MATH 1: INTRODUCTION TO CALCULUS
MIDTERM EXAM #1 SOLUTIONS

Problem 1. [12 points]

(a) Simplify $2\ln(e^3)$.

$$2\ln(e^3) = 2(3) = \boxed{6}$$

(b) Let $f(x) = \cos(x)$ and $g(x) = e^{x^2}$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

$$(g \circ f)(x) = g(\cos(x)) = e^{\cos(x)^2} = \boxed{e^{\cos^2(x)}}$$

$$(f \circ g)(x) = f(e^{x^2}) = \boxed{\cos(e^{x^2})}$$

(c) Solve the equation $\ln(\sqrt{x-5}) = 1$ for x.

$$\begin{aligned} \ln(\sqrt{x-5}) &= 1 \\ \sqrt{x-5} &= e^1 \\ x-5 &= e^2 \\ \boxed{x = e^2+5} \end{aligned}$$

(d) Solve the equation $9^{(3^x)} = 3^{(27^x)}$ for x.

$$\begin{aligned} 9^{(3^x)} &= 3^{(27^x)} \\ 3^{(2*3^x)} &= 3^{27^x} \\ 2\cdot 3^x &= 27^x \\ 2\cdot 3^x &= 3^{3x} \\ \log_3 2 + x &= 3x \\ \log_3 2 &= 2x \\ \boxed{x = \frac{1}{2}\log_3 2} \end{aligned}$$

Date: Thursday, 10 October 2019.

Problem 2. [9 points]

(a) Find all solutions of $2\cos^2(\theta)\tan(\theta) = \sin(\theta)$ for θ in the interval $[0, 2\pi]$.

$$\begin{aligned}
2\cos^2(\theta)\tan(\theta) &= \sin(\theta) \\
2\cos^2(\theta)\frac{\sin(\theta)}{\cos(\theta)} &= \sin(\theta) \\
2\cos(\theta)\sin(\theta) &= \sin(\theta) \\
2\cos(\theta)\sin(\theta) - \sin(\theta) &= 0 \\
\sin(\theta)(2\cos(\theta) - 1) &= 0 \\
\sin(\theta) = 0 \text{ or } 2\cos(\theta) - 1 &= 0 \\
\sin(\theta) = 0 \text{ or } \cos(\theta) &= \frac{1}{2}
\end{aligned}$$

$$\boxed{\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}}$$

(b) Let $f(x) = (x-3)^2$ for $x \leq 3$. Find $f^{-1}(x)$.

$$\begin{aligned}
y &= (x-3)^2 \\
y &= (x-3)^2 \\
\sqrt{y} &= \pm(x-3) \\
3 \pm \sqrt{y} &= x
\end{aligned}$$

$$\boxed{f^{-1}(x) = 3 - \sqrt{x}}$$

(c) Solve the equation $\log_2(\sqrt{x}) + \log_2(\sqrt[3]{x}) = 2$ for x.

$$\begin{aligned}
\log_2(\sqrt{x}) + \log_2(\sqrt[3]{x}) &= 2 \\
\log_2(\sqrt{x} \cdot \sqrt[3]{x}) &= 2 \\
\sqrt{x} \cdot \sqrt[3]{x} &= 2^2 \\
x^{\frac{5}{6}} &= 4
\end{aligned}$$

$$\boxed{x = 4^{\frac{6}{5}}}$$

Problem 3. [8 points] Mark the following statements as true or false.

False $\ln(e^{-2}) = e^{\ln(-2)}$

False If k is positive, 10^{-k} is negative.

False $\log_5(xy) = \log_5(x)\log_5(y)$

True If a is a positive, constant, then $\ln(a^r) = r\ln(a)$.

False $\log_3 0 = 1$

False The domain of $f(x) = 2^x + 1$ is $(2, \infty)$.

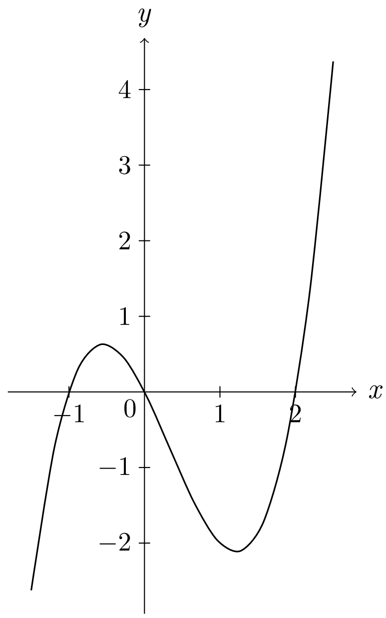
False The graph of $y = e^{\ln x}$ is a parabola.

True The slope of a secant line passing through the point $(a, f(a))$ and a nearby point on the graph of $f(x)$ approximates the instantaneous rate of change of $f(x)$ at a.

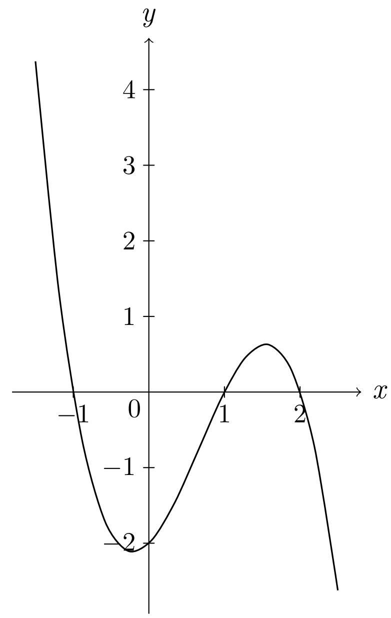
Problem 4. [7 points]

(1) Match the transformations of $f(x)$ with their graphs.

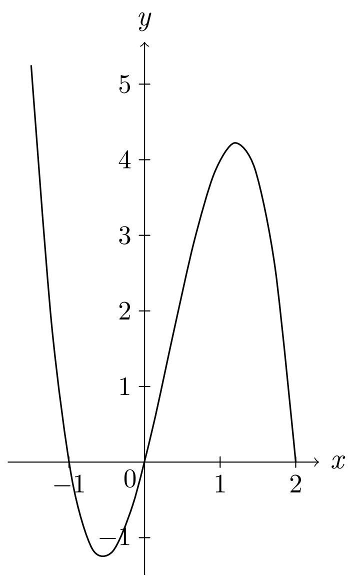
Options: $-2f(x)$, $f(x/2)$, $f(1-x)$



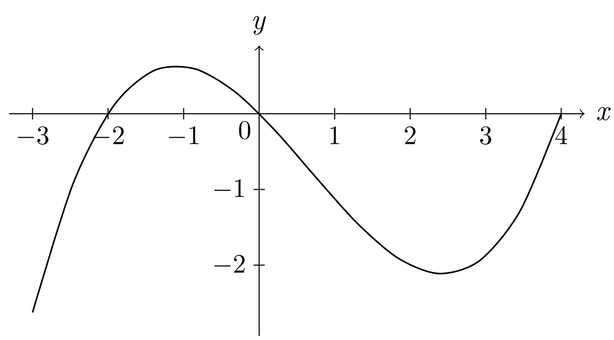
$f(x)$



$f(1-x)$



$-2f(x)$



$f(x/2)$

(2) Is $f(x)$ even, odd, or neither?

$\boxed{\text{neither}}$

Problem 5. [12 points]

(a) Plot the graph of $f(x) = e^x$ and its inverse function $f^{-1}(x) = \ln(x)$.

(b) Let $g(x) = e^{2x}$ and $h(x) = e^{x+1}$. What are $g^{-1}(x)$ and $h^{-1}(x)$?

$$g^{-1}(x) = \frac{1}{2}\ln x$$
$$h^{-1}(x) = \ln x - 1$$

(c) Describe which **basic** transformation we need to

- transform the graph of $f(x) = e^x$ into the graph of $g(x) = e^{2x}$.
  $\boxed{\text{horizontal compression by a factor of 2}}$

- transform the graph of $f^{-1}(x) = \ln(x)$ into the graph of $g^{-1}(x)$, as found above.
  $\boxed{\text{vertical compression by a factor of 2}}$

(d) Describe which **basic** transformation we need to

- transform the graph of $f(x) = e^x$ into the graph of $h(x) = e^{x+1}$.
  $\boxed{\text{shift left by 1}}$

- transform the graph of $f^{-1}(x) = \ln(x)$ into the graph of $h^{-1}(x)$ as found above.
  $\boxed{\text{shift down by 1}}$

(e) Fill in the blanks below with options from the bank of terms.

If we horizontally compress the original function by a factor of a, its corresponding inverse function will be vertically compressed by a factor of a ($a > 1$).
(There are other correct answers)

Term Bank: horizontally compress, horizontally stretch, vertically compress, vertically stretch

If we horizonally shift left the original function a ($a > 0$) units, its corresponding inverse function will be vertically shifted down a units ($a > 0$).
(There are other correct answers)

Term Bank: horizontally shift left, horizontally shift right, vertically shift up, vertically shift down

Problem 6. [8 points]

The number of hours of daylight in a northeastern city is modeled by the function

$$N(t) = 12 + 3\sin[\frac{2\pi}{365}(t - 79)]$$

where t is the number of days after January 1.

(a) Find the amplitude and period of $N(t)$.

$\boxed{\text{The amplitude is 3, and the period is 365.}}$

(b) How many hours of sunlight does the model predict on the longest day of the year?

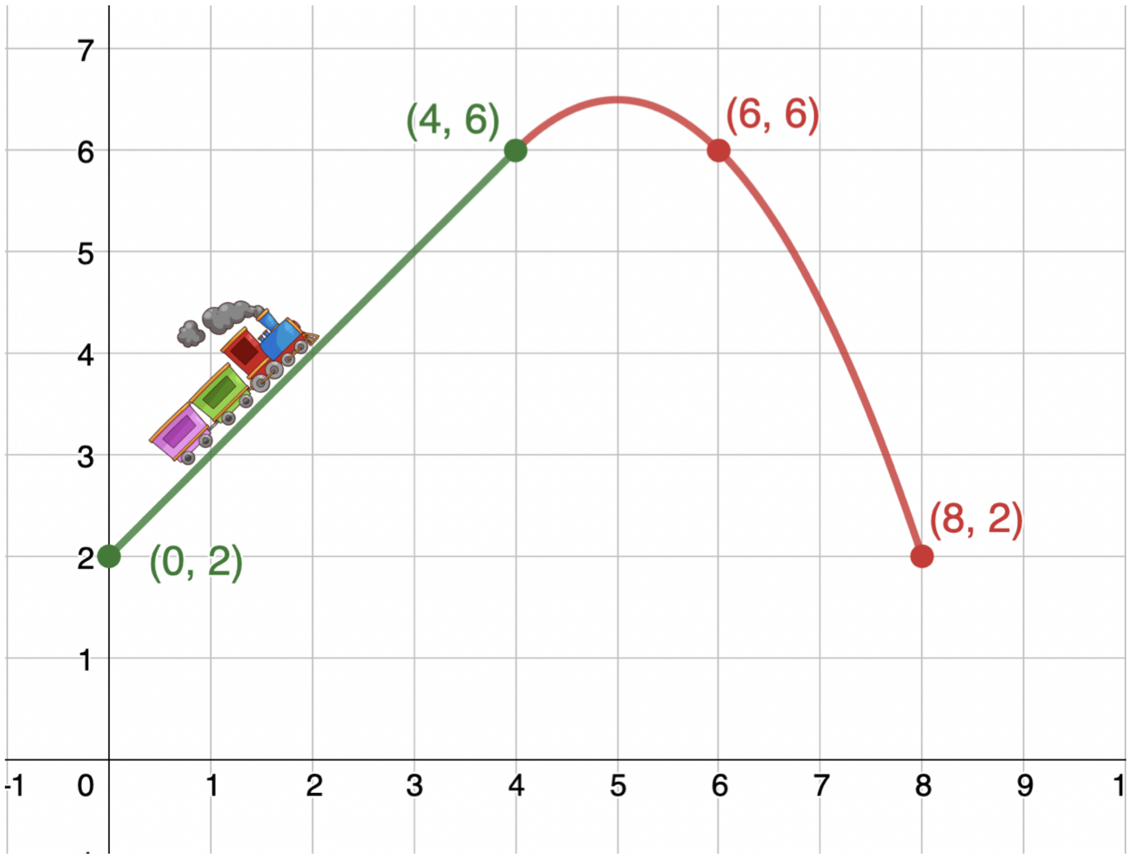
The sine function fluctuates between -1 and 1, so the maximum value of $\sin(x)$ is 1. Therefore, the maximum value of $N(t)$ is $12 + 3(1) = 15$. On the longest day of the year, the model predicts $\boxed{\text{15 hours of sunlight}}$.

(c) How many hours of sunlight does the model predict 90 days after January 1?

$$\begin{aligned} N(90) &= 12 + 3\sin[\frac{2\pi}{365}(90 - 79)] = 12 + 3\sin[\frac{2\pi}{365} \cdot 11)] \\ &= \boxed{12 + 3\sin[\frac{22\pi}{365}]} \end{aligned}$$

Problem 7. [9 points]

The figure below shows a track of the Green-Red Mountain. The first part of the mountain is linear, while the rest is part of the quadratic curve $-\frac{1}{2}x^2 + 5x - 6$.



(a) Suppose the figure depicts the entire mountain, modeled by a function $f(x)$. Using the given coordinates and the formula of the quadratic curve, write down a piecewise definition of $f(x)$.

$$f(x) = \begin{cases} x+2 & x \leq 4 \\ -\frac{1}{2}x^2 + 5x - 6 & x \geq 4 \end{cases}$$

(b) Engineers are working to decide if it is safe for a train to travel on the mountain or if they need to build a tunnel through the mountain. If the absolute value of the slope at the foot of the mountain, which is the point $(8, 2)$, is greater than 2, it is too dangerous to run a train, and they need to build a tunnel. Compute the slope of the secant line through $(7, \frac{9}{2})$ and $(8, 2)$, and the slope of the secant line through $(\frac{15}{2}, \frac{27}{8})$ and $(8, 2)$. Use these slopes to estimate the slope of the tangent line at $(8, 2)$ and help the engineers make the decision.

$$m_1 = \frac{2 - \frac{9}{2}}{8 - 7} = \frac{-\frac{5}{2}}{1} = -\frac{5}{2}$$

$$m_2 = \frac{2 - \frac{27}{8}}{8 - \frac{15}{2}} = \frac{-\frac{11}{8}}{\frac{1}{2}} = -\frac{22}{8} = -\frac{11}{4}$$

Based on these two slopes, we estimate that the slope of the tangent line at $(8, 2)$ is around -3, or at least steeper than -2, and thus not safe for the train.

Problem 8. [6 points]

Show algebraically whether the following functions are even, odd, or neither:

(a) $f(x) = e^{|x|}$

$$f(-x) = e^{|-x|} = e^{|x|} = f(x)$$

Therefore, f must be $\boxed{\text{even}}$.

(b) $g(x) = \sin x \cos x$

$$g(-x) = \sin(-x)\cos(-x) = -\sin(x)\cos(x) = -g(x)$$

Therefore, g must be $\boxed{\text{odd}}$.

(c) $h(x) = x \sin x$

$$h(-x) = (-x)\sin(-x) = (-x) \cdot (-\sin(x)) = x\sin(x) = h(x)$$

Therefore, h must be $\boxed{\text{even}}$.