Problem 1.

Find the limit of each of the following sequences, if it exists. If the limit does not exist, say if the sequence diverges to infinity, negative infinity, or neither.

(a) For $a_n = \frac{-2}{n}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-2) \cdot \lim_{n\to\infty}(\frac{1}{n}) = -2 \cdot 0 = 0$.

(b) For $a_n = \frac{(-1)^n}{n^2+2}$, we see the first few terms are $-1/3, 1/6, -1/11, 1/18, -1/27, 1/38, \ldots$. We can see that these terms approach zero, so $\lim_{n\to\infty}(-1)^n/(n^2+2) = 0$.

(c) For $a_n = cos(n^4)$, we recall that cos is a trigonometric wave, so plugging in integer values does not approach any single value. Thus $\lim_{n\to\infty} \cos(n^4)$ does not exist.

(d) For $a_n = -1 + (\frac{1}{3})^n$, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1) + \lim_{n\to\infty}(\frac{1}{3})^n = -1 + 0 = -1$.

(e) Factor out n^5 from the top and the bottom of $a_n = \frac{n^5+n^4+n^3+n^2+n+1}{n^5+n^3+1}$ to get $a_n = \frac{1+1/n+1/n^2+1/n^3+1/n^4+1/n^5}{1+1/n^2+1/n^5}$, so

$$\begin{aligned}\lim_{n\to\infty} a_n &= \frac{\lim_{\mathrm{n}\to\infty} 1 + \lim_{\mathrm{n}\to\infty} 1/n + \lim_{\mathrm{n}\to\infty} 1/n^2 + \lim_{\mathrm{n}\to\infty} 1/n^3 + \lim_{\mathrm{n}\to\infty} 1/n^4 + \lim_{\mathrm{n}\to\infty} 1/n^5}{\lim_{\mathrm{n}\to\infty} 1 + \lim_{\mathrm{n}\to\infty} 1/n^2 + \lim_{\mathrm{n}\to\infty} 1/n^5} \\ &= \frac{1+0+0+0+0+0}{1+0+0} = 1\end{aligned}$$

(f) For $a_n = 5 + n/5$, we see that this sequence is arithmetic, meaning it diverges to either infinity or negative infinity. Since each term is larger than the term before, it diverges to infinity.

Problem 2.

Use the Squeeze Theorem to evaluate the following limits.

(a) We see since sin is bounded below by -1 and above by 1 that if $b_n = \frac{-1}{n^3}$ and $c_n = \frac{1}{n^3}$ then $b_n \le a_n \le c_n$, and $\lim_{\mathrm{n}\to\infty} b_n = 0 = \lim_{\mathrm{n}\to\infty} c_n$, so by the Squeeze Theorem, $\lim_{\mathrm{n}\to\infty} a_n = 0$.

(b) Again $(-1)^n$ is bounded below by -1 and above by 1, so if $b_n = \frac{-11}{n!}$ and $c_n = \frac{1}{n!}$, then $b_n \le a_n \le c_n$ and $\lim_{\mathrm{n}\to\infty} b_n = 0 = \lim_{\mathrm{n}\to\infty} c_n$, so by the Squeeze Theorem, $\lim_{\mathrm{n}\to\infty} a_n = 0$.