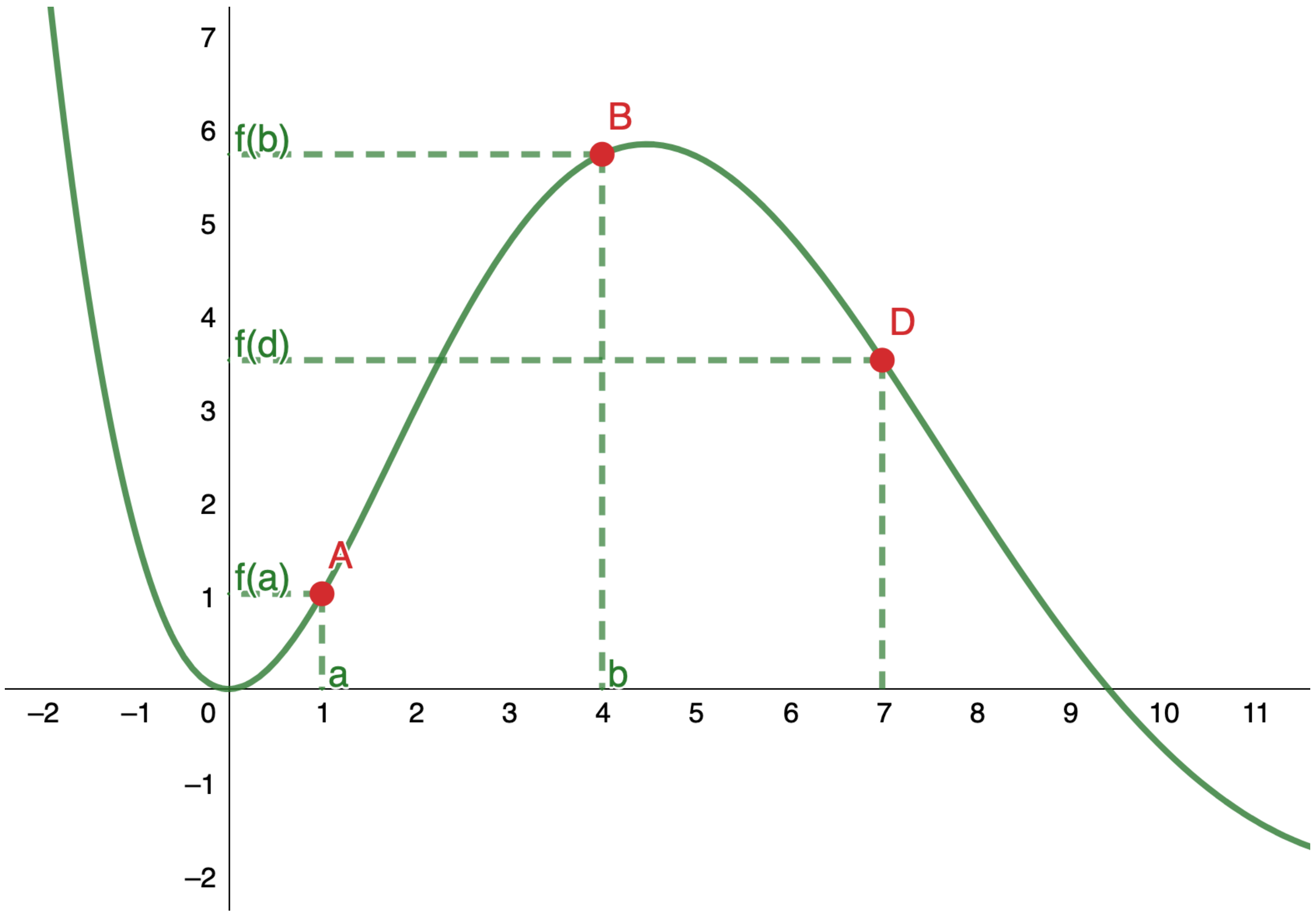


Evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$.

- $\sin^2\theta = 1-\cos^2\theta = (1+\cos\theta)(1-\cos\theta)$
- $\frac{1-\cos\theta}{\theta} = \frac{(1+\cos\theta)(1-\cos\theta)}{\theta(1+\cos\theta)} = \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \frac{\sin\theta}{\theta}\frac{\sin\theta}{1+\cos\theta}$
- $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta} = 0$
- Hence, $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{\sin\theta}{1+\cos\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta} = 0$

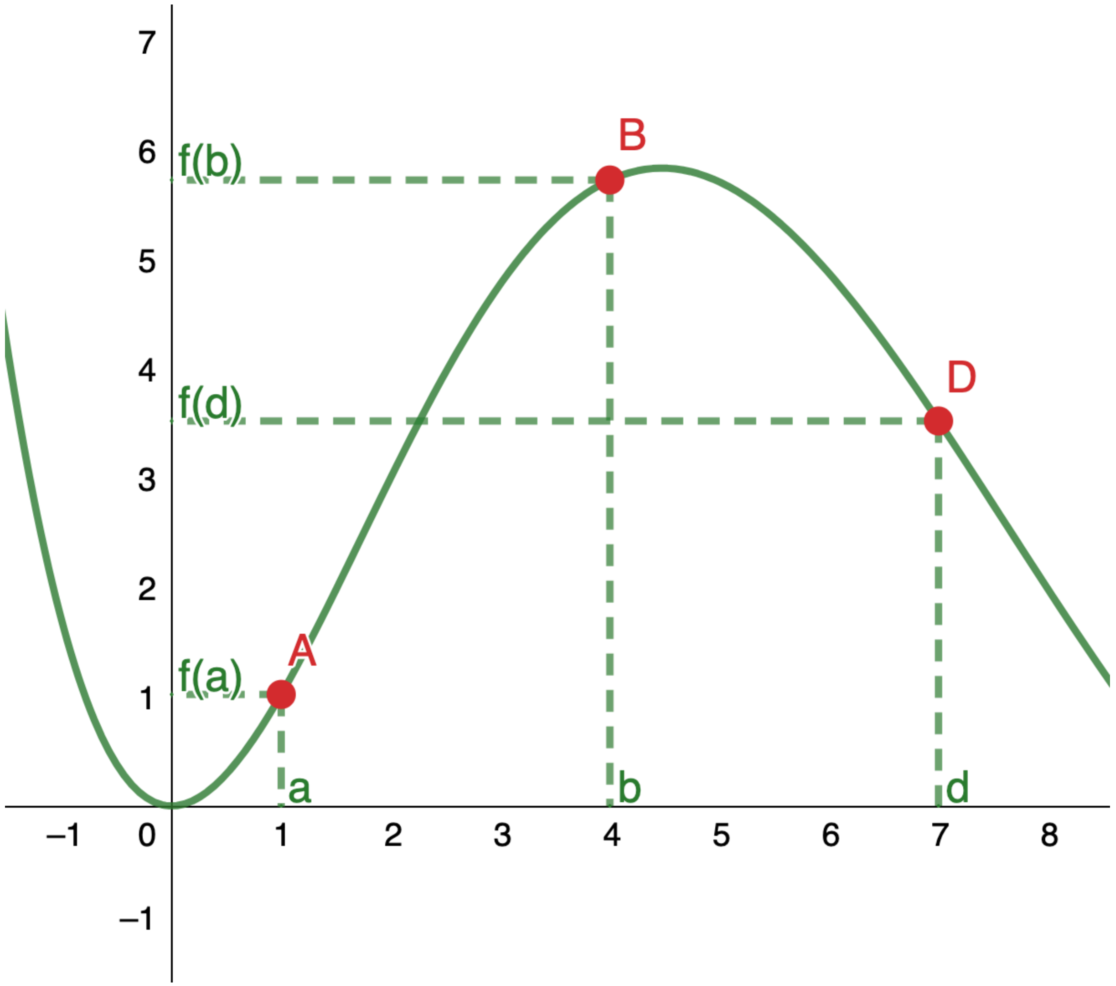
INTERMEDIATE VALUE THEOREM

- Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties.
- Throughout our study of calculus, we will encounter many powerful theorems concerning such functions.



INTERMEDIATE VALUE THEOREM

- The first of these theorems is the **Intermediate Value Theorem.**



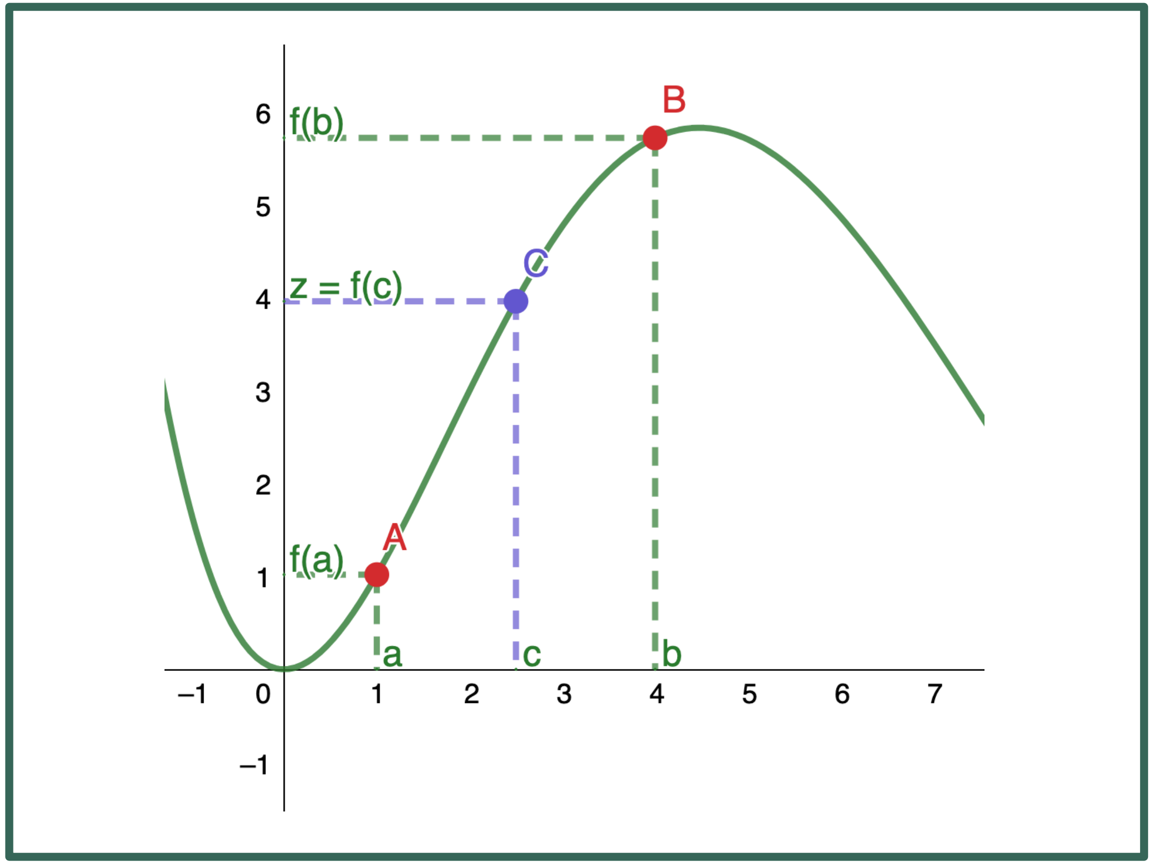
THEOREM 2.11

The Intermediate Value Theorem

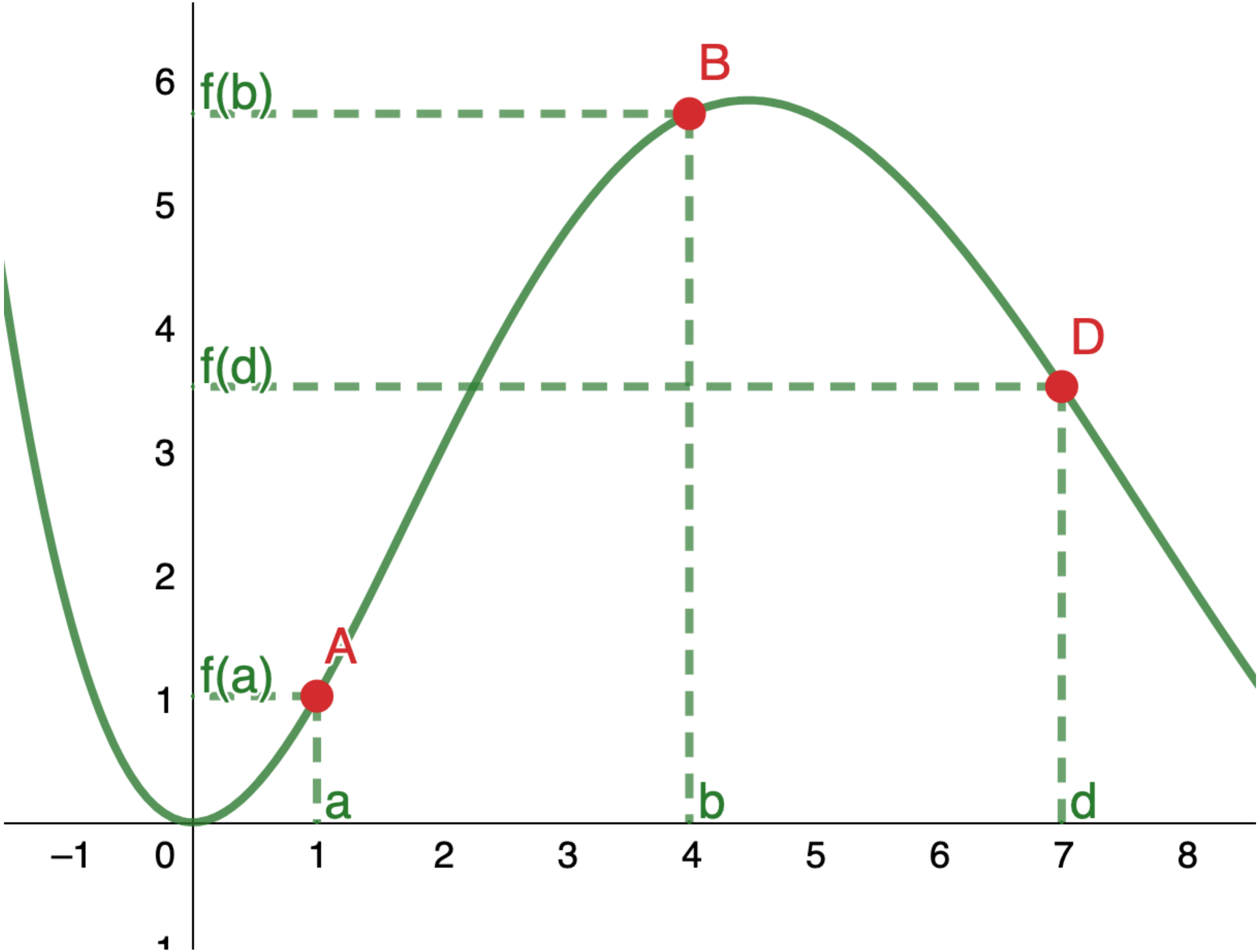
Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in Figure 2.38.

INTERMEDIATE VALUE THEOREM

INTERMEDIATE VALUE THEOREM



- f is continuous over a **closed**, **bounded** interval $[a, b]$.
- z is any real number **between** $f(a)$ and $f(b)$.
- There is a number c in $[a, b]$, satisfying $f(c) = z$.



INTERMEDIATE VALUE THEOREM (MORE TO SAY)

APPLICATION ZERO

Show that $f(x) = x^3 + x^2 + 1$ has at least one zero.

Hint

- Find a **closed**, **bounded** interval $[a, b]$.
- 0 is a real number **between** $f(a)$ and $f(b)$.
- There is a number c in $[a, b]$, satisfying $f(c) = 0$.

APPLICATION ONE

Show that $f(x) = e^x \sin x - 1$ has at least one zero.

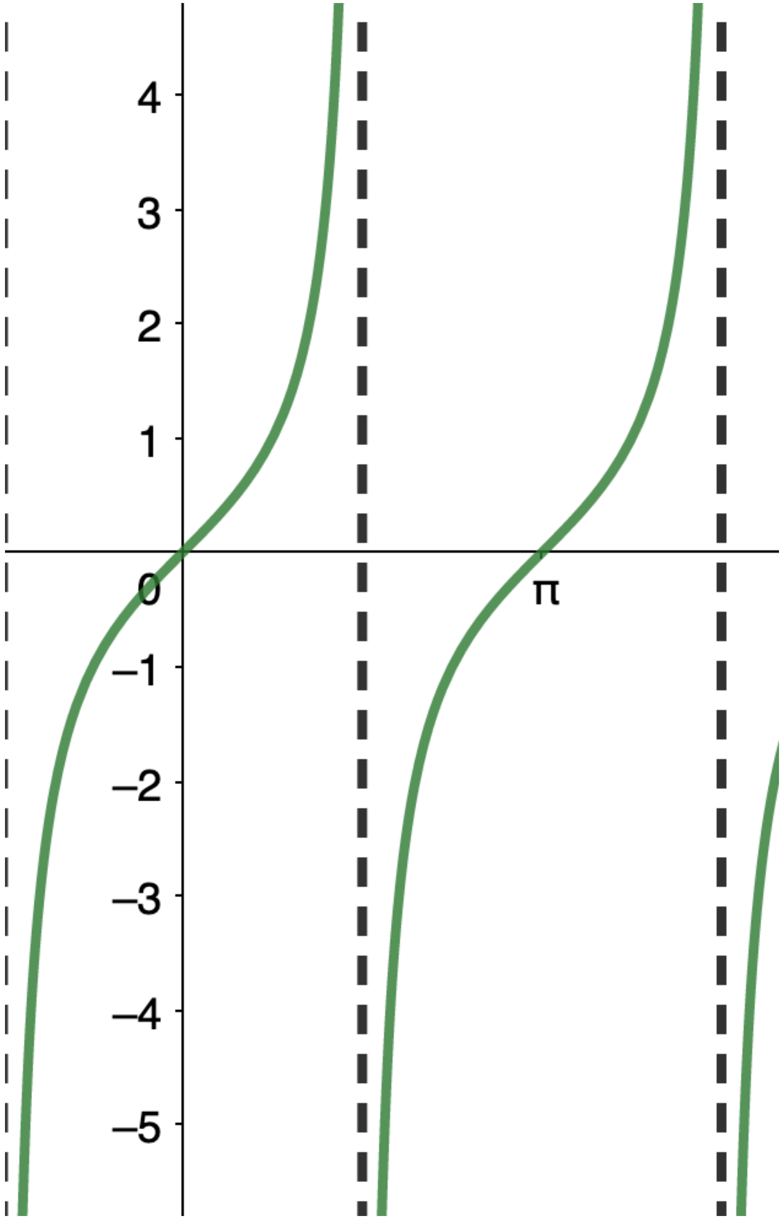
Hint

- Find a **closed**, **bounded** interval $[a, b]$.
- 0 is a real number **between** $f(a)$ and $f(b)$.
- There is a number c in $[a, b]$, satisfying $f(c) = 0$.

Show that $f(x) = e^x \sin x - 1$ has at least one zero.

- Find a **closed**, **bounded** interval $[0, \frac{\pi}{2}]$.
- 0 is a real number **between** $f(0) = -1$ and $f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} - 1$.
- There is a number c in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, satisfying $f(c) = 0$.

APPLICATION ONE



APPLICATION TWO

When Can You Apply the Intermediate Value Theorem?

- For $f(x) = \tan x,\ f\left(\frac{\pi}{4}\right) = 1 > 0$ and $f\left(\frac{3\pi}{4}\right) = -1 < 0$.
- Can we conclude that $f(x)$ has a zero in the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$?

APPLICATION THREE

What Does the Intermediate Value Theorem tell us?

- If $f(x)$ is continuous over $[7, 22]$, $f(7) > 0$ and $f(22) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[7, 22]$?

APPLICATION THREE