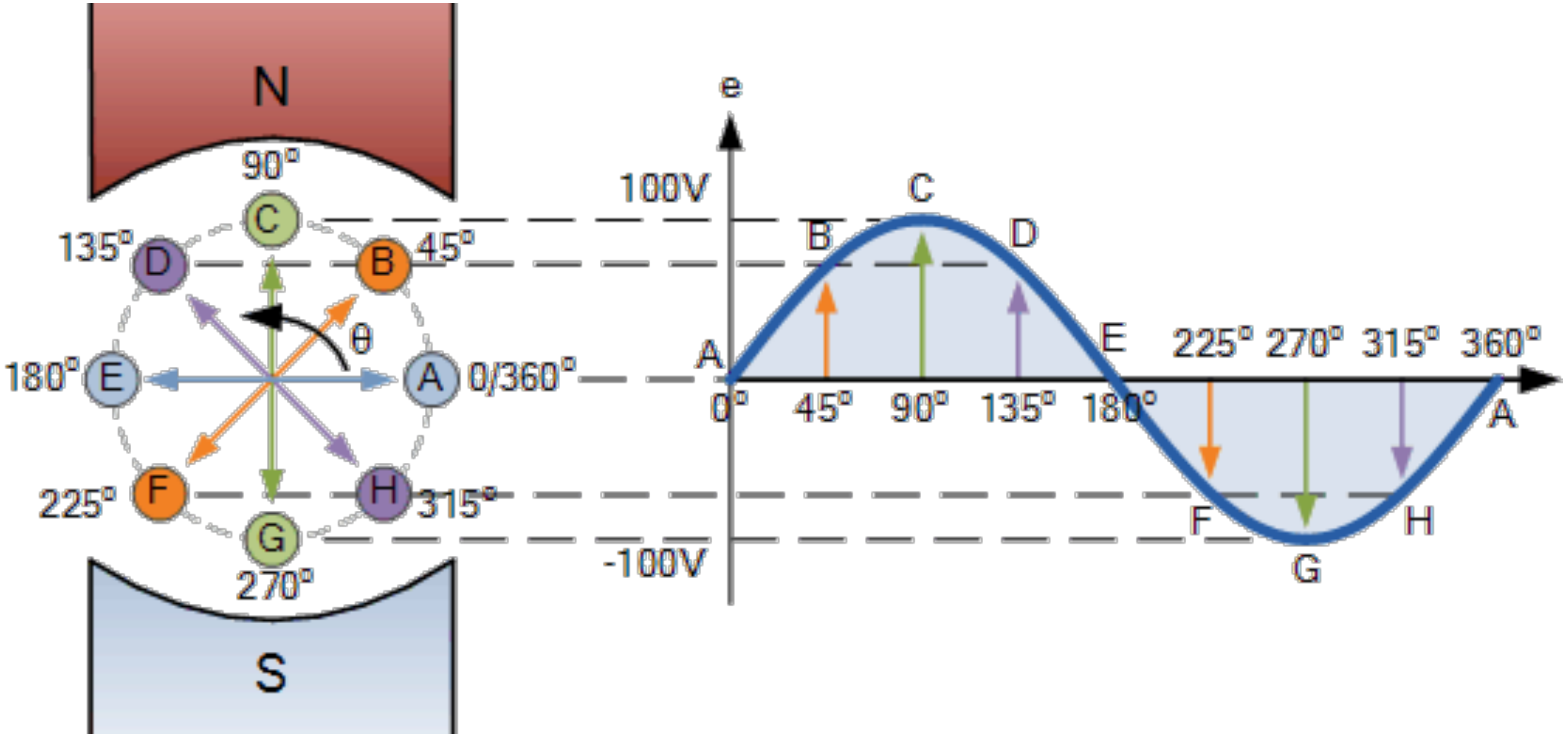
TRIGONOMETRIC FUNCTIONS

INTRODUCTION TO CALCULUS

TRIGONOMETRIC FUNCTIONS

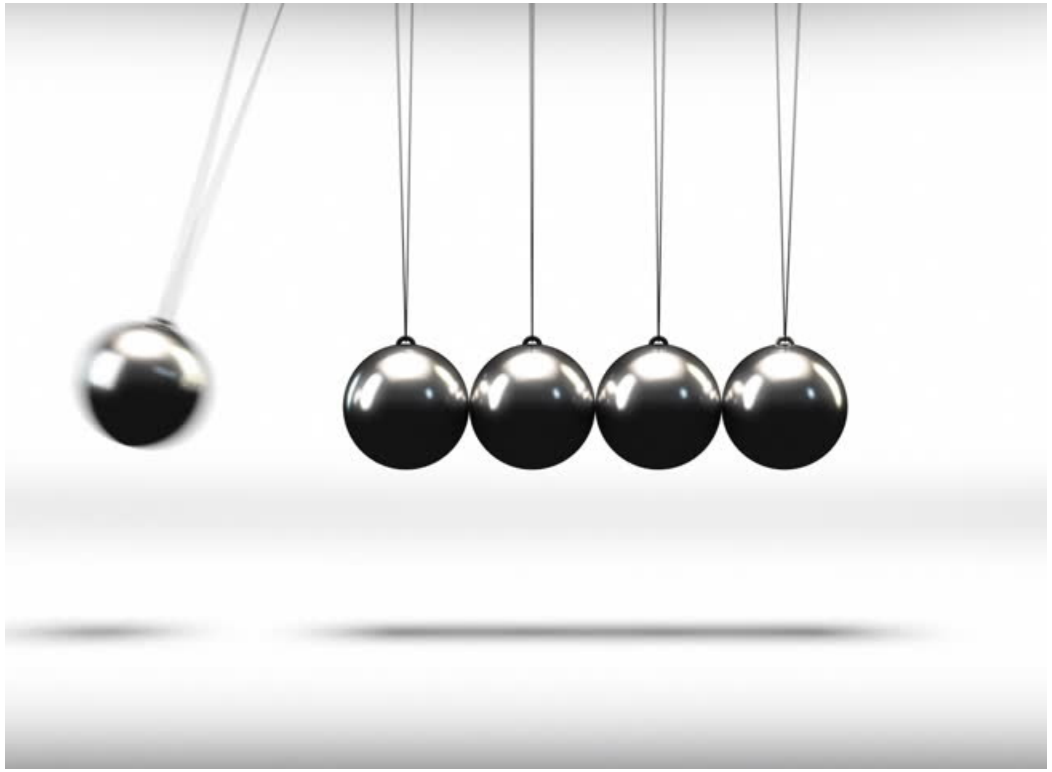
- **Vibration** is a mechanical phenomenon whereby oscillations occur about an equilibrium point.



TRIGONOMETRIC FUNCTIONS

THE USUAL WAVEFORM OF ALTERNATING **CURRENT** IN MOST **ELECTRIC** POWER CIRCUITS IS A **SINE WAVE**

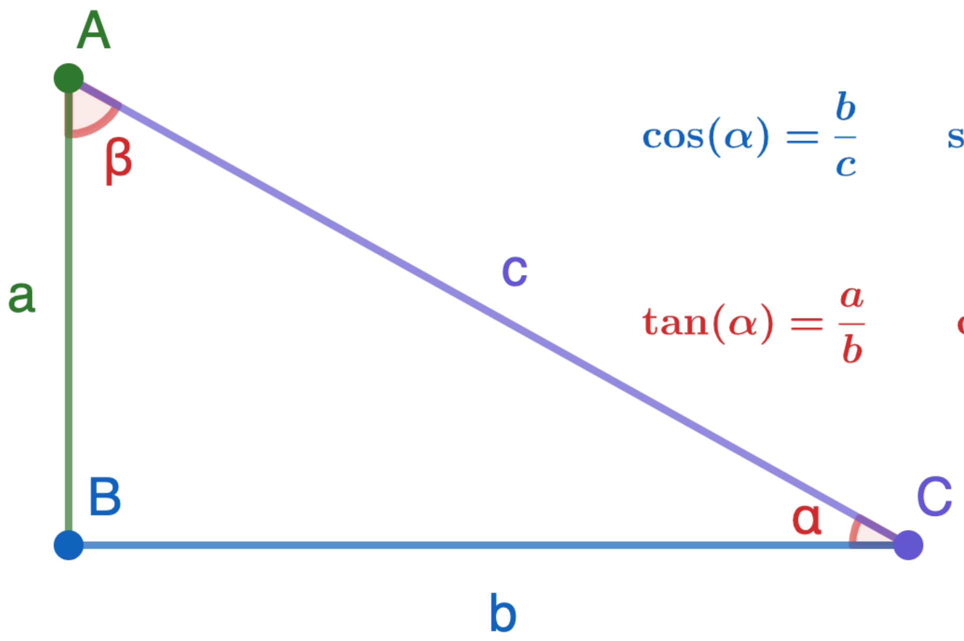
TRIGONOMETRIC FUNCTIONS



- When a pendulum is displaced sideways from its resting, equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position.

TRIGONOMETRIC FUNCTIONS

- The six basic trigonometric functions
- The main identities involving these functions

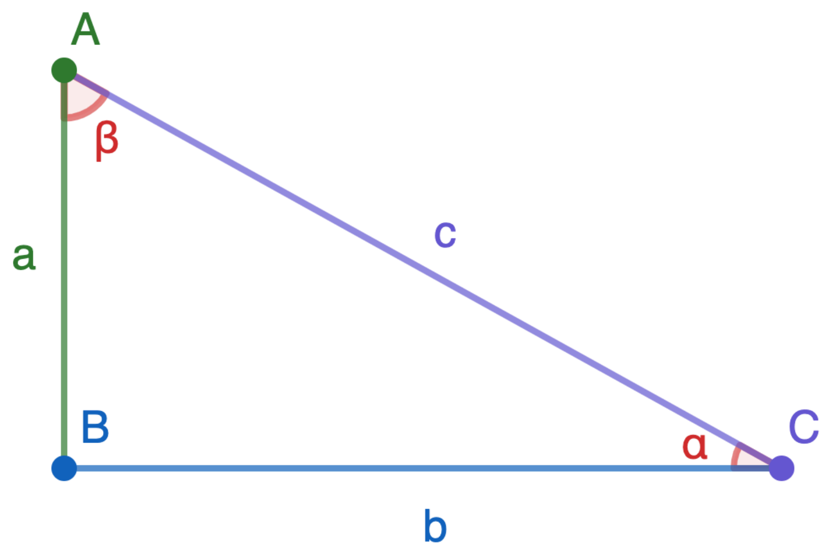


$$\sin(\alpha) = \frac{a}{c} \qquad \csc(\alpha) = \frac{c}{a}$$

$$\cos(\alpha) = \frac{b}{c} \qquad \sec(\alpha) = \frac{c}{b}$$

$$\tan(\alpha) = \frac{a}{b} \qquad \cot(\alpha) = \frac{b}{a}$$

RADIAN MEASURE

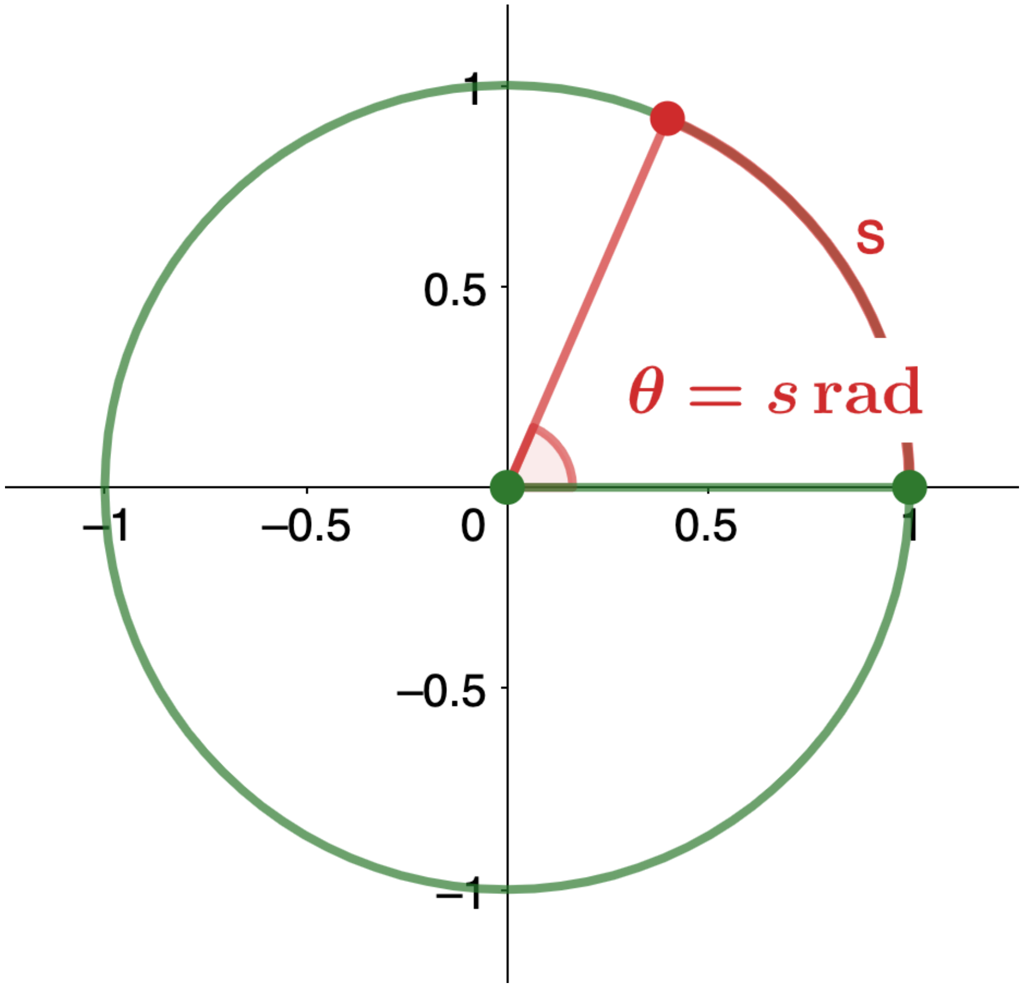


$\alpha = 30°$ $\beta = 60°$

- How to measure the angles?
  - Degree
  - ?

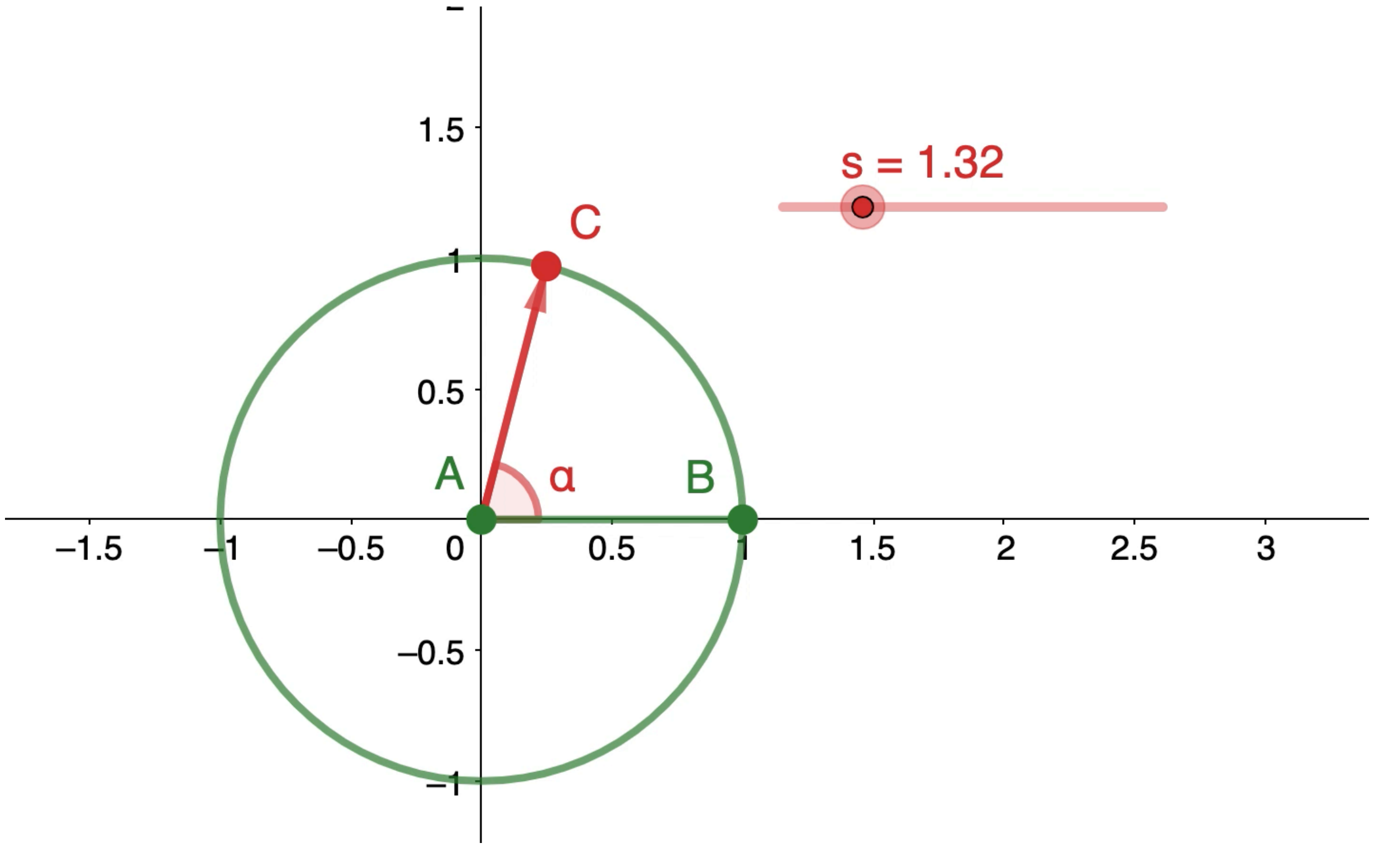
RADIAN MEASURE

- Given an angle θ, let s be the length of the corresponding arc on the **unit circle**.
- The angle corresponding to the arc of length 1 has radian measure 1.



RADIAN MEASURE

- An angle of 360° corresponds to the circumference of a circle, or an arc of length 2π.
- An angle with a degree measure of 360° has a radian measure of 2π.
- Similarly, we see that 180° is equivalent to π radians.



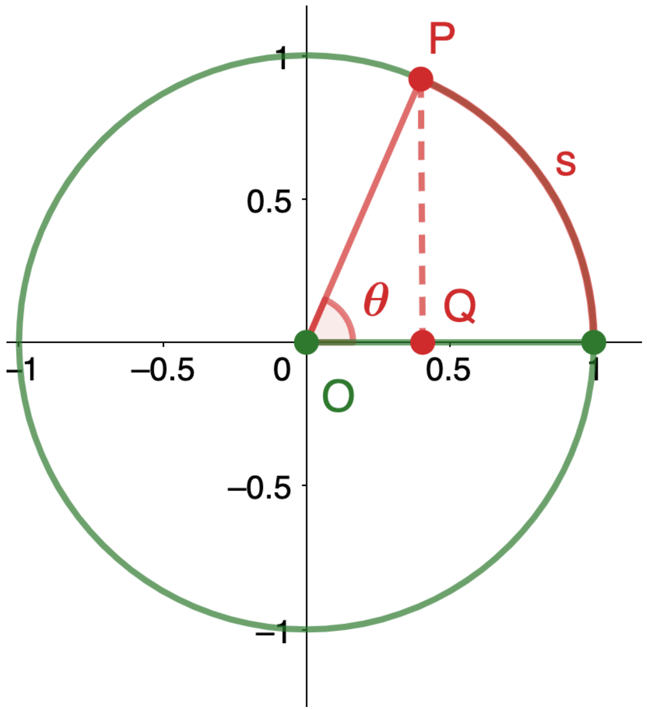
RADIAN MEASURE

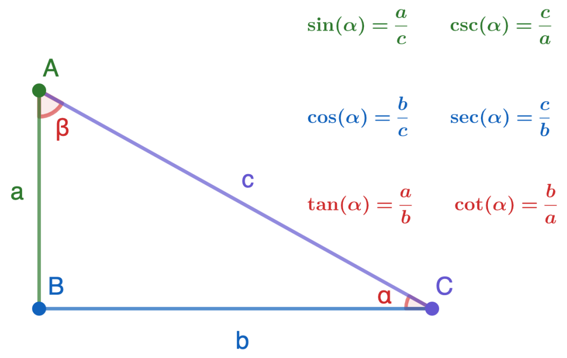
THE RELATIONSHIP BETWEEN COMMON DEGREE AND RADIAN VALUES

Degree	0	30	45	60	90	120	135	150	180
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

THE SIX BASIC TRIGONOMETRIC FUNCTIONS

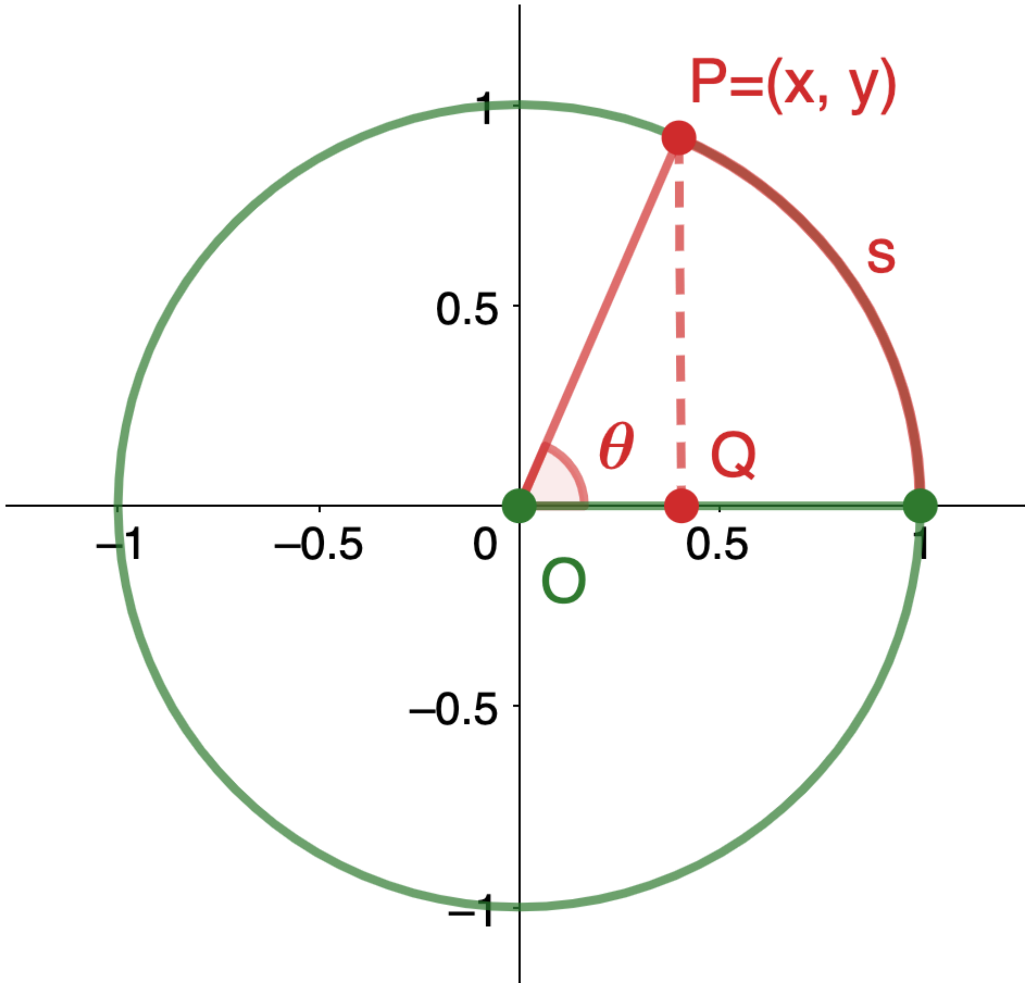
- Why we need trigonometric functions?
  - To use angle measures (in radians or degrees).
  - To find the coordinates of a point on any circle (not only on a unit circle).
  - To find an angle given a point on a circle.
  - To define the relationship among the sides and angles of a triangle.





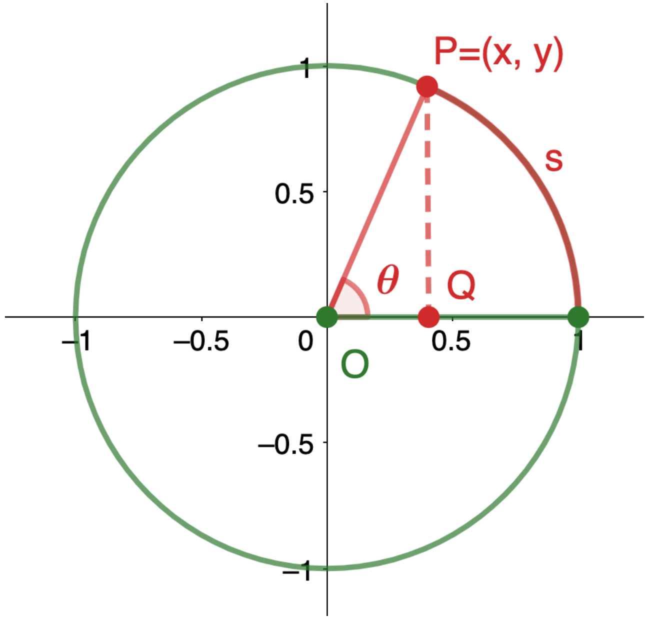
DEFINE TRIGONOMETRIC FUNCTIONS

- The unit circle centered at the origin
- A point $P = (x, y)$ on the unit circle
- An angle θ
  - an initial side that lies along the positive x-axis
  - a terminal side that is the line segment OP
- An angle in this position is said to be in *standard position.*



DEFINE TRIGONOMETRIC FUNCTIONS

DEFINE THE VALUES OF THE SIX TRIGONOMETRIC FUNCTIONS FOR θ IN TERMS OF THE COORDINATES x AND y.



DEFINITION

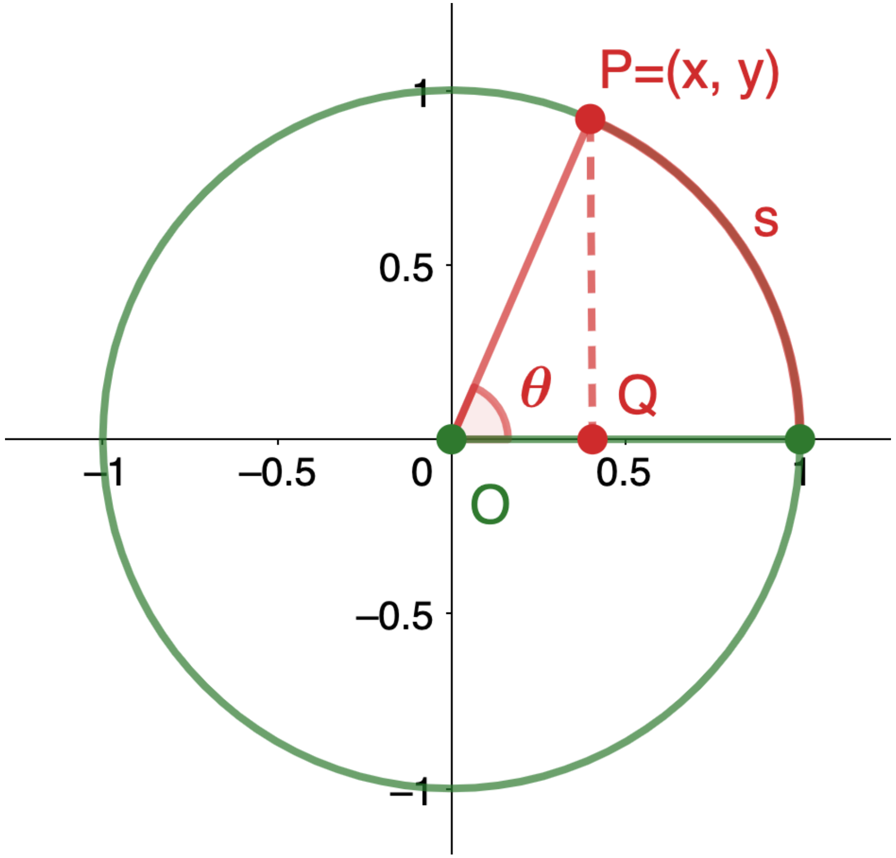
Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x-axis and a terminal side given by the line segment OP. The **trigonometric functions** are then defined as

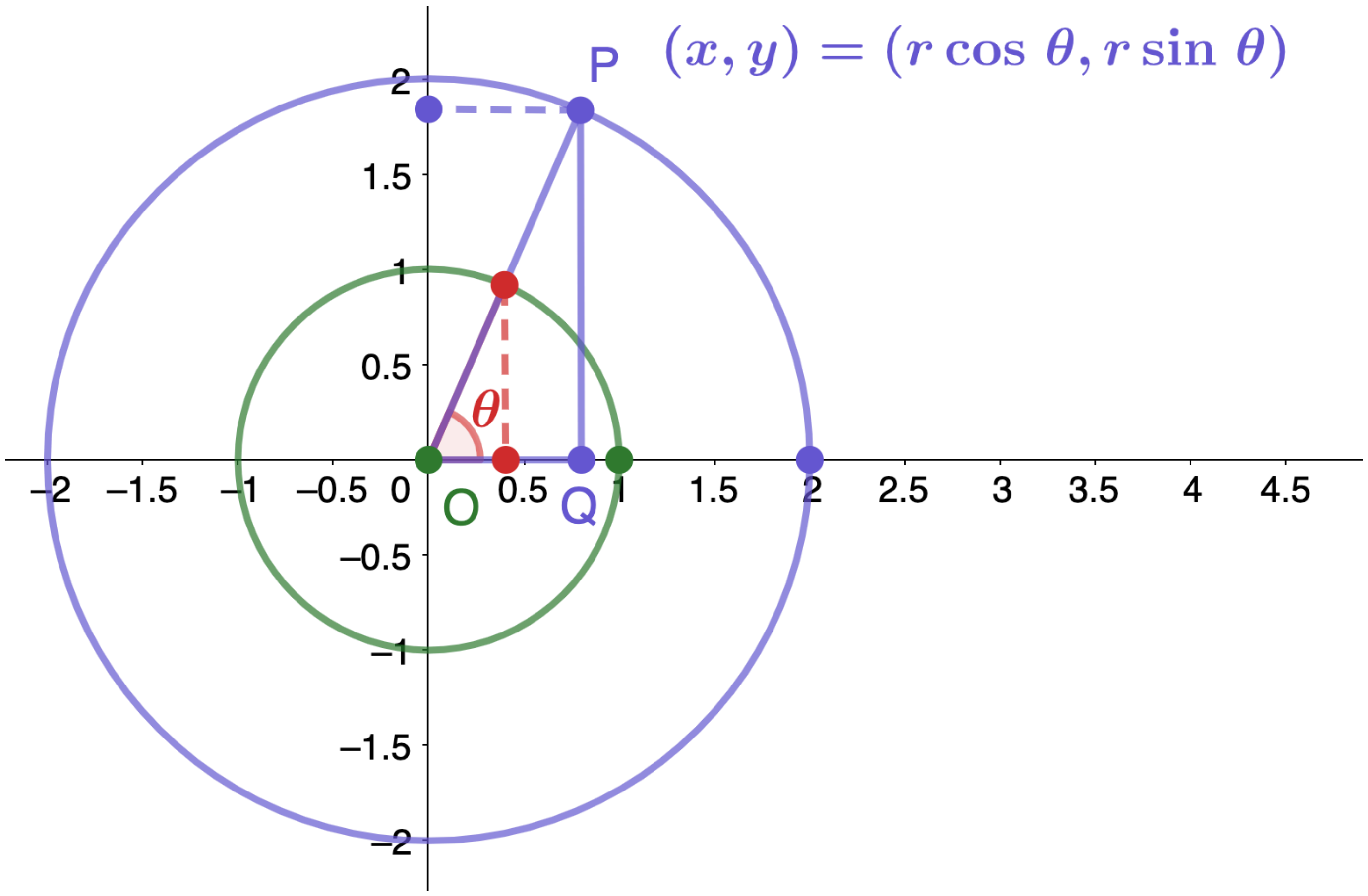
$$\begin{aligned} \sin\theta &= y & \csc\theta &= \tfrac{1}{y} \\ \cos\theta &= x & \sec\theta &= \tfrac{1}{x} \\ \tan\theta &= \tfrac{y}{x} & \cot\theta &= \tfrac{x}{y} \end{aligned} \tag{1.9}$$

If $x = 0$, $\sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

VALUES OF SINE AND COSINE (AT THE MAJOR ANGLES IN THE FIRST QUADRANT)

$\boldsymbol{\theta}$	$\boldsymbol{sin\ (\theta)}$	$\boldsymbol{cos\ (\theta)}$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0





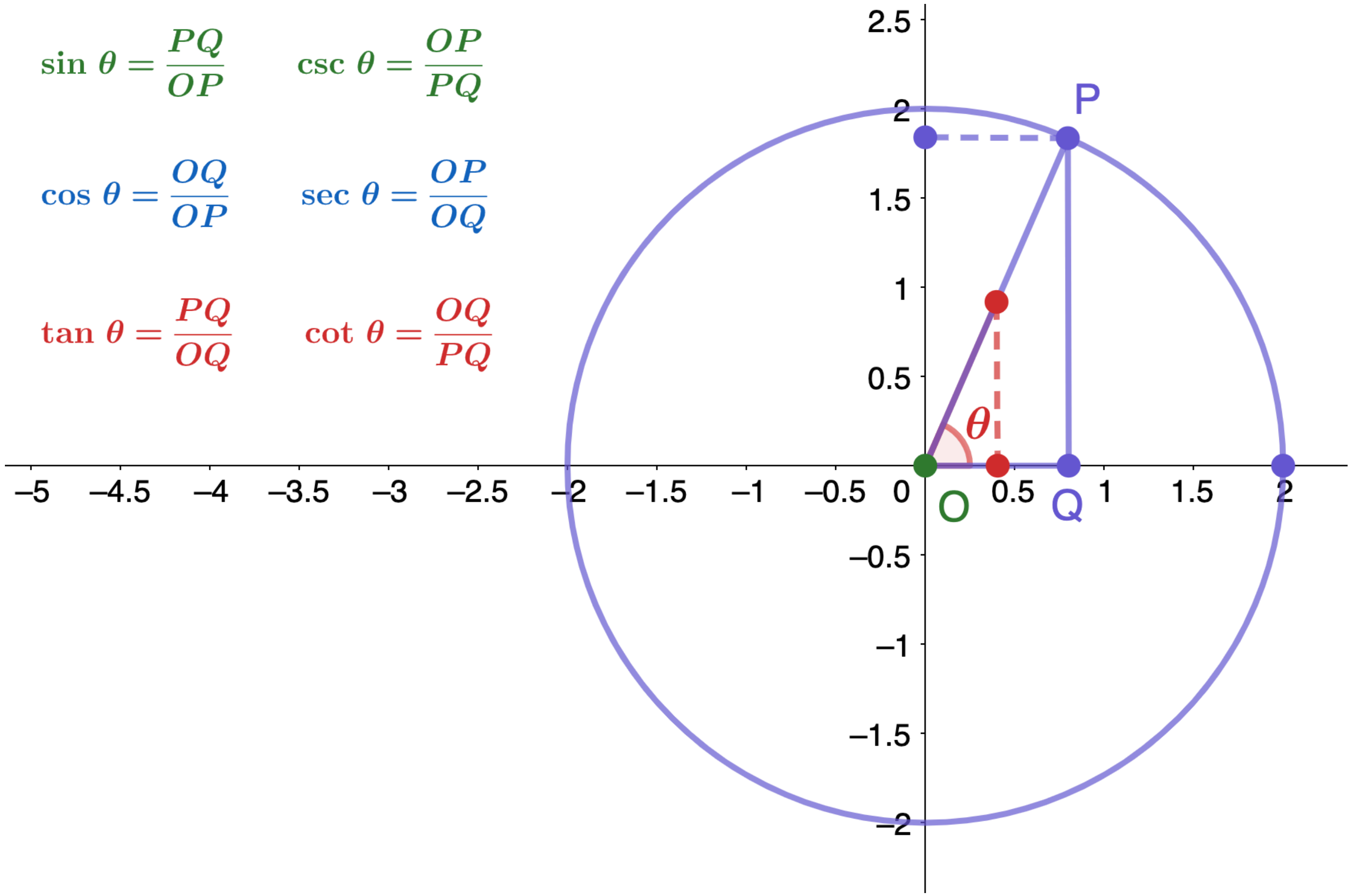
GENERALIZATION TO ANY CIRCLE

GENERALIZATION TO ANY RIGHT TRIANGLE

$\sin\theta = \dfrac{PQ}{OP}$ $\quad$ $\csc\theta = \dfrac{OP}{PQ}$

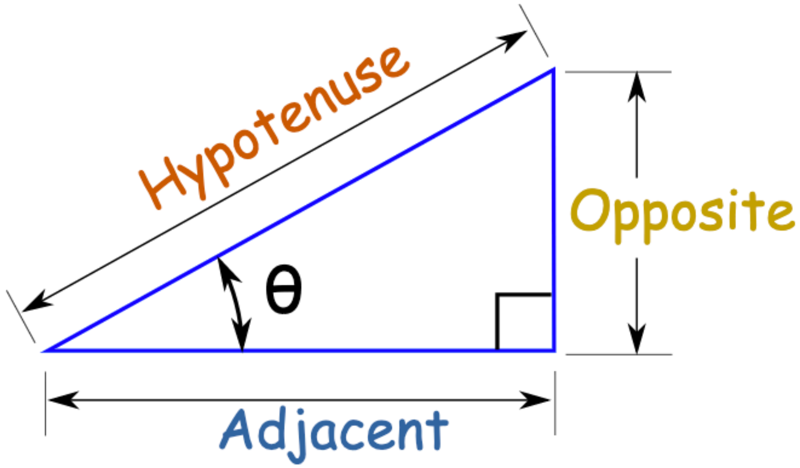
$\cos\theta = \dfrac{OQ}{OP}$ $\quad$ $\sec\theta = \dfrac{OP}{OQ}$

$\tan\theta = \dfrac{PQ}{OQ}$ $\quad$ $\cot\theta = \dfrac{OQ}{PQ}$



A RIGHT TRIANGLE

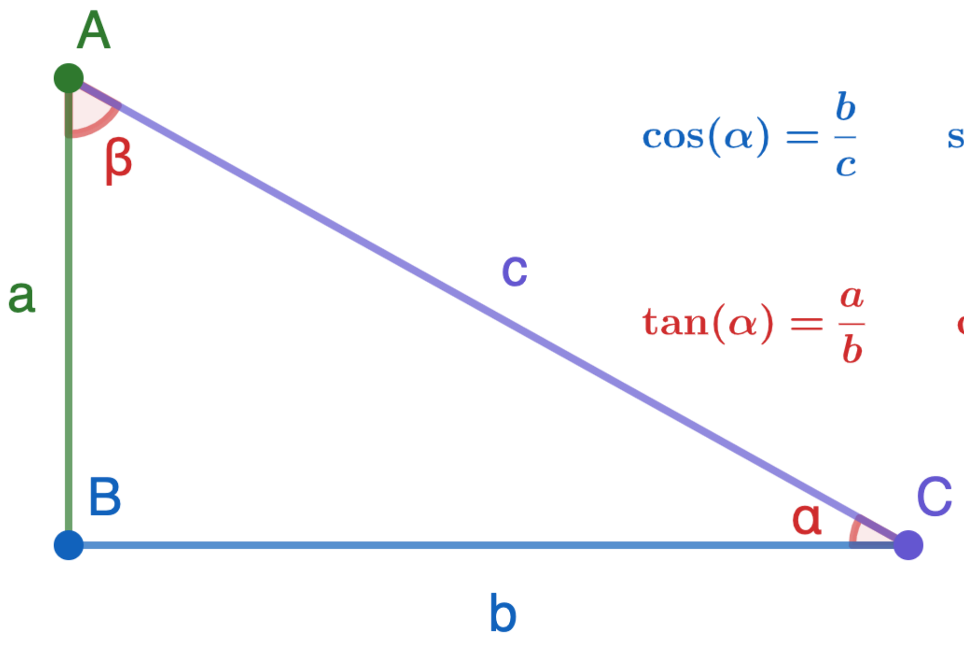
Firstly, the names **Opposite, Adjacent and Hypotenuse** come from the right triangle:



- "Opposite" is opposite to the angle θ
- "Adjacent" is adjacent (next to) to the angle θ
- "Hypotenuse" is the long one

GENERALIZATION TO ANY RIGHT TRIANGLE

- The ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle.



$$\sin(\alpha) = \frac{a}{c} \qquad \csc(\alpha) = \frac{c}{a}$$

$$\cos(\alpha) = \frac{b}{c} \qquad \sec(\alpha) = \frac{c}{b}$$

$$\tan(\alpha) = \frac{a}{b} \qquad \cot(\alpha) = \frac{b}{a}$$

HOW TO MEMORIZE: **SOHCAHTOA!**

sin α	soh…	opposite/hypotenuse
cos α	…cah…	adjacent/hypotenuse
tan α	…toa	opposite/adjacent

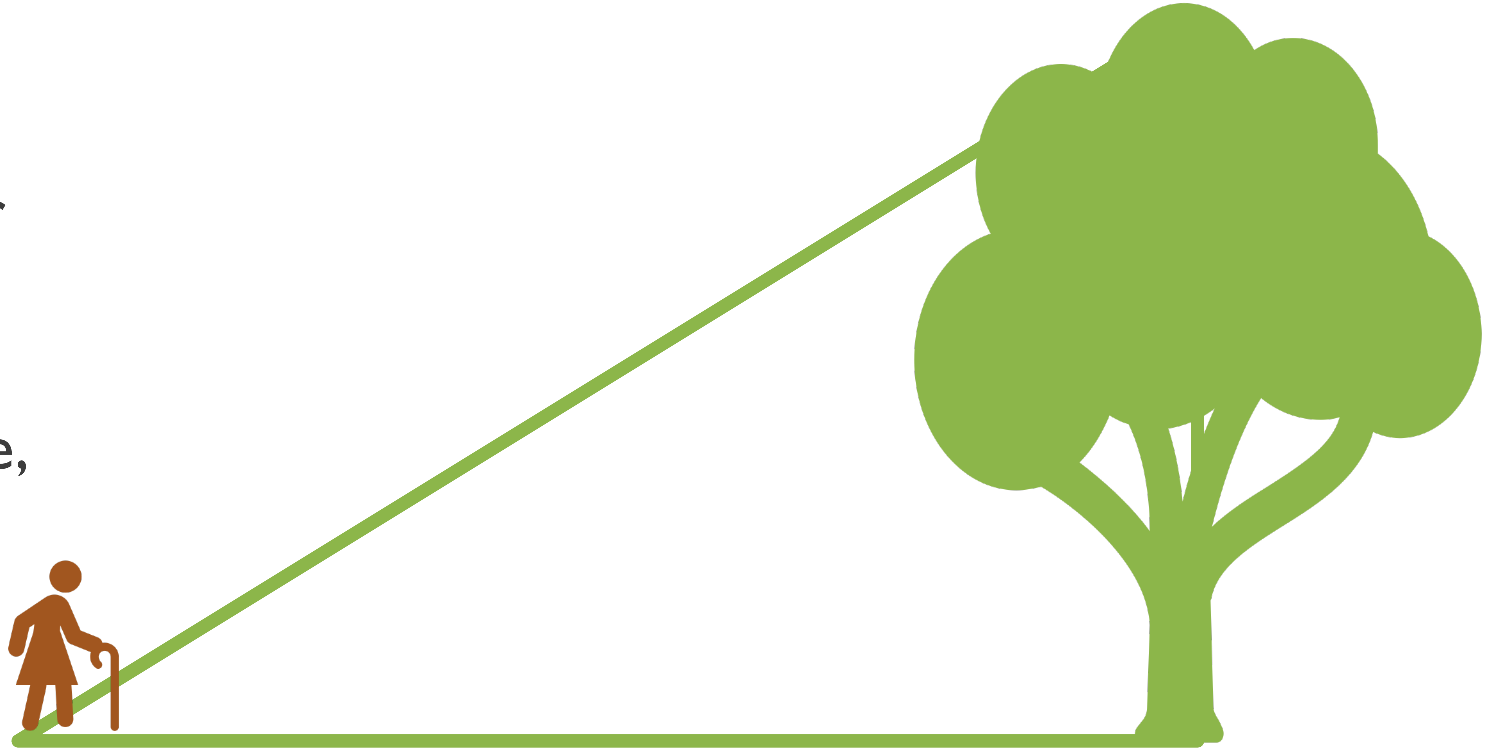
EXAMPLE: ESTIMATE THE HEIGHT OF A TREE

Scientists studying a forest ecosystem in Hanover over a long period of time may record measurements of trees for a number of variables, including each tree's diameter at breast height, height of the lowest living branch, canopy cover, etc.

One aspect of a tree's growth that can be hard to measure is tree height.

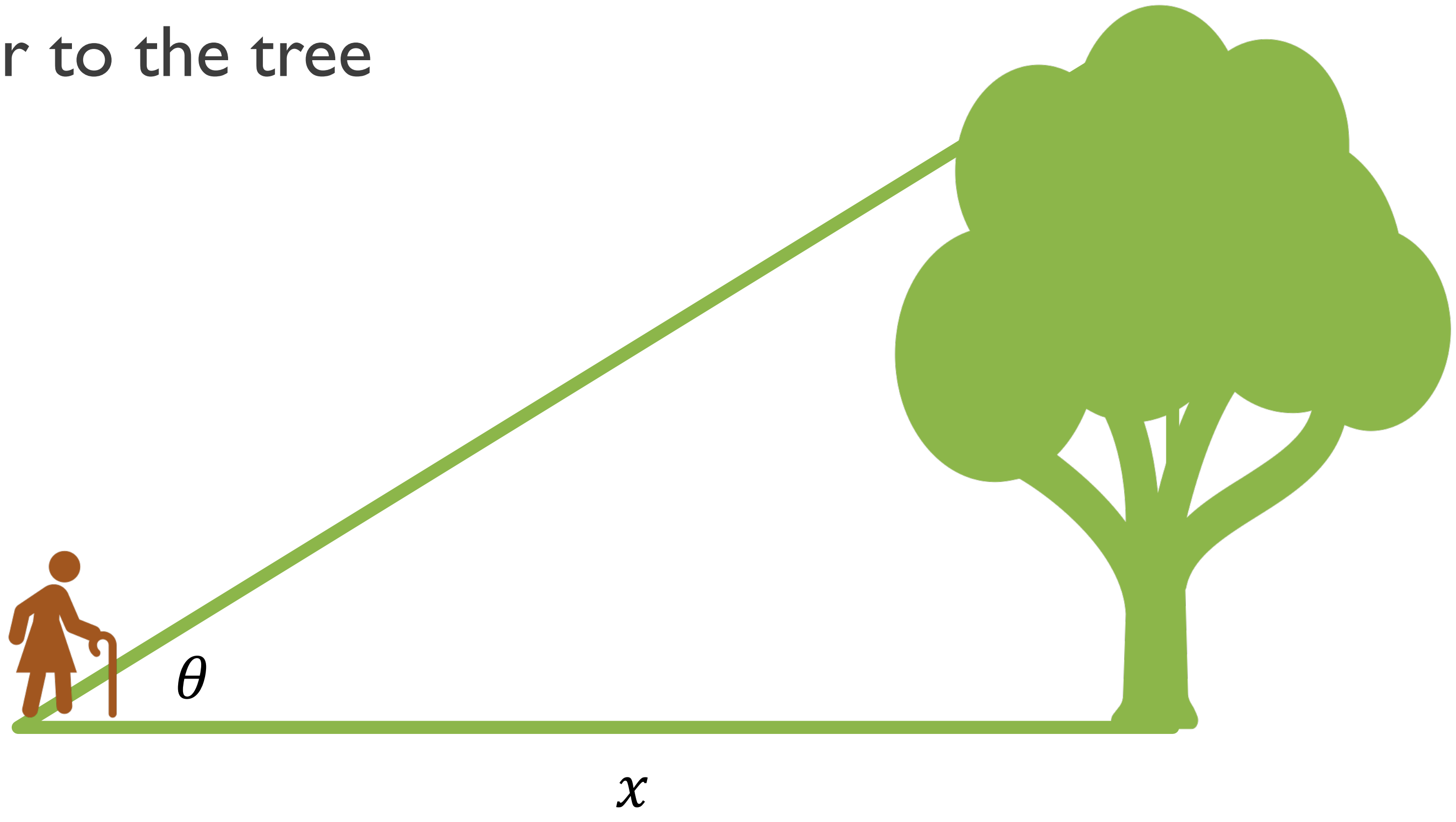
EXAMPLE: ESTIMATE THE HEIGHT OF A TREE

- Assuming that the tree is at a right angle to the plane on which the forester is standing, the base of the tree, the top of the tree, and the forester form the vertices (or corners) of a right triangle.
- The forester measures his or her distance from the base of the tree, and then uses a clinometer (a small instrument that measures inclination, or angle of elevation) to look at the top of the tree and determine the angle.



EXAMPLE: ESTIMATE THE HEIGHT OF A TREE

- Distance from the forester to the tree
  - x
- Angle of measurement
  - θ
- Hight of the tree
  - $h = \cdots$



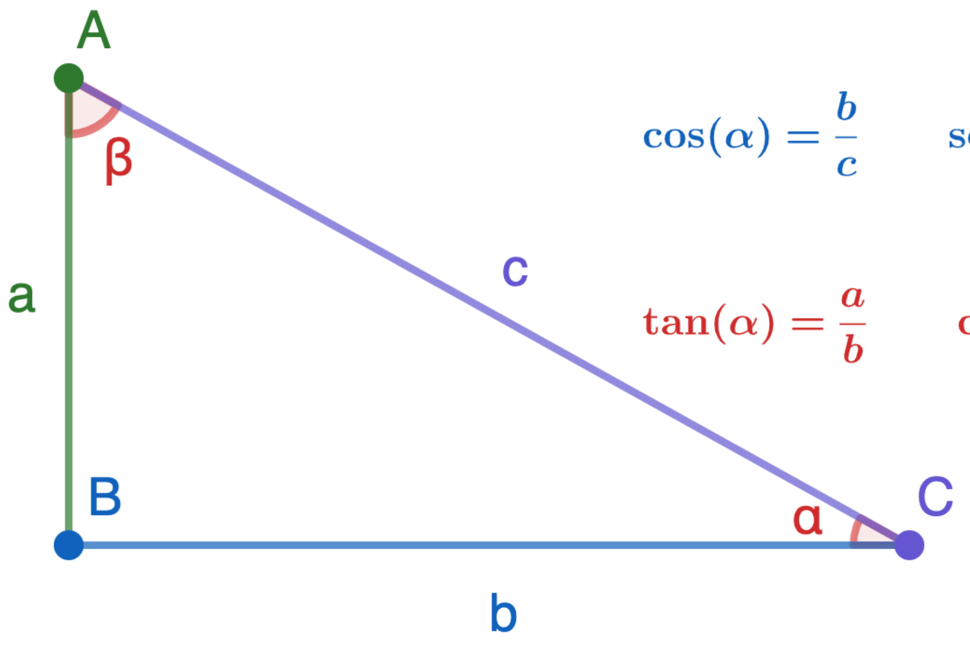
TRIGONOMETRIC IDENTITIES

- A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined.

$$\sin(\alpha) = \frac{a}{c} \qquad \csc(\alpha) = \frac{c}{a}$$

$$\cos(\alpha) = \frac{b}{c} \qquad \sec(\alpha) = \frac{c}{b}$$

$$\tan(\alpha) = \frac{a}{b} \qquad \cot(\alpha) = \frac{b}{a}$$



RULE: TRIGONOMETRIC IDENTITIES

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

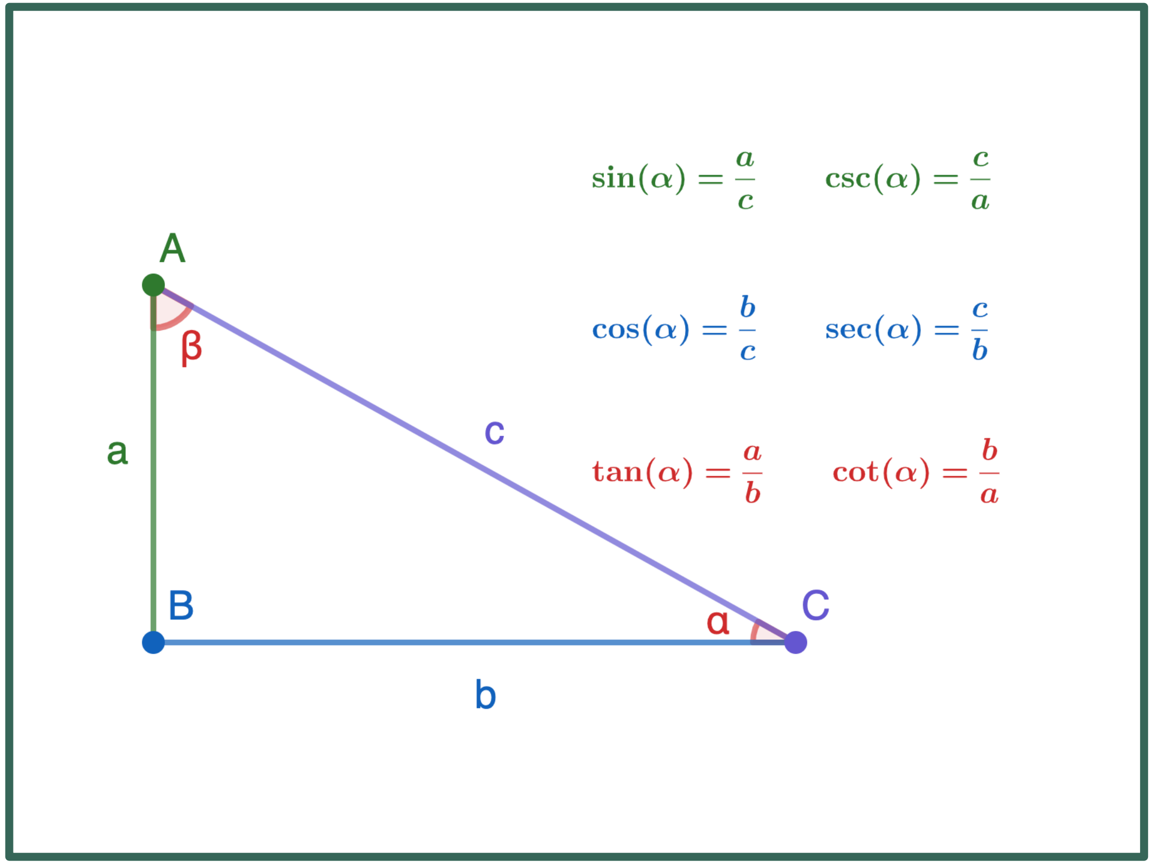
Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

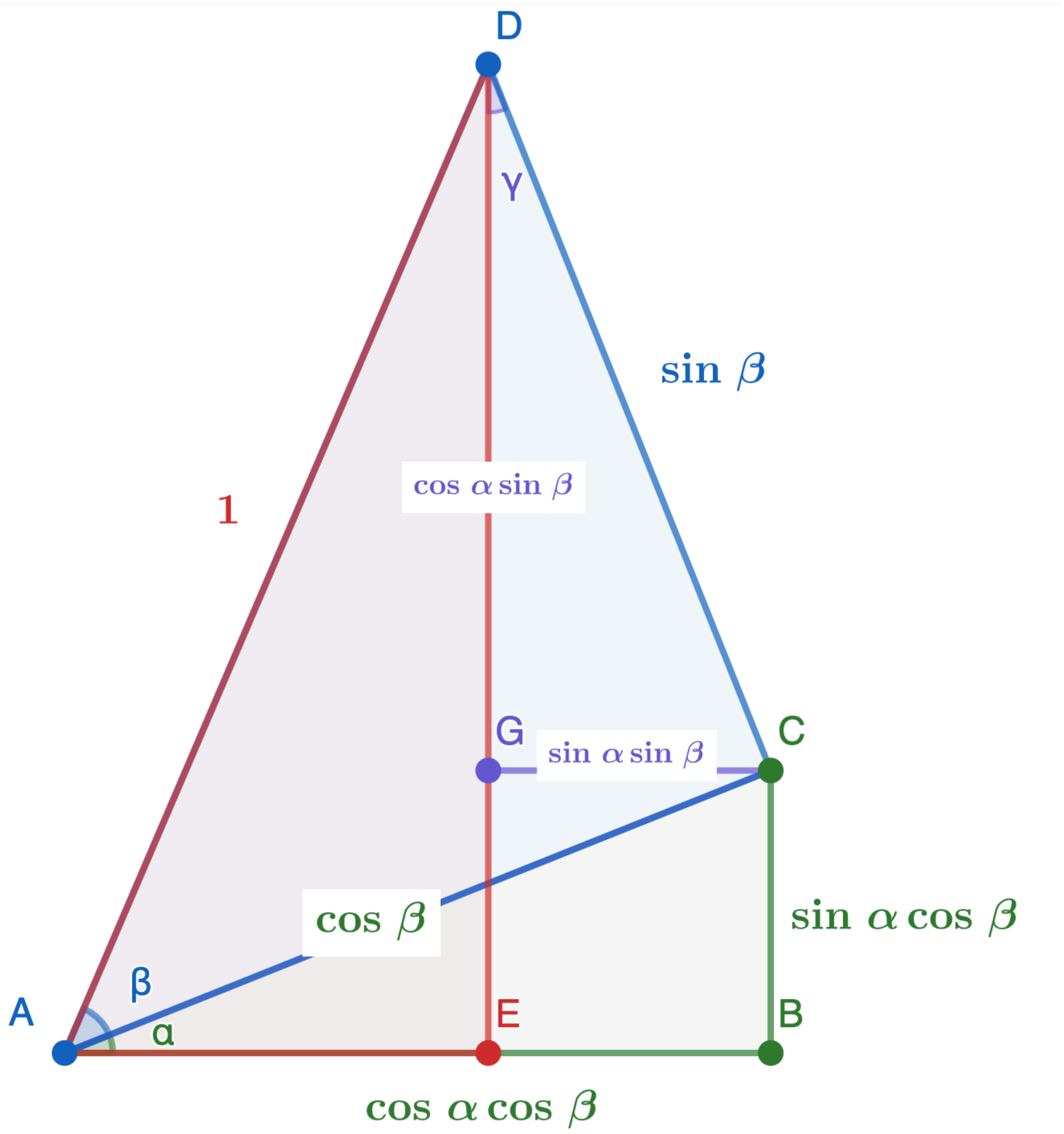
$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

TRIGONOMETRIC IDENTITIES

TRIGONOMETRIC IDENTITIES



- How to prove the Pythagorean identities?
  - $\sin^2 x + \cos^2 x = 1$
  - $1 + \tan^2 x = \sec^2 x$
  - $1 + \cot^2 x = \csc^2 x$

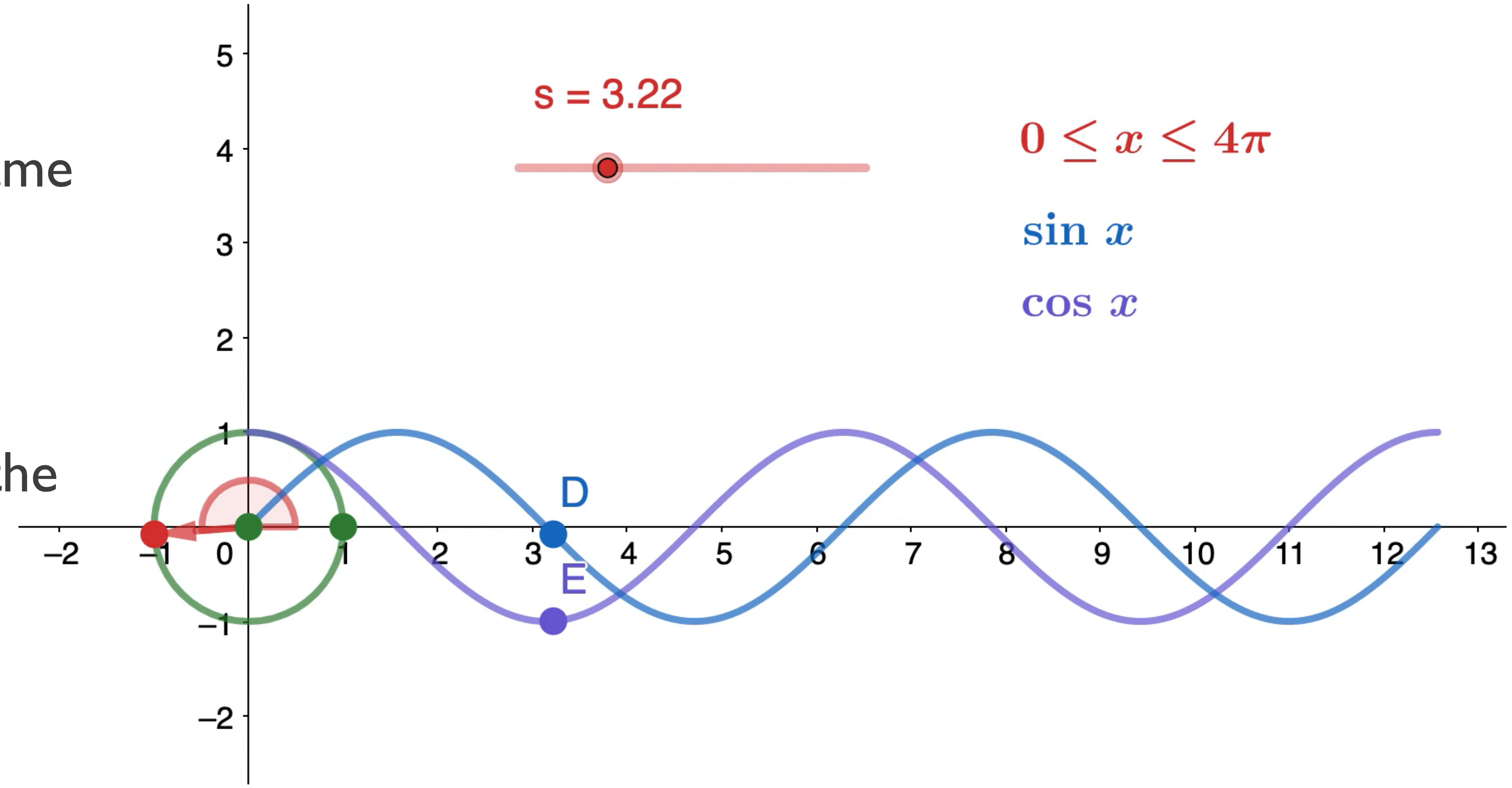


HOW TO PROVE THE ADDITION AND SUBTRACTION FORMULAS

HINT :D

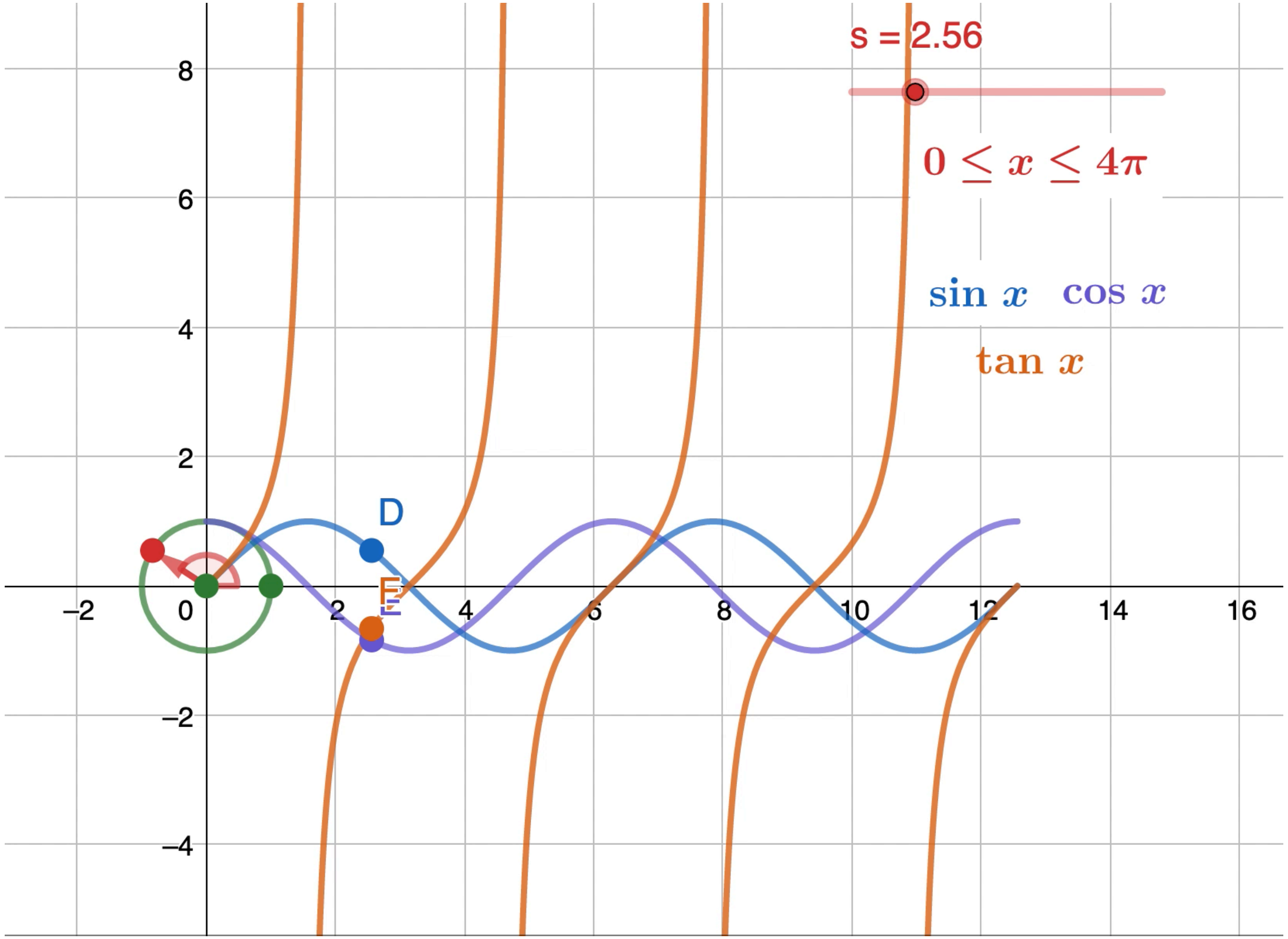
GRAPHS AND PERIODS OF THE TRIGONOMETRIC FUNCTIONS

- The angle θ and $\theta + 2\pi$ correspond to the same point P.
- The values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same.
- Consequently, the trigonometric functions are **periodic functions.**



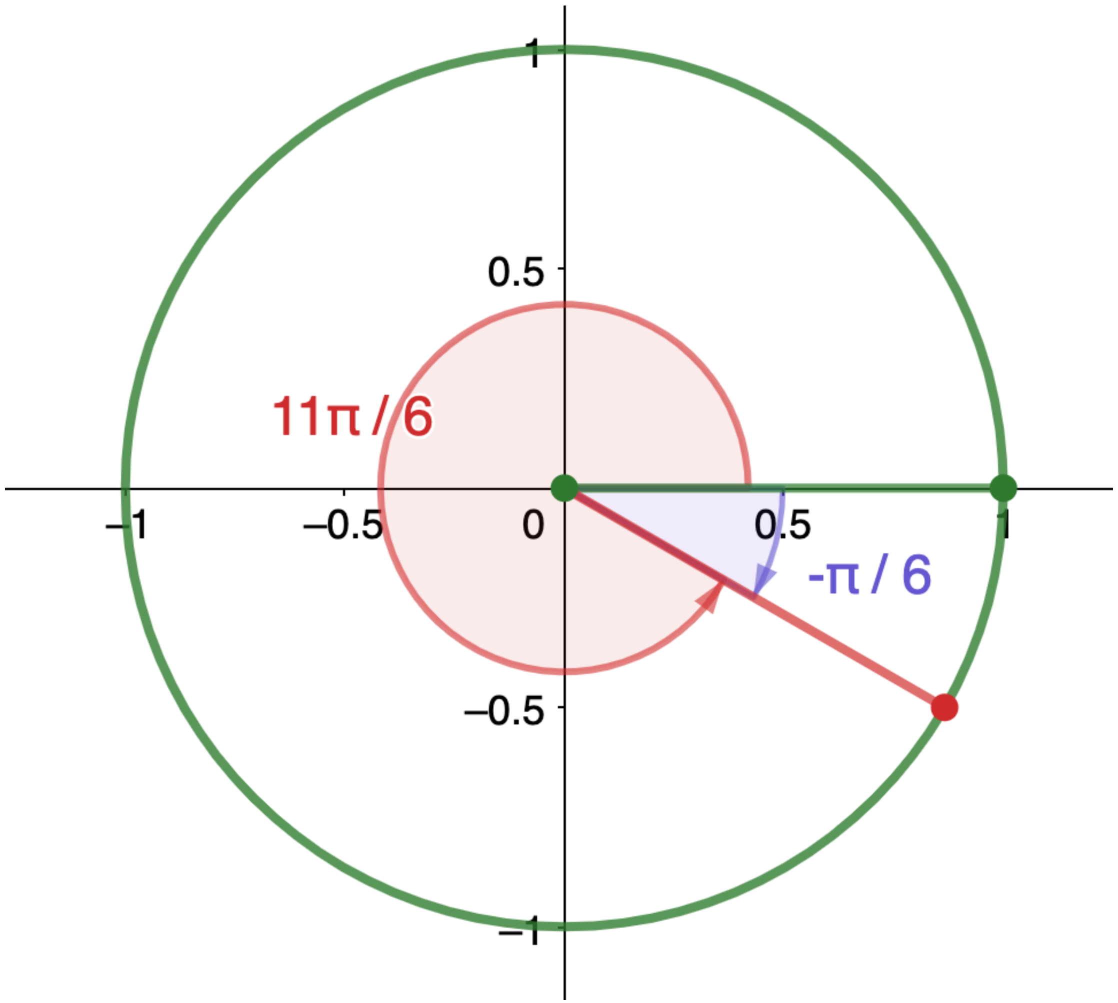
PERIODIC FUNCTIONS

- The period of a function f is defined to be the smallest positive value p such that $f(x+p) = f(x)$ for all values x in the domain of f.
- The sine, cosine, secant, and cosecant functions have a period of 2π.
- Since the tangent and cotangent functions repeat on an interval of length π, their period is π.



WORK SHEET ONE: PROBLEM ONE

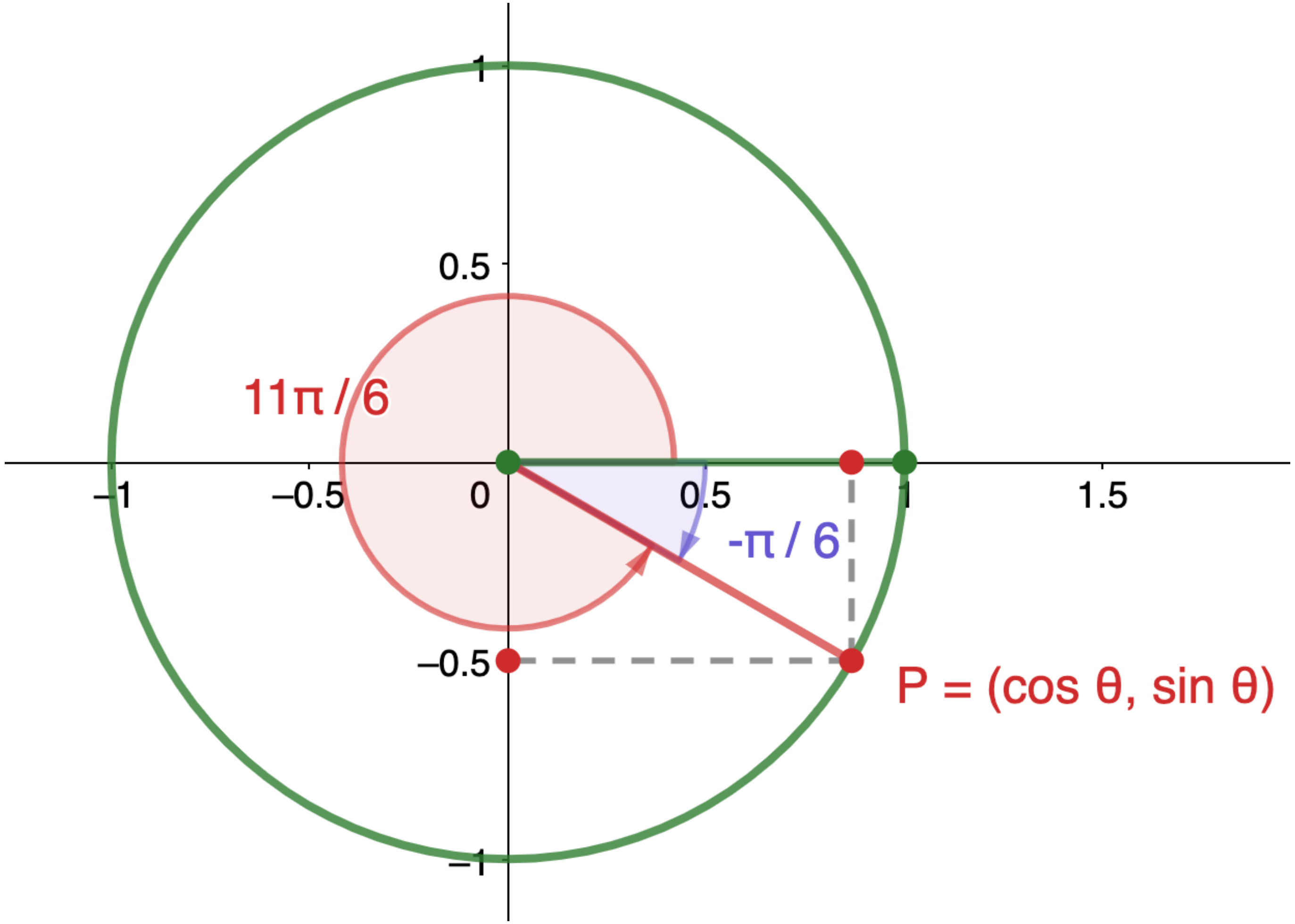
$\sin(-\frac{\pi}{6})$	$\sin(\frac{\pi}{6})$	$\cot(\frac{\pi}{2})$	$\tan(\frac{2\pi}{3})$	$\cos(\frac{3\pi}{4})$	$\csc(\frac{3\pi}{2})$	$\sec(\frac{7\pi}{3})$	$\cos(-\frac{2\pi}{3})$
$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\sqrt{3}$	$-\frac{\sqrt{2}}{2}$	-1	2	$-\frac{1}{2}$



PROBLEM ONE

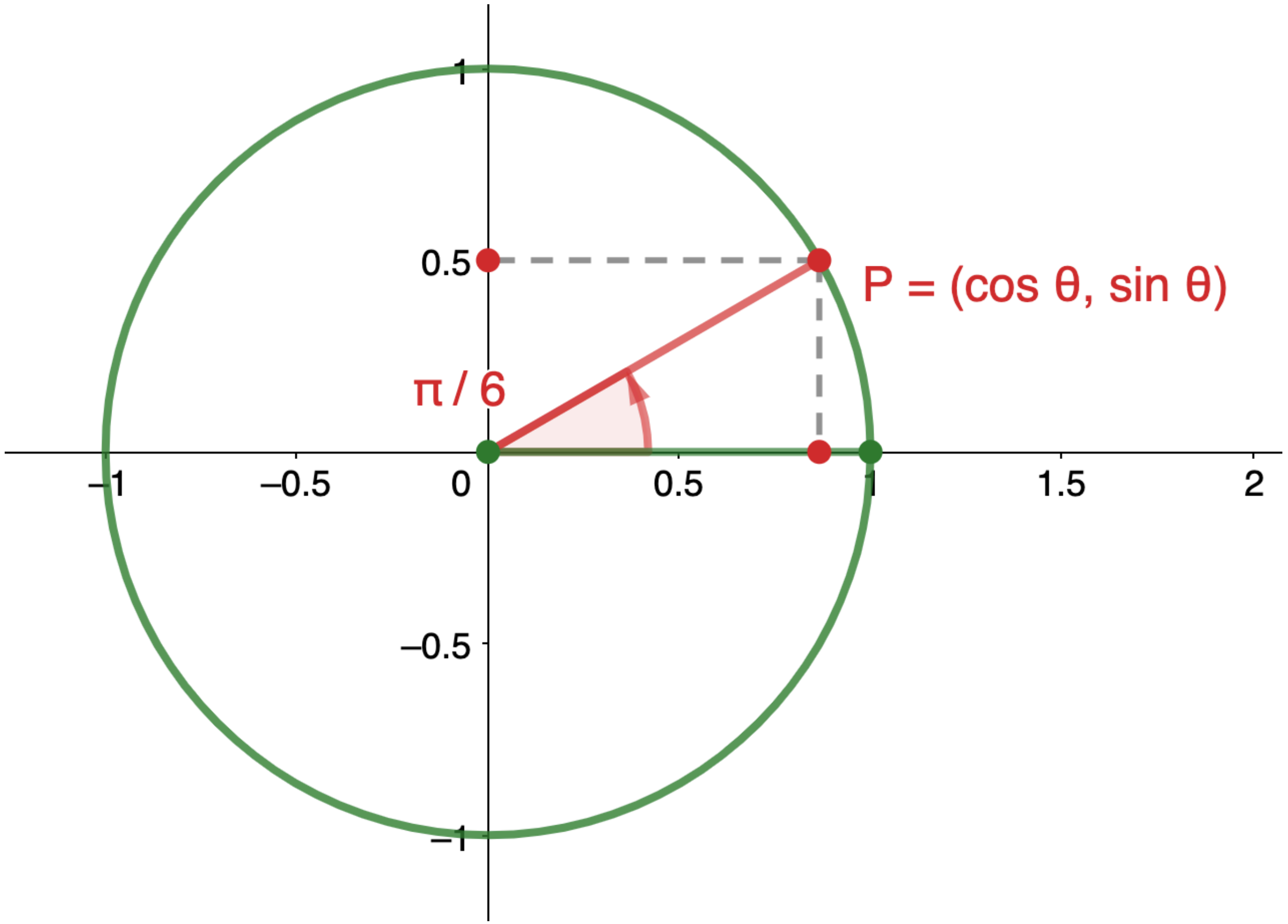
PROBLEM ONE

- $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$



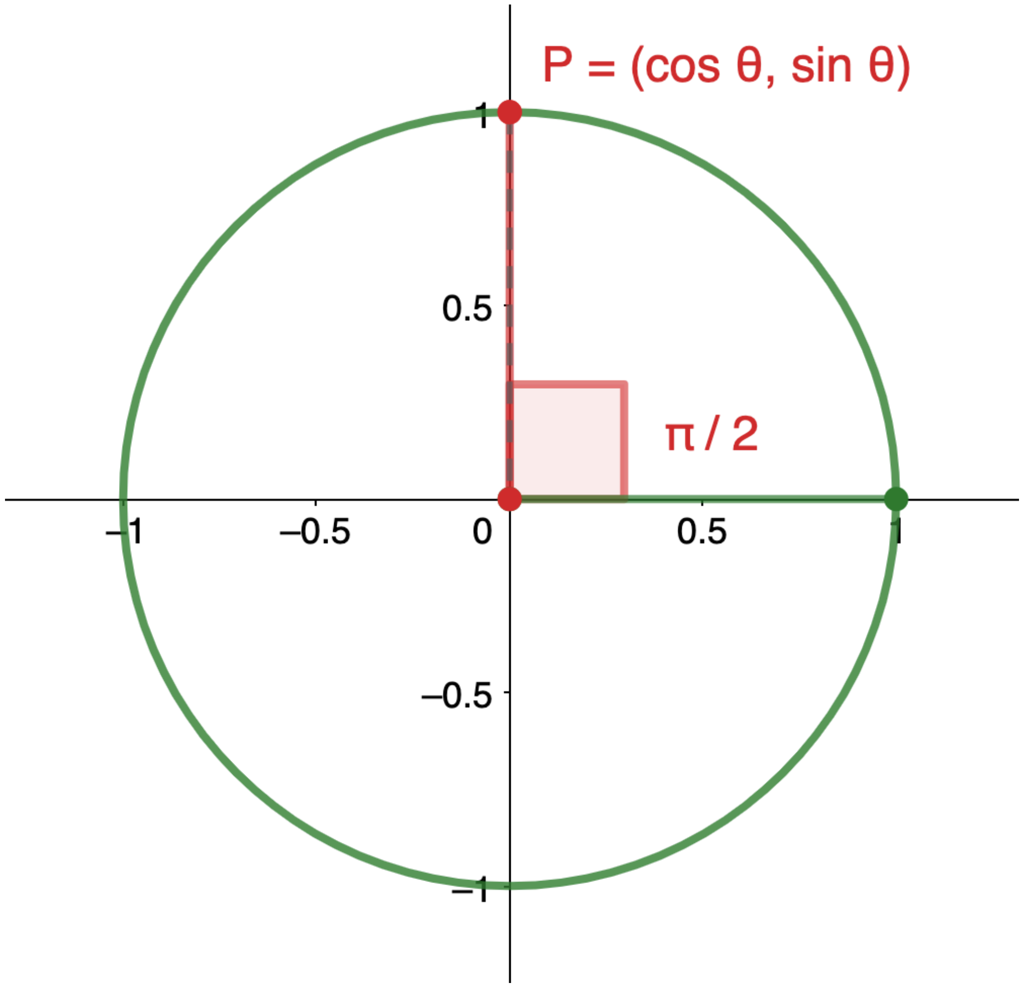
PROBLEM ONE

- $\sin(\frac{\pi}{6}) = \frac{1}{2}$



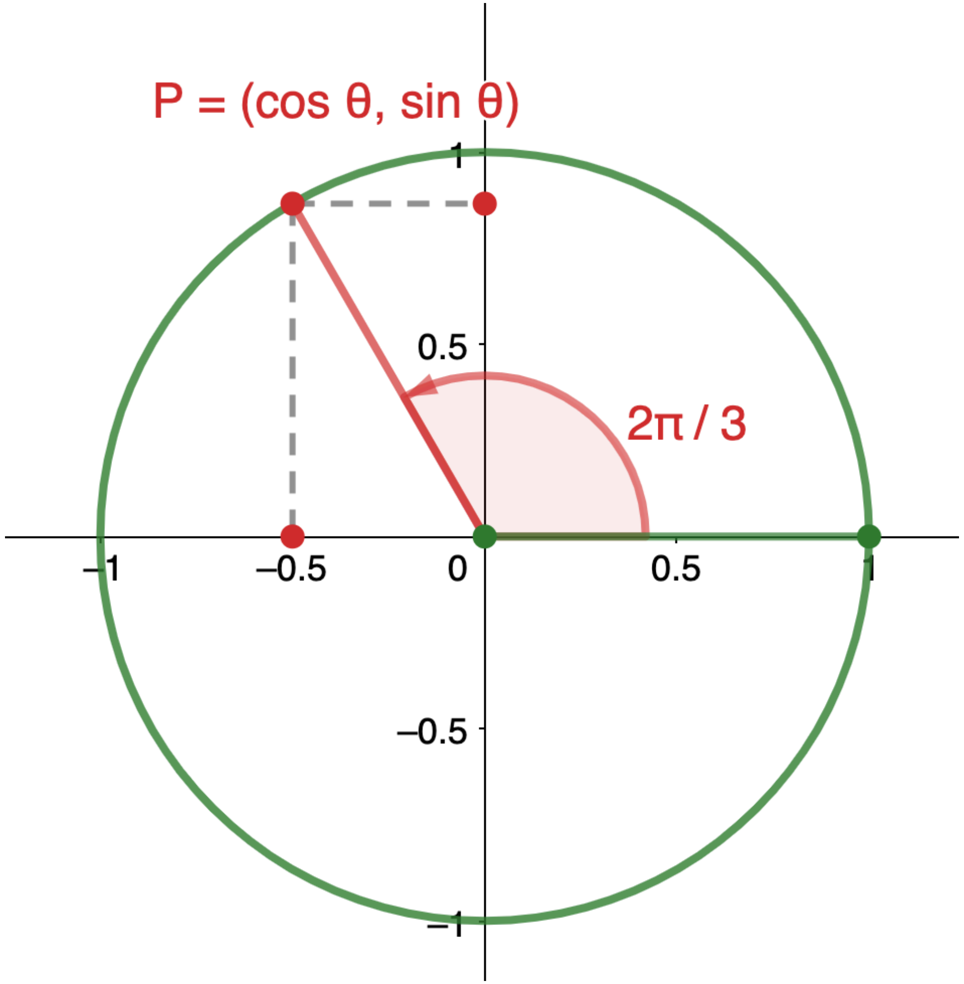
PROBLEM ONE

- $\cot(\frac{\pi}{2}) = \frac{1}{\tan(\frac{\pi}{2})} = 0$



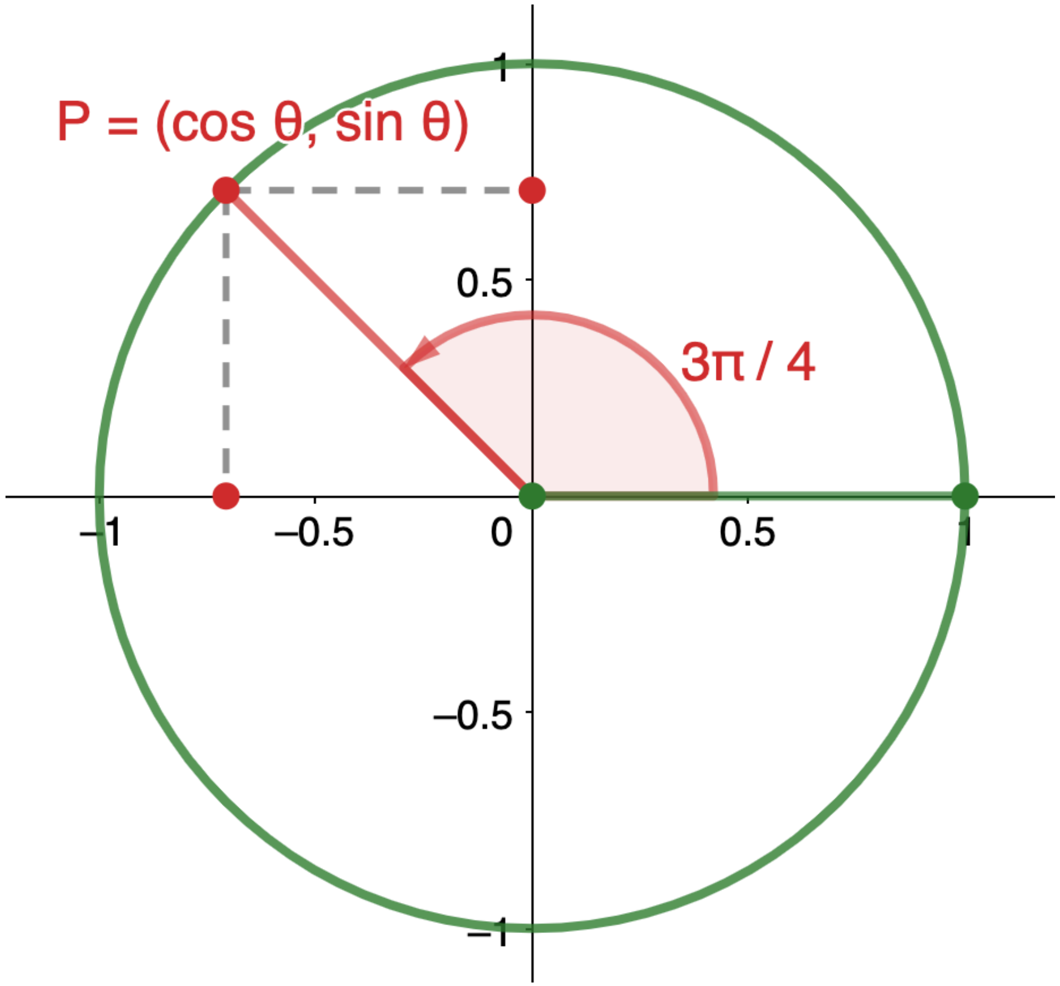
PROBLEM ONE

- $\tan(\frac{2\pi}{3}) = \frac{\sin\frac{2\pi}{3}}{\cos\frac{2\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$



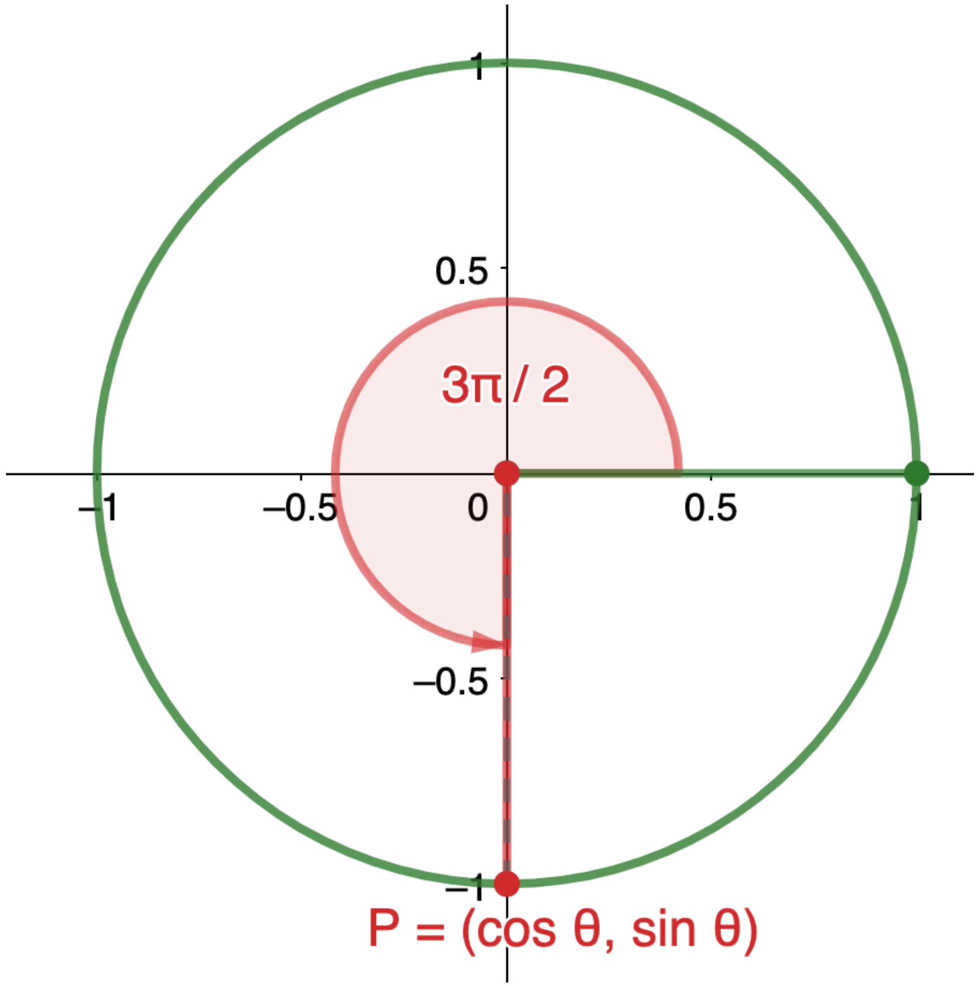
PROBLEM ONE

- $\cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$



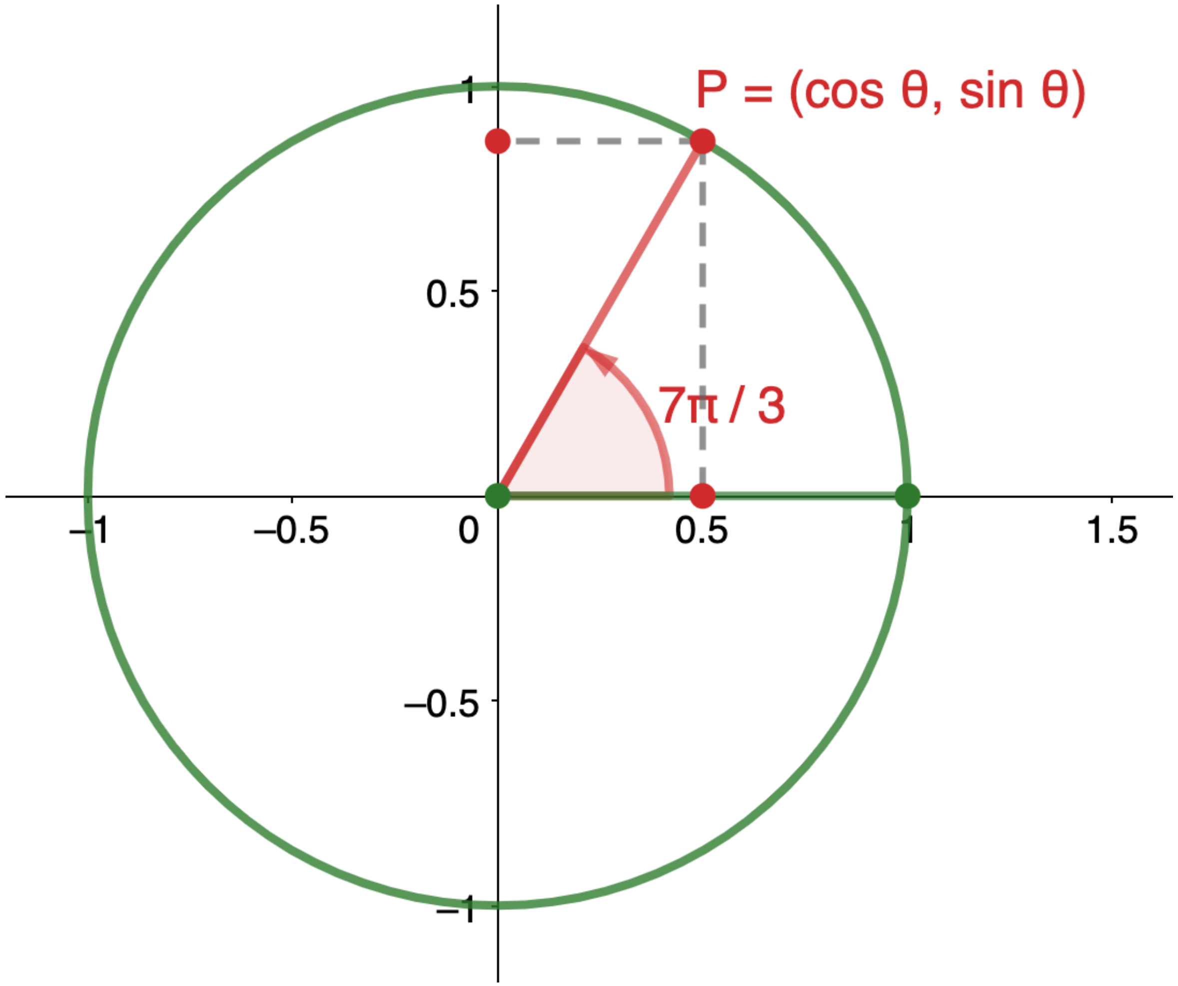
PROBLEM ONE

- $\csc(\frac{3\pi}{2}) = \frac{1}{\sin(\frac{3\pi}{2})} = \frac{1}{-1} = -1$



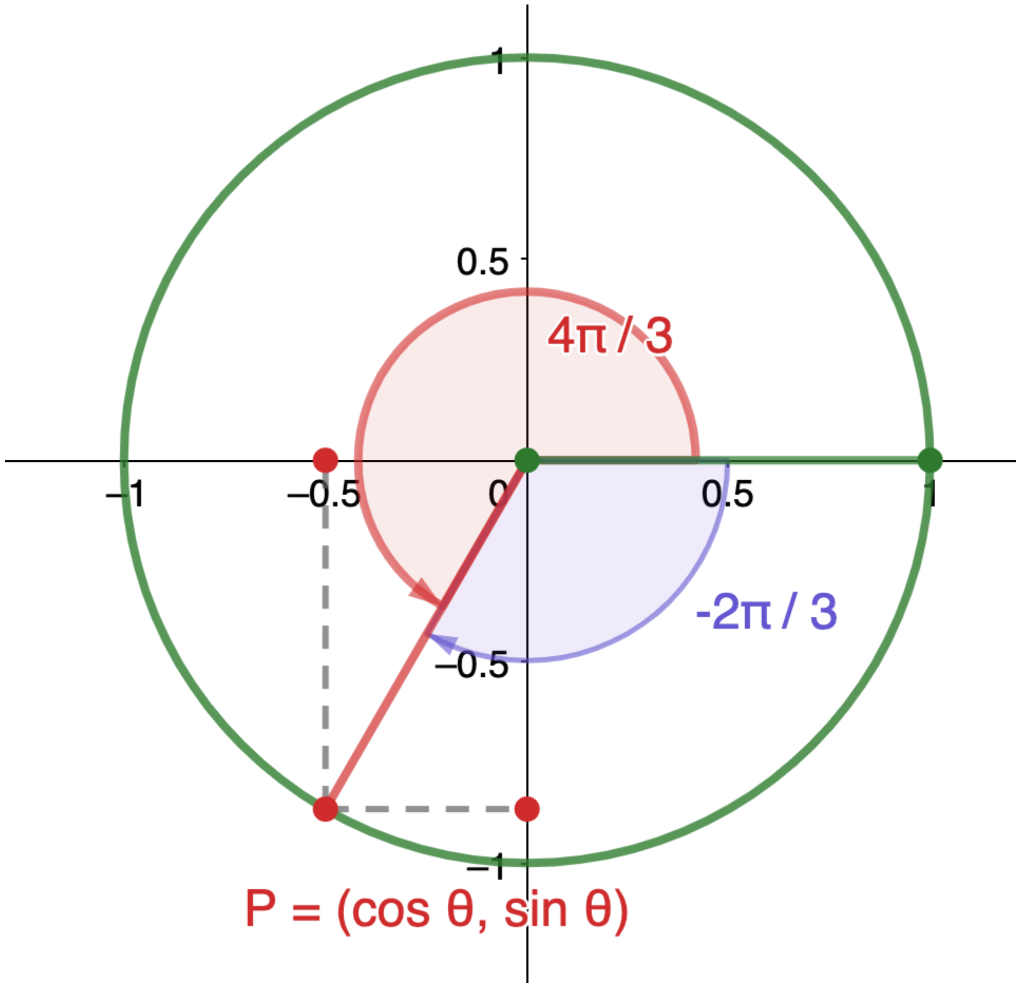
PROBLEM ONE

- $\sec(\frac{7\pi}{3}) = \sec(\frac{7\pi}{3} - 2\pi) = \sec(\frac{\pi}{3}) = \frac{1}{\cos(\frac{\pi}{3})} = \frac{1}{\frac{1}{2}} = 2$



PROBLEM ONE

- $\cos(-\frac{2\pi}{3}) = -\frac{1}{2}$



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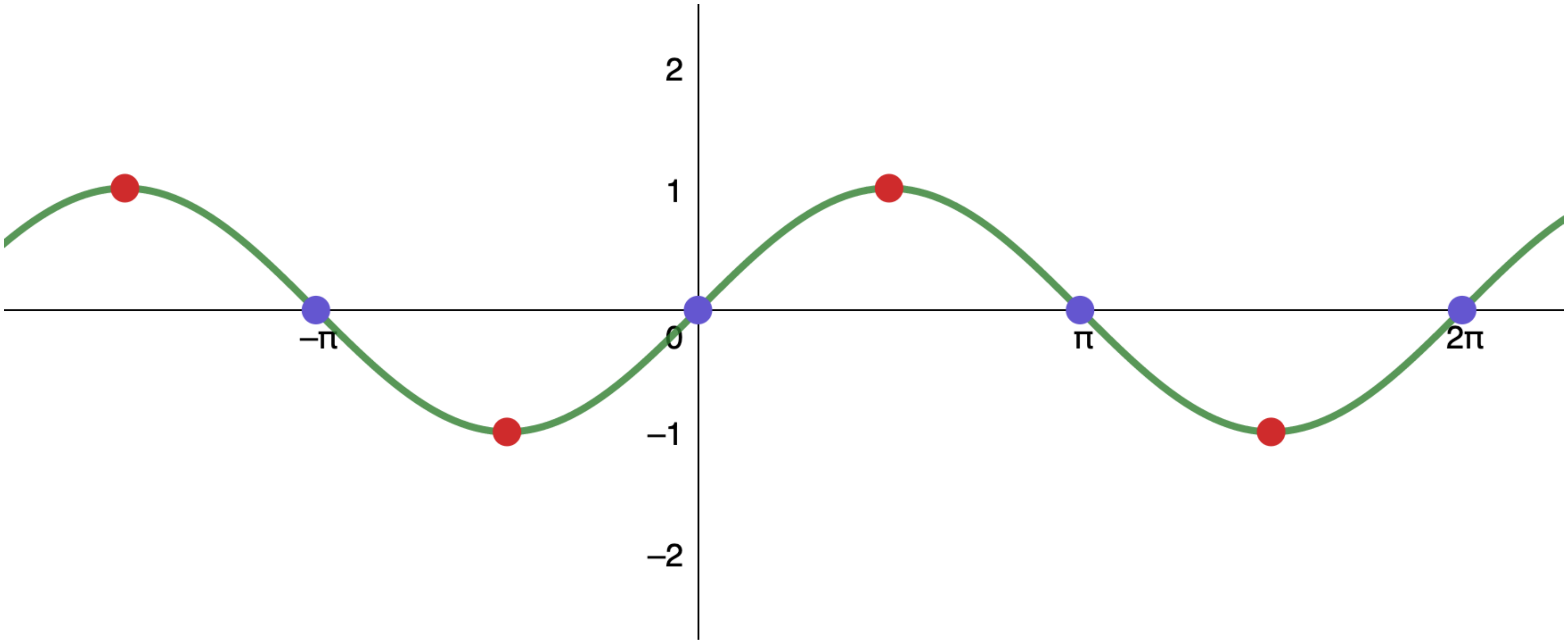

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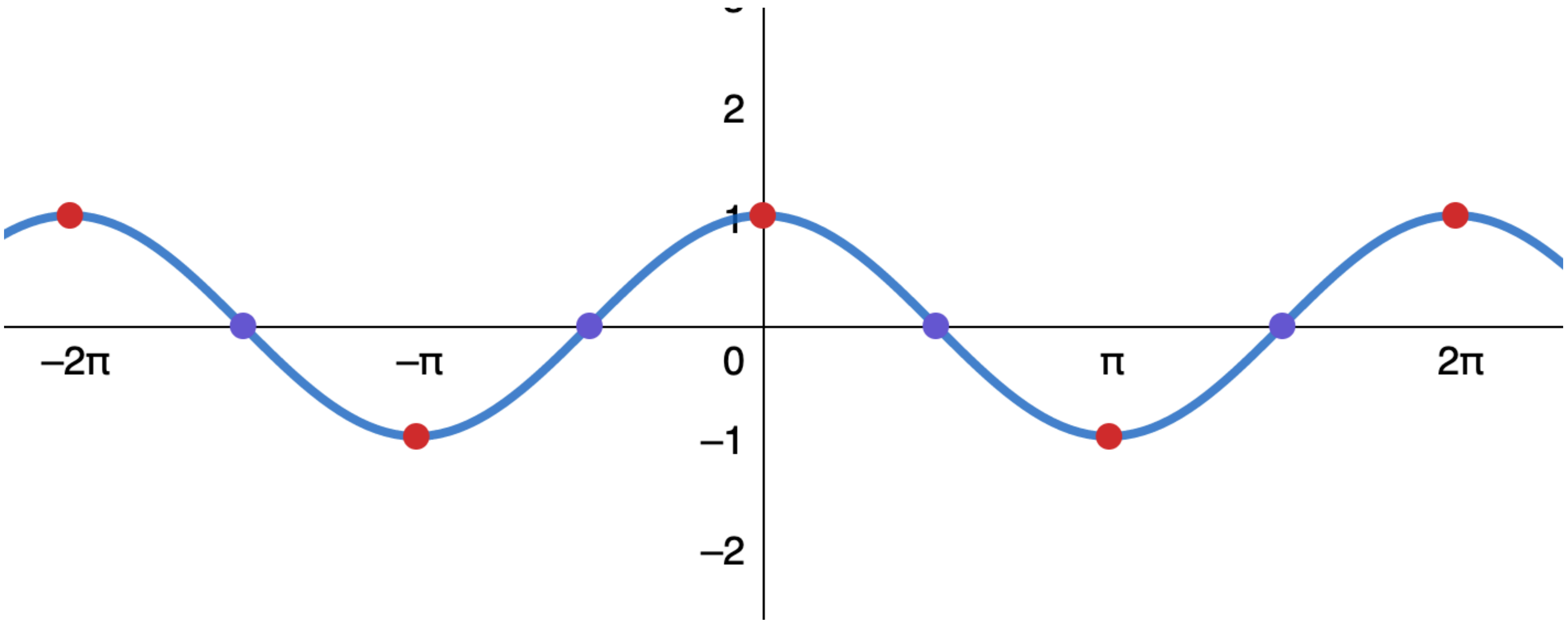
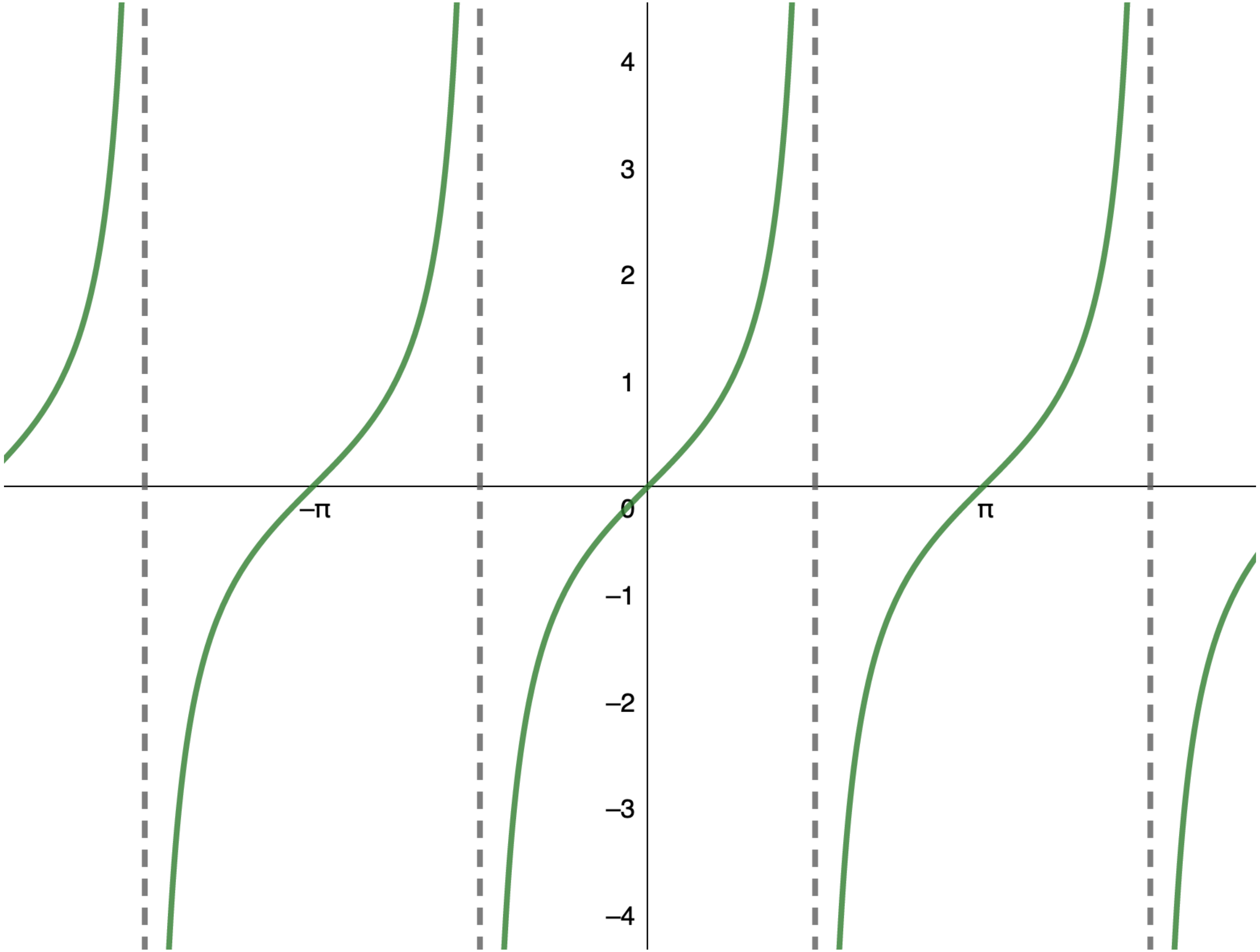
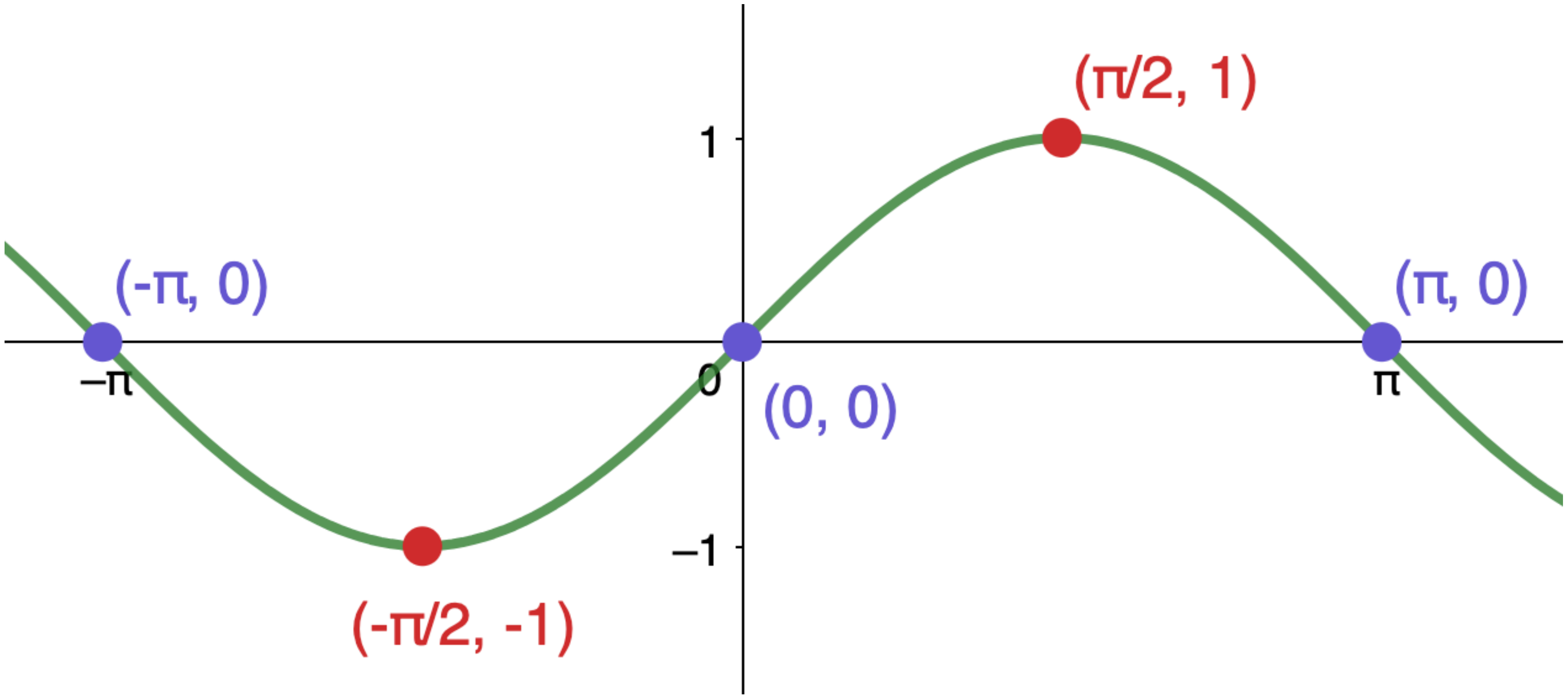




PROBLEM TWO

PROBLEM THREE



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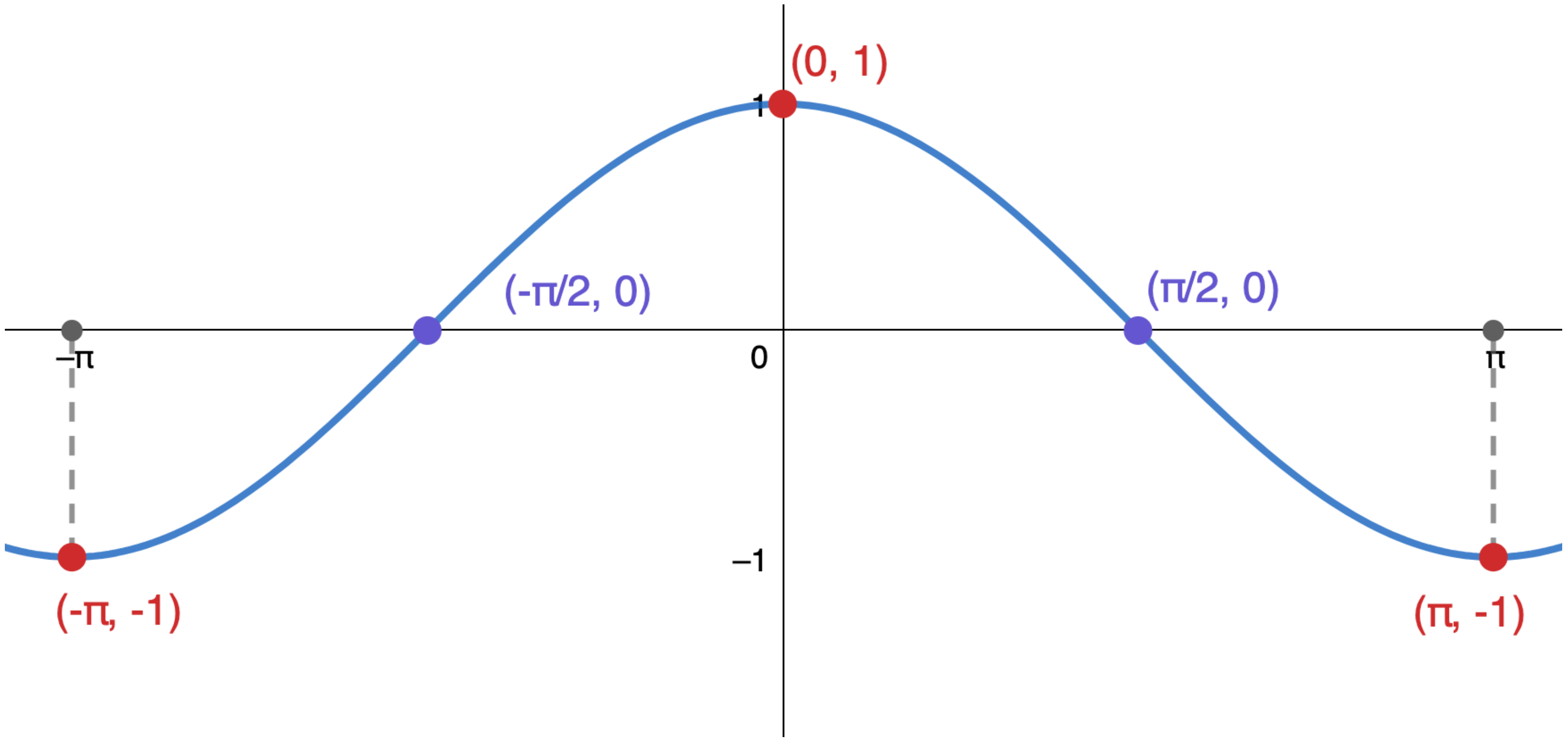
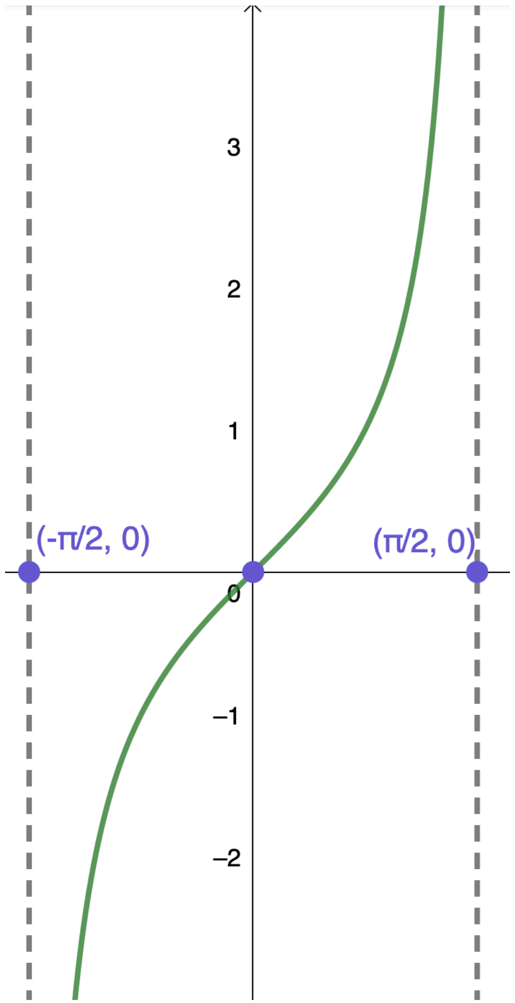
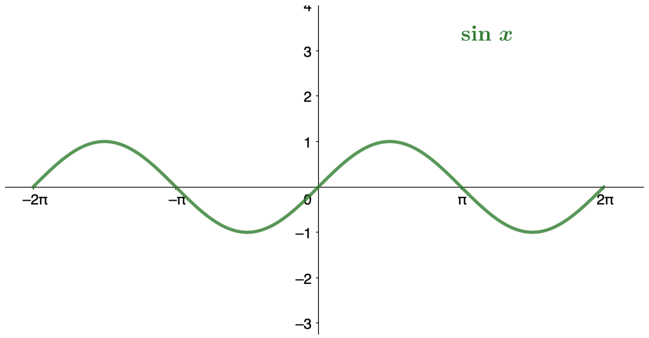
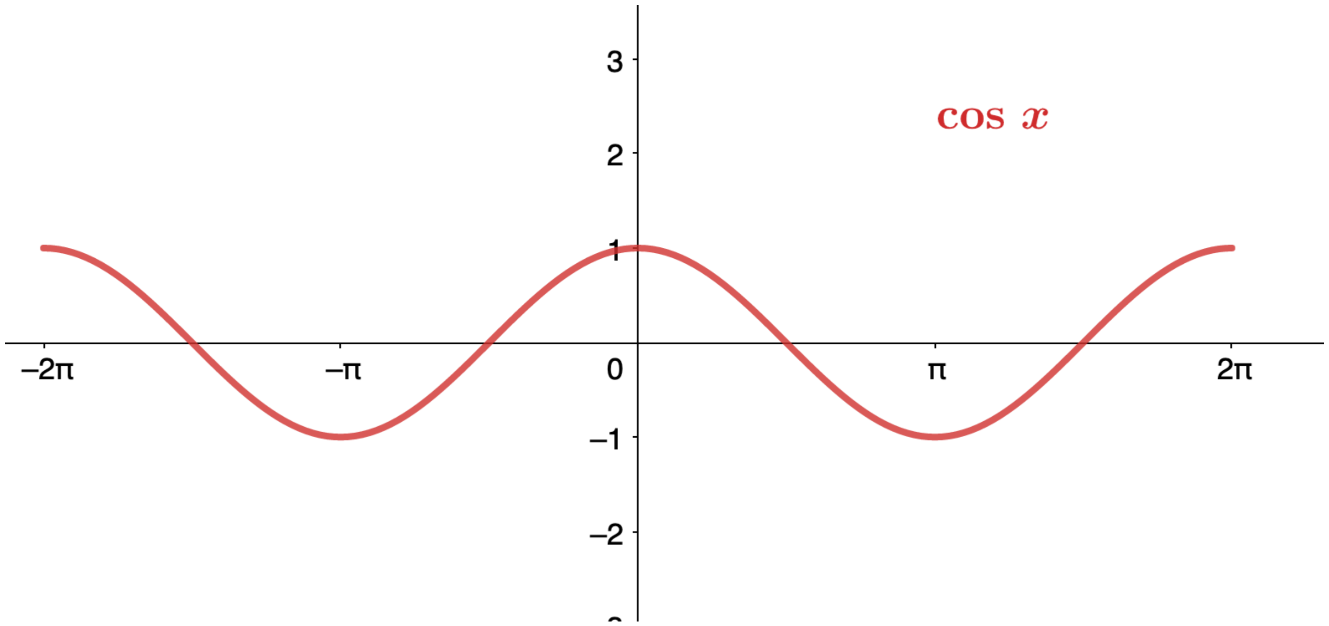


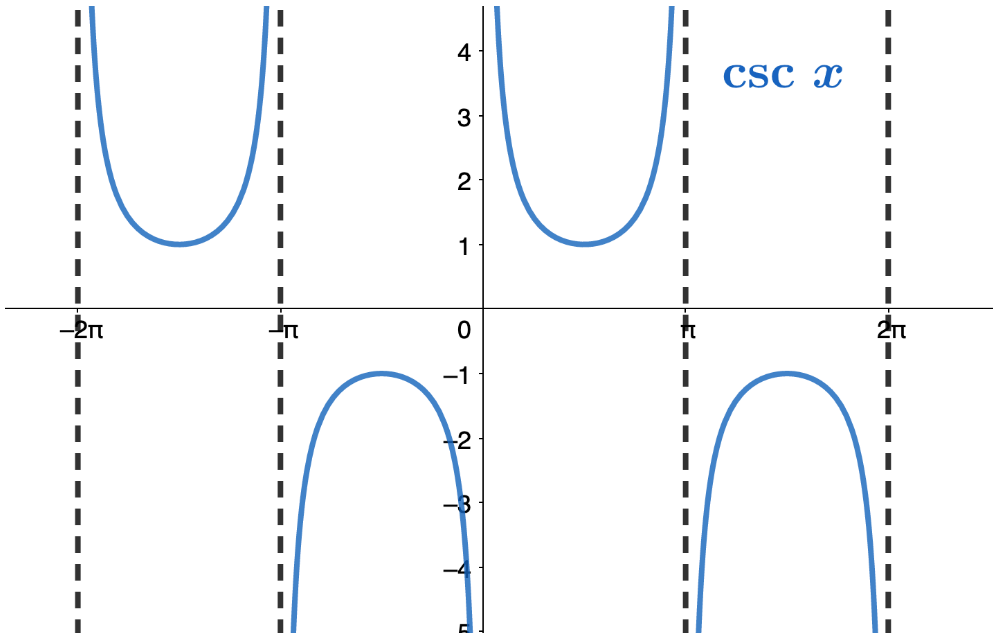


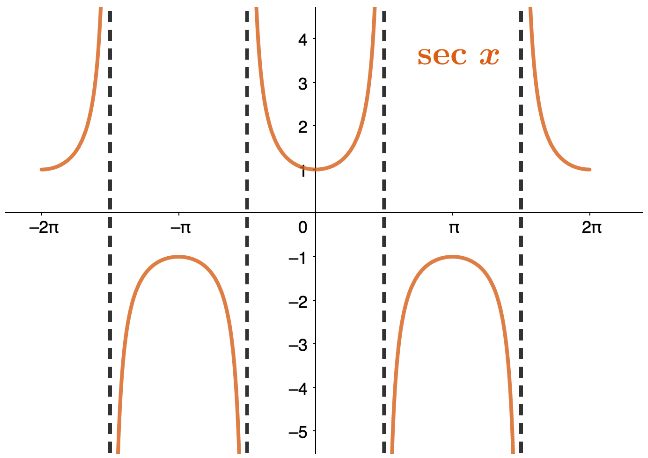
PROBLEM THREE

TRIGONOMETRIC FUNCTIONS: SINE AND COSINE



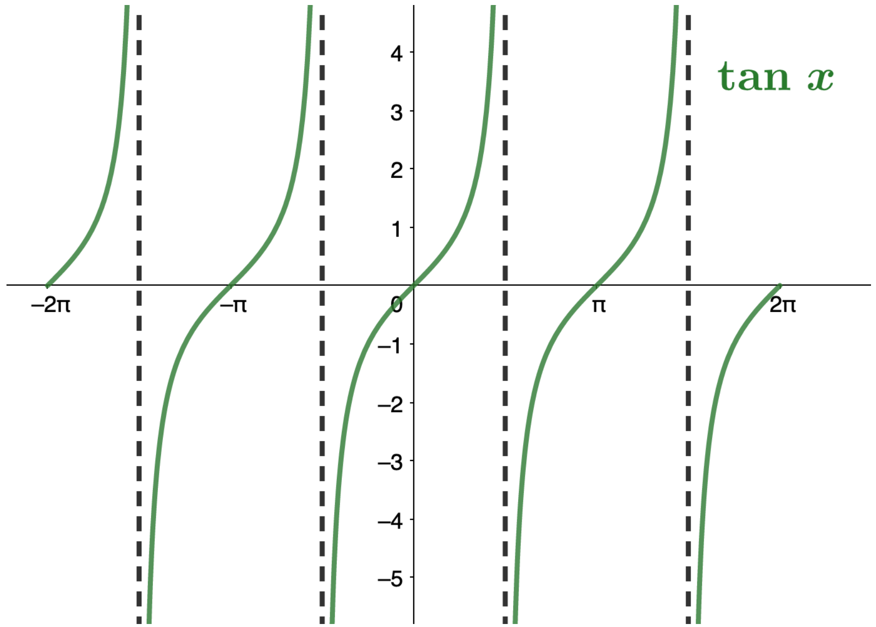


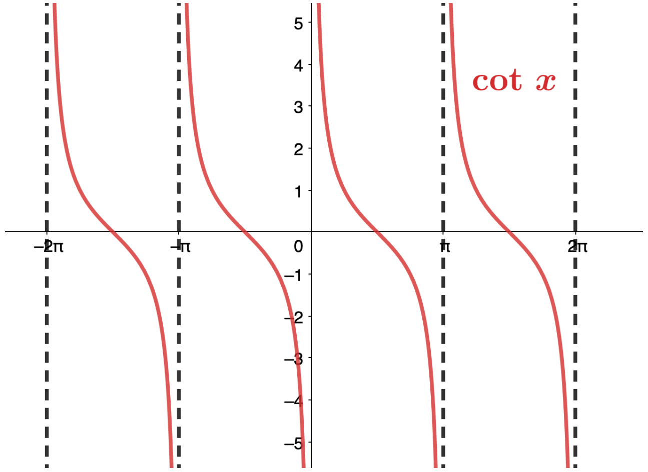




TRIGONOMETRIC FUNCTIONS: SECANT AND COSECANT

TRIGONOMETRIC FUNCTIONS: TANGENT AND COTANGENT





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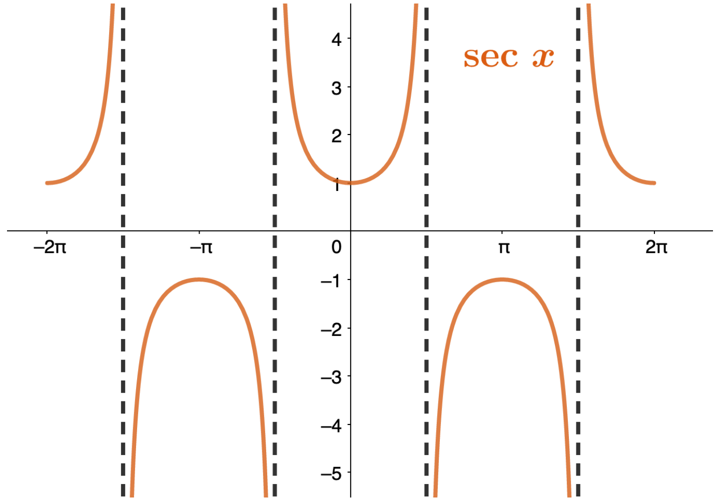
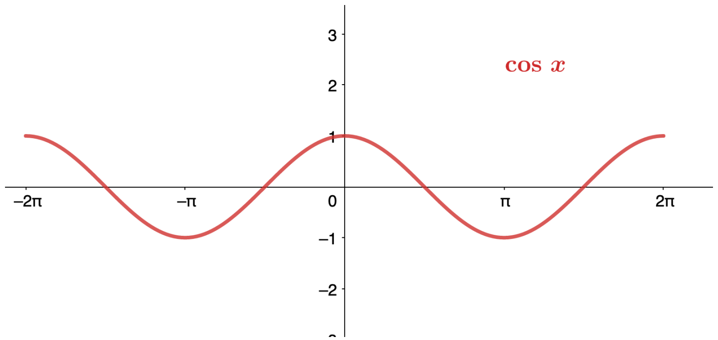
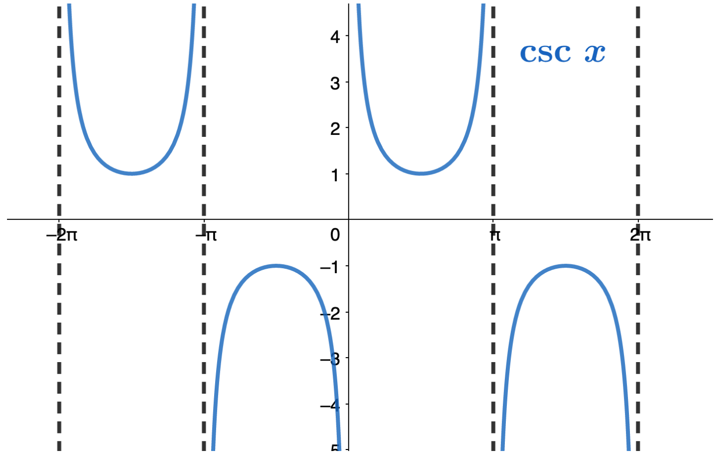
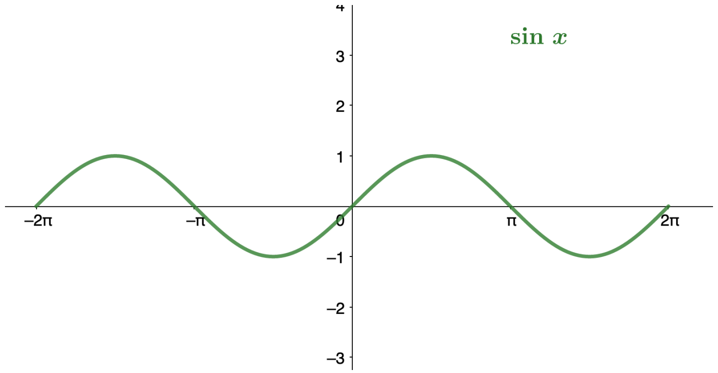
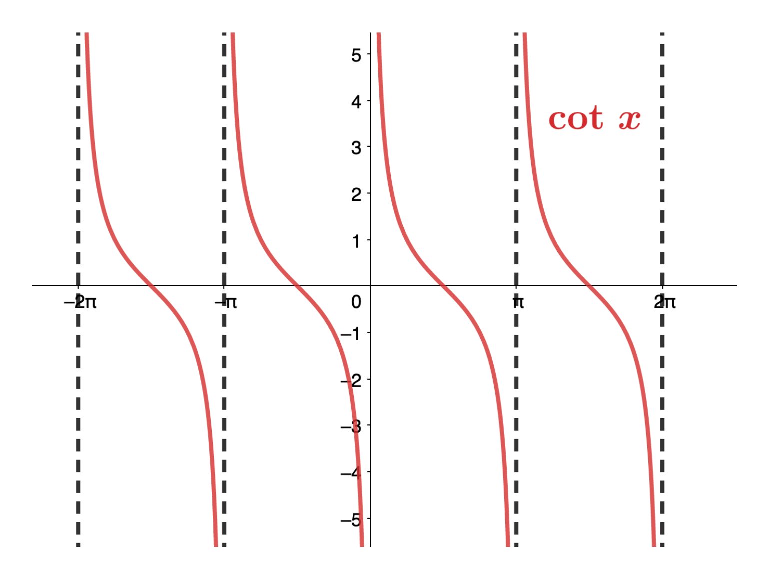


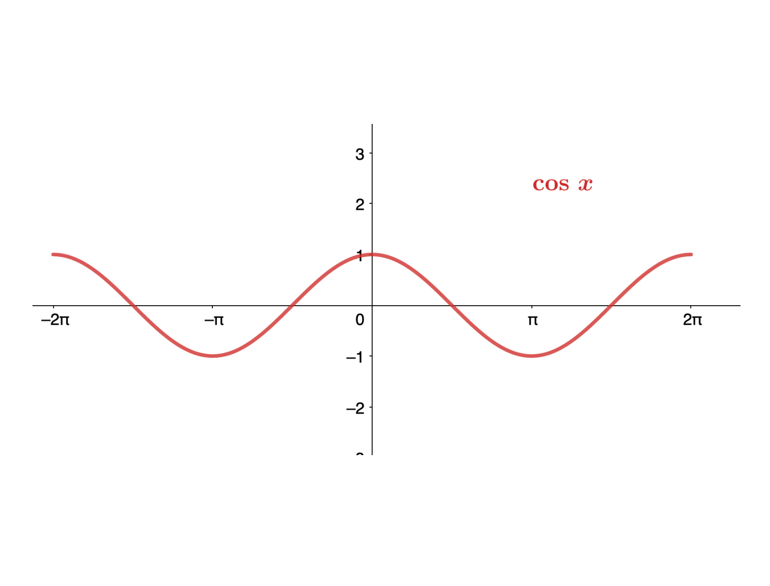


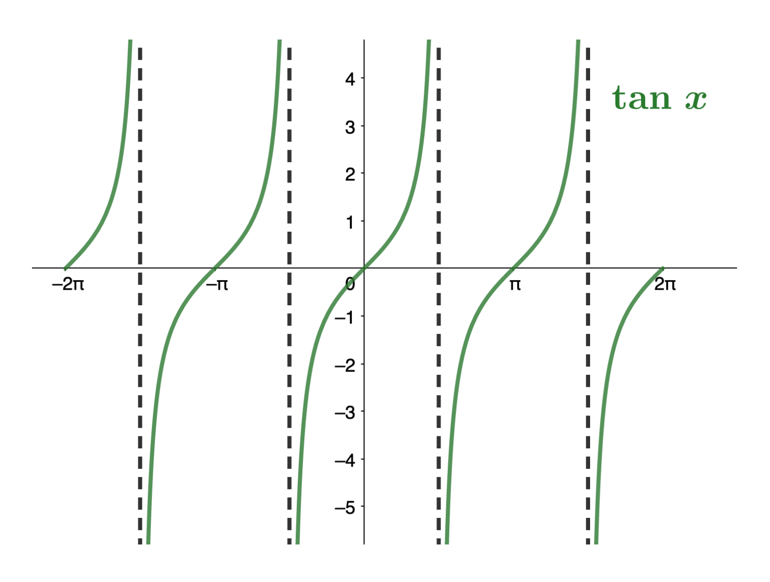


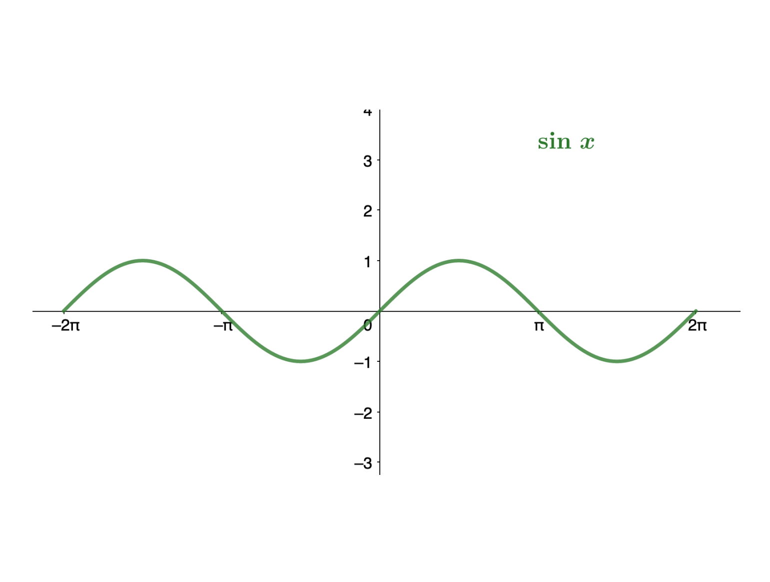


TRIGONOMETRIC FUNCTIONS



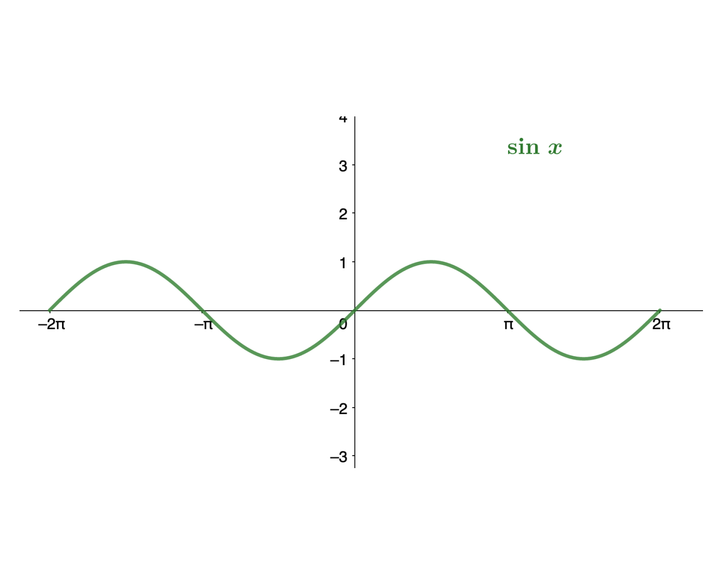


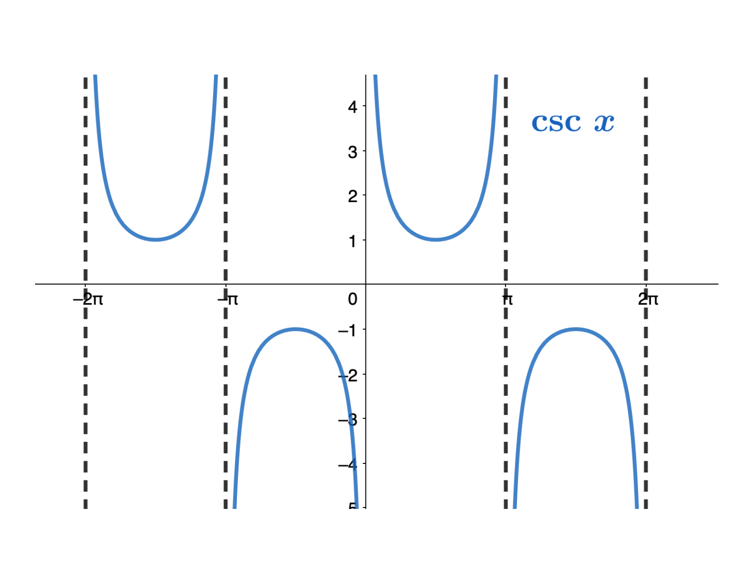


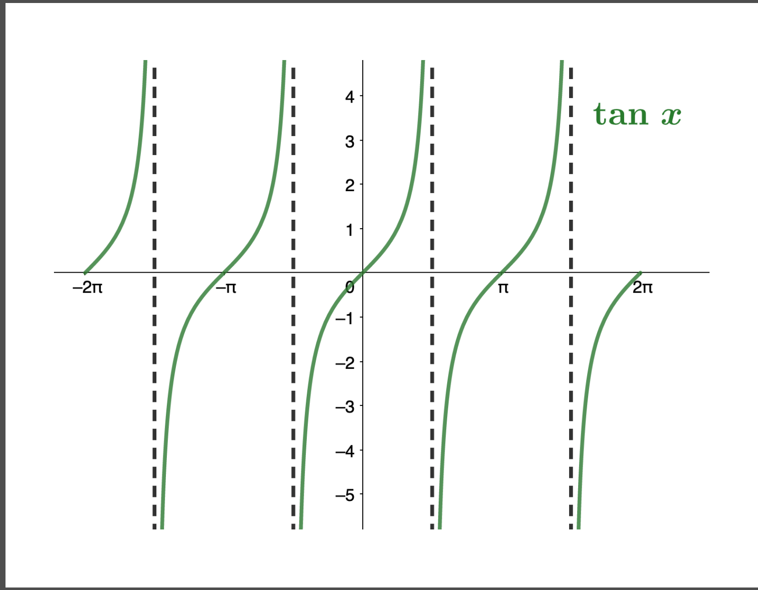


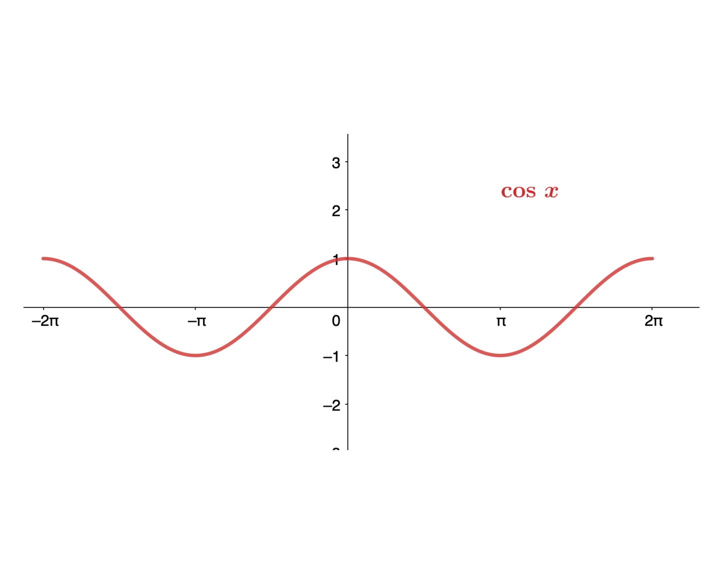
function	$\sin x$	$\cos x$	$\csc x$	$\sec x$	$\tan x$	$\cot x$
range	$[-1, 1]$	$[-1, 1]$	$(-\infty, -1] \cup [1, +\infty)$	$(-\infty, -1] \cup [1, +\infty)$	$\mathbb{R}$	$\mathbb{R}$
symmetry	Odd	Even	Odd	Even	Odd	Odd
period	2π	2π	2π	2π	π	π
equivalence	$\cos(x - \frac{\pi}{2})$	$\sin(x + \frac{\pi}{2})$	$\frac{1}{\sin x}$	$\frac{1}{\cos x}$	$\frac{\sin x}{\cos x}$	$\frac{\cos x}{\sin x}$

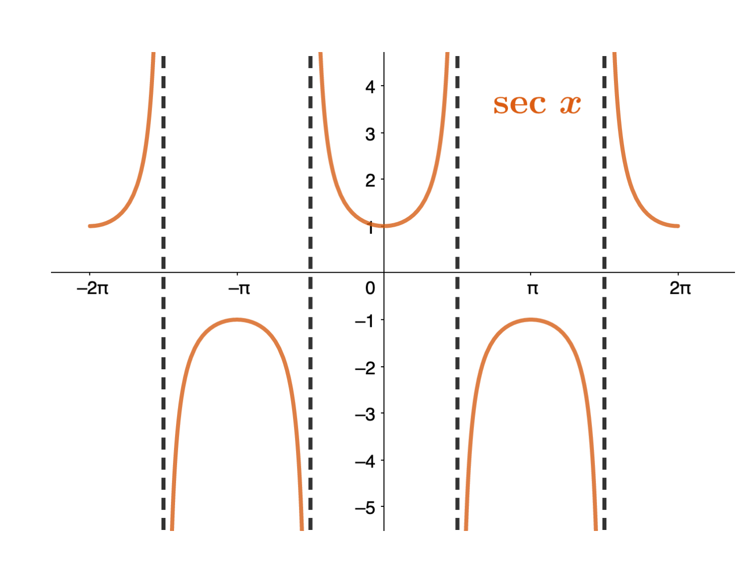
TRIGONOMETRIC FUNCTIONS

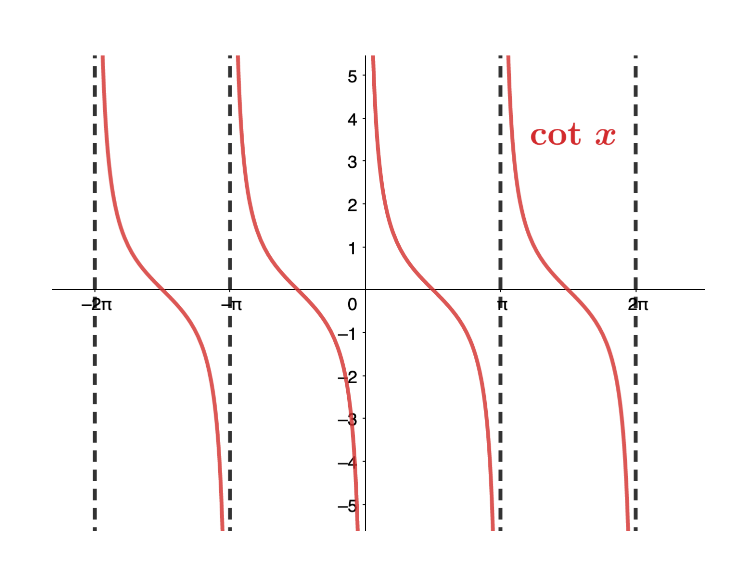

sin x
−2π
−π
0
π
2π
3
2
1
−1
−2
−3


csc x
−2π
−π
0
π
2π
4
3
2
1
−1
−2
−3
−4

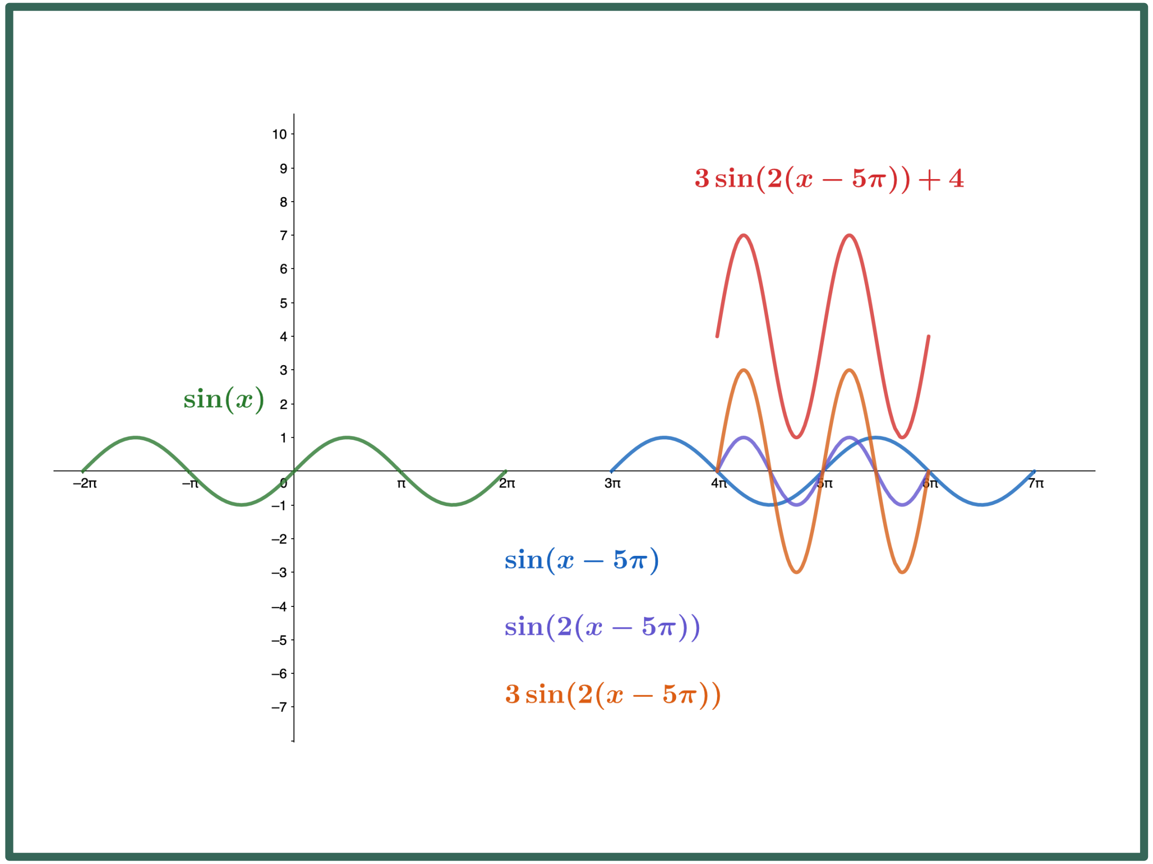

tan x
−2π
−π
0
π
2π
4
3
2
1
−1
−2
−3
−4
−5


cos x
−2π
−π
0
π
2π
3
2
1
−1
−2


sec x
−2π
−π
0
π
2π
4
3
2
1
−1
−2
−3
−4
−5

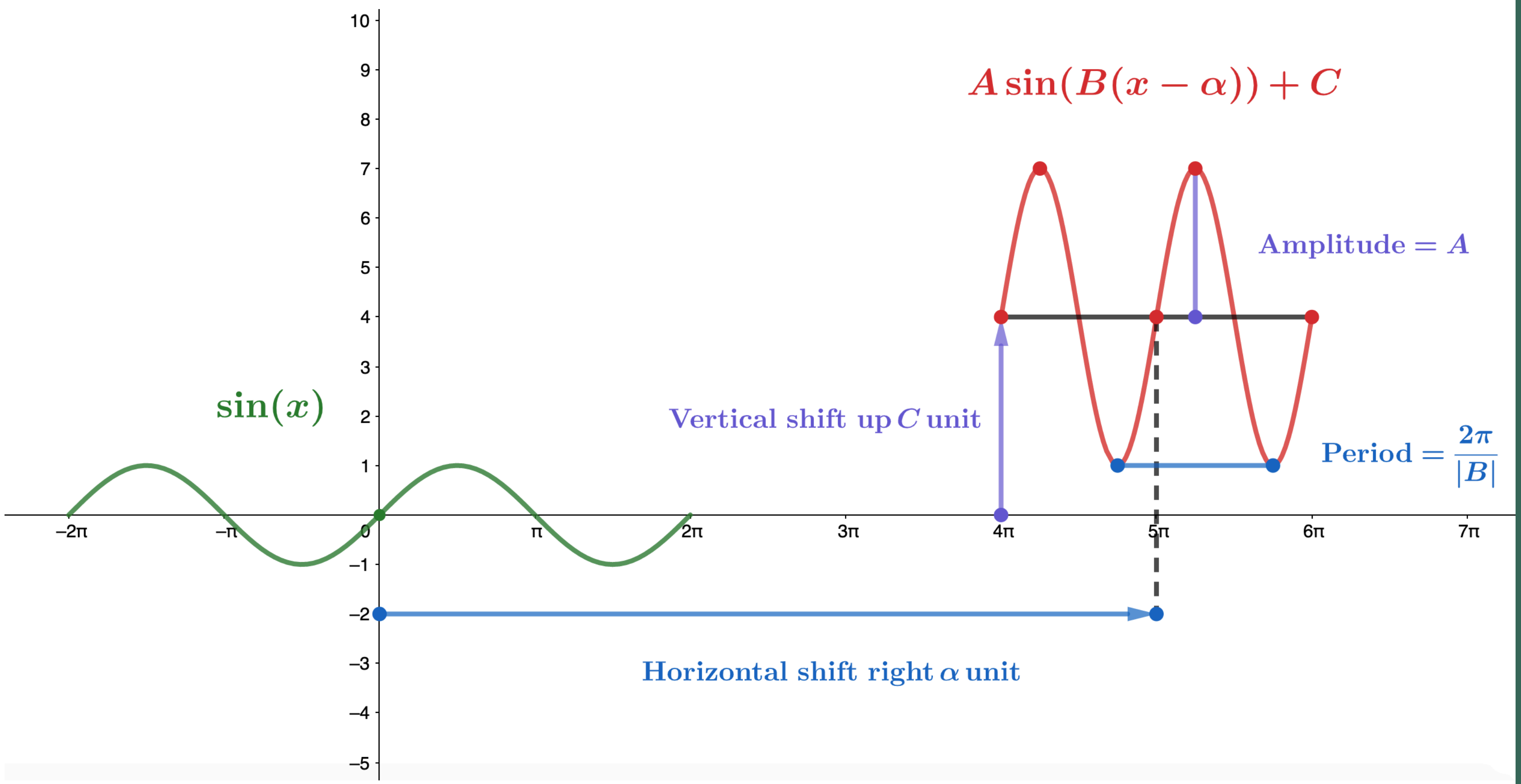

cot x
−2π
−π
0
π
2π
5
4
3
2
1
−1
−2
−3
−4
−5

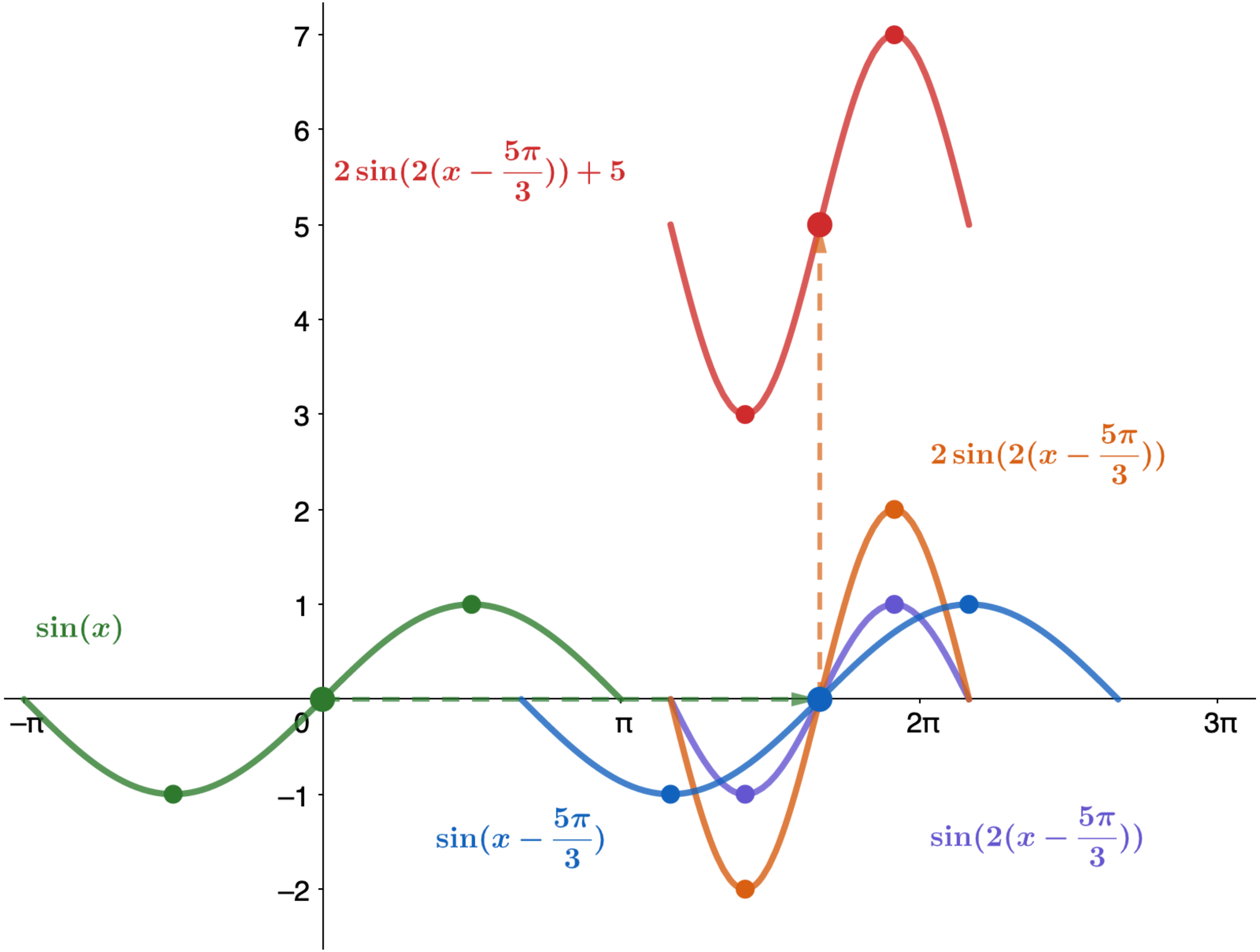
APPLY TRANSFORMATIONS TO TRIGONOMETRIC FUNCTIONS



- Recall what we have learnt about transformations!
- How to get $f(x) = A\sin\big(B(x-a)\big) + C$ from $\sin(x)$ in general?

A GENERAL TRIGONOMETRIC FUNCTION





$2\sin(2(x-\frac{5\pi}{3}))+5$
$2\sin(2(x-\frac{5\pi}{3}))$
$\sin(x)$
$\sin(x-\frac{5\pi}{3})$
$\sin(2(x-\frac{5\pi}{3}))$
7
6
5
4
3
2
1
0
−1
−2
−π
π
2π
3π

PROBLEM TWO

RULE: TRIGONOMETRIC IDENTITIES

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

PROBLEM 4