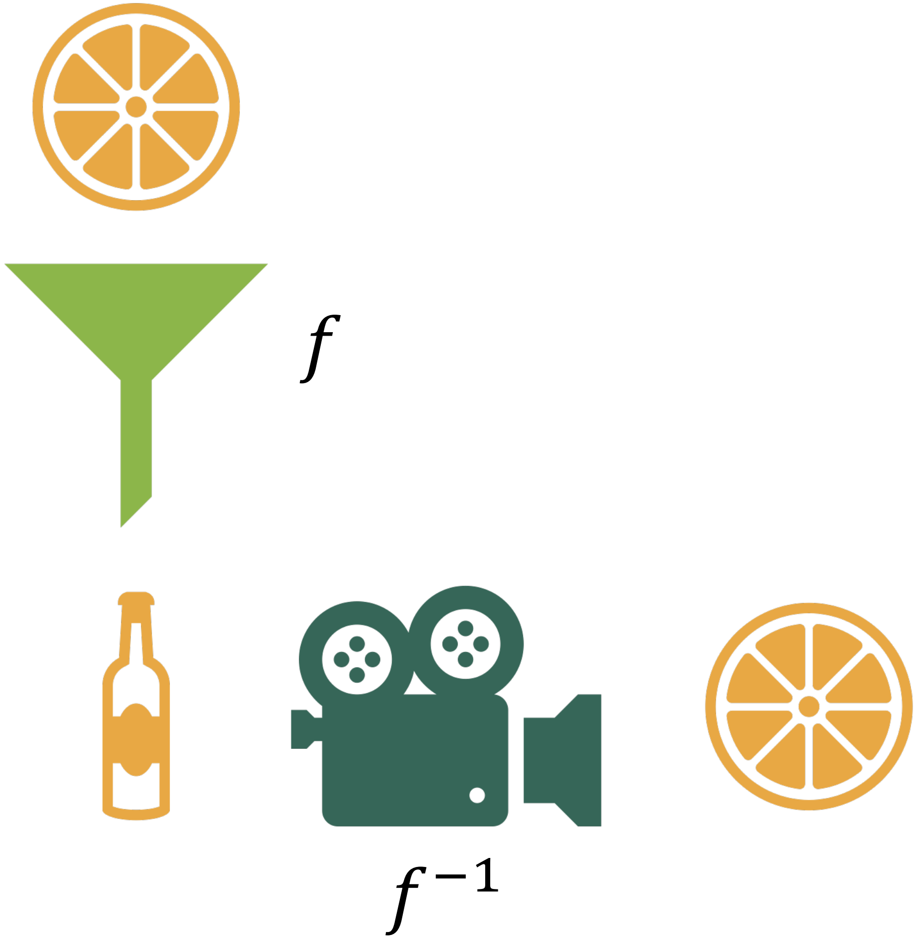
INTRODUCTION TO CALCULUS

INVERSE FUNCTIONS

INVERSE FUNCTIONS

- An inverse function reverses the operation done by a particular function.
- In other words, whatever a function does, the inverse function undoes it.



EXISTENCE OF AN INVERSE FUNCTION

$$f(x) = y = 2x - 5$$

$$x = f^{-1}(y) = \frac{y+5}{2}$$

$$f^{-1}\big(f(x)\big) = f^{-1}(2x-5) = x$$

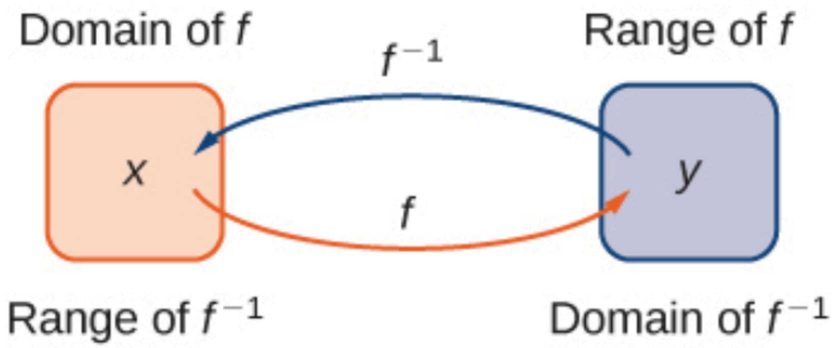
- This new function f^{-1} undid what the original function f did.
- A function with this property is called the inverse function of the original function.

f INVERSE (f^{-1})

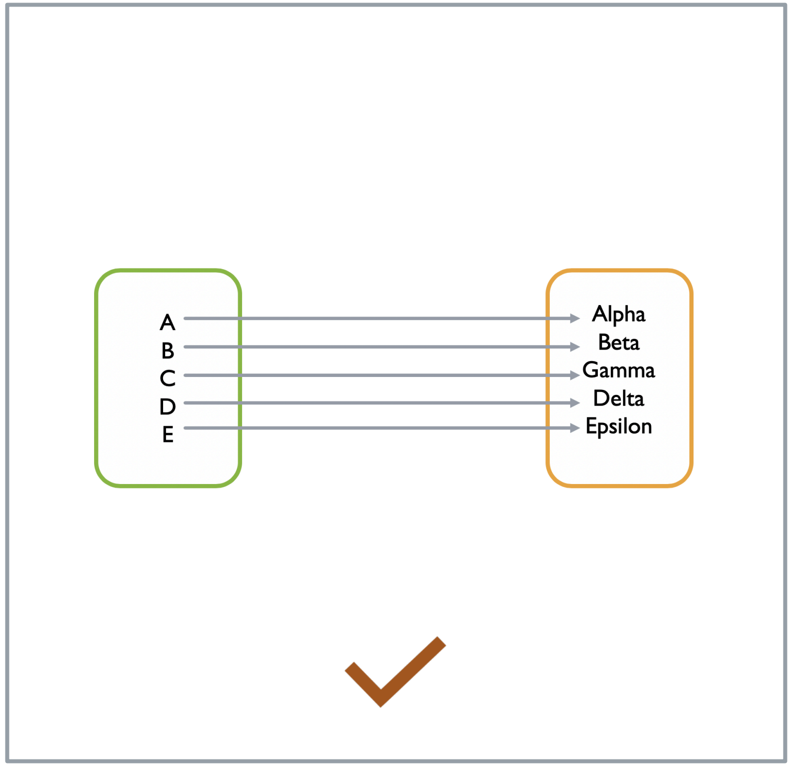
DEFINITION

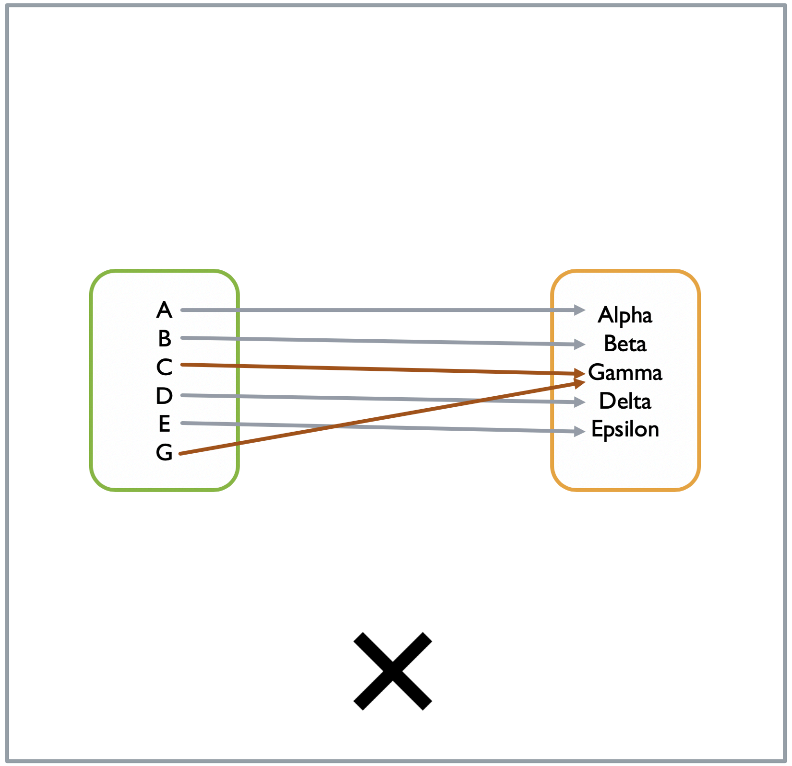
Given a function f with domain D and range R, its **inverse function** (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if $f(x) = y$. In other words, for a function f and its inverse f^{-1},

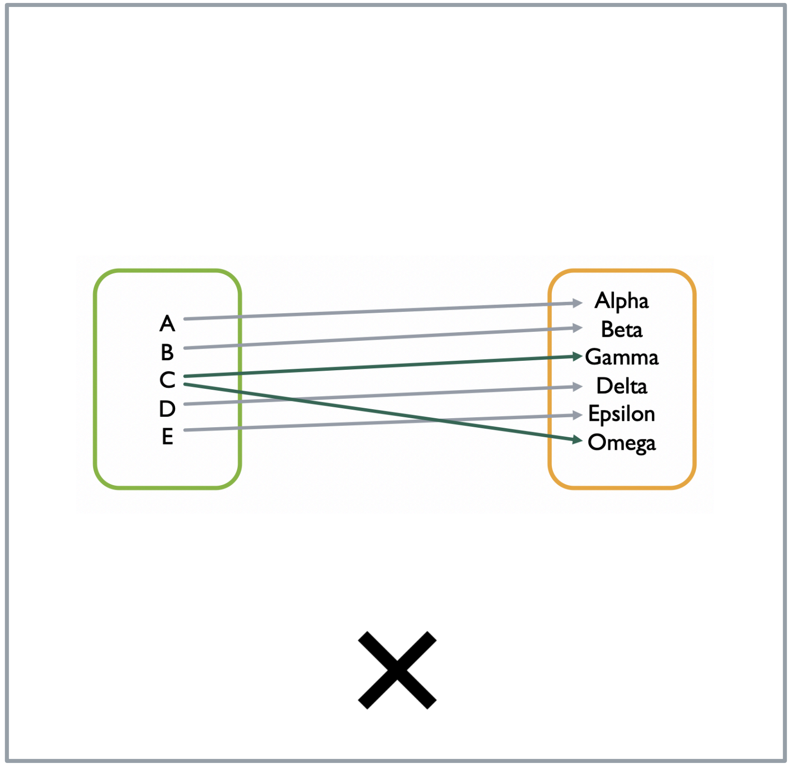
$$f^{-1}(f(x)) = x \text{ for all } x \text{ in } D, \text{ and } f\left(f^{-1}(y)\right) = y \text{ for all } y \text{ in } R. \tag{1.11}$$



EXISTENCE OF AN INVERSE FUNCTION

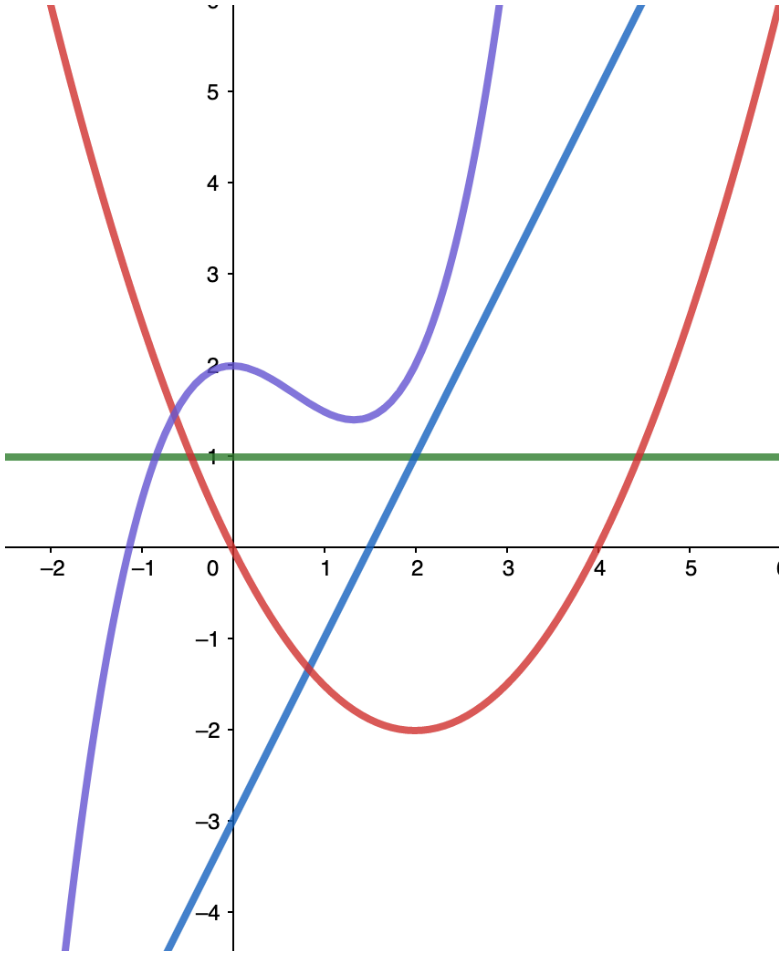






NOT EVERY FUNCTION HAS AN INVERSE FUNCTION

- Does $f(x) = x^2$ have an inverse function? Why or why not?
- Is there any function in the graph has an inverse function?



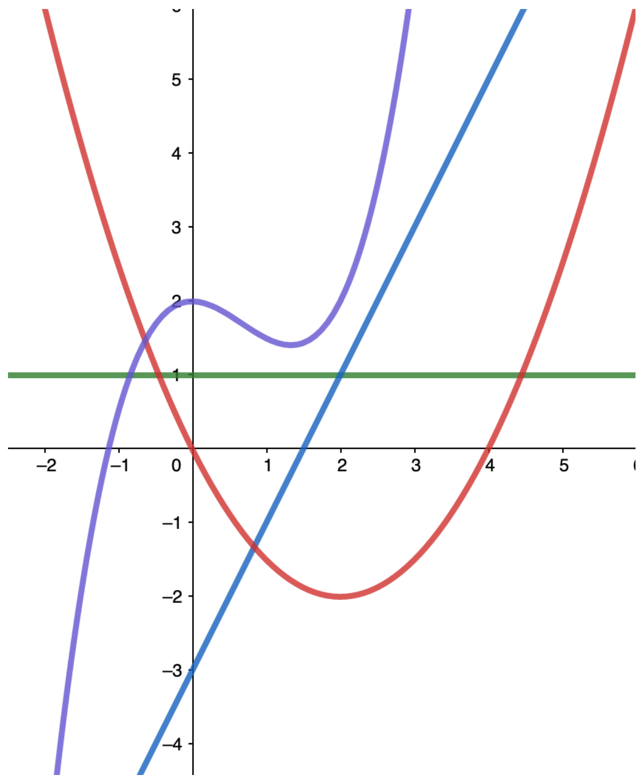
ONE-TO-ONE FUNCTION

A FUNCTION THAT SENDS EACH INPUT TO A *DIFFERENT* OUTPUT IS CALLED A ONE-TO-ONE FUNCTION.

DEFINITION

We say a f is a **one-to-one function** if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

HOW TO DETERMINE WHETHER A FUNCTION IS ONE-TO-ONE



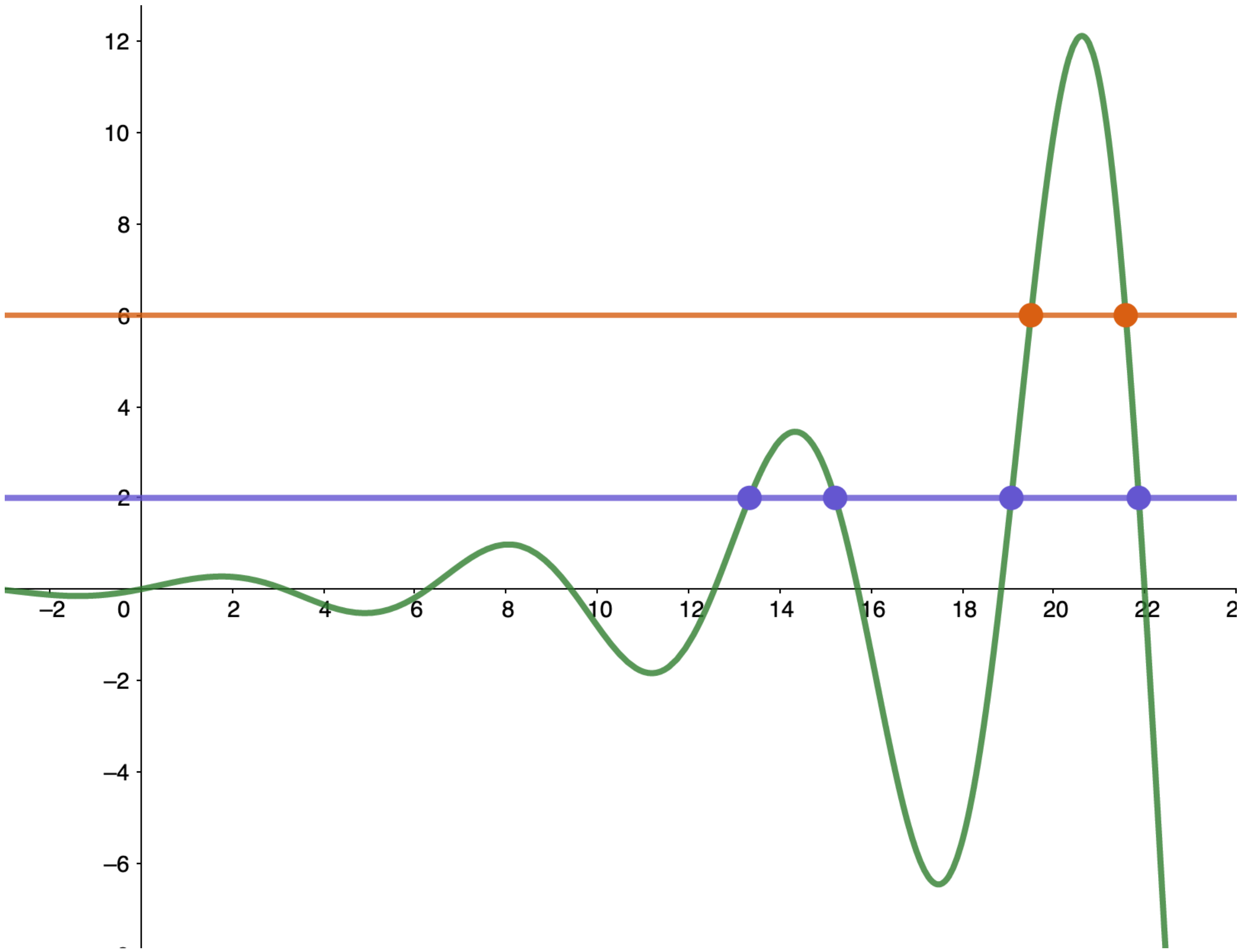
- Is there any one to one function in the graph?

HORIZONTAL LINE TEST

IF A FUNCTION IS ONE-TO-ONE, THEN NO TWO INPUTS CAN BE SENT TO THE SAME OUTPUT.

RULE: HORIZONTAL LINE TEST

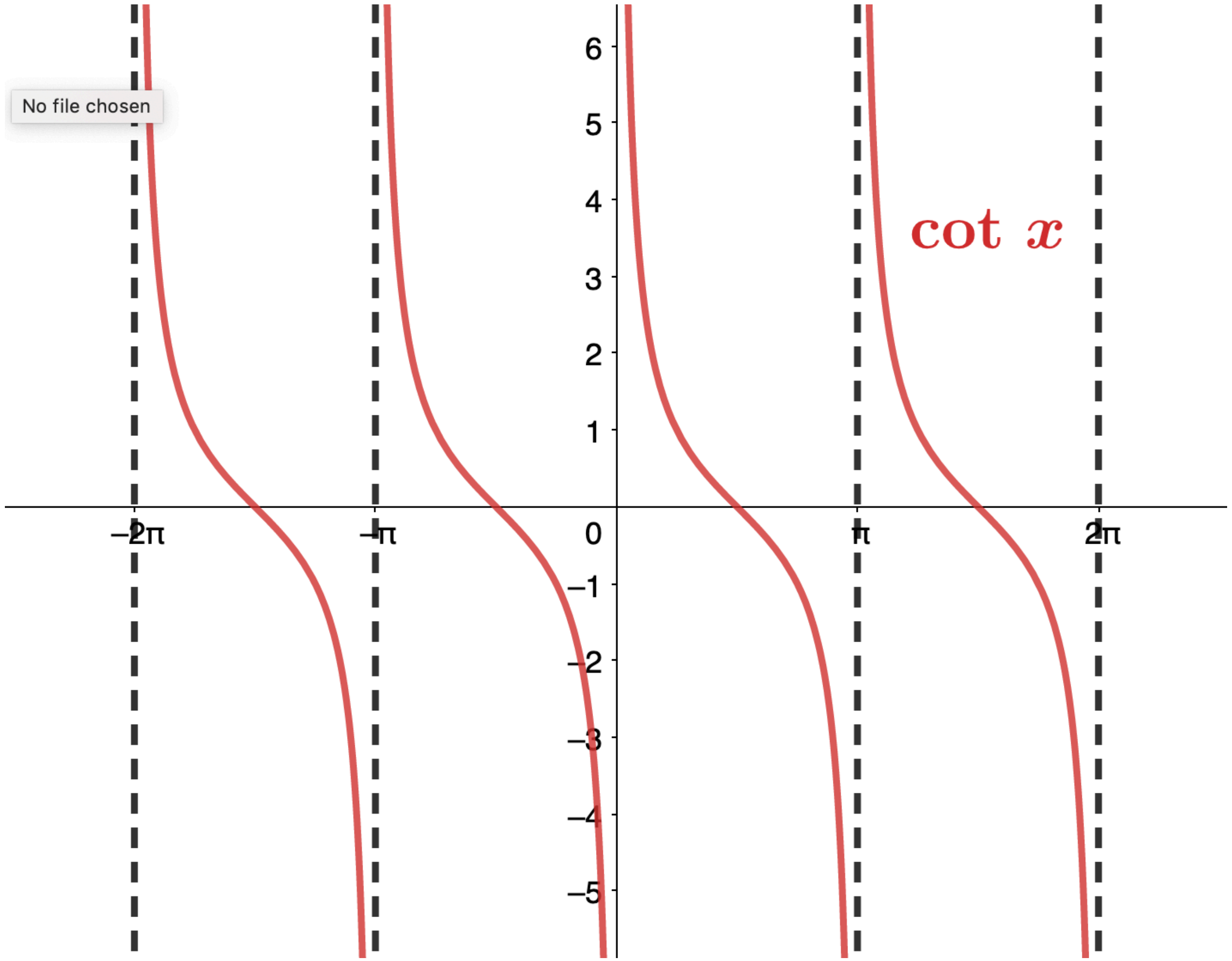
A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.



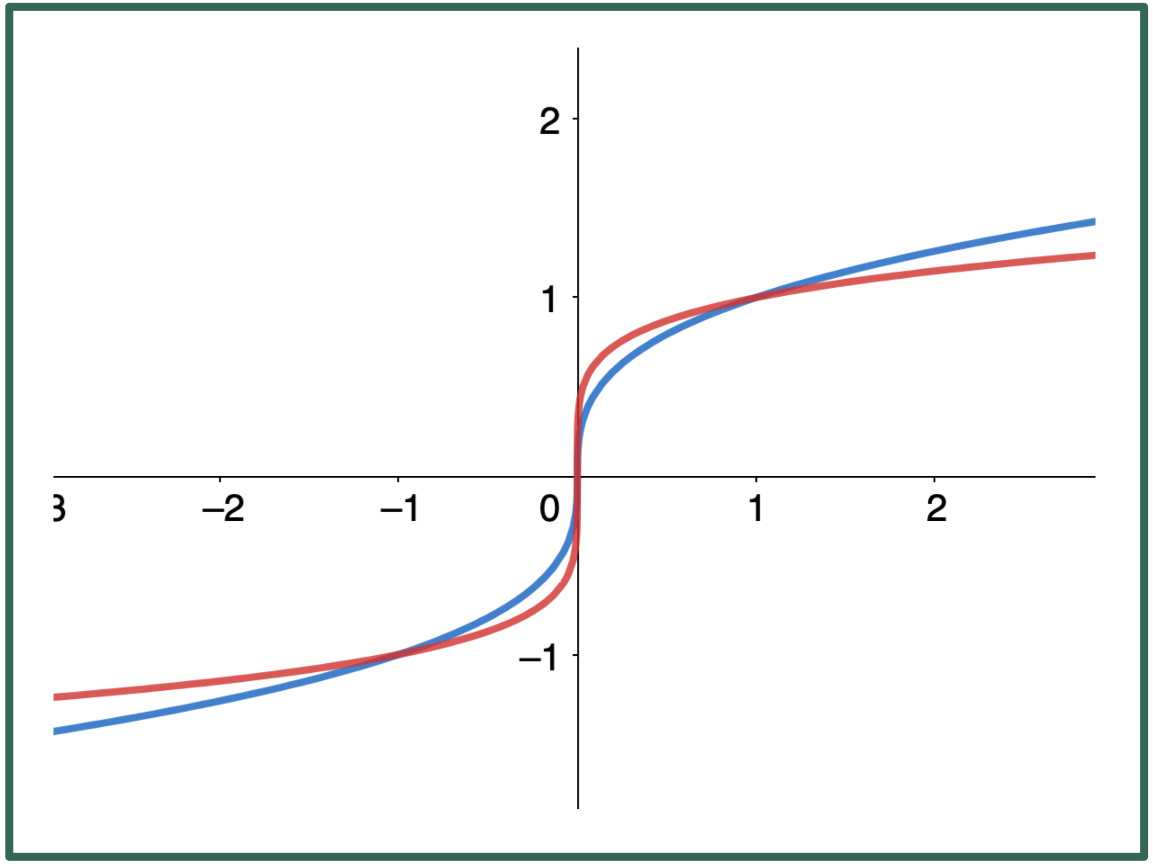
HORIZONTAL LINE TEST

EXERCISE ONE

- Is $\cot x$ a one-to-one function?

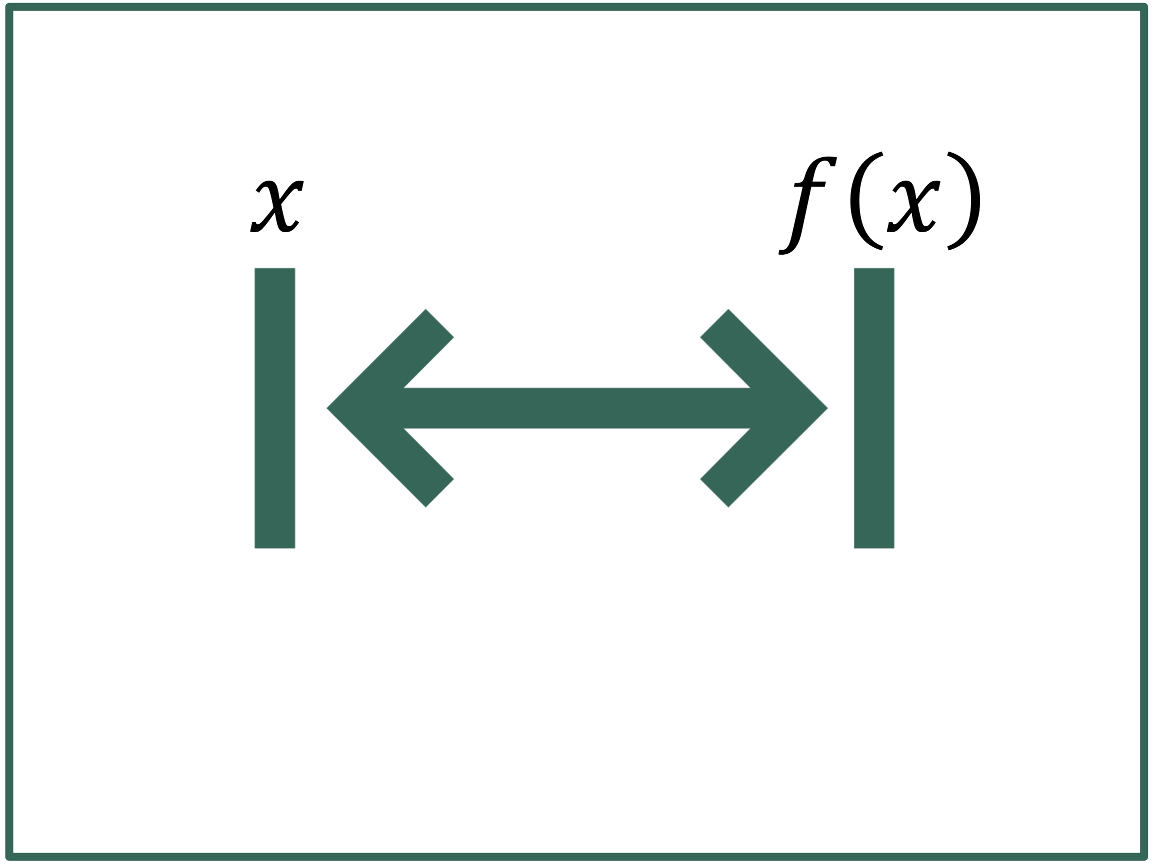


EXERCISE TWO



- Is $\sqrt[3]{x}$ a one-to-one function?
- Is $\sqrt[5]{x}$ a one-to-one function?

FIND A FUNCTION'S INVERSE



- Recall that a function maps elements in the domain of f to elements in the range of f.
- Hence the inverse function maps each element from the range of f back to its corresponding element from the domain of f.
- We can find that value x by solving the equation $\boldsymbol{f(x) = y}$ for x.

FIND A FUNCTION'S INVERSE

- Write x as a function of y.
  - the inverse of f
  - $x = f^{-1}(y)$
- The domain of this function is the range of f and the range of this new function is the domain of f.

$$y = 2x - 5$$

- original

$$x = \frac{1}{2}(y + 5)$$

- inverse (pre)

FIND A FUNCTION'S INVERSE

- Since we **typically** use the variable x to denote **the independent variable** and y to denote **the dependent variable**, we often interchange the roles of x and y.
  - $y = f^{-1}(x)$

$$x = \frac{1}{2}(y+5)$$

- inverse (pre)

$$y = \frac{1}{2}(x+5)$$

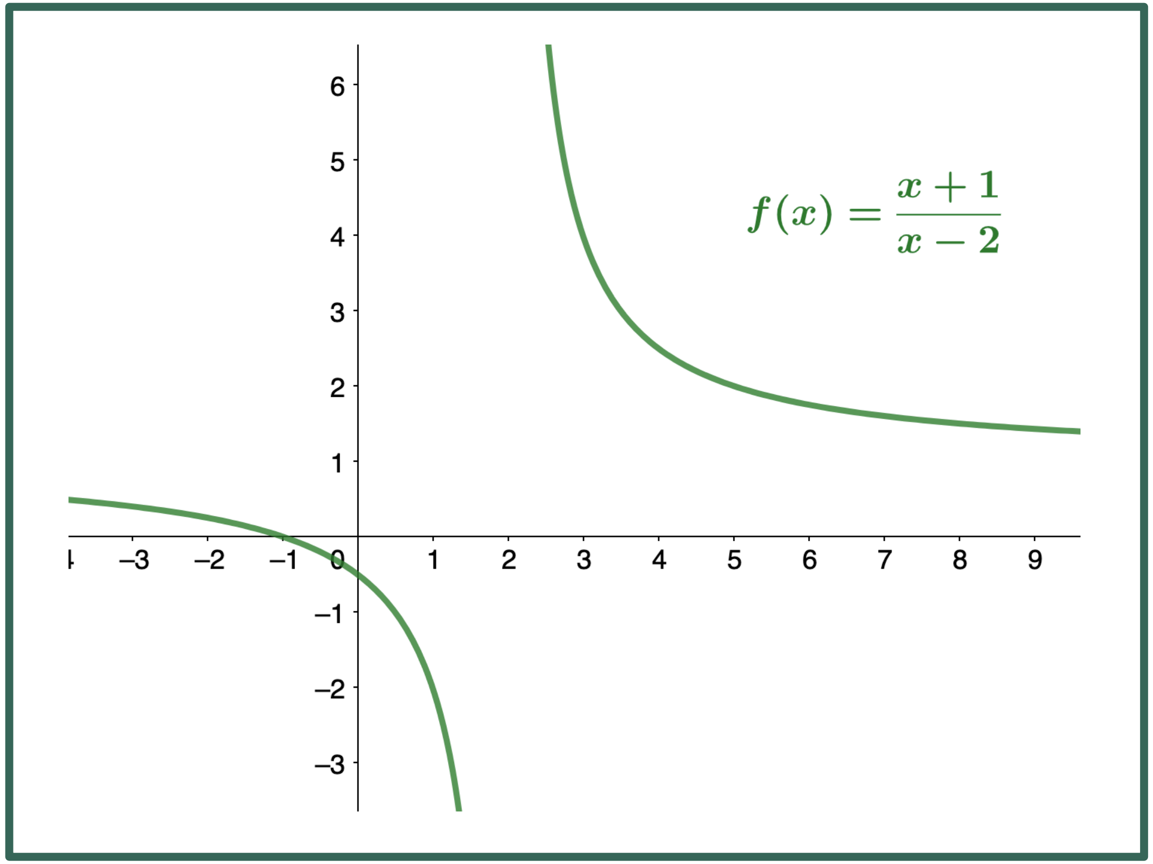
- inverse

FIND A FUNCTION'S INVERSE

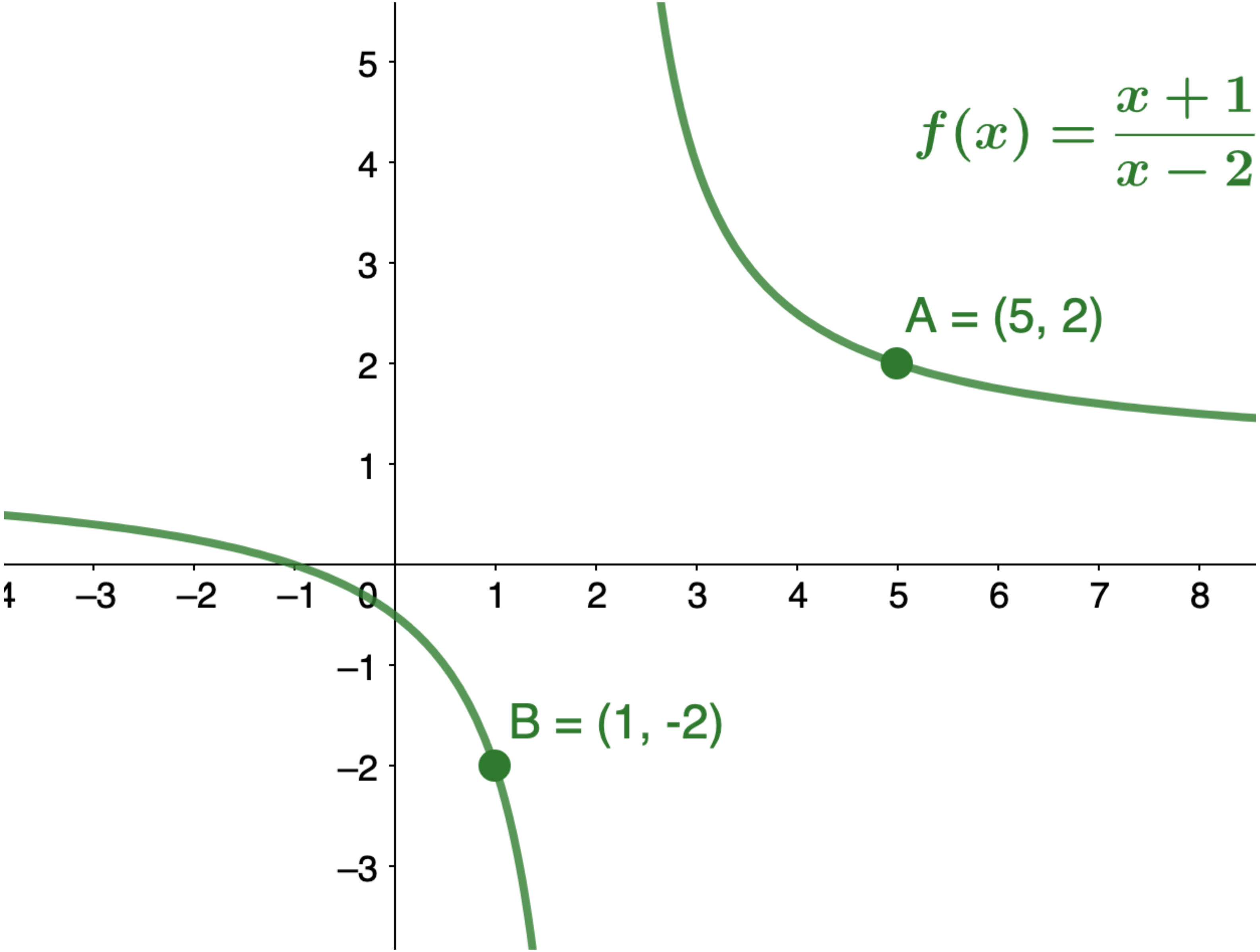
PROBLEM-SOLVING STRATEGY: FINDING AN INVERSE FUNCTION

1. Solve the equation $y = f(x)$ for x.
2. Interchange the variables x and y and write $y = f^{-1}(x)$.

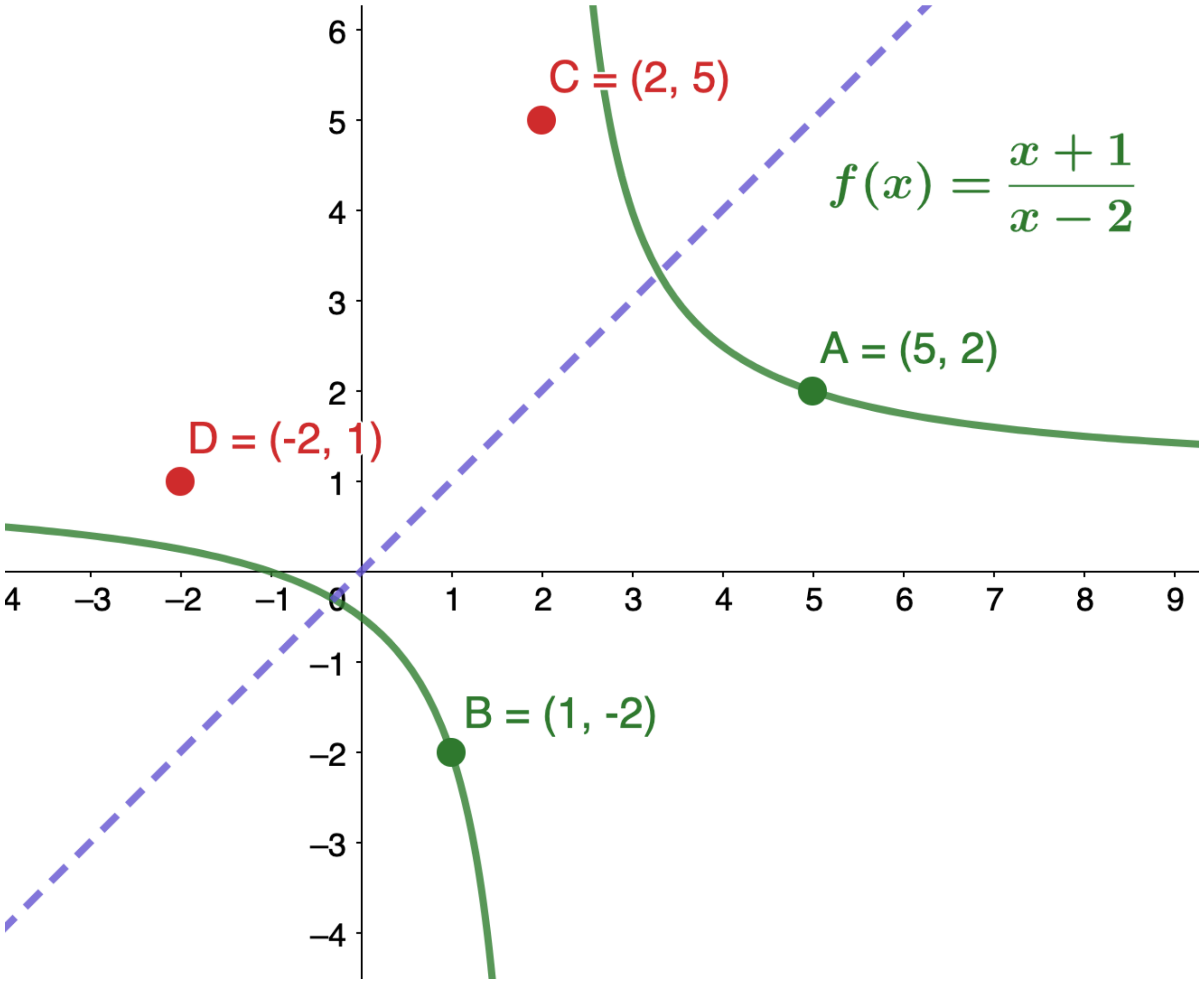
EXERCISE



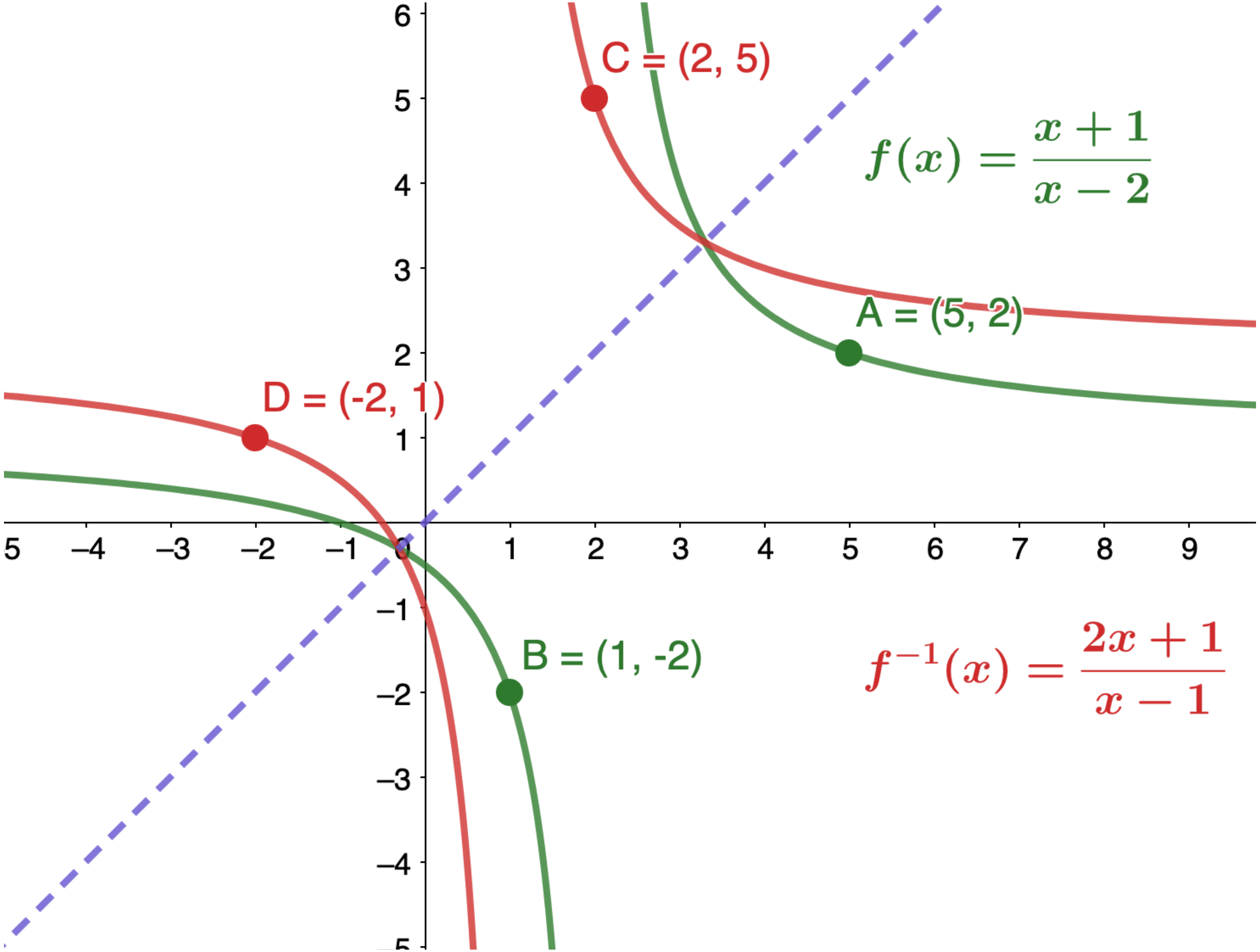
- Find the inverse of the function $f(x) = \frac{x+1}{x-2}$.
- State the domain and range of the inverse function.



GRAPH INVERSE FUNCTIONS



GRAPH INVERSE FUNCTIONS



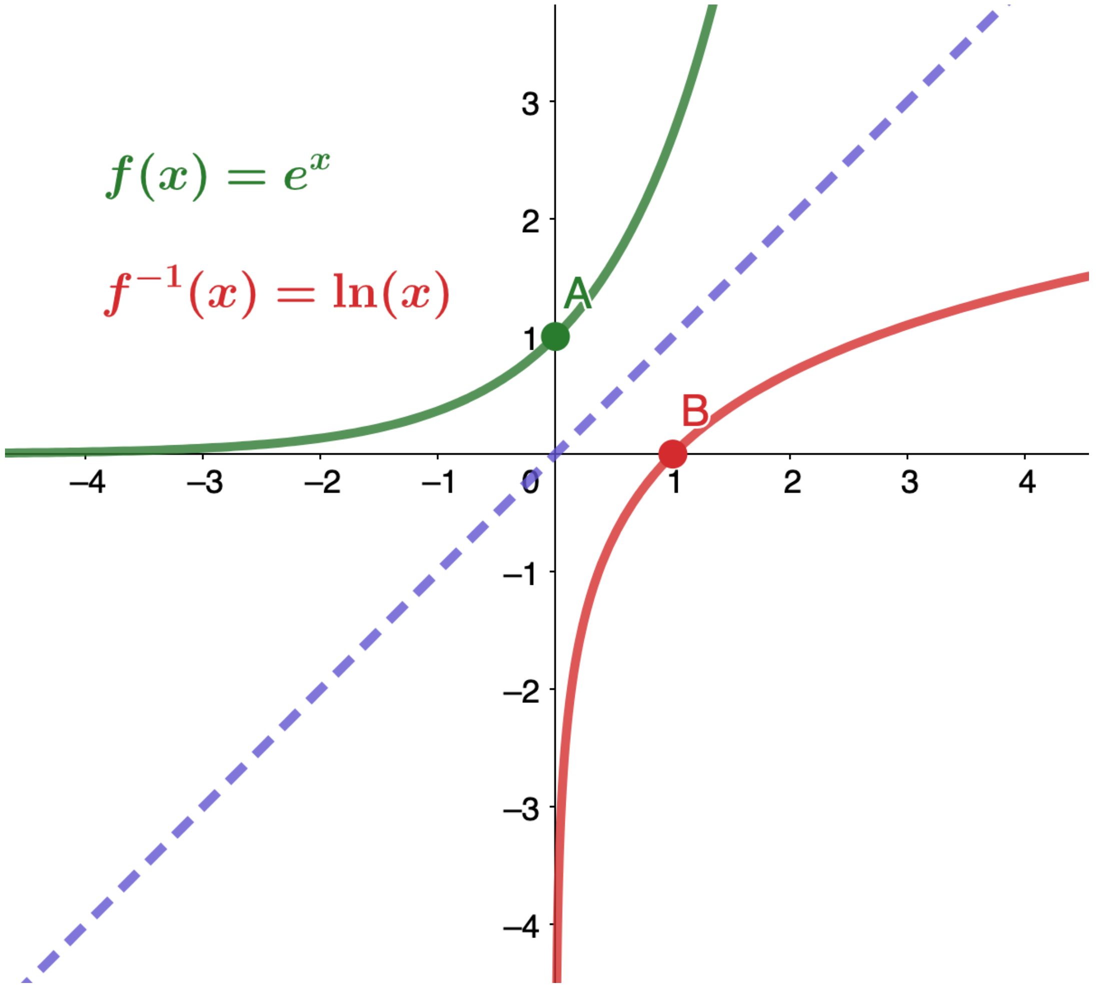
GRAPH INVERSE FUNCTIONS

GRAPH INVERSE FUNCTIONS

- A point (a, b) on the graph of f.
  - $b = f(a)$.
  - $a = f^{-1}(b)$.
- A point (b, a) on the graph of f^{-1}.
- As a result, the graph of f^{-1} is **a reflection of the graph** of f about **the line $y = x$**.

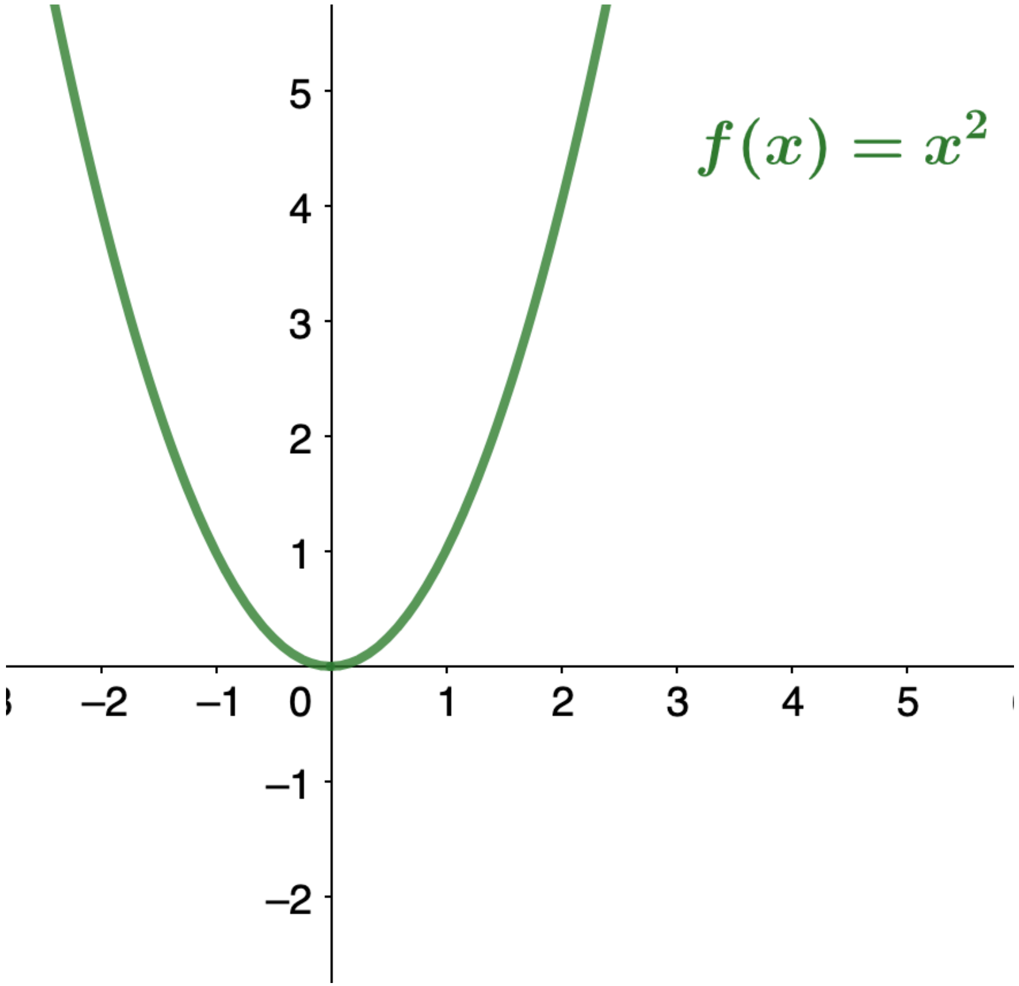
EXAMPLE

- Exponential and logarithmic functions

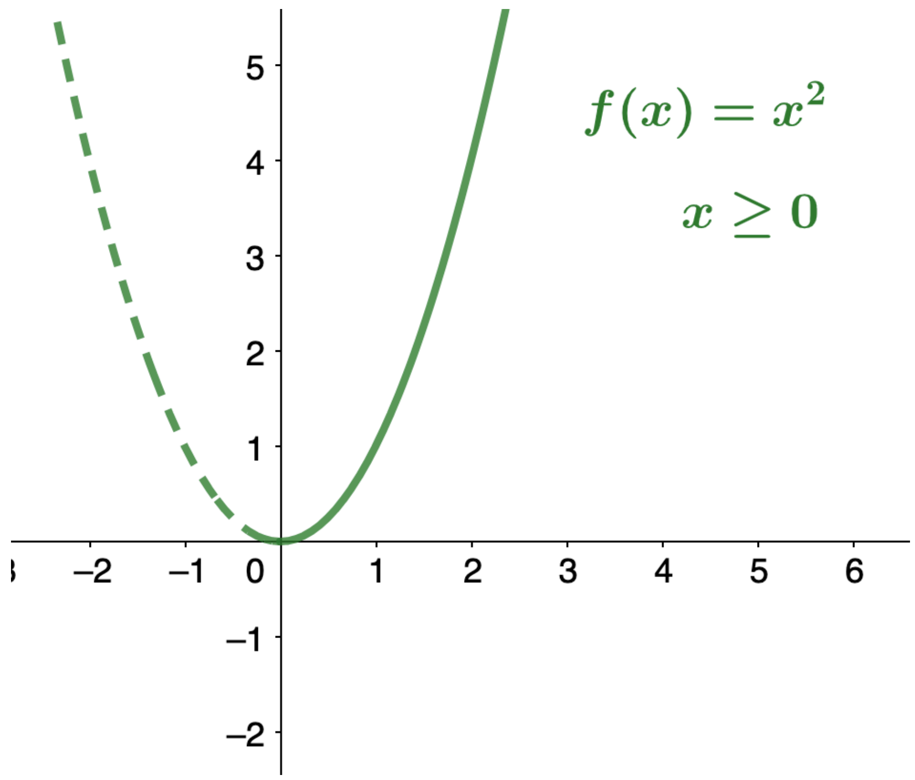


RESTRICT DOMAIN

- When a function does not have an inverse function because it is not one-to-one, what can we do?



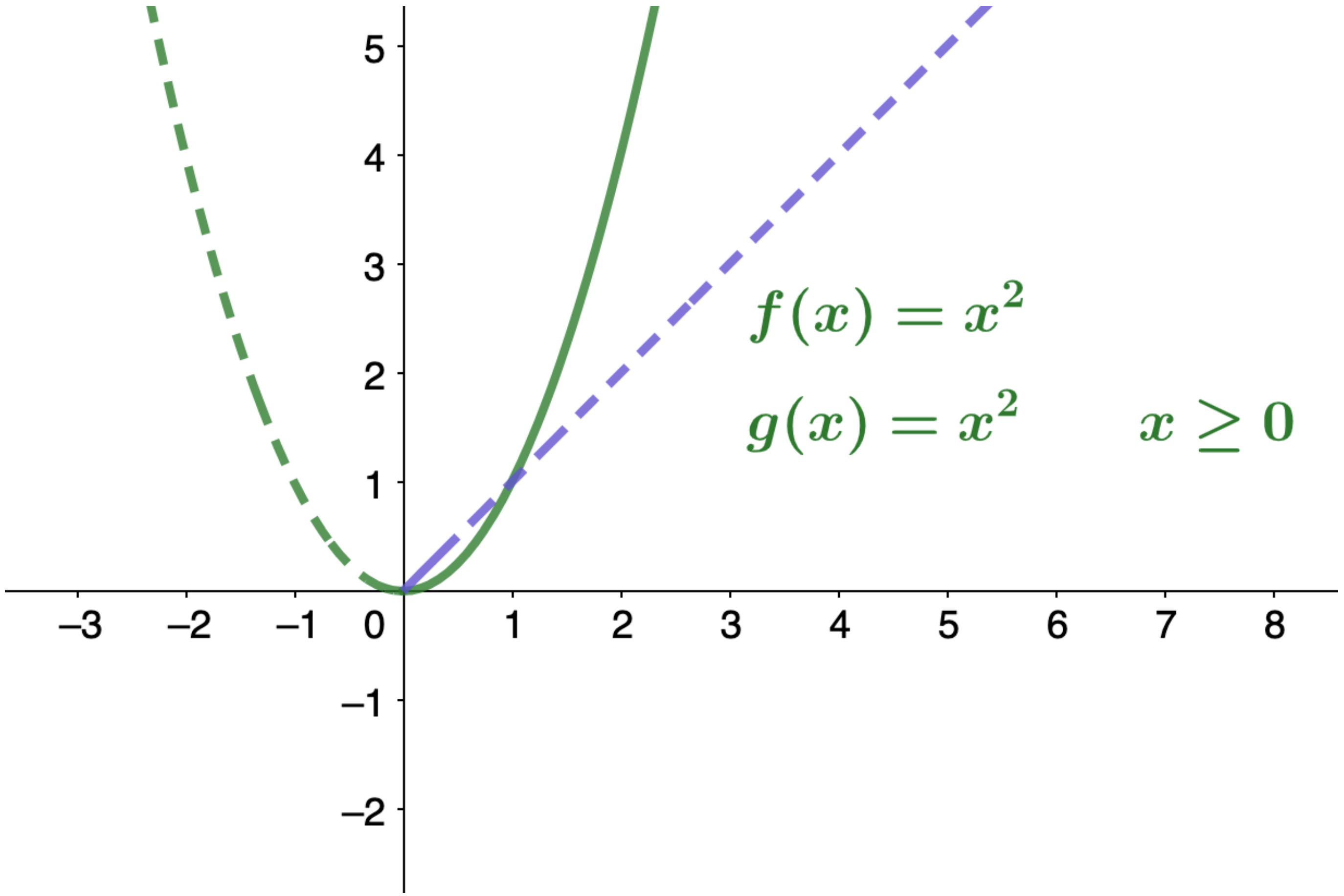
RESTRICT DOMAIN



- We can choose a subset of the domain of f such that the function is one-to-one.
- This subset is called a **restricted domain**.

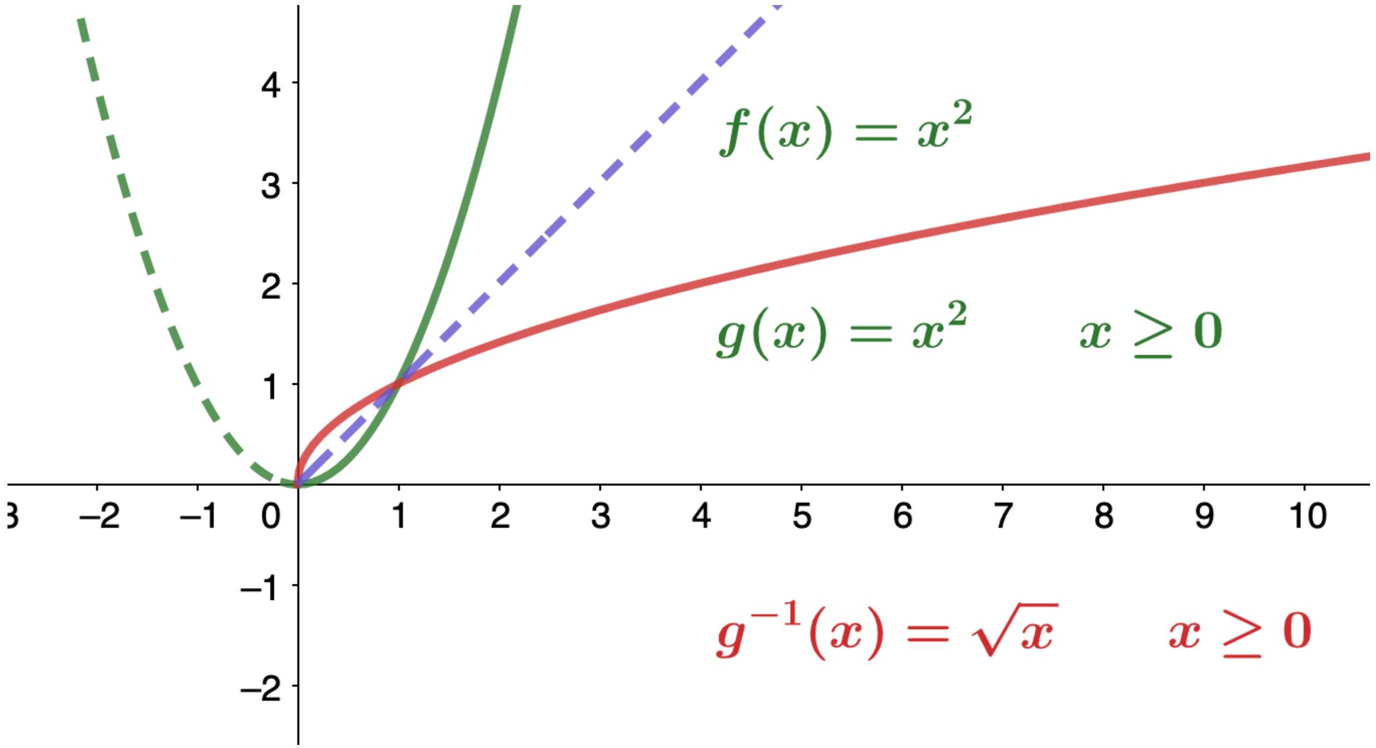
RESTRICT DOMAIN

- By restricting the domain of f, we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g.
- Then we can define an inverse function for g on that domain.



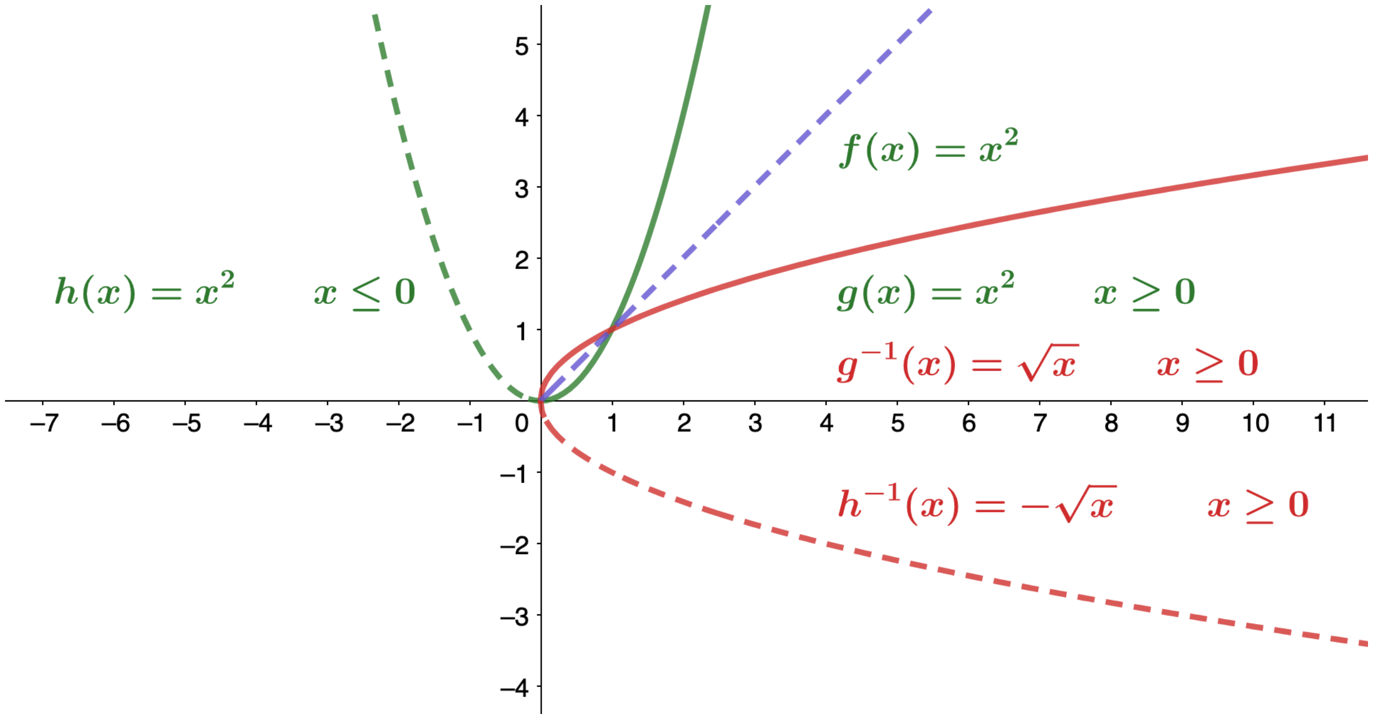
RESTRICT DOMAIN

- For the inverse function
  - What is the domain?
  - What is the range?

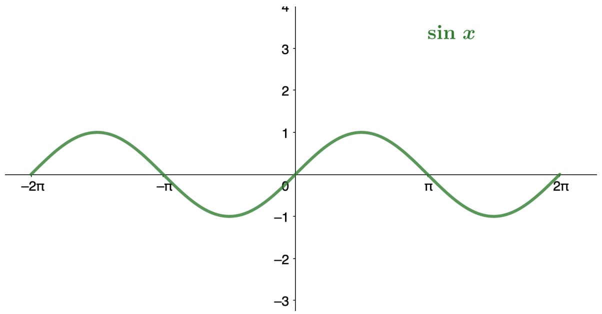


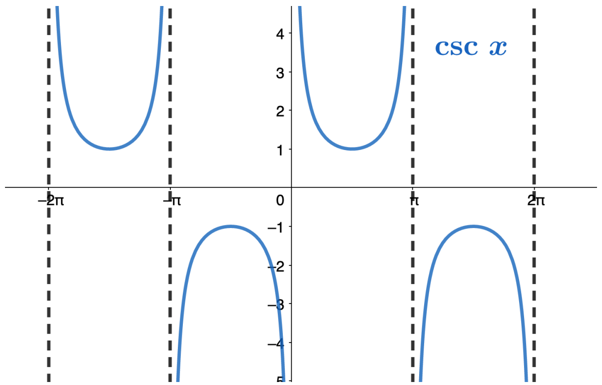
RESTRICT DOMAIN

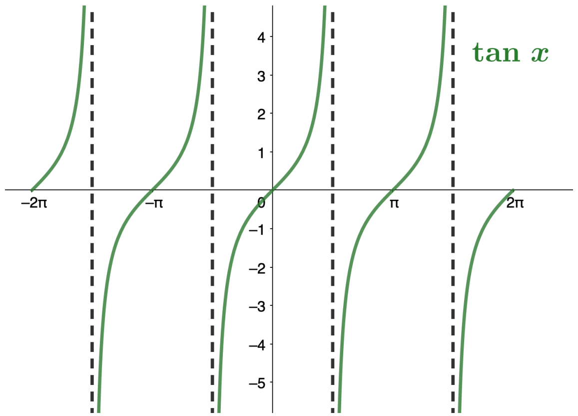
- For the inverse function
  - What is the domain?
  - What is the range?

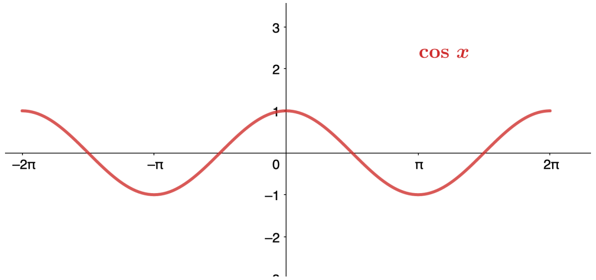


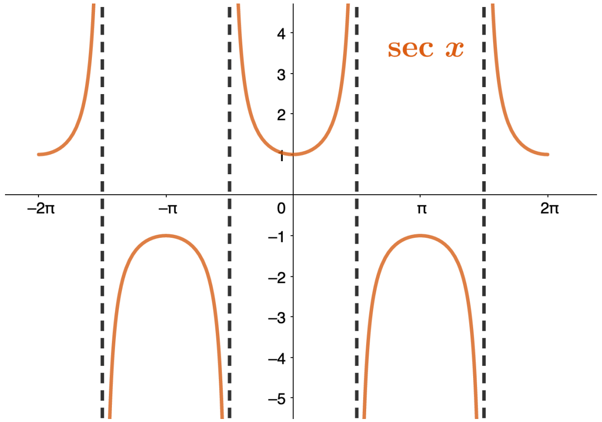
LOOK BACK ON TRIGONOMETRIC FUNCTIONS

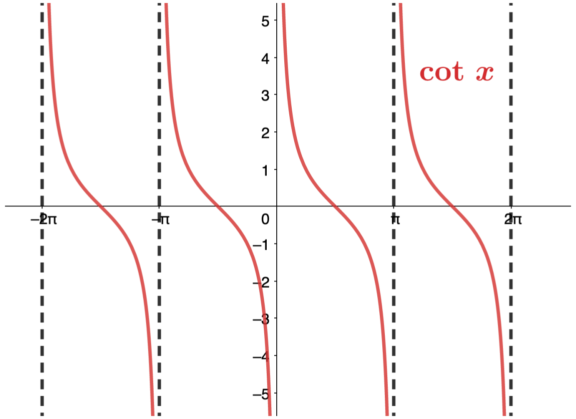




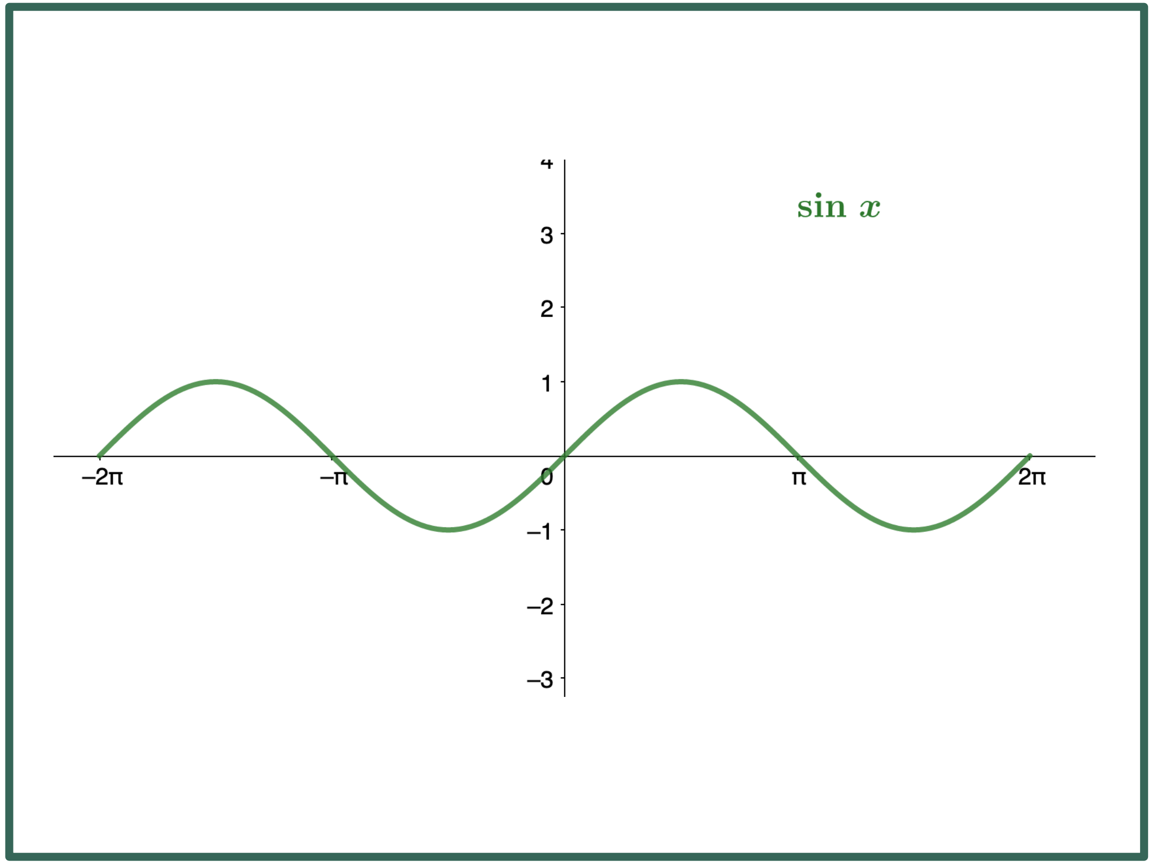






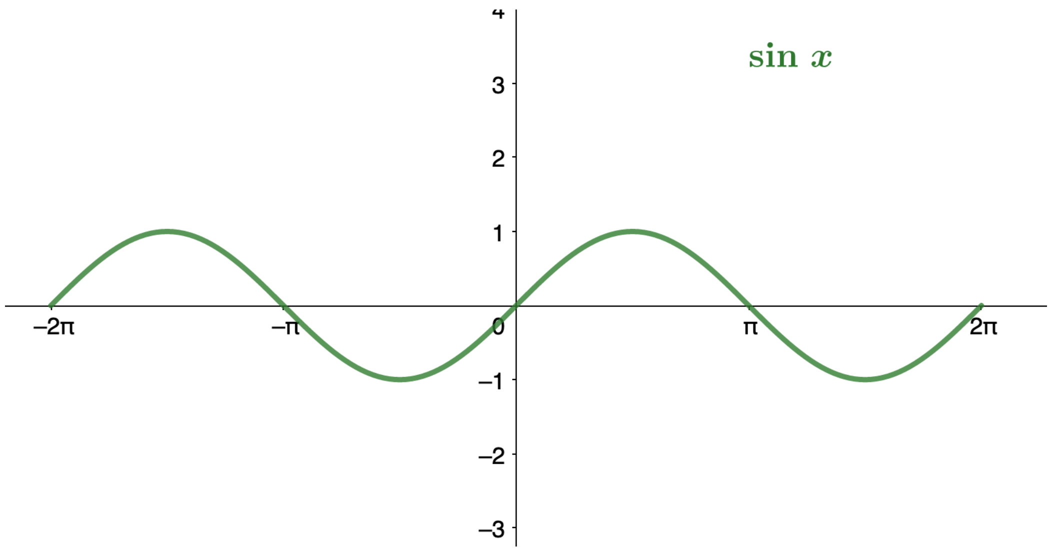


INVERSE TRIGONOMETRIC FUNCTIONS



- The six basic trigonometric functions are periodic.
- Therefore they are not one-to-one. (Why periodic functions are not one-to-one?)
- However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse.

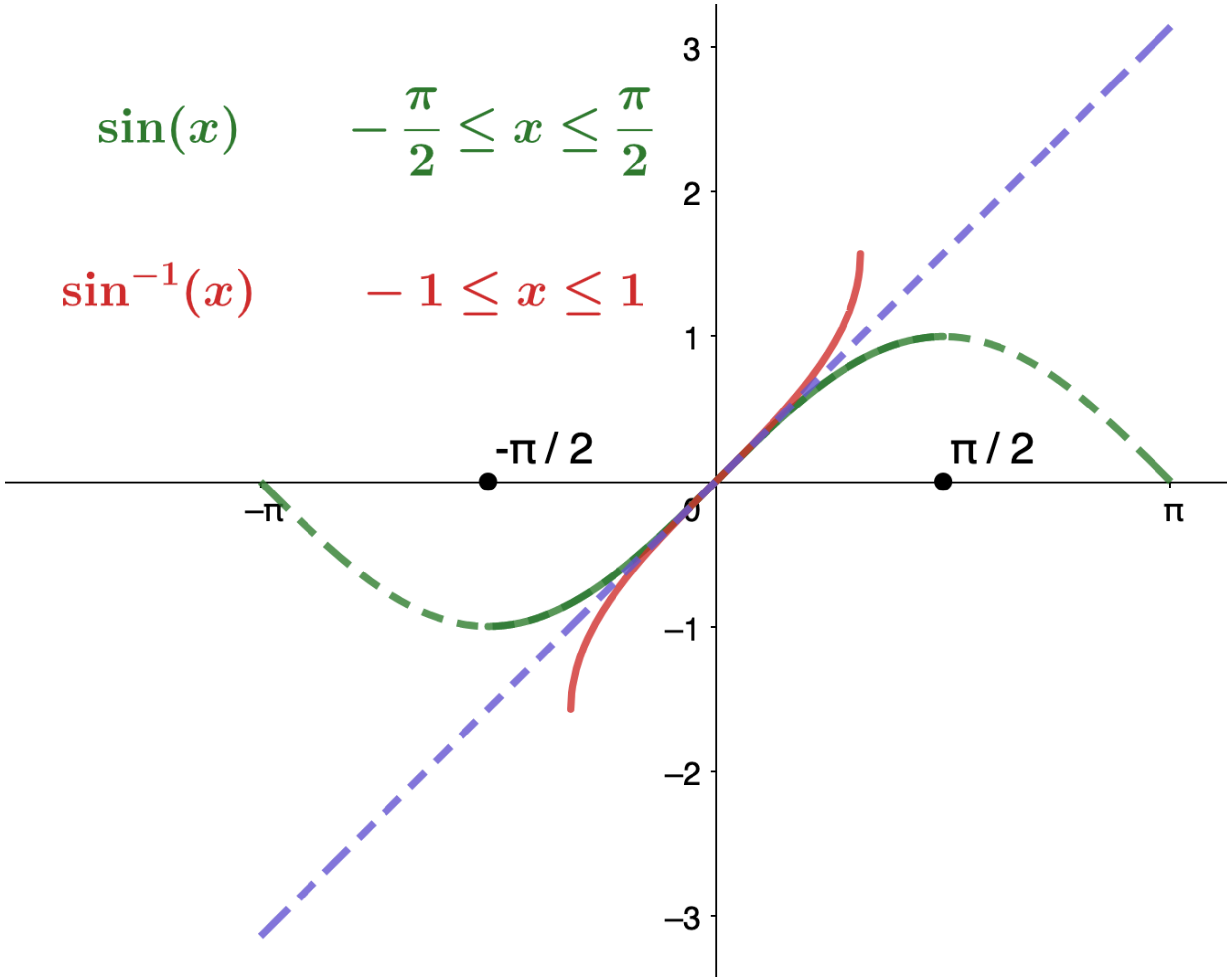
INVERSE SINE FUNCTION



- The sine function is one-to-one on an infinite number of intervals.
- The standard convention is to restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

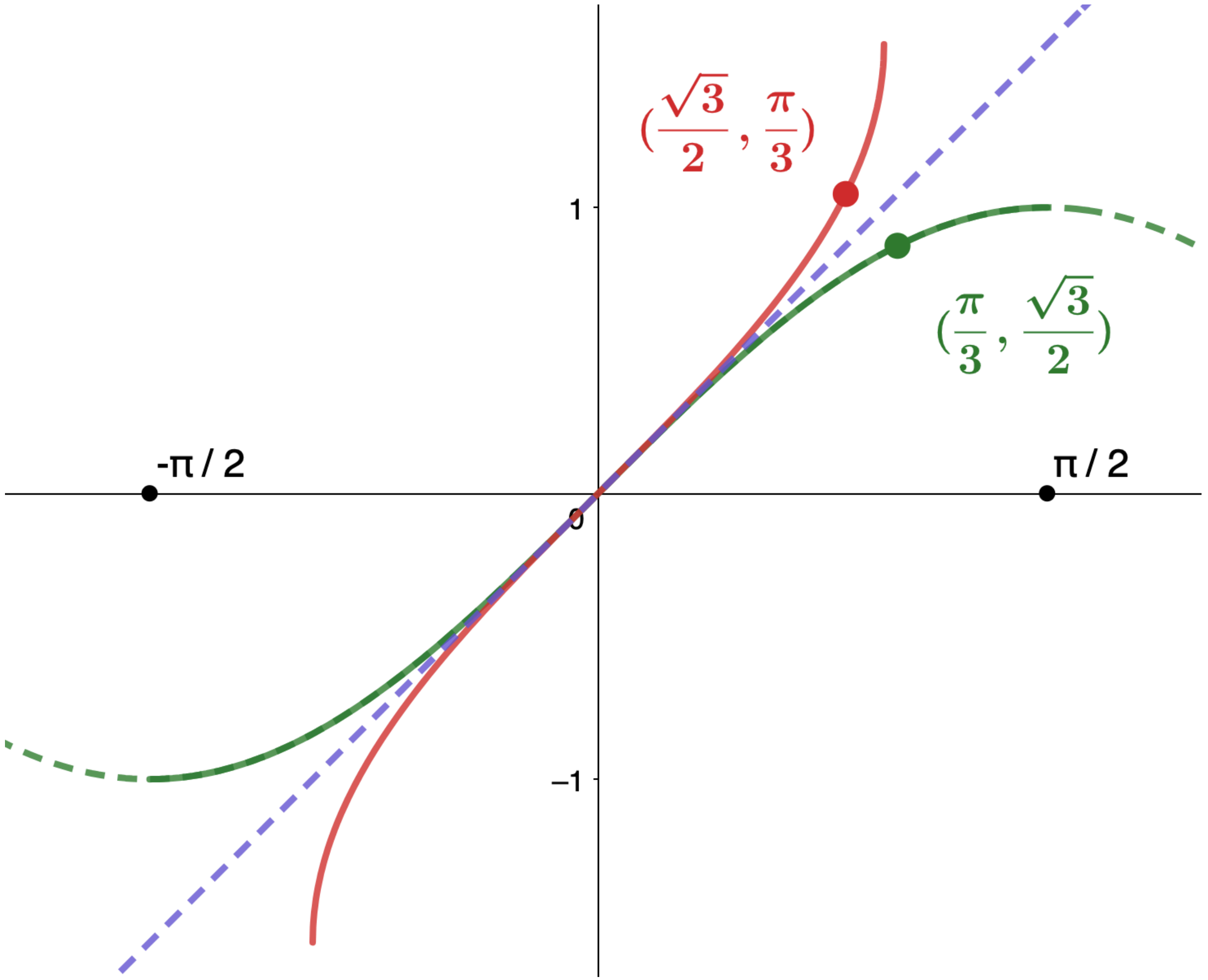
INVERSE SINE FUNCTION

- We define the inverse sine function on the domain $[-1,1]$.



INVERSE SINE FUNCTION

- For any x in the interval $[-1,1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ satisfies $\sin\theta = x$.



INVERSE TRIGONOMETRIC FUNCTIONS

- Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us **which angle in a certain interval has a specified trigonometric value**.

DEFINITION

The inverse sine function, denoted $\sin^{-1}$ or arcsin, and the inverse cosine function, denoted $\cos^{-1}$ or arccos, are defined on the domain $D = \{x| -1 \leq x \leq 1\}$ as follows:

$$\begin{aligned} &\sin^{-1}(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\tfrac{\pi}{2} \leq y \leq \tfrac{\pi}{2}; \\ &\cos^{-1}(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \leq y \leq \pi. \end{aligned} \tag{1.12}$$

The inverse tangent function, denoted $\tan^{-1}$ or arctan, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain $D = \{x| -\infty < x < \infty\}$ as follows:

$$\begin{aligned} &\tan^{-1}(x) = y \text{ if and only if } \tan(y) = x \text{ and } -\tfrac{\pi}{2} < y < \tfrac{\pi}{2}; \\ &\cot^{-1}(x) = y \text{ if and only if } \cot(y) = x \text{ and } 0 < y < \pi. \end{aligned} \tag{1.13}$$

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{x| \, |x| \geq 1\}$ as follows:

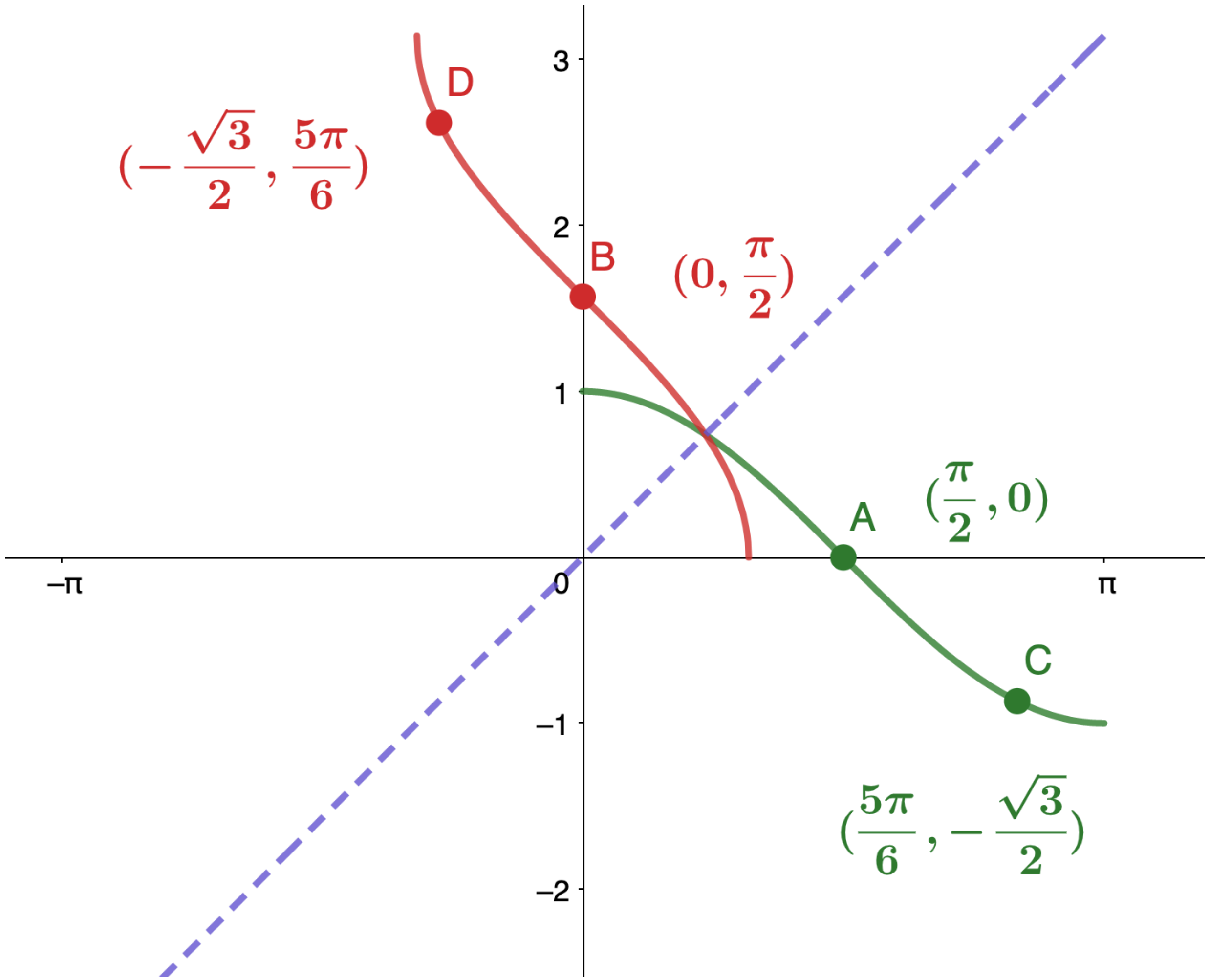
$$\begin{aligned} &\csc^{-1}(x) = y \text{ if and only if } \csc(y) = x \text{ and } -\tfrac{\pi}{2} \leq y \leq \tfrac{\pi}{2}, y \neq 0; \\ &\sec^{-1}(x) = y \text{ if and only if } \sec(y) = x \text{ and } 0 \leq y \leq \pi, y \neq \pi/2. \end{aligned} \tag{1.14}$$

INVERSE SINE AND COSINE FUNCTIONS

The inverse sine function, denoted $\sin^{-1}$ or arcsin, and the inverse cosine function, denoted $\cos^{-1}$ or arccos, are defined on the domain $D = \{x| -1 \le x \le 1\}$ as follows:

$$\sin^{-1}(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2};$$
$$\cos^{-1}(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \le y \le \pi. \tag{1.12}$$

INVERSE COSINE FUNCTION

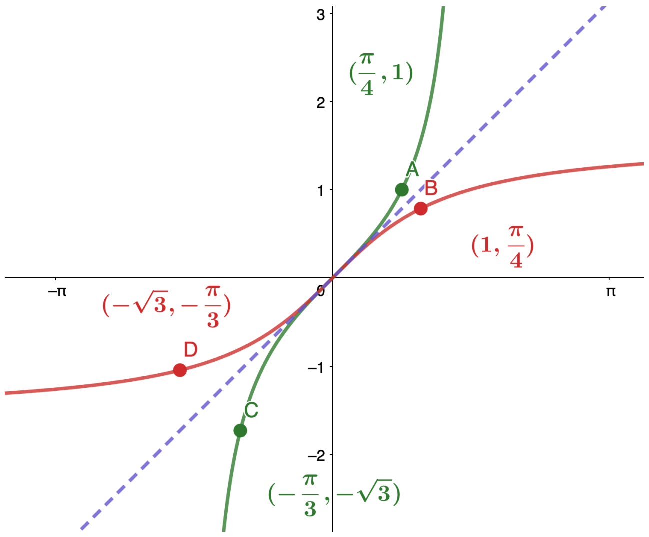


INVERSE TANGENT AND COTANGENT FUNCTIONS

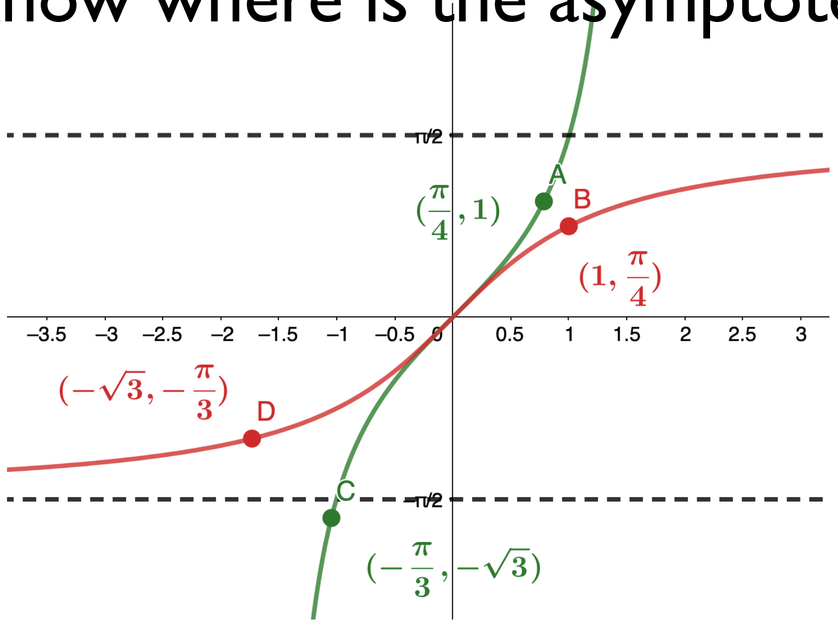
The inverse tangent function, denoted $\tan^{-1}$ or arctan, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain $D = \{x \mid -\infty < x < \infty\}$ as follows:

$$\tan^{-1}(x) = y \text{ if and only if } \tan(y) = x \text{ and } \boxed{-\tfrac{\pi}{2} < y < \tfrac{\pi}{2};}$$
$$\cot^{-1}(x) = y \text{ if and only if } \cot(y) = x \text{ and } \boxed{0 < y < \pi.} \qquad (1.13)$$

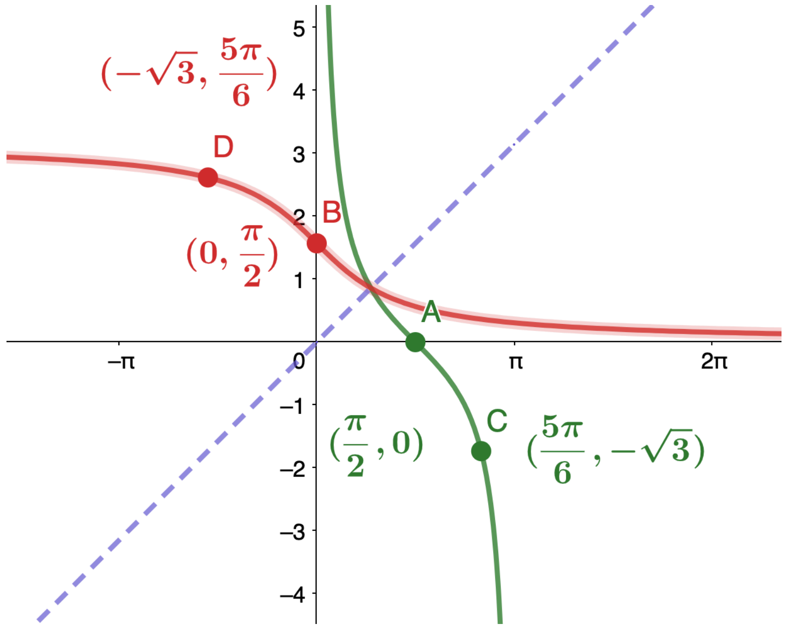
INVERSE TANGENT FUNCTION

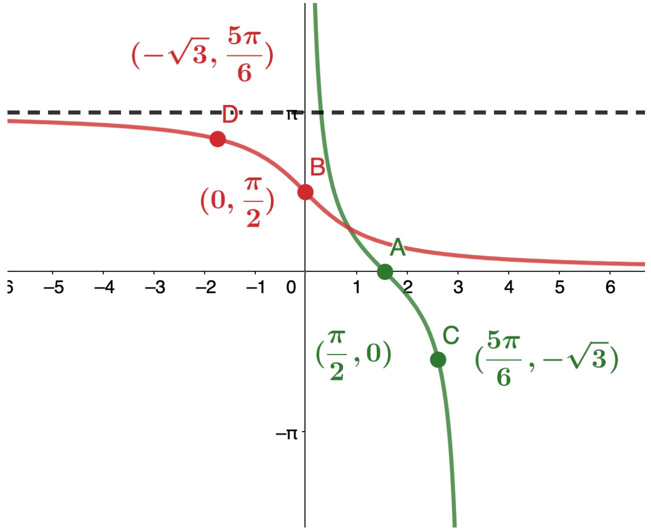


Know where is the asymptote!!!



INVERSE COTANGENT FUNCTION



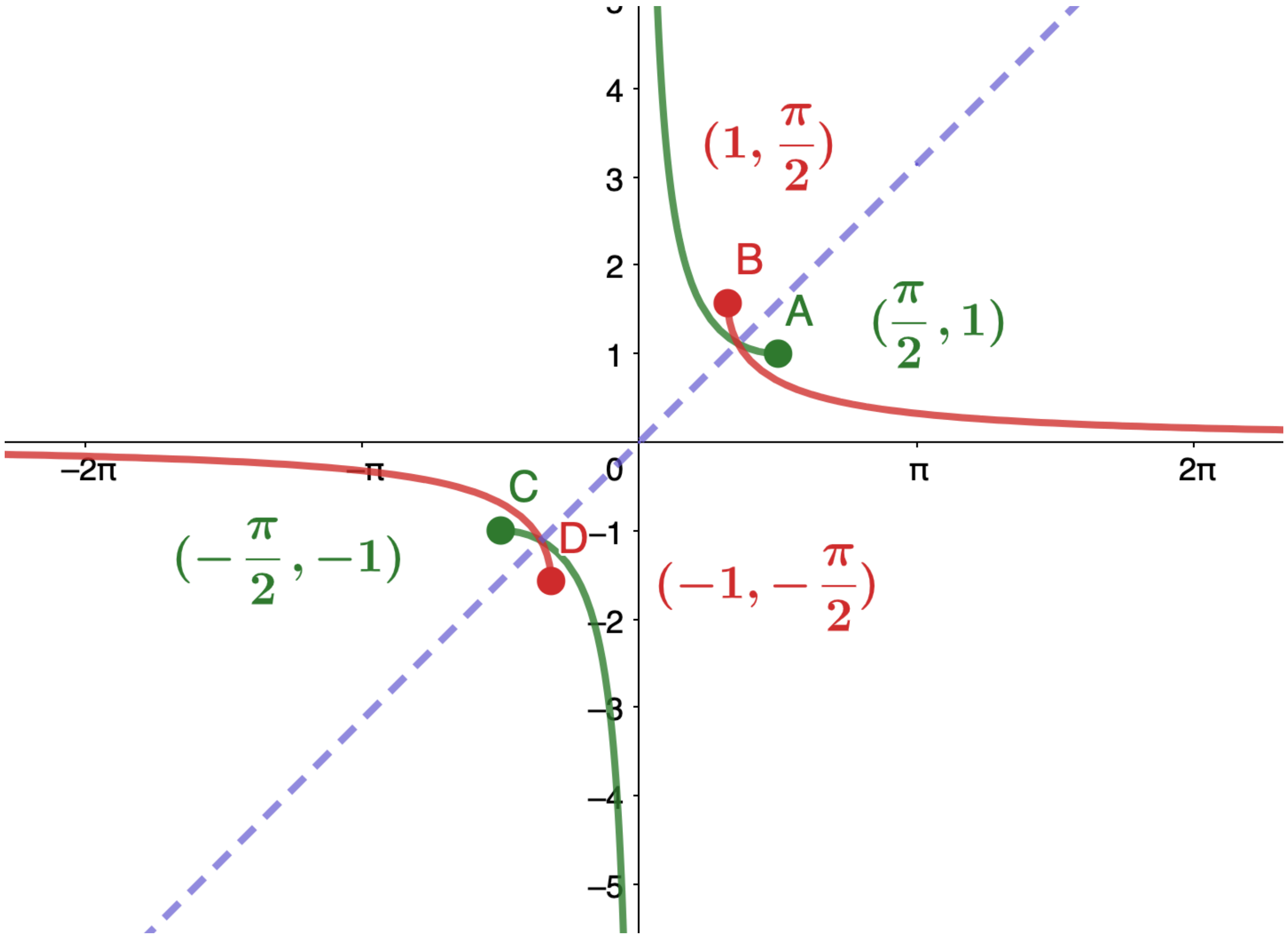


INVERSE COSECANT AND SECANT FUNCTIONS

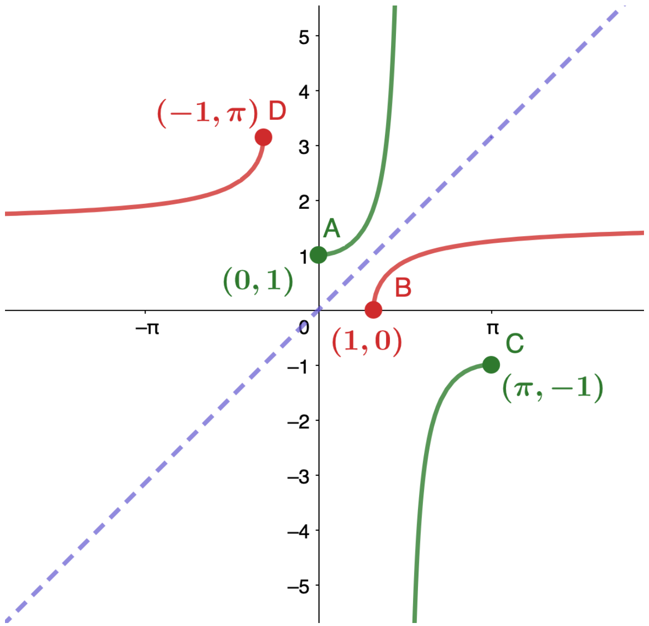
The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{x \mid |x| \geq 1\}$ as follows:

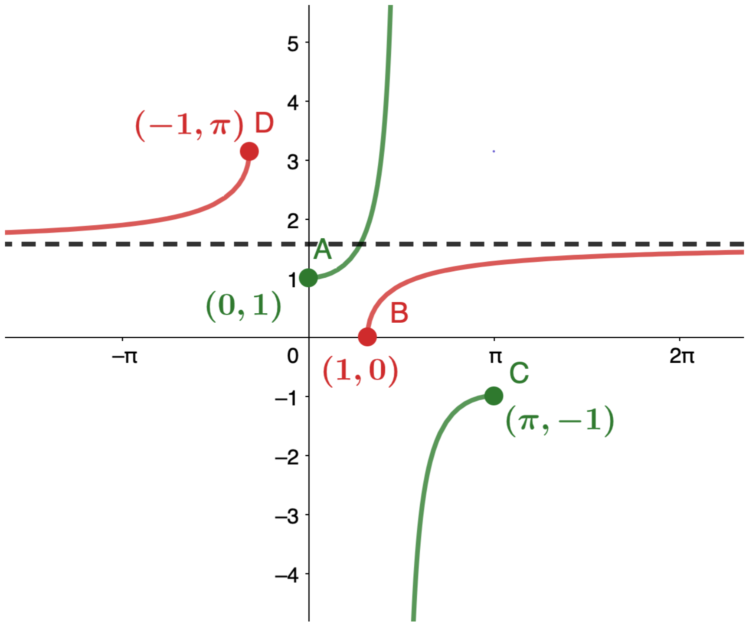
$$\begin{aligned} &\csc^{-1}(x) = y \text{ if and only if } \csc(y) = x \text{ and } -\tfrac{\pi}{2} \leq y \leq \tfrac{\pi}{2}, y \neq 0; \\ &\sec^{-1}(x) = y \text{ if and only if } \sec(y) = x \text{ and } 0 \leq y \leq \pi, y \neq \pi/2. \end{aligned} \tag{1.14}$$

INVERSE COSECANT FUNCTION



INVERSE SECANT FUNCTION





ATTENTION!!!

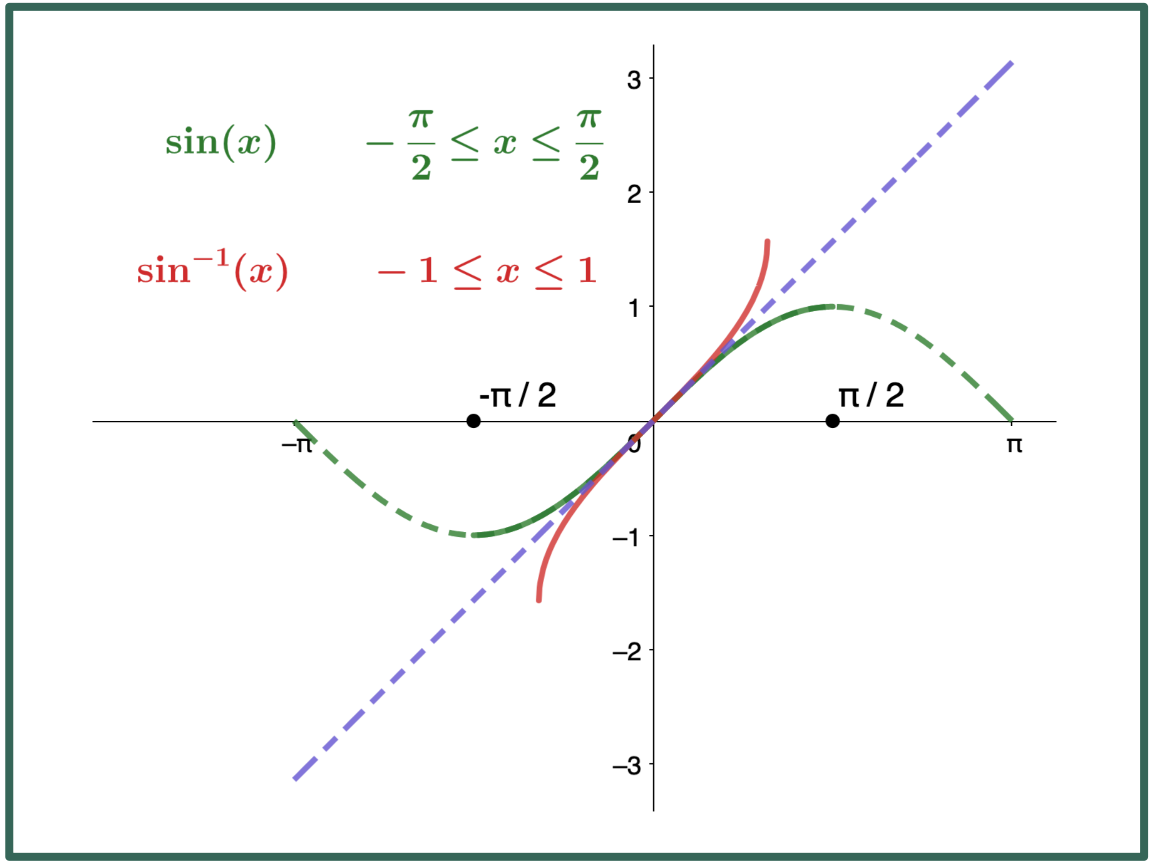
WHEN EVALUATING AN INVERSE TRIGONOMETRIC FUNCTION, THE OUTPUT IS AN ANGLE.

The inverse sine function, denoted $\sin^{-1}$ or arcsin, and the inverse cosine function, denoted $\cos^{-1}$ or arccos, are defined on the domain $D = \{x| -1 \leq x \leq 1\}$ as follows:

$$\sin^{-1}(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2};$$
$$\cos^{-1}(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \leq y \leq \pi. \tag{1.12}$$

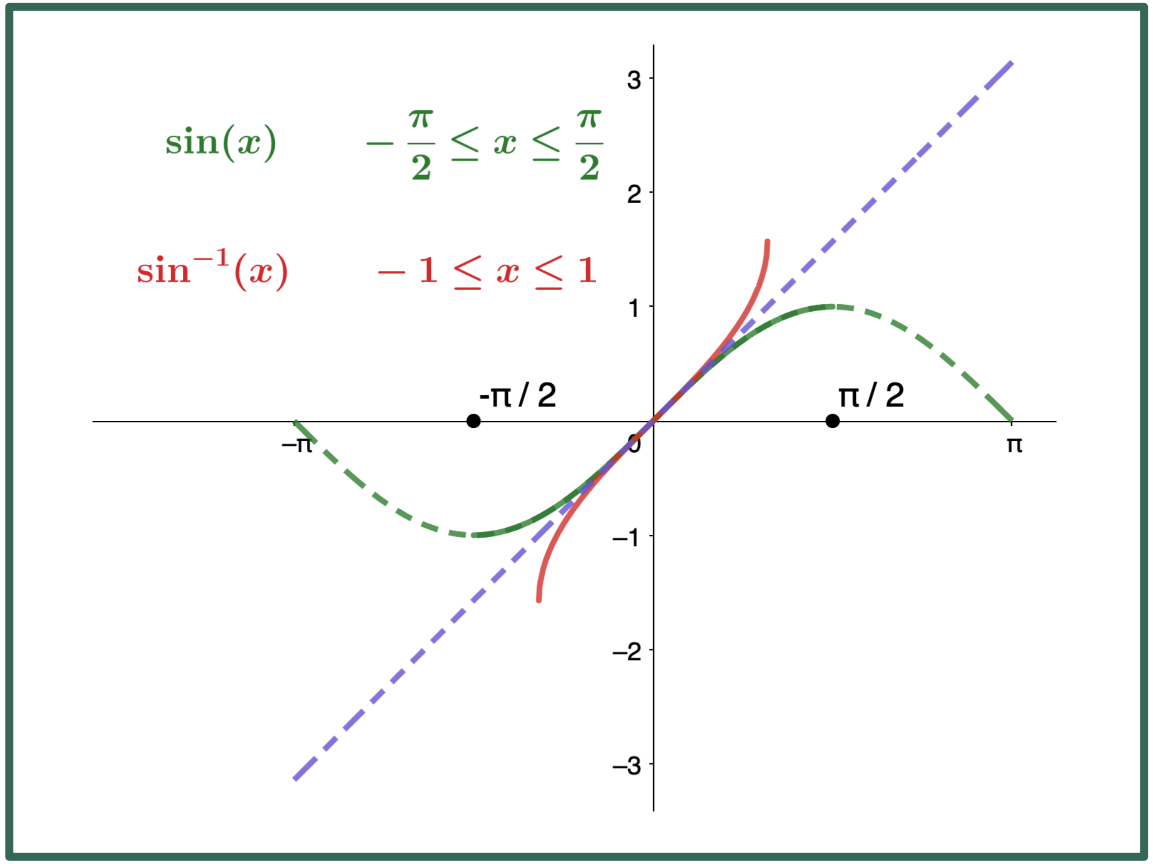
$$\sin^{-1}\frac{1}{2} = ?$$
$$\cos^{-1}(-\frac{1}{2}) = ?$$

ATTENTION!!!



- Now consider a composition of a trigonometric function and its inverse.
- $\sin(\sin^{-1}(\frac{\sqrt{2}}{2})) = ?$
- $\sin^{-1}(\sin(\pi)) = ?$

ATTENTION!!!



- $\sin(\sin^{-1}(\frac{\sqrt{2}}{2})) = \frac{\sqrt{2}}{2}$

  $\sin(\sin^{-1}(x)) = x?$
- $\sin^{-1}(\sin(\pi)) = 0$
- How can we make $\sin^{-1}(\sin(\theta)) = \theta?$
- (Hint) The inverse sine function is the inverse of the *restricted* sine function defined on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- $\sin^{-1}(\sin(\theta)) = \theta$ if …

REVISIT FUNCTION COMPOSITION

- $f^{-1}\left(f(x)\right) =?$
- $f\left(f^{-1}(x)\right) =?$

EXERCISE

- Use **composition** to determine which pairs of functions are inverses.

- $f(x) = \frac{2}{3}x + 2, g(x) = \frac{3}{2}x + 3.$
- $f(x) = \frac{1}{x-1}, x \neq 1, g(x) = \frac{1}{x} + 1, x \neq 0.$