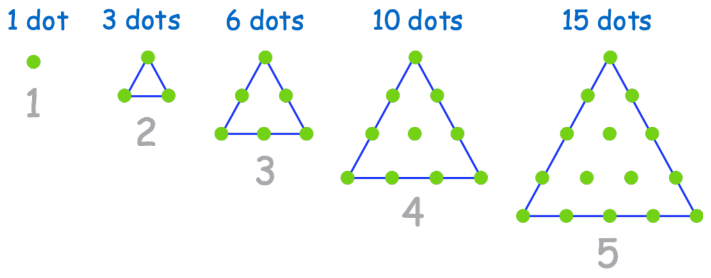
SEQUENCES

INTRODUCTION TO CALCULUS

RECALL THE RICE AND CHESSBOARD STORY

- How can we represent the number of grains in the nth square?

INTRODUCTION



- In this section, we introduce sequences and define what it means for a sequence to **converge** or **diverge**.
- We show how to find **limits of sequences that converge**, often by using the properties of limits for functions discussed earlier.
- We close this section with the **Monotone Convergence Theorem**, a tool we can use to prove that certain types of sequences converge.

TERMINOLOGY OF SEQUENCES

An infinite sequence is an ordered list of numbers of the form

- $a_1, a_2, a_3, \cdots\cdots, a_n, \cdots\cdots.$

A term

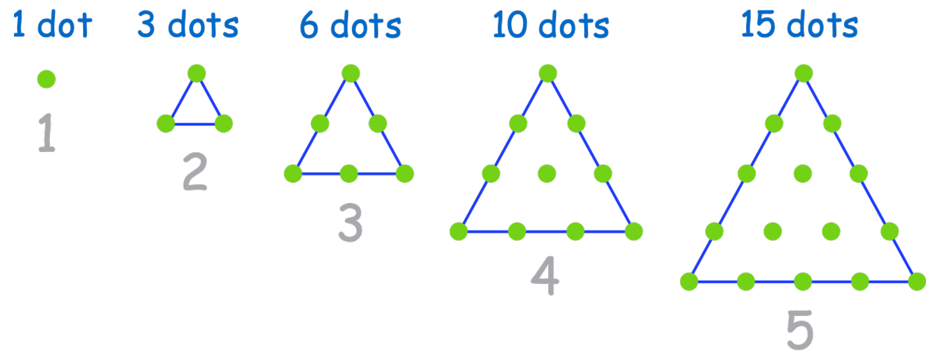
- a_n

The index variable

- n

We use the notation

- $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$.

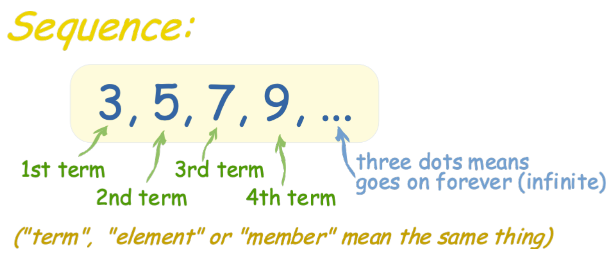


EXAMPLE

Dartmouth Football 2018
Dartmouth Game Results (as of Nov 17, 2018)
All games

Date	Opponent		Score	Overall	Conference	Time	Attend
Sep 15, 2018	GEORGETOWN	W	41-0	1-0	0-0	2:55	4815
Sep 22, 2018	at Holy Cross	W	34-14	2-0	0-0	2:51	7175
* Sep 29, 2018	PENN	W	37-14	3-0	1-0	2:58	3692
* Oct 05, 2018	at Yale	W	41-18	4-0	2-0	3:19	10176
Oct 13, 2018	SACRED HEART	W	42-0	5-0	2-0	2:51	3138
* Oct 20, 2018	at Columbia	W	28-12	6-0	3-0	3:03	12506
* Oct 27, 2018	HARVARD	W	24-17	7-0	4-0	3:05	5814
* Nov 3, 2018	at #14 Princeton	L	9-14	7-1	4-1	2:55	8014
* Nov 10, 2018	at Cornell	W	35-24	8-1	5-1	3:11	3604
* Nov 17, 2018	BROWN	W	49-7	9-1	6-1	3:16	2575

IS THE RELATION ONE-TO-ONE?



- A particular number a_n exists for each positive integer n.
- We can also define a sequence as a function whose domain is the set of positive integers.

GO BACK TO OUR RICE AND CHESSBOARD EXAMPLE

- $a_1 = 1, a_2 = 2, a_3 = 4, \cdots$
- $a_n = 2^{n-1}$

Using this notation, we can write this sequence as

- $\{2^{n-1}\}_{n=1}^{\infty}$ or simply $\{2^{n-1}\}$.

DEFINE THE SEQUENCE IN A DIFFERENT WAY (RECURRENT RELATION)

Since each term is twice the previous term, this sequence can be defined **recursively** by expressing the nth term a_n in terms of the previous term a_{n-1}.

In particular, we can define this sequence as the sequence $\{a_n\}$ where $\boldsymbol{a_1 = 1}$ and for all $n \geq 2$ each term a_n is defined by the recurrence relation

- $\boldsymbol{a_n = 2a_{n-1}}$.

FORMAL DEFINITION

Definition

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots .$$

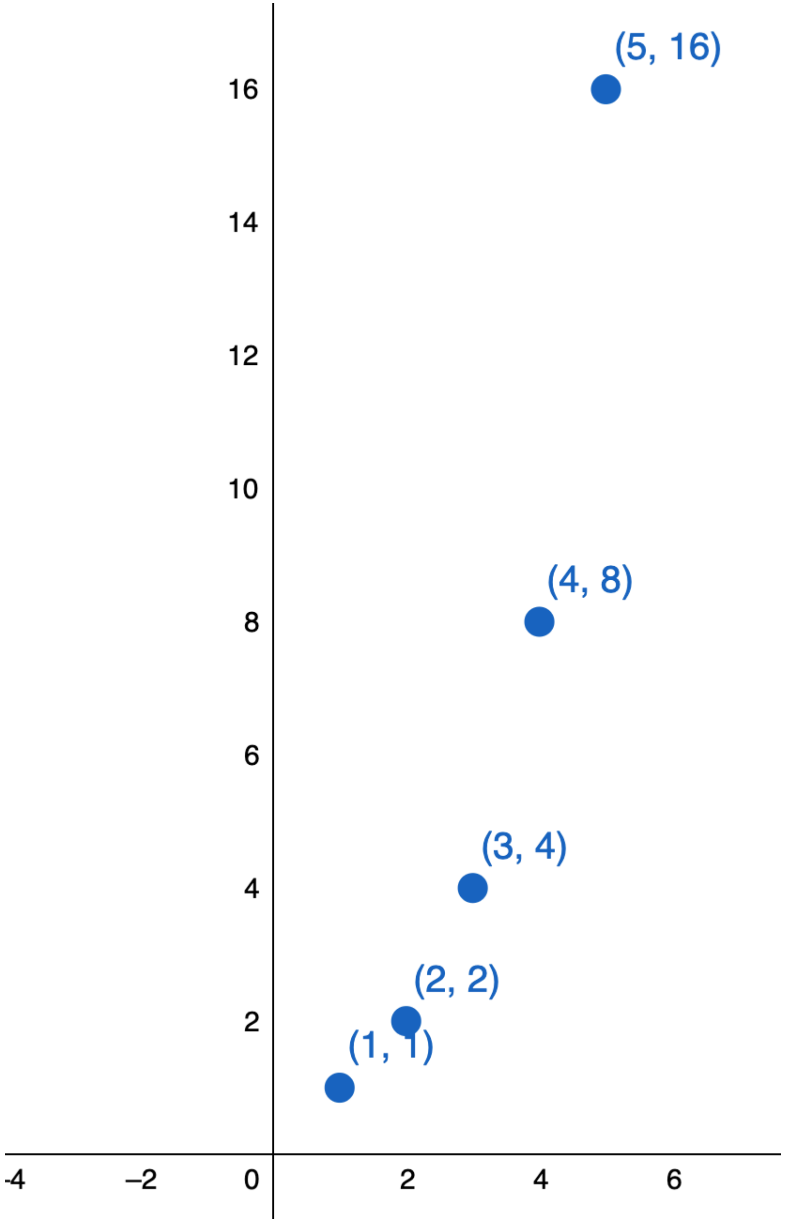
The subscript n is called the **index variable** of the sequence. Each number a_n is a **term** of the sequence. Sometimes sequences are defined by **explicit formulas**, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

NOTICE

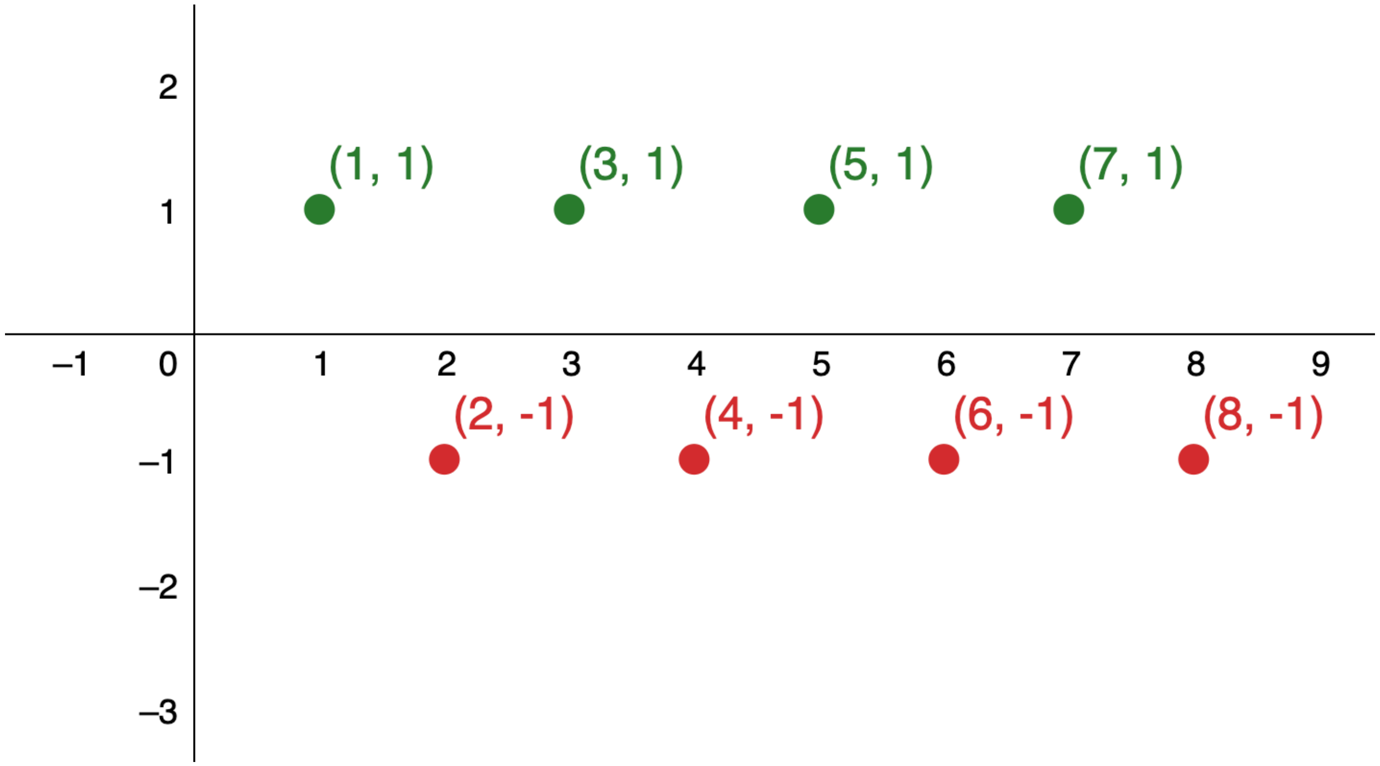
- The index does not have to start at $n = 1$ but could start with other integers.
- For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$ in which case the sequence would be
  - $a_0, a_1, a_2, \cdots$
- Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}.

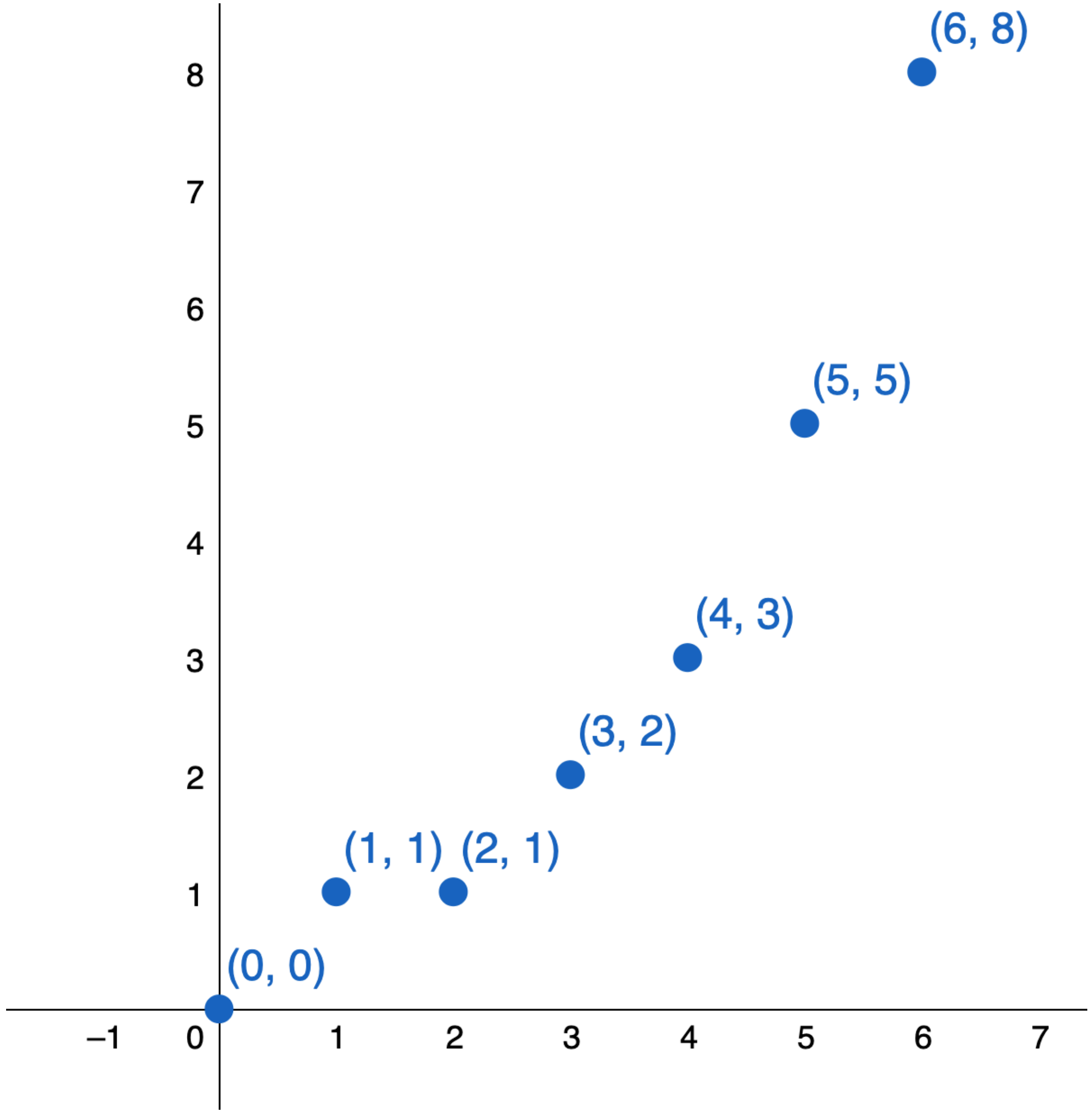
THE GRAPH OF A SEQUENCE

- Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers.
- As a result, it makes sense to discuss the graph of a sequence.
- The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n.



THE GRAPH OF A SEQUENCE





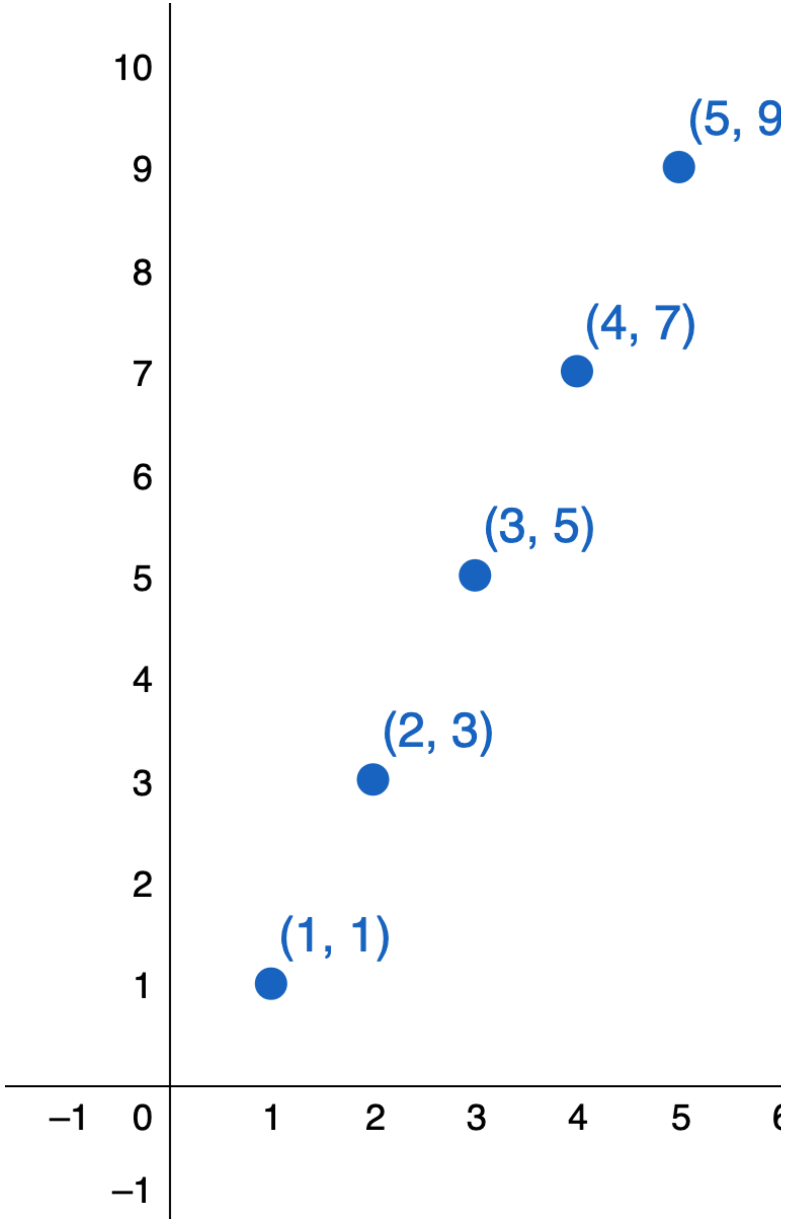
THE GRAPH OF A SEQUENCE

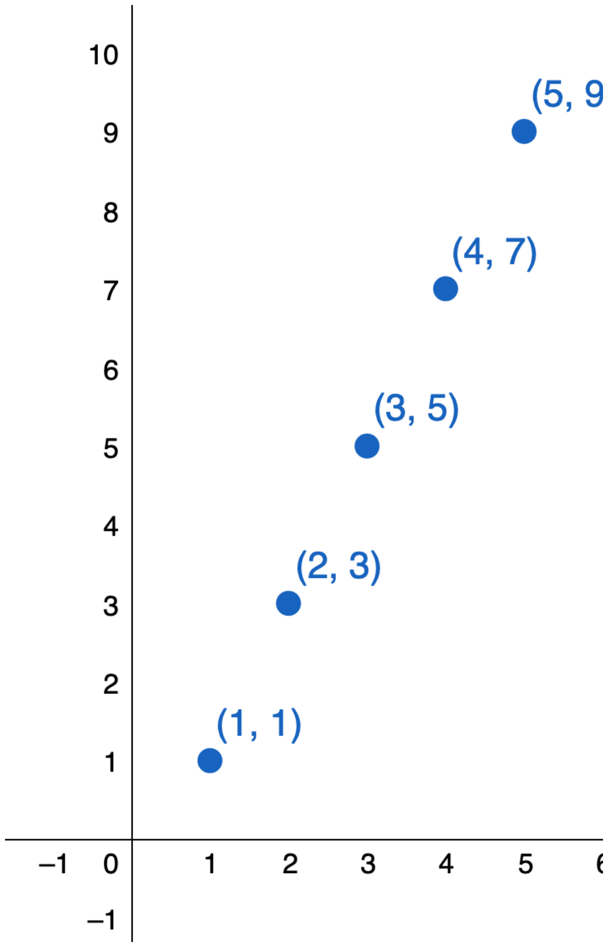
SPECIAL SEQUENCES

- Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences.

ARITHMETIC SEQUENCE

- In an arithmetic sequence, the difference between every pair of consecutive terms is the same.
- For example, consider the sequence
  - $1, 3, 5, 7, 9, \cdots\cdots$
- You can see that the difference between every consecutive pair of terms is 2.





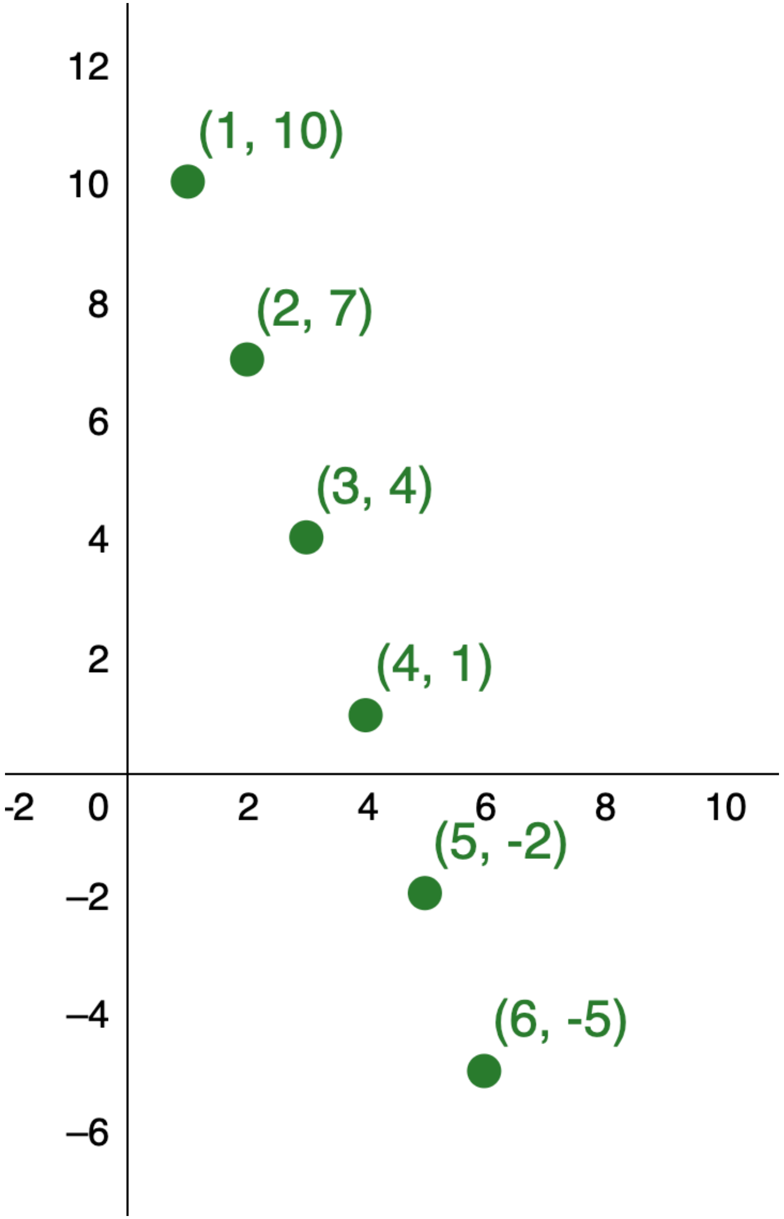
ARITHMETIC SEQUENCE

The sequence can be described by using the recurrence relation

- $$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 2, n \geq 2 \end{cases}$$

It can also be described using the explicit formula

- $a_n = 2n - 1$



ARITHMETIC SEQUENCE: EXAMPLE 2

The sequence can be described by using the recurrence relation

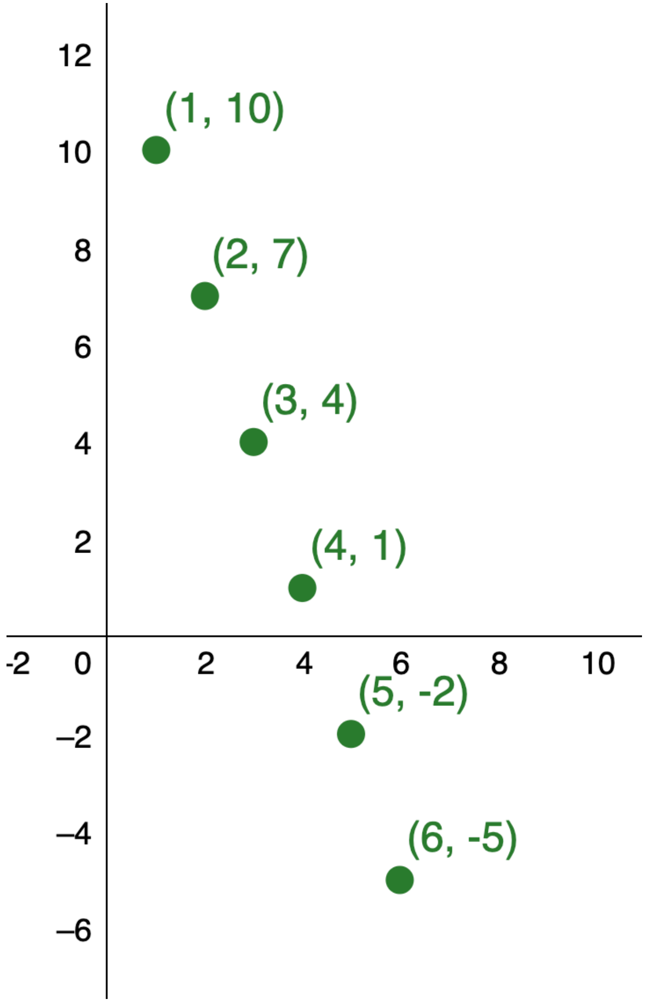
- $\begin{cases} a_1 = 10 \\ a_n = a_{n-1} - 3, n \geq 2 \end{cases}$

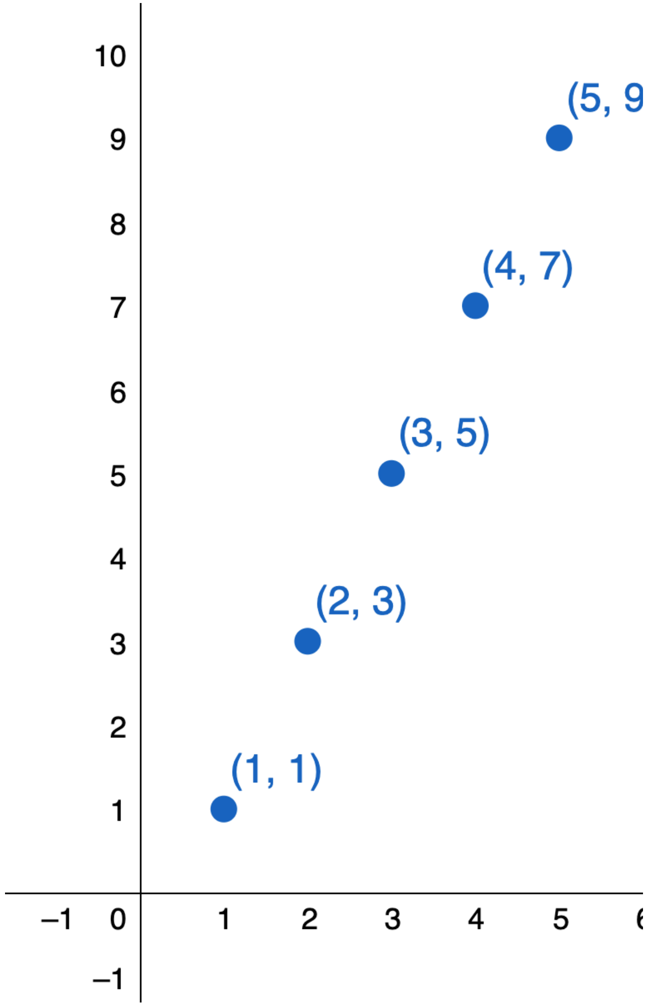
It can also be described using the explicit formula

- $a_n = -3n + 13$

ARITHMETIC SEQUENCE

- In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$.
- What is c?



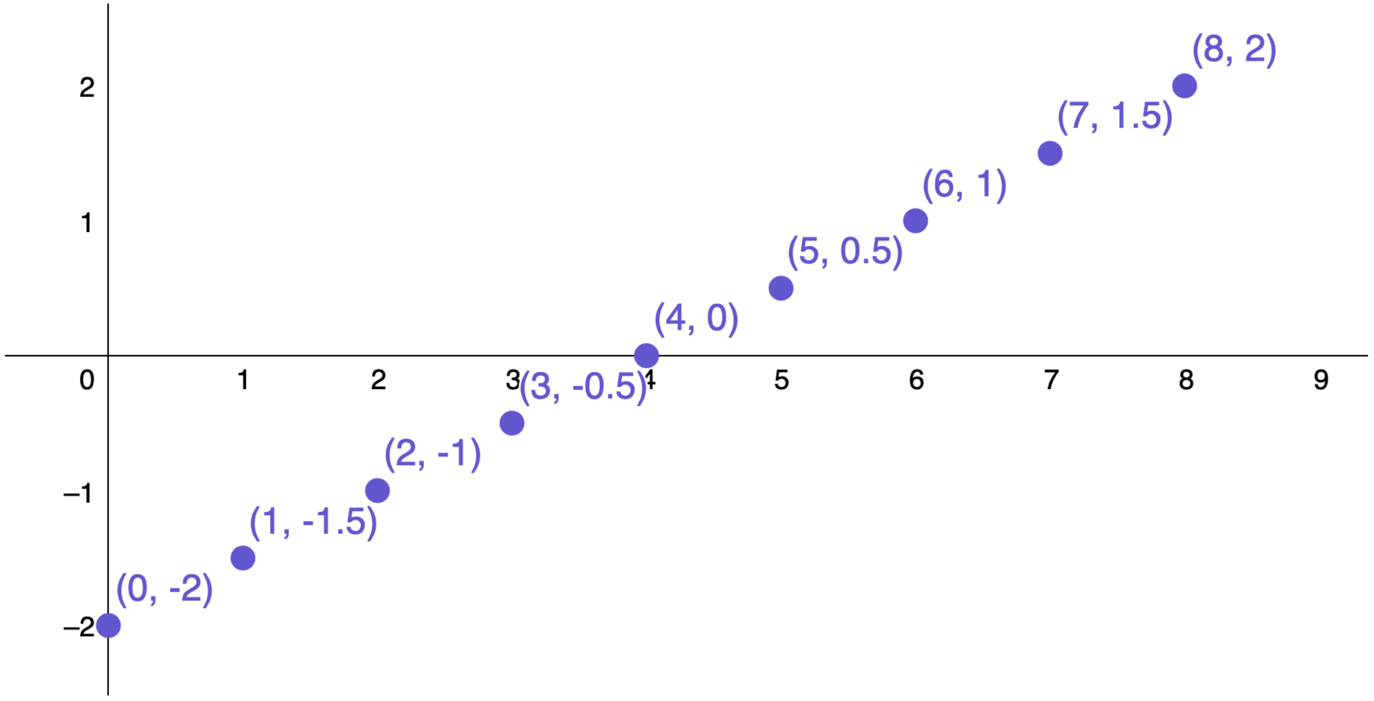


EXERCISE ONE

The sequence can be described by using the recurrence relation

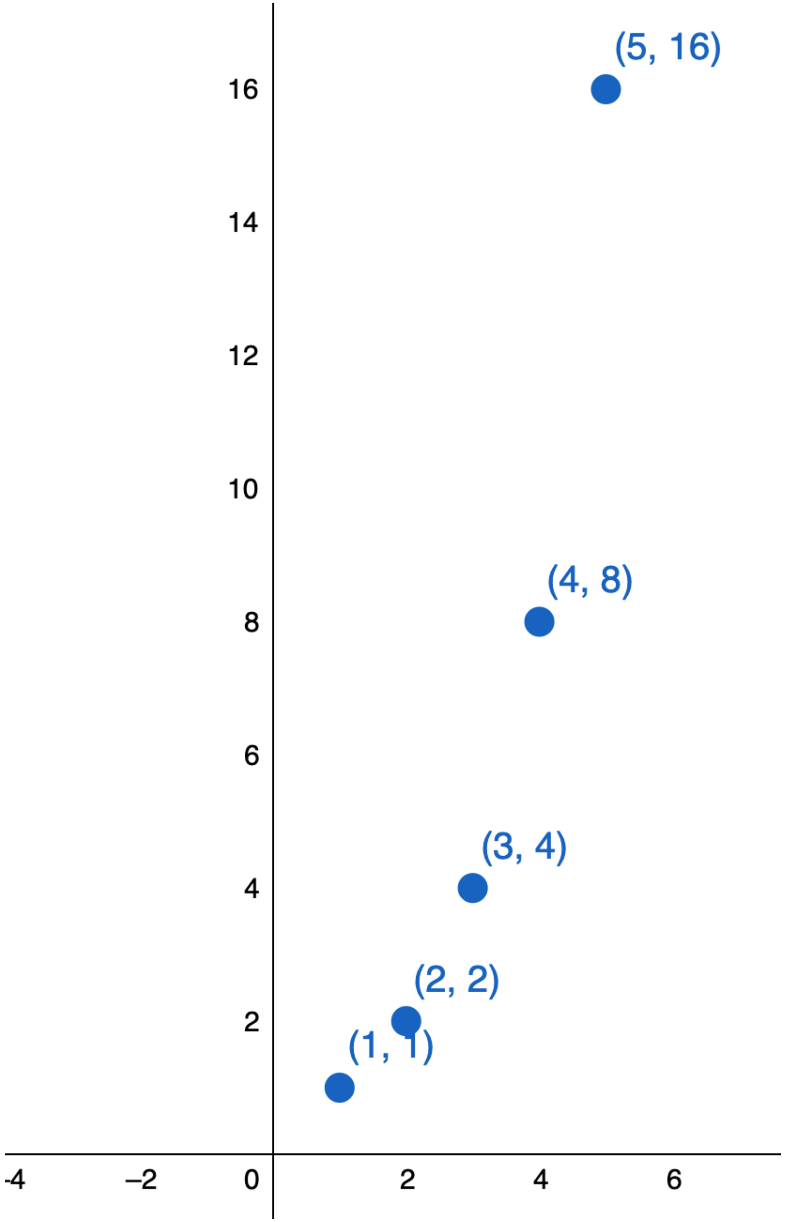
_____

It can also be described using the explicit formula _____



GEOMETRIC SEQUENCE

- In a geometric sequence, the ratio of every pair of consecutive terms is the same.
- For example, consider the sequence
  - $1, 2, 4, 8, 16, \cdots\cdots$
- You can see that the ratio of any term to the preceding term is 2.



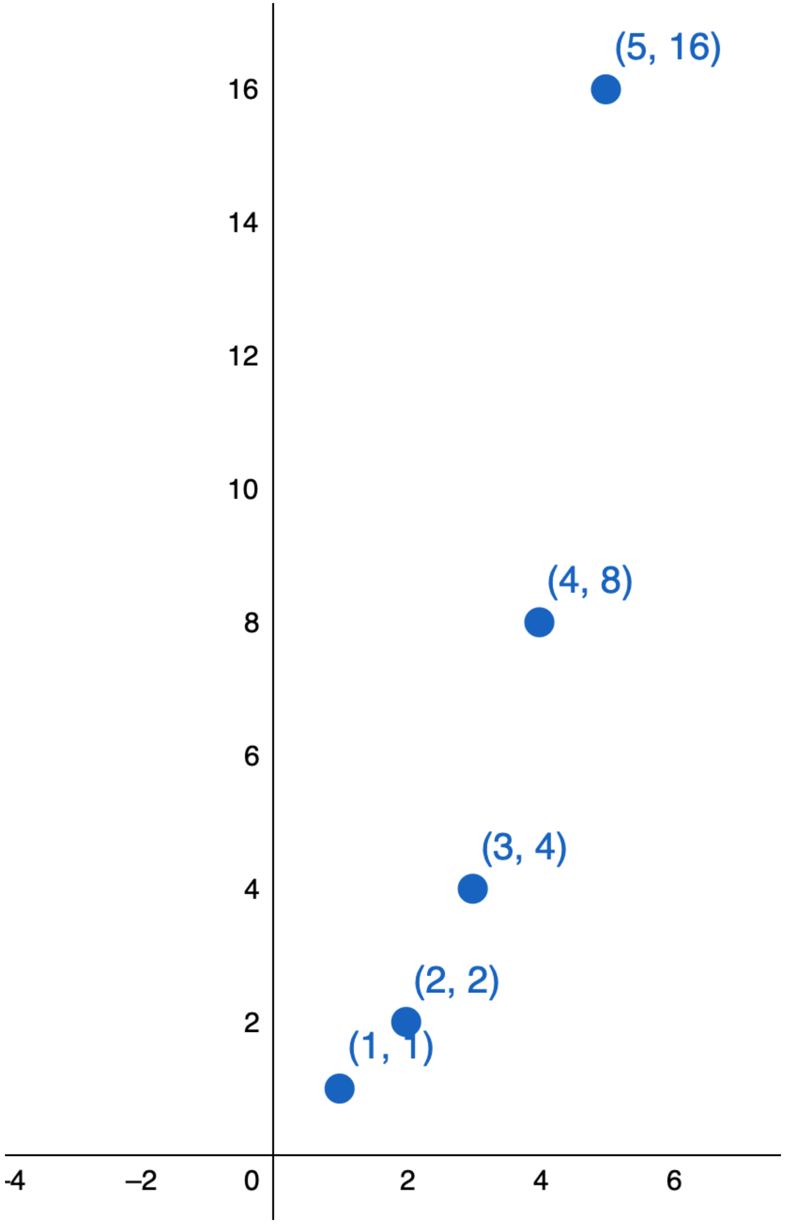
GEOMETRIC SEQUENCE

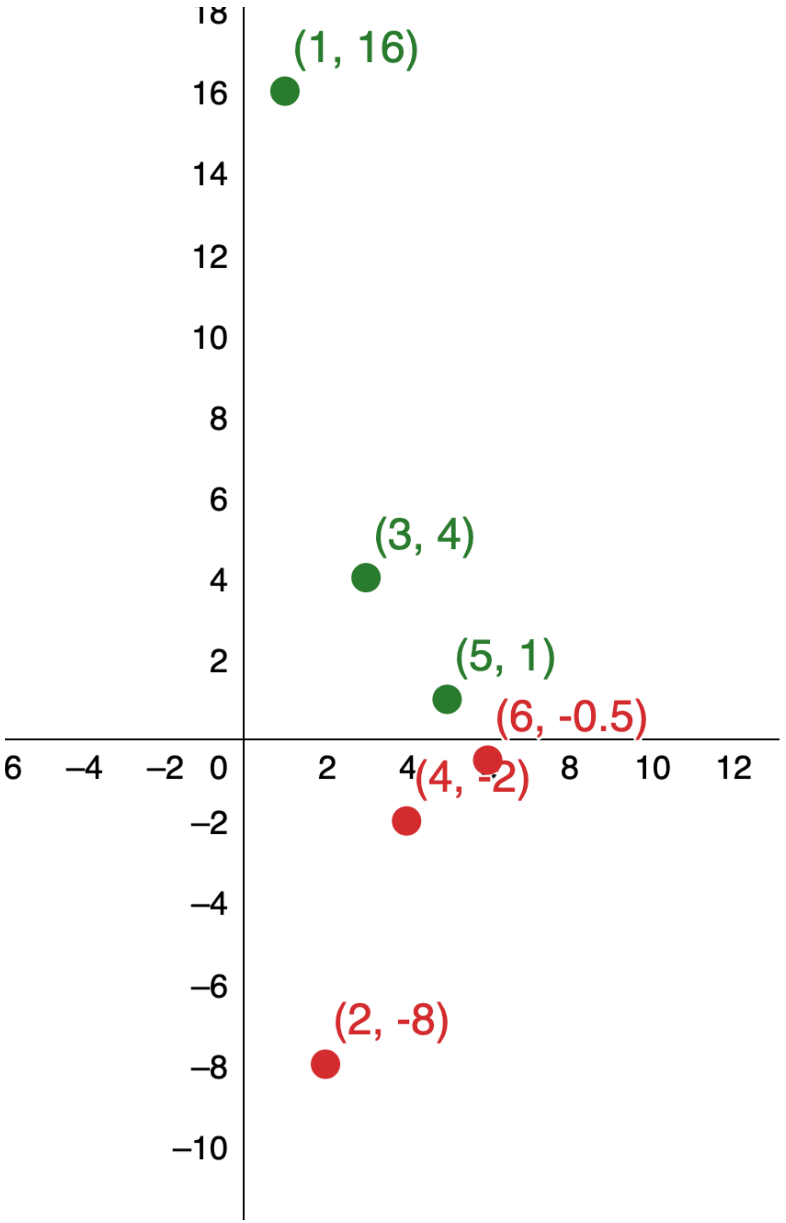
The sequence can be described by using the recurrence relation

- $\begin{cases} a_1 = 1 \\ a_n = 2a_{n-1}, n \geq 2 \end{cases}$

It can also be described using the explicit formula

- $a_n = 2^{n-1}$





GEOMETRIC SEQUENCE: EXAMPLE 2

The sequence can be described by using the recurrence relation

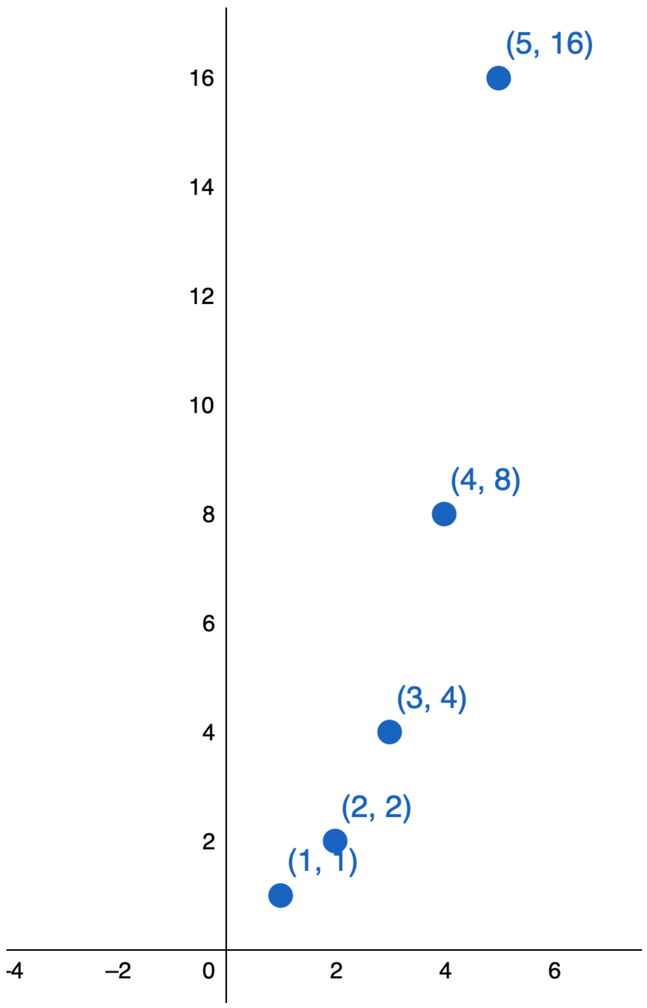
- $$\begin{cases} a_1 = 16 \\ a_n = -\frac{1}{2} a_{n-1}, n \geq 2 \end{cases}$$

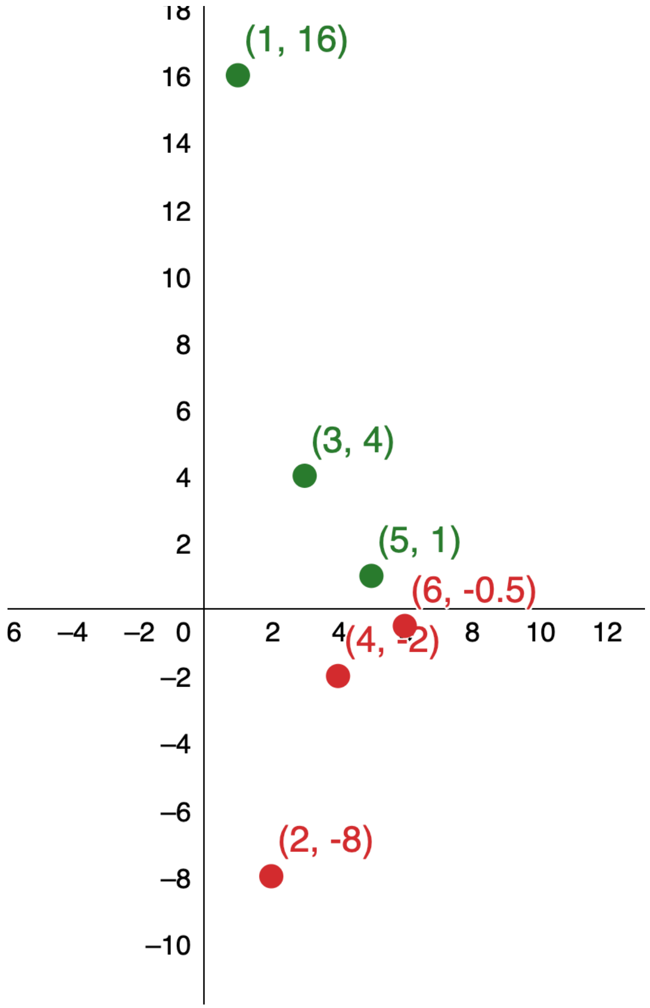
It can also be described using the explicit formula

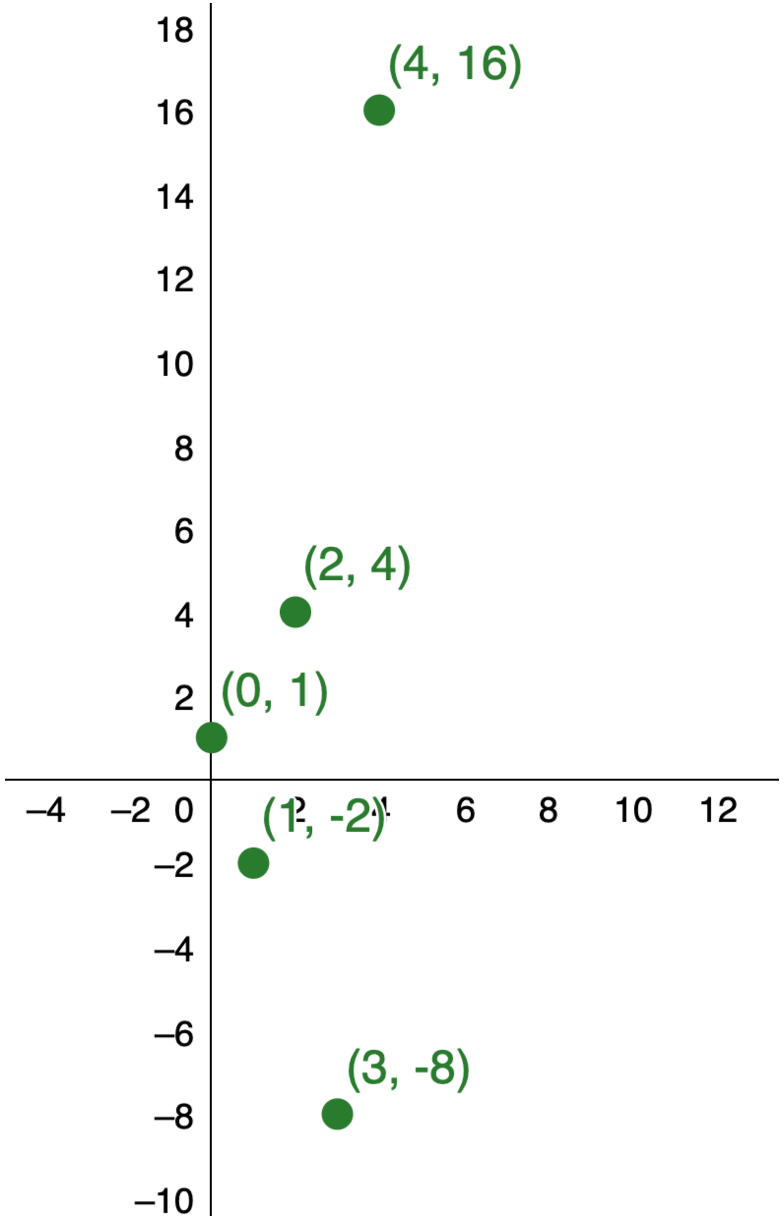
- $a_n = 16(-\frac{1}{2})^{n-1}$

GEOMETRIC SEQUENCE

- In general, a geometric sequence is any sequence of the form $a_n = cr^n$.
- What is r?







EXERCISE TWO

The sequence can be described by using the recurrence relation _____

It can also be described using the explicit formula _____