INTRODUCTION TO CALCULUS

THE CHAIN RULE

State	State **the chain rule** for the **composition of two functions**.
Apply	Apply the chain rule together with the power rule.
Apply	Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
Recognize	Recognize the chain rule for a composition of three or more functions.
Describe	Describe the proof of the chain rule.

OUTLINE

FROM BRUTAL FORCE TO HIGH TECHNIQUES

WHY WE NEED MORE TECHNIQUES?

- We have seen the techniques for differentiating basic functions (x^n, $\sin(x)$, $\cos(x)$, etc.) as well as sums, differences, products, quotients, and constant multiples of these functions.

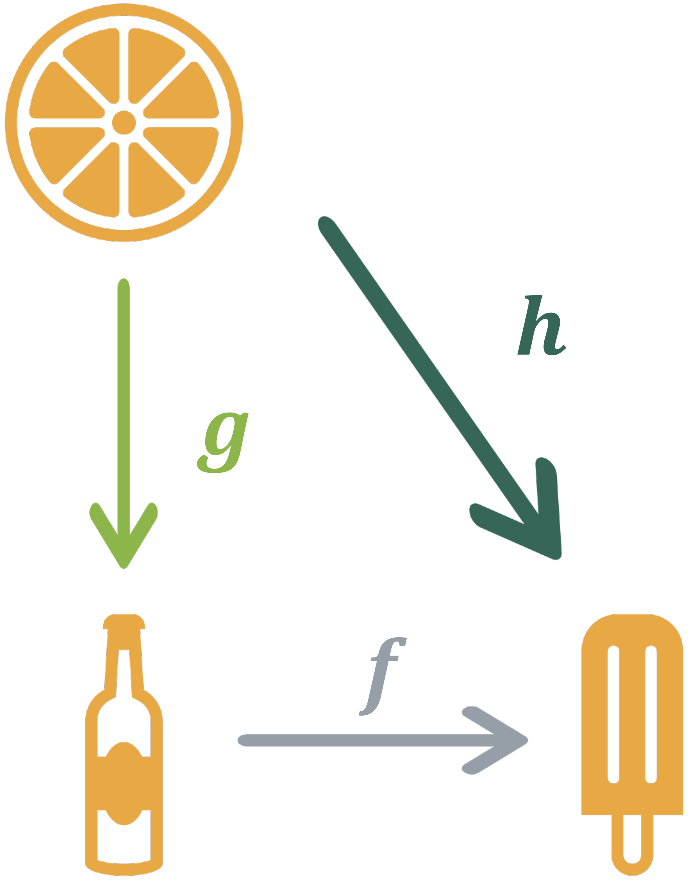
DIFFERENTIATE COMPOSITE FUNCTIONS

- $f(x) = \sin^2 x$
- $g(x) = \sin(x^2)$
- $h(x) = \sqrt{x^{2019} + 1984}$

In this section, we study the rule for finding the derivative of the composition of two or more functions.

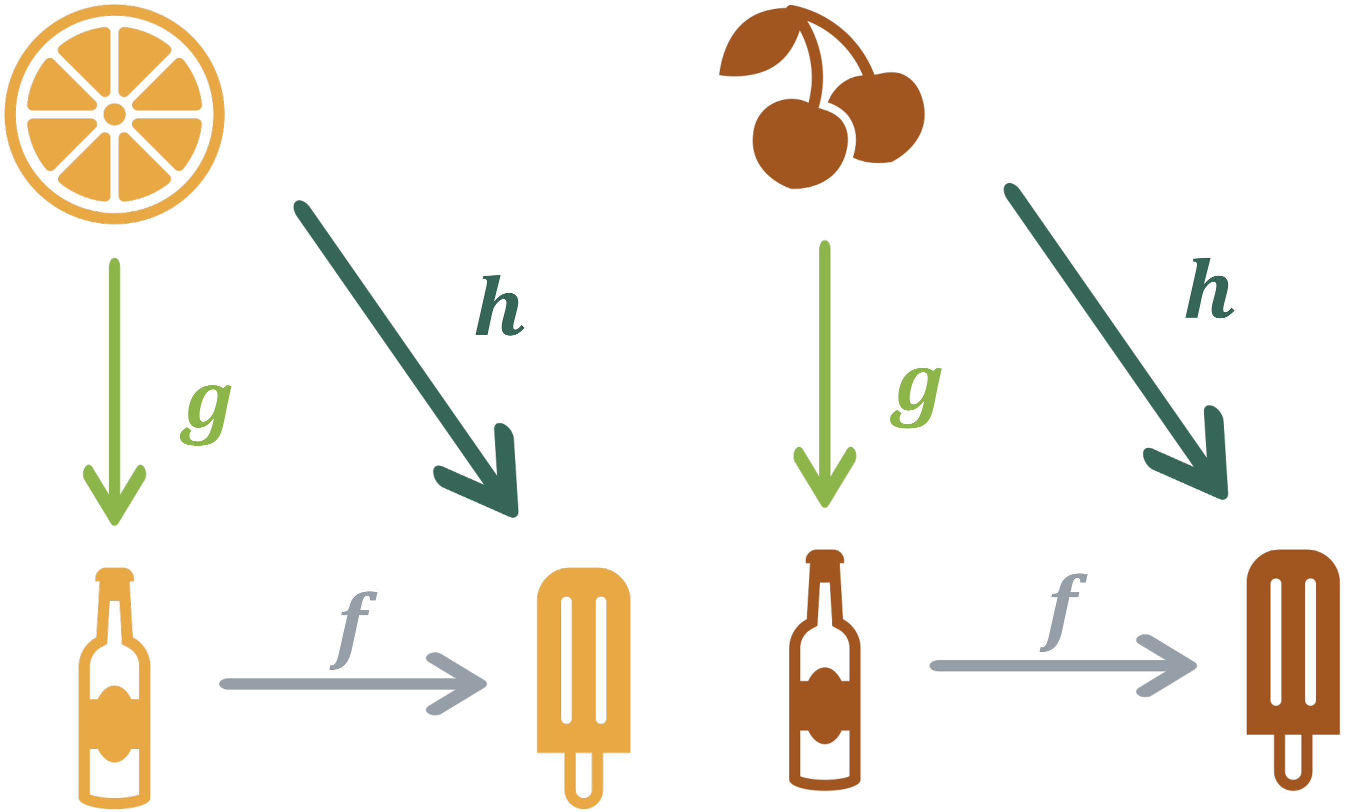
DERIVING THE CHAIN RULE

- $h(x) = (f \circ g)(x) = f\big(g(x)\big)$
- $h'(x) = f'(g(x))g'(x)$



EXAMPLE

- $h(x) = (f \circ g)(x) = f\big(g(x)\big)$
- $h(x) = \sin(x^2)$
- $f(x) = \sin(x)$
- $g(x) = x^2$



EXAMPLE

- $h(x) = \sin(x^2), f(x) = \sin(x), g(x) = x^2$
- $h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{\sin(x^2)-\sin(a^2)}{x-a} = ?$
- **What we have known?**
- $f'(a^2) = \lim_{u\to a^2} \frac{f(u)-f(a^2)}{u-a^2} = \lim_{u\to a^2} \frac{\sin(u)-\sin(a^2)}{u-a^2} = \cos(a^2)$
- $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = 2a$

$$\boxed{f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}}$$

EXAMPLE

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{\sin(x^2) - \sin(a^2)}{x - a}$$

$$= \lim_{x \to a} \frac{\sin(x^2) - \sin(a^2)}{x^2 - a^2} \lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

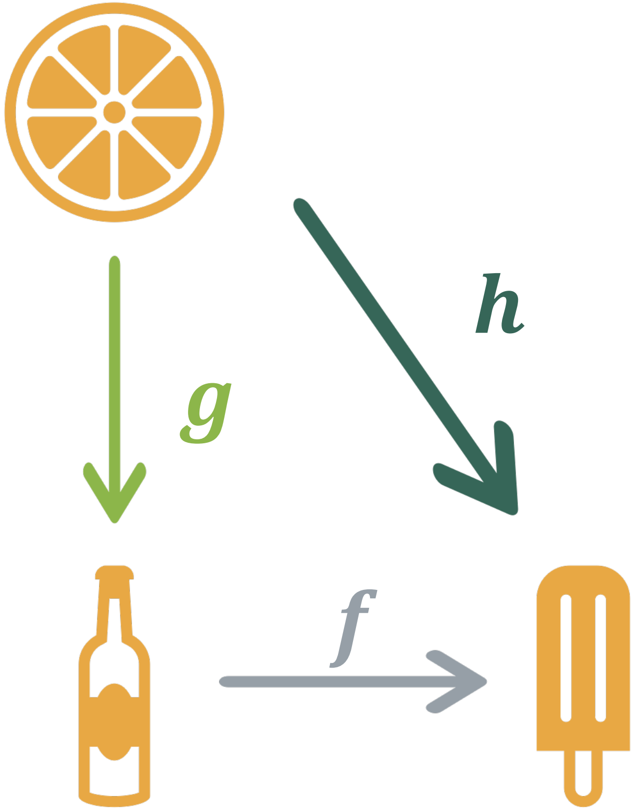
$$= \lim_{u \to a^2} \frac{\sin(u) - \sin(a^2)}{u - a^2} \lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

$$= \cos(a^2)\, 2a = f'\left(a^2\right) g'(a) = f'(g(a))\, g'(a)$$

EXAMPLE

- $h'(a) = f'\left(a^2\right) g'(a) = f'(g(a))\, g'(a)$

The derivative of $h(x) = \sin(x^2)$ is the product of the derivative of $g(x) = x^2$ and the derivative of the function $f(x) = \sin(x)$ evaluated at the function $g(x) = x^2$.



THE CHAIN RULE

RULE: THE CHAIN RULE

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$\boxed{h'(x) = f'(g(x))\, g'(x).} \qquad (3.17)$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.}$$

PROBLEM-SOLVING STRATEGY

PROBLEM-SOLVING STRATEGY: APPLYING THE CHAIN RULE

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

THE CHAIN AND POWER RULES COMBINED

$h(x) = (g(x))^n$

- $f(x) = ?$
- $g(x)$

$h(x) = (g(x))^n$

- $f(x) = x^n$
- $g(x)$

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.

THE CHAIN AND POWER RULES COMBINED

$h(x) = (g(x))^n$

$f(x) = x^n$

$g(x)$

$f'(x) = nx^{n-1}$

$f'(g(x)) = n(g(x))^{n-1}$

2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.

THE CHAIN AND POWER RULES COMBINED

$h(x) = (g(x))^n$

$f(x) = x^n$

$f'(x) = nx^{n-1}$

$f'(g(x)) = n(g(x))^{n-1}$

$g(x)$

$g'(x)$

3. Find $g'(x)$.

THE CHAIN AND POWER RULES COMBINED

$h(x) = (g(x))^n$

$f(x) = x^n$

$f'(x) = nx^{n-1}$

$f'(g(x)) = n(g(x))^{n-1}$

$g(x)$

$g'(x)$

$h'(x) = n(g(x))^{n-1} g'(x)$

4. Write $h'(x) = f'(g(x)) \cdot g'(x)$.

RULE: POWER RULE FOR COMPOSITION OF FUNCTIONS

For all values of *x* for which the derivative is defined, if

$$h(x) = (g(x))^n.$$

Then

$$\boxed{h'(x) = n(g(x))^{n-1} g'(x).} \tag{3.18}$$

THE CHAIN AND POWER RULES COMBINED

EXERCISE ONE

Using the Chain and Power Rules

- Find the derivative of $f(x) = \sin^2(x)$
- Find the derivative of $f(x) = (2x^2 + 3x + 2)^3$
- Find the derivative of $f(x) = \frac{1}{(x^2+3)^3}$

EXERCISE TWO

Finding the Equation of a Tangent Line

- Find the equation of a line tangent to the graph of $f(x) = (2\sqrt{x} + 1)^2$ at $x = 4$.
- Find the equation of a line tangent to the graph of $f(x) = \frac{1}{(\sin(x)+\cos(x))^2}$ at $x = \frac{\pi}{2}$.

$$\boxed{h'(x) = f'(g(x))\, g'(x)\,.}$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.}$$

COMBINE THE CHAIN RULE WITH OTHER RULES

EXAMPLE ON TRIGONOMETRIC FUNCTIONS

- Find the derivative of $h(x) = \sin(g(x))$.
- Find the derivative of $h(x) = \cos(g(x))$.

EXERCISE THREE

- Find the derivative of $h(x) = \sin(2019x + 1984)$.
- Find the derivative of $h(x) = \cos(\frac{1}{x})$.
- Find the derivative of $h(x) = \tan(\sqrt{x})$.

CHAIN-RULE AND TRIGONOMETRIC FUNCTIONS

THEOREM 3.10

Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$$\begin{aligned}
\frac{d}{dx}(\sin(g(x)) &= \cos\,(g\,(x))\,g'(x)\\
\frac{d}{dx}(\cos(g(x)) &= -\sin\,(g\,(x))\,g'(x)\\
\frac{d}{dx}(\tan(g(x)) &= \sec^2\,(g\,(x))\,g'(x)\\
\frac{d}{dx}(\cot(g(x)) &= -\csc^2\,(g\,(x))\,g'(x)\\
\frac{d}{dx}(\sec(g(x)) &= \sec(g\,(x)\ \tan\,(g\,(x))\,g'(x)\\
\frac{d}{dx}(\csc(g(x)) &= -\csc(g\,(x))\cot\,(g\,(x))\,g'(x)
\end{aligned}$$

COMBINE THE CHAIN RULE WITH THE PRODUCT RULE

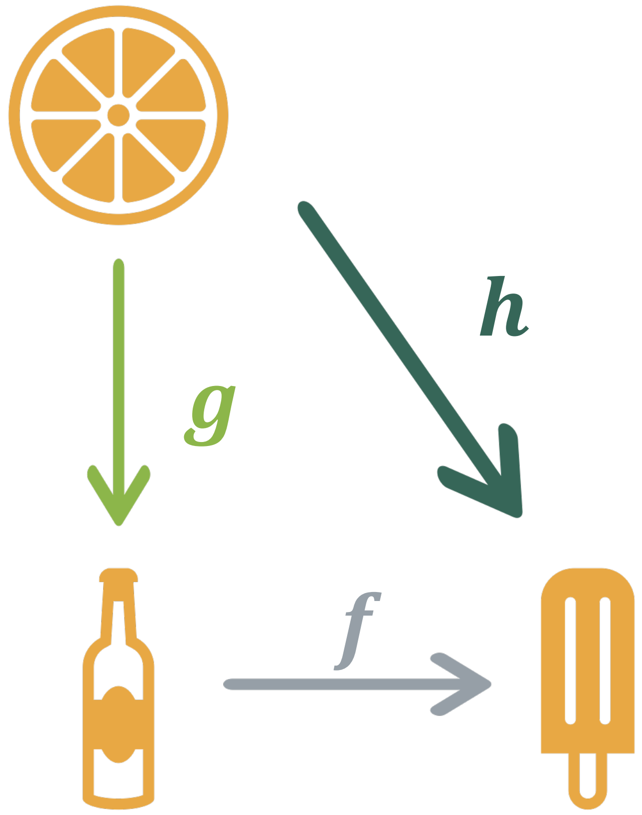
- Find the derivative of $h(x) = (x+1)^2(2x^2+x+2)^2$.

COMBINE CHAIN RULE WITH THE QUOTIENT RULE

- Find the derivative of $h(x) = \frac{2x}{(\sin(x)+\cos(x))^2}$.

COMPOSITES OF THREE OR MORE FUNCTIONS

- How to differentiate the composition of three or more functions?
- We can simply apply the chain rule multiple times!!!



COMPOSITES OF THREE OR MORE FUNCTIONS

In general terms, first we let

- $k(x) = h(f(g(x)))$

Then, applying the chain rule once we obtain

- $k'(x) = \frac{d}{dx} h\left(f\left(g(x)\right)\right) = h'(f(g(x))) \frac{d}{dx} f(g(x))$

Applying the chain rule again, we obtain

- $k'(x) = h'\left(f\left(g(x)\right)\right) \frac{d}{dx} f\left(g(x)\right) = h'\left(f\left(g(x)\right)\right) f'(g(x)) g'(x)$

CHAIN RULE FOR A COMPOSITION OF THREE FUNCTIONS

RULE: CHAIN RULE FOR A COMPOSITION OF THREE FUNCTIONS

For all values of *x* for which the function is differentiable, if

$$k(x) = h(f(g(x))),$$

then

$$k'(x) = h'(f(g(x)))f'(g(x))g'(x).$$

In other words, we are applying the chain rule twice.

CHAIN RULE FOR A COMPOSITION OF THREE FUNCTIONS

- $\boldsymbol{k}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})))$
- $\boldsymbol{k}'(\boldsymbol{x}) = \boldsymbol{h}'\left(\boldsymbol{f}\left(\boldsymbol{g}(\boldsymbol{x})\right)\right)\boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{x}))\boldsymbol{g}'(\boldsymbol{x})$

EXERCISE FOUR

- Find the derivative of $k(x) = \sin^4(2x^2 + x + 2)$.
- Find the derivative of $k(x) = \sqrt{\cos(2x)}$.

EXERCISE FIVE

Using the Chain Rule with Functional Values

- Let $h(x) = f(g(x))$. If $g(3) = 20, g'(3) = 3$, and $f'(20) = 15$, find $h'(3)$.
- Given $k(x) = h(f(g(x)))$. If $g(2) = -3,\ g'(2) = 4, f(-3) = 5, f'(-3) = 7$, and $h'(5) = 6$, find $k'(2)$.

THE CHAIN RULE USING LEIBNIZ'S NOTATION

- This notation for the chain rule is used heavily in physics applications.
- For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = g(u)$. Thus,
- $h'(x) = \frac{dy}{dx}$, $f'\big(g(x)\big) = \frac{dy}{du}$, and $g'(x) = \frac{du}{dx}$.
- Consequently,
- $\frac{dy}{dx} = h'(x) = f'\big(g(x)\big)g'(x) = \frac{dy}{du}\frac{du}{dx}$.

EXERCISE FIVE

Taking a Derivative Using Leibniz's Notation

- Find the derivative of $y = \left(\frac{x}{2x+3}\right)^4$.
- Find the derivative of $y = \tan(\sqrt{x})$.