INTRODUCTION TO CALCULUS

BASIC RULES FOR DERIVATIVES

OUTLINE

State	State the constant, constant multiple, and **power rules**.
Apply	Apply **the sum and difference rules** to combine derivatives.
Use	Use **the product rule** for finding the derivative of a product of functions.
Use	Use **the quotient rule** for finding the derivative of a quotient of functions.
Extend	Extend the power rule to functions with negative exponents.
Combine	Combine the differentiation rules to find **the derivative of a polynomial or rational function**.

FROM ARMLESS STATE TO BRUTAL FORCE

THE BASIC RULES

Constant function

- $f(x) = c$

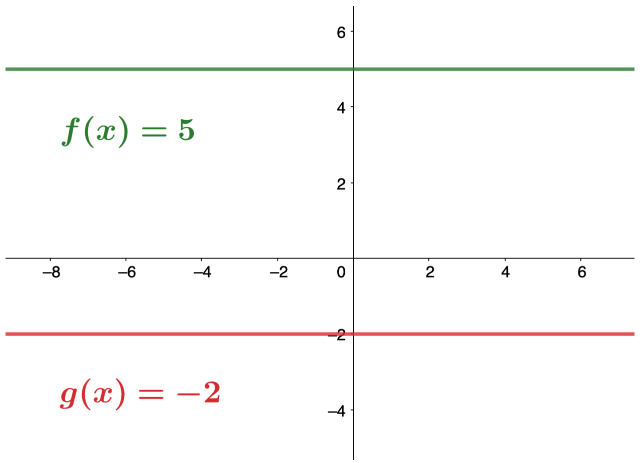
Power function

- $g(x) = x^n$

They are the building blocks from which all polynomials and rational functions are constructed.

To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

THE CONSTANT RULE



$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{c-c}{h} \\
&= \lim_{h\to 0}\frac{0}{h} \\
&= \lim_{h\to 0} 0 = 0.
\end{aligned}
$$

- The derivative of a constant function is zero.
- Since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0.

THEOREM 3.2

The Constant Rule

Let c be a constant.

If $f(x) = c$, then $f'(c) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

THE CONSTANT RULE

APPLYING THE CONSTANT RULE

- Find the derivative of $f(x) = 2019$.

THE POWER RULE

Positive integer powers → Negative integer powers → Rational powers → arbitrary powers

$$\frac{d}{dx}\left(x^2\right) = 2x \text{ and } \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

POSITIVE INTEGER POWERS

- $\frac{d}{dx}(x^3) =?$

Use the limit definition of derivative.

- $\frac{d}{dx}(x^3) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} =$
  $\lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} =$
  $\lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2.$

POSITIVE INTEGER POWERS

- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^2) = 2x$
- $\frac{d}{dx}(x^3) = 3x^2$
- $\dots$

$$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}$$

POSITIVE INTEGER POWERS

THEOREM 3.3

The Power Rule

Let n be a positive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}x^n = nx^{n-1}.$$

PROOF

- Still we start from the limit definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}.$$

According to the theorem, it is possible to expand any power of $x + y$ into a sum of the form

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

PROOF

HINT: BINOMIAL THEOREM

EXERCISE

- Find the derivative of the function $f(x) = x^{2019}$ by applying the power rule.

THE SUM, DIFFERENCE, AND CONSTANT MULTIPLE RULES

Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) + g(x), j'(x) = f'(x) + g'(x).$$

THE SUM, DIFFERENCE, AND CONSTANT MULTIPLE RULES

Difference Rule. The derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of g:

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) - g(x), j'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant *k* multiplied by a function *f* is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x));$$

that is,

$$\text{for } j(x) = kf(x), j'(x) = kf'(x).$$

THE SUM, DIFFERENCE, AND CONSTANT MULTIPLE RULES

PROOF

For differentiable functions $f(x)$ and $g(x)$, we set $j(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$j'(x) = \lim_{h \to 0} \frac{j(x+h) - j(x)}{h}.$$

EXERCISE ONE

Find the derivatives of the following functions.

- $f(x) = x^2 + 2019x + 1984$
- $g(x) = 2019x^2$
- $h(x) = x^3 - 1984x^2 + 2019$

EXERCISE TWO

Find the equation of the line tangent to the graph of $f(x) = -2x^2 + 2x + 1$ at $x = 2$. Use the point slope form.