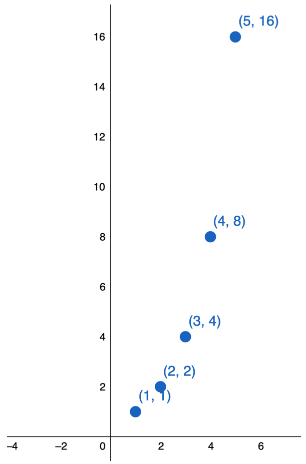
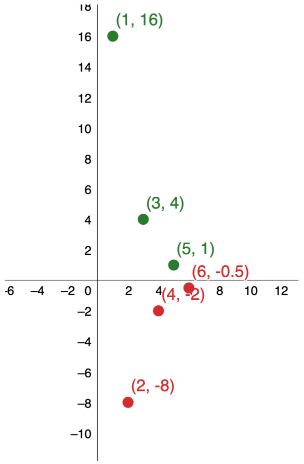
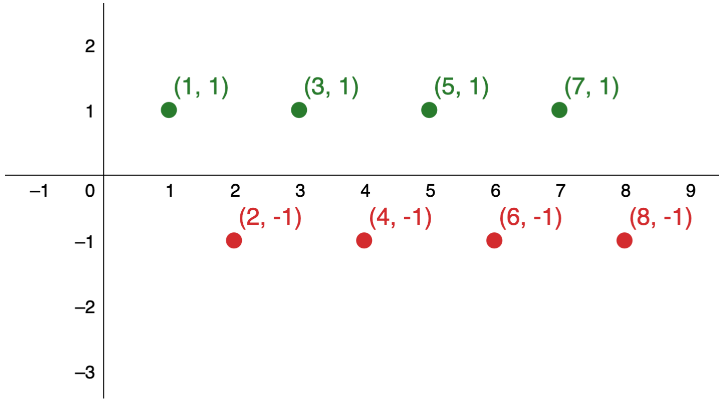
INTRODUCTION TO CALCULUS

LIMIT OF A SEQUENCE







WHY SHOULD WE CARE?

- A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger.
- Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$.

EXAMPLES

A

$$\{2^{n-1}\} = \{1, 2, 4, 8, \cdots\cdots\}$$

B

$$\{(-1)^{n-1}\}$$
$$= \{1, -1, 1, -1, \cdots\cdots\}$$

C

$$\left\{1 - \left(\frac{2}{3}\right)^n\right\}$$
$$= \left\{\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \cdots\cdots\right\}$$

D

$$\left\{16(-\frac{1}{2})^{n-1}\right\}$$
$$= \{16, -8, 4, -2, \cdots\cdots\}$$

The terms $\rightarrow 1$ as $n \rightarrow \infty$

The terms alternate but do not approach one single value as $n \rightarrow \infty$

The terms $\rightarrow 0$ as $n \rightarrow \infty$

The terms $\rightarrow \infty$ as $n \rightarrow \infty$

EXAMPLES

A

$$\{2^{n-1}\} = \{1, 2, 4, 8, \cdots\cdots\}$$

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D

$$\left\{16\left(-\frac{1}{2}\right)^{n-1}\right\}$$
$$= \{16, -8, 4, -2, \cdots\cdots\}$$

C

The terms $\to 1$ as $n \to \infty$

B

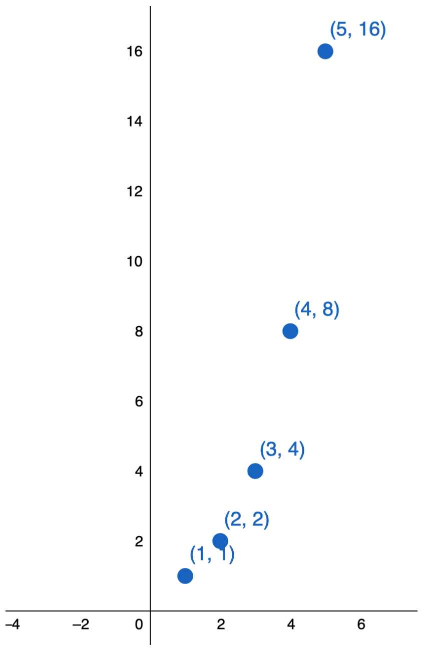
The terms alternate but do not approach one single value as $n \to \infty$

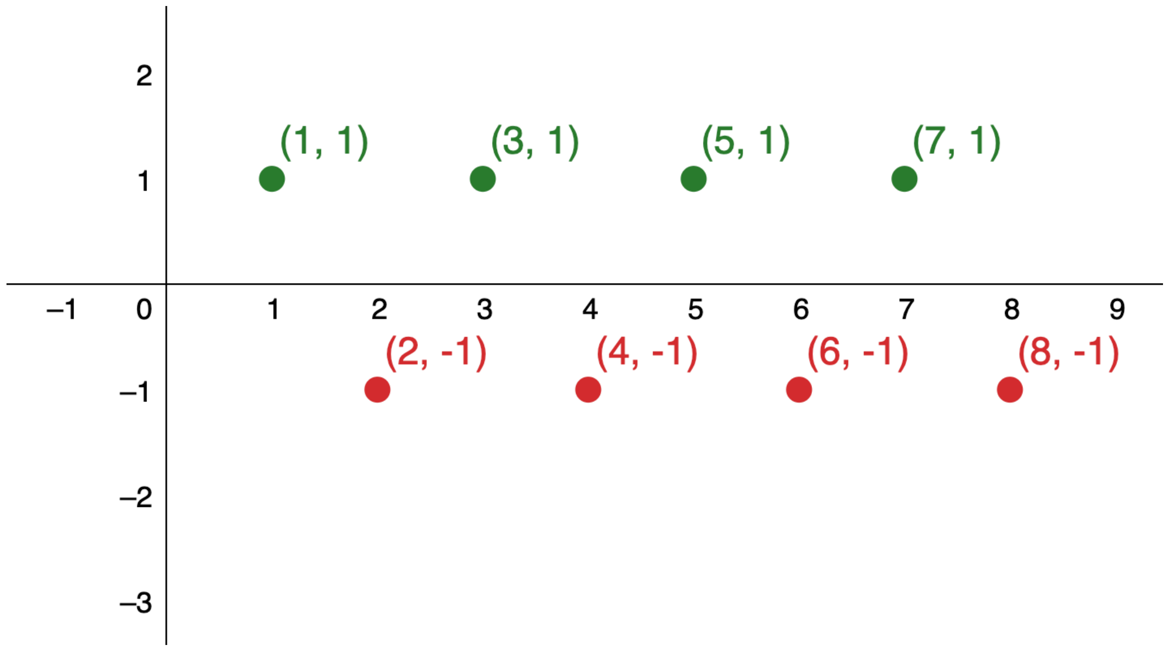
D

The terms $\to 0$ as $n \to \infty$

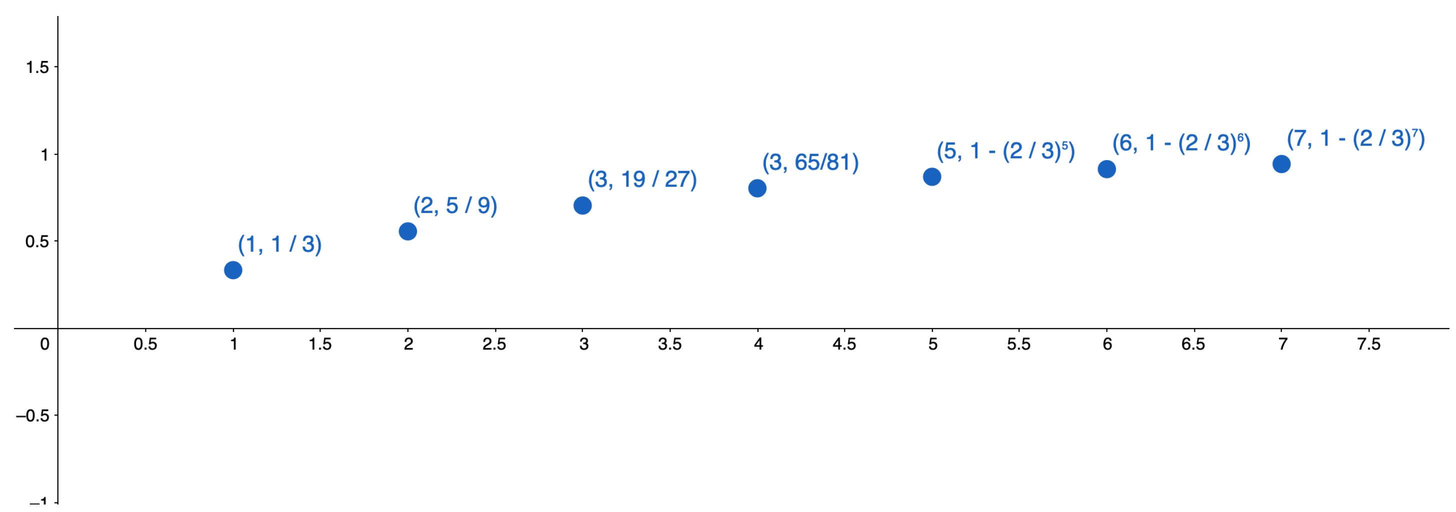
A

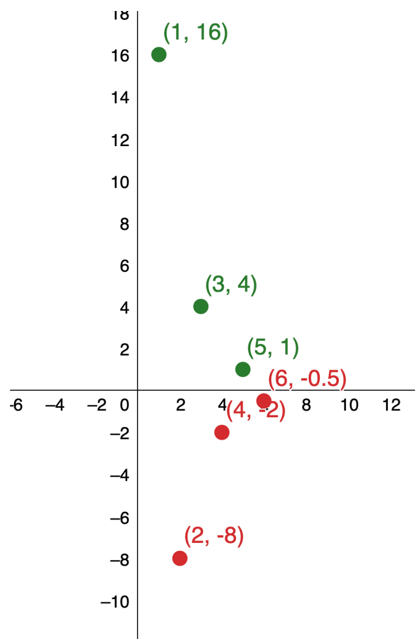
The terms $\to \infty$ as $n \to \infty$





GRAPHS OF THE FIRST TWO EXAMPLES





GRAPHS THE LAST TWO EXAMPLES

POSSIBILITIES FOR THE BEHAVIOR OF THE TERMS OF A SEQUENCE

As $n \to \infty$

- the terms approach a finite number
- <u>the terms do not approach a finite number</u>
  - $\to \infty$
  - Alternating

If the terms of a sequence approach **a finite number** L as $n \to \infty$ we say that the sequence is a convergent sequence and the real number L is the limit of the sequence.

CONVERGENT/DIVERGENT SEQUENCES

Definition

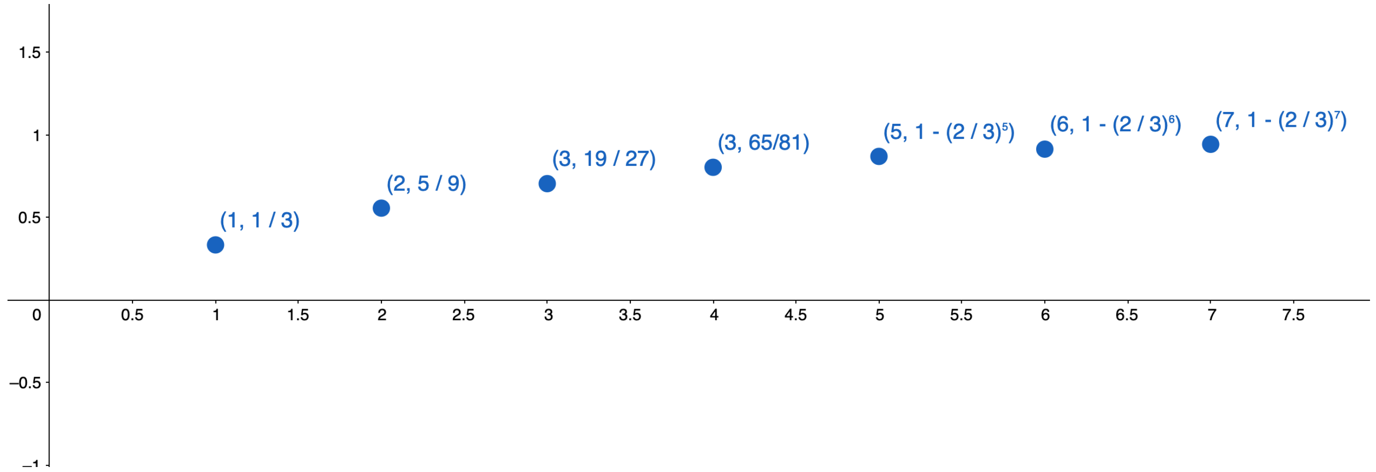
Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. In this case, we write

$$\boxed{\lim_{n \to \infty} a_n = L.}$$

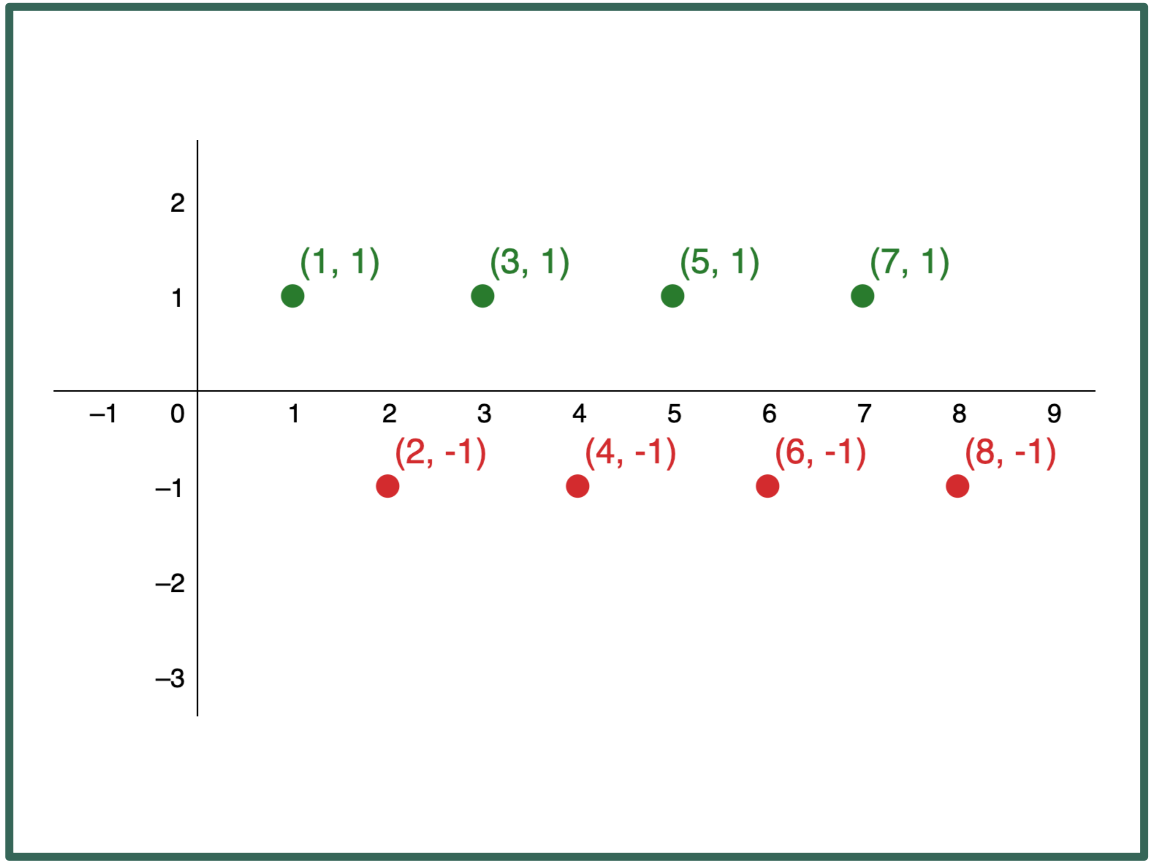
If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence**.

EXAMPLE C

- $\left\{1-\left(\frac{2}{3}\right)^{n}\right\} = \left\{\frac{1}{3},\frac{5}{9},\frac{19}{27},\frac{65}{81},\cdots\cdots\right\}$ is a convergent sequence and its limit is 1.



EXAMPLE B



- $\{(-1)^{n-1}\} = \{1, -1, 1, -1, \cdots\cdots\}$ are not approaching a finite number as n becomes larger.
- We say that it is a divergent sequence.

FROM INFORMAL TO FORMAL

- In the informal definition for the limit of a sequence, we used the terms "**arbitrarily close**" and "**sufficiently large**".
- Although these phrases help illustrate the meaning of a converging sequence, they are somewhat **vague**.
- To be more precise, we now present **the more formal definition** of limit for a sequence and show these ideas graphically.

FORMAL DEFINITION OF THE LIMIT

Definition

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0,$ there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geq N.$ The number L is the limit of the sequence and we write

$$\lim_{n \to \infty} a_n = L \, or \, a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

EXPLANATION BY A FIGURE

FOR ALL $\varepsilon > 0$, THERE EXISTS AN INTEGER N SUCH THAT $|a_n - L| < \varepsilon$ IF $n \geq N$.

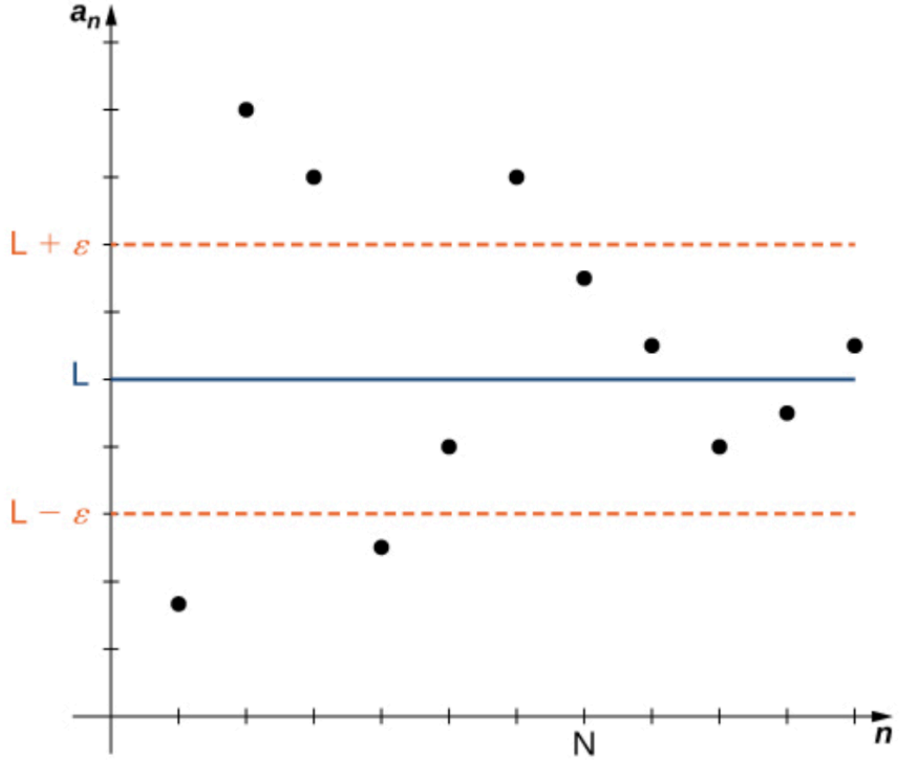


Figure 5.4 As n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

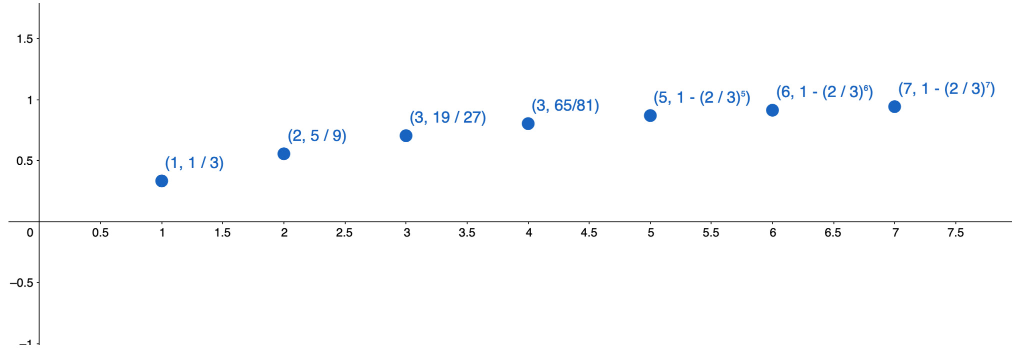
EXPLANATION BY AN EXAMPLE

For all $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geq N$.

An example

$$\left\{1-\left(\frac{2}{3}\right)^n\right\} = \left\{\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \ldots\ldots\right\}$$

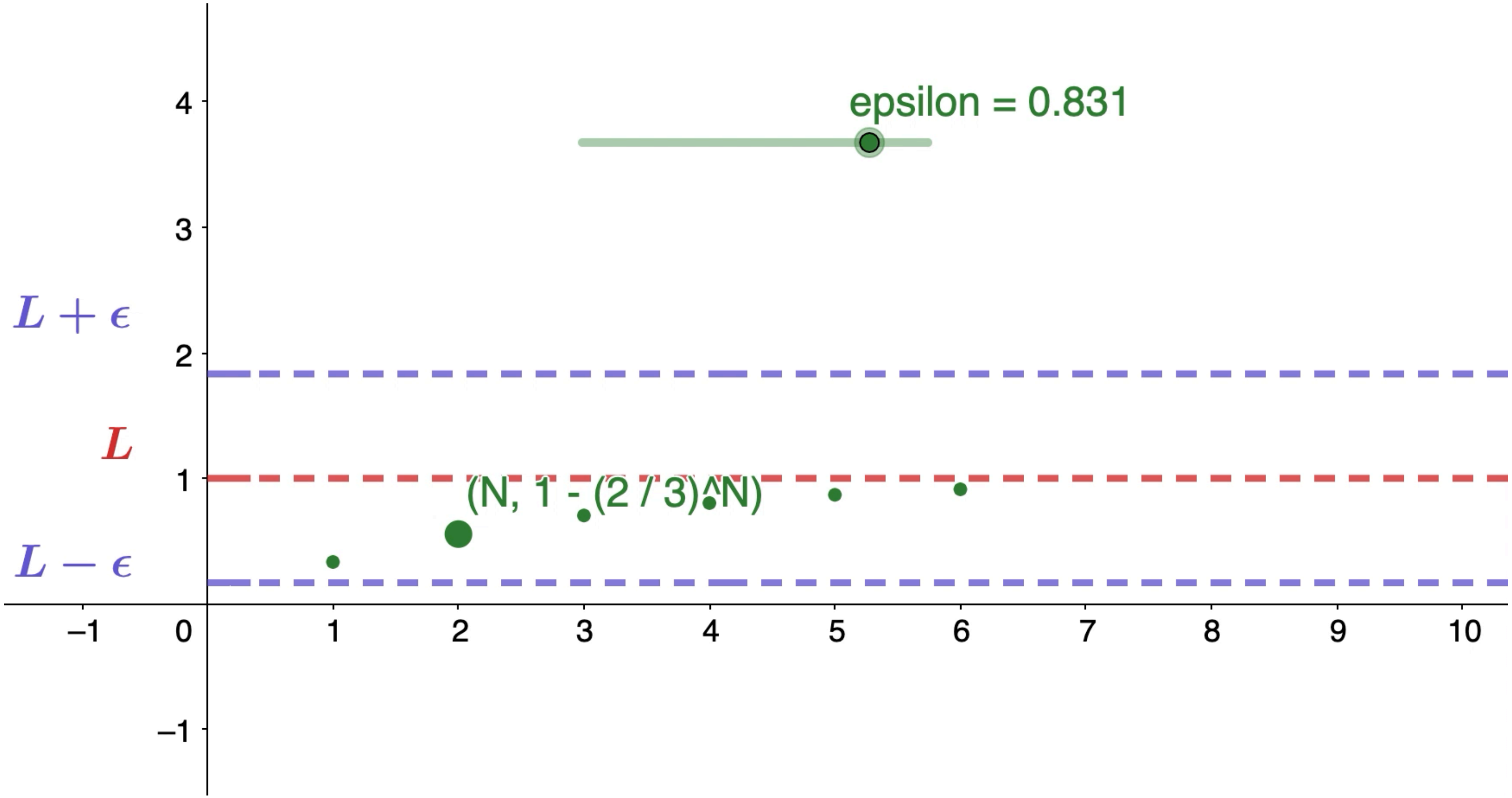
- $\varepsilon = 1$
- $\varepsilon = \frac{1}{10}$
- $\varepsilon = \frac{1}{100}$



EXPLANATION BY AN EXAMPLE

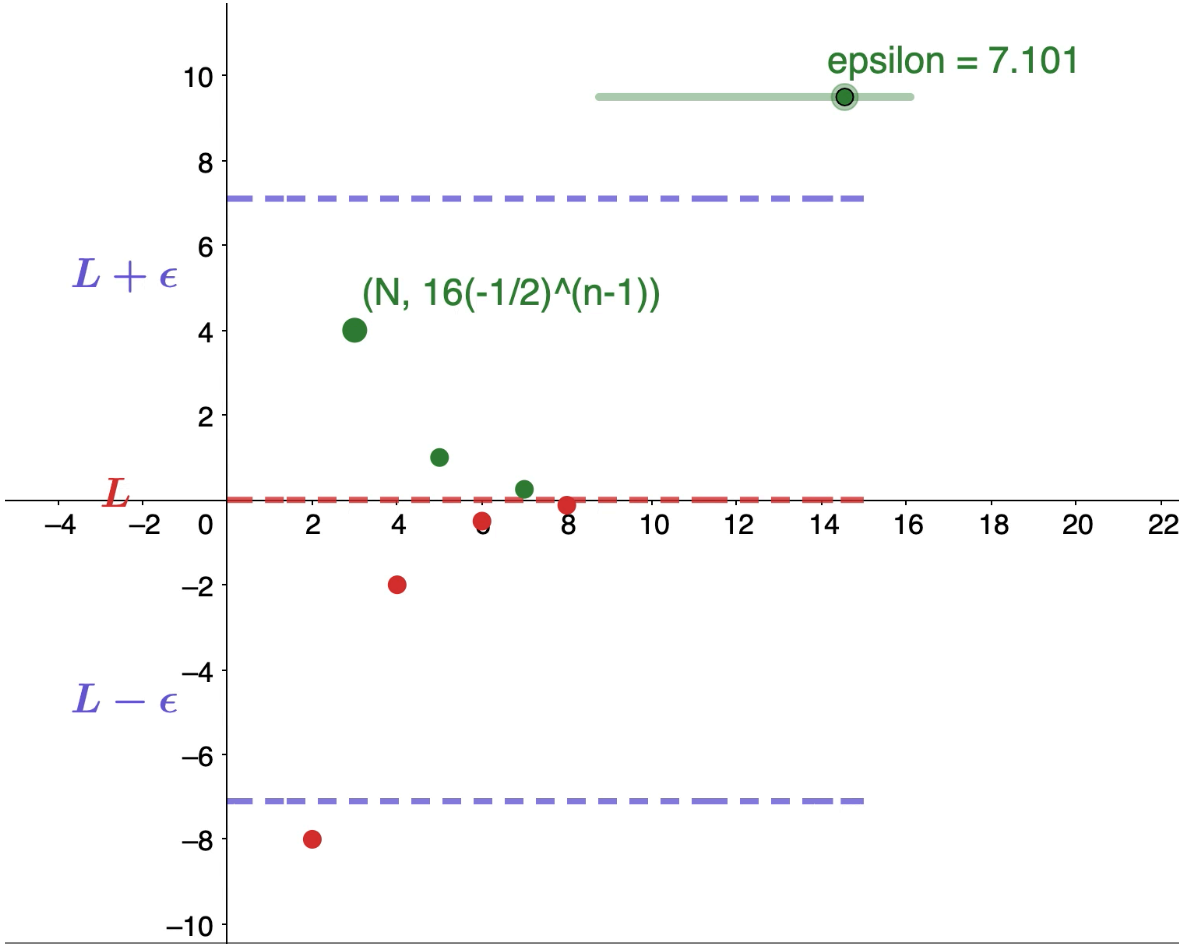
$$\left\{1 - \left(\frac{2}{3}\right)^n\right\}$$

$$= \left\{\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, \cdots\cdots\right\}$$



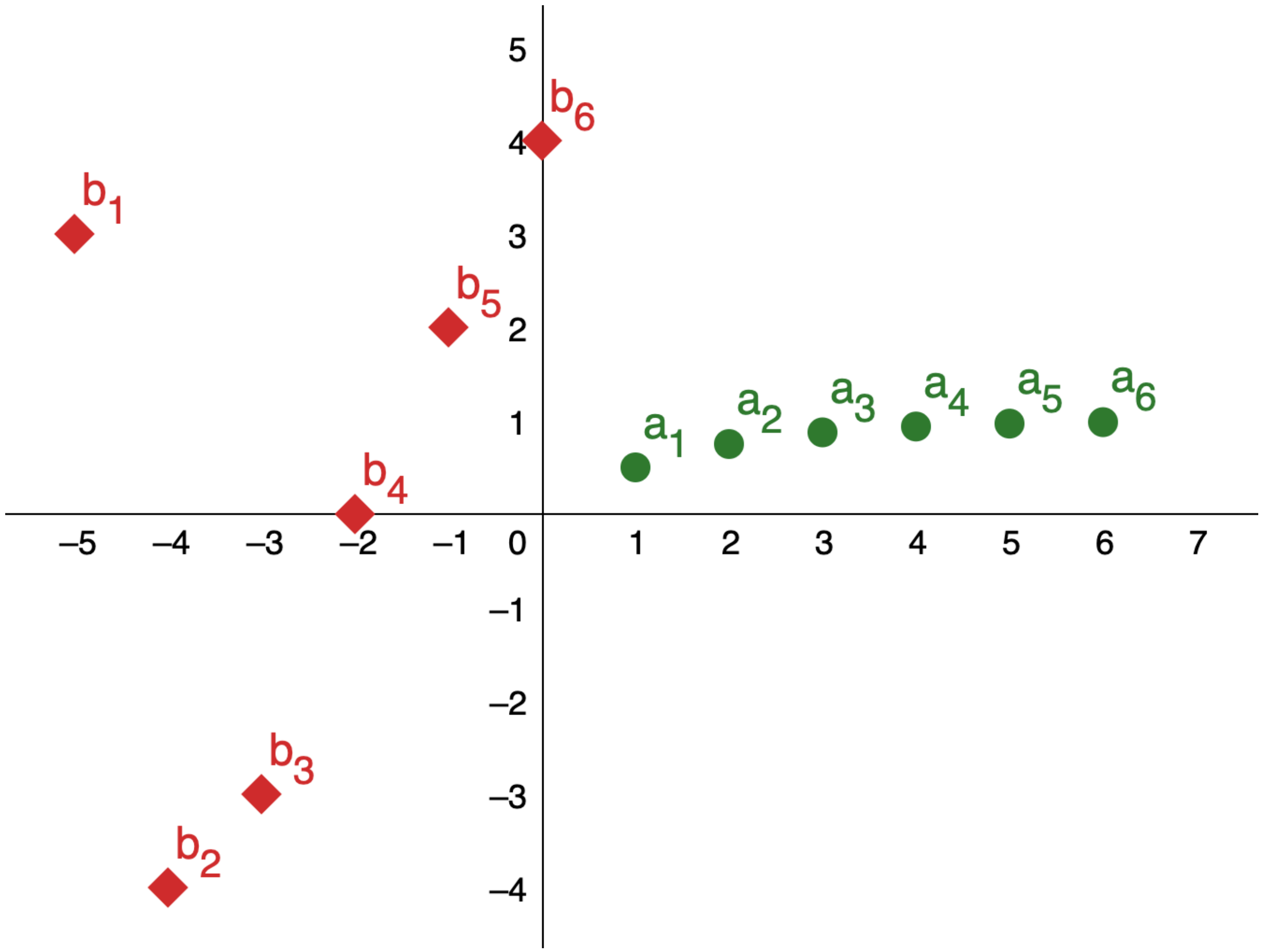
EXPLANATION BY ANOTHER EXAMPLE

- $\left\{16(-\frac{1}{2})^{n-1}\right\} = \{16, -8, 4, -2, \cdots\cdots\}$



REMARK

- We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$.
- Therefore, if a finite number of terms $b_1, b_2, \cdots, b_n$ are placed before a_1 to create a new sequence $b_1, b_2, \cdots, b_n, a_1, a_2, \cdots$, this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges.
- Further, if the sequence $\{a_n\}$ converges to L this new sequence will also converge to L.



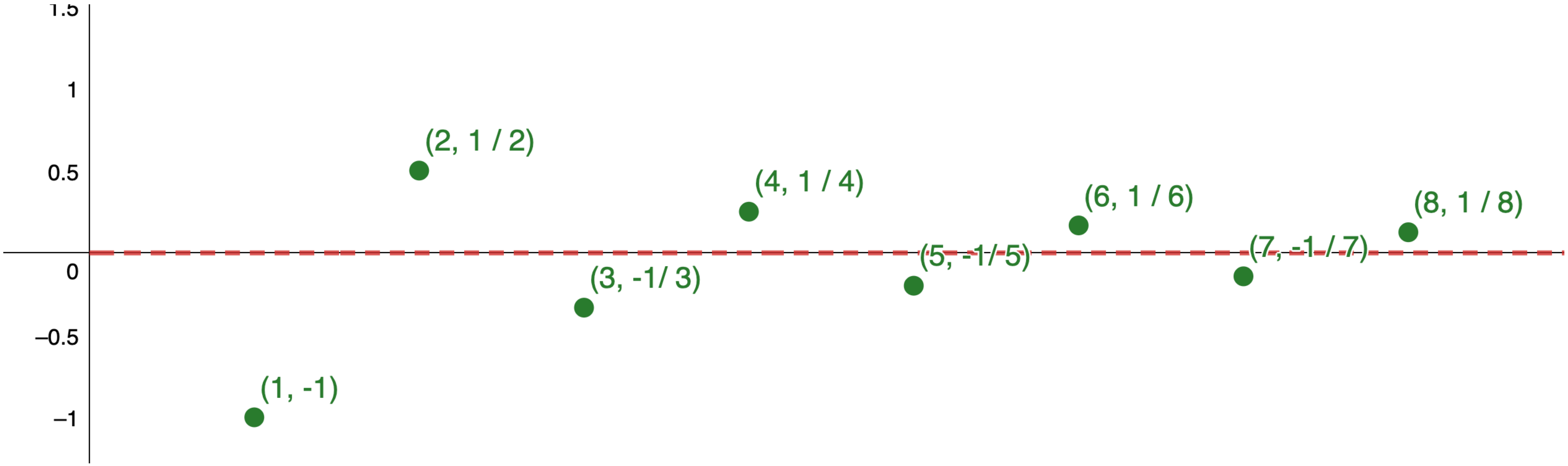
REMARK

MORE ABOUT DIVERGENT SEQUENCES

- Different sequences can diverge in different ways.
  - $\to +\infty$
  - $\to -\infty$
  - Alternating
- It is important to recognize that this notation $\to +\infty$ or $\to -\infty$ does not imply that the limit of the sequence exists.
- Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent.

HOW TO SHOW THAT A SEQUENCE IS CONVERGENT?

$\left\{\frac{(-1)^n}{n}\right\}$



AN EXTREMELY USEFUL METHOD

Theorem 5.4: Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$\boxed{a_n \le b_n \le c_n \text{ for all } n \ge N.}$$

If there exists a real number L such that

$$\boxed{\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n,}$$

then $\{b_n\}$ converges and $\lim_{n \to \infty} b_n = L$ (Figure 5.6).

SQUEEZE THEOREM FOR CALCULUS

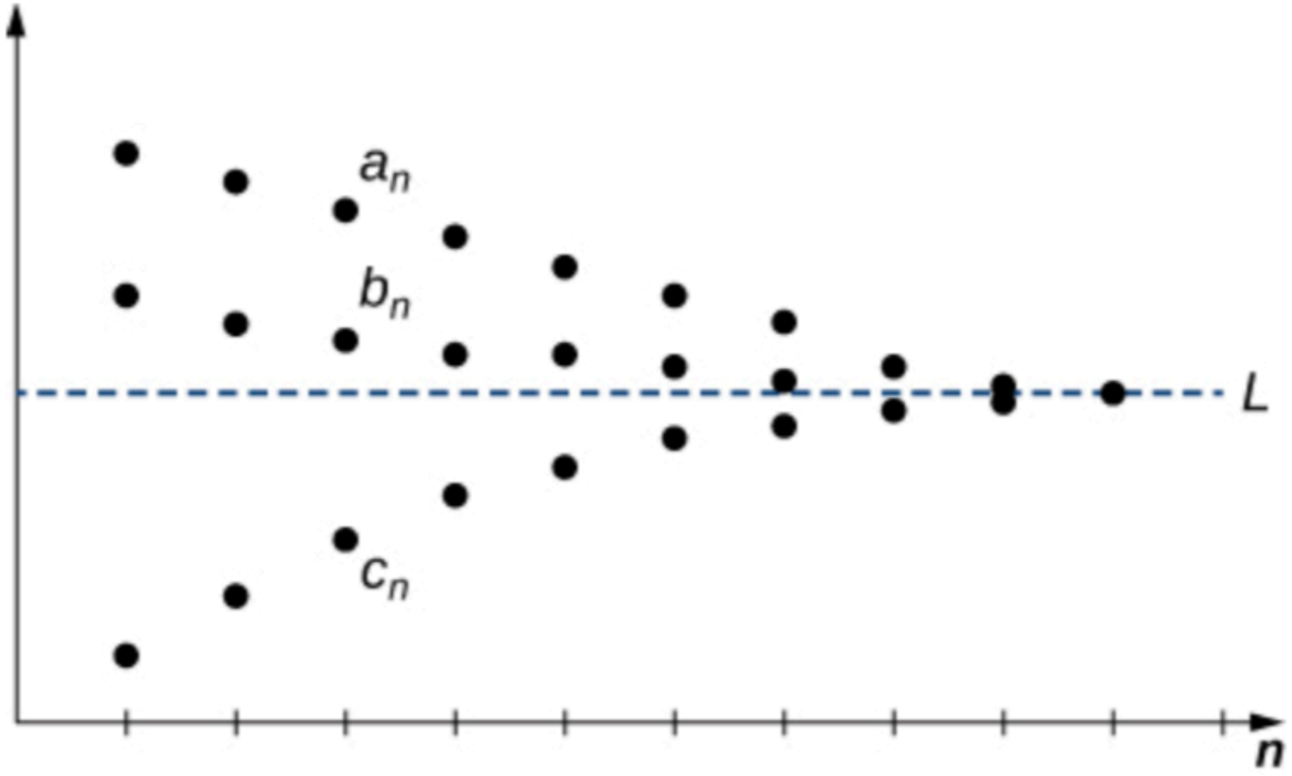


Figure 5.6 Each term b_n satisfies $a_n \leq b_n \leq c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

PROOF

Hint!

- For all $\varepsilon > 0$,
  - There exists an integer N_1 such that $|a_n - L| < \varepsilon$ if $n \geq N_1$.
  - There exists an integer N_2 such that $|c_n - L| < \varepsilon$ if $n \geq N_2$.
- There exists an integer N such that $a_n \leq b_n \leq c_n$ if $n \geq N$.

PROOF

Proof

Let $\varepsilon > 0$. Since the sequence $\{a_n\}$ converges to L, there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \geq N_1$. Similarly, since $\{c_n\}$ converges to L, there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \geq N_2$. By assumption, there exists an integer N such that $a_n \leq b_n \leq c_n$ for all $n \geq N$. Let M be the largest of $N_1, N_2,$ and N. We must show that $|b_n - L| < \varepsilon$ for all $n \geq M$. For all $n \geq M$,

$$\boxed{-\varepsilon < -|a_n - L| \leq a_n - L \leq b_n - L \leq c_n - L \leq |c_n - L| < \varepsilon.}$$

Therefore, $-\varepsilon < b_n - L < \varepsilon$, and we conclude that $|b_n - L| < \varepsilon$ for all $n \geq M$, and we conclude that the sequence $\{b_n\}$ converges to L.

□

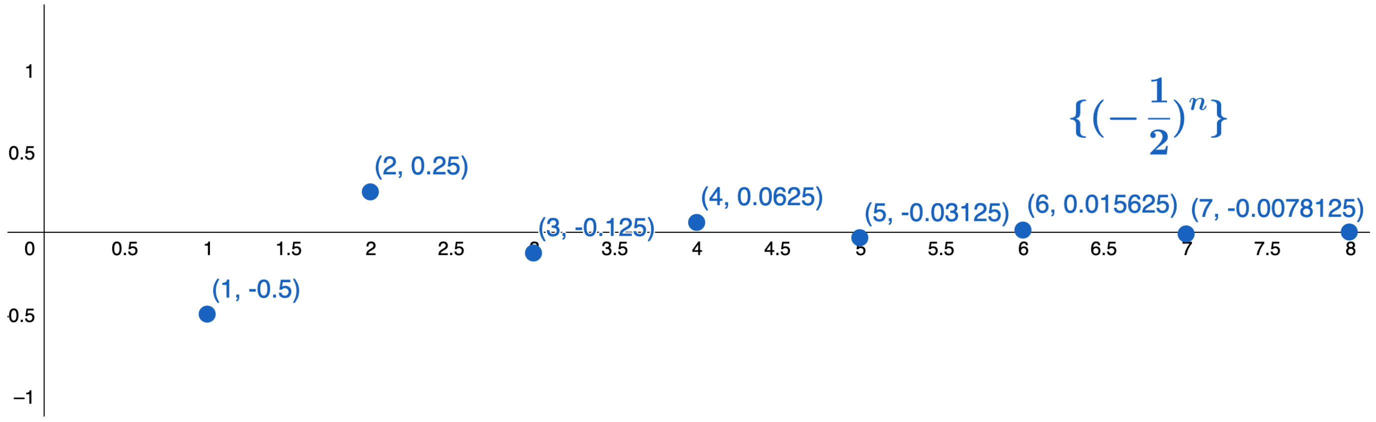
APPLICATION

Use the Squeeze Theorem to find the limit of each of the following sequences.

- $\left\{\frac{\sin n}{n}\right\}$
- $\left\{\left(-\frac{1}{2}\right)^n\right\}$
- $\left\{\frac{2n-\cos 2n}{n}\right\}$

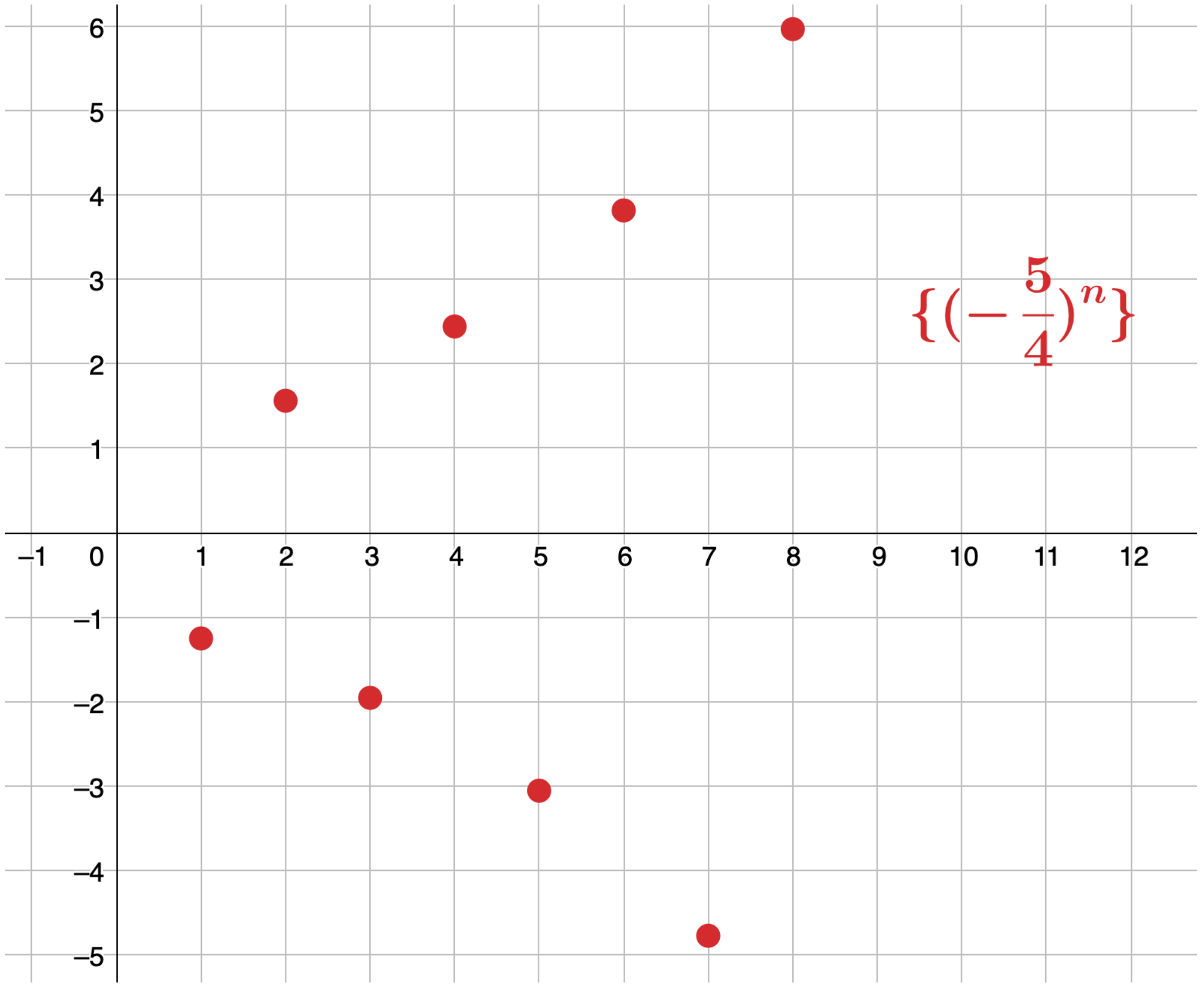
LOOK BACK ON GEOMETRIC SEQUENCES

- Using the idea from $\left\{(-\frac{1}{2})^n\right\}$, we conclude that
- $r^n \to 0$ for any real number such that $-1 < r < 0$.



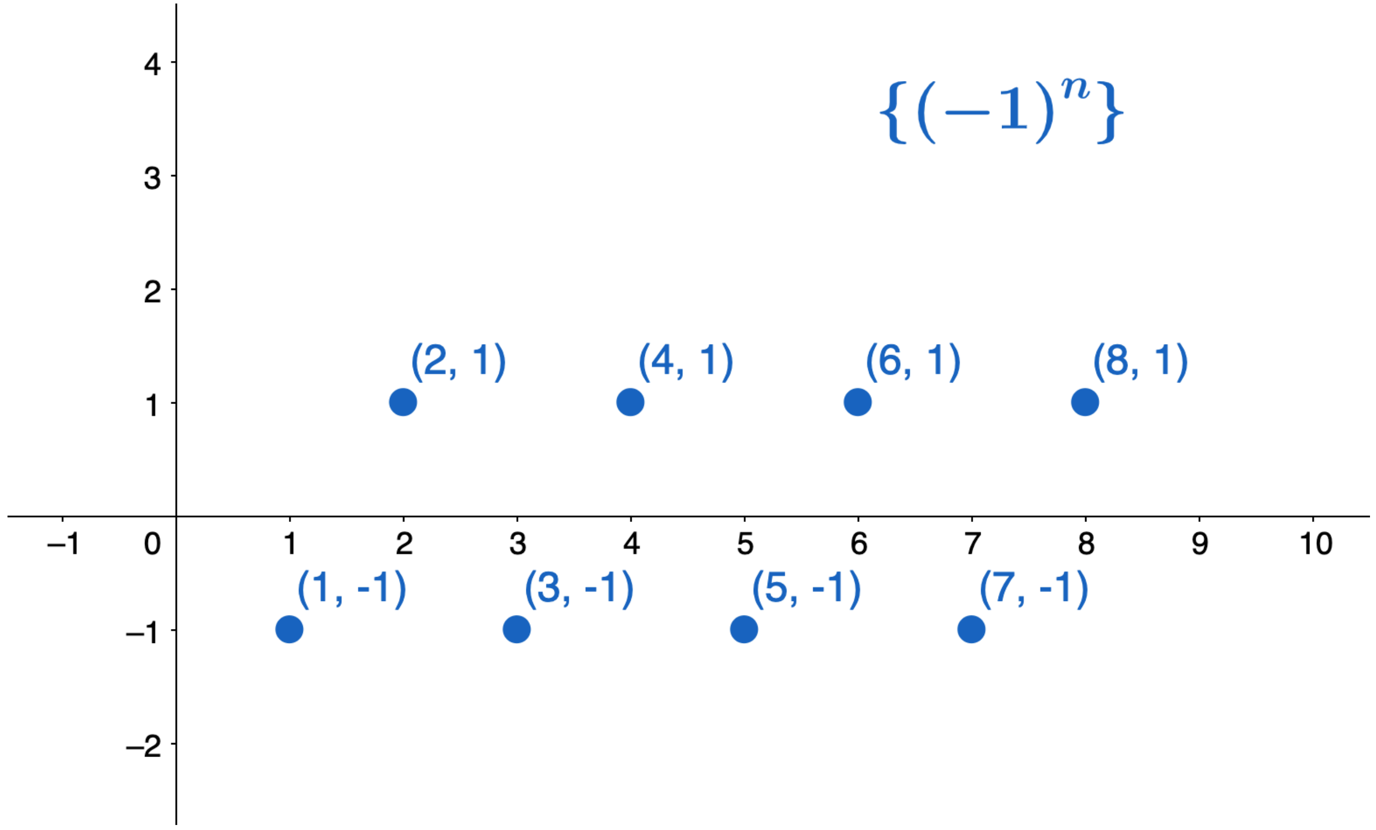
LOOK BACK ON GEOMETRIC SEQUENCES

- If $r < -1$, the sequence diverges because the terms oscillate and become arbitrarily large in magnitude.



LOOK BACK ON GEOMETRIC SEQUENCES

- If $r = -1$, the sequence $\{((-1)^n\}$ diverges, as discussed earlier.



LOOK BACK ON GEOMETRIC SEQUENCES

- Using the idea from $\left\{(-\frac{1}{2})^n\right\}$, we conclude that $r^n \to 0$ for any real number such that $-1 < r < 0$.
- If $r < -1$, the sequence diverges because the terms oscillate and become arbitrarily large in magnitude.
- If $r = -1$, the sequence $\{((-1)^n\}$ diverges, as discussed earlier.

A SUMMARY OF THE PROPERTIES FOR GEOMETRIC SEQUENCES

$$r^n \to 0 \text{ if } |r| < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1$$

$$\{r^n\} \text{ diverges if } r \leq -1$$