INTRODUCTION TO CALCULUS

AVERAGE RATE OF CHANGE, CONSTRUCTING A FUNCTION WHICH DESCRIBES A MODEL

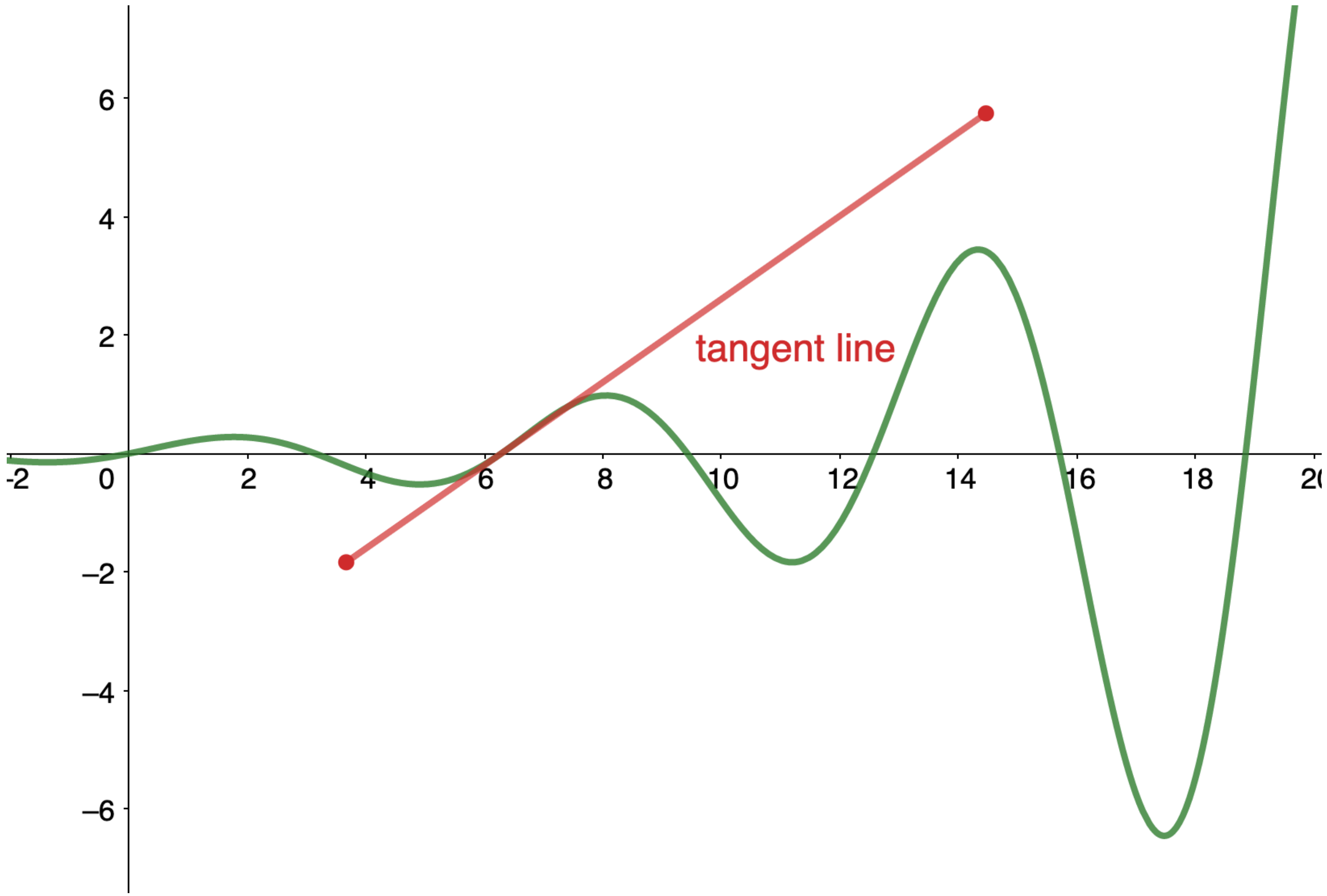
OUTLINE

Average rate of change (1.2)

- Describe the **tangent** (**rate of change**) and the **secant** (**average rate of change**) problems.
- Recognize a tangent to a curve at a point as the **limit** of secant lines.
- Identify **instantaneous velocity** as the limit of **average velocity** over a small time interval.

Constructing a function which describes a model (2.1)

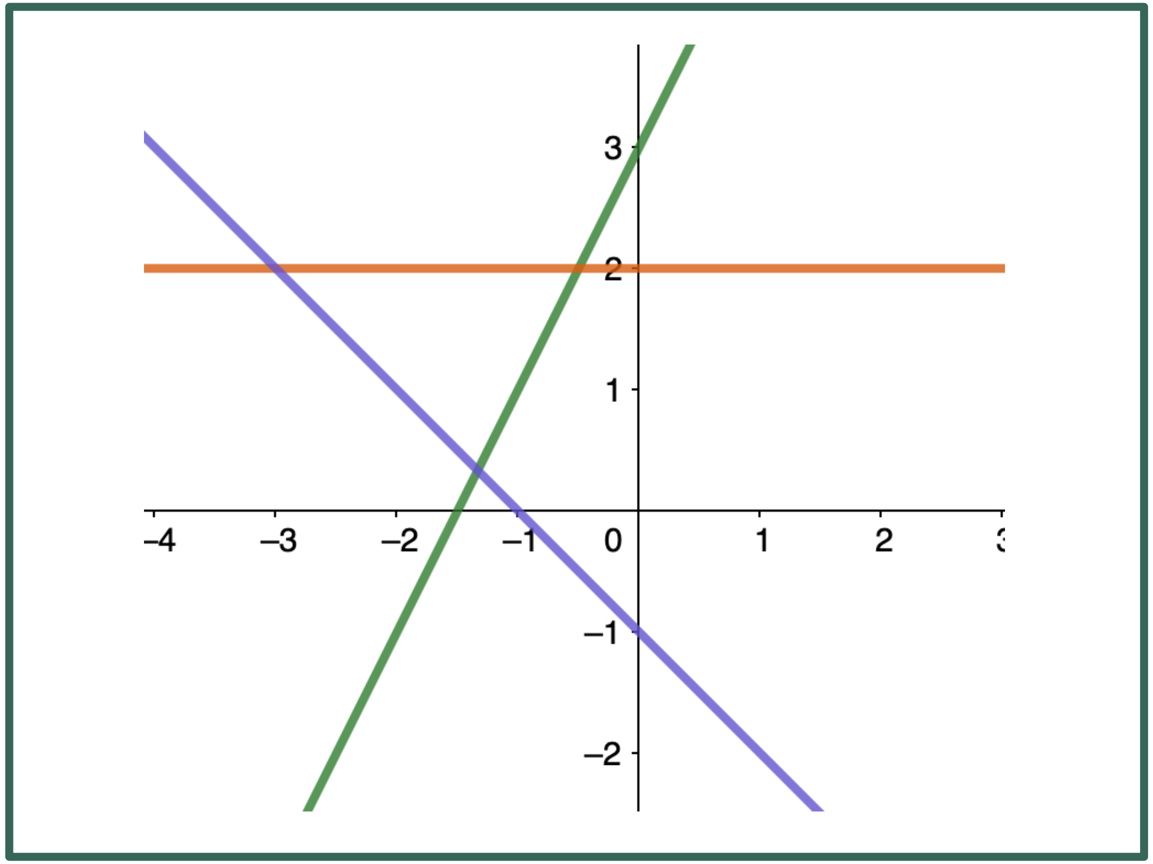
- Linear functions
- Quadratic functions
- General polynomials, rational functions as well as other algebraic functions



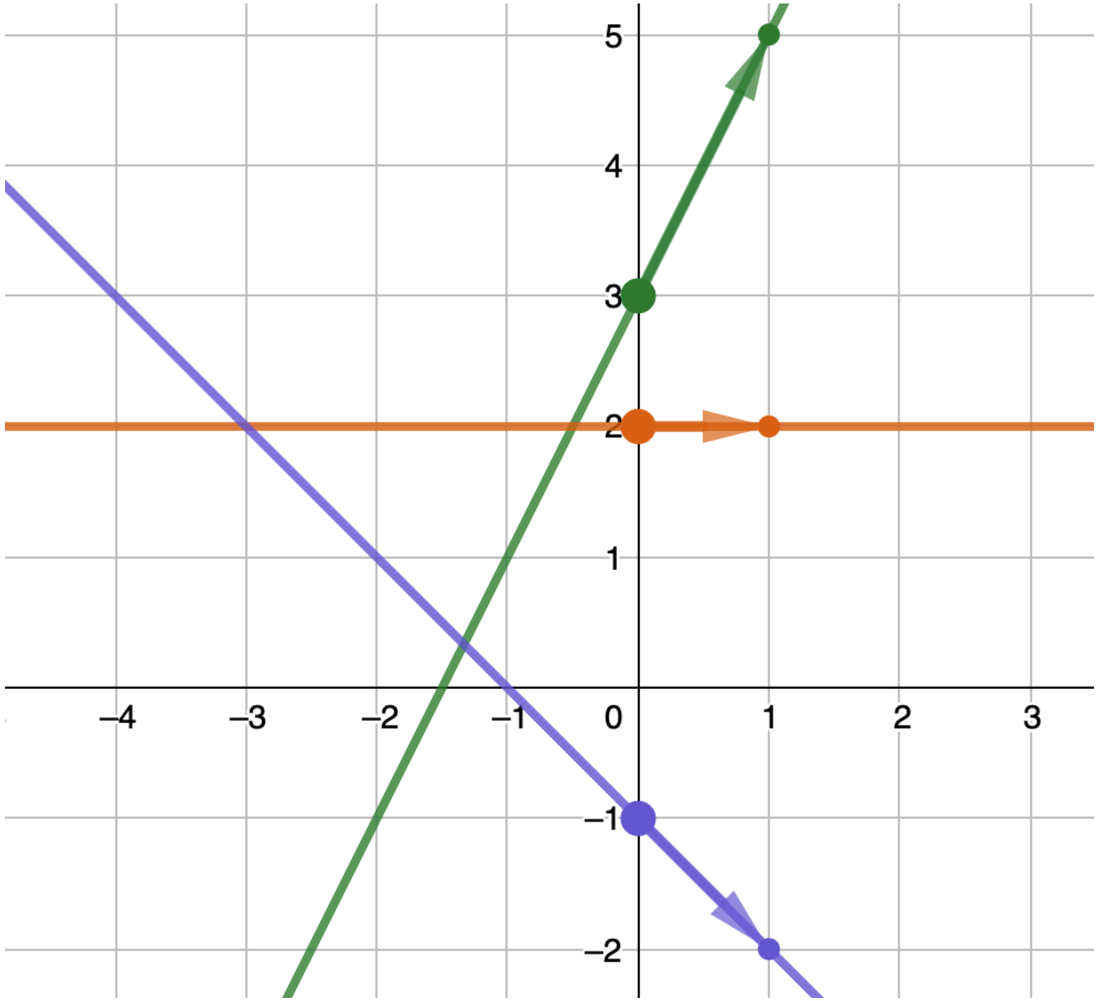
THE TANGENT PROBLEM

HOW TO DETERMINE THE SLOPE OF A LINE TANGENT TO A CURVE AT A POINT?

THE TANGENT LINE AND RATE OF CHANGE



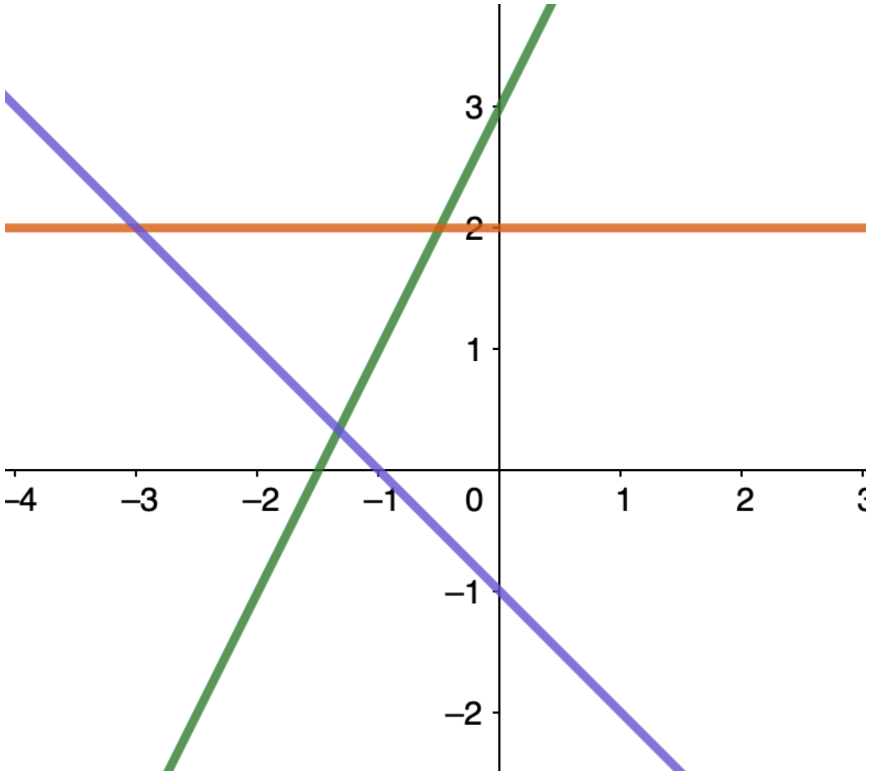
- For linear functions
  - $f(x) = 2x + 3$
  - $g(x) = 2$
  - $h(x) = -x - 1$
- **The tangent line coincides with the curve of the linear function.**



LINEAR FUNCTION: SLOPE

THE SLOPE IS THE CHANGE OF y FOR EACH UNIT CHANGE IN x.

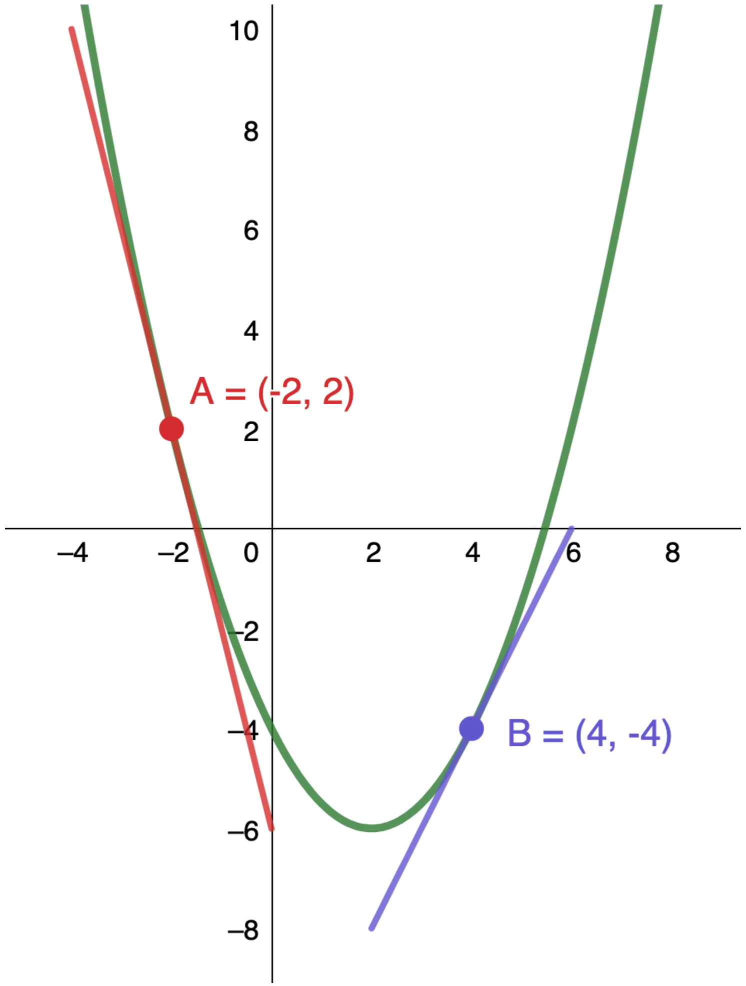
THE TANGENT LINE AND RATE OF CHANGE: LINEAR FUNCTIONS



- For linear functions
  - $f(x) = 2x + 3$
  - $g(x) = 2$
  - $h(x) = -x - 1$
- **The slope of each linear function indicates the rate of change of the function.**

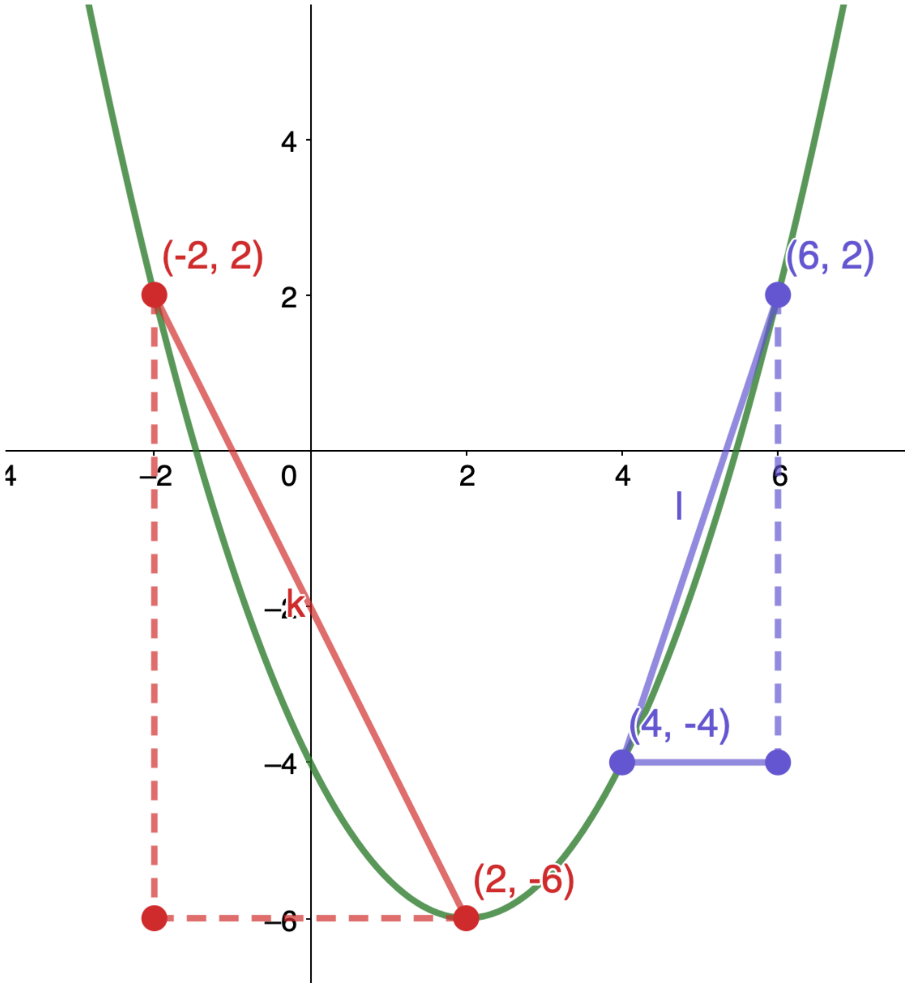
THE TANGENT LINE AND RATE OF CHANGE: QUADRATIC FUNCTIONS

- For quadratic functions
  - $f(x) = \frac{1}{2}x^2 - 2x - 4$
- Unlike a linear function, no single number represents the rate of change for this function.

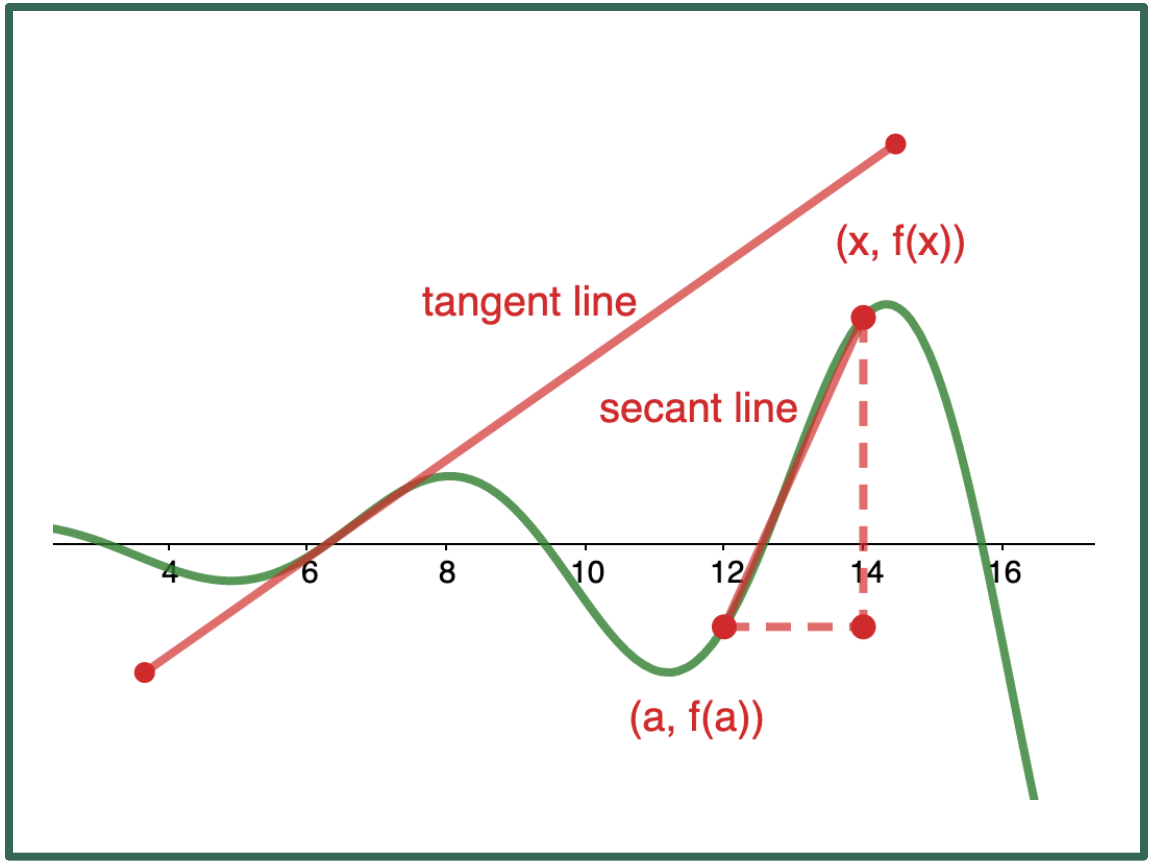


AVERAGE RATE OF CHANGE

- For quadratic functions
  - $f(x) = \frac{1}{2}x^2 - 2x - 4$
- Before learning how to derive the rate of change, we can first estimate it with the average rate of change.



AVERAGE RATE OF CHANGE



- For a general function
  - $f(x)$
- The rate of change at point
  - $(a, f(a))$
- Approximation
  - Take another point $(x, f(x))$
  - Draw a line through the tow points
  - Calculate the slope

DEFINITION

The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{\text{sec}} = \boxed{\frac{f(x) - f(a)}{x - a}}. \tag{2.1}$$

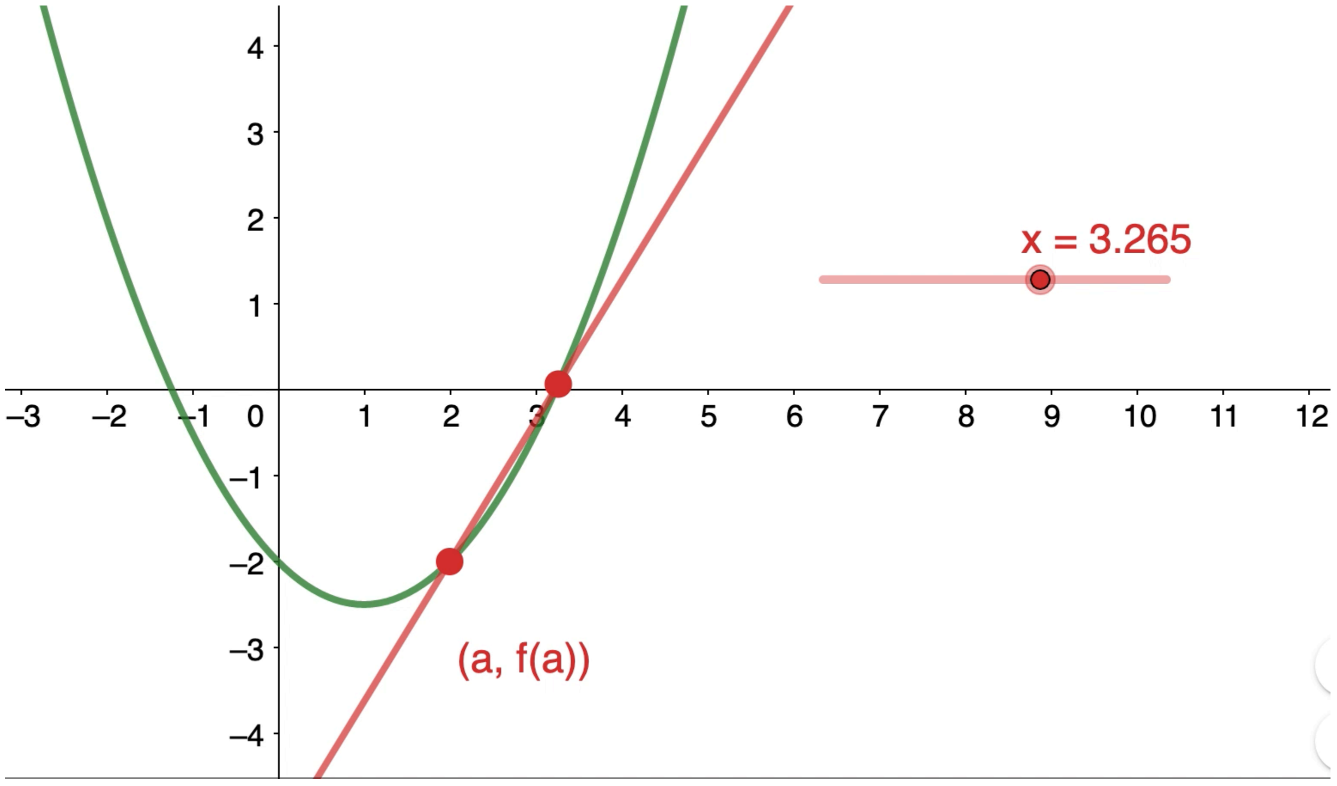
AVERAGE RATE OF CHANGE AND SECANT LINE

FROM SECANT LINE TO TANGENT LINE (A PREVIEW OF THE LIMIT)

$$x \to a$$

The tangent line is the limit of secant lines.

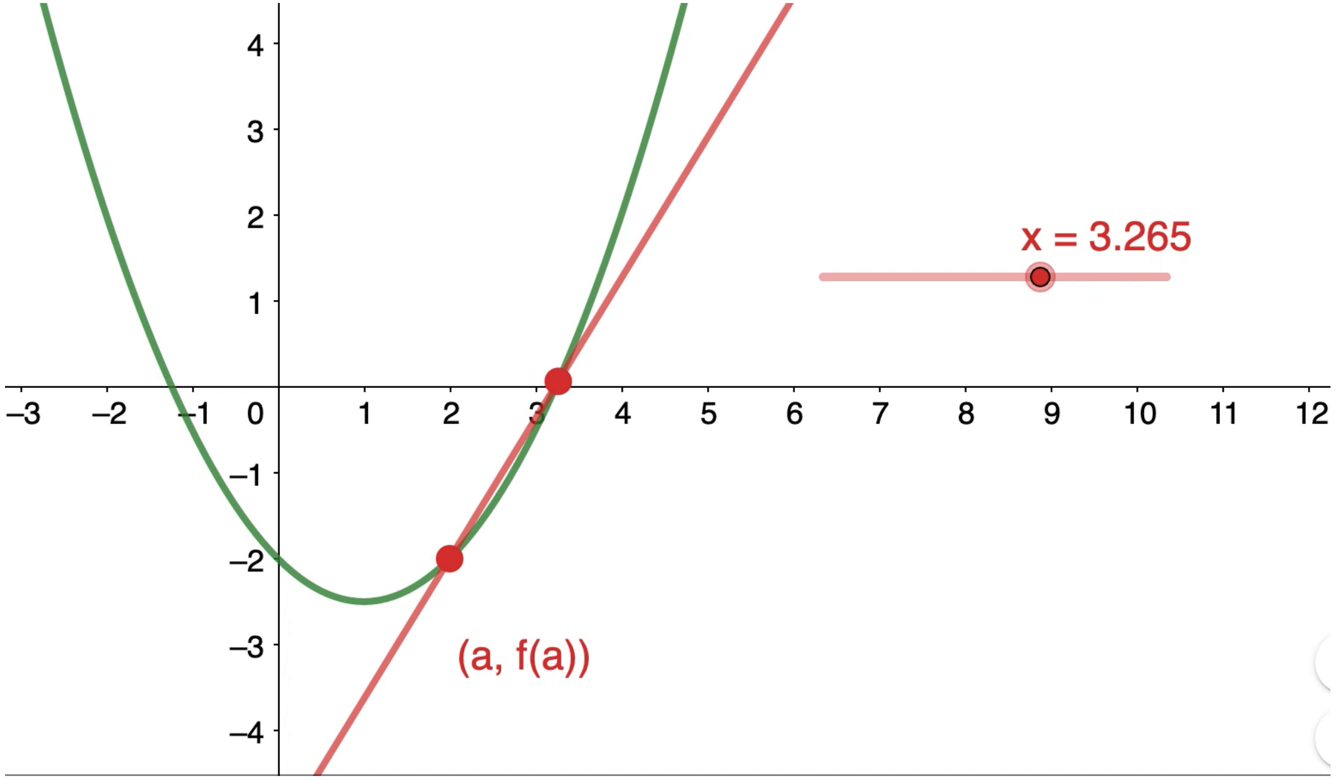
- The slope of the tangent line
- The rate of change of the function at a
- The **derivative** of the function $f(x)$ at a



FROM SECANT LINE TO TANGENT LINE (A PREVIEW OF THE LIMIT)

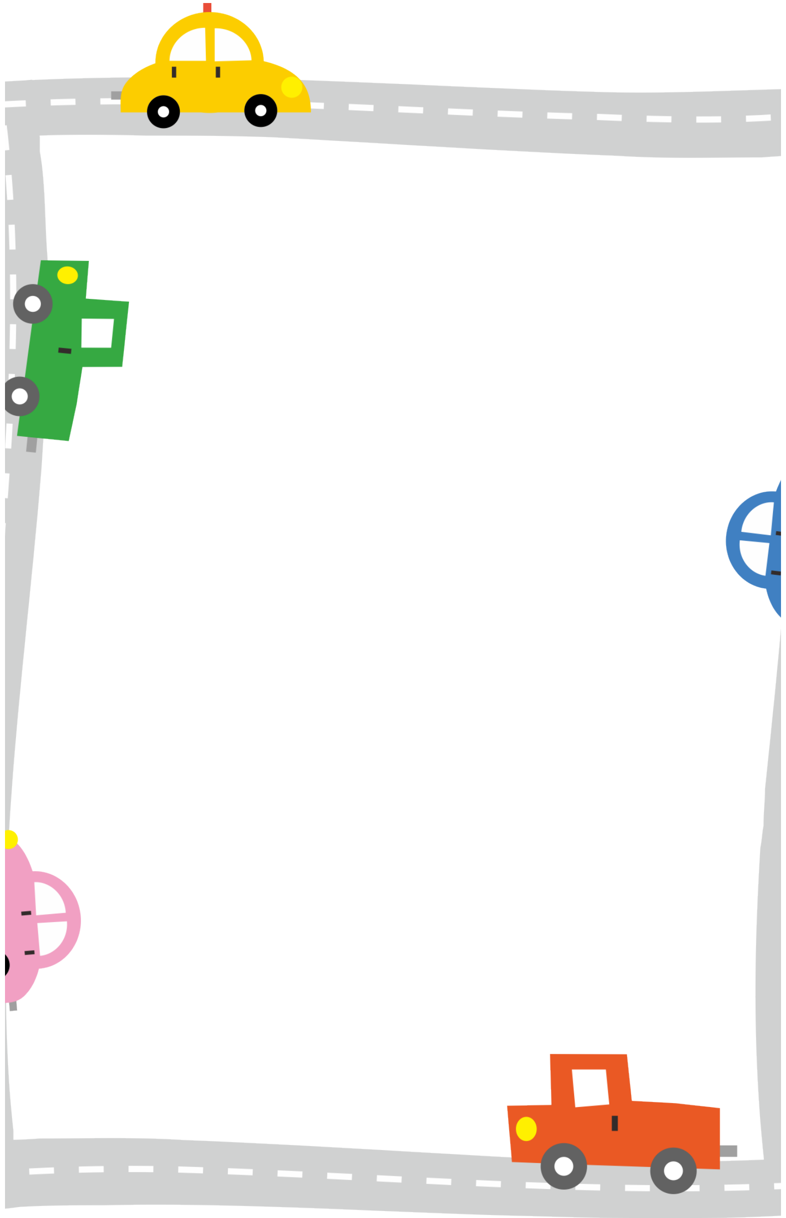
$f(x) = \frac{1}{2}x^2 - x - 2$ and $(a, f(a)) = (2, -2)$

- $(x, f(x)) = (4,2)$
  - slope is 2
- $(x, f(x)) = (3, -\frac{1}{2})$
  - slope is $\frac{3}{2}$
- $(x, f(x) = (\frac{5}{2}, -\frac{11}{8})$
  - slope is $\frac{5}{4}$
- $\ldots$
- $g(x) = x - 4$
  - slope is 1



AVERAGE RATE OF CHANGE: AN APPLICATION

- We introduce a function $s(t)$, that gives the **position** of an object along a coordinate axis at any given time t.
- **Velocity** may be thought of as the rate of change of position.
- How to create a reasonable definition of the instantaneous velocity at a given time $t = a$?



AVERAGE RATE OF CHANGE: AN APPLICATION

DEFINITION

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$\boxed{v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.} \tag{2.2}$$

FROM AVERAGE VELOCITY TO INSTANTANEOUS VELOCITY

DEFINITION

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

- Taking a limit
  - $x \to a$
  - $t \to a$

AN EXERCISE

- Two zebras are wandering in the wood, seeking some little shelter from the wind and rain. They set off 10 days ago from the Maryland Zoo in Baltimore ($t = 0$), passing through Washington DC and happen to meet each other again in Hanover. Treating DC as the origin ($S = 0$), their positions are

$$S_1(t) = t^2 - 2t - 8 \text{ and } S_2(t) = 8t - 8.$$

- Use this information to decide which one of them are faster when they arrived at DC. What about now?

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

AN EXERCISE

- The positions of the two zebras are

$$S_1(t) = t^2 - 2t - 8 \text{ and } S_2(t) = 8t - 8.$$

- Arrived at DC ($S = 0$)
- $S_1(a) = a^2 - 2a - 8 = (a-4)(a+2) = 0 \Longrightarrow a = 4$
  - $v_{\text{ave}} = \dfrac{S_1\left(\frac{9}{2}\right) - S_1(4)}{\frac{9}{2} - 4} = \dfrac{\frac{81}{4} - 9 - 8 - 0}{\frac{1}{2}} = \dfrac{13}{2}$
- $S_2(a) = 8a - 8 = 0 \Longrightarrow a = 1$
  - $v_{\text{ins}} = v_{\text{ave}} = 8$ (Why?)
- $\dfrac{13}{2} < 8$

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

AN EXERCISE

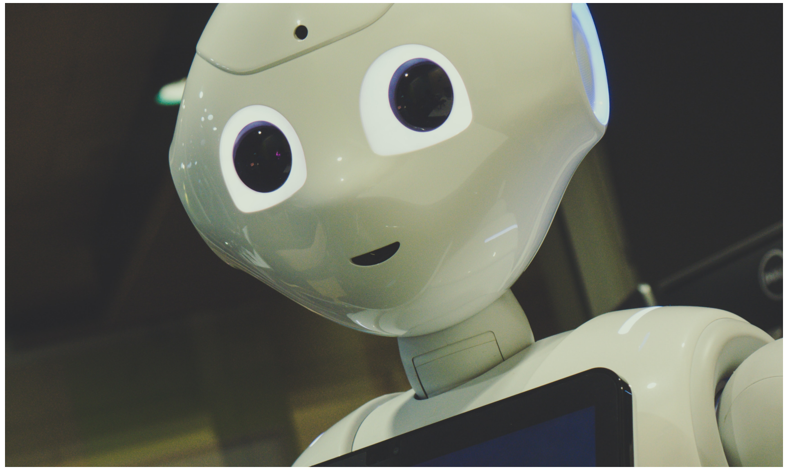
- The positions of the two zebras are

$$S_1(t) = t^2 - 2t - 8 \text{ and } S_2(t) = 8t - 8.$$

- Arrived at Hanover ($a = 10$)
- For the first zebra
  - $v_{\text{ave}} = \frac{S_1\left(\frac{19}{2}\right) - S_1(10)}{\frac{19}{2} - 10} = \frac{\frac{361}{4} - 19 - 8 - (100 - 20 - 8)}{-\frac{1}{2}} = \frac{35}{2}$
- For the second zebra
  - $v_{\text{ins}} = v_{\text{ave}} = 8$
- $\frac{35}{2} > 8$

MATHEMATICAL MODELS

- Math 11 (Accelerated Multivariable Calculus)
- Math 20 (Discrete Probability)
- Math 22 (Linear Algebra with Applications)
- Math 23 (Differential Equations)
- What can Math 1 (Calculus with Algebra) do?

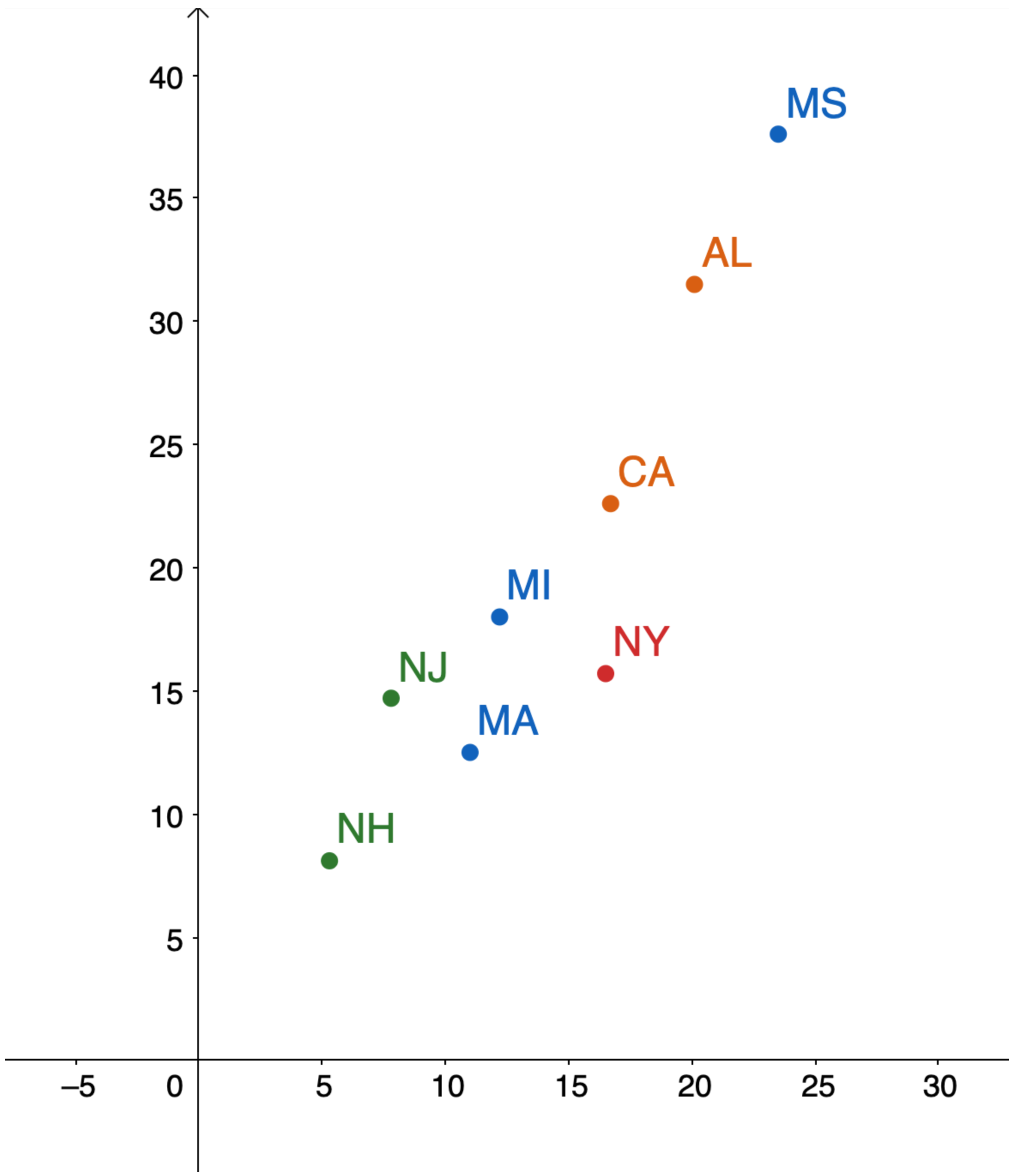


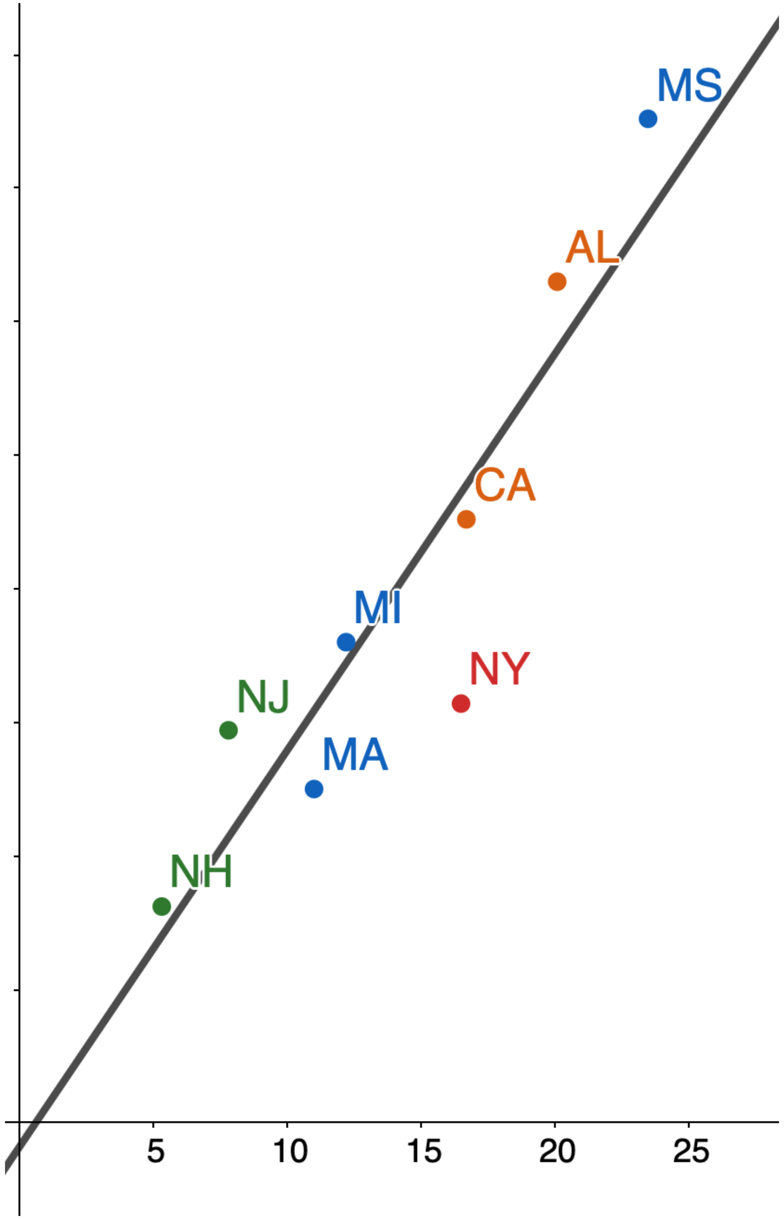
EXAMPLE ONE: LINEAR REGRESSION (YEAR 2002)

State	Alabama	California	Massachusetts	Michigan	Mississippi	New Hampshire	New Jersey	New York
Poverty rate	20.1	16.7	11	12.2	23.5	5.3	7.8	16.5
Birth rate (per 1000 females 15 to 17)	31.5	22.6	12.5	18	37.6	8.1	14.7	15.7

EXAMPLE ONE: LINEAR REGRESSION (YEAR 2002)

- Poverty rate
  - x
- Birth rate per 1000 females 15 to 17
  - y





EXAMPLE ONE: LINEAR REGRESSION (YEAR 2002)

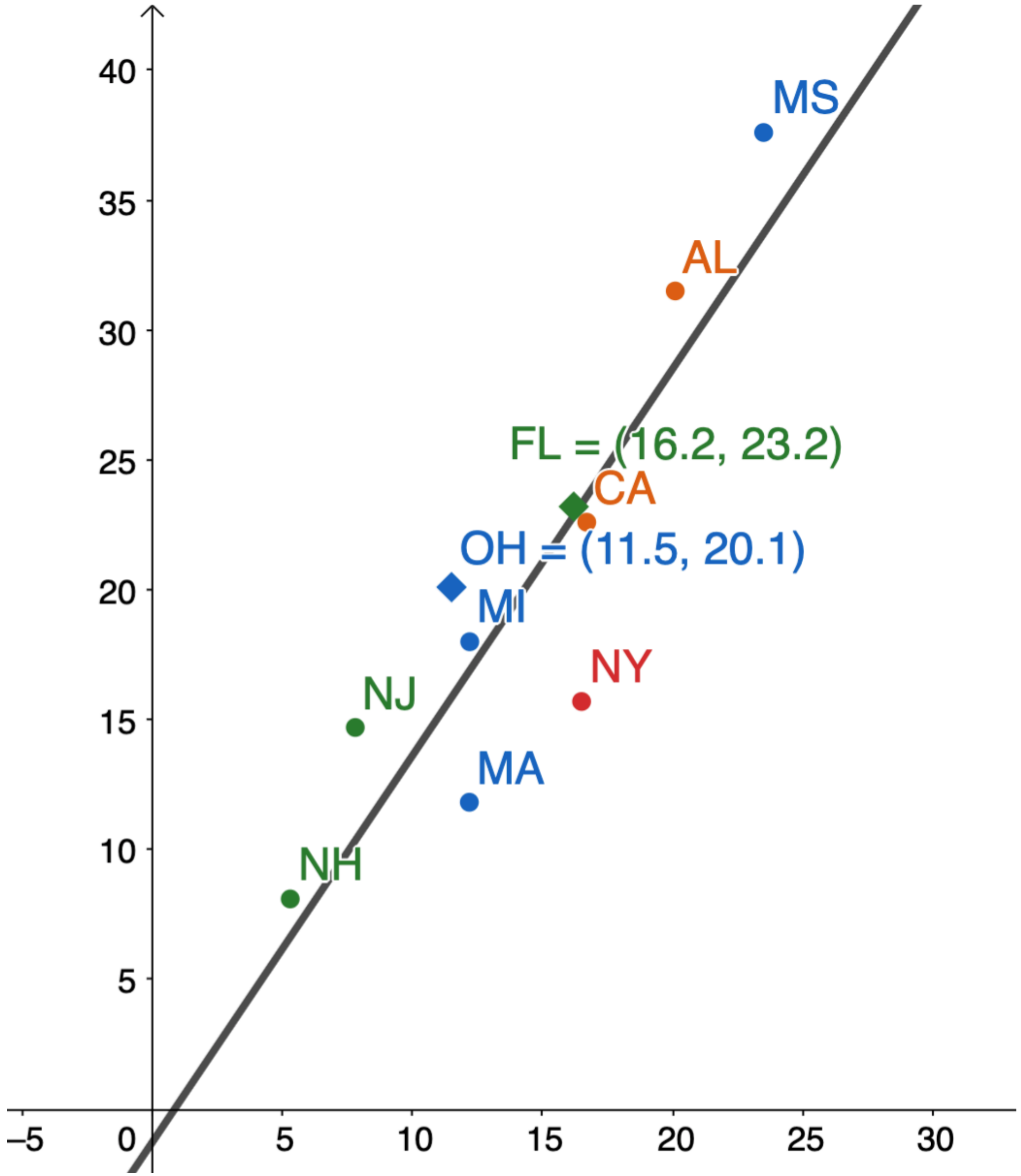
- Poverty rate
  - x
- Birth rate per 1000 females 15 to 17
  - y
- $y = 1.488x - 1.258$

Iowa	12.2	16.4	55.4	1.8	32.5
Kansas	10.8	21.4	74.2	6.2	43
Kentucky	14.7	26.5	84.8	7.2	51
Louisiana	19.7	31.7	96.1	17	58.1
Maine	11.2	11.9	45.2	2	25.4
Maryland	10.1	20	59.6	11.8	35.4
Massachusetts	11	12.5	39.6	3.6	23.3
Michigan	12.2	18	60.8	8.5	34.8
Minnesota	9.2	14.2	47.3	3.9	27.5
Mississippi	23.5	37.6	103.3	12.9	64.7
Missouri	9.4	22.2	76.6	8.8	44.1
Montana	15.3	17.8	63.3	3	36.4
Nebraska	9.6	18.3	64.2	2.9	37
Nevada	11.1	28	96.7	10.7	53.9
New_Hampshire	5.3	8.1	39	1.8	20
New_Jersey	7.8	14.7	46.1	5.1	26.8
New_[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
New_York	16.5	15.7	50.1	8.5	29.5
North_Carolina	12.6	28.6	89.3	9.4	52.2
North_Dakota	12	11.7	48.7	0.9	27.2
Ohio	11.5	20.1	69.4	5.4	39.5
Oklahoma	17.1	30.1	97.6	12.2	58

EXAMPLE ONE: LINEAR REGRESSION (YEAR 2002)

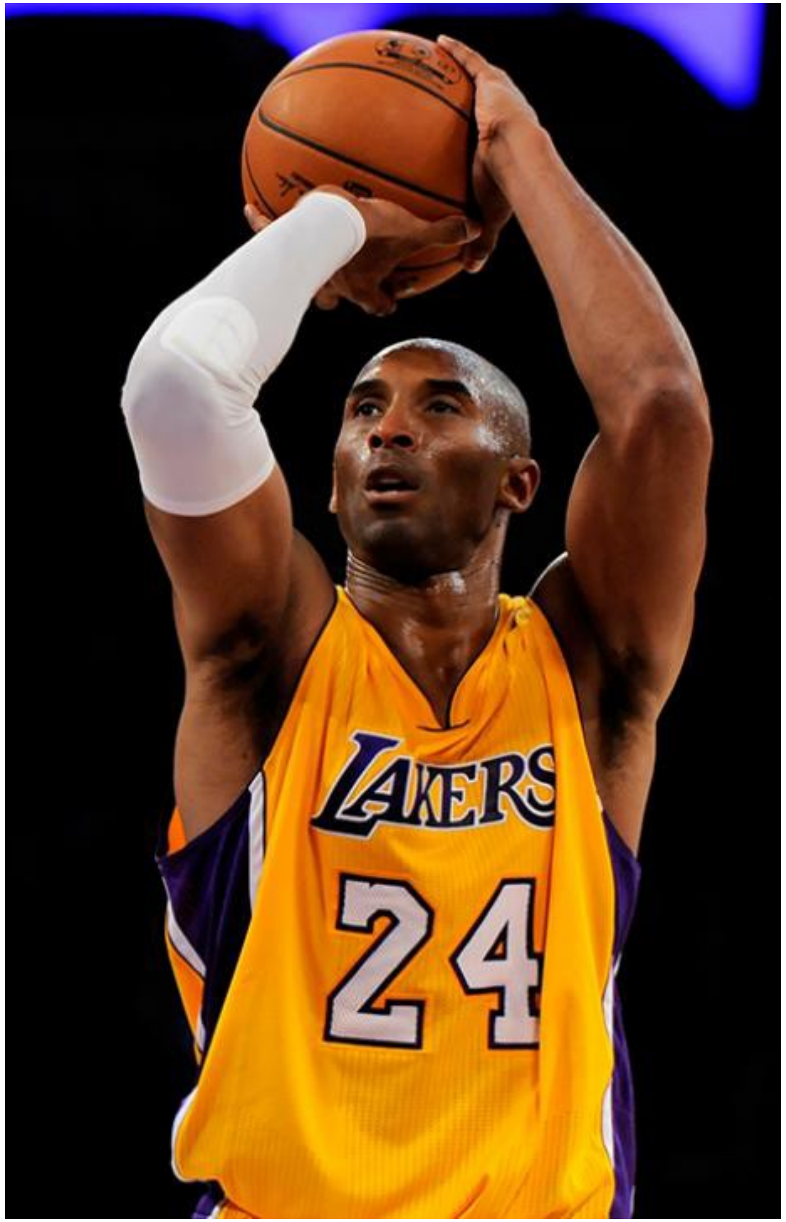
EXAMPLE ONE: LINEAR REGRESSION (YEAR 2002)

- $y = 1.488x - 1.258$
  - Florida $y = 1.488 \cdot 16.2 - 1.258 = 22.8476$
  - Ohio $y = 1.488 \cdot 11.5 - 1.258 = 15.845$



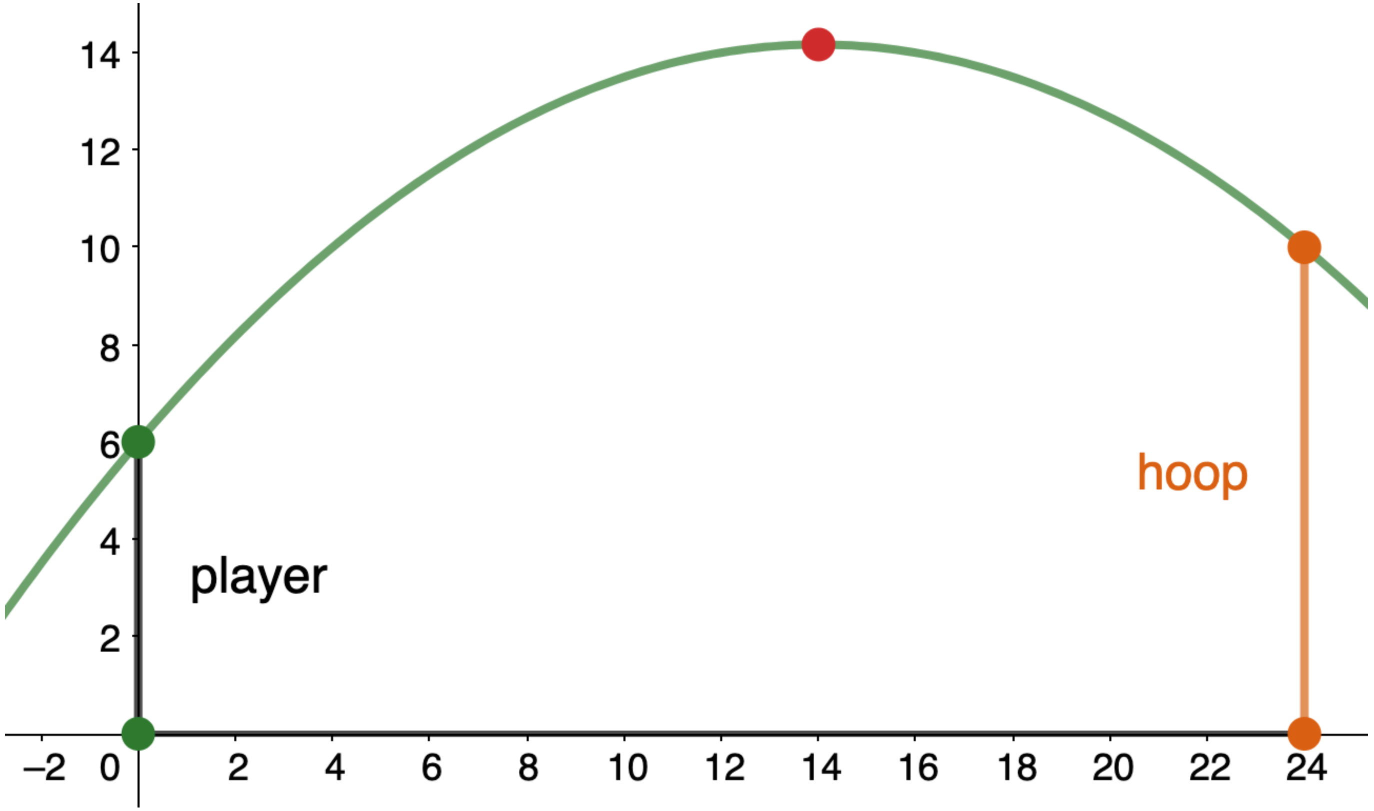
EXAMPLE TWO: BASKETBALL PENALTY SHOT

- The trace of the basketball is **A Parabola!**
- In other words, we can describe its movement using **A Quadratic Function**!
- How can we set up the coordinate?

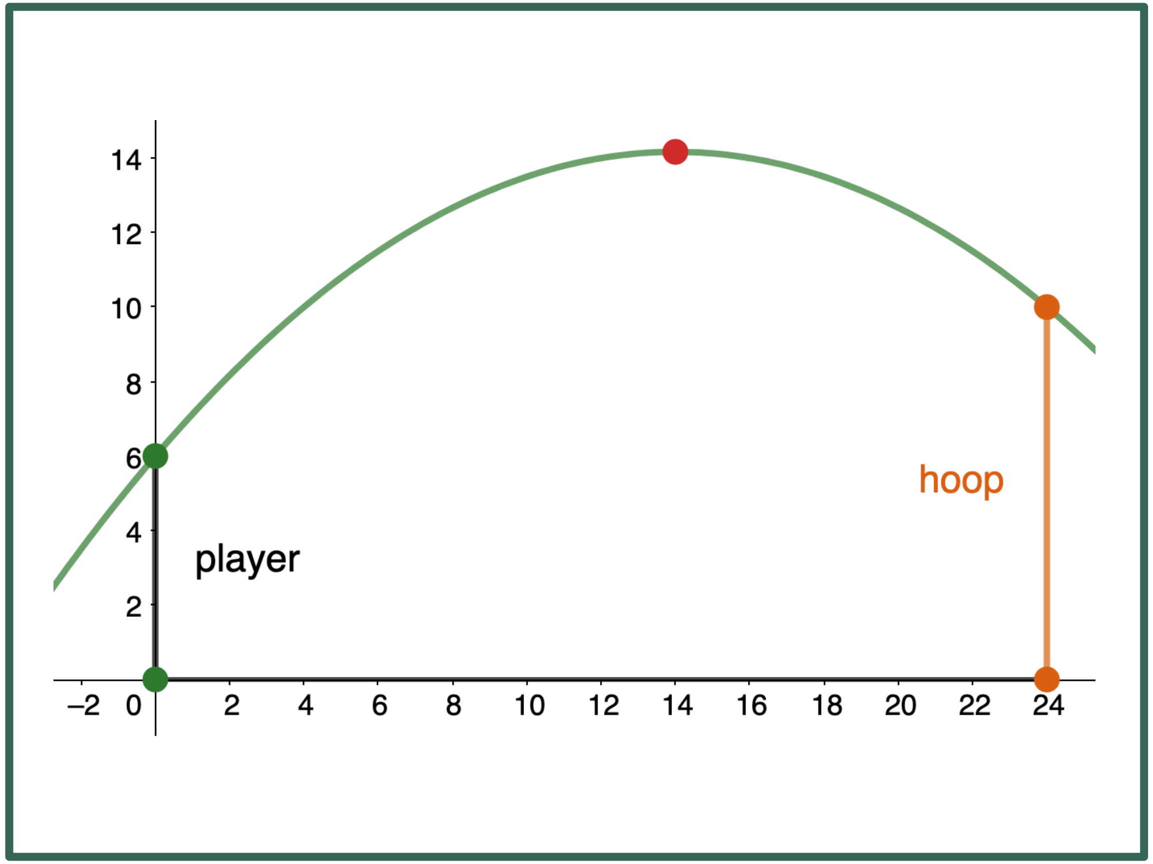


EXAMPLE TWO: BASKETBALL PENALTY SHOT

- Height of a player is 6 feet.
- Height of the basketball hoop is 10 feet.
- Distance from the three-point line to the hoop is 24 feet.



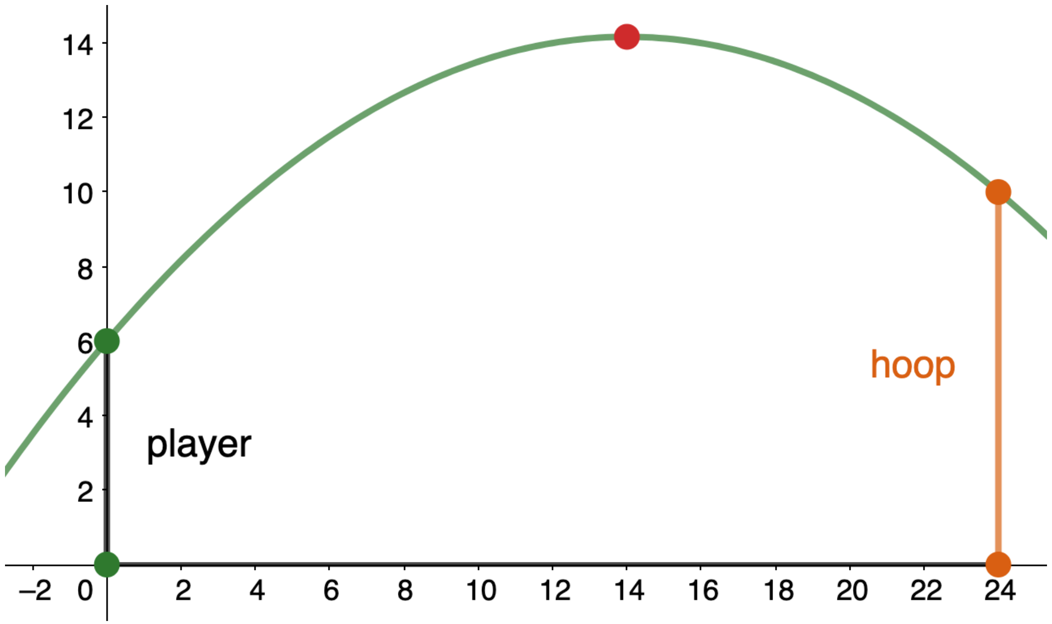
EXAMPLE TWO: BASKETBALL PENALTY SHOT



- The trace of the basketball is $f(x) = -\frac{1}{24}x^2 + \frac{7}{6}x + 6.$

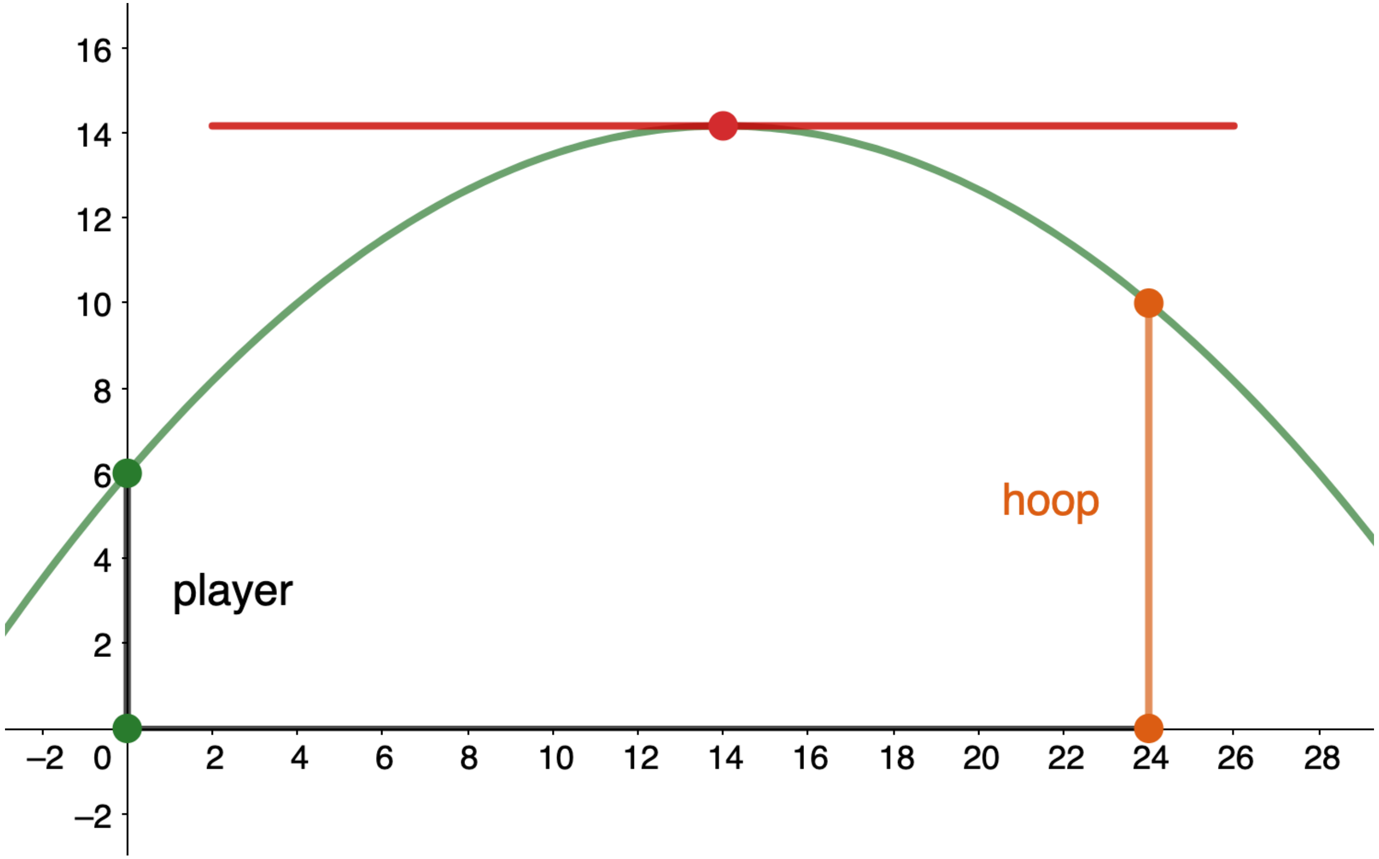
1. Find the y-intercept and the zeros of this function and interpret the meaning of them.
2. Use the graph to determine the value of x that maximizes height.
3. Can you come up with a more mathematical method for the second problem?
   - **Hint: Axis** of **Symmetry** in a **Parabola**

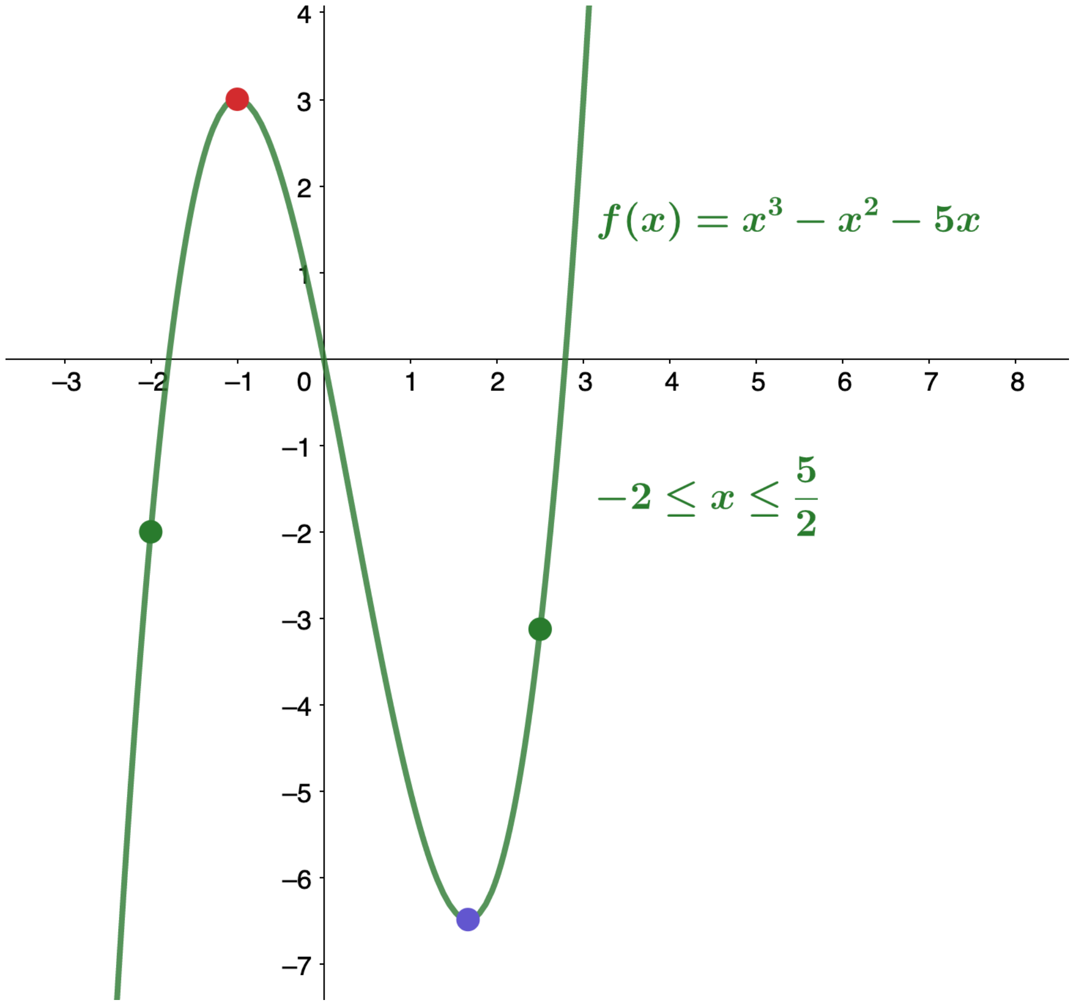
EXAMPLE TWO: BASKETBALL PENALTY SHOT



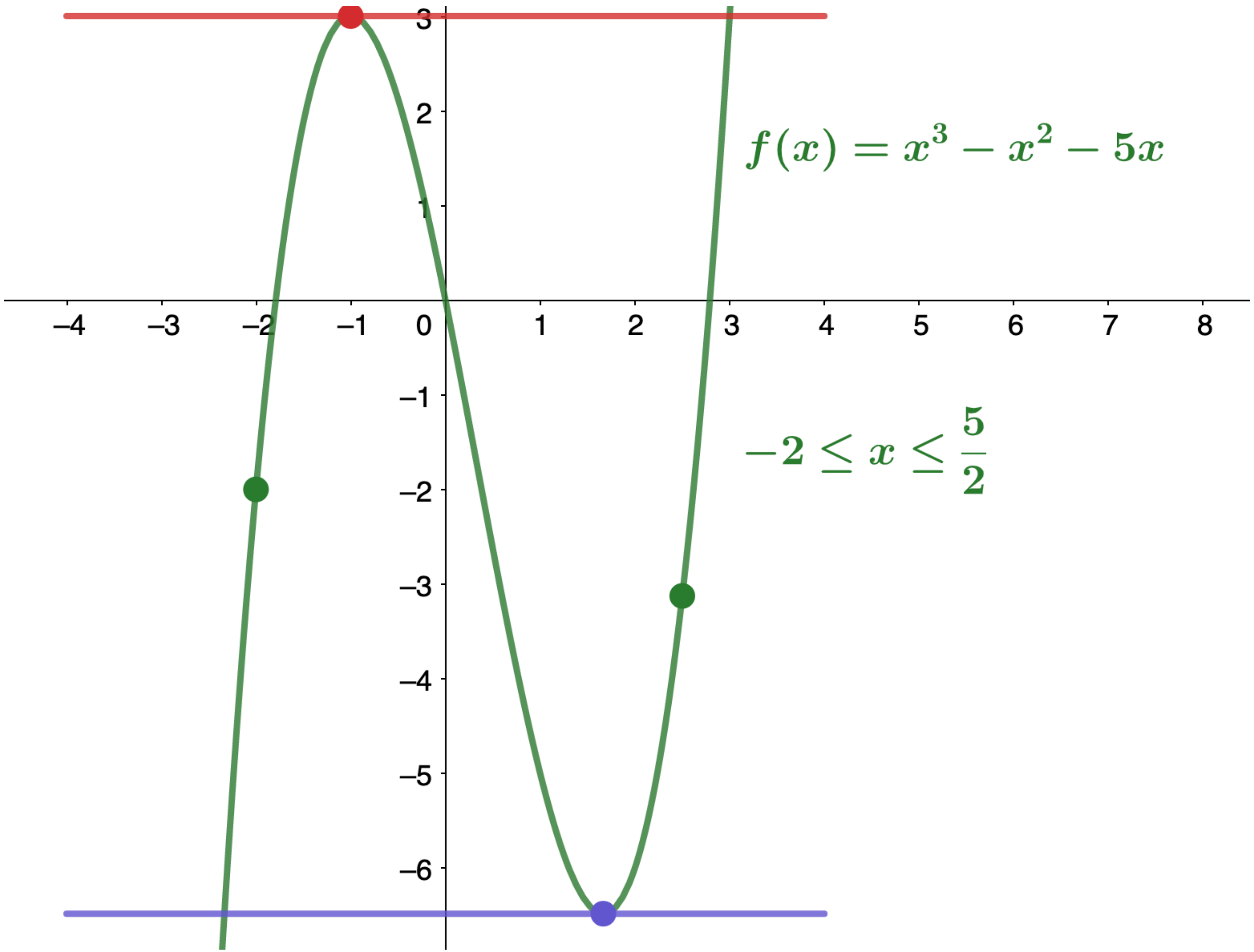
- The trace of the basketball is $f(x) = -\frac{1}{24}x^2 + \frac{7}{6}x + 6.$
- Can you come up with a more mathematical method for the second problem?
  - **Hint: Axis** of **Symmetry** in a **Parabola**
  - **Hint: Tangent line**

EXAMPLE TWO: BASKETBALL PENALTY SHOT





EXAMPLE TWO: EXTENSION



EXAMPLE TWO: EXTENSION

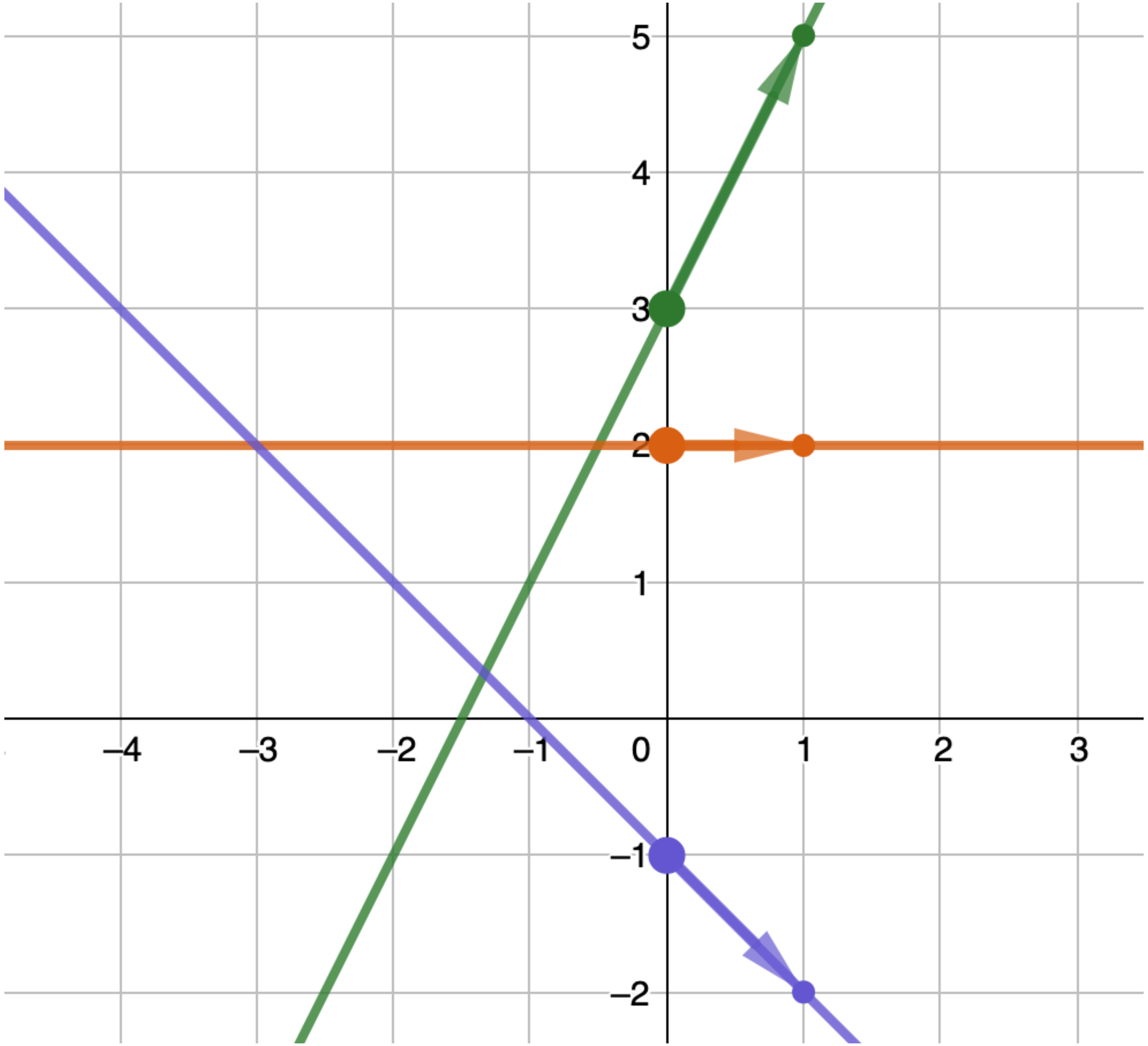
EXAMPLE THREE: RATE OF CHANGE OF PROFIT

- Morano Gelato determines that the daily profit on ice cream obtained by charging s dollars per cup is $P(s) = -50s^2 + 300s - 20$. The shop currently charges $\$4.00$ per cup. Estimate the rate of change of profit when the price is $\$4.00$ and decide whether or not the gelato shop should consider raising or lowering its prices.

EXAMPLE THREE: RATE OF CHANGE OF PROFIT

- The daily profit on ice cream obtained by charging s dollars per cup is $P(s) = -50s^2 + 300s - 20$. The shop currently charges $\$4.00$ per cup.

- $r_{\text{ave}} = \dfrac{P\left(\frac{9}{2}\right) - P(4)}{\frac{9}{2} - 4} = \dfrac{-\frac{4050}{4} + 1350 - 20 - (-800 + 1200 - 20)}{\frac{1}{2}} = -75.$



LINEAR FUNCTION: SLOPE

THE SLOPE IS THE CHANGE OF y FOR EACH UNIT CHANGE IN x.

EXAMPLE THREE: RATE OF CHANGE OF PROFIT

- $P(s) = -50s^2 + 300s - 20.$
- $r_{\text{ave}} = \dfrac{P\left(\frac{9}{2}\right) - P(4)}{\frac{9}{2} - 4} = \dfrac{-\frac{4050}{4} + 1350 - 20 - (-800 + 1200 - 20)}{\frac{1}{2}} = -75.$
- $P(4) = \$380$
- $P(4.5) = \$317.5$
- $P(3.5) = \$417.5$

OTHER MODELS AND APPLICATIONS

- Recall what we discussed on the library of functions last class!