INTRODUCTION TO CALCULUS

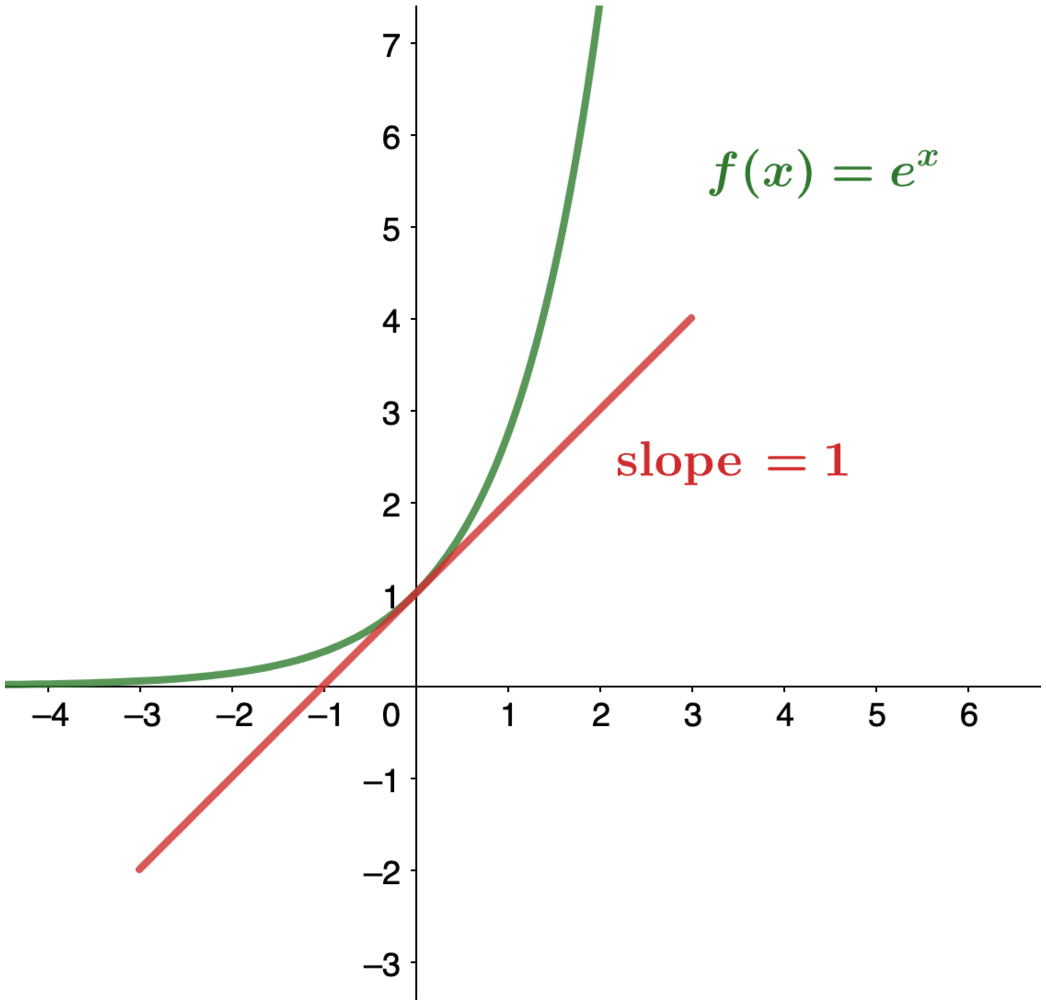
DERIVATIVES OF EXP AND LOG FUNCTIONS

OUTLINE

Find — Find the derivative of **exponential functions**.

Find — Find the derivative of **logarithmic functions**.

Use — Use **logarithmic differentiation** to determine the derivative of a function.



RECALL WHEN WE LEARNED EXPONENTIAL AND LOGARITHMIC FUNCTIONS

THEOREM 3.14

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x).$$

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTIONS

EXERCISE ZERO

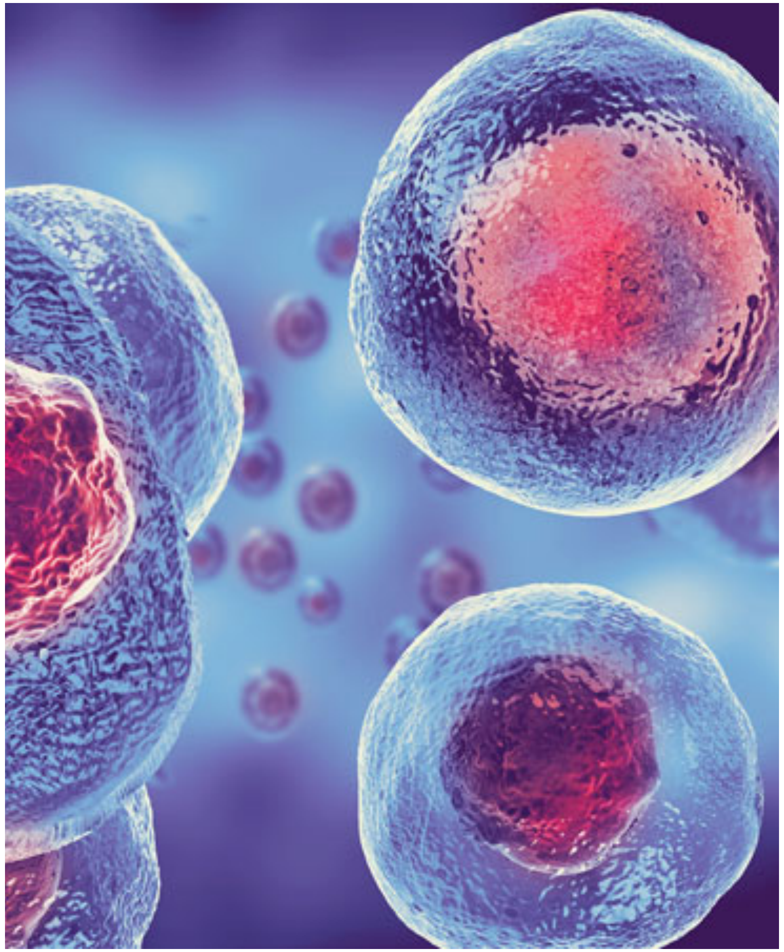
Uninhibited Growth: If a population increases according to The Law of Uninhibited Growth, the number of organisms N at time t is given by the formula

$$N(t) = N_0 e^{kt},$$

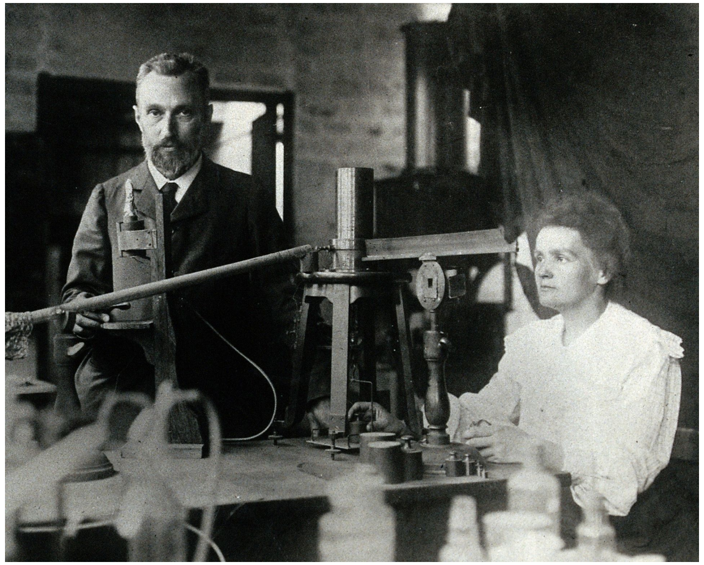
where $N(0) = N_0$ is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = kN(t).$$

- Prove the equation above.



EXERCISE ZERO



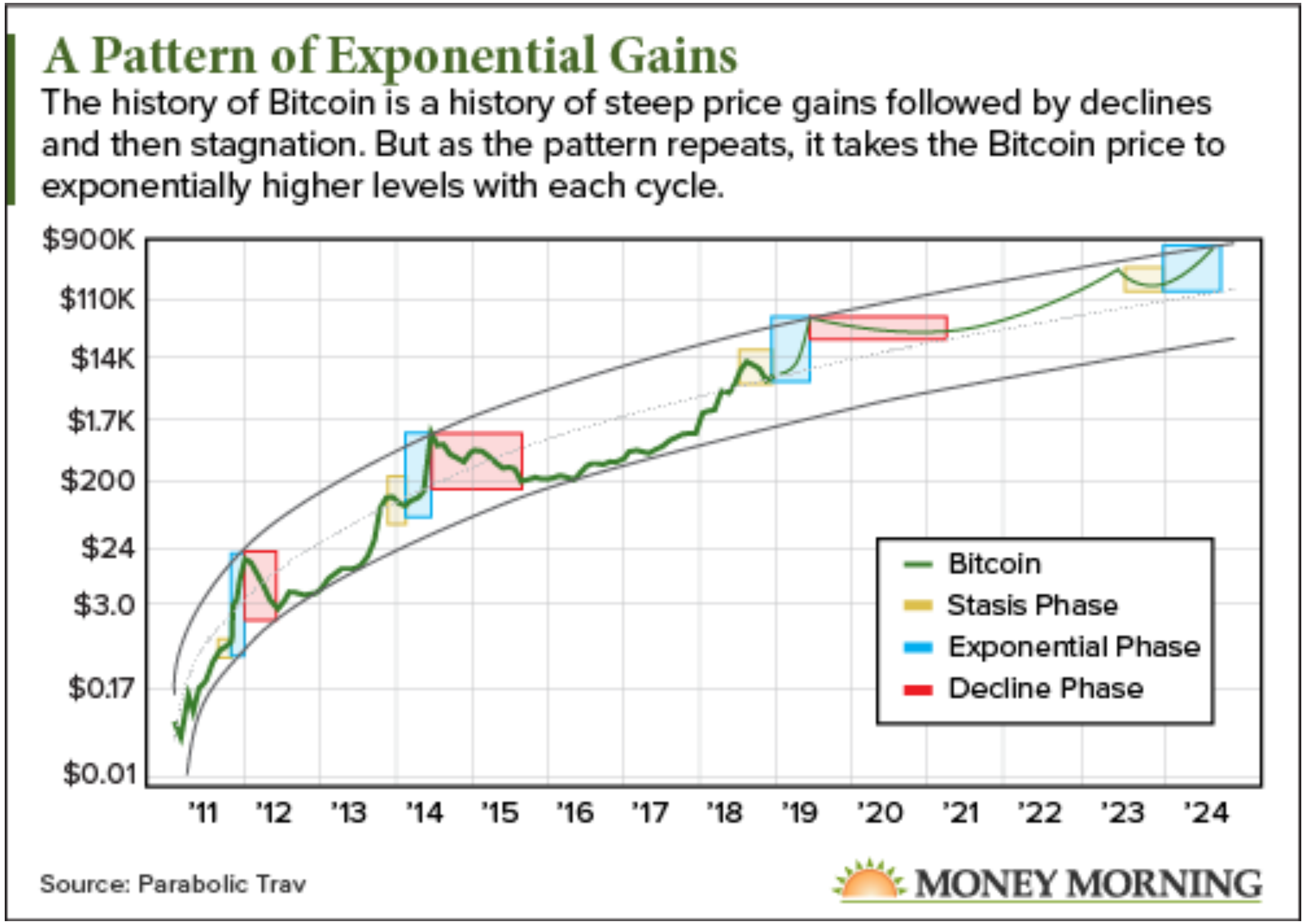
Radioactive Decay: The amount of a radioactive element A at time t is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element $k < 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } A(t) \text{ at time } t) = kA(t).$$

- Prove the equation above.



EXERCISE ZERO

EXERCISE ONE

$$\boxed{\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x).}$$

Derivative of an Exponential Function

- Find the derivative of $f(x) = e^{\tan(2x)}$.

EXERCISE TWO

$$\boxed{E'(x) = e^x.}$$

$$\boxed{\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x).}$$

Combining Differentiation Rules

- Find the derivative of $f(x) = xe^{x^2}$.
- Find the derivative of $f(x) = \dfrac{e^{2x}}{x}$.

DERIVATIVE OF THE LOGARITHMIC FUNCTION

$$\boxed{g'(x) = \frac{1}{f'(g(x))}.}$$

Now that we have the derivative of the natural exponential function, we can use **implicit differentiatio**n to find the derivative of **its inverse**, the natural logarithmic function.

THEOREM 3.15

The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\boxed{\frac{dy}{dx} = \frac{1}{x}.} \tag{3.30}$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

$$\boxed{h'(x) = \frac{1}{g(x)} g'(x).} \tag{3.31}$$

DERIVATIVE OF THE LOGARITHMIC FUNCTION

ANOTHER PROOF

$$\boxed{\frac{dy}{dx} = \frac{1}{x}.}$$

If $x > 0$ and $y = \ln(x)$, then $e^y = x$.

Differentiating both sides of the equation results in the equation

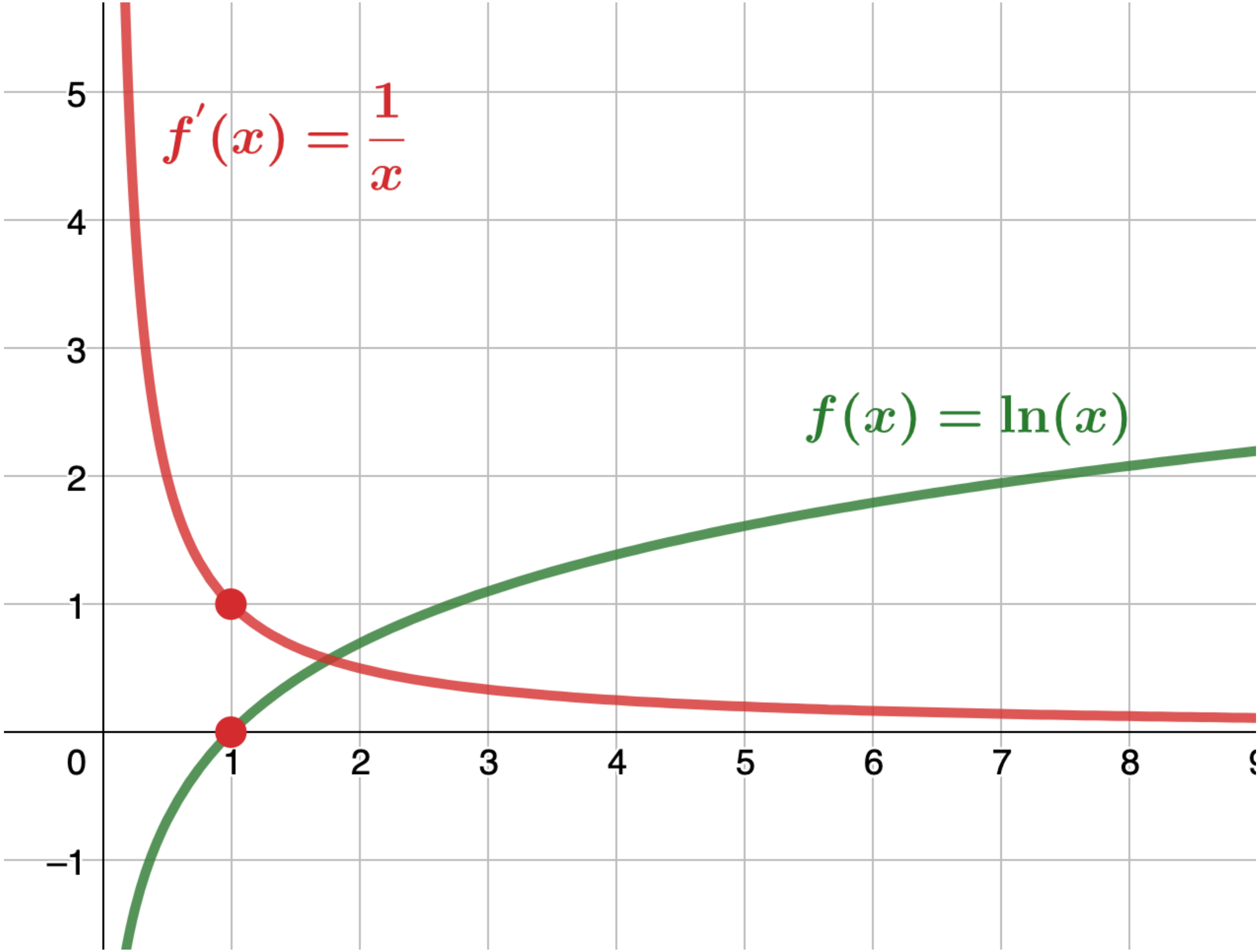
$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$



DERIVATIVE OF THE LOGARITHMIC FUNCTION

EXERCISE ONE

Taking a Derivative of a Natural Logarithm

- Find the derivative of $f(x) = \ln(2x^2 + 3x + 1)$.

EXERCISE TWO

Using Properties of Logarithms in a Derivative

- Find the derivative of $f(x) = \ln(\frac{\sin(x)}{3x+2})$.

EXERCISE THREE

- Find the derivative of $f(x) = [\ln(3x+2)]^3$.

FROM NATURAL EXPONENTIAL AND NATURAL LOGARITHMIC FUNCTIONS TO …

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

$$\frac{d}{dx} b^x = ?$$

$$\frac{d}{dx} \log_b(x) = ?$$

$$\frac{d}{dx} b^x = b^x \ln(b)$$

$$\frac{d}{dx} \log_b(x) = \frac{1}{x \ln(b)}$$

DERIVATIVES OF GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Let $b > 0, b \neq 1$, and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x$, then

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b (g(x))$, then for all values of x for which $g(x) > 0$,

$$\boxed{h'(x) = \frac{g'(x)}{g(x) \ln b}.}$$

ii. If $y = b^x$, then

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then

$$\boxed{h'(x) = b^{g(x)} g'(x) \ln b.}$$

CHANGE-OF-BASE FORMULAS

USING THIS CHANGE OF BASE, WE TYPICALLY WRITE A GIVEN EXPONENTIAL OR LOGARITHMIC FUNCTION IN TERMS OF THE NATURAL EXPONENTIAL AND NATURAL LOGARITHMIC FUNCTIONS.

RULE: CHANGE-OF-BASE FORMULAS

Let $a > 0, b > 0$, and $a \neq 1, b \neq 1$.

1. $a^x = b^{x \log_b a}$ for any real number x.
   If $b = e$, this equation reduces to $a^x = e^{x \log_e a} = e^{x \ln a}$.
2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.
   If $b = e$, this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

DERIVATIVES OF GENERAL EXPONENTIAL FUNCTIONS

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^{\boldsymbol{g(x)}} = e^{g(x)}\frac{d}{dx}\boldsymbol{g(x)}$$

$$\frac{d}{dx}b^x = b^x \ln(\boldsymbol{b})$$

$$\frac{d}{dx}b^{\boldsymbol{g(x)}} = b^{g(x)}\frac{d}{dx}g(x)\ln(\boldsymbol{b})$$

PROOF

$$\frac{d}{dx}e^{x} = e^{x}$$

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}\frac{d}{dx}\boldsymbol{g(x)}$$

$$\boldsymbol{b^{x}} = \boldsymbol{e^{x\,\ln(b)}}$$

$$\longrightarrow$$

$$\frac{d}{dx}b^{x} = b^{x}\,\mathbf{ln}(\boldsymbol{b})$$

$$\frac{d}{dx}b^{g(x)} = b^{g(x)}\frac{d}{dx}\boldsymbol{g(x)}\,\mathbf{ln}(\boldsymbol{b})$$

DERIVATIVES OF GENERAL LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln(\boldsymbol{g}(\boldsymbol{x})) = \frac{1}{\boldsymbol{g}(\boldsymbol{x})}\frac{d}{dx}\boldsymbol{g}(\boldsymbol{x})$$

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\,\mathbf{ln}(\boldsymbol{b})}$$

$$\frac{d}{dx}\log_b(g(x)) = \frac{1}{\boldsymbol{g}(\boldsymbol{x})\,\mathbf{ln}(\boldsymbol{b})}\frac{d}{dx}\boldsymbol{g}(\boldsymbol{x})$$

PROOF

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)}\frac{d}{dx}g(x)$$

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$$

$$\frac{d}{dx}\log_b(g(x)) = \frac{1}{g(x)\ln(b)}\frac{d}{dx}g(x)$$

Applying Derivative Formulas

- Find the derivative of $h(x) = \frac{2^x}{2^x+2}$.

EXERCISE ONE

EXERCISE TWO

Finding the Slope of a Tangent Line

- Find the slope of the line tangent to the graph of $y = \log(x^2 + 2x + 3)$ at $x = 1$.

LOGARITHMIC DIFFERENTIATION

A technique called **logarithmic differentiation** allows us to differentiate any function of the form

$$h(x) = g(x)^{f(x)}.$$

LOGARITHMIC DIFFERENTIATION

PROBLEM-SOLVING STRATEGY: USING LOGARITHMIC DIFFERENTIATION

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

EXERCISE ONE

Extending the Power Rule

- Find the derivative of $y = x^r$ where r is an arbitrary real number.

EXERCISE THREE

Using Logarithmic Differentiation

- Find the derivative of $y = x^x$.

EXERCISE THREE

Using Logarithmic Differentiation

- Find the derivative of $y = (2x^2 + 3x + 2)^{\sin(x)}$.

EXERCISE FOUR

Using Logarithmic Differentiation

- Find the derivative of $y = \frac{\sqrt{x}}{e^x \sin^2 x}$.