PRODUCT AND QUOTIENT RULES

INTRODUCTION TO CALCULUS

THE PRODUCT RULE

The derivative of the product is the product of the derivatives?

- $f(x) = x^2 = x \cdot x.$
- $\frac{d}{dx}f(x) = \frac{d}{dx}x^2 = 2x$
- $\left(\frac{d}{dx}x\right)\left(\frac{d}{dx}x\right) = 1 \cdot 1 = 1$

THE PRODUCT RULE

THEOREM 3.5

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\boxed{\frac{d}{dx}(f(x)\,g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).}$$

That is,

$$\text{if } j(x) = f(x)\,g(x), \text{ then } j'(x) = f'(x)\,g(x) + g'(x)\,f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

PROOF

USE THE LIMIT DEFINITION OF THE DERIVATIVE.

$$\boxed{\frac{d}{dx}(f(x)\,g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).}$$

PROOF: STEP 1

By applying the limit definition of the derivative to $j(x) = f(x)\,g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}.$$

PROOF: STEP 2 🚑🚑🚑

By adding and subtracting $\boxed{f(x)\,g(x+h)}$ in the numerator, we have

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}.$$

PROOF: STEP 3

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$j'(x) = \lim_{h\to 0}\left(\frac{f(x+h)\,g(x+h) - f(x)\,g(x+h)}{h}\right) + \lim_{h\to 0}\left(\frac{f(x)g(x+h) - f(x)\,g(x)}{h}\right).$$

PROOF: STEP 4

Rearranging, we obtain

$$j'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\cdot g(x+h)\right) + \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\cdot f(x)\right).$$

PROOF: STEP 5

By using the continuity of $g\,(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$j'\,(x) = f'\,(x)\,g\,(x) + g'\,(x)f(x).$$

EXERCISE ONE

Applying the Product Rule to Functions at a Point

- For $j(x) = f(x)g(x)$, use the product rule to find $j'(2019)$ if $f(2019) = 1$, $f'(2019) = 2019$, $g(2019) = 0$ and $g'(2019) = 1984$.

EXERCISE TWO

Applying the Product Rule to Binomials

- For $j(x) = (x^2 + 2x + 1)(2x^3 - x + 4)$, find $j'(x)$ by applying the product rule.
- Check the result by first finding the product and then differentiating.

EXERCISE THREE

Applying the Product Rule to find the derivative.

- $f(x) = (x^2 + 3x - 4)(x^3 - 1)$.
- $f(x) = (x^3 - x)^2$.
- $f(x) = (x + 1)(x + 2)(x + 3)$.

THE QUOTIENT RULE

The derivative of the quotient is the quotient of the derivatives?

- $f(x) = x^2 = \frac{x^3}{x}$.
- $\frac{d}{dx}f(x) = \frac{d}{dx}x^2 = 2x$.
- $\frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2$.

THE QUOTIENT RULE

THEOREM 3.6

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.}$$

That is,

$$\text{if } j(x) = \frac{f(x)}{g(x)}, \text{ then } j'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

PROOF

USE THE LIMIT DEFINITION OF THE DERIVATIVE.

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.}$$

EXERCISE ONE

Applying the Quotient Rule

- Use the quotient rule to find the derivative of $k(x) = \frac{x+1}{x^2+1}$.

EXERCISE TWO

Applying the Quotient Rule

- Use the quotient rule to find the derivative of $k(x) = \frac{1}{x^2}$.
- Use the quotient rule to find the derivative of $k(x) = \frac{1}{x^3}$.
- Use the quotient rule to find the derivative of $k(x) = \frac{1}{x^n}$.

EXTEND POWER RULE

THEOREM 3.7

Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

EXERCISE

Using the Extended Power Rule and the Constant Multiple Rule

- Find the derivative of $f(x) = \frac{2}{x^{2019}}$.

COMBINING DIFFERENTIATION RULES

- The power rule
- The Sum, Difference, and Constant Multiple Rules
- The product rule
- The quotient rule

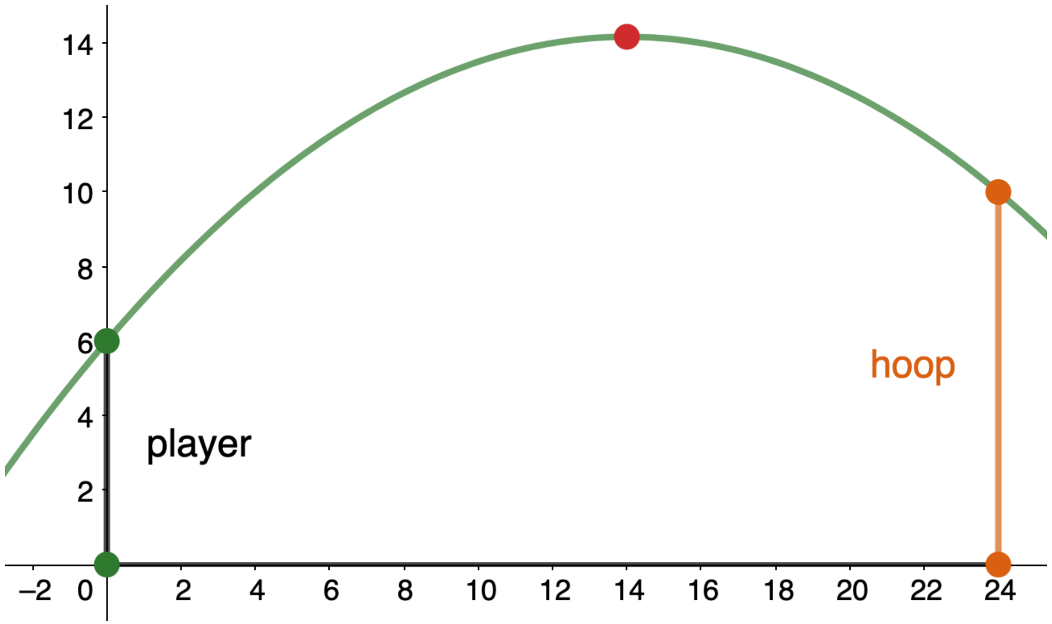
REVISIT THE ZEBRAS

- Two zebras are wandering in the wood, seeking some little shelter from the wind and rain. They set off 10 days ago from the Maryland Zoo in Baltimore ($t = 0$), passing through Washington DC and happen to meet each other again in Hanover. Treating DC as the origin ($S = 0$), their positions are

$$S_1(t) = t^2 - 2t - 8 \text{ and } S_2(t) = 8t - 8.$$

- Use this information to decide when the first zebra has the same velocity as the second one.

REVISIT THE BASKETBALL PENALTY SHOT



- The trace of the basketball is $f(x) = -\frac{1}{24}x^2 + \frac{7}{6}x + 6.$
- Can you come up with a more mathematical method for the second problem?
  - **Hint: Axis** of **Symmetry** in a **Parabola**
  - **Hint: Tangent line**

EXERCISE TWO

Extend the Product Rule

- For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x), g(x), h(x)$, and their derivatives.

EXERCISE TWO

Combining the Quotient Rule and the Product Rule

- For $k(x) = \frac{f(x)g(x)}{h(x)}$, find $k'(x)$.