THE DERIVATIVE AS A FUNCTION

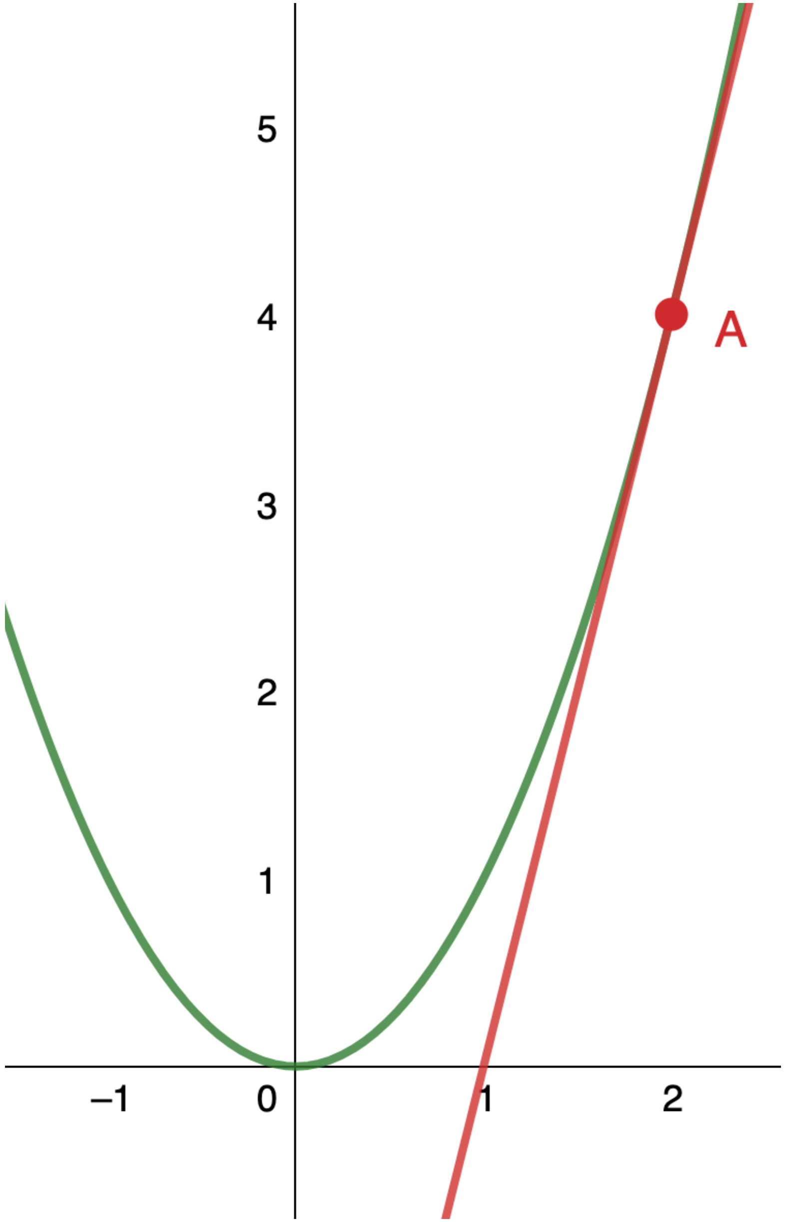
INTRODUCTION TO CALCULUS

Define	Define **the derivative function** of a given function.
Graph	Graph a derivative function **from the graph of** a given function.
State	State the connection between **derivatives and continuity**.
Describe	Describe three conditions for when a function does not have a derivative.
Explain	Explain the meaning of a **higher-order** derivative.

OUTLINE

REVIEW A LITTLE BIT

- The derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function **at that point**.
- If we differentiate a position function at a given time, we obtain the velocity **at that time**.
- Knowing the derivative of the function **at every point** would produce valuable information about the behavior of the function.
- The process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious.



DEFINITION

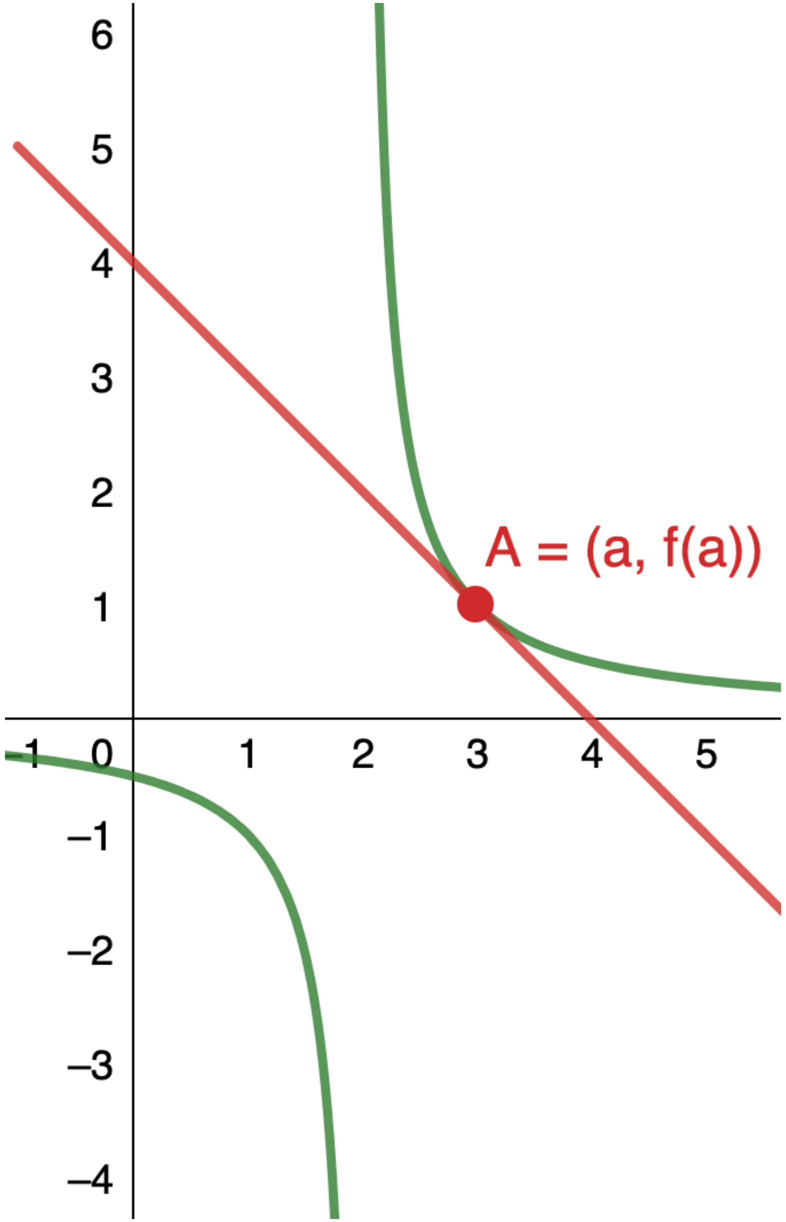
Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.} \tag{3.9}$$

DERIVATIVE FUNCTIONS

REMARKS

- A function $f(x)$ is said to be **differentiable at a** if $f'(a)$ exists.
- More generally, a function is said to be **differentiable on S** if it is differentiable at every point in an open set S, and **a differentiable function** is one in which $f'(x)$ exists on its domain.



NOTATIONS

$$f(x) = x^2 - 2x - 3$$

$$f'(x) = \frac{df(x)}{dx} = 2x - 2$$

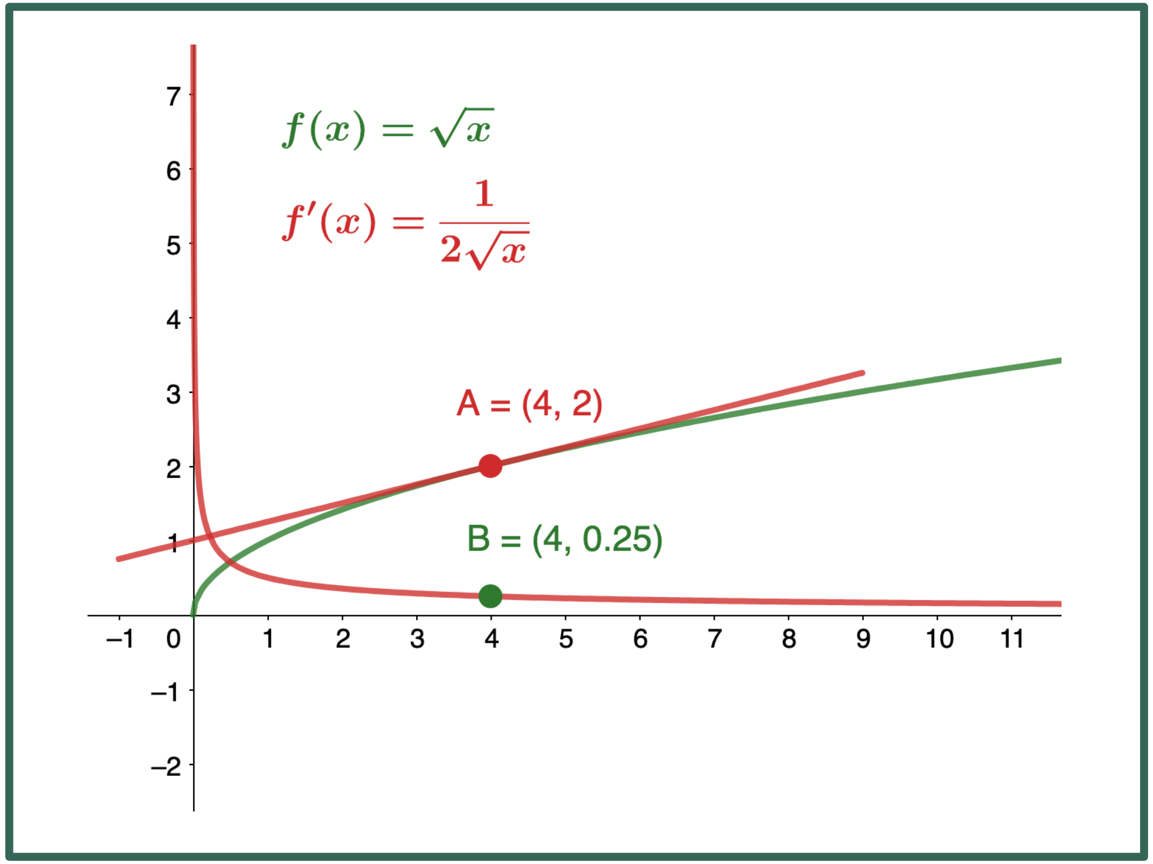
$$f'(2) = 2$$

$$y = x^2 - 2x - 3$$

$$y' = \frac{dy}{dx} = 2x - x$$

$$\frac{dy}{dx}|_{x=2} = 2$$

EXAMPLE ONE: **FINDING THE DERIVATIVE OF A SQUARE-ROOT FUNCTION**



- Find the derivative of $f(x) = \sqrt{x}$.
- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} =$

$$\lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} =$$

$$\lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0} \frac{1}{2\sqrt{x}}.$$

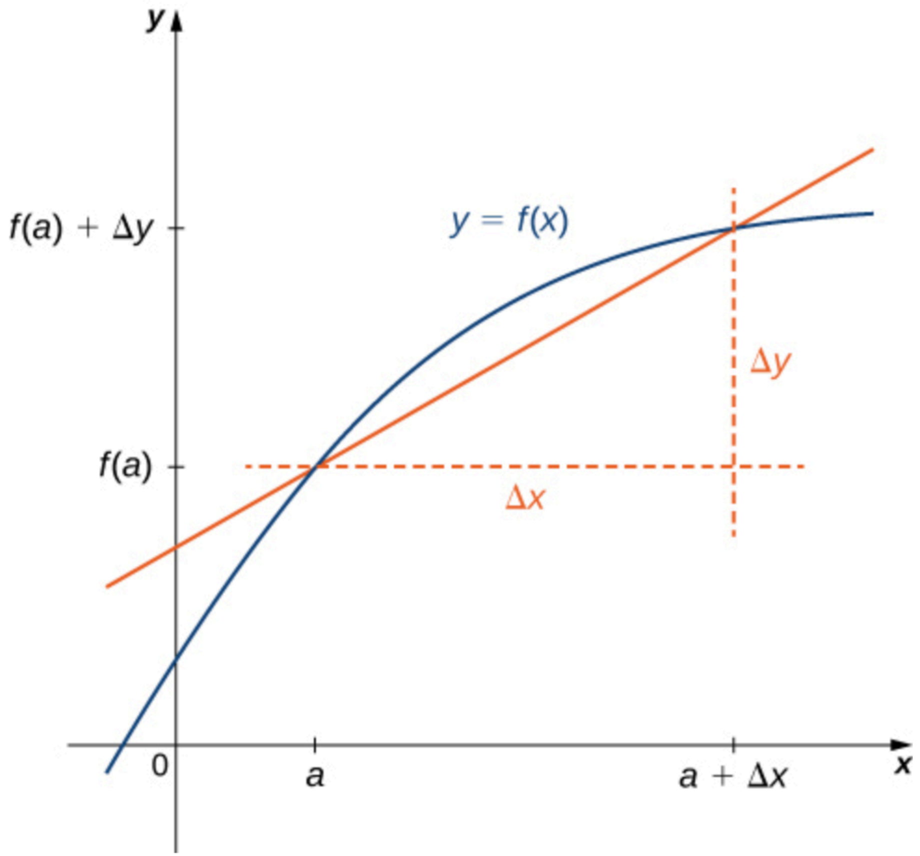


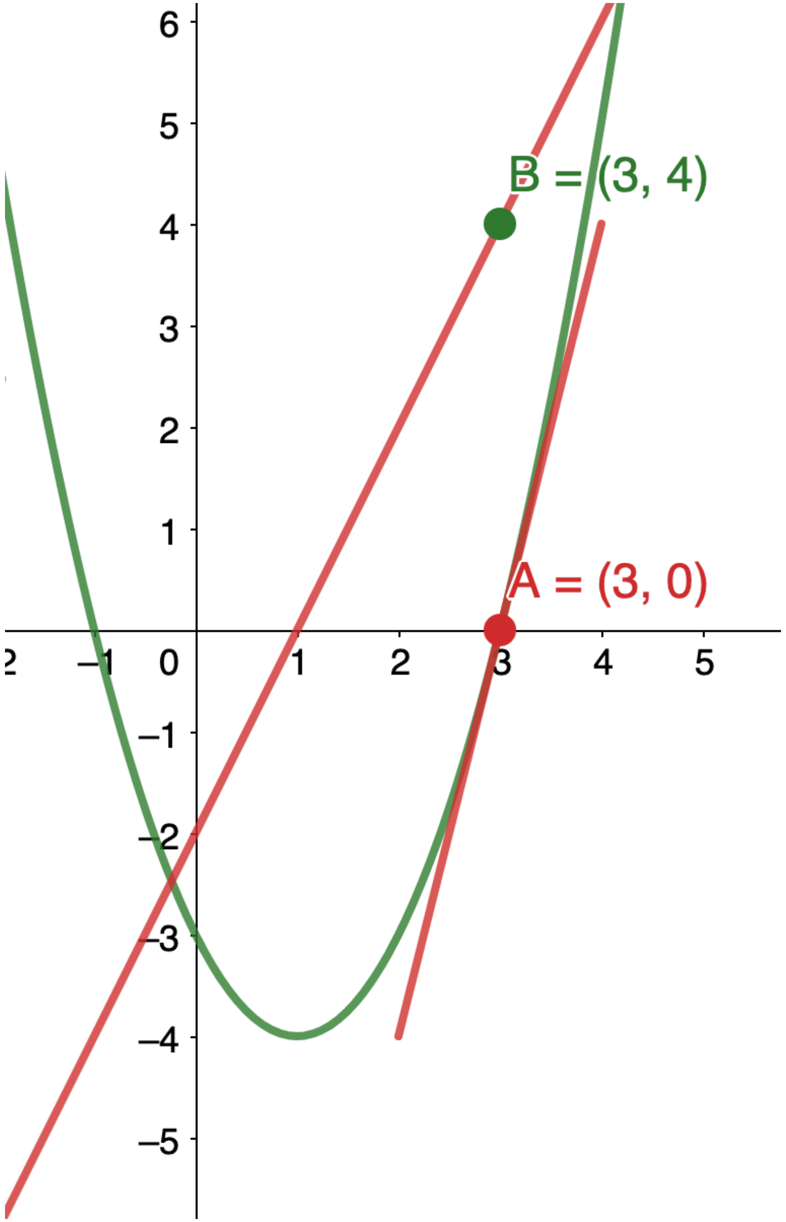
Figure 3.11 The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

NOTATIONS: FROM SECANT LINES TO THE TANGENT LINE

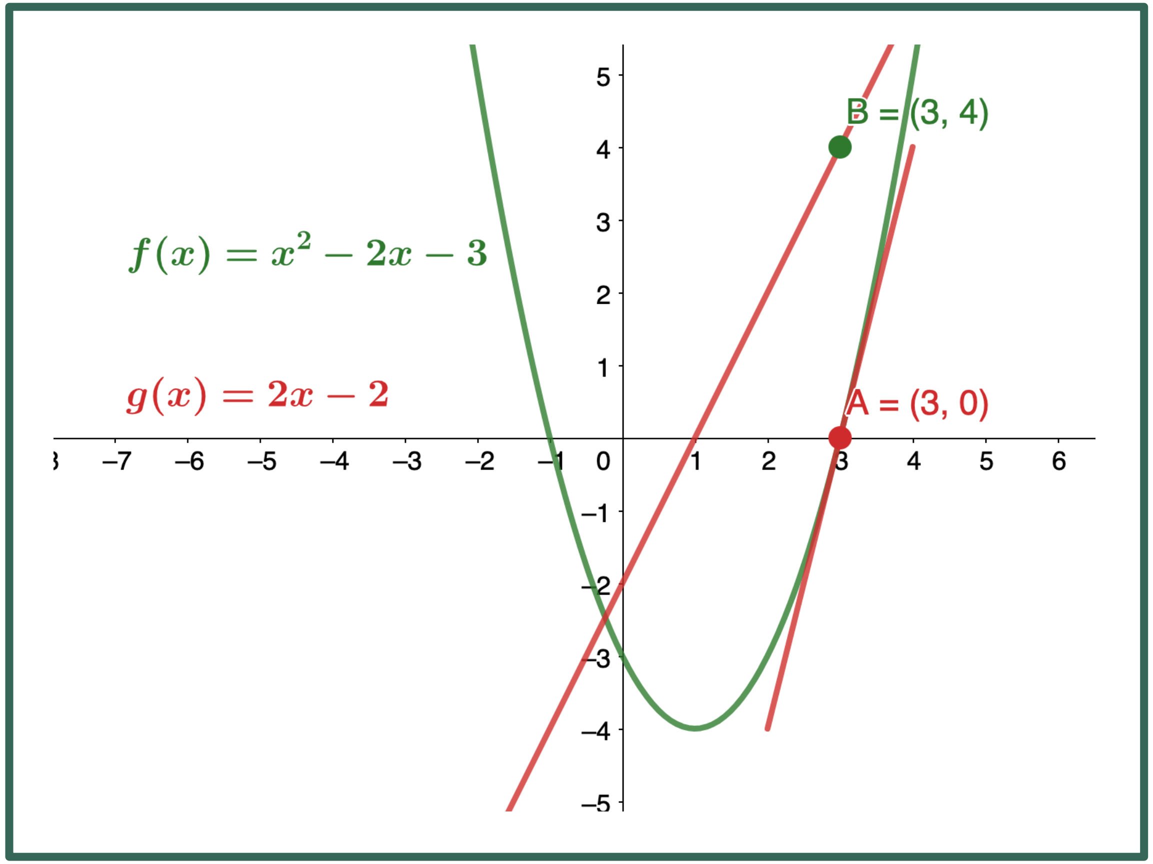
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

GRAPH A DERIVATIVE

- We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it.
- Given both, we would expect to see **a correspondence between the graphs of these two functions**, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).



GRAPH A DERIVATIVE



- Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$.
- For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$.
- Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

GRAPH A FUNCTION

$f(x)$

increasing

decreasing

Horizontal tangent line

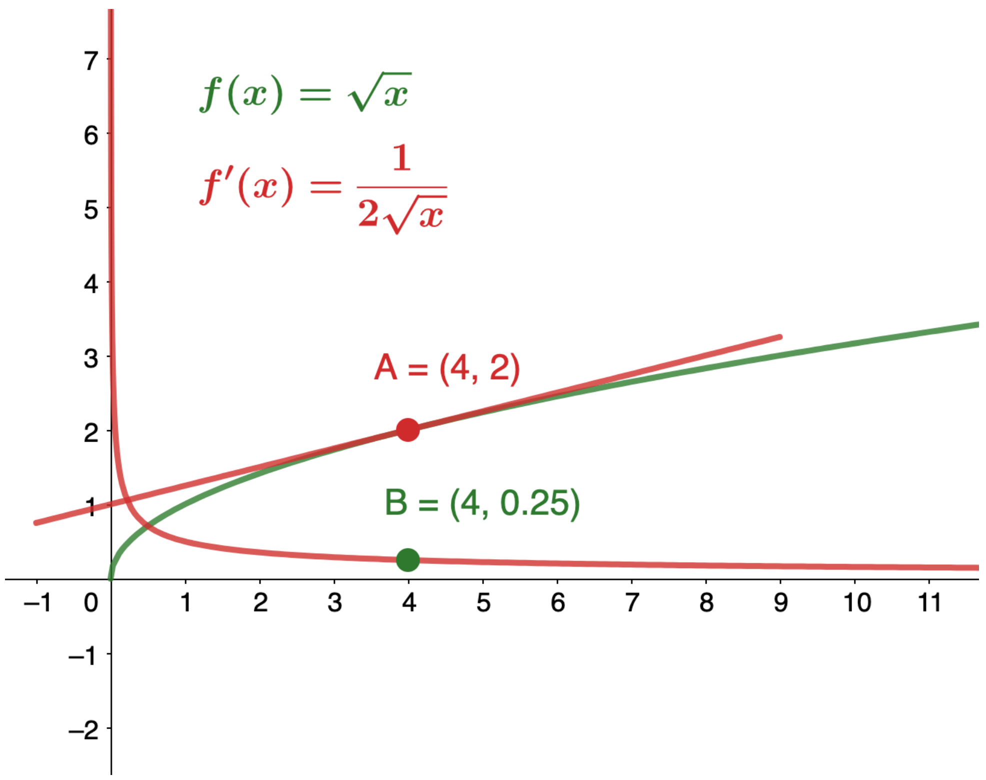
$f'(x)$

> 0

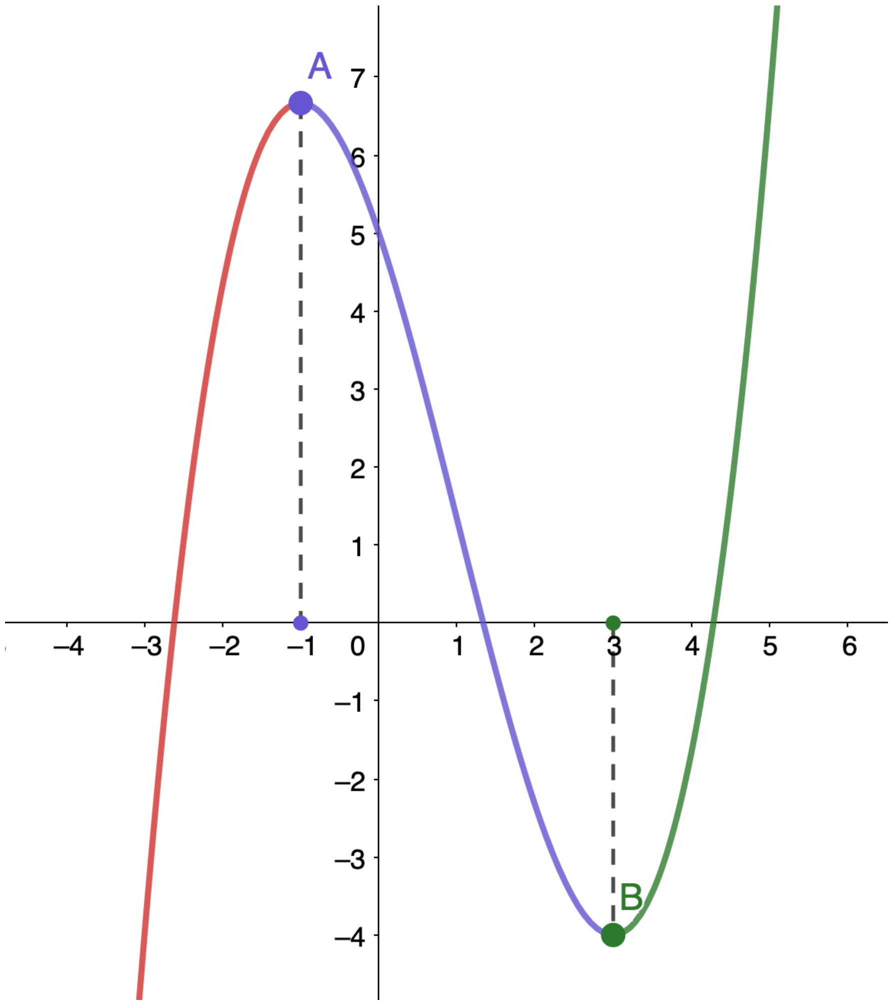
< 0

$= 0$

GRAPH A DERIVATIVE

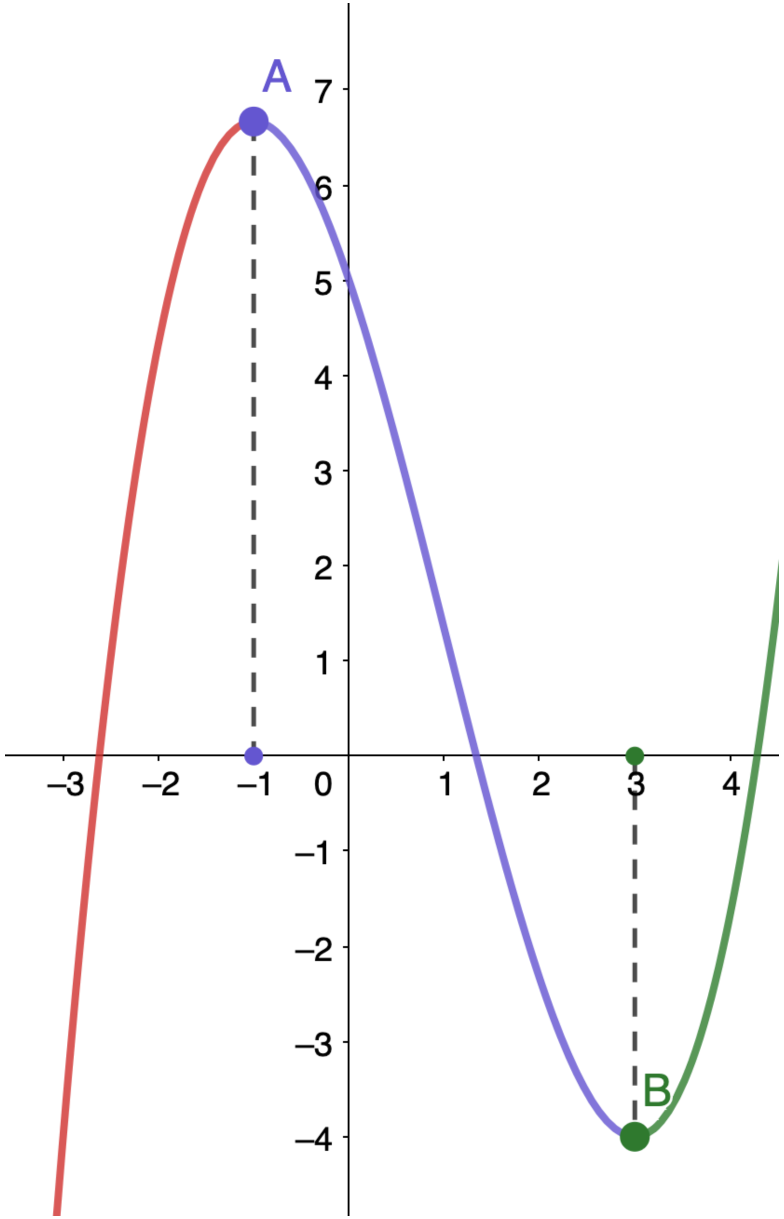


- First, $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive.
- Consequently, we expect $f'(x) > 0$ for all values of x in its domain.
- Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$.
- We also observe that $f'(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.



EXERCISE: SKETCHING A DERIVATIVE USING A FUNCTION

USE THE FOLLOWING GRAPH OF $f(x)$ TO SKETCH A GRAPH OF $f'(x)$.

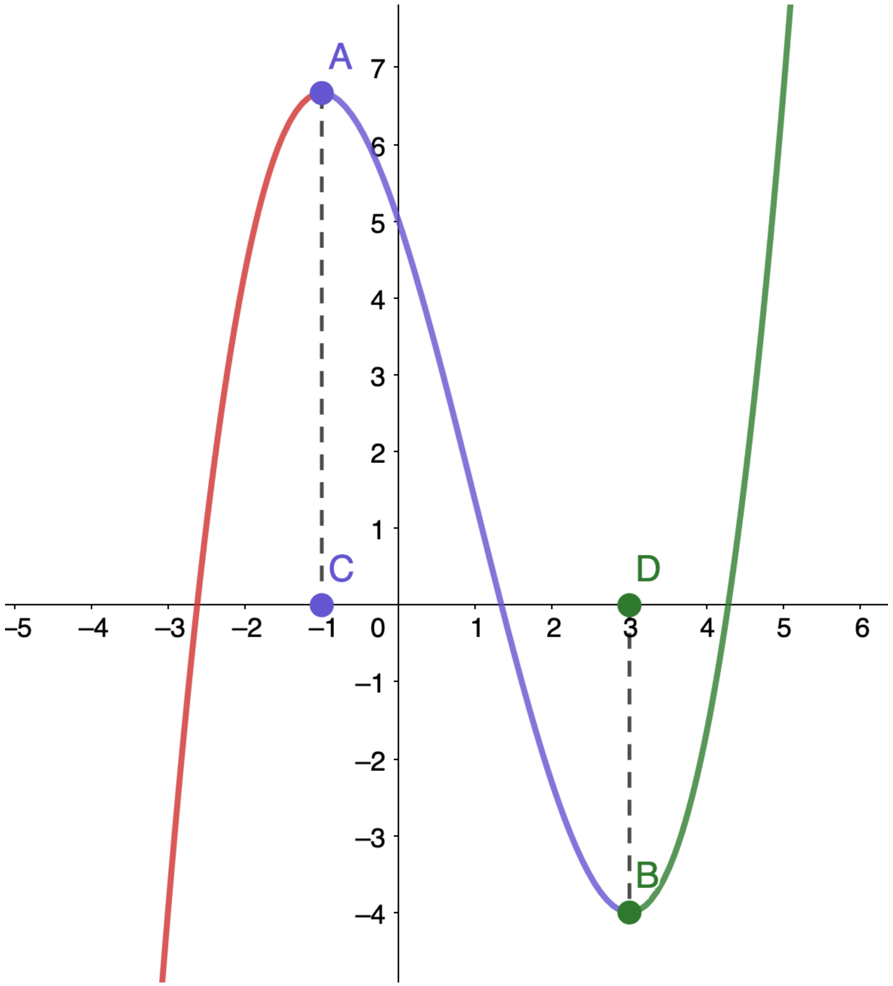


EXERCISE: SKETCHING A DERIVATIVE USING A FUNCTION

- Observe that $f(x)$ is decreasing and $f'(x) < 0$ on $(-1, 3)$.
- Also, $f(x)$ is increasing and $f'(x) > 0$ and on $(-\infty, -1)$ and $(3, +\infty)$.
- Also note that $f(x)$ has horizontal tangents at -1 and 3, and $f'(-1) = f'(3) = 0$.

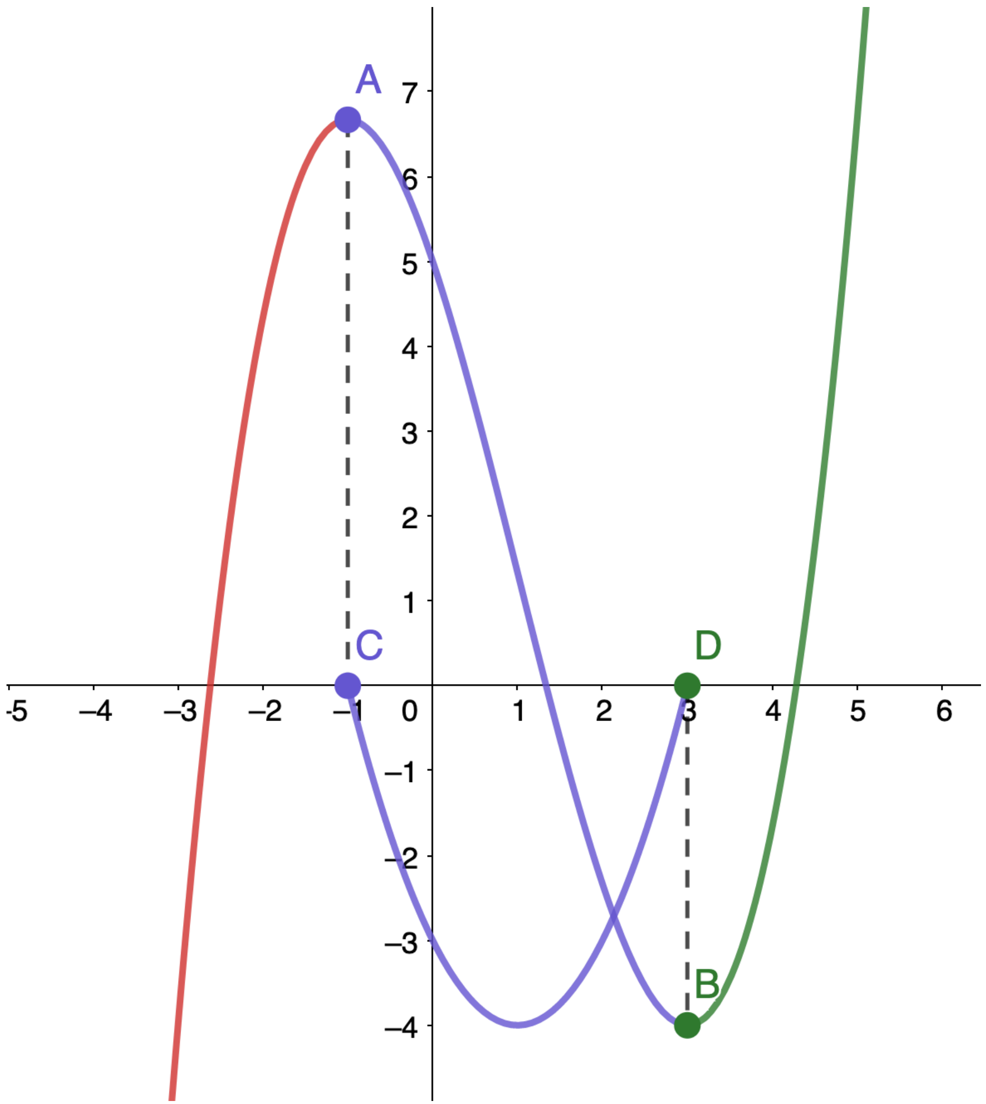
EXERCISE: SKETCHING A DERIVATIVE USING A FUNCTION

- $f(x)$ has horizontal tangents at -1 and 3, and $f'(-1) = f'(3) = 0$.



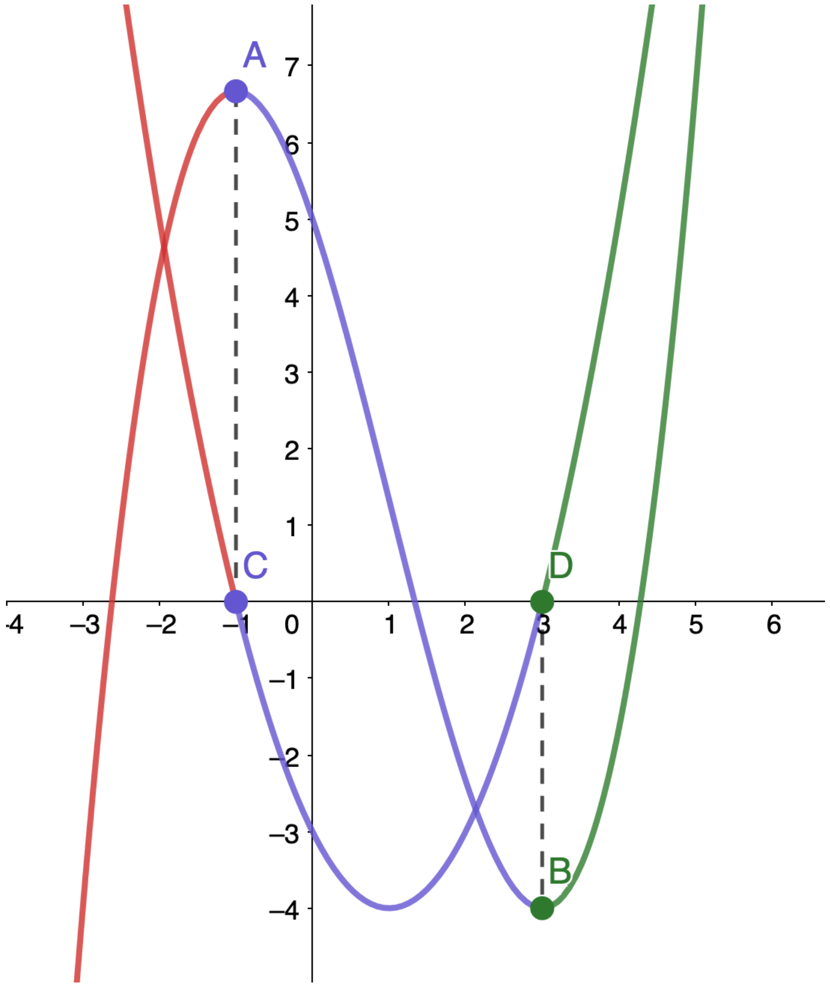
EXERCISE: SKETCHING A DERIVATIVE USING A FUNCTION

- $f(x)$ is decreasing and $f'(x) < 0$ on $(-1, 3)$.



EXERCISE: SKETCHING A DERIVATIVE USING A FUNCTION

- $f(x)$ is increasing and $f'(x) > 0$ and on $(-\infty, -1)$ and $(3, +\infty)$.



HIGHER ORDER DERIVATIVES

$s(t)$
- position

$v(t)$
- velocity

$a(t)$
- accelera
tion

- The derivative of a function is itself a function, so we can find the derivative of a derivative.
- For example, the derivative of a position function is the rate of change of position, or velocity.
- The derivative of velocity is the rate of change of velocity, which is acceleration.

HIGHER ORDER DERIVATIVE

$$f''(x), f'''(x), f^{(4)}(x), \dots, f^{(n)}(x)$$

$$y''(x), y'''(x), y^{(4)}(x), \dots, y^{(n)}(x)$$

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \dots, \frac{d^ny}{dx^n}.$$

- The new function obtained by differentiating the derivative is called **the second** derivative.
- Furthermore, we can continue to take derivatives to obtain **the third** derivative, **fourth** derivative, and so on.
- Collectively, these are referred to as **higher-order derivatives**.

EXERCISE

For $f(x) = x^2 + 4x + 5$, find $f''(x)$.

First find the derivative

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 + 4(x+h) + 5] - (x^2 + 4x + 5)}{h} =$
  $\lim_{h \to 0} \frac{2xh + h^2 + 4h}{h} = \lim_{h \to 0} 2x + h + 4 = 2x + 4$

Then find the second derivative

- $f''(x) = 2$

EXERCISE

$$f(x) = x^2 + 4x + 5$$

- Original function, quadratic

$$f'(x) = 2x + 4$$

- First derivative, linear

$$f''(x) = 2$$

- Second derivative, constant