INTRODUCTION TO CALCULUS

THE LIMIT LAWS

OUTLINE

Recognize	Recognize **the basic limit laws**.
Use	Use the limit laws to evaluate the limit of a function.
Evaluate	Evaluate the limit of a function by factoring.
Use	Use the limit laws to evaluate **the limit of a polynomial or rational function**.
Evaluate	Evaluate the limit of a function by **factoring** or by **using conjugates**.
Evaluate	Evaluate the limit of a function by using **the squeeze theorem**. (the class after the next)

EVOLUTION!

- Evaluate limits by looking at **graphs** or by constructing **a table of values**.
- Establish laws for calculating limits and apply these laws.

THE FIRST TWO LIMIT LAWS

THEOREM 2.4

Basic Limit Results

For any real number *a* and any constant *c*,

i. $$\lim_{x \to a} x = a \tag{2.14}$$

ii. $$\lim_{x \to a} c = c \tag{2.15}$$

THE FIRST TWO LIMIT LAWS

- $\lim_{x \to 1000} x$
- $\lim_{x \to 1984} -2019$

THEOREM 2.5

Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing *a*. Assume that *L* and *M* are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let *c* be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$

Difference law for limits: $\lim\limits_{x \to a} (f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim\limits_{x \to a} cf(x) = c \cdot \lim\limits_{x \to a} f(x) = cL$

Product law for limits: $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim\limits_{x \to a} (f(x))^n = \left(\lim\limits_{x \to a} f(x)\right)^n = L^n$ for every positive integer *n*.

Root law for limits: $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} = \sqrt[n]{L}$ for all *L* if *n* is odd and for $L \geq 0$ if *n* is even.

MORE LIMIT LAWS: THE INDIVIDUAL PROPERTIES OF LIMITS

Sum	$\lim_{x \to a}(🥦 + 🍅) = \lim_{x \to a} 🥦 + \lim_{x \to a} 🍅$
Difference	$\lim_{x \to a}(🥦 - 🍅) = \lim_{x \to a} 🥦 - \lim_{x \to a} 🍅$
Constant multiple	$\lim_{x \to a}(c🥦) = c \lim_{x \to a} 🥦$
Product	$\lim_{x \to a}(🥦🍅) = \lim_{x \to a} 🥦 \lim_{x \to a} 🍅$
Quotient	$\lim_{x \to a}\left(\frac{🥦}{🍅}\right) = \frac{\lim_{x \to a} 🥦}{\lim_{x \to a} 🍅}$
Power	$\lim_{x \to a}(🥦^n) = (\lim_{x \to a} 🥦)^n$
Root	$\lim_{x \to a}(\sqrt[n]{🍅}) = \sqrt[n]{\lim_{x \to a} 🍅}$

LIMIT LAWS

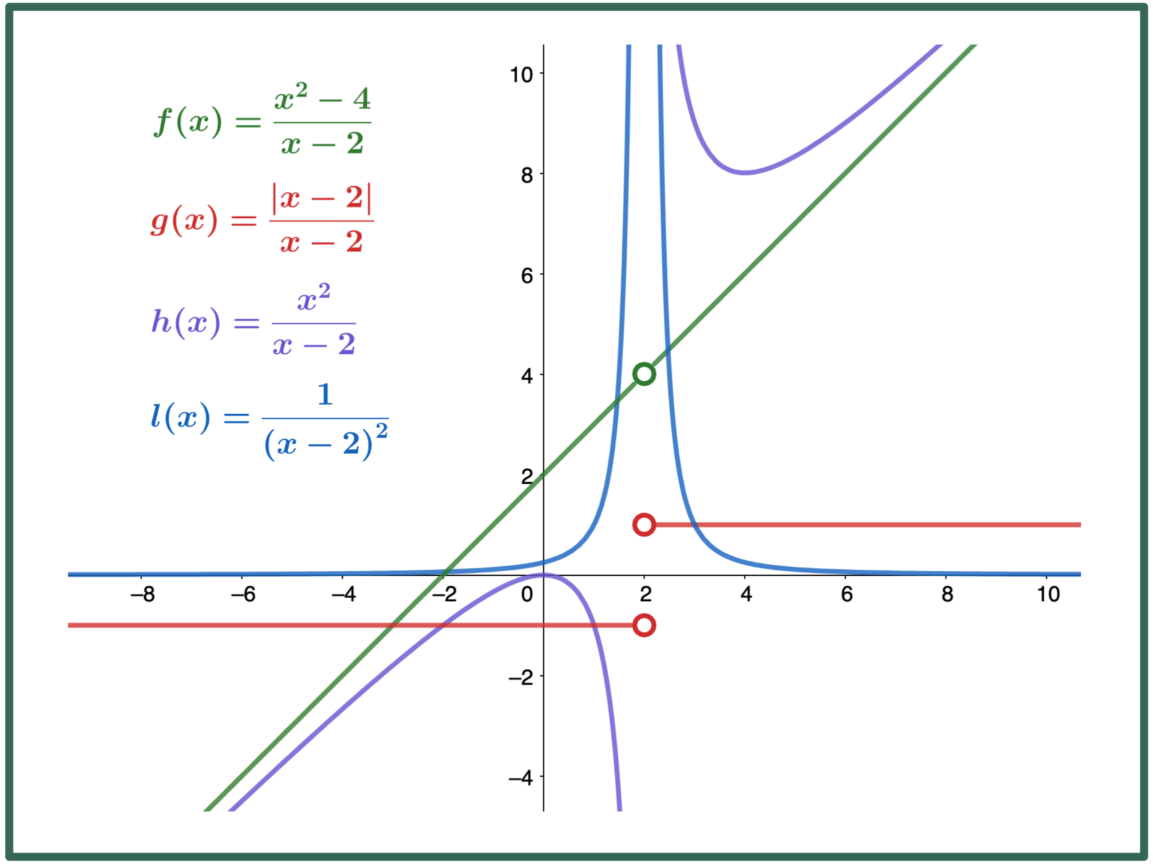
- $\lim_{x \to -1} (x+1)(x-1984)$

- $\lim_{x \to 2} \frac{x+2}{x^2+x+1}$

- $\lim_{x \to 10} \sqrt[3]{x-2} + 2017$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS



In each of the previous examples, it has been the case that $\lim_{x \to a} f(a)$.

This is **not** always true, but it does hold for

- all **polynomials** for any choice of a
- all **rational functions** at all values of a for which the rational function is **defined**.

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS

THEOREM 2.6

Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let *a* be a real number. Then,

$$\lim_{x \to a} p(x) = p(a)$$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

WHY?

$$\lim_{x\to a}(🥦 + 🍅) = \lim_{x\to a} 🥦 + \lim_{x\to a} 🍅$$

$$\lim_{x\to a}(🥦 - 🍅) = \lim_{x\to a} 🥦 - \lim_{x\to a} 🍅$$

$$\lim_{x\to a}(c🥦) = c\lim_{x\to a} 🥦$$

$$\lim_{x\to a}(🥦🍅) = \lim_{x\to a} 🥦 \lim_{x\to a} 🍅$$

$$\lim_{x\to a}\left(\frac{🥦}{🍅}\right) = \frac{\lim_{x\to a} 🥦}{\lim_{x\to a} 🍅}$$

$$\lim_{x\to a}(🥦^{n}) = (\lim_{x\to a} 🥦)^{n}$$

$$\lim_{x\to a}(\sqrt[n]{🍅}) = \sqrt[n]{\lim_{x\to a} 🍅}$$

$$\lim_{x\to a}(c🥦^{n}) = c\,(\lim_{x\to a} 🥦)^{n}$$

$$🥗 = c_n🥦^{n} + c_{n-1}🥦^{n-1} + \cdots + c_1🥦 + c_0$$

$$🍲 = d_m🥦^{m} + d_{m-1}🥦^{m-1} + \cdots + d_1🥦 + d_0$$

$$\lim_{x\to a}(🥗),\ \lim_{x\to a}(🍲),\ \lim_{x\to a}\left(\frac{🥗}{🍲}\right)$$

LIMIT LAWS

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

- $\lim\limits_{x \to -2} x^3 + x^2 - 2x - 10$

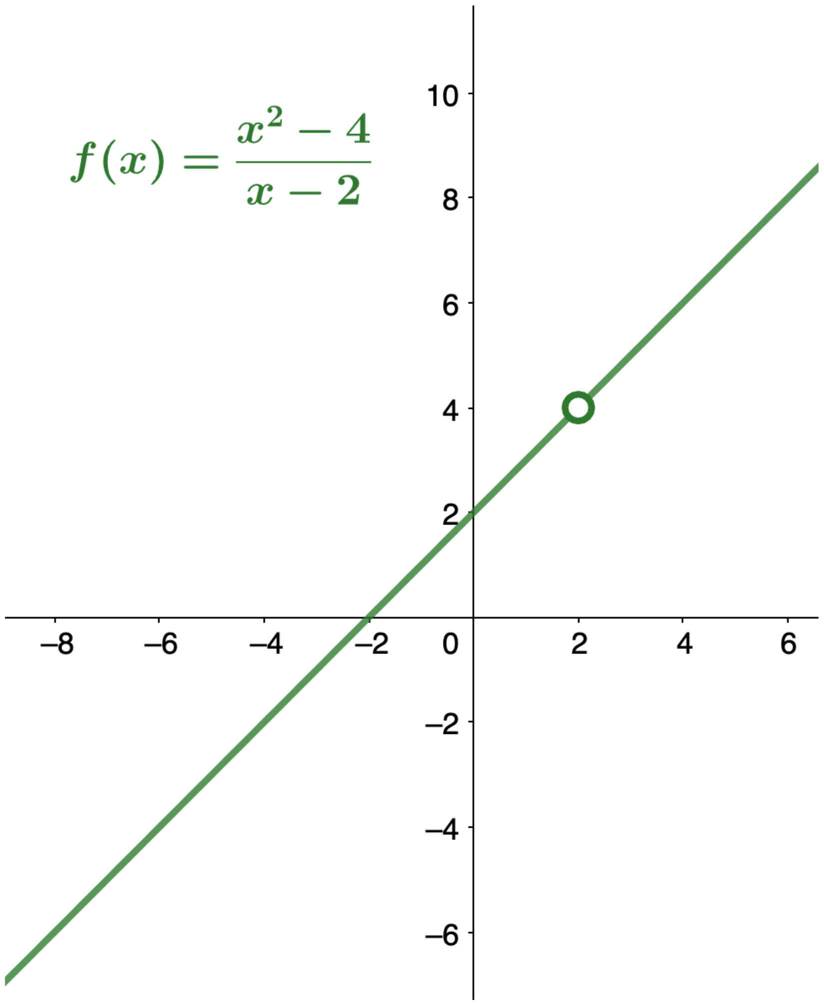
- $\lim\limits_{x \to 2} \frac{x^2}{x^2+x+1}$

- $\lim\limits_{x \to 1} (x+1)(x-1)(x-3)$

IS THAT ENOUGH? NO!

- Now we may evaluate easily the limits of **polynomials** and limits of some (but not all) **rational functions** by direct substitution.
- However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x \to a} f(x)$ to exist when $f(a)$ is **undefined**.

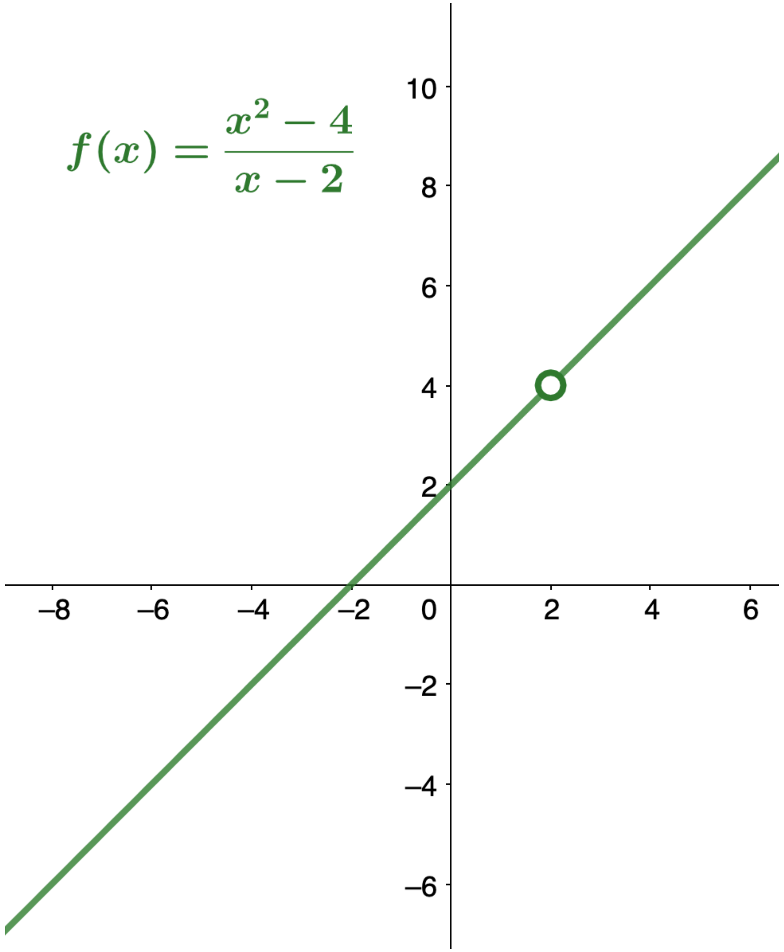
RECALL OUR OLD FRIEND FROM LAST LECTURE



ADDITIONAL LIMIT EVALUATION TECHNIQUES

If for all $x \neq a$, $\boldsymbol{f(x) = g(x)}$ over some open interval **containing** a, then $\lim\limits_{x \to a} f(x) = g(a)$.

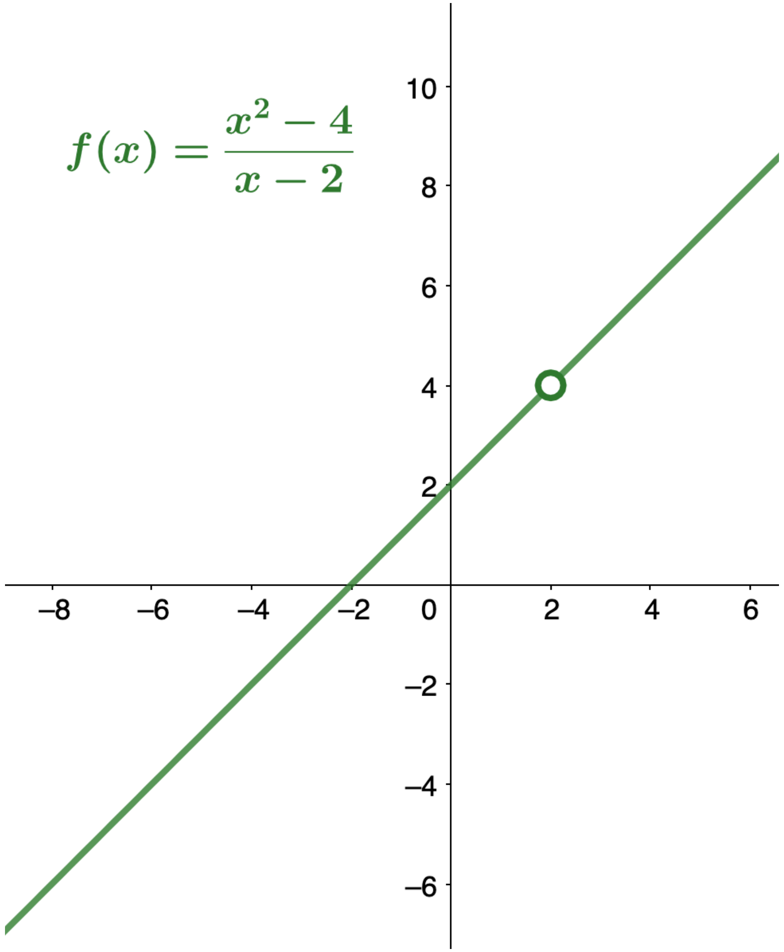
- $f(x) = \frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2}$
- $g(x) = x + 2$



THE INDETERMINATE FORM 0/0

- $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2}(x+2) = 4$

- $\lim_{x \to 2} \frac{0}{0}$



PROBLEM-SOLVING STRATEGY: CALCULATING A LIMIT WHEN $f(x)/g(x)$ HAS THE INDETERMINATE FORM 0/0

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing *a*. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.

THE INDETERMINATE FORM $0/0$

HOW TO FIND $g(x)$?

THE FACTOR-AND-CANCEL TECHNIQUE

MULTIPLYING BY A CONJUGATE

SIMPLIFYING A COMPLEX FRACTION

THE FACTOR-AND-CANCEL TECHNIQUE

THE INDETERMINATE FORM $0/0$

- $\lim\limits_{x \to -1} \frac{x+1}{x^2+4x+3}$

- $\lim\limits_{x \to 3} \frac{x^2-9}{2x^2-3x-9}$

MULTIPLE BY A CONJUGATE

THE INDETERMINATE FORM $0/0$

- $$\lim_{x \to 4} \frac{\sqrt{x-3}-1}{x-4}$$
- $$\lim_{x \to -5} \frac{\sqrt{x+9}-2}{x+5}$$

SIMPLIFY A COMPLEX FUNCTION

THE INDETERMINATE FORM 0/0

- $\lim_{x \to 1} \dfrac{\frac{1}{x} - \frac{2}{x(x+1)}}{x-1}$

- $\lim_{x \to 2019} \dfrac{\frac{1}{x} - \frac{1}{2019}}{x-2019}$

IT IS NOT THAT EASY TO BECOME A LION KING…

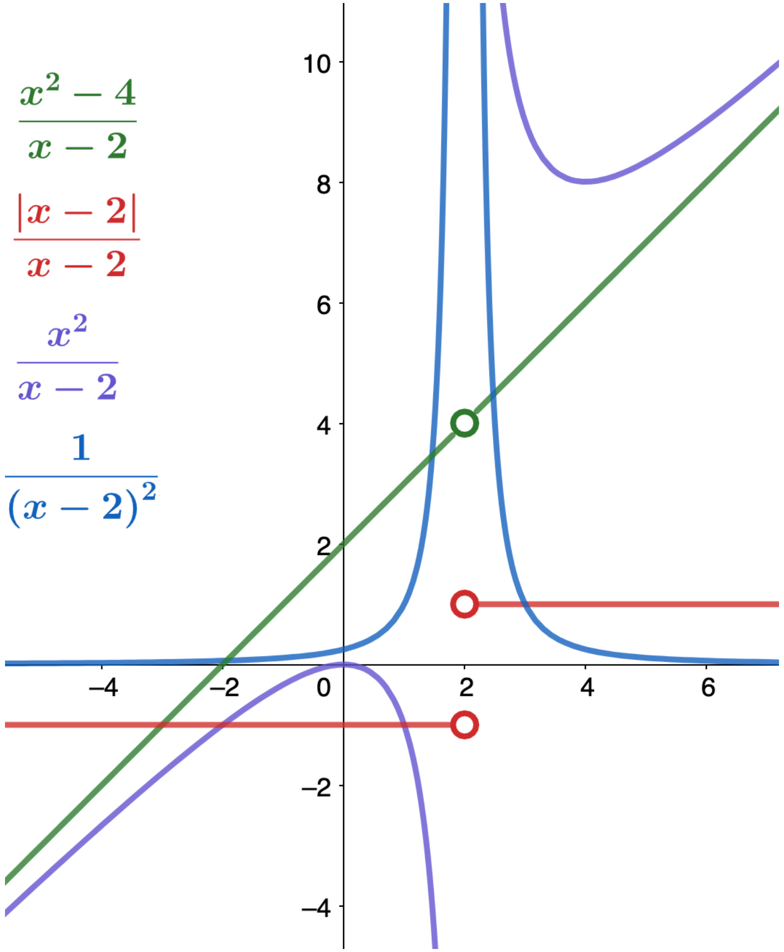
- $\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)$

This one does not fall neatly into any of the patterns established in the previous examples.

However, with a little creativity, we can still use these same techniques.

REVISIT ONE-SIDED LIMITS

- To apply the limit laws to a limit of the form $\lim_{x\to a^-} g(x)$, we require the function $g(x)$ to be defined over an open interval of the form (b, a).
- For a limit of the form $\lim_{x\to a^+} g(x)$, we require the function $g(x)$ to be defined over an open interval of the form (a, c).



REVISIT ONE-SIDED LIMITS

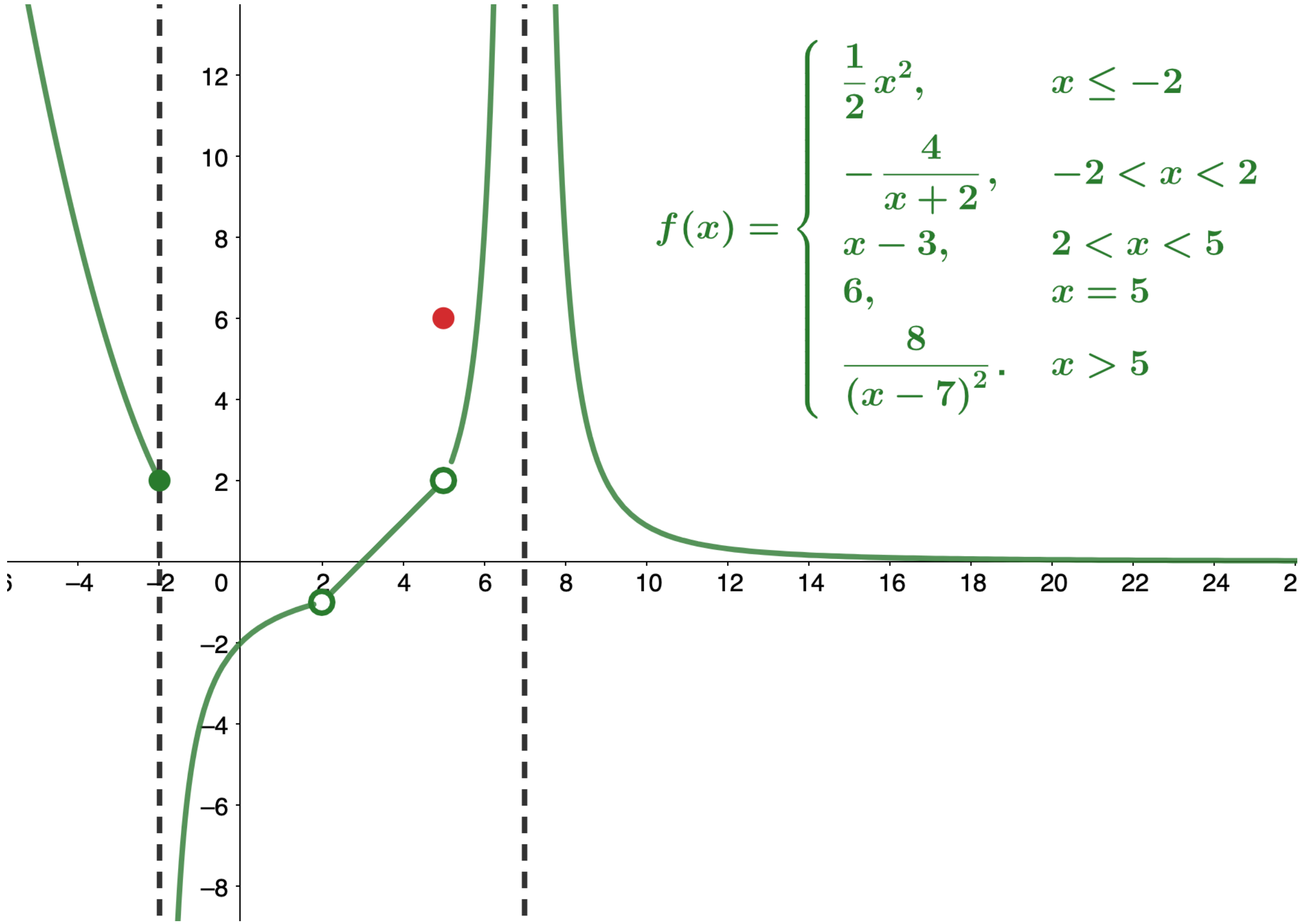
$\lim_{x\to a^-} g(x)$, we require $g(x)$ to be defined over (b, a).

$\lim_{x\to a^+} g(x)$, we require $g(x)$ to be defined over (a, c).

- $\lim_{x\to 2019^-} \sqrt{x - 2019}$
- $\lim_{x\to 2019^+} \sqrt{x - 2019}$
- $\lim_{x\to 2019^-} \sqrt[3]{x - 2019}$
- $\lim_{x\to 2019^+} \sqrt[3]{x - 2019}$

REVISIT ONE-SIDED LIMITS

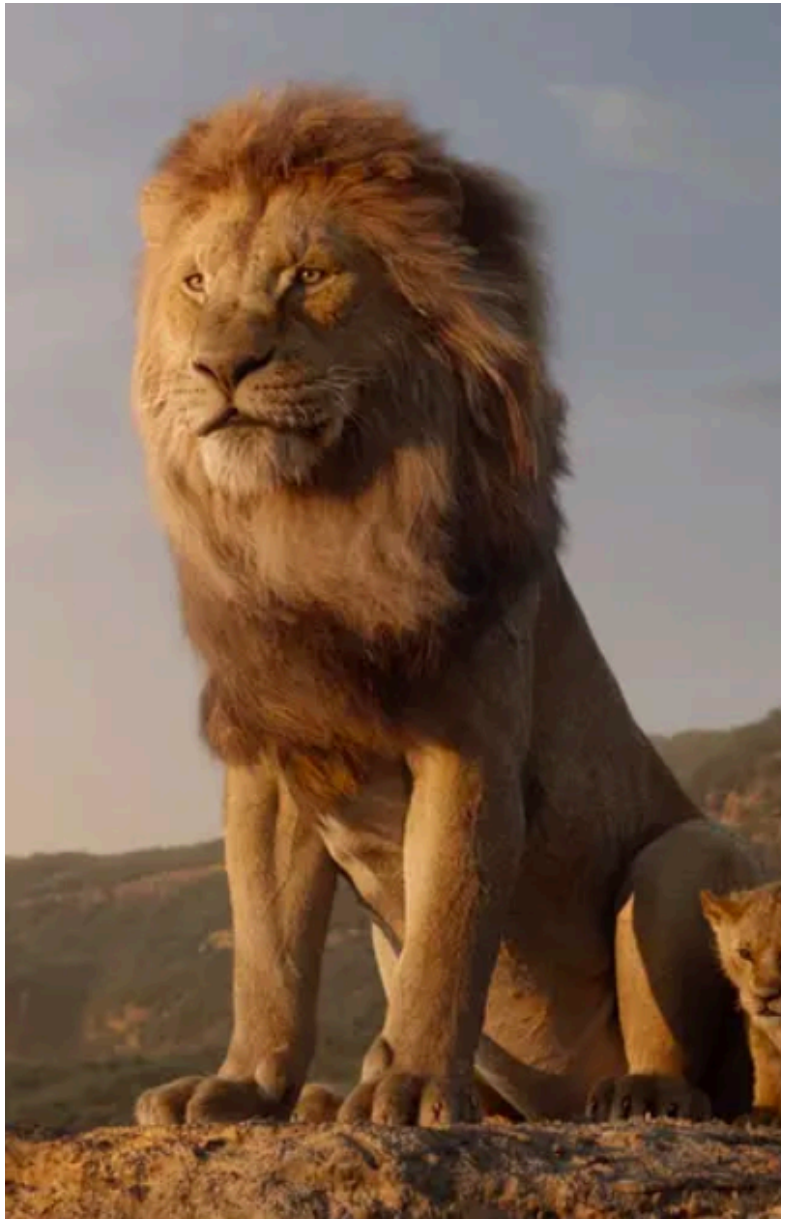
- $\lim_{x \to -2^-} f(x)$
- $\lim_{x \to -2^+} f(x)$
- $\lim_{x \to 2^-} f(x)$
- $\lim_{x \to 5^+} f(x)$



THE INDETERMINATE FORM $K/0$

- We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$.

- That is, $\frac{f(x)}{g(x)}$ has the form $\frac{K}{0}$ at a.



THE INDETERMINATE FORM $K/0$

$$\lim_{x\to 2019^-} \frac{x}{x-2019}$$

$$\lim_{x\to 2019^+} \frac{x}{x-2019}$$

$$\lim_{x\to 1^-} \frac{x+3}{x(x-1)^2}$$

$$\lim_{x\to 1^+} \frac{x+3}{x(x-1)^2}$$

$+\infty$

$-\infty$