INTRODUCTION TO CALCULUS

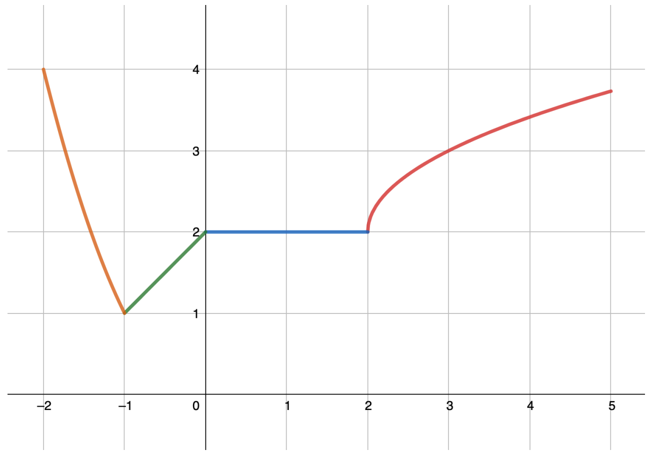
CONTINUITY

OUTLINE

Explain	Explain the **three conditions** for continuity at a point.
Describe	Describe **three kinds of discontinuities**.
Define	Define continuity on an interval.
State	State the theorem for **limits of composite functions**.

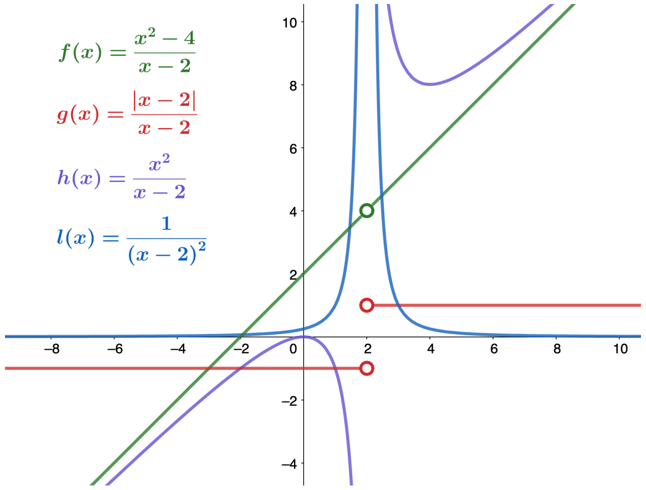
WHAT IS CONTINUITY?

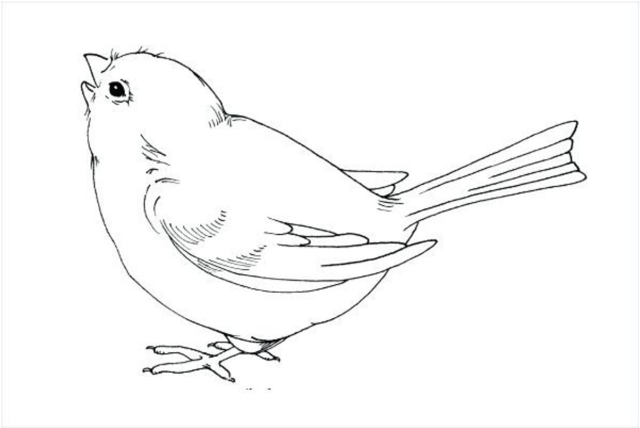
- Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called ***continuous***.



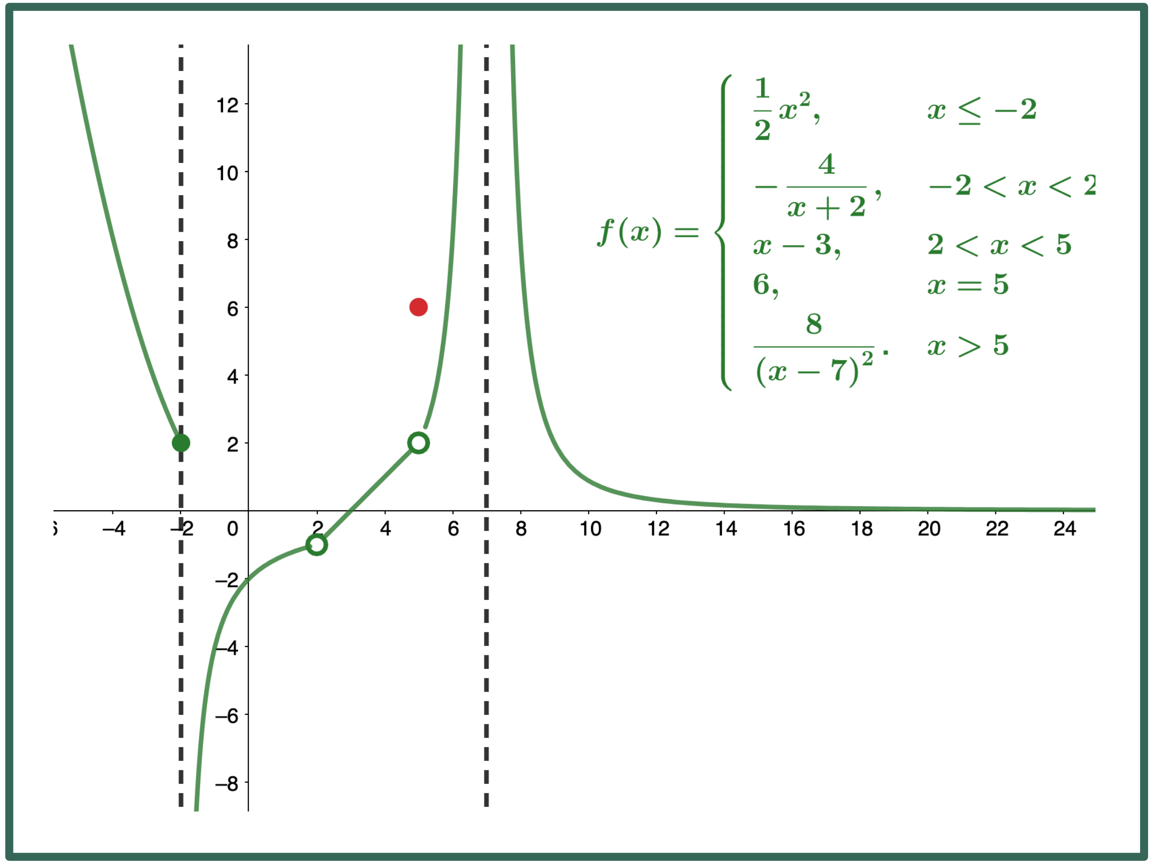
WHAT IS DISCONTINUITY?

- Other functions have points at which **a break** in the graph occurs, but satisfy this property over intervals contained in their domains.
- They are continuous on these intervals and are said to have a ***discontinuity*** *at a point* where a break occurs.

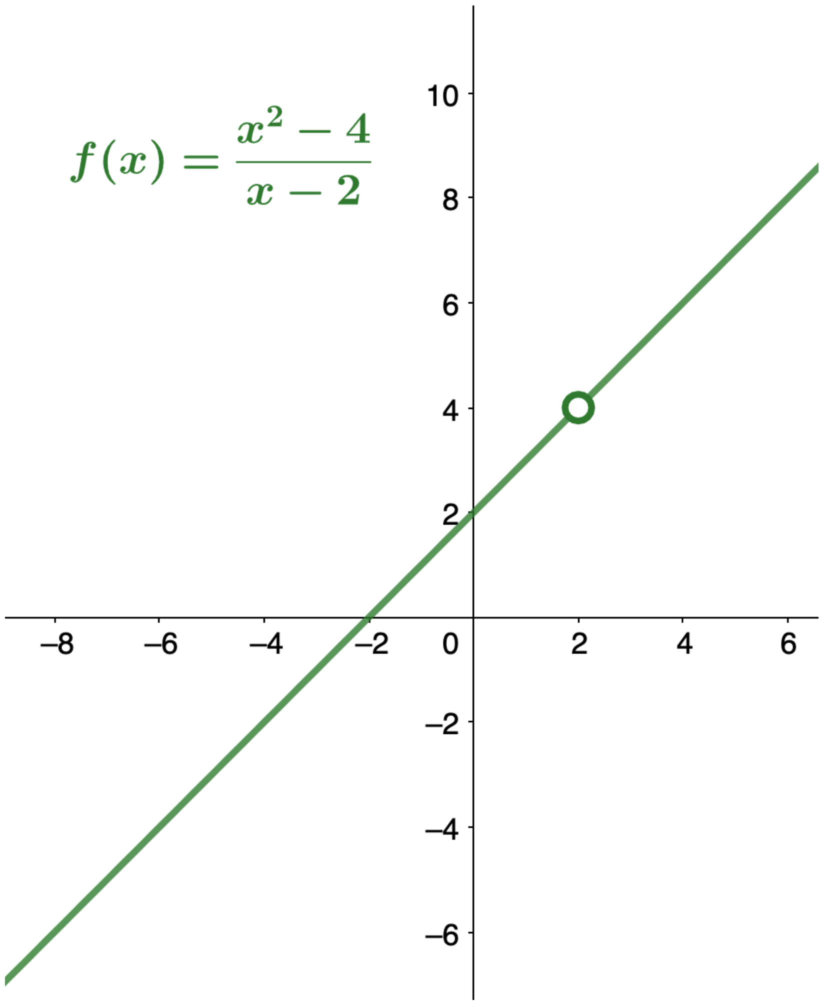




CONTINUITY AT A POINT



- Before we look at a formal definition of what it means for a function to be continuous at a point, let’s consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point.

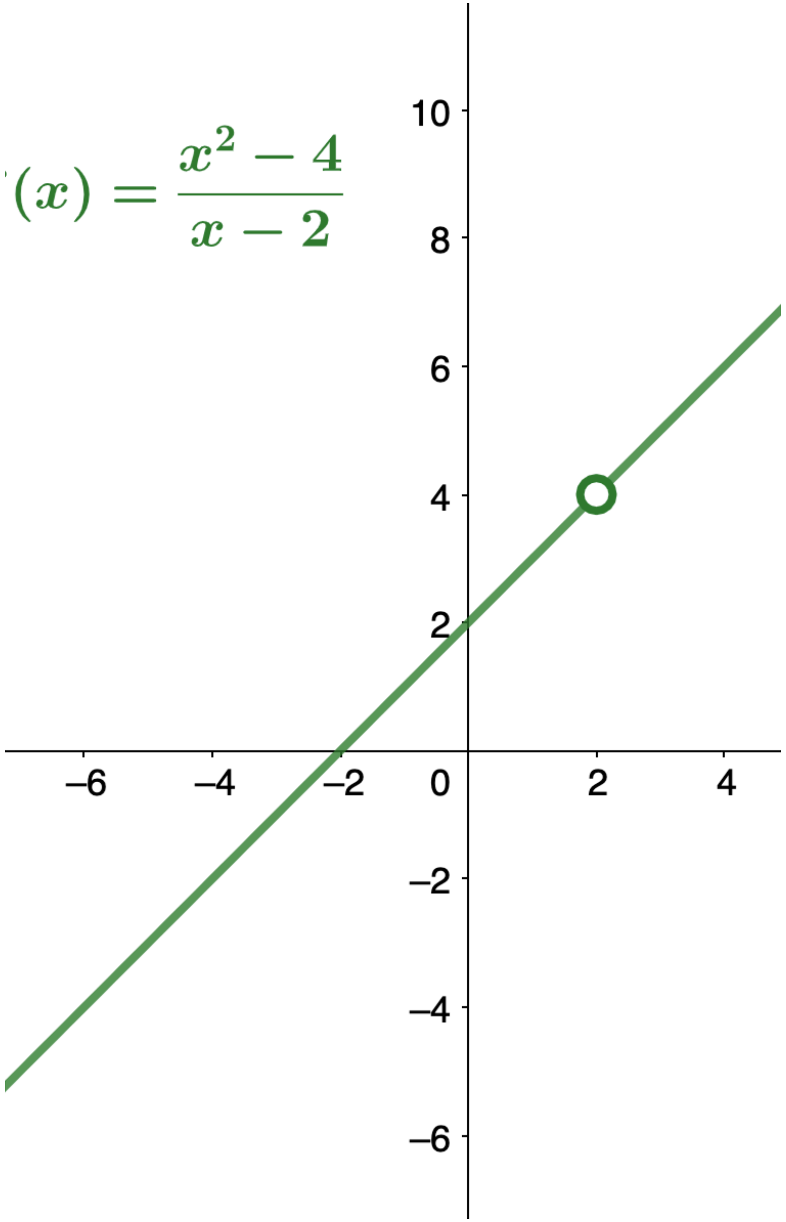


CONTINUITY AT A POINT: 1ST REQUIREMENT

CONTINUITY AT A POINT: 1ST REQUIREMENT

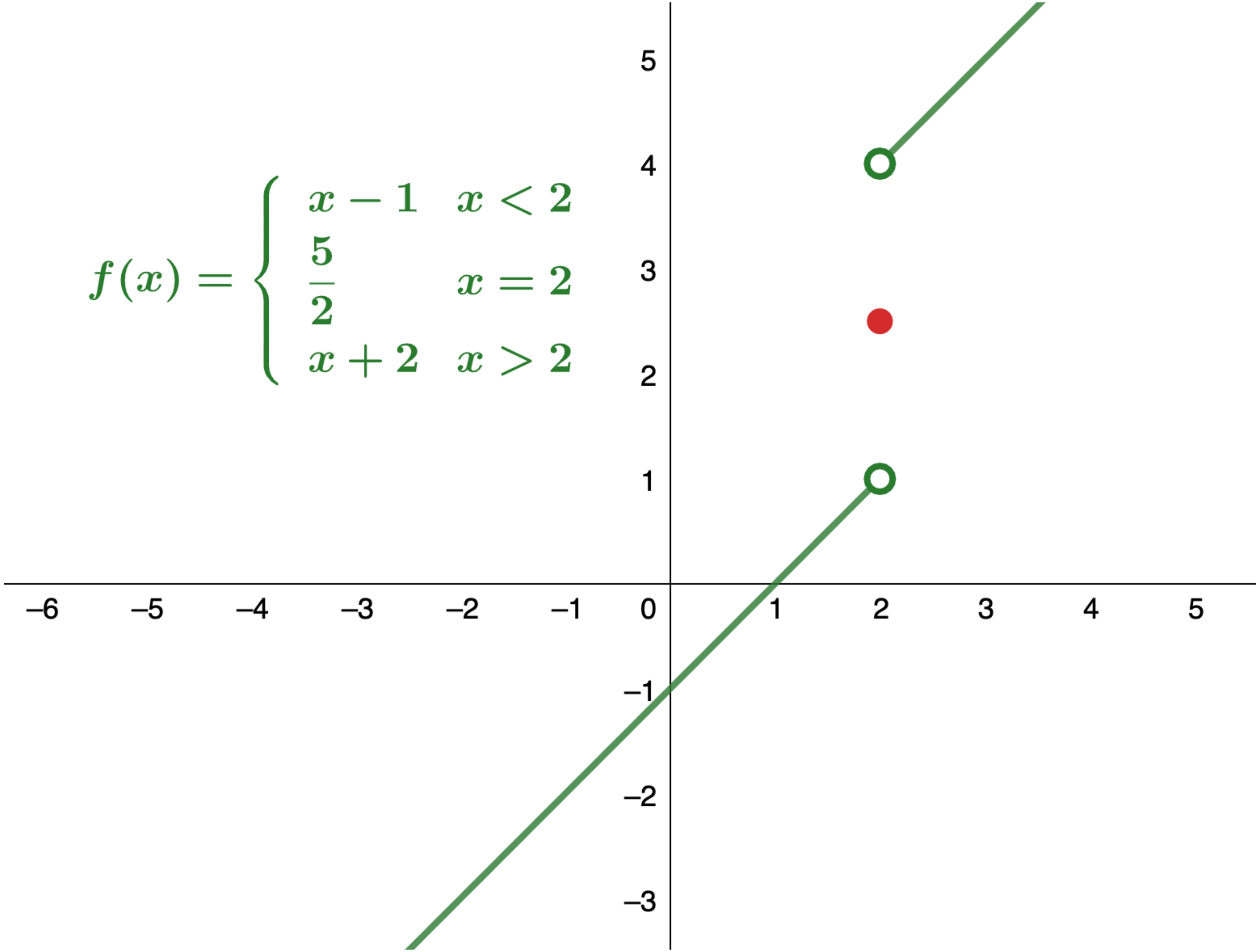
At the very least, for $f(x)$ to be continuous at a, we need the following condition:

- **$f(a)$ is defined.**

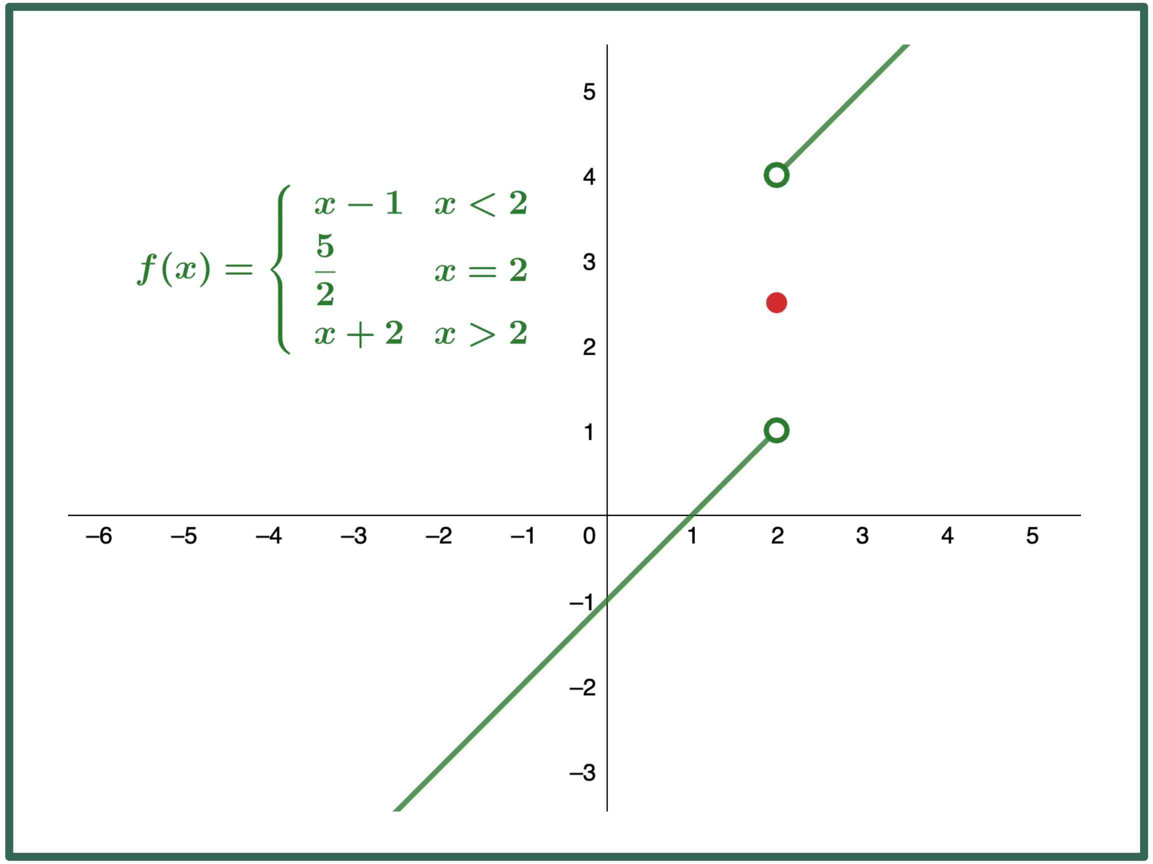


CONTINUITY AT A POINT: 2^{ND} REQUIREMENT

$$f(x) = \begin{cases} x - 1 & x < 2 \\ \frac{5}{2} & x = 2 \\ x + 2 & x > 2 \end{cases}$$



CONTINUITY AT A POINT: 2ND REQUIREMENT

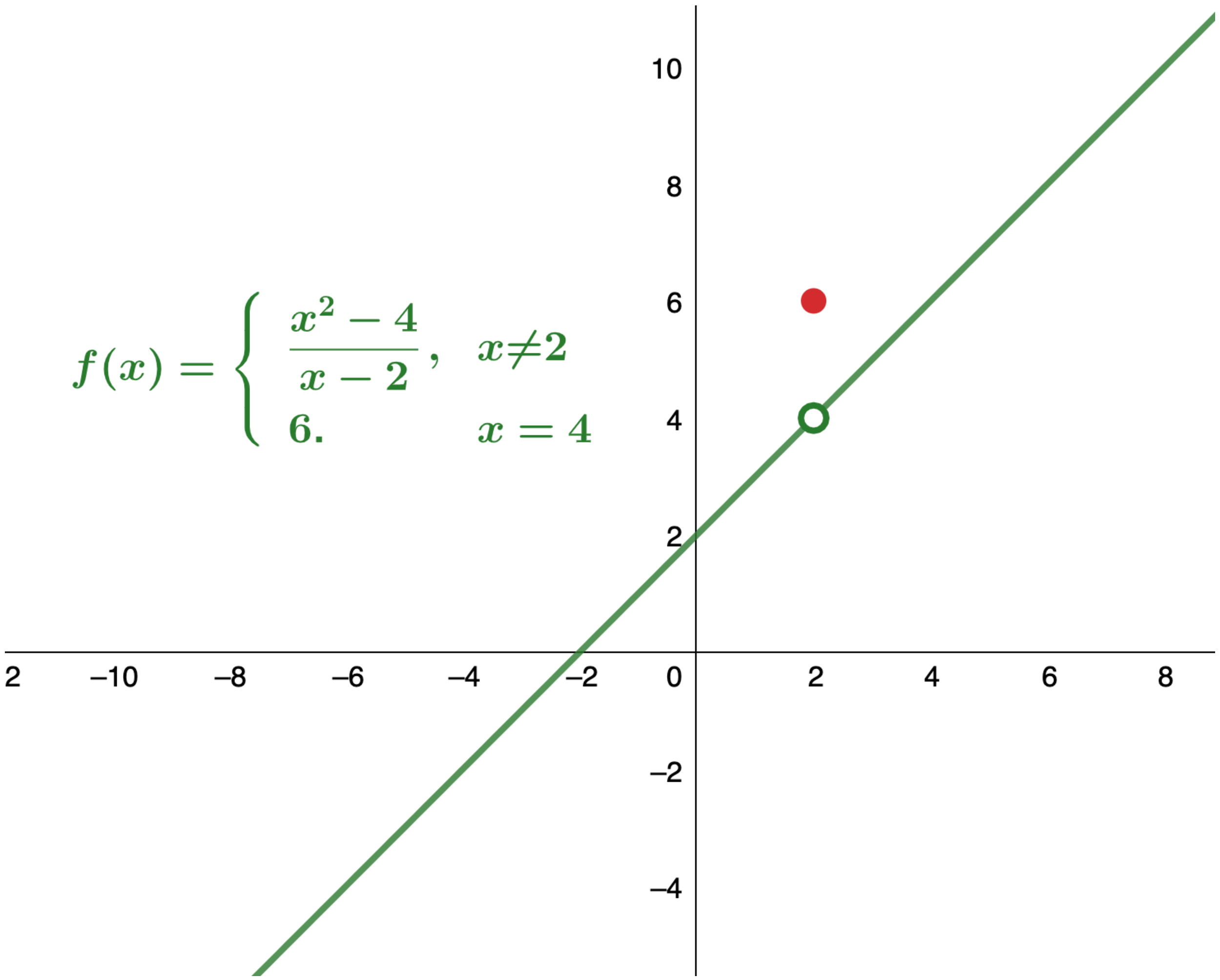


- Although $f(a)$ is defined, the function has a gap at a.
- In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist.

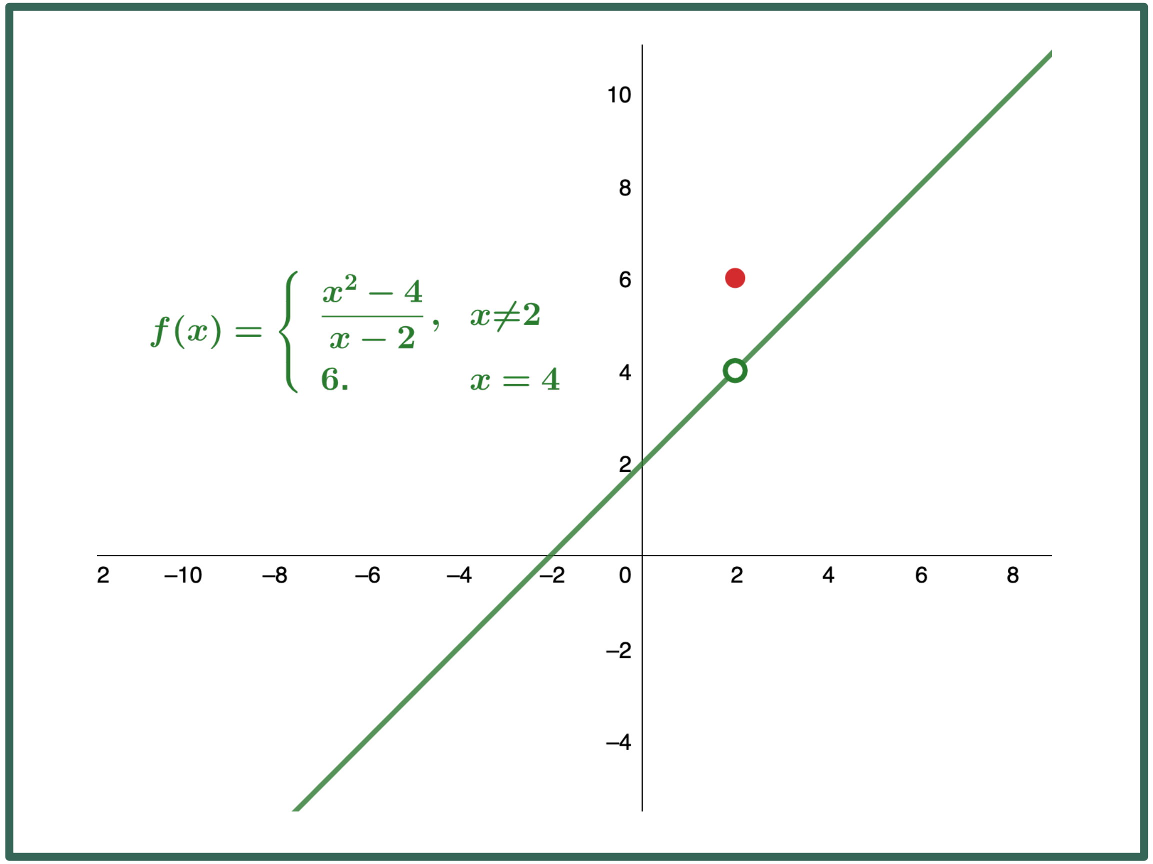
Therefore, for $f(x)$ to be continuous at a, we also need the following condition:

- $\lim_{x \to a} f(x)$ **exist.**

CONTINUITY AT A POINT: 3^{RD} REQUIREMENT



CONTINUITY AT A POINT: 3RD REQUIREMENT



- The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:
- $\lim_{x \to a} f(x) = f(a).$

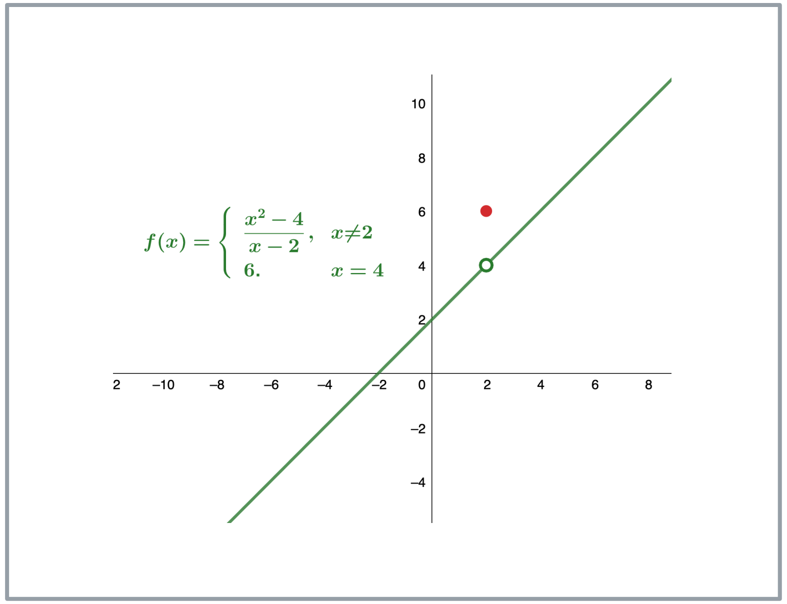
CONTINUITY AT A POINT

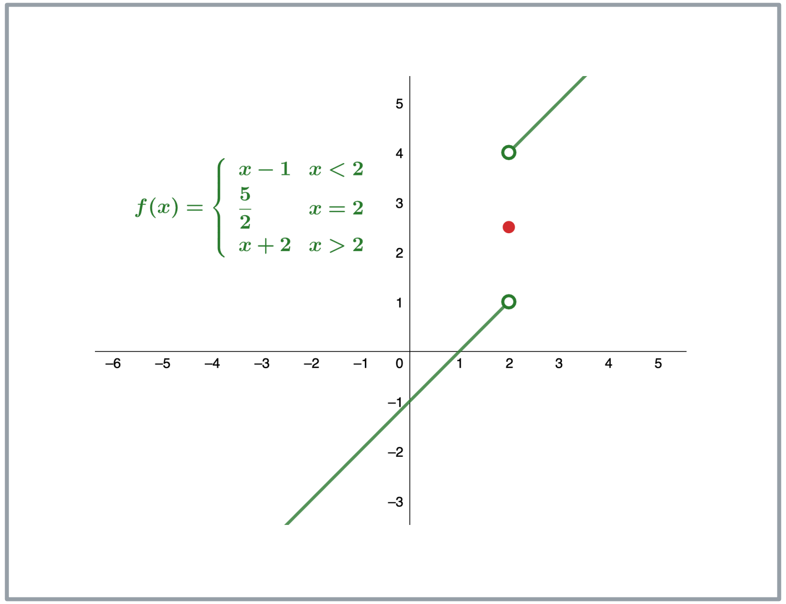
DEFINITION

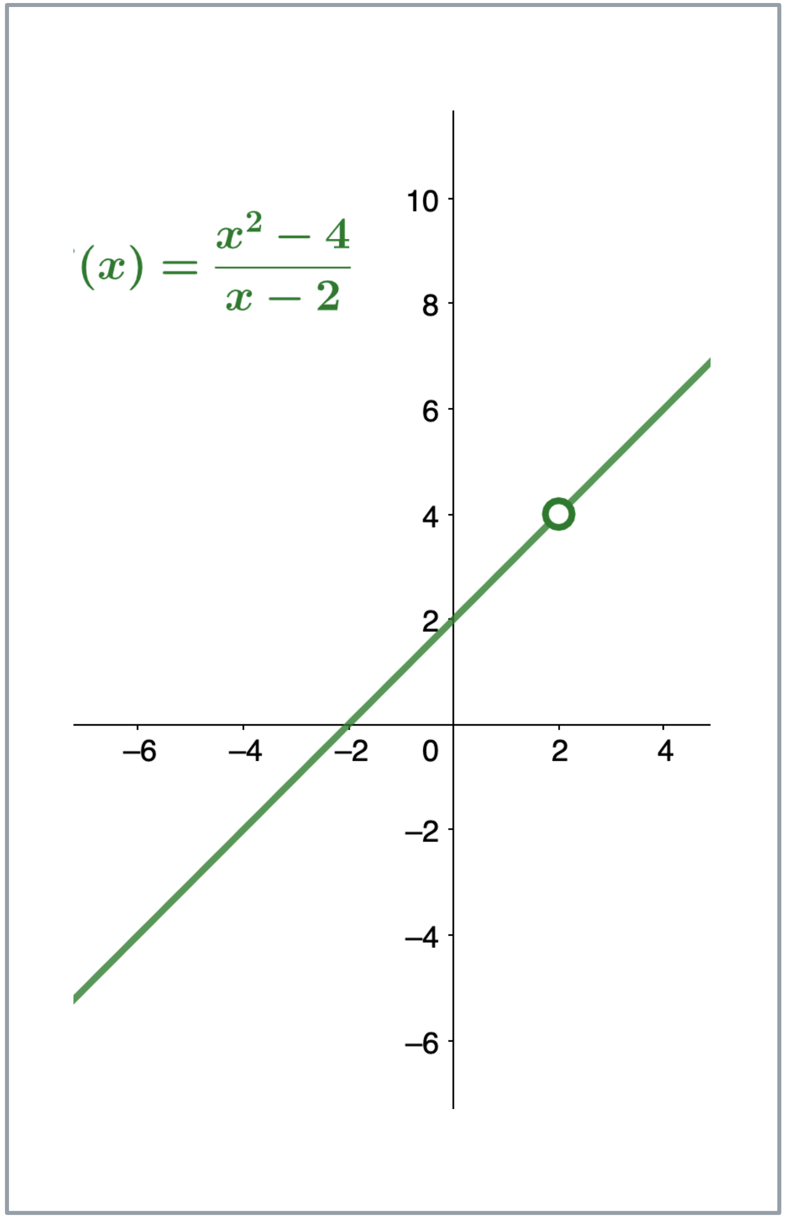
A function $f(x)$ is **continuous at a point** *a* if and only if the following three conditions are satisfied:

i. $f(a)$ is defined
ii. $\lim_{x \to a} f(x)$ exists
iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** *a* if it fails to be continuous at *a*.







CONTINUITY AT A POINT

- $f(a)$ is defined.
- $\lim_{x \to a} f(x)$ exist.
- $\lim_{x \to a} f(x) = f(a)$.

CONTINUITY AT A POINT

- $f(a)$ is defined.
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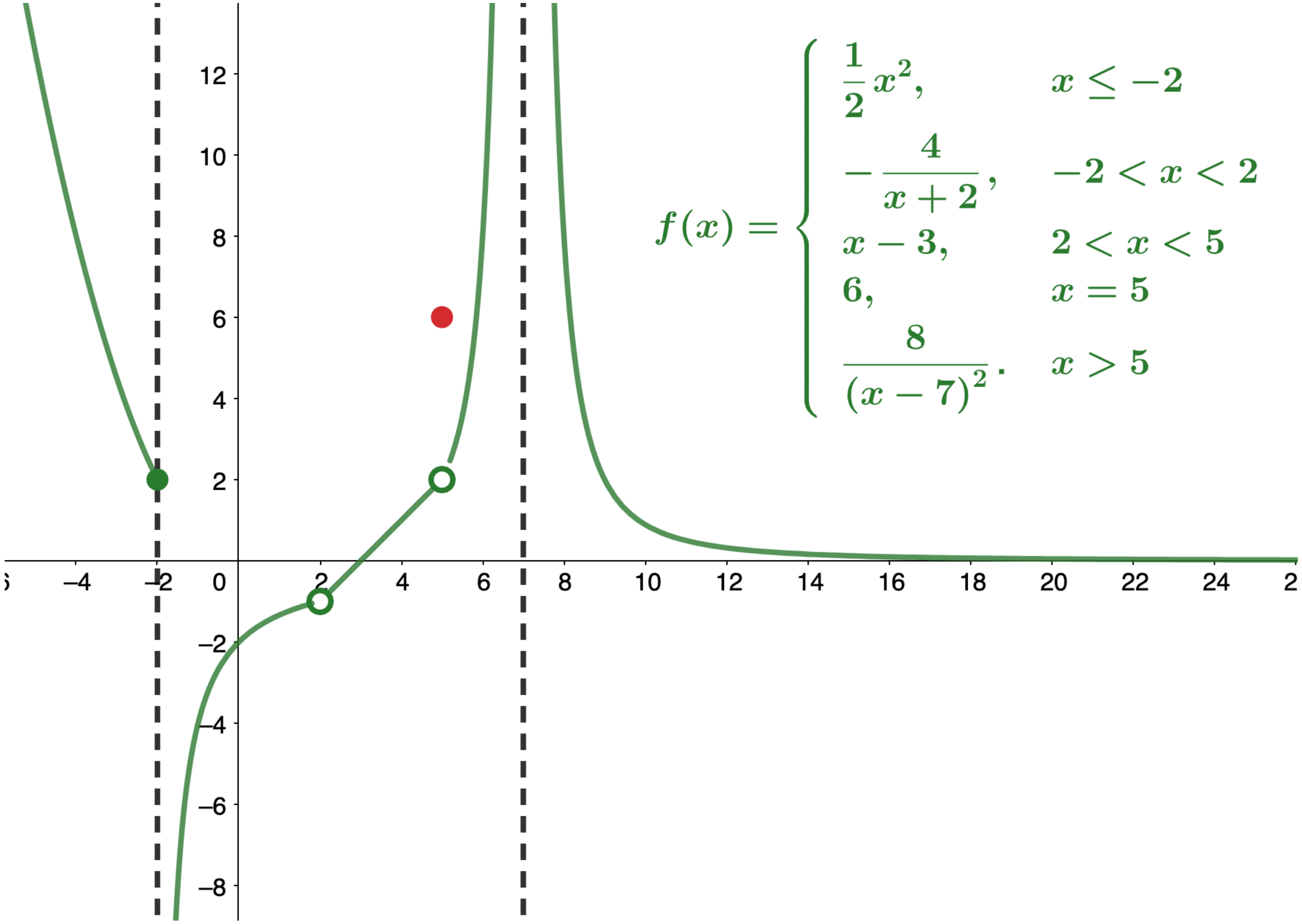
- $\lim_{x \to a} f(x) = f(a)$.

PROBLEM-SOLVING STRATEGY: DETERMINING CONTINUITY AT A POINT

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at *a*. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at *a* and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at *a*. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at *a*.



$$f(x) = \begin{cases} \frac{1}{2}x^2, & x \le -2 \\ -\frac{4}{x+2}, & -2 < x < 2 \\ x-3, & 2 < x < 5 \\ 6, & x = 5 \\ \frac{8}{(x-7)^2}. & x > 5 \end{cases}$$

EXERCISE

- $f(a)$ is defined.
- $\lim_{x \to a} f(x)$ exist.
- $\lim_{x \to a} f(x) = f(a)$.

WHEN IT COMES TO POLYNOMIALS AND RATIONAL FUNCTIONS

THEOREM 2.8

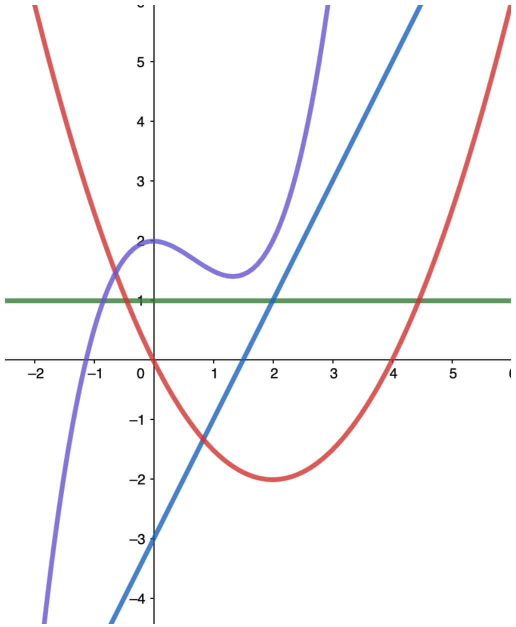
Continuity of Polynomials and Rational Functions

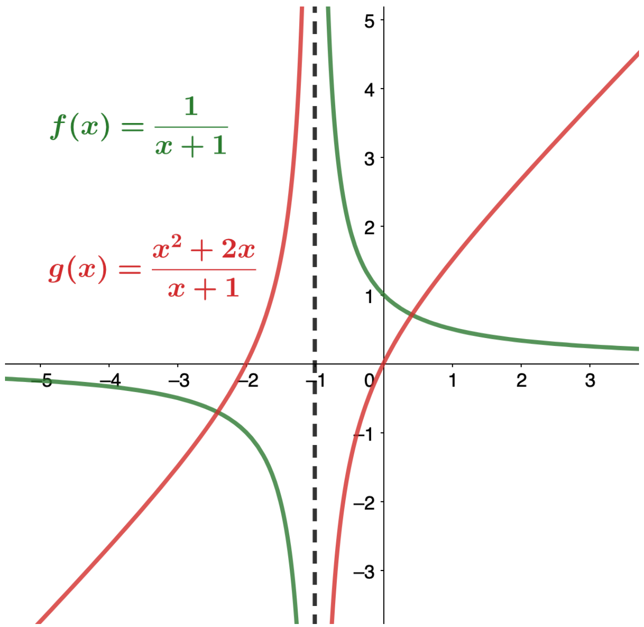
Polynomials and rational functions are continuous at every point in their domains.

WHEN IT COMES TO POLYNOMIALS AND RATIONAL FUNCTIONS

WHY?

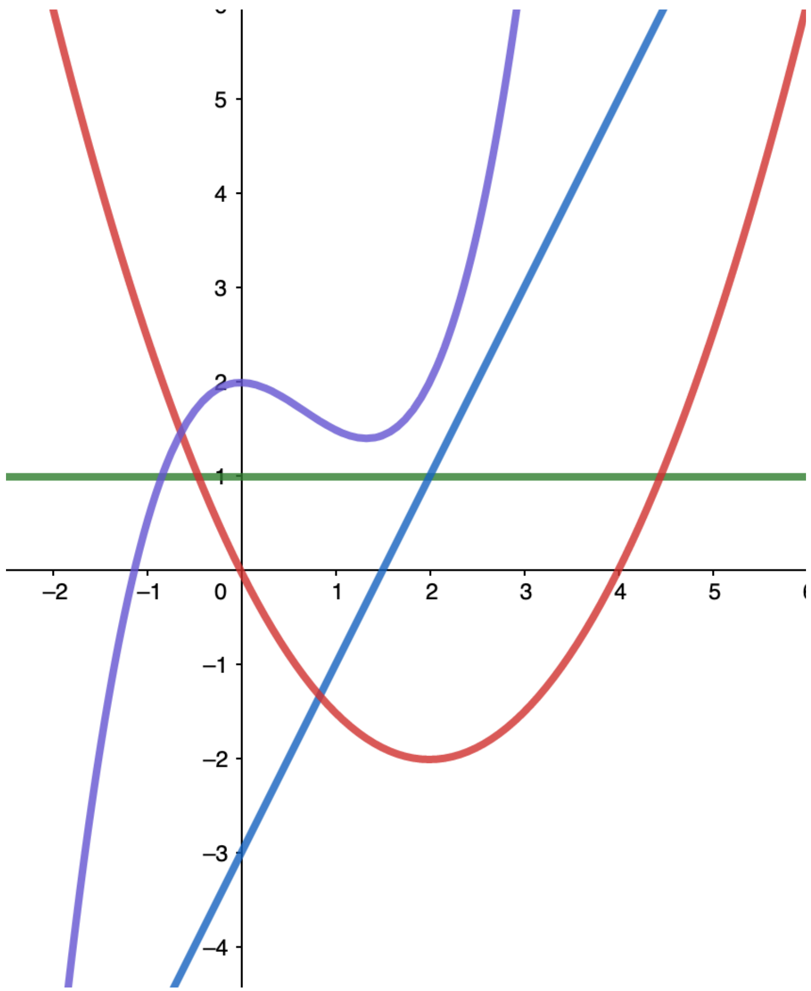
Recall what we have learnt about their limits.





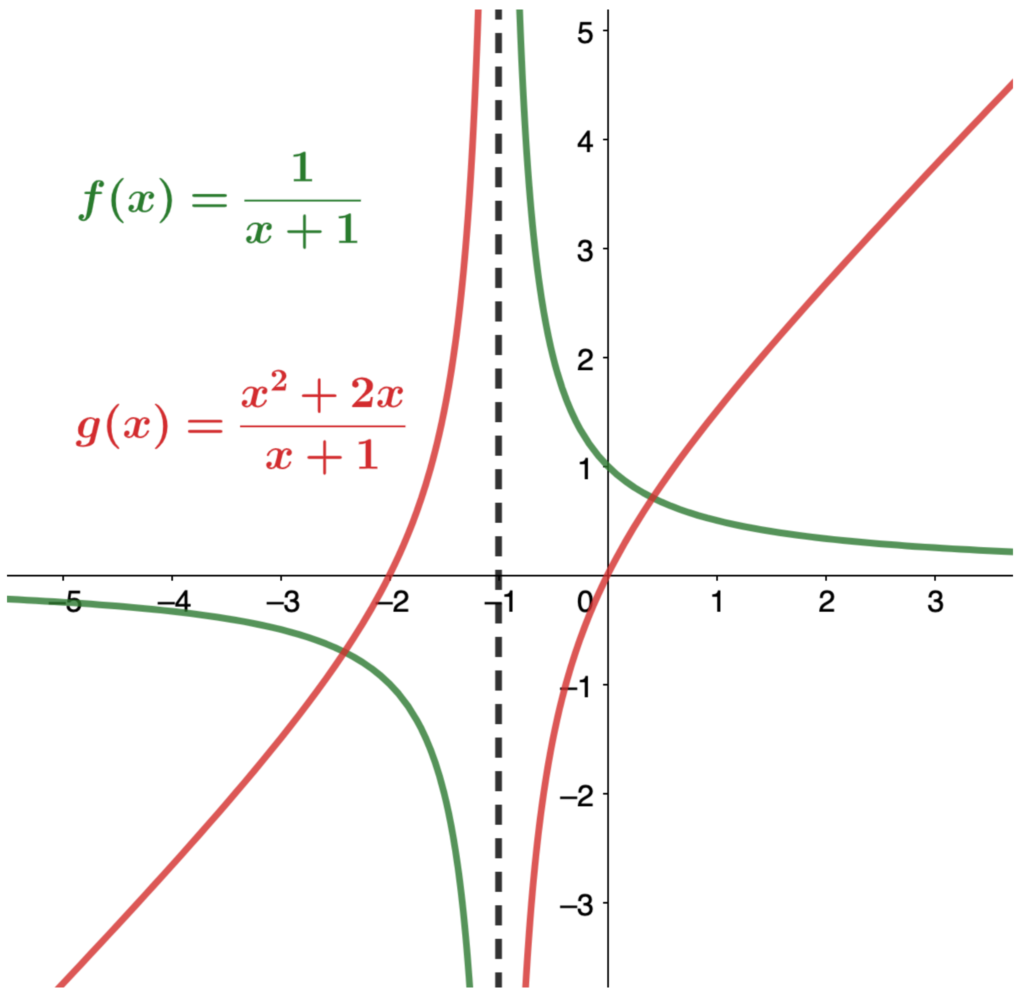
CONTINUITY OF A POLYNOMIAL FUNCTION

- For what values of x is $f(x)$ continuous?



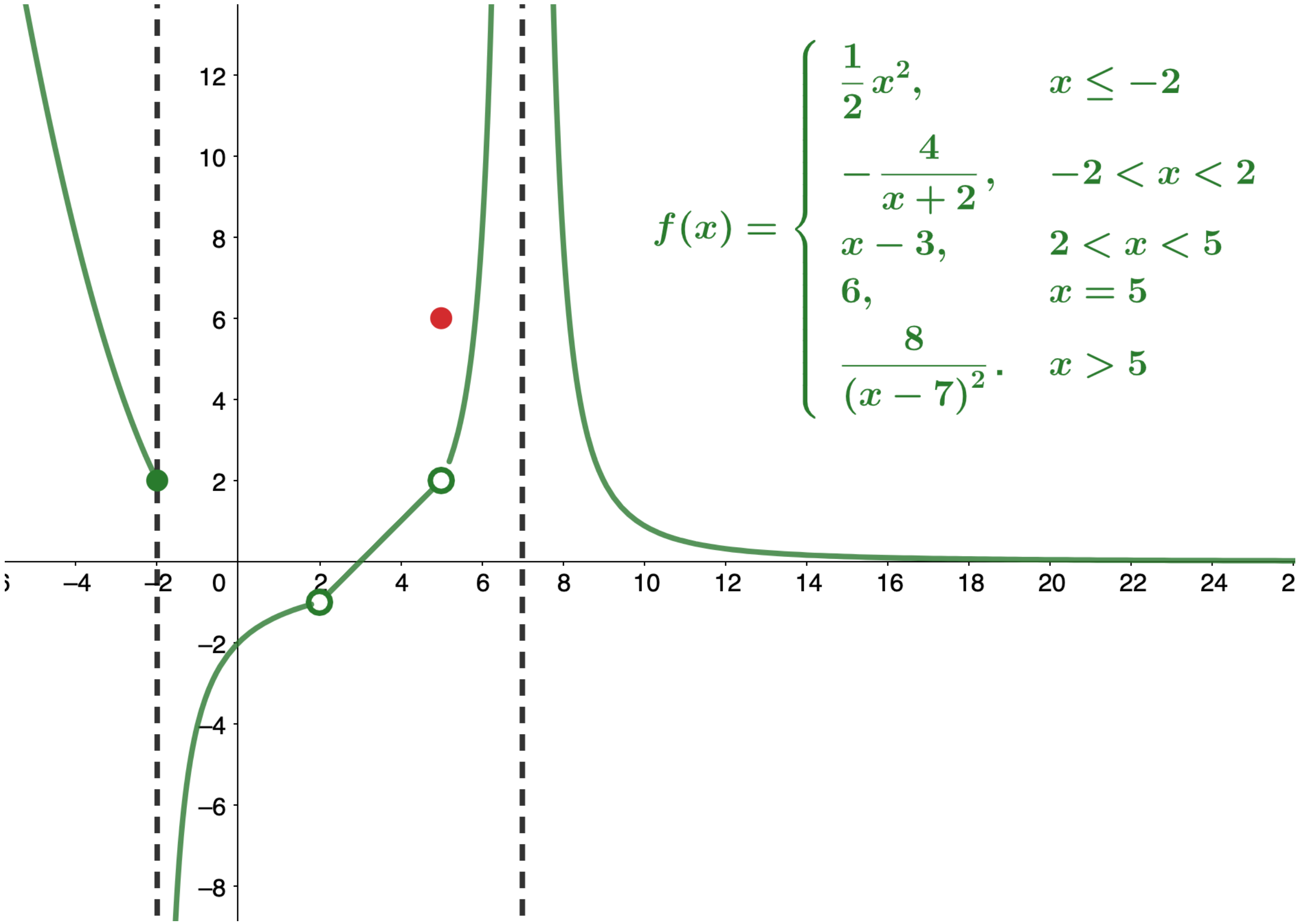
CONTINUITY OF A RATIONAL FUNCTION

- For what values of x is $f(x)$ continuous?
- For what values of x is $g(x)$ continuous?



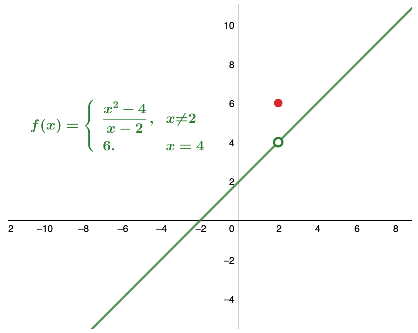
TYPES OF DISCONTINUITIES

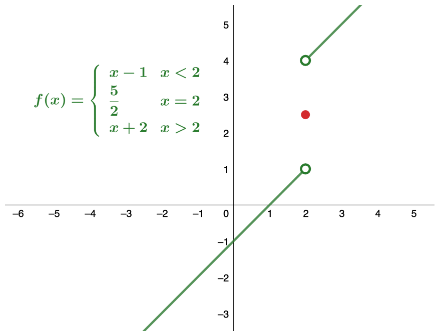
- Discontinuities take on several different appearances.

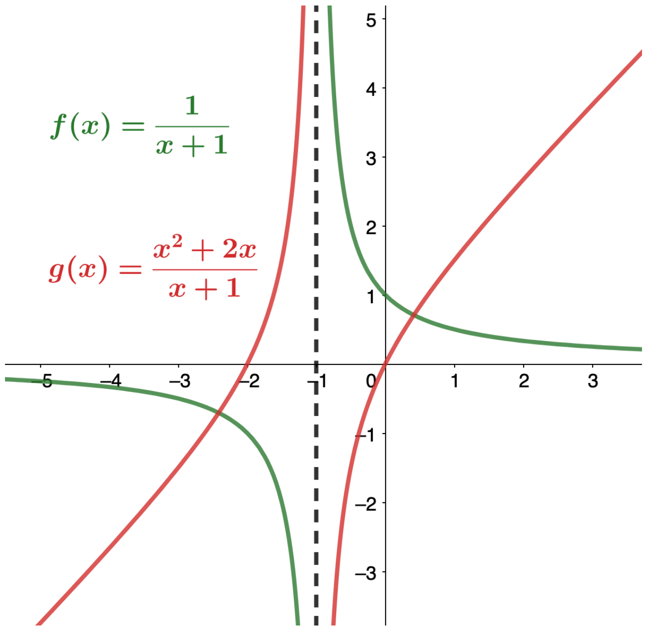


THREE COMMON TYPES OF DISCONTINUITIES

- a **removable discontinuity** is a discontinuity for which there is **a hole** in the graph.
- a **jump discontinuity** is a **noninfinite** discontinuity for which the sections of the function **do not meet up**.
- an **infinite discontinuity** is a discontinuity located at a **vertical asymptote**.







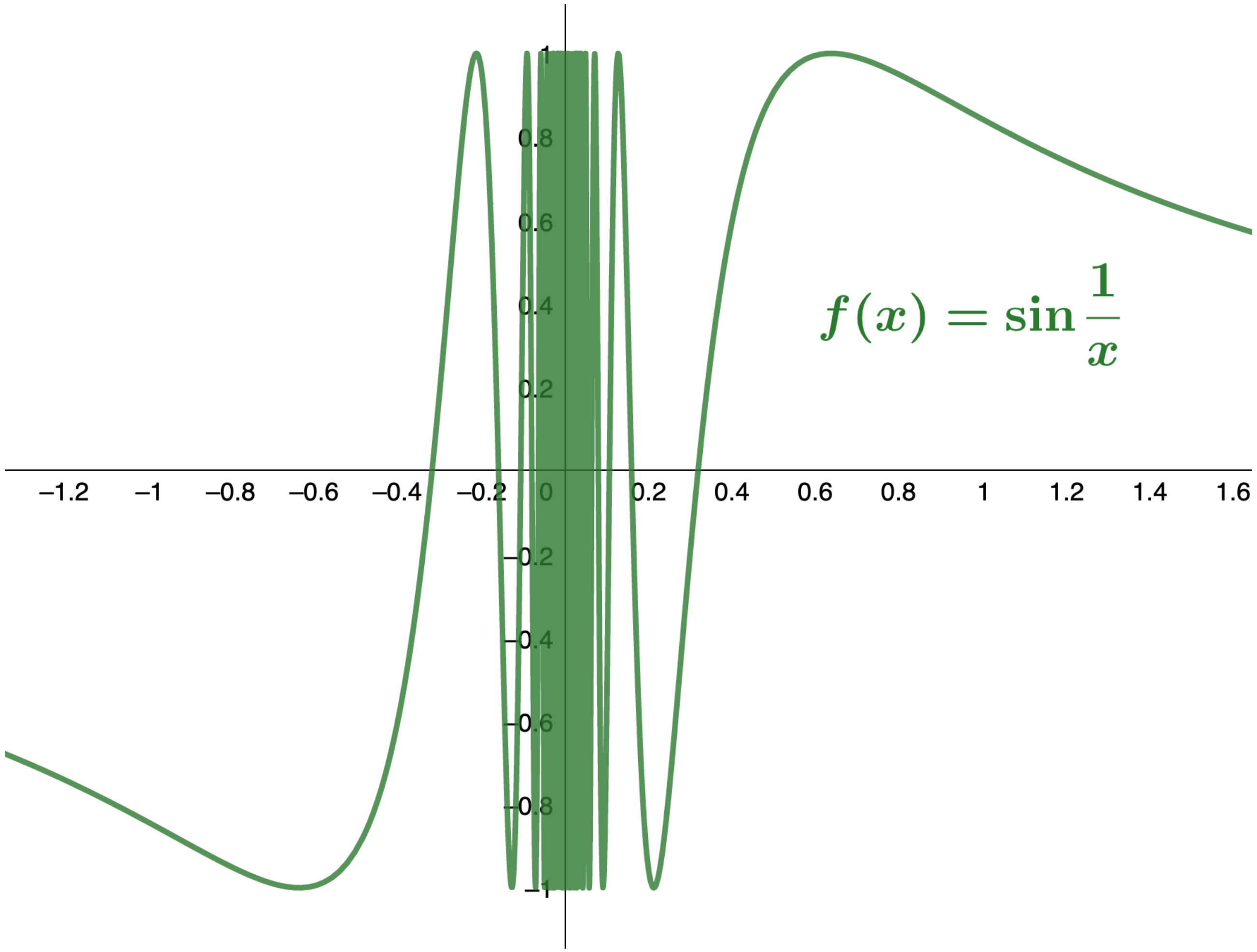
THREE COMMON TYPES OF DISCONTINUITIES

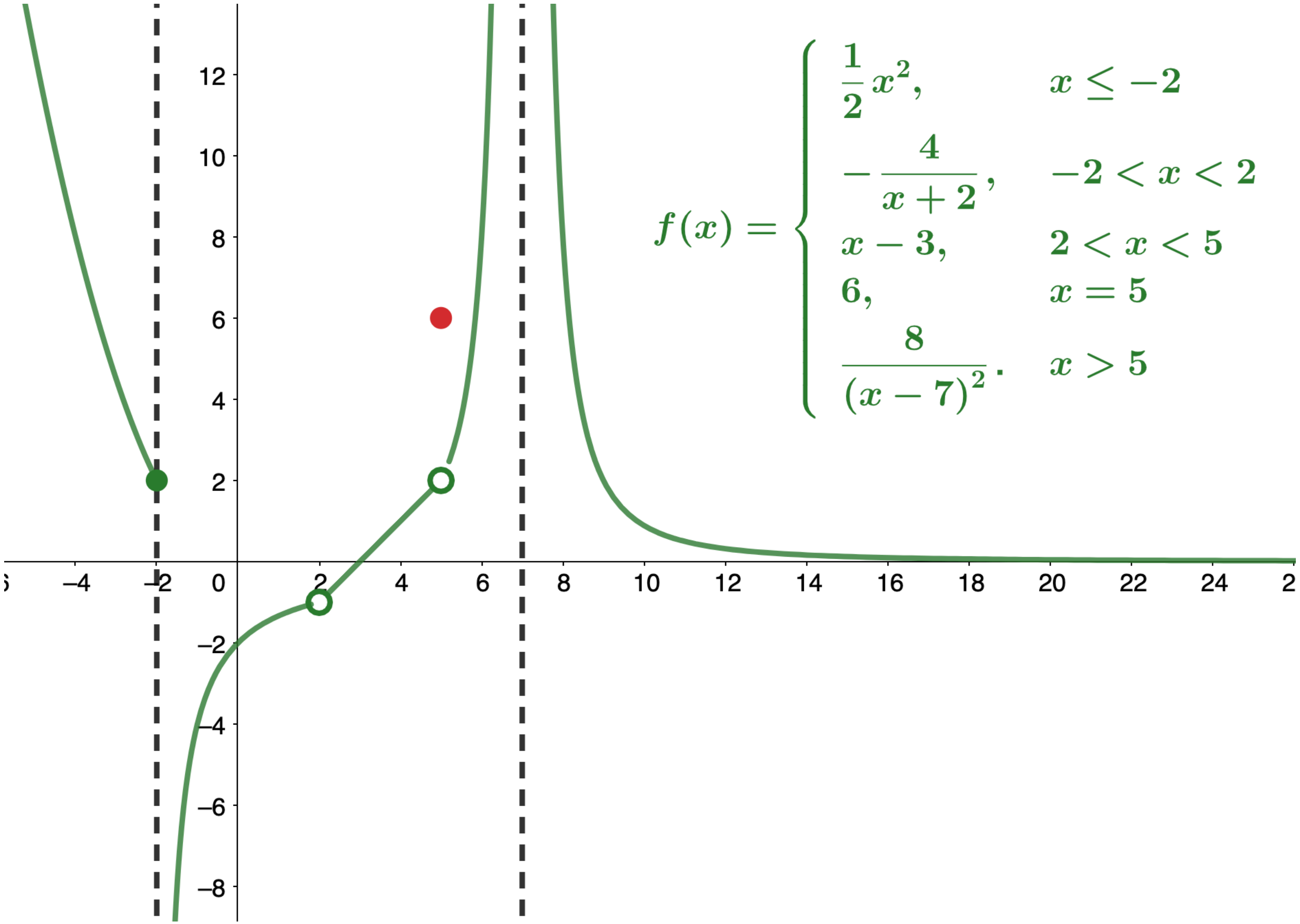
If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists. (Note: When we state that $\lim_{x\to a} f(x)$ exists, we mean that $\lim_{x\to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. (Note: When we state that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

REMARKS

- Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that **not all discontinuities fit neatly into these categories**.





$$f(x) = \begin{cases} \frac{1}{2}x^2, & x \le -2 \\ -\frac{4}{x+2}, & -2 < x < 2 \\ x - 3, & 2 < x < 5 \\ 6, & x = 5 \\ \frac{8}{(x-7)^2}. & x > 5 \end{cases}$$

EXERCISE ONE

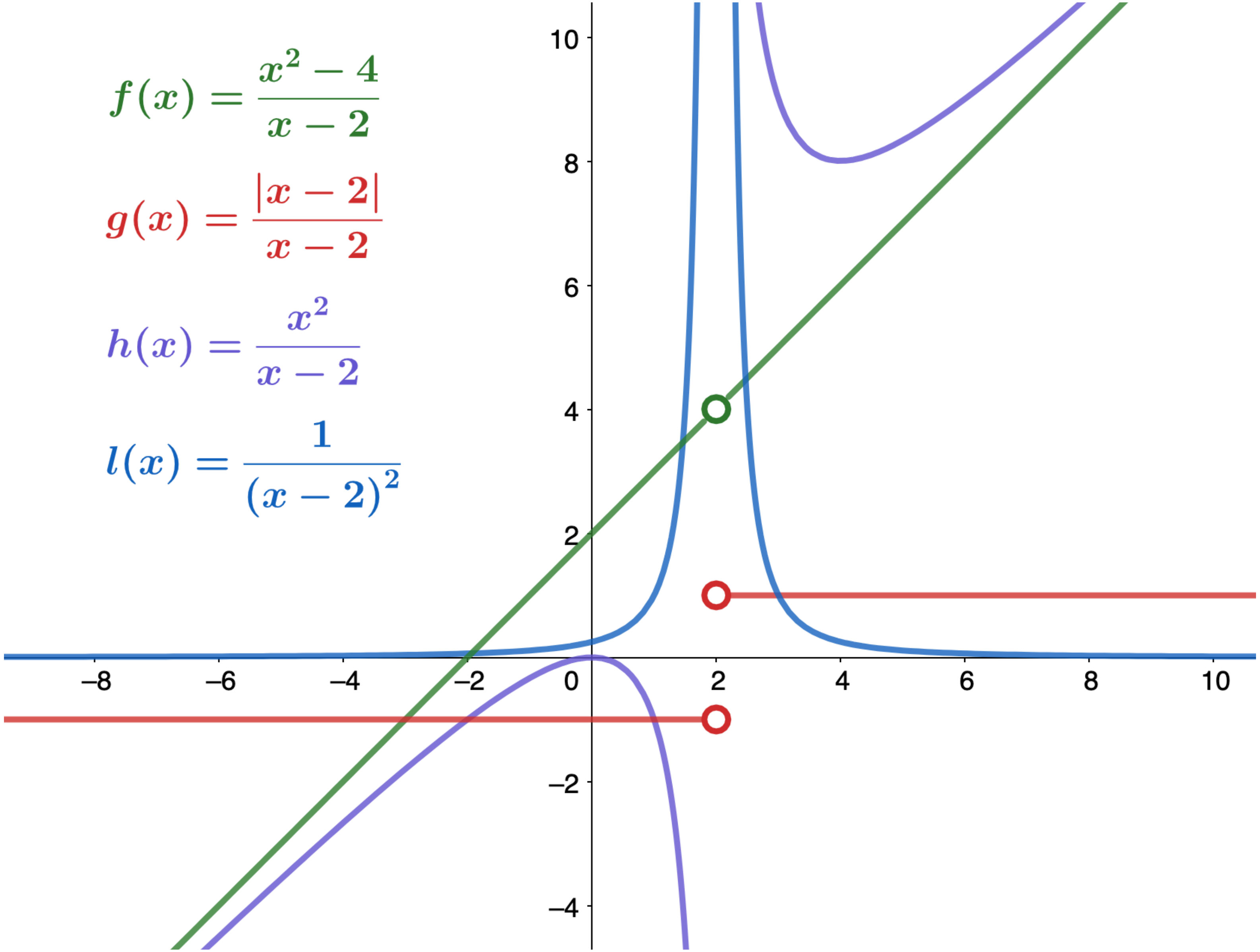
- a **removable discontinuity** is a discontinuity for which there is a hole in the graph.
- a **jump discontinuity** is a **noninfinite** discontinuity for which the sections of the function do not meet up.
- an **infinite discontinuity** is a discontinuity located at a **vertical asymptote**.

$$f(x) = \frac{x^2 - 4}{x - 2}$$

$$g(x) = \frac{|x - 2|}{x - 2}$$

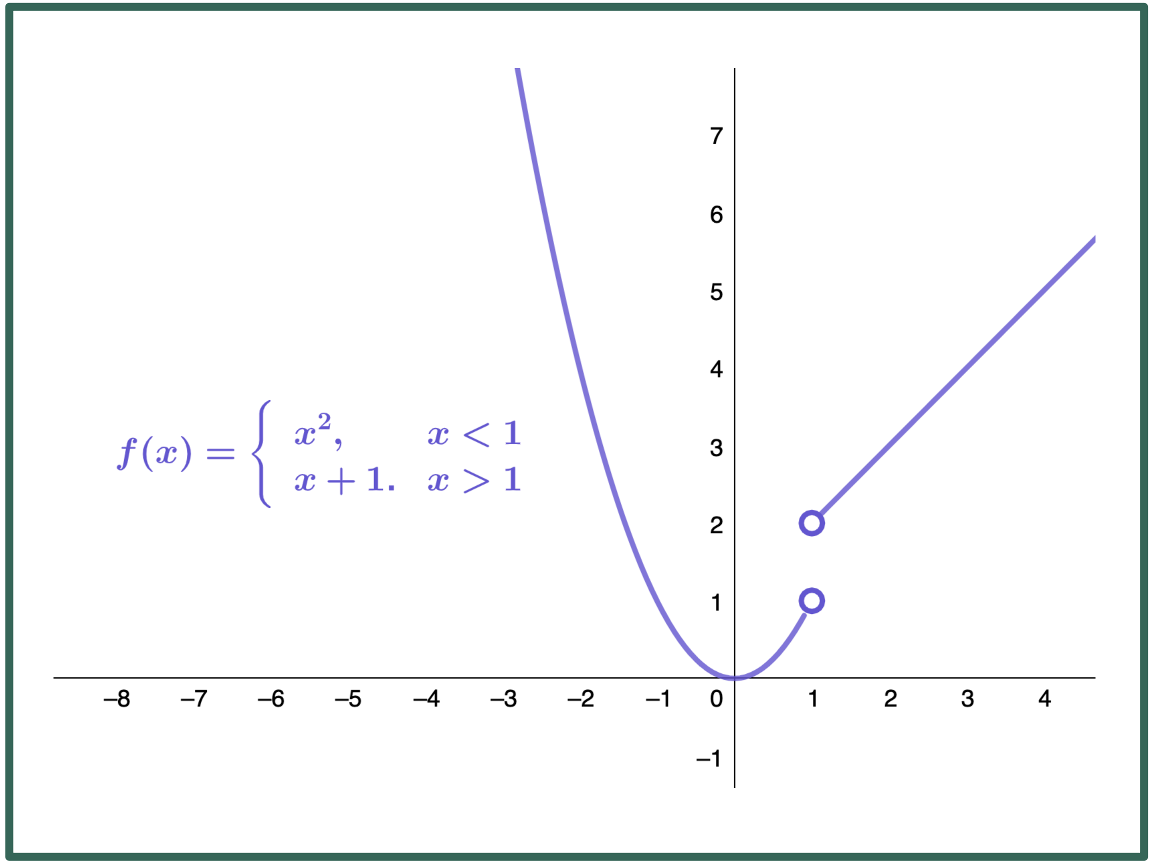
$$h(x) = \frac{x^2}{x - 2}$$

$$l(x) = \frac{1}{(x - 2)^2}$$



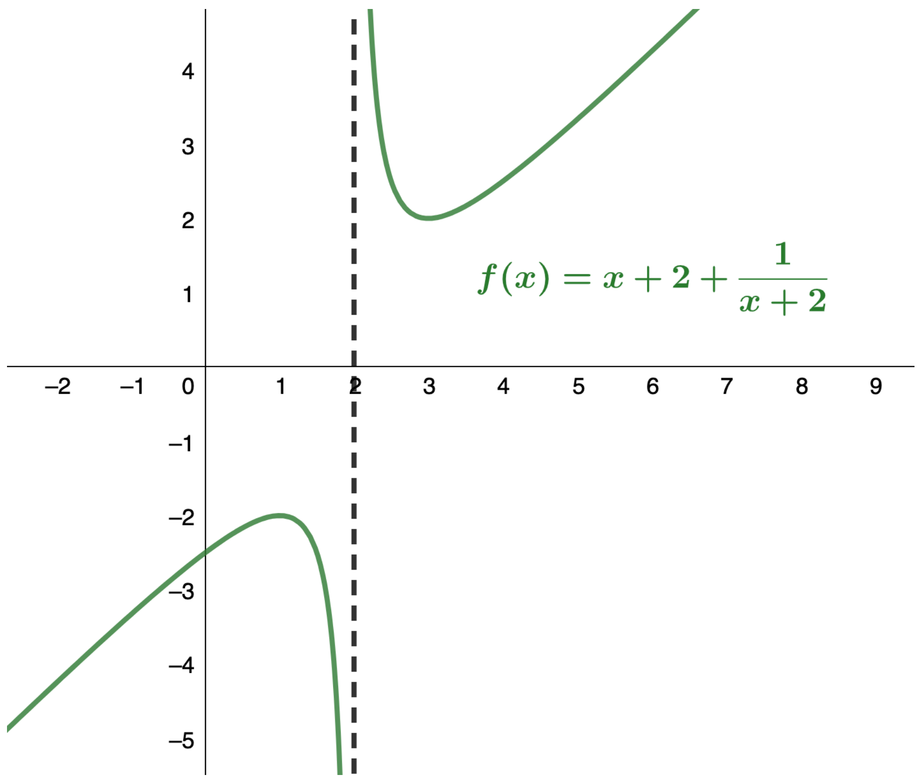
EXERCISE TWO

EXERCISE THREE



- $f(x) = \begin{cases} x^2, \; x < 1 \\ x + 1. \, x > 1 \end{cases}$

EXERCISE FOUR



- $f(x) = x + 2 + \frac{1}{x+2}$.

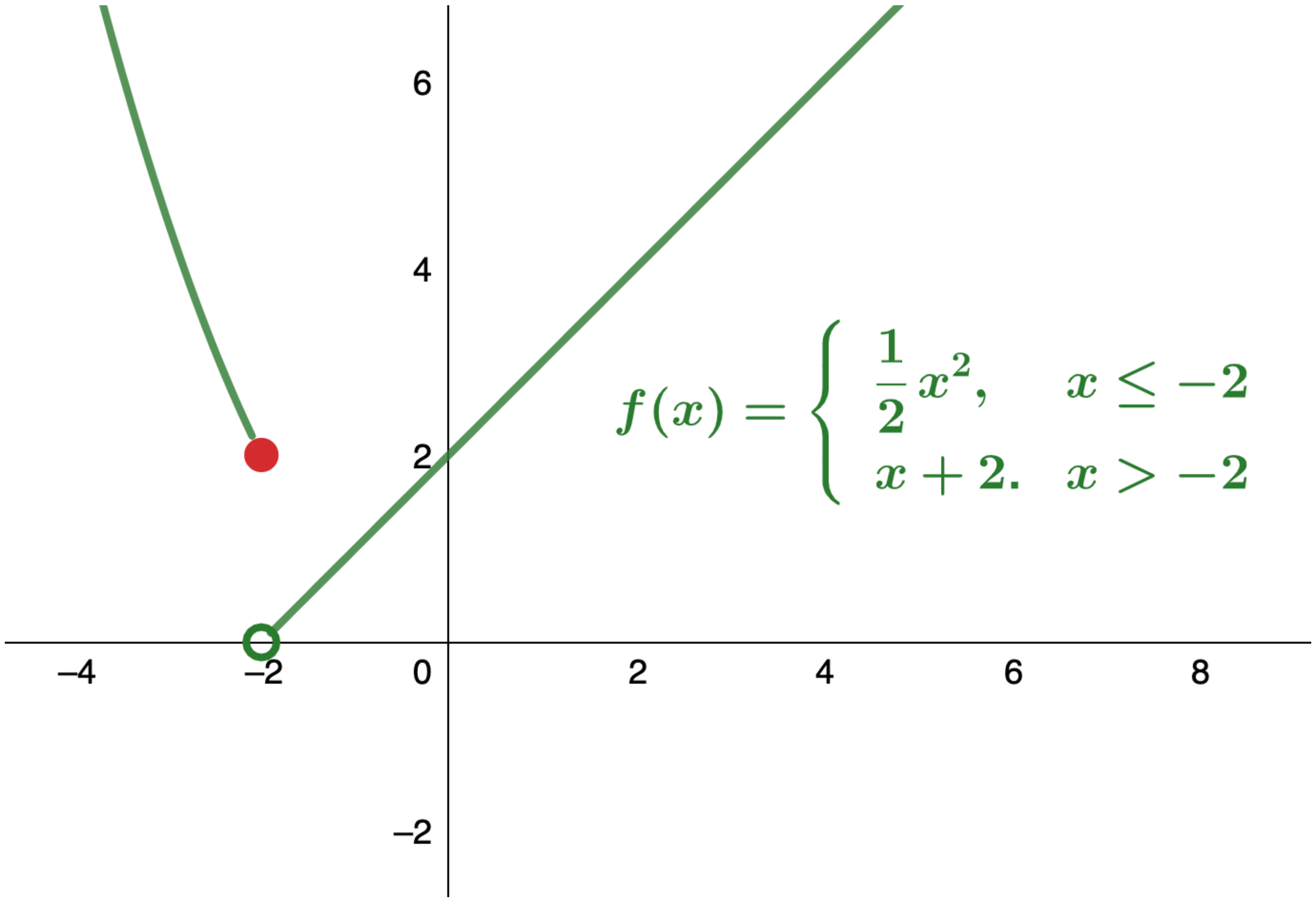
CONTINUITY OVER AN INTERVAL

CONTINUITY FROM THE RIGHT AND FROM THE LEFT

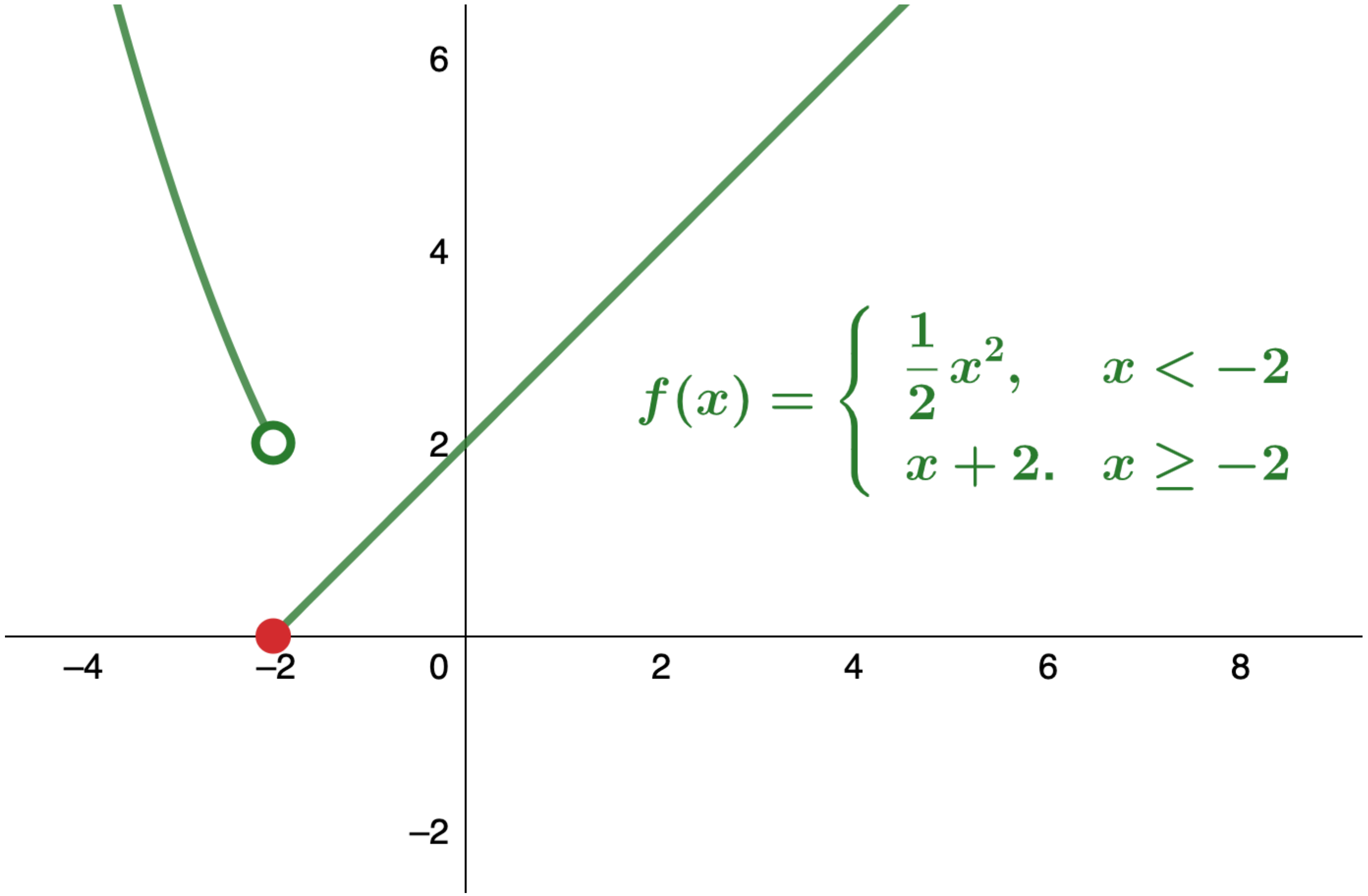
CONTINUITY FROM THE RIGHT AND FROM THE LEFT

A function $f\,(x)$ is said to be **continuous from the right** at *a* if $\lim\limits_{x \to a^+} f\,(x) = f\,(a)$.

A function $f\,(x)$ is said to be **continuous from the left** at *a* if $\lim\limits_{x \to a^-} f\,(x) = f\,(a)$.



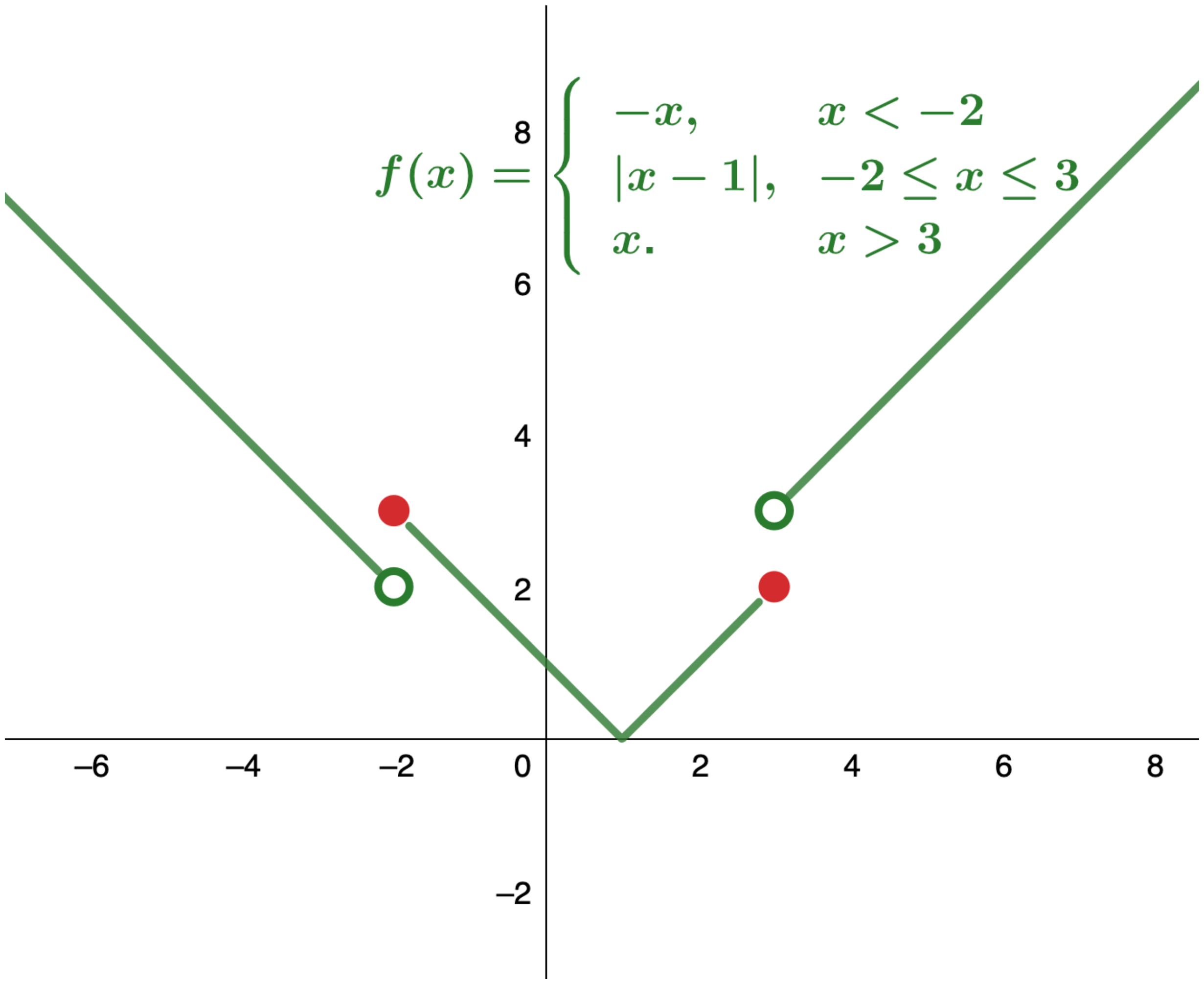
RIGHT OR LEFT?



RIGHT OR LEFT?

CONTINUITY OVER AN INTERVAL

- A function is continuous over **an open interval** if it is continuous at every point in the interval.
- A function $f(x)$ is continuous over **a closed interval** of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous **from the right at a** and is continuous **from the left at b**.
- Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous **from the left at b**.
- Continuity over other types of intervals are defined in a similar fashion.



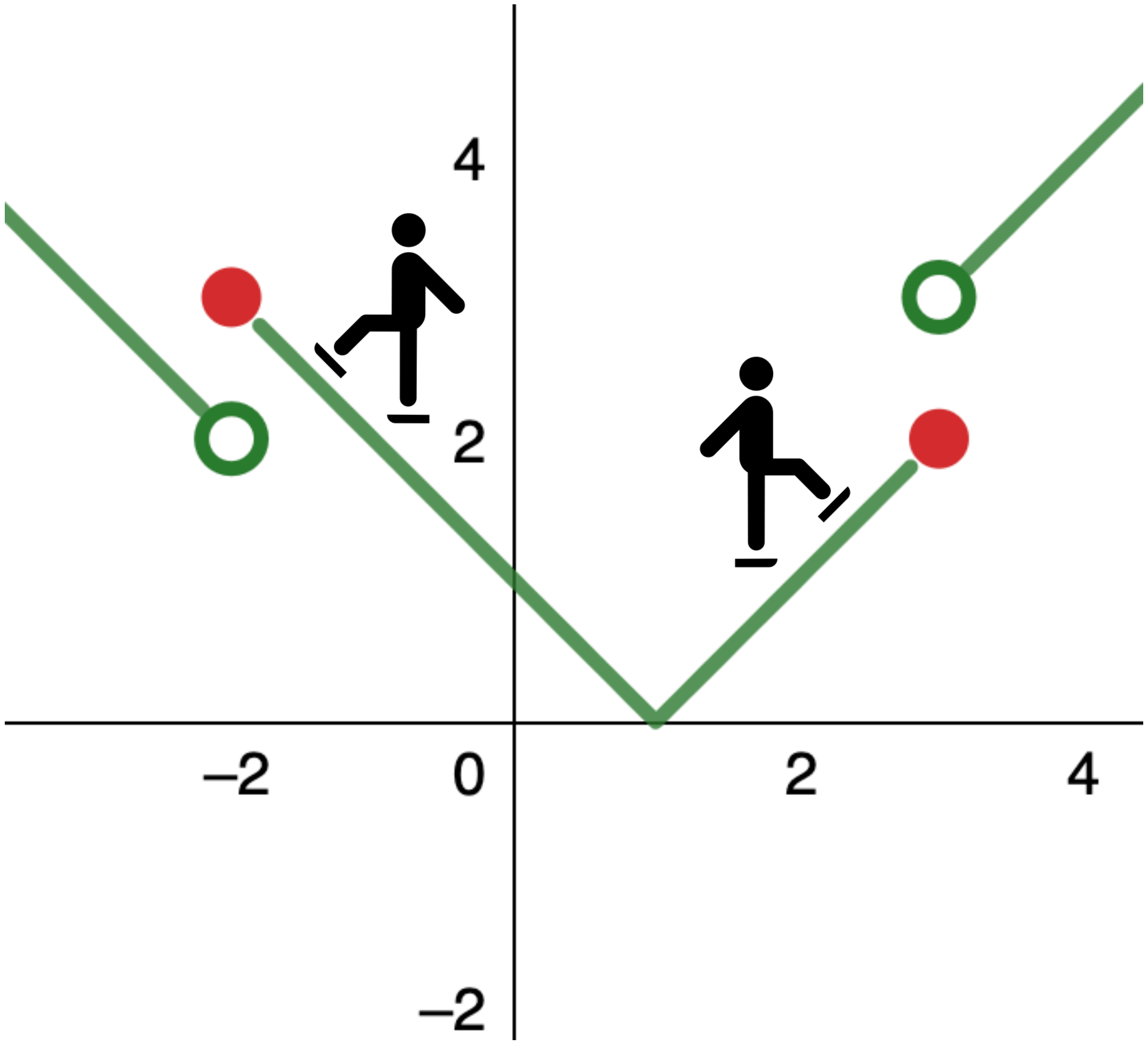
CONTINUITY OVER AN INTERVAL

A function $f(x)$ is continuous over **a closed interval** of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous **from the right at a** and is continuous **from the left at b**.

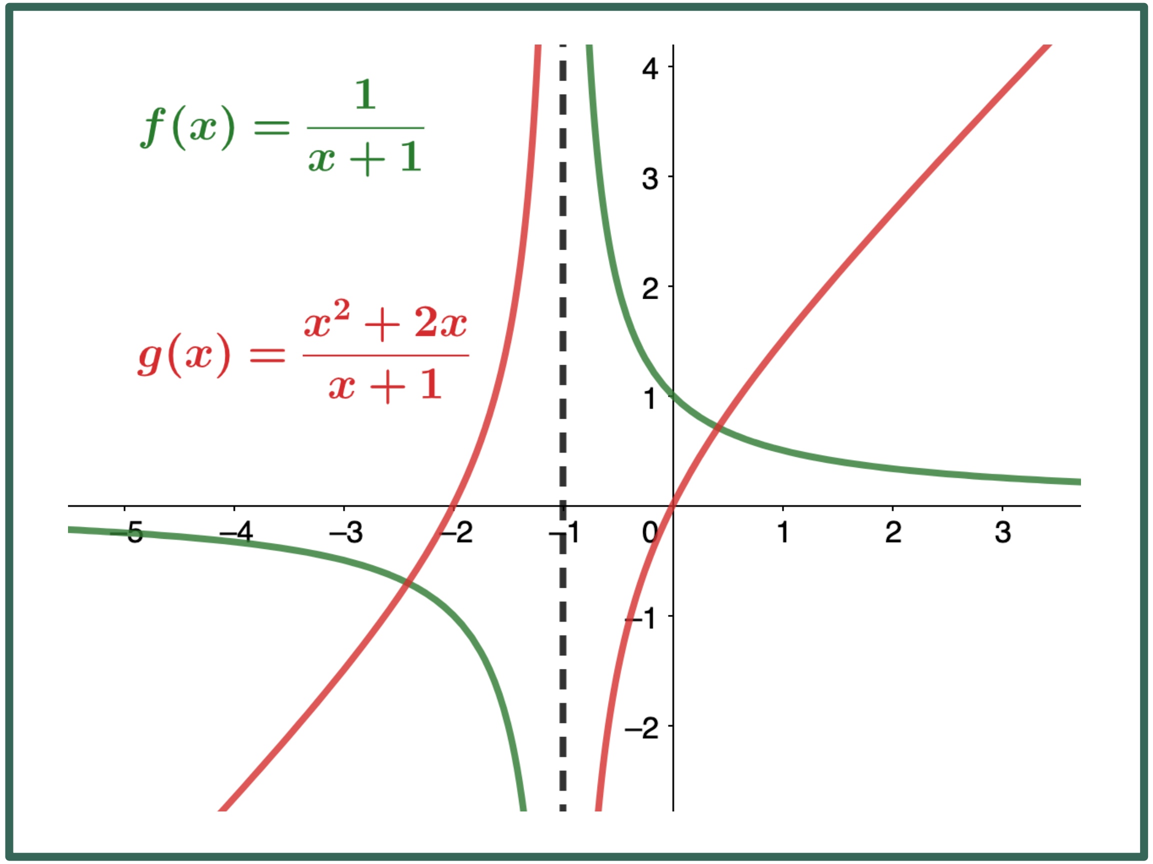
CONTINUOUS OVER **A CLOSED INTERVAL**

Continuous from

- the right at a
- the left at b



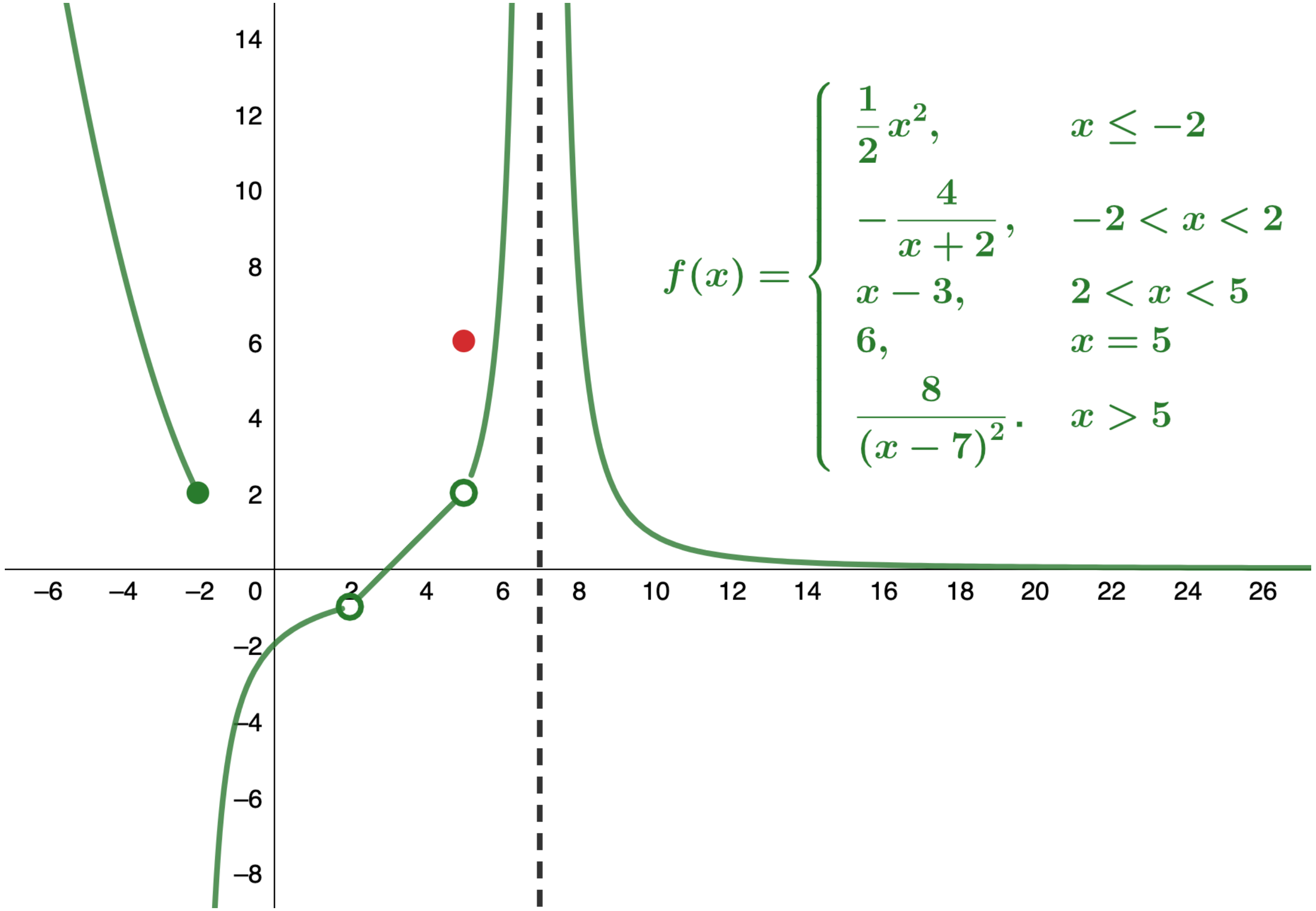
EXERCISE ONE



- State the interval(s) over which the function is continuous.

EXERCISE TWO

- State the interval(s) over which the function is continuous.



$$f(x) = \begin{cases} \frac{1}{2}x^2, & x \le -2 \\ -\frac{4}{x+2}, & -2 < x < 2 \\ x - 3, & 2 < x < 5 \\ 6, & x = 5 \\ \frac{8}{(x-7)^2}. & x > 5 \end{cases}$$

REVISIT COMPOSITE FUNCTION

THEOREM 2.9

Composite Function Theorem

If $f(x)$ is continuous at L and $\lim\limits_{x \to a} g(x) = L$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

HOW TO UNDERSTAND COMPOSITE FUNCTION THEOREM

$f(g(x))$

Does $g(x)$ need to be continuous at a?

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$$

$f(x)$ is continuous at L

$\lim_{x\to a} g(x) = L$

MORE TO SAY ABOUT TRIGONOMETRIC FUNCTIONS

THEOREM 2.10

Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

PROOF

$$\begin{aligned}
\lim_{x\to a}\cos x &= \lim_{x\to a}\cos\left((x-a)+a\right) \\
&= \lim_{x\to a}\left(\cos(x-a)\cos a - \sin(x-a)\sin a\right) \\
&= \cos\left(\lim_{x\to a}(x-a)\right)\cos a - \sin\left(\lim_{x\to a}(x-a)\right)\sin a \\
&= \cos(0)\cos a - \sin(0)\sin a \\
&= 1\cdot\cos a - 0\cdot\sin a = \cos a.
\end{aligned}$$