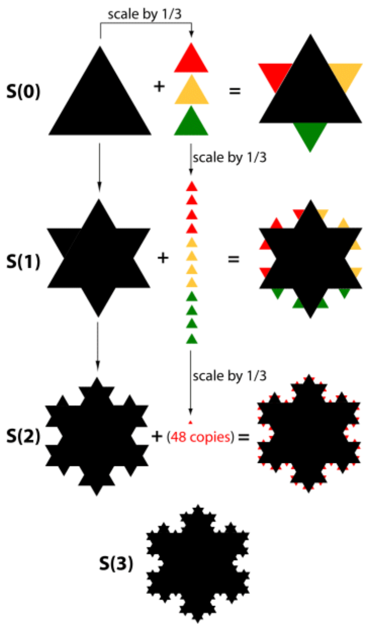
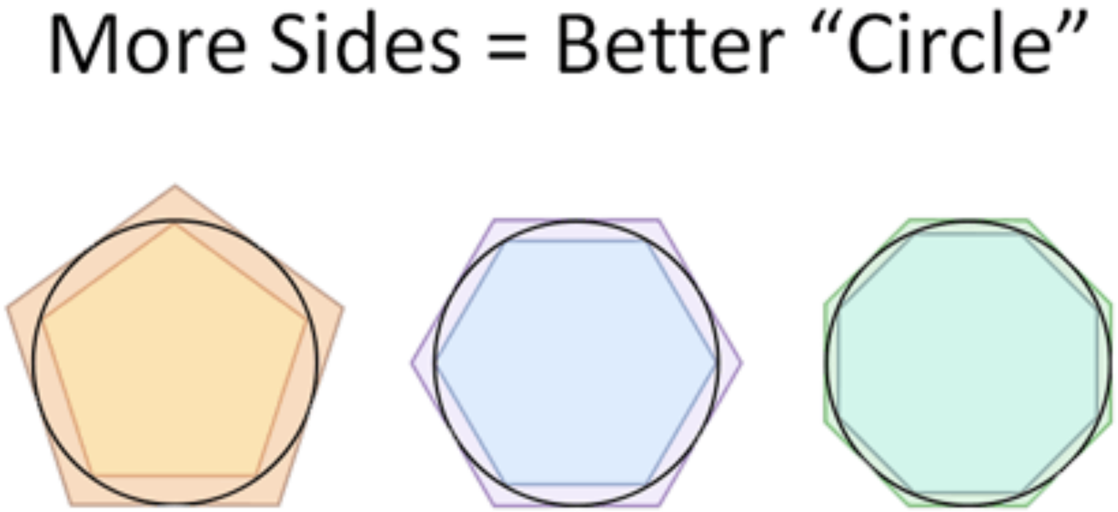
THE LIMIT OF A FUNCTION

INTRODUCTION TO CALCULUS

Use	• Use correct notation to describe the limit of a function.
Use	• Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
Use	• Use a graph to estimate the limit of a function or to identify when the limit does not exist.
Define	• Define one-sided limits and provide examples.
Explain	• Explain the relationship between one-sided and two-sided limits.
Use	• Use correct notation to describe an infinite limit.
Define	• Define a vertical asymptote.

WHAT IS A LIMIT? WHEN DO WE NEED A LIMIT?





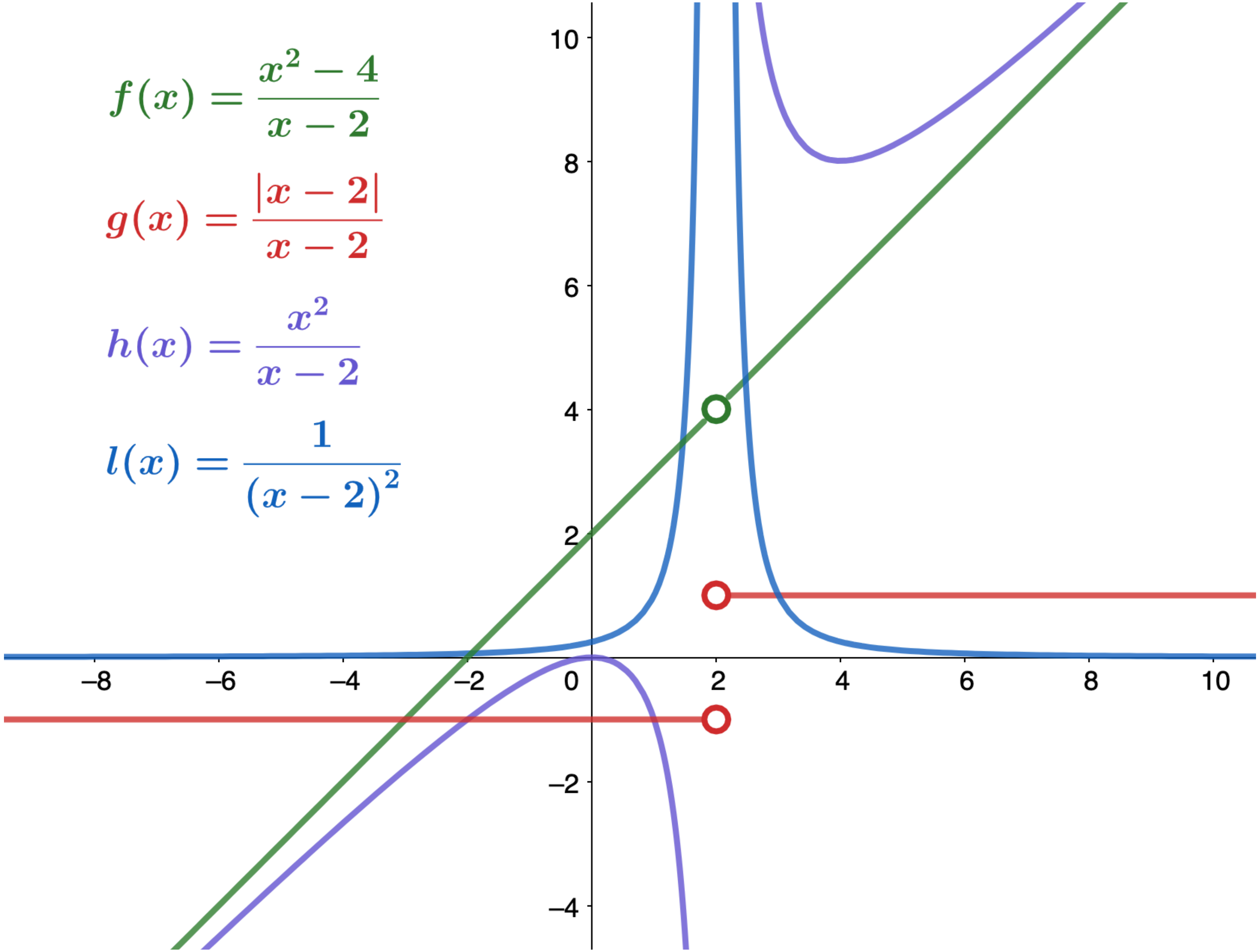
THE GRAPHS OF FOUR FUNCTIONS

$$f(x) = \frac{x^2 - 4}{x - 2}$$

$$g(x) = \frac{|x - 2|}{x - 2}$$

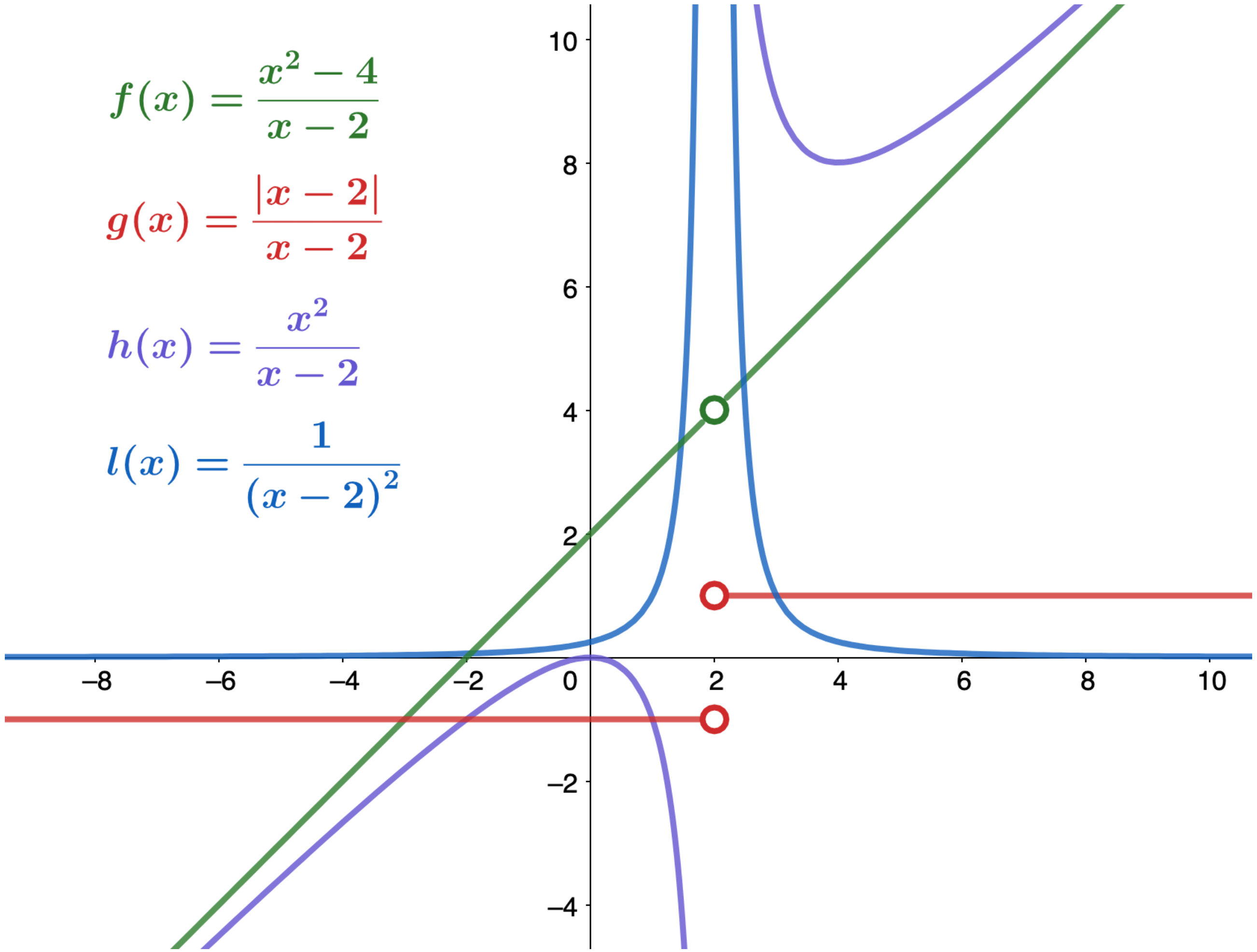
$$h(x) = \frac{x^2}{x - 2}$$

$$l(x) = \frac{1}{(x - 2)^2}$$

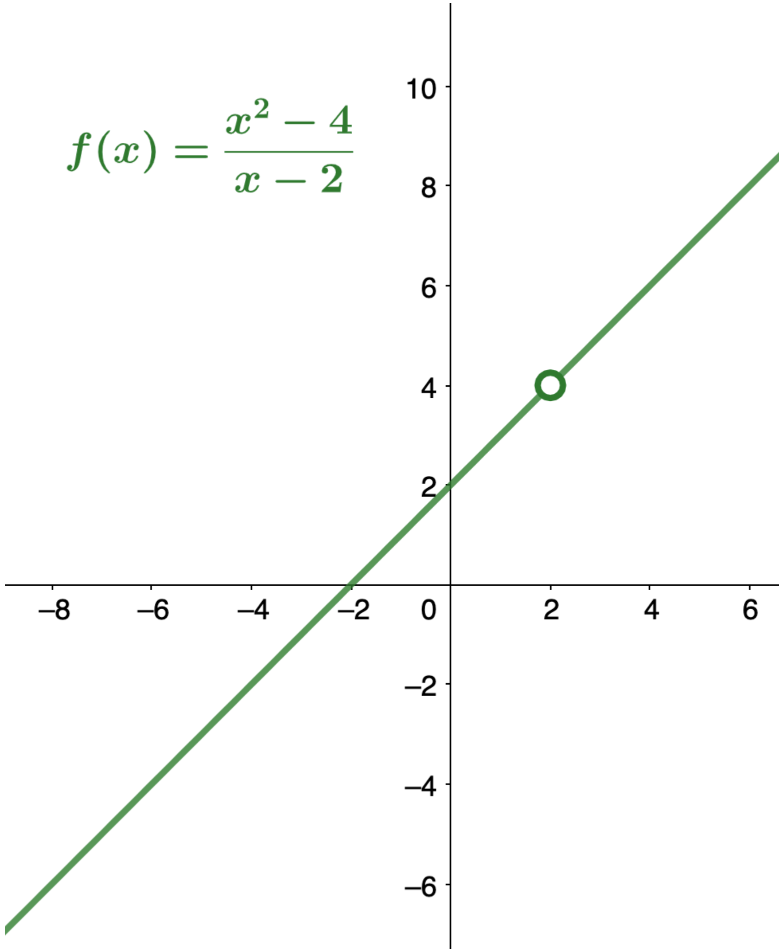


BEHAVIOR AT $x = 2$

- Each of the four functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$.
- To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.



INTUITIVE DEFINITION OF A LIMIT



- As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4.
- Mathematically, we say that the limit of $f(x)$ as x approach 2 is 4.
- Symbolically, we express this limit as $\lim_{x \to 2} f(x) = 4$.

A MORE CAREFUL STATEMENT

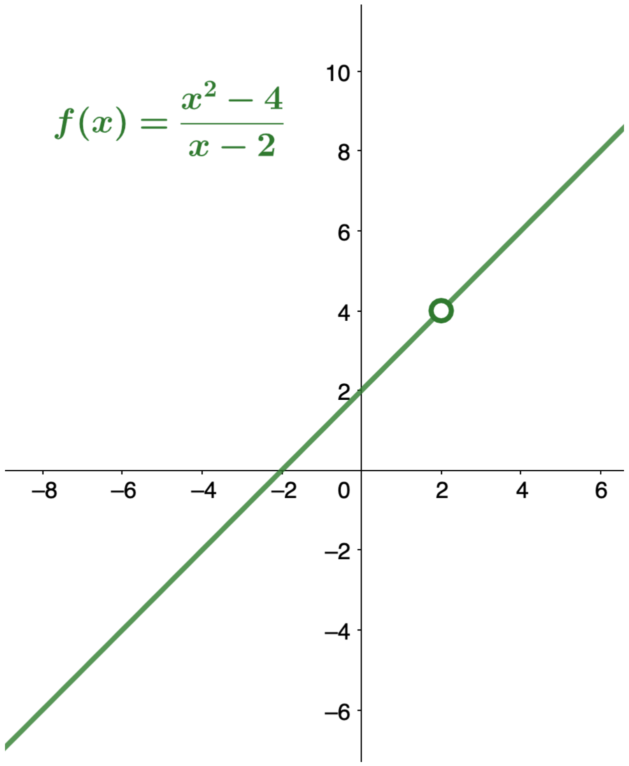
DEFINITION

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. If *all* values of the function $f(x)$ approach the real number *L* as the values of $x\,(\neq a)$ approach the number *a*, then we say that the limit of $f(x)$ as *x* approaches *a* is *L*. (More succinct, as *x* gets closer to *a*, $f(x)$ gets closer and stays close to *L*.) Symbolically, we express this idea as

$$\boxed{\lim_{x \to a} f(x) = L.} \tag{2.3}$$

HOW TO ESTIMATE LIMITS

WE CAN ESTIMATE LIMITS BY CONSTRUCTING TABLES OF FUNCTIONAL VALUES AND BY LOOKING AT THEIR GRAPHS.



x	$f(x)$
1.9	
1.99	
1.999	
1.9999	

x	$f(x)$
2.1	
2.01	
2.001	
2.0001	

EVALUATING A LIMIT USING A TABLE OF FUNCTIONAL VALUES

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of *x*-values —one set of values approaching *a* and less than *a*, and another set of values approaching *a* and greater than *a*. Table 2.1 demonstrates what your tables might look like.

x	$f(x)$		x	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.			Use additional values as necessary.	

Table 2.1 Table of Functional Values for $\lim_{x \to a} f(x)$

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a-0.1), f(a-0.01), f(a-0.001).,f(a-0.0001)$, and so on, and $f(a+0.1), f(a+0.01), f(a+0.001), f(a+0.0001)$, and so on. (*Note*: Although we have chosen the *x*-values $a \pm 0.1, a \pm 0.01, a \pm 0.001, a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)
3. If both columns approach a common *y*-value *L*, we state $\lim\limits_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

EXAMPLE ONE

- $f(x) = \frac{x^2-4}{x-2}$
- $x \to 2$

x	$f(x)$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x)$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

EXAMPLE TWO

- $f(x) = \frac{\sin x}{x}$
- $x \to 0$

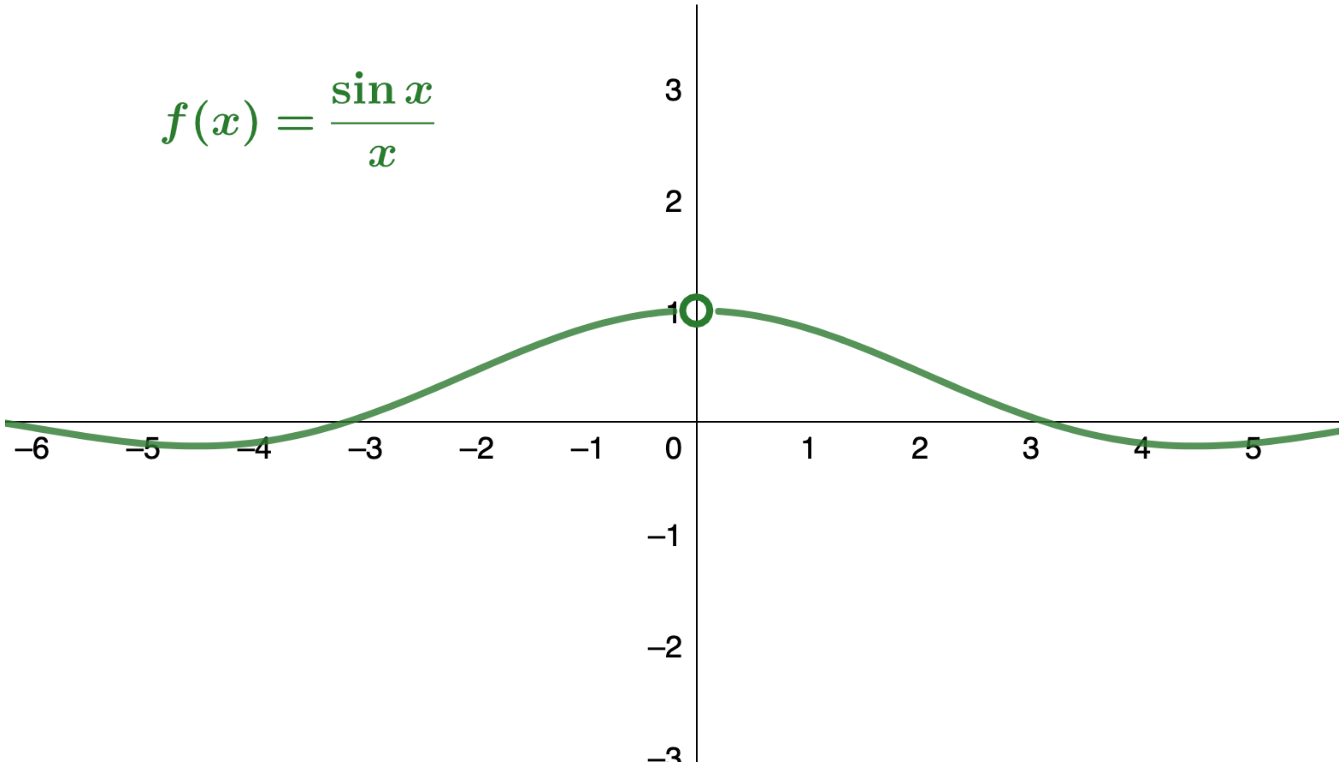
x	$f(x)$
-0.1	0.998
-0.01	0.99998
-0.001	0.9999998
-0.0001	0.999999998

x	$f(x)$
0.1	0.998
0.01	0.99998
0.001	0.9999998
0.0001	0.999999998

EVALUATING A LIMIT USING A GRAPH

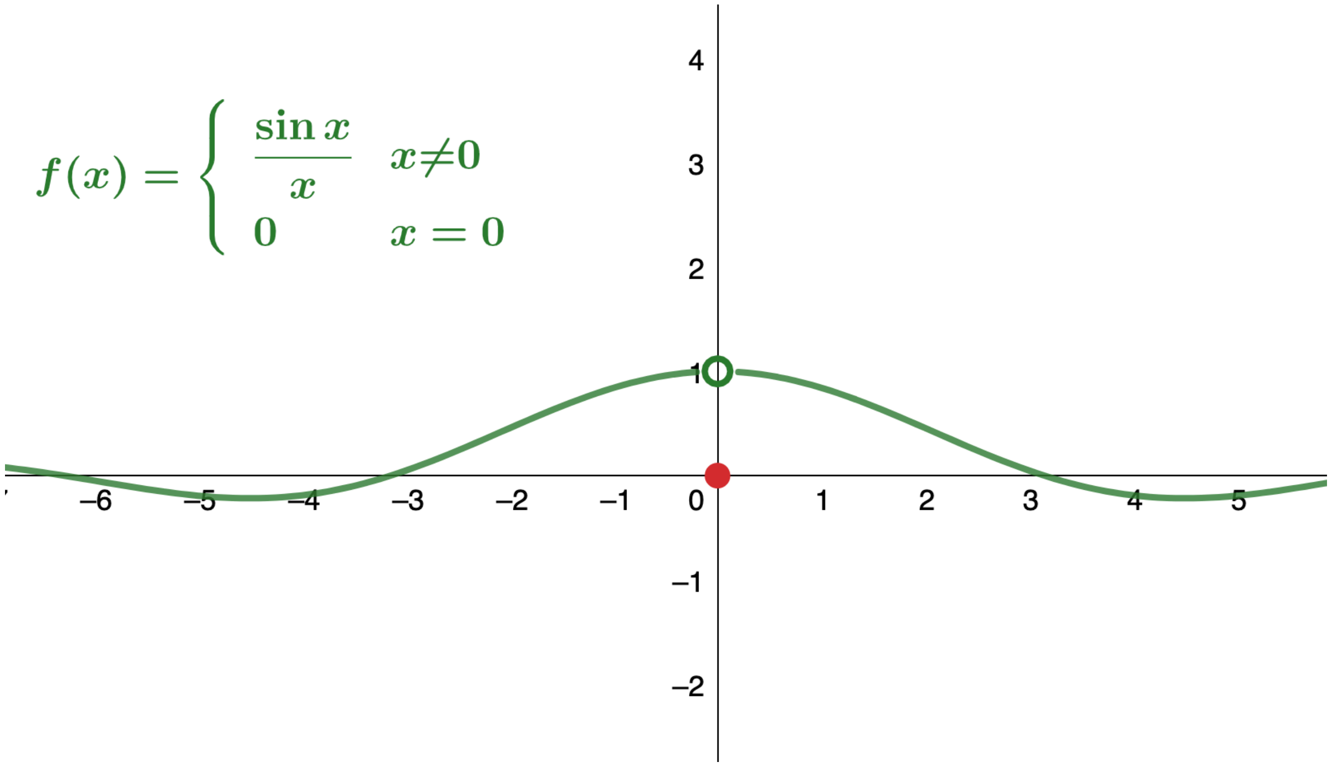
4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for *x*-values near *a* are in our window. We can use the trace feature to move along the graph of the function and watch the *y*-value readout as the *x*-values approach *a*. If the *y*-values approach *L* as our *x*-values approach *a* from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

EXAMPLE TWO



EXAMPLE THREE

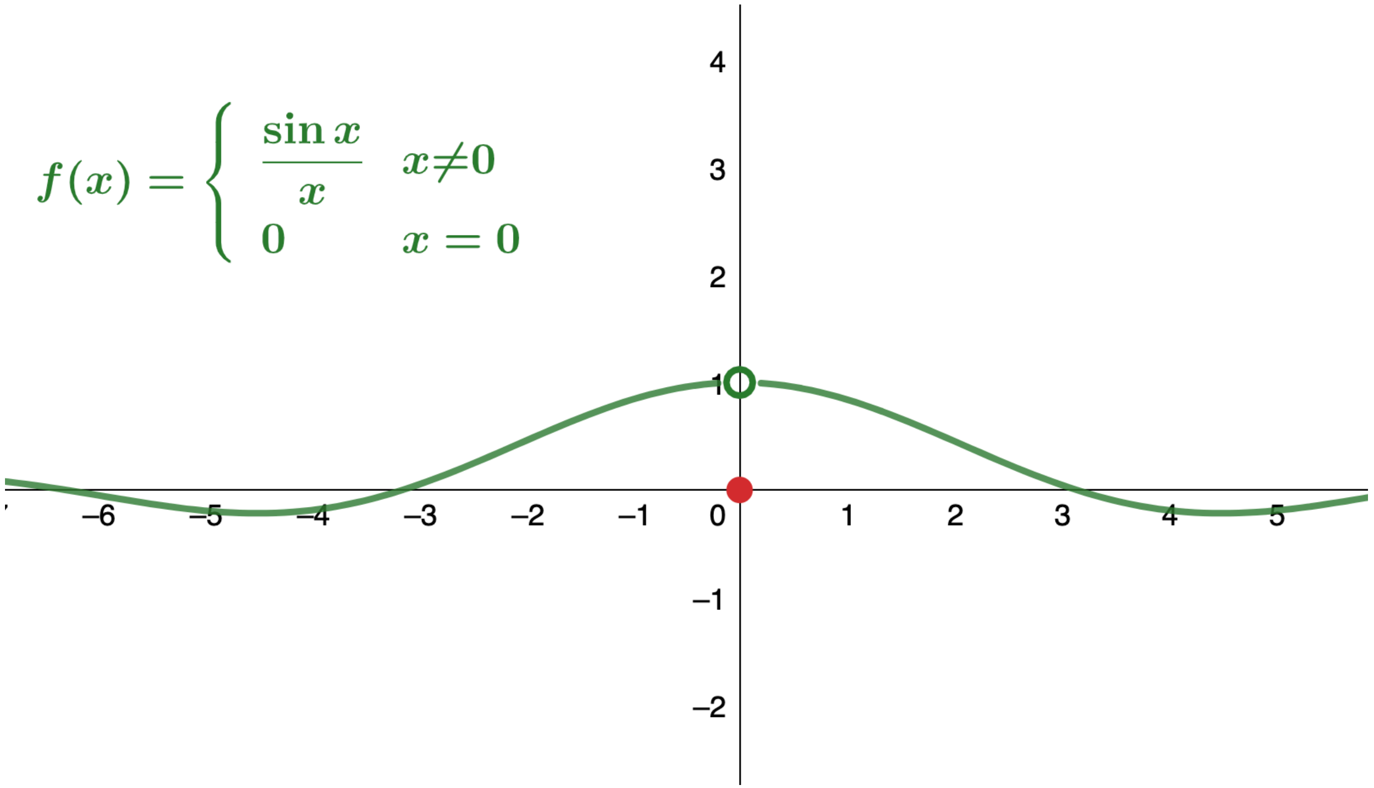
$$f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



EXAMPLE THREE

It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be **different**.

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



REMARKS

Looking at a **table** of functional values or looking at the **graph** of a function provides us with useful insight into the value of the limit of a function at a given point.

However, these techniques **rely too much on guesswork**.

We eventually need to develop alternative methods of evaluating limits.

These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce **two special limits** that are foundational to the techniques to come.

TWO SPECIAL LIMITS

THEOREM 2.1

Two Important Limits

Let *a* be a real number and *c* be a constant.

i. $$\boxed{\lim_{x \to a} x = a} \qquad (2.4)$$

ii. $$\boxed{\lim_{x \to a} c = c} \qquad (2.5)$$

TWO SPECIAL LIMITS

a is a real number

c is a constant

1. $\lim_{x \to a} x = a$
   - $f(x) = x$
2. $\lim_{x \to a} c = c$
   - $f(x) = c$

TWO SPECIAL LIMITS

We can make the following observations about these two limits.

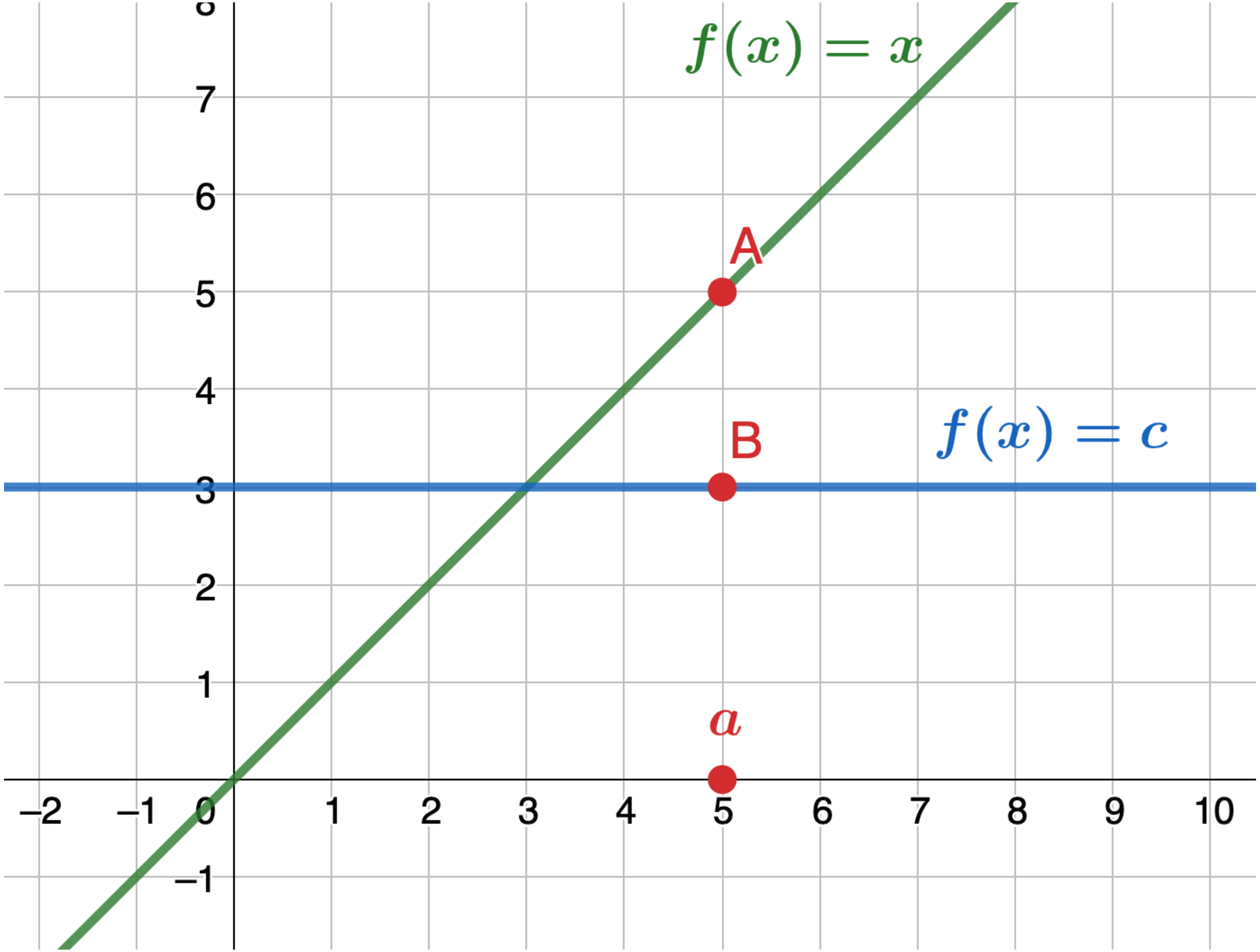
i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.

ii. For the second limit, consider Table 2.4.

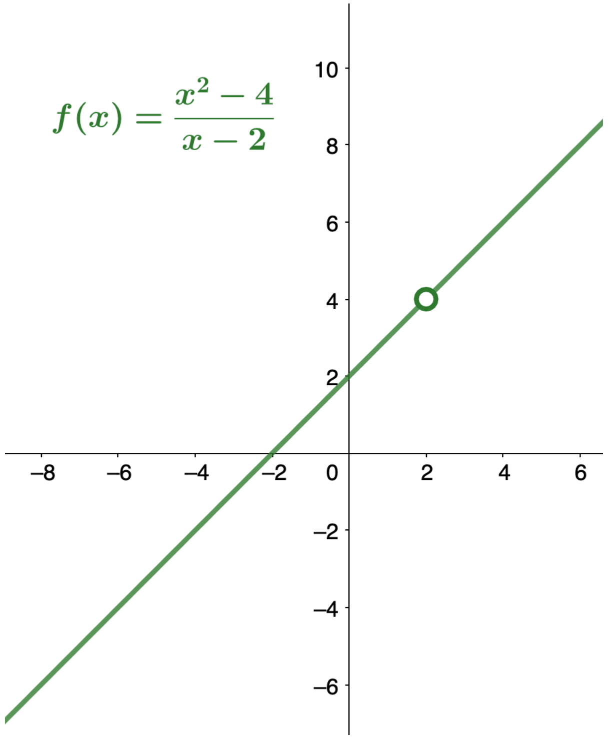
x	$f(x) = c$		x	$f(x) = c$
$a - 0.1$	c		$a + 0.1$	c
$a - 0.01$	c		$a + 0.01$	c
$a - 0.001$	c		$a + 0.001$	c
$a - 0.0001$	c		$a + 0.0001$	c

Table 2.4 Table of Functional Values for $\lim_{x \to a} c = c$

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim_{x \to a} c = c$.



TWO SPECIAL LIMITS



THE EXISTENCE OF A LIMIT

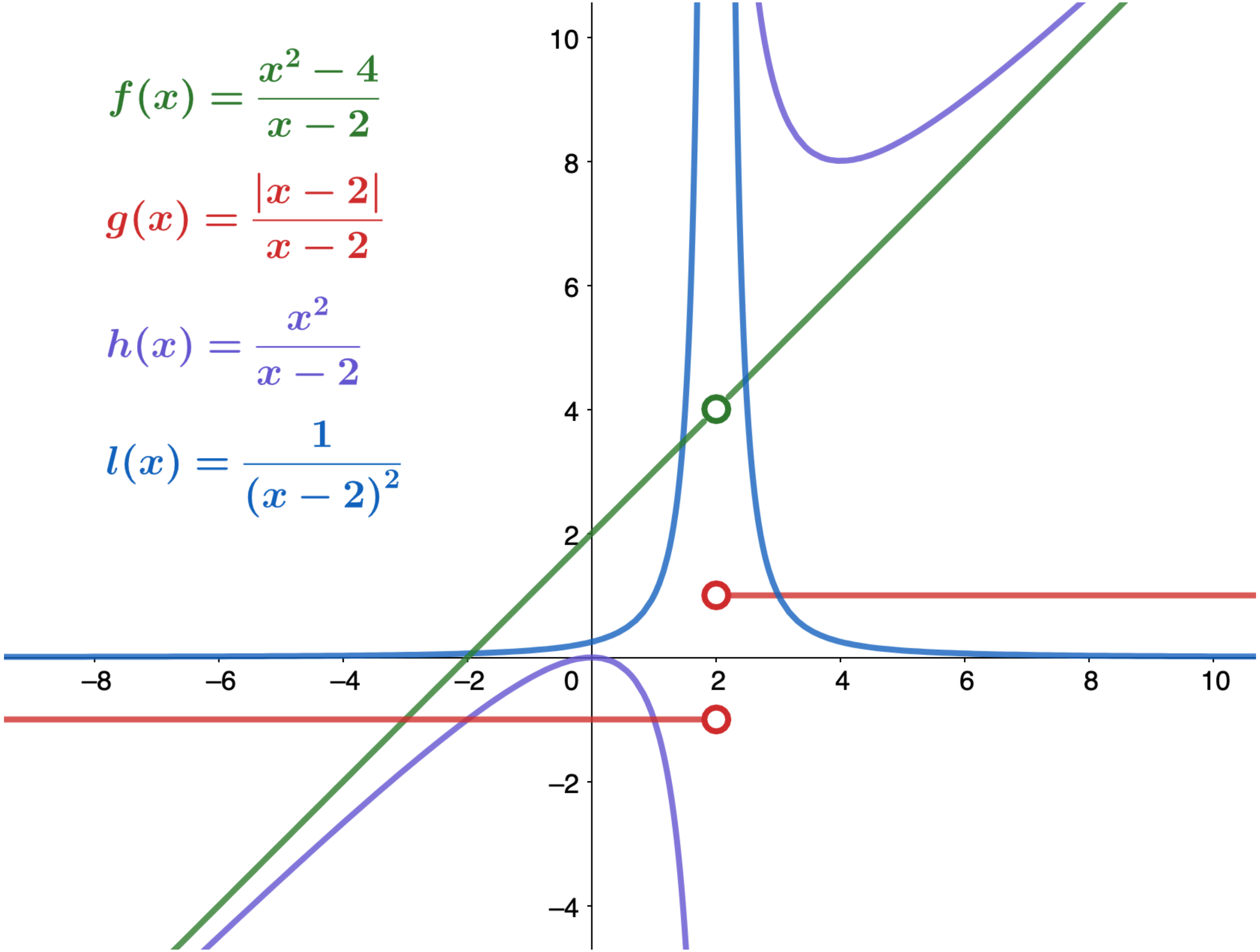
- As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach **a** **single** **real-number** **value** at that point.
- If the functional values do not approach a single value, then the limit does not exist.

$$f(x) = \frac{x^2 - 4}{x - 2}$$

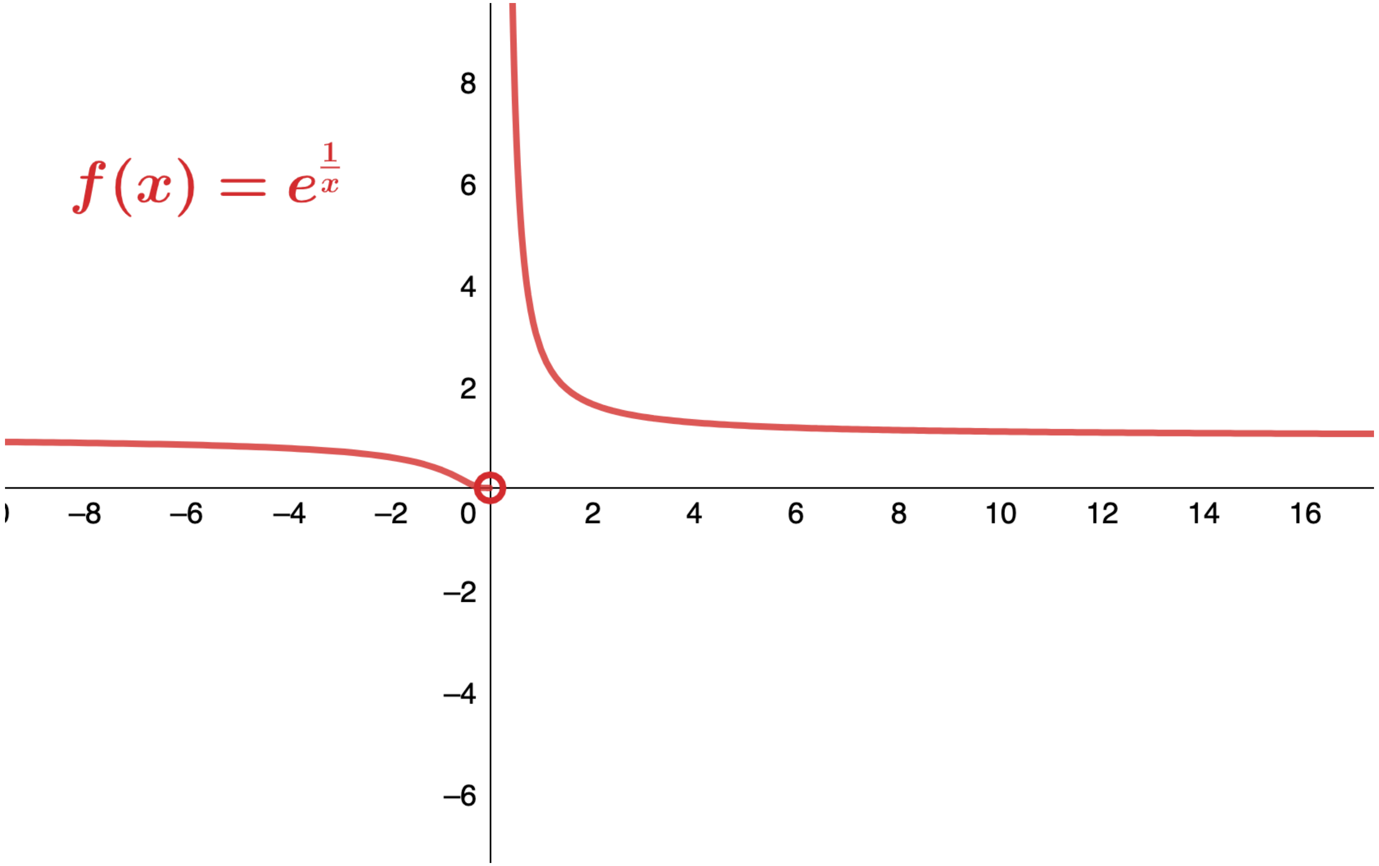
$$g(x) = \frac{|x - 2|}{x - 2}$$

$$h(x) = \frac{x^2}{x - 2}$$

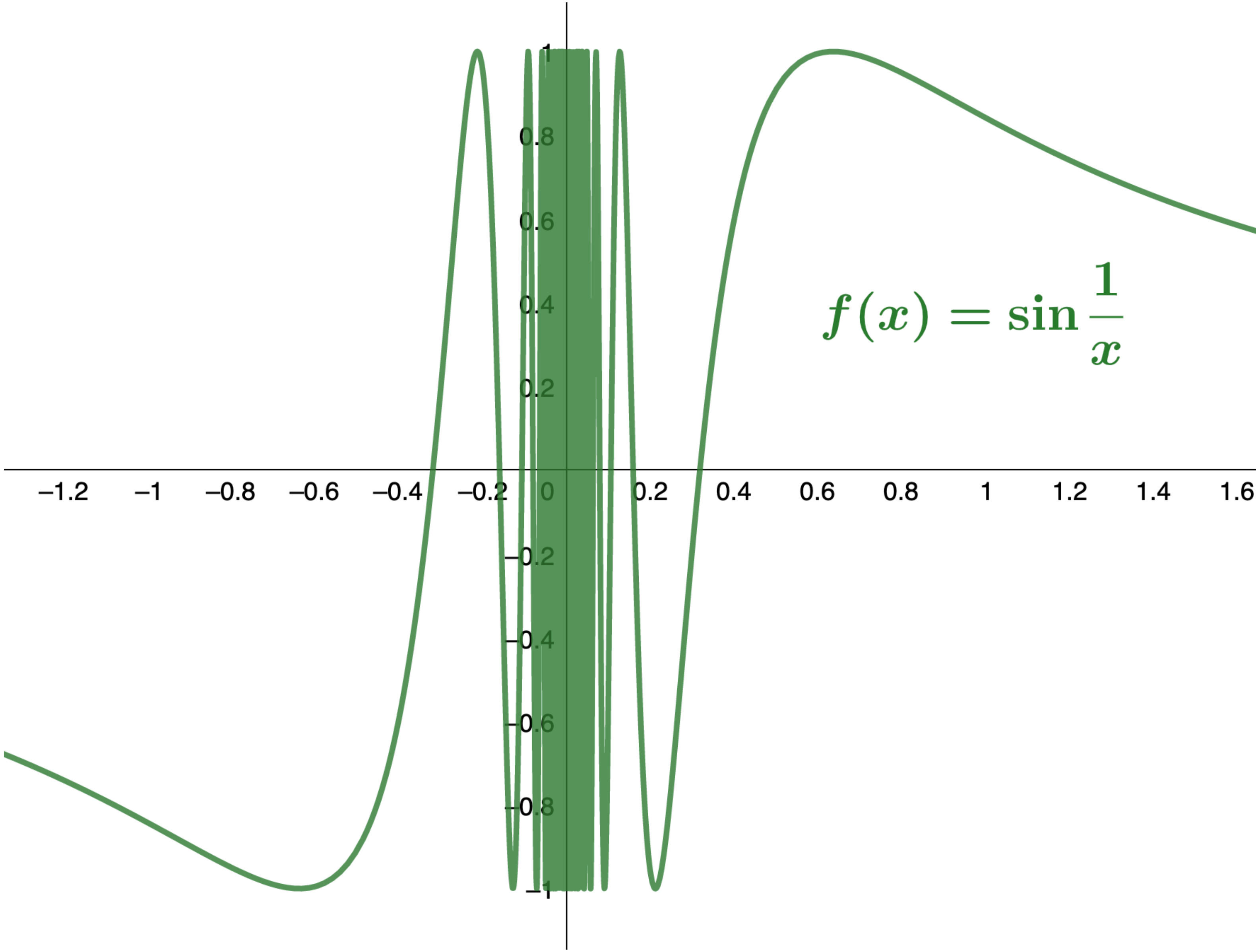
$$l(x) = \frac{1}{(x - 2)^2}$$



EXAMPLE ZERO (LIMIT EXISTS OR NOT?)

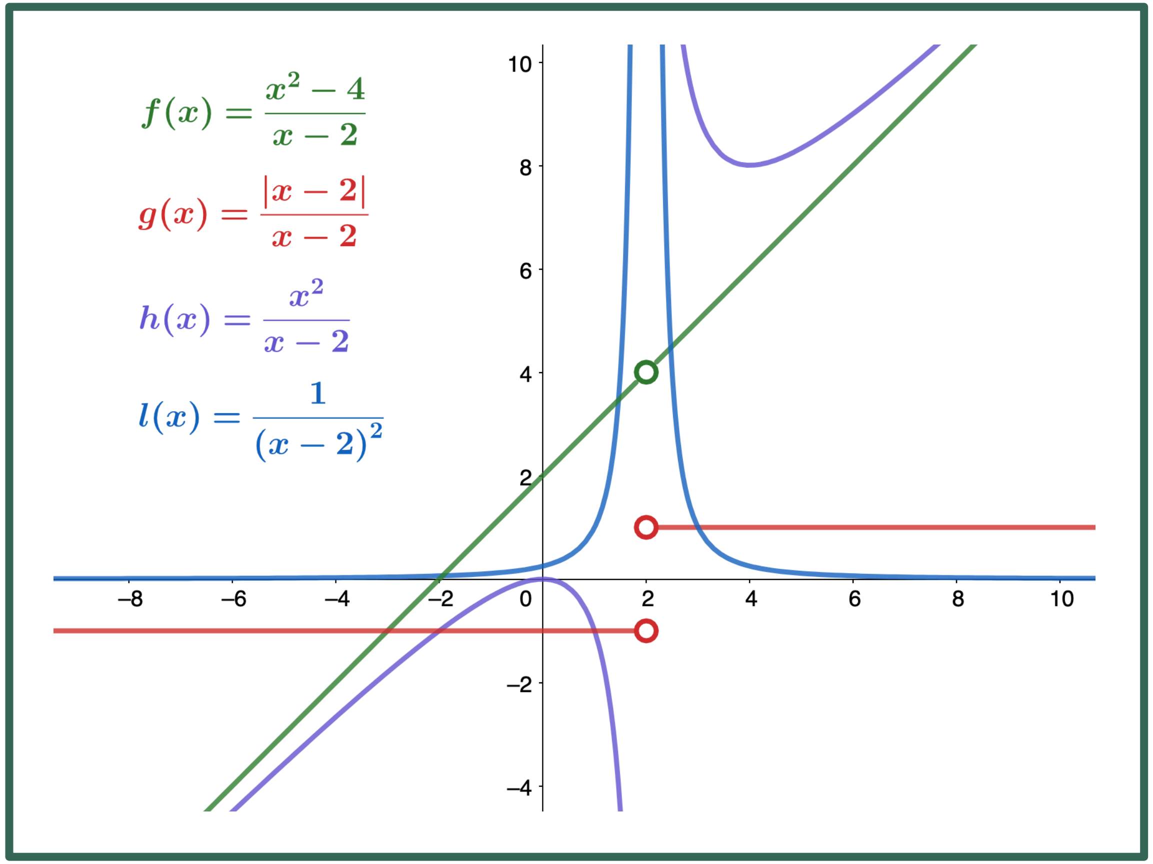


EXAMPLE FOUR



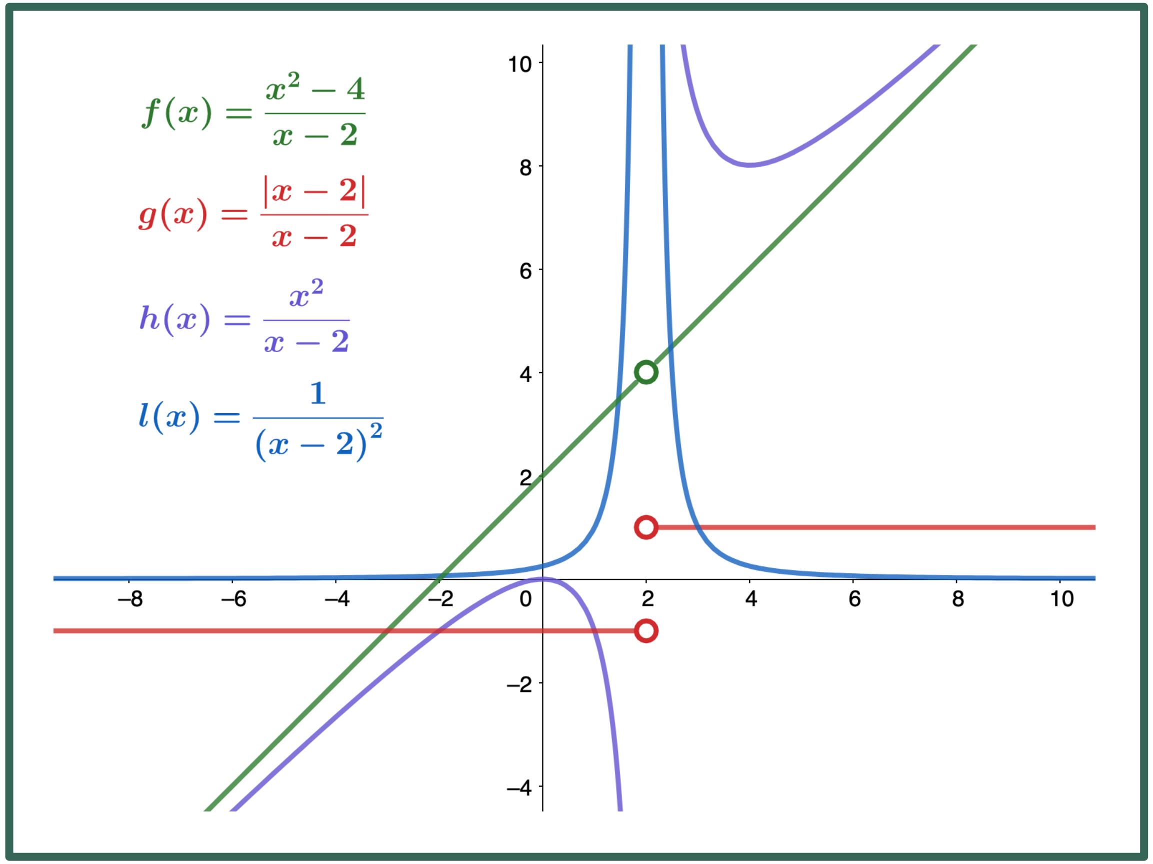
EXAMPLE FIVE

HOW TO GET A COMPLETE PICTURE OF THE BEHAVIOR OF THE FUNCTION



Sometimes indicating that the limit of a function fails to exist at a point **does not provide us with enough information** about the behavior of the function at that particular point.

ONE-SIDED LIMITS



As x approaches 2 from the left, $g(x)$ approaches -1.

Mathematically, we say that the limit as x approaches 2 from the left is -1.

Symbolically, we express this idea as

- $\lim\limits_{x \to 2^-} g(x) = -1.$

Similarly, as x approaches 2 from the right (or *from the positive side*), $g(x)$ approaches 1.

Symbolically, we express this idea as

- $\lim\limits_{x \to 2^+} g(x) = 1.$

ONE-SIDED LIMITS

DEFINITION

We define two types of **one-sided limits**.

Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\boxed{\lim_{x \to a^-} f(x) = L.} \tag{2.6}$$

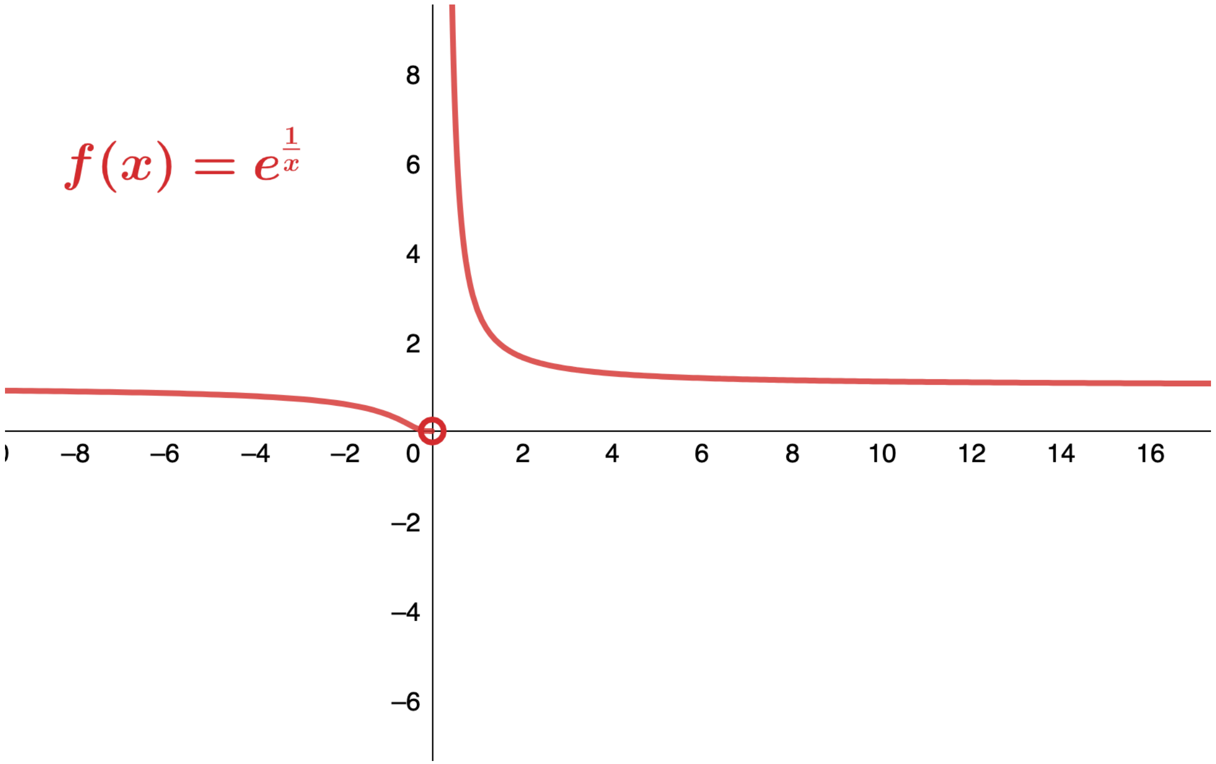
Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\boxed{\lim_{x \to a^+} f(x) = L.} \tag{2.7}$$

EXAMPLE FOUR

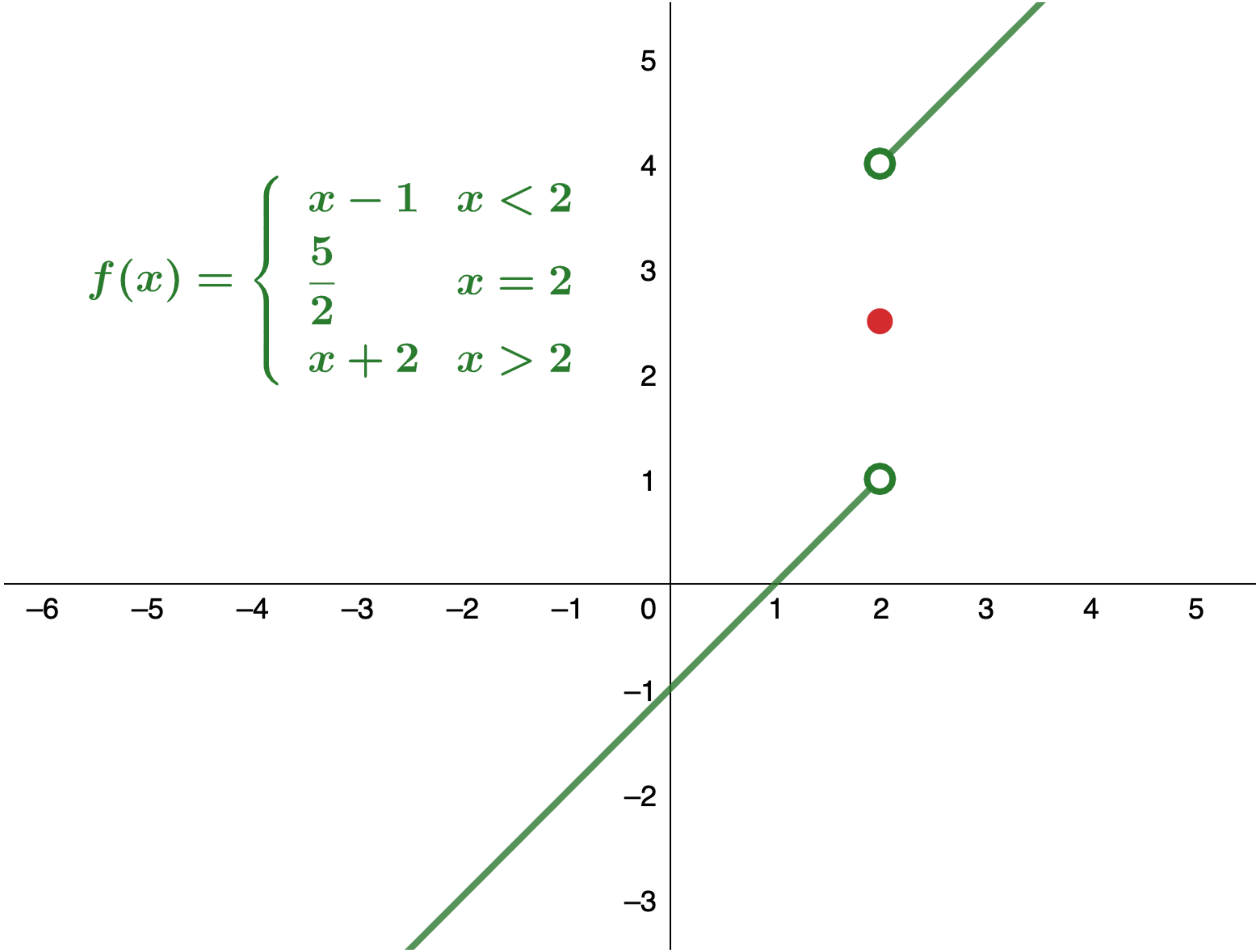
APPROACH FROM THE LEFT

$$f(x) = e^{\frac{1}{x}}$$



EXAMPLE SIX

$$f(x) = \begin{cases} x - 1 & x < 2 \\ \dfrac{5}{2} & x = 2 \\ x + 2 & x > 2 \end{cases}$$



RELATE ONE-SIDED AND TWO-SIDED LIMITS

THEOREM 2.2

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. Then,

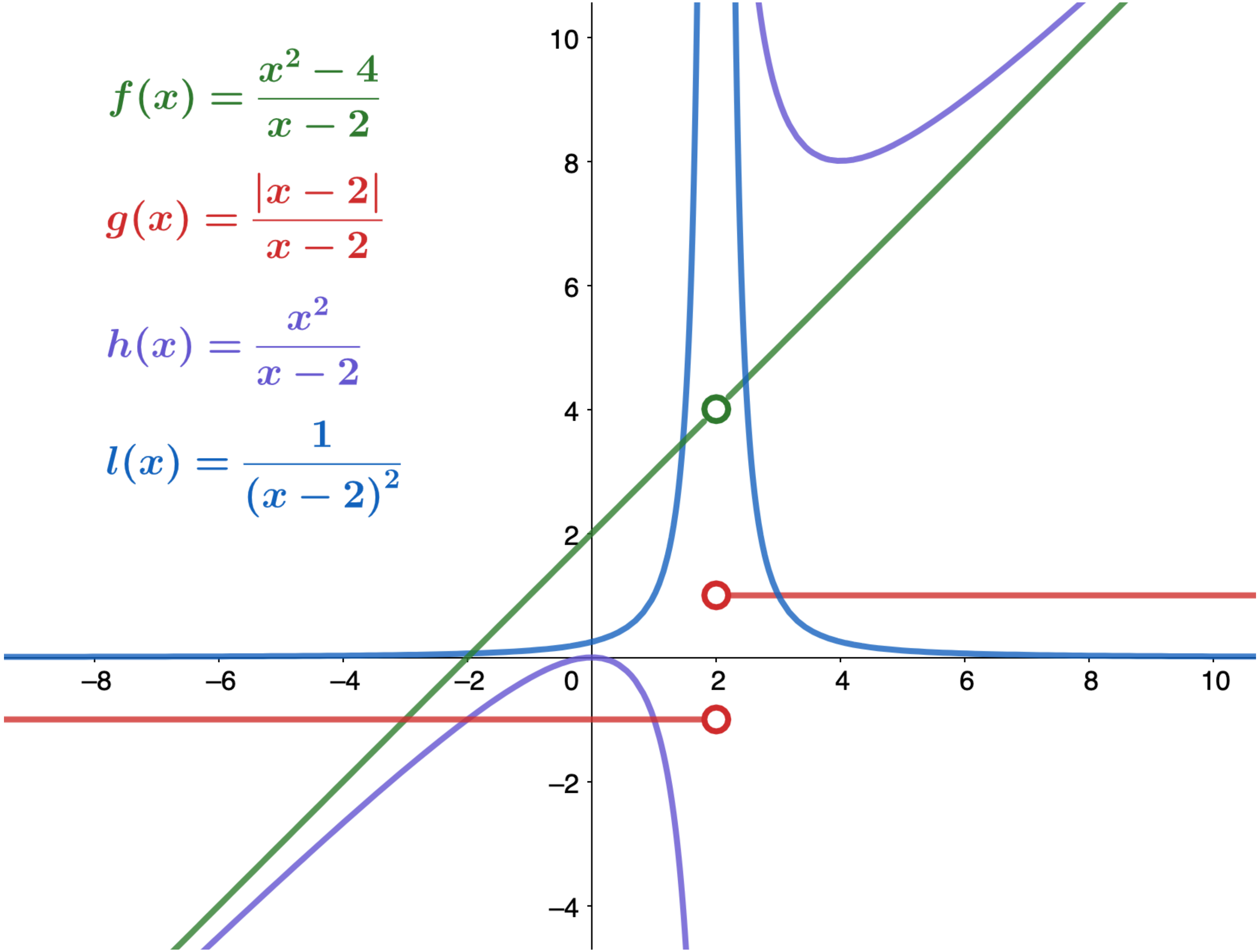
$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

$$f(x) = \frac{x^2 - 4}{x - 2}$$

$$g(x) = \frac{|x - 2|}{x - 2}$$

$$h(x) = \frac{x^2}{x - 2}$$

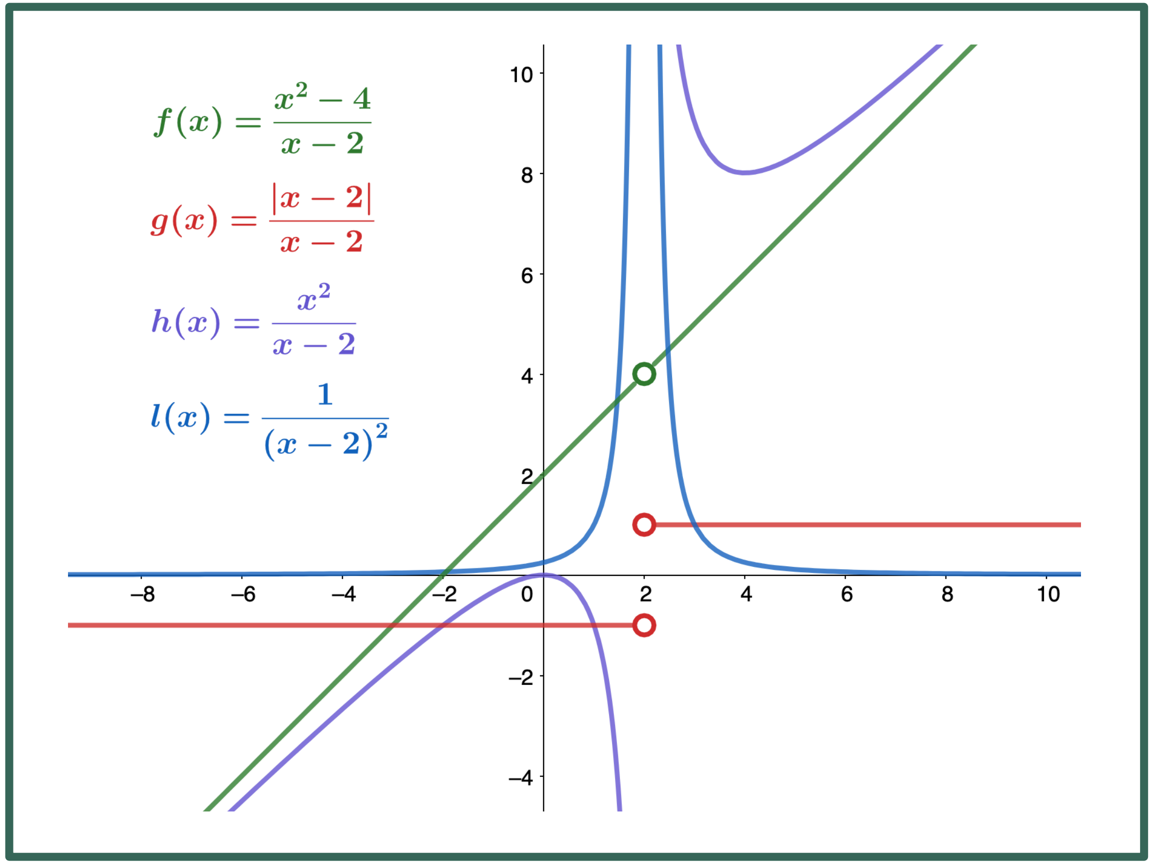
$$l(x) = \frac{1}{(x - 2)^2}$$



INFINITE LIMITS

WE CAN ALSO DESCRIBE THE BEHAVIOR OF FUNCTIONS THAT DO NOT HAVE FINITE LIMITS.

INFINITE LIMITS



As the values of x approach 2, the values of $l(x) = \frac{1}{(x-2)^2}$ become larger and larger and, in fact, become infinite.

Mathematically, we say that the limit of $l(x)$ as x approach 2 is positive infinity.

Symbolically, we express this idea as

- $\lim_{x \to 2} l(x) = +\infty.$

THREE TYPES OF INFINITE LIMITS: FROM THE LEFT

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of *x* (where $x < a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty. \tag{2.8}$$

ii. If the values of $f(x)$ decrease without bound as the values of *x* (where $x < a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty. \tag{2.9}$$

THREE TYPES OF INFINITE LIMITS: FROM THE RIGHT

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of *x* (where $x > a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the left is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty. \tag{2.10}$$

ii. If the values of $f(x)$ decrease without bound as the values of *x* (where $x > a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the left is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty. \tag{2.11}$$

THREE TYPES OF INFINITE LIMITS: TWO-SIDED

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing *a*.

i. If the values of $f(x)$ increase without bound as the values of *x* (where $x \neq a$) approach the number *a*, then we say that the limit as *x* approaches *a* is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty. \tag{2.12}$$

ii. If the values of $f(x)$ decrease without bound as the values of *x* (where $x \neq a$) approach the number *a*, then we say that the limit as *x* approaches *a* is negative infinity and we write

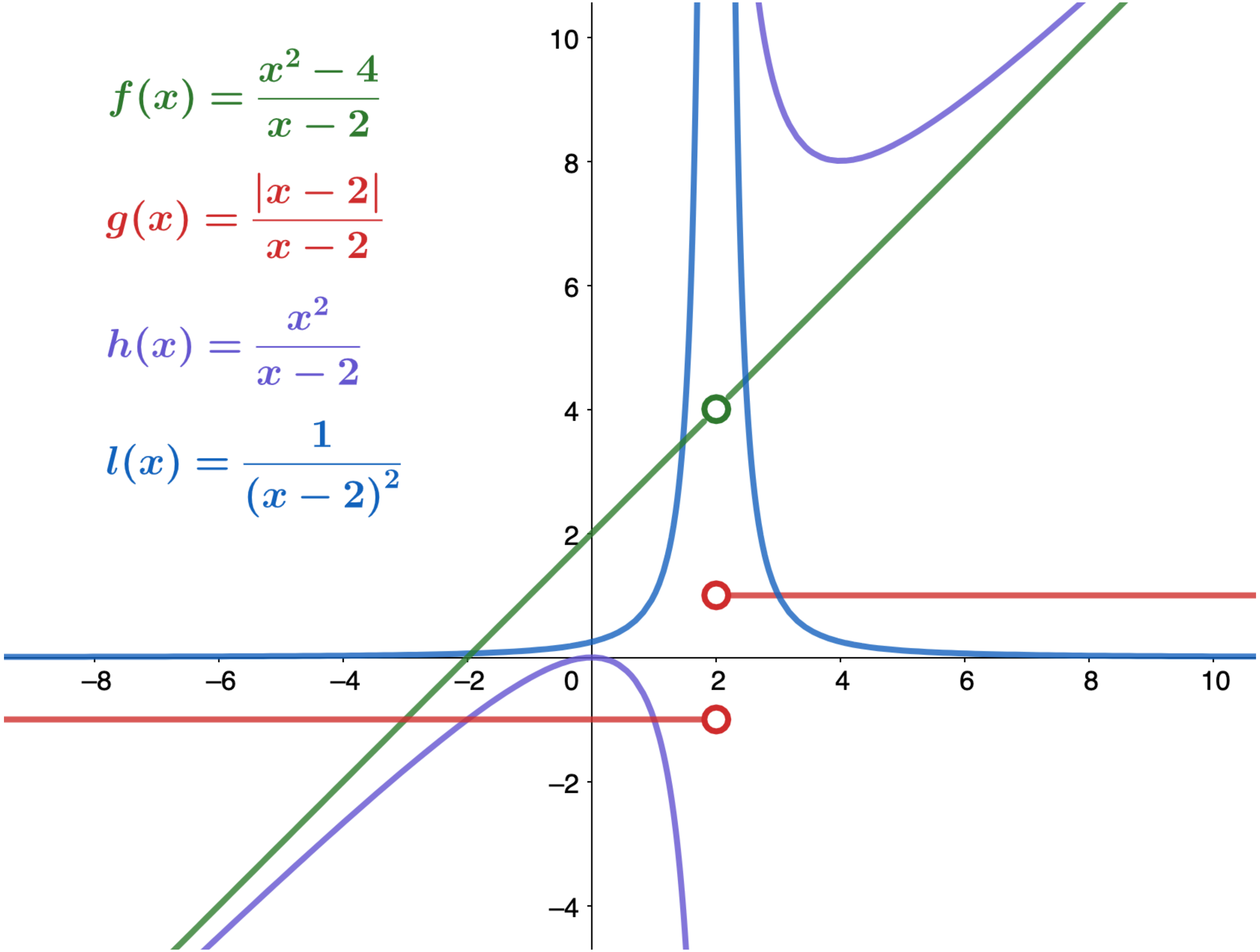
$$\lim_{x \to a} f(x) = -\infty. \tag{2.13}$$

$$f(x) = \frac{x^2 - 4}{x - 2}$$

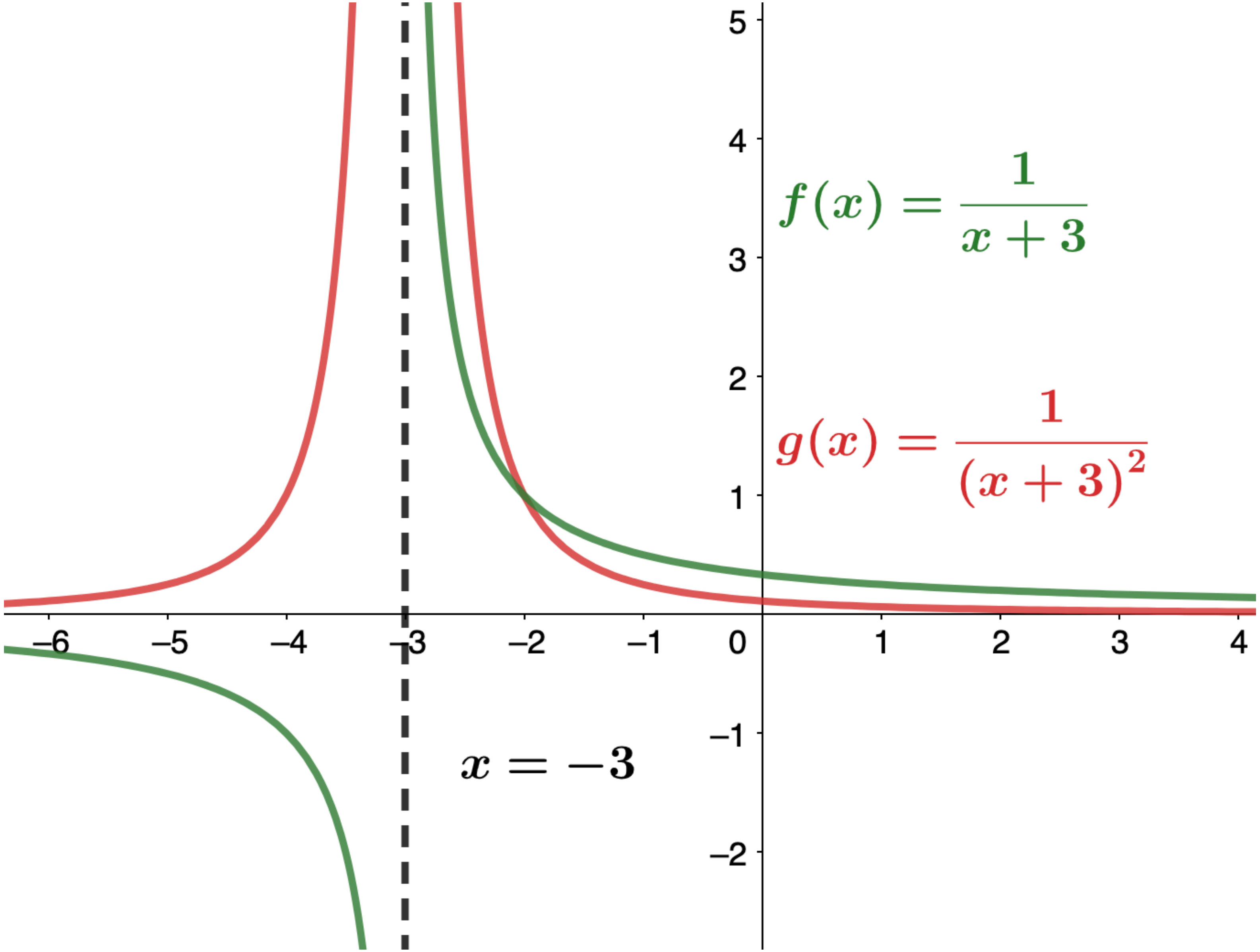
$$g(x) = \frac{|x - 2|}{x - 2}$$

$$h(x) = \frac{x^2}{x - 2}$$

$$l(x) = \frac{1}{(x - 2)^2}$$



EXAMPLE ZERO



EXAMPLE SEVEN

INFINITE LIMITS FROM POSITIVE INTEGERS

THEOREM 2.3

Infinite Limits from Positive Integers

If *n* is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If *n* is a positive odd integer, then

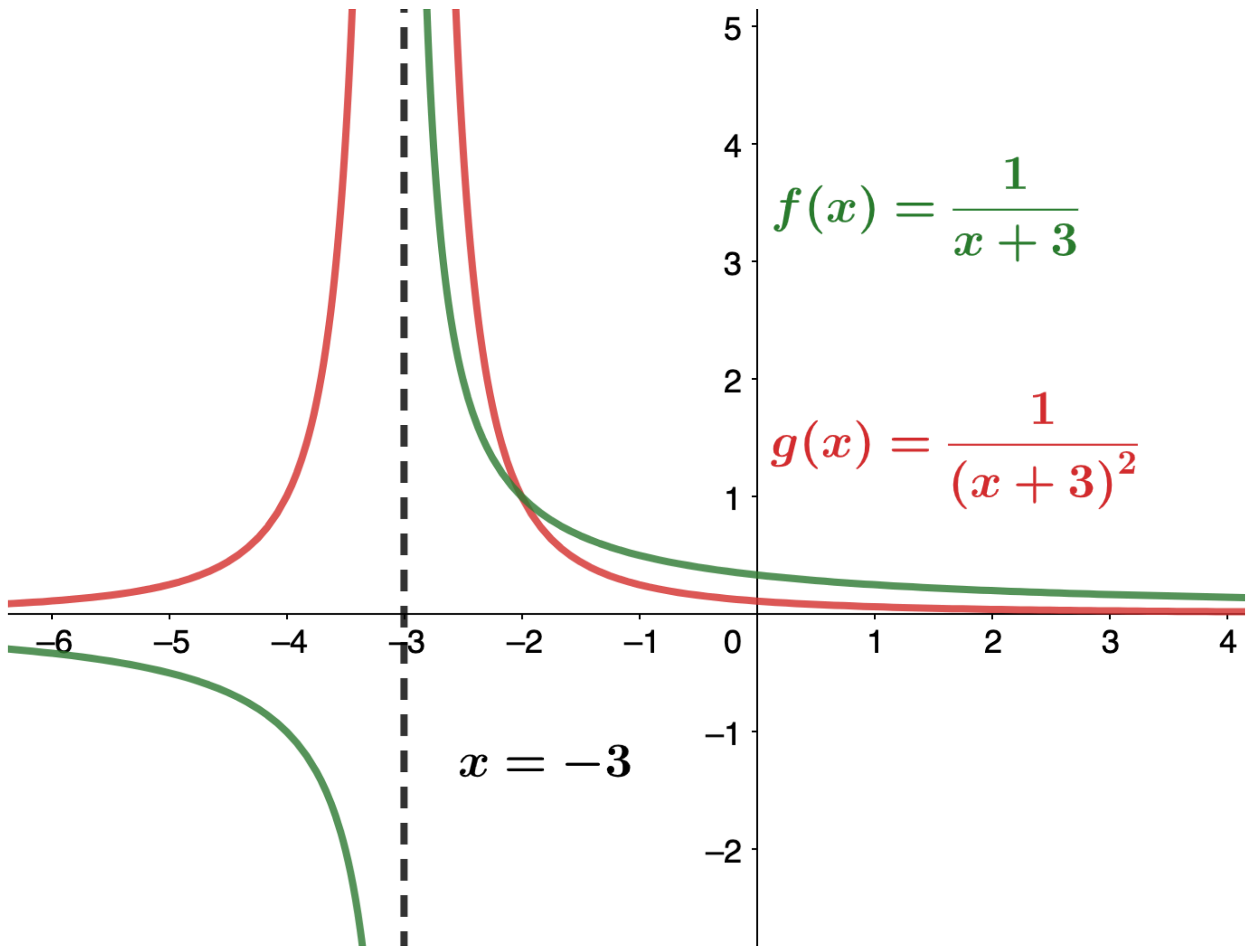
$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

A VERTICAL ASYMPTOTE

- In the graphs of $f(x) = \frac{1}{(x-a)^n}$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$.
- That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph.



A VERTICAL ASYMPTOTE

DEFINITION

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

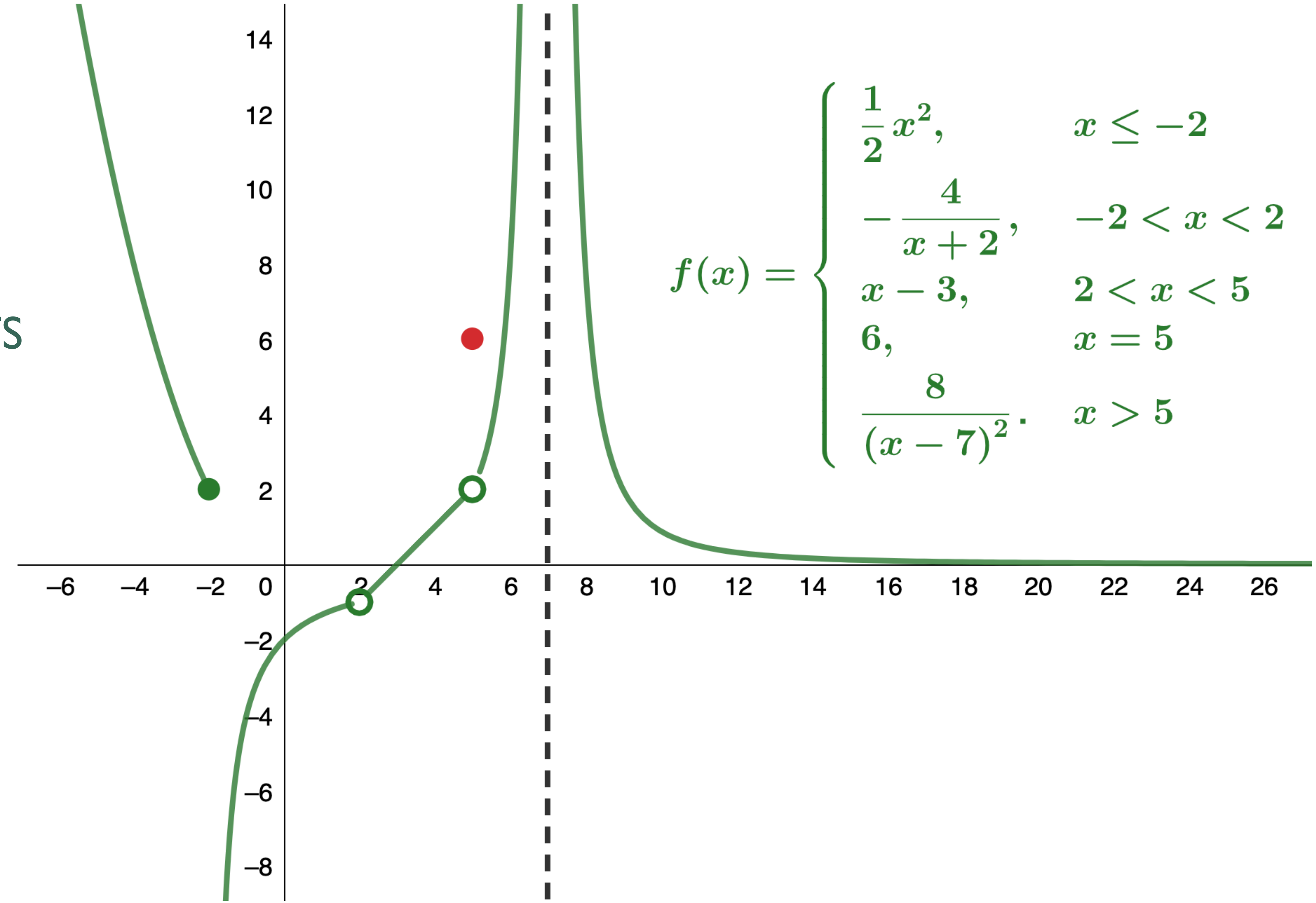
$$\lim_{x \to a^-} f(x) = +\infty \text{ or } -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty \text{ or } -\infty$$

or

$$\lim_{x \to a} f(x) = +\infty \text{ or } -\infty$$

EXAMPLE TERMINATOR: BEHAVIOR OF A FUNCTION AT DIFFERENT POINTS



$$f(x) = \begin{cases} \frac{1}{2}x^2, & x \leq -2 \\ -\frac{4}{x+2}, & -2 < x < 2 \\ x-3, & 2 < x < 5 \\ 6, & x = 5 \\ \frac{8}{(x-7)^2}. & x > 5 \end{cases}$$