DERIVATIVES AND CONTINUITY

- Consider the relationship between differentiability and continuity.
- If a function is differentiable at a point, it **must be** continuous there.
- However, a function that is continuous at a point **need not be** differentiable at that point.
- In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Differentiability $\Rightarrow$ Continuity $\nRightarrow$ Differentiability

DERIVATIVES AND CONTINUITY

DERIVATIVES AND CONTINUITY

THEOREM 3.1

Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

PROOF

$f(x)$ differentiable

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f(x)$ continuous

i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

PROOF

i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

- To show $\lim_{x \to a} f(x) = f(a)$.
- The Left Hand Side is
- $\lim_{x \to a} f(x) = \lim_{x \to a}\big(f(x) - f(a)\big) + f(a)) =$
  $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}(x - a) + f(a)$

PROOF

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

- To show $\lim_{x \to a} f(x) = f(a)$.
- The Left Hand Side is
- $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) + f(a) = \lim_{x \to a} f'(a)(x - a) + f(a) = \lim_{x \to a} f'(a)(x - a) + \lim_{x \to a} f(a) = 0 + f(a) = f(a)$
- The Right Hand Side is
- $f(a)$

REMARKS

- Mathematatic technique
- $\lim_{x \to a} f(x) = \lim_{x \to a}\big(f(x) - f(a)\big) + f(a))$

ON THE OTHER HAND

$f(x)$ differentiable

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f(x)$ continuous

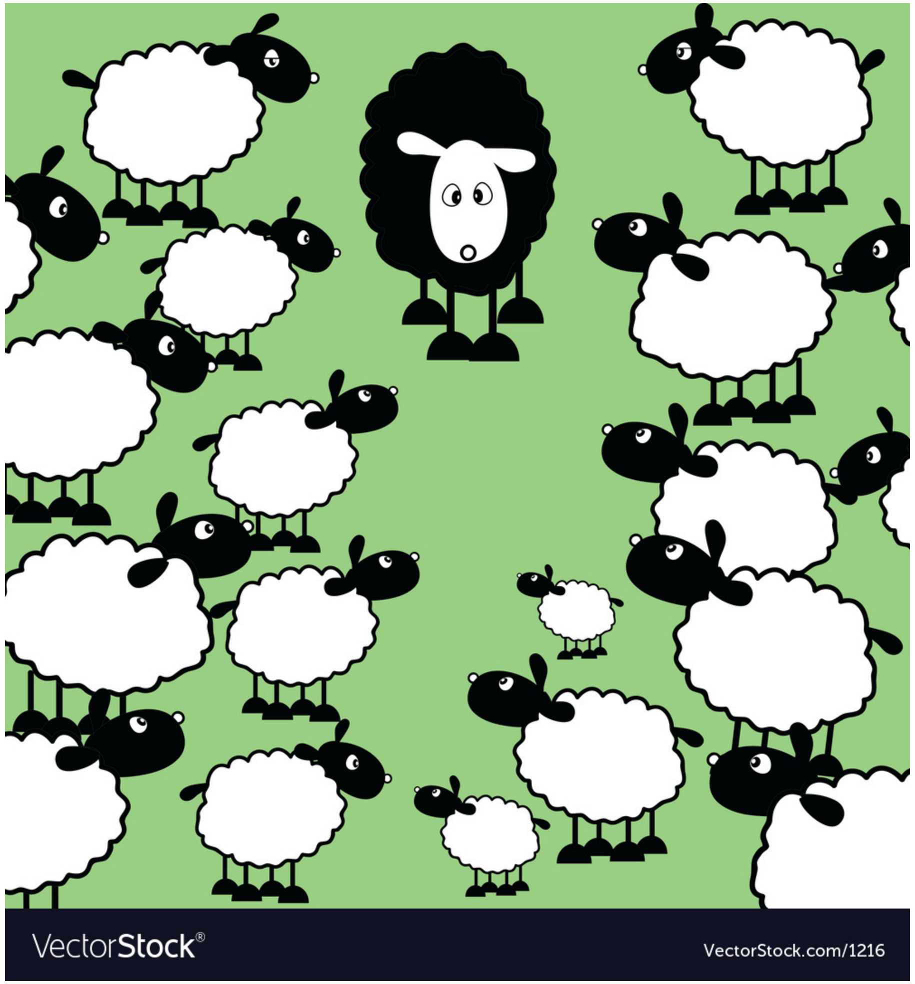
i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

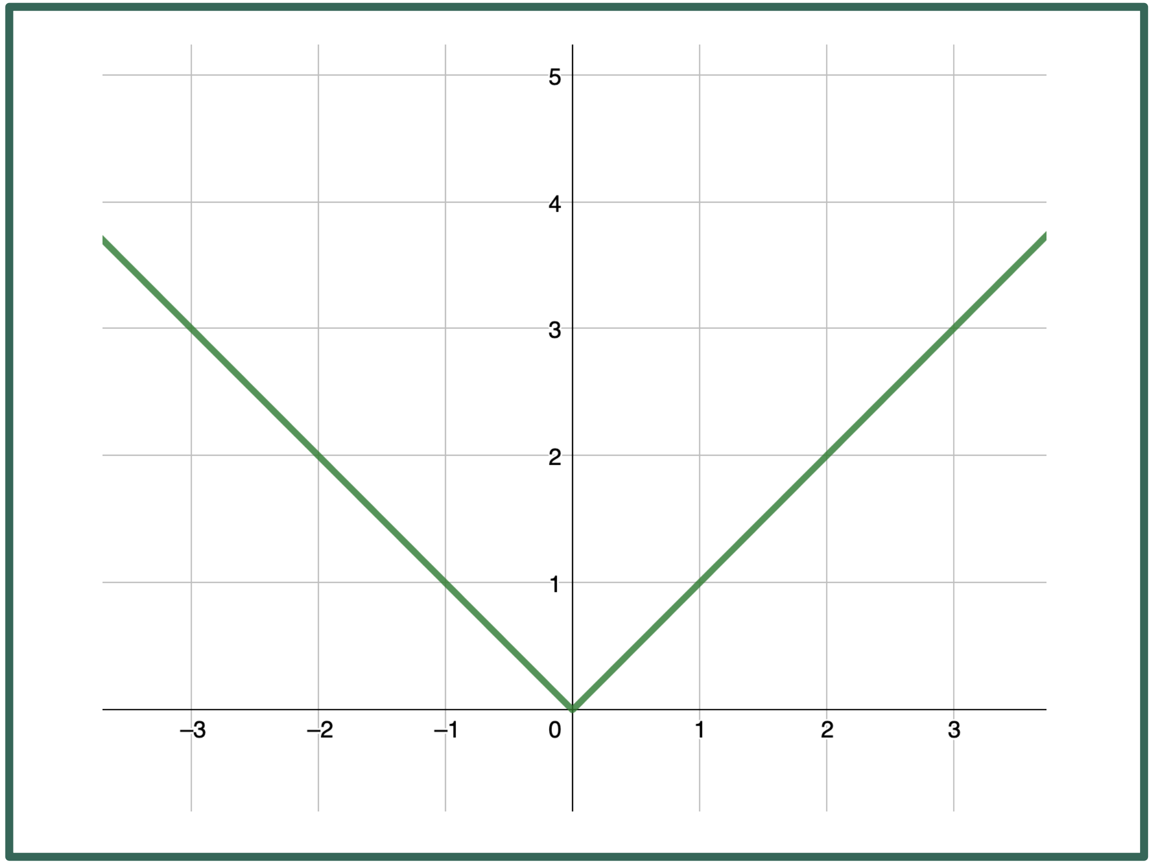
iii. $\lim_{x \to a} f(x) = f(a)$

HOW TO SHOW THAT?

Use **counterexamples** to disprove statements that are false!



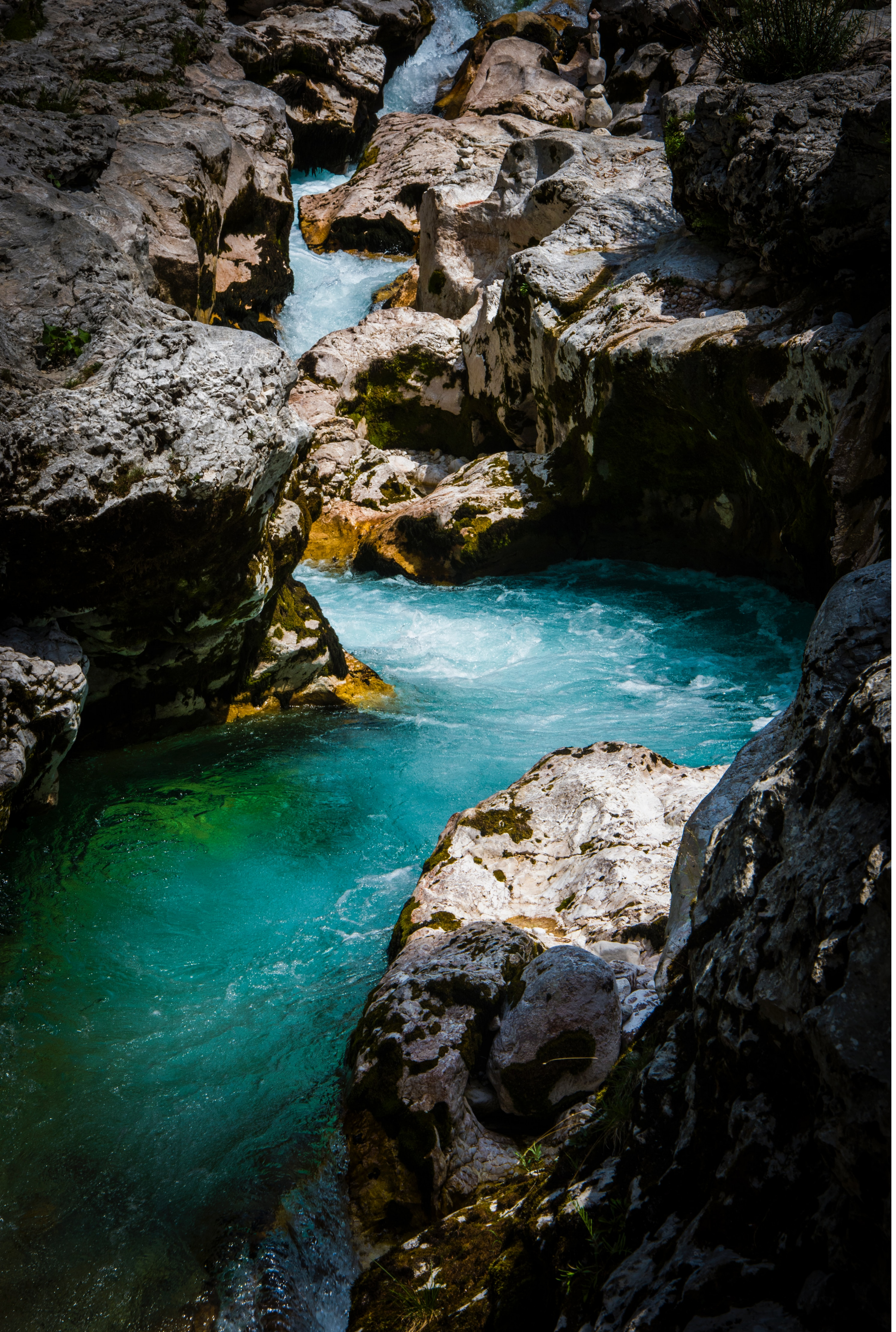
FIRST COUNTER EXAMPLE

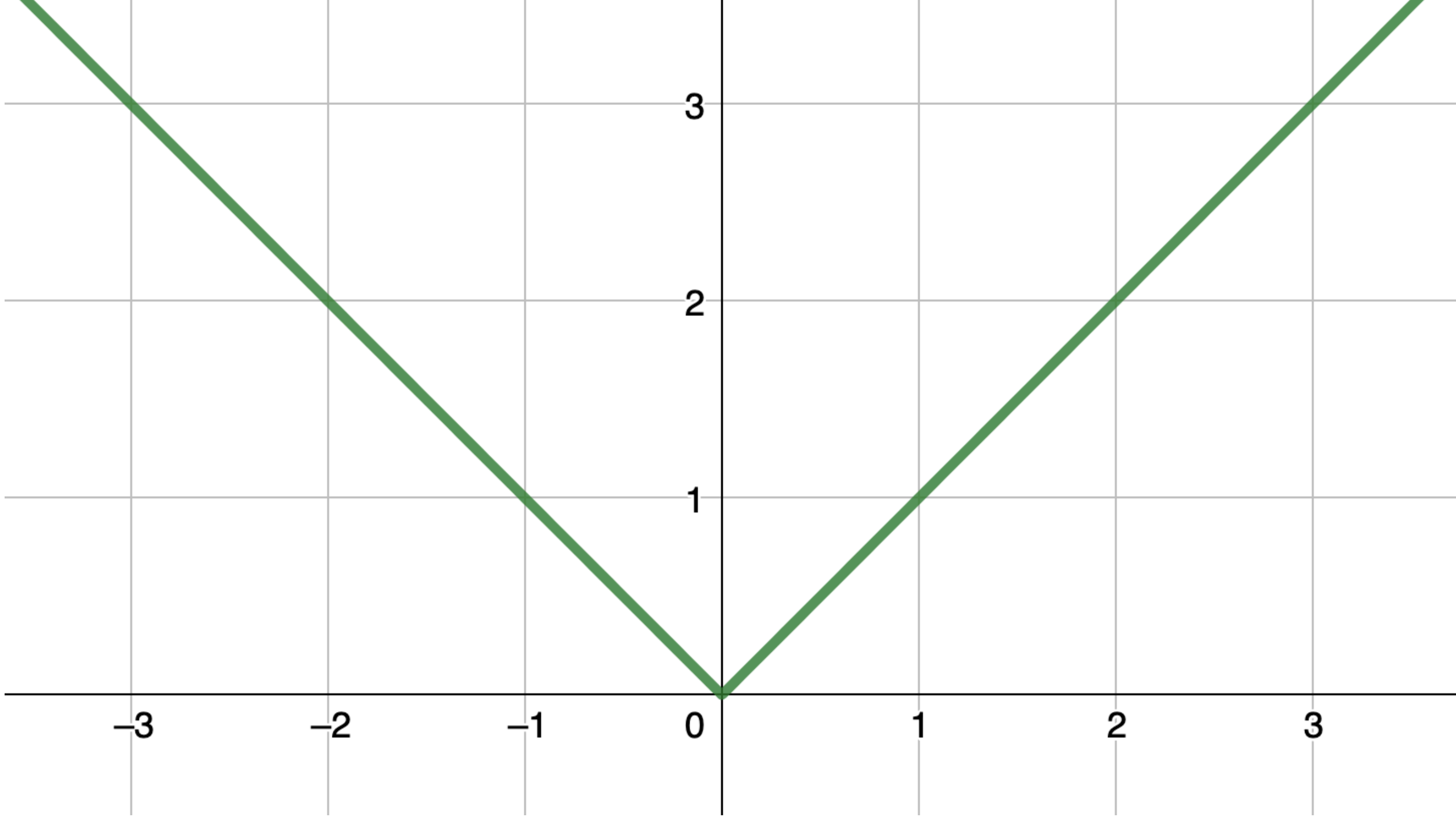


$$f(x) = |x|$$

Is it continuous at $x = 0$?

- $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$
- $\lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$
- $\lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$
- $-1 \neq 1$





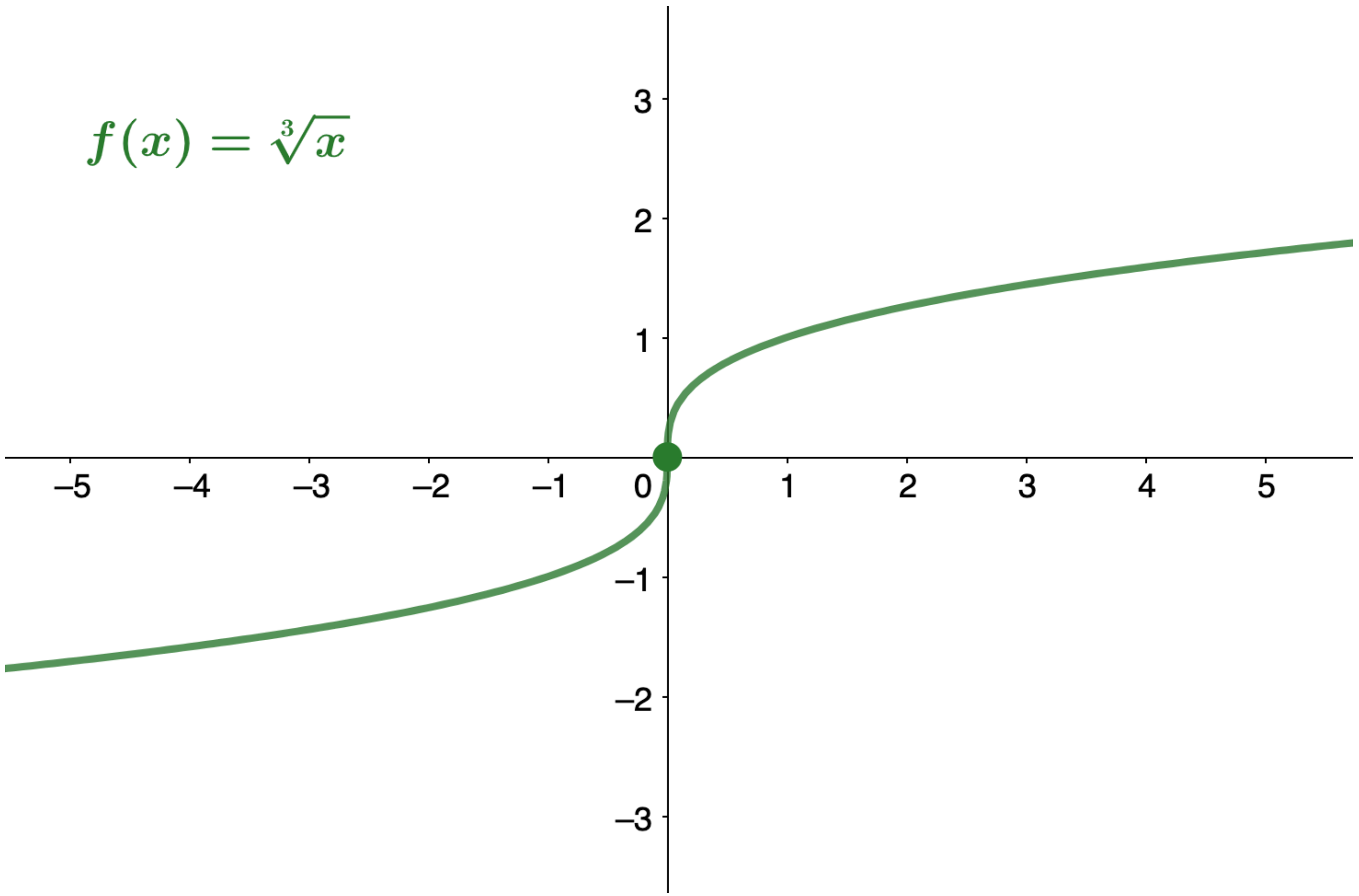
FIRST COUNTER EXAMPLE

SECOND COUNTER EXAMPLE

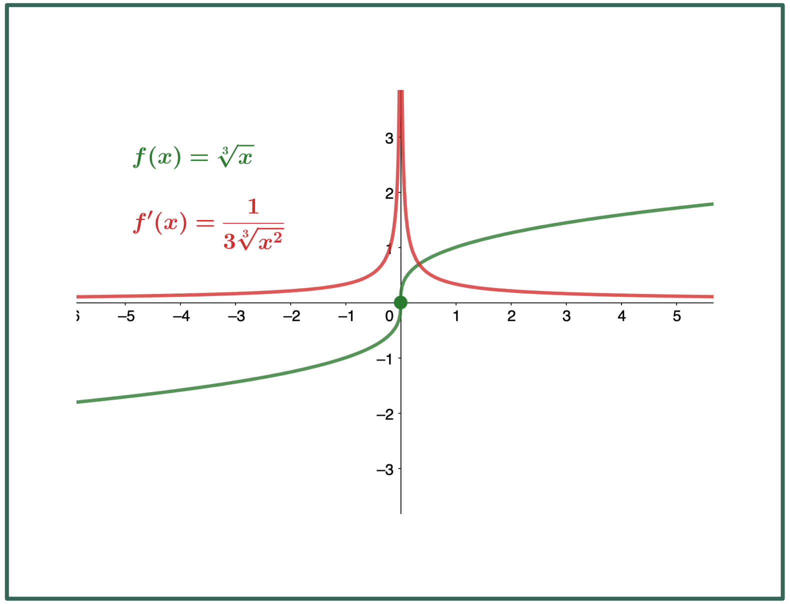
- $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} =$

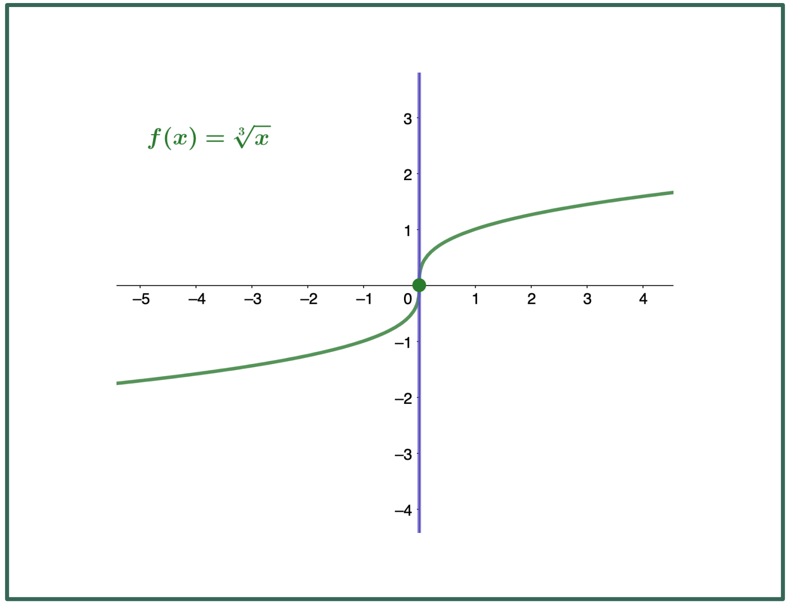
  $\lim_{x\to 0} \frac{\sqrt[3]{x}}{x} =$

  $\lim_{x\to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$



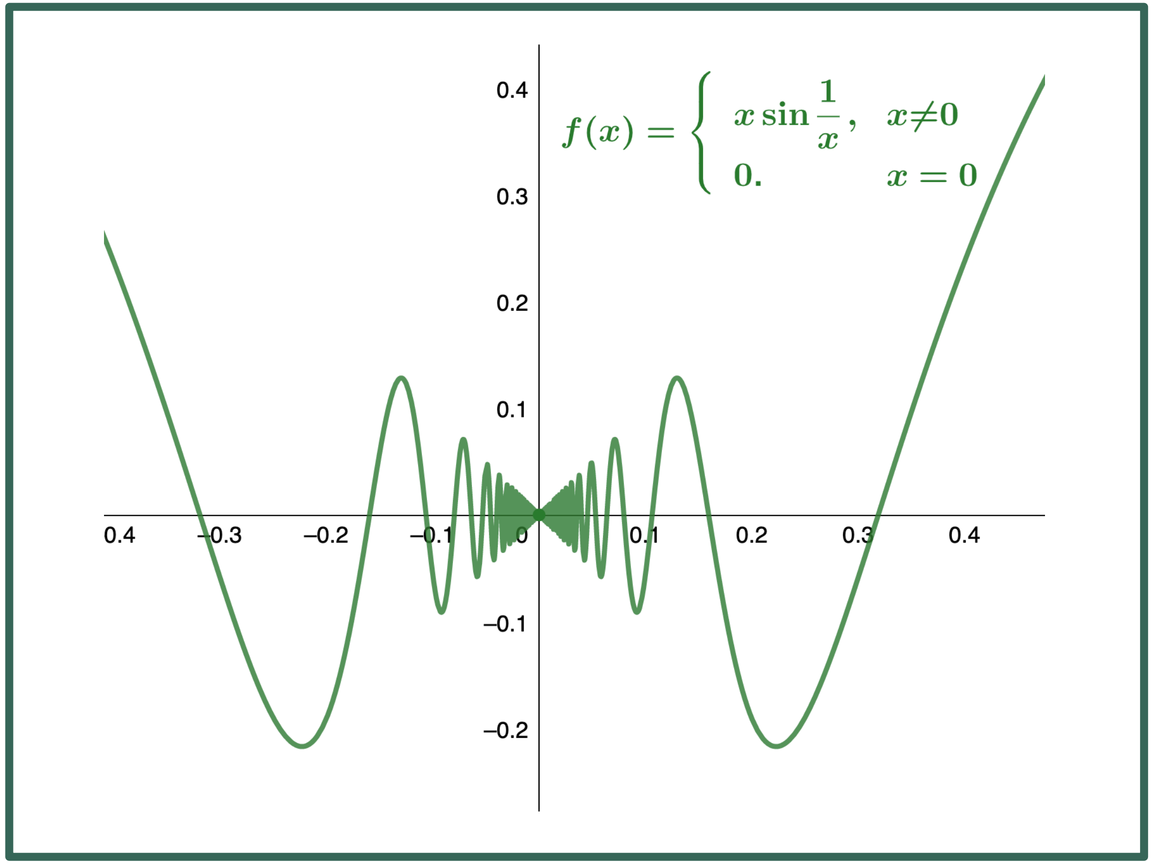
SECOND COUNTER EXAMPLE



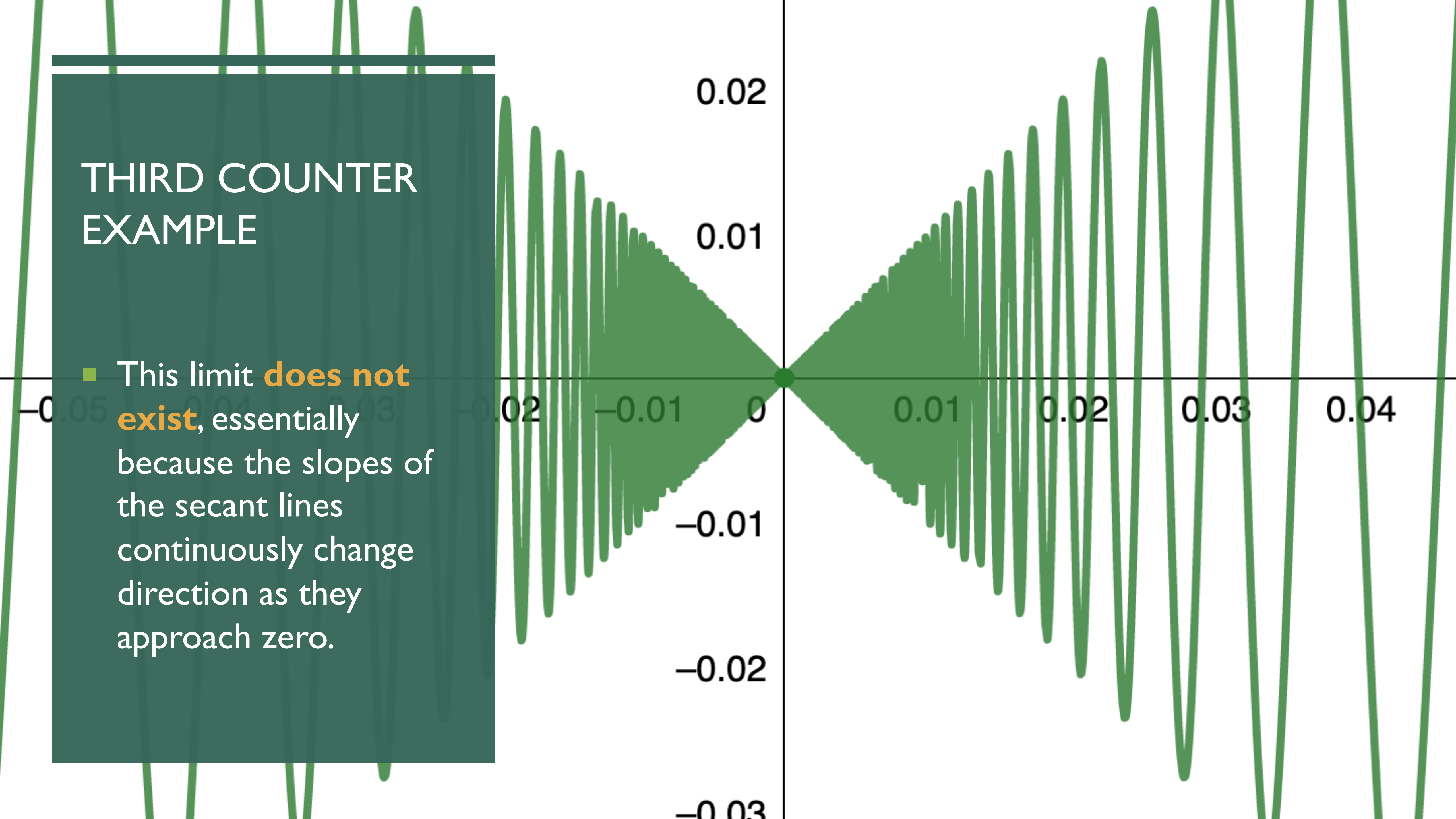


- $f'(0)$ does not exist.
- Function has **a vertical tangent line** at 0.

THIRD COUNTER EXAMPLE



- $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x\sin\frac{1}{x}}{x} = \lim_{x\to 0}\sin\frac{1}{x}.$
- $x = \frac{1}{2n\pi}$, where $n = 1, 10, 100, 1000, \cdots$
- $\sin\frac{1}{x} = \sin 2n\pi = 0$
- $x = \frac{1}{(2n+\frac{1}{2})\pi}$, where $n = 1, 10, 100, 1000, \cdots$
- $\sin\frac{1}{x} = \sin(2n+\frac{1}{2})\pi = 1$



THIRD COUNTER EXAMPLE

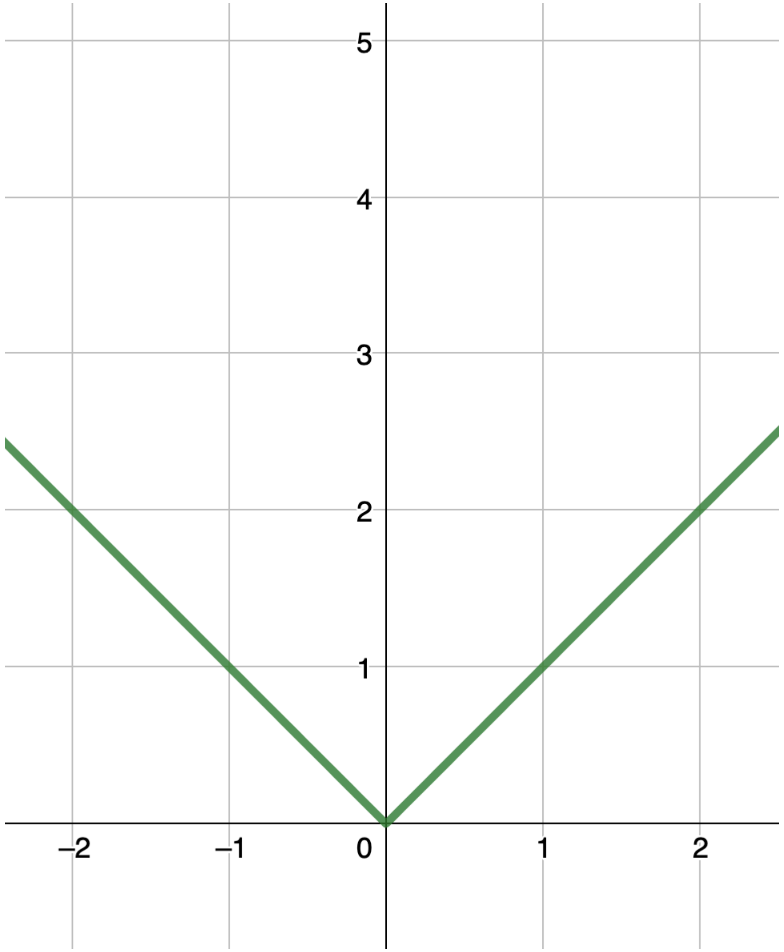
- This limit **does not exist**, essentially because the slopes of the secant lines continuously change direction as they approach zero.

SUMMARY

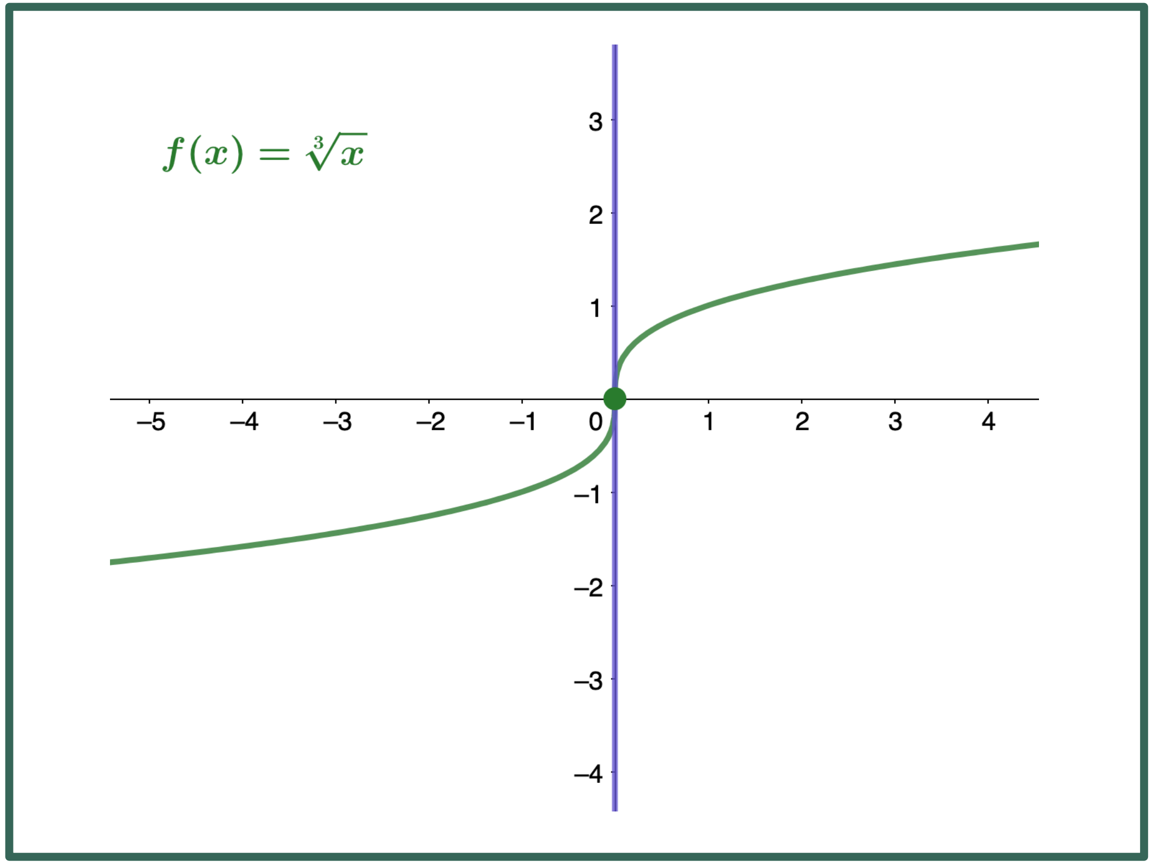
We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.

SUMMARY

- $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same.
- Visually, this resulted in **a sharp corner** on the graph of the function at 0.
- In order to be differentiable at a point, a function must be "**smooth**" at that point.



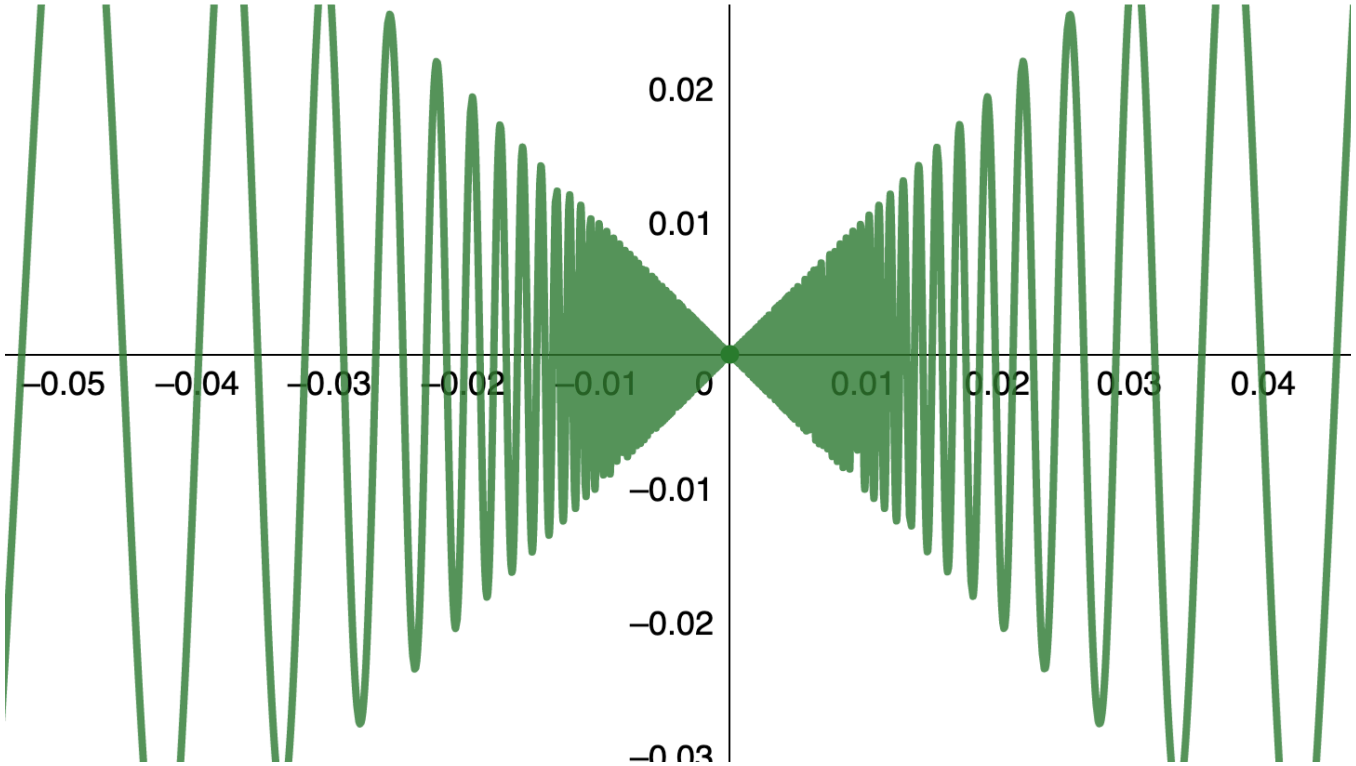
SUMMARY



- As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is **a vertical tangent line**.

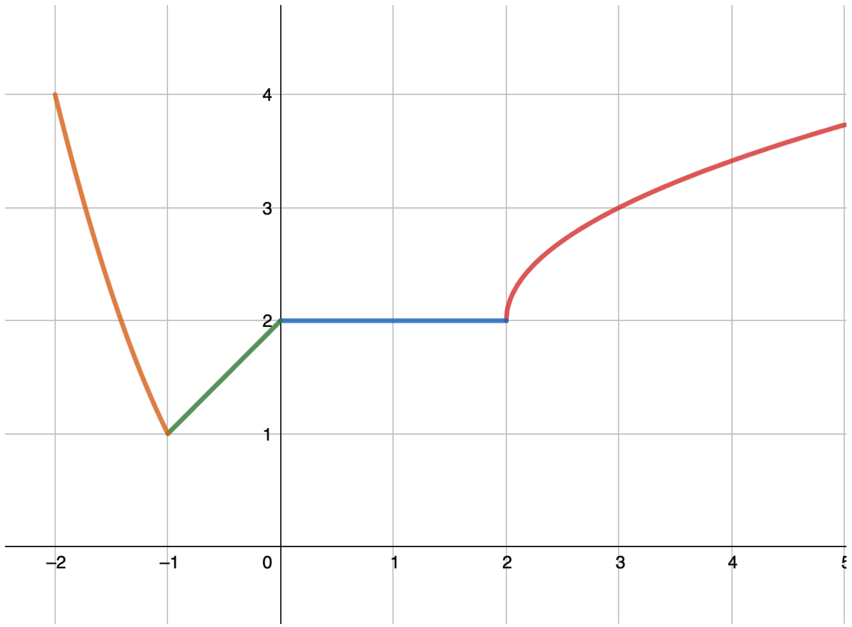
SUMMARY

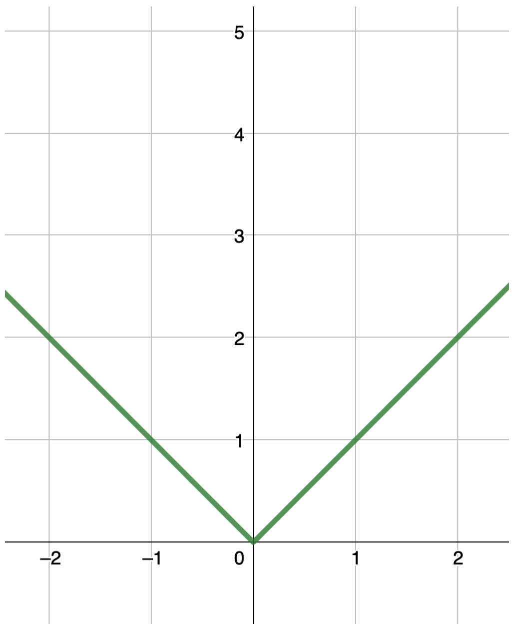
A FUNCTION MAY FAIL TO BE DIFFERENTIABLE AT A POINT IN MORE COMPLICATED WAYS AS WELL.



REVISIT PIECEWISE FUNCTIONS

PIECEWISE FUNCTIONS THAT ARE NOT CONTINUOUS AND DIFFERENTIABLE.

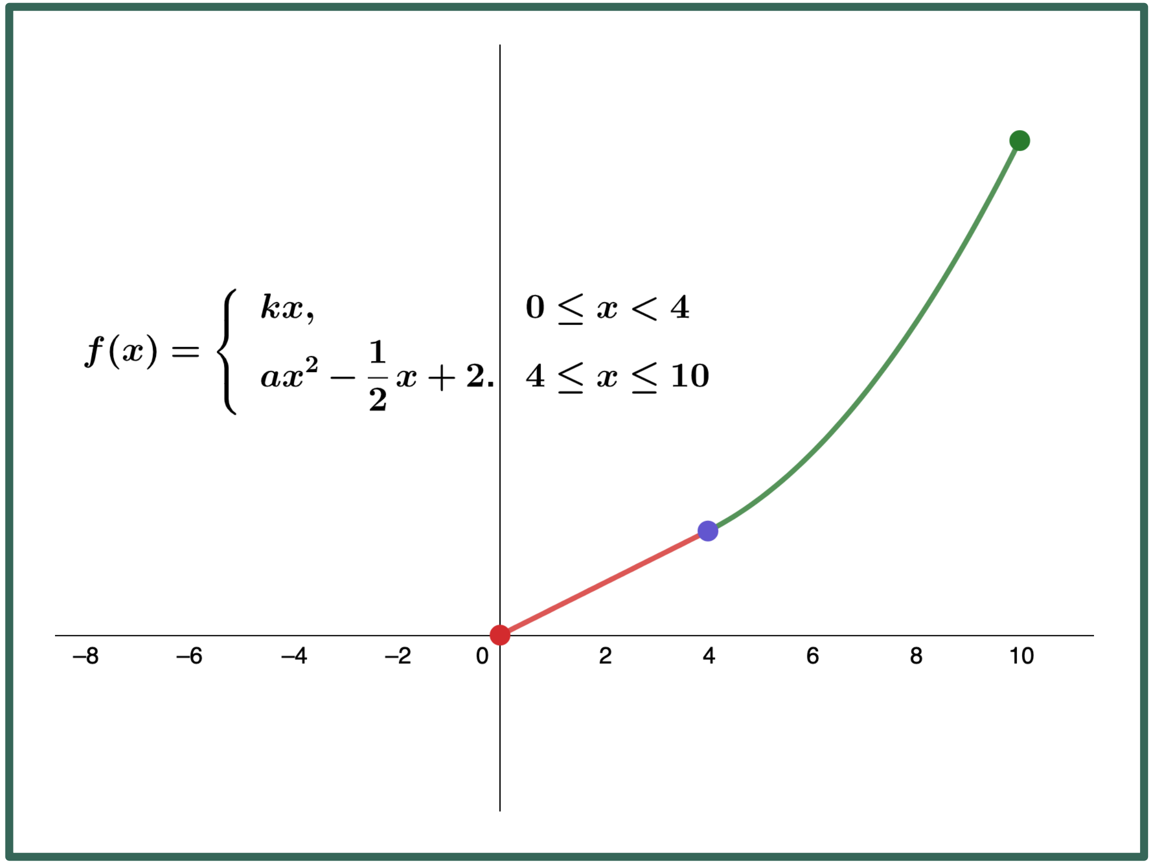




EXERCISE: A PIECEWISE FUNCTION THAT IS CONTINUOUS AND DIFFERENTIABLE

- A ***piste*** (/piːst/)[1] is a marked ski run or path down a mountain for snow skiing, snowboarding, or other mountain sports.
- Pistes are usually maintained using tracked vehicles known as snowcats to compact or "groom" the snow to even out trail conditions, remove moguls, and redistribute snow to extend the ski season.

EXERCISE: A PIECEWISE FUNCTION THAT IS CONTINUOUS AND DIFFERENTIABLE



Dartmouth Skiway wants to design a new piste this year that starts out along a straight line and then converts to a parabolic curve. The function that describes the track is to have the form

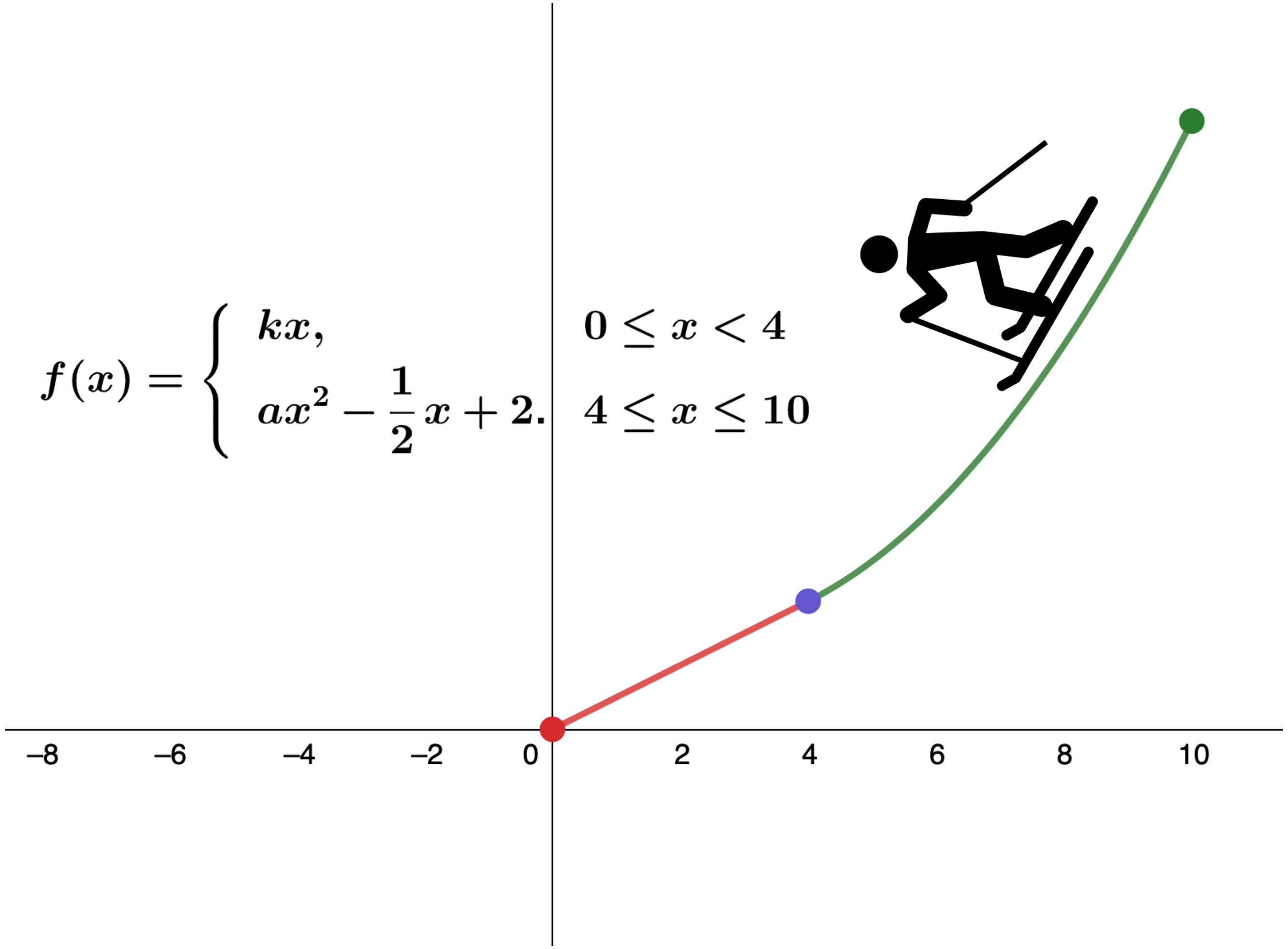
$$f(x) = \begin{cases} kx, & 0 \le x < 4 \\ ax^2 - \frac{1}{2}x + 2, & 4 \le x \le 10 \end{cases}$$

where x and $f(x)$ are in inches.

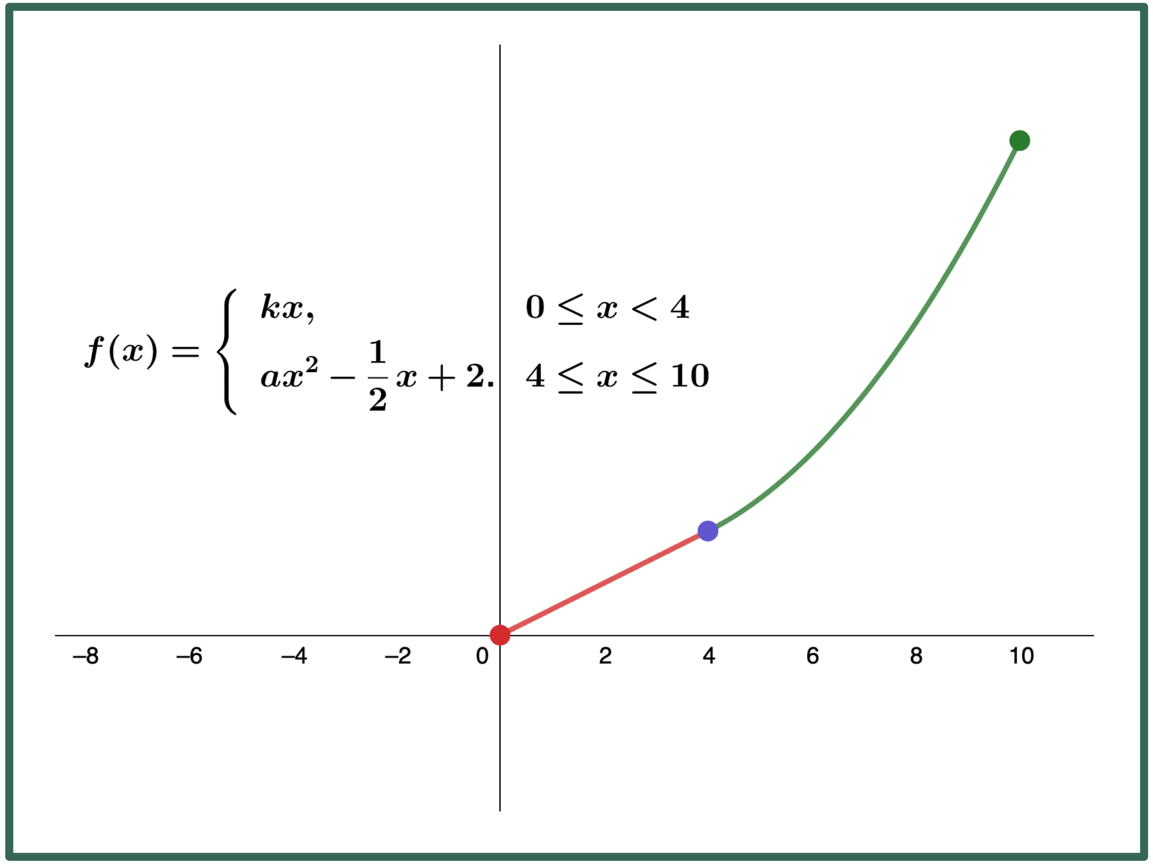
EXERCISE: A PIECEWISE FUNCTION THAT IS CONTINUOUS AND DIFFERENTIABLE

- For the skier to move smoothly along the piste, the function $f(x)$ must be both continuous and differentiable at 4.
- Find values of k and a that make $f(x)$ both continuous and differentiable.

$$f(x) = \begin{cases} kx, & 0 \leq x < 4 \\ ax^2 - \frac{1}{2}x + 2. & 4 \leq x \leq 10 \end{cases}$$



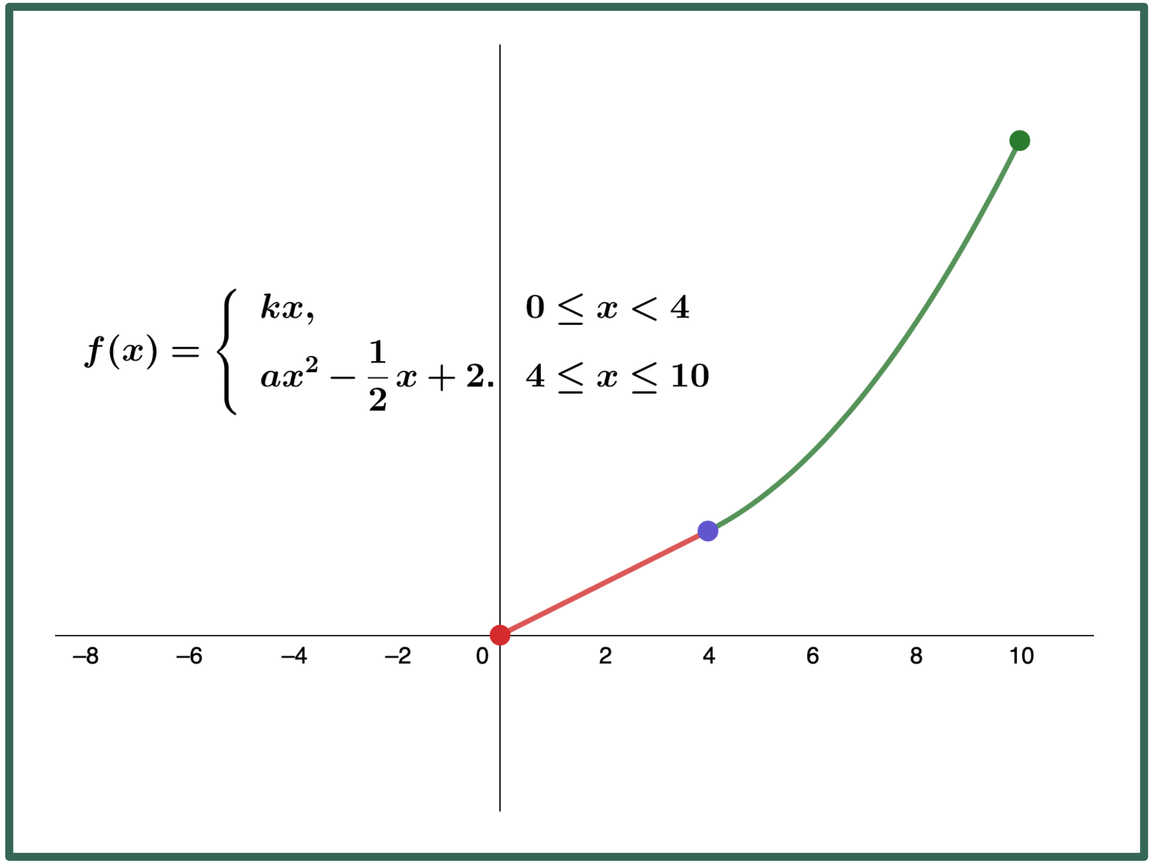
EXERCISE: A PIECEWISE FUNCTION THAT IS CONTINUOUS AND DIFFERENTIABLE



$f(x)$ be continuous at 4.

- $k \cdot 4 = a \cdot 4^2 - \frac{1}{2} \cdot 4 + 2 = a \cdot 16$
- $k = 4a$

EXERCISE: A PIECEWISE FUNCTION THAT IS CONTINUOUS AND DIFFERENTIABLE



$f(x)$ be differentiable at 4.

- $\lim_{x \to 4^-} \frac{f(x)-f(4)}{x-4} = k = 4a$
- $\lim_{x \to 4^+} \frac{f(x)-f(4)}{x-4} = \lim_{x \to 4^+} \frac{ax^2-\frac{1}{2}x+2-(16a-2+2)}{x-4} =$ $\lim_{x \to 4^+} \frac{ax^2-\frac{1}{2}x+2-16a}{x-4} = \lim_{x \to 4^+} \frac{a(x^2-16)-\frac{1}{2}x+2}{x-4} =$ $\lim_{x \to 4^+} \frac{a(x+4)(x-4)-\frac{1}{2}(x-4)}{x-4} = \lim_{x \to 4^+} a(x+4) - \frac{1}{2} = 8a - \frac{1}{2}$
- $4a = 8a - \frac{1}{2}$
- $a = \frac{1}{8}$