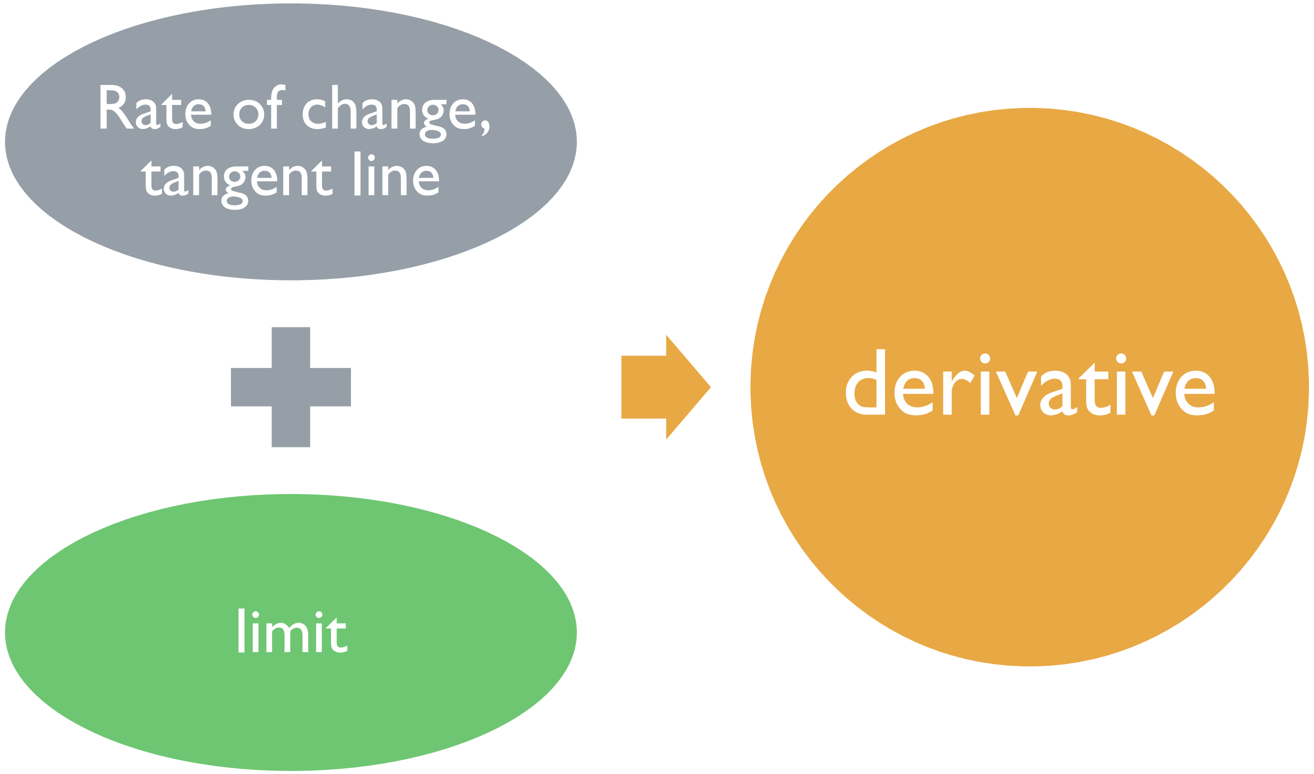
INTRODUCTION TO CALCULUS

DEFINING THE DERIVATIVE

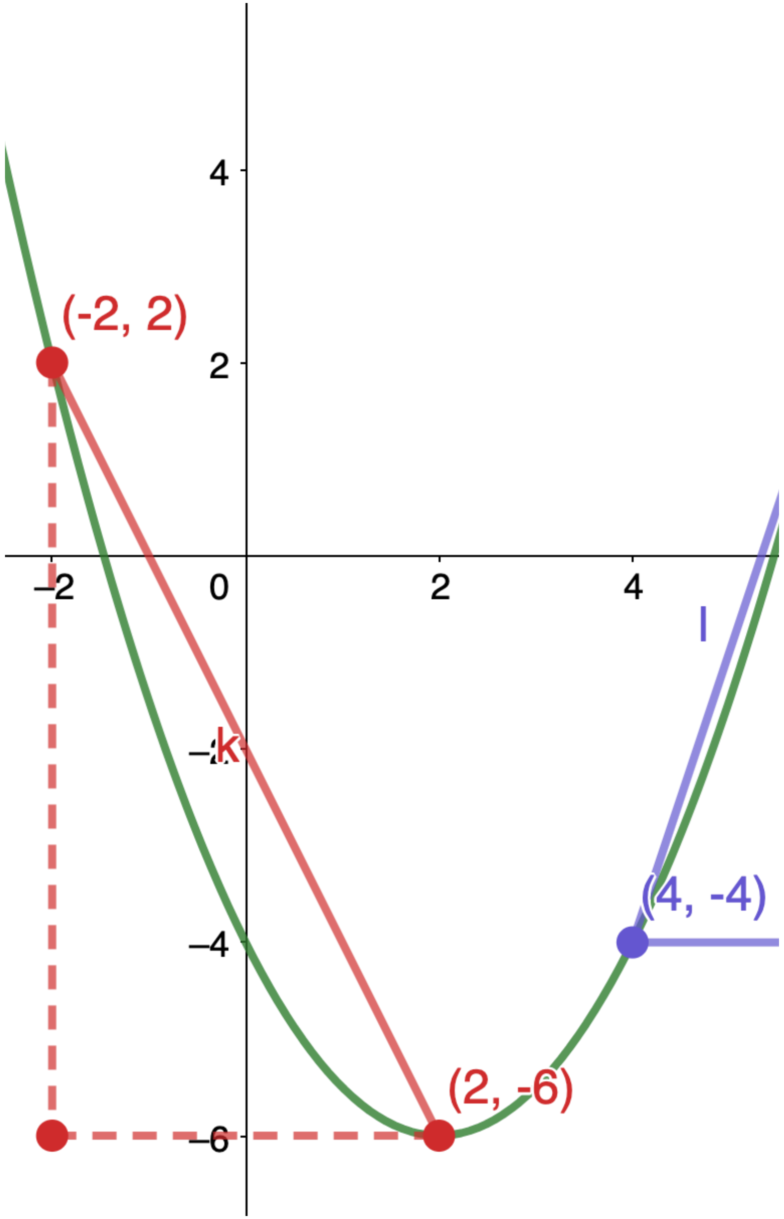
OUTLINE

Recognize	Recognize the meaning of **the tangent to a curve at a point**.
Calculate	Calculate **the slope of a tangent line**.
Identify	Identify the derivative as **the limit of a difference quotient**.
Calculate	**Calculate the derivative** of a given function at a point.
Describe	Describe **the velocity** as a rate of change.
Explain	Explain the difference between average velocity and instantaneous velocity.
Estimate	Estimate the derivative from a table of values.

HOW TO GET THE DERIVATIVE?



THANKS TO THESE GREAT MATHEMATICIANS

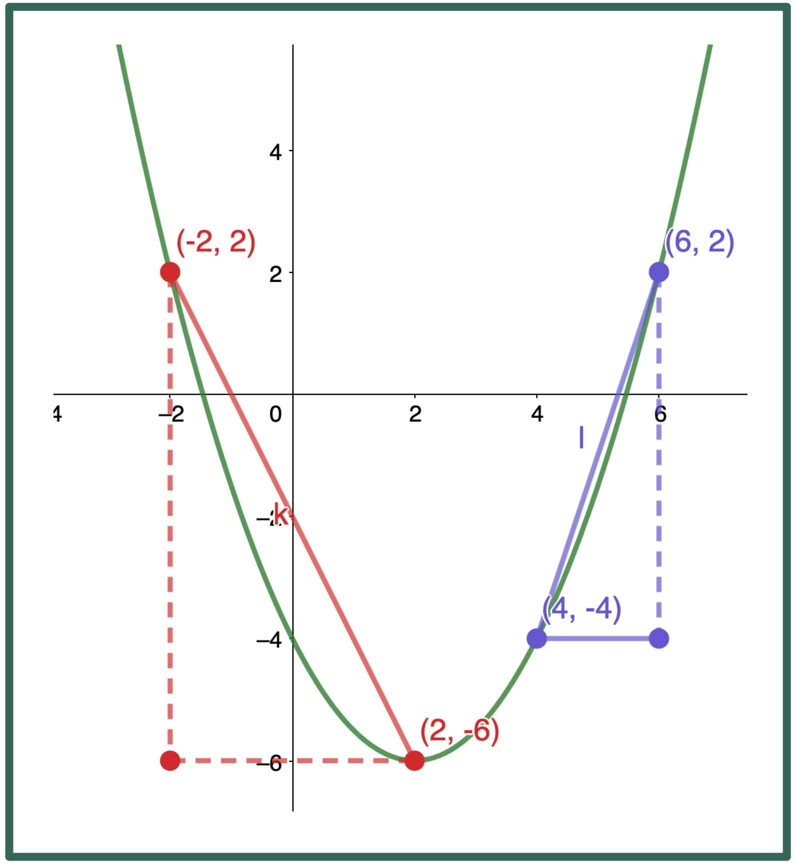


REVISIT THE NOTION OF SECANT LINES AND TANGENT LINES

We used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change.

- $m_{sec} = \frac{f(x) - f(a)}{x - a}$.

REVISIT THE NOTION OF SECANT LINES AND TANGENT LINES



Replacing x with $a+h$, where h is a value close to 0.

- $m_{sec} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$.

DEFINE THE DIFFERENCE QUOTIENT

DEFINITION

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a} \qquad \boxed{3.1}$$

is a **difference quotient**.

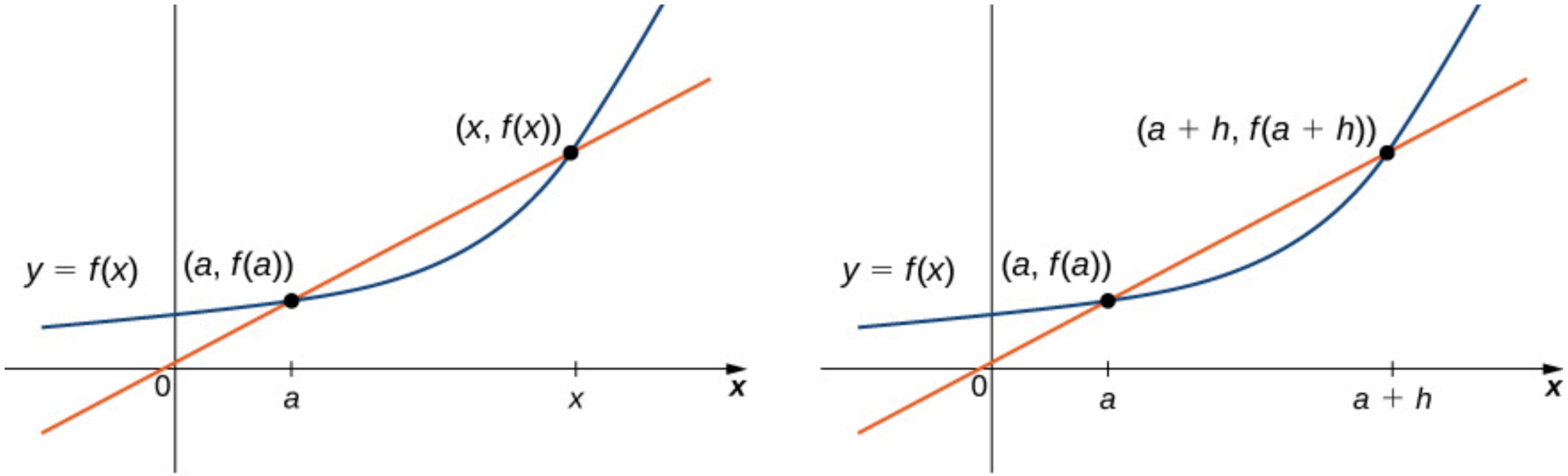
Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{h} \qquad \boxed{3.2}$$

is a difference quotient with increment h.

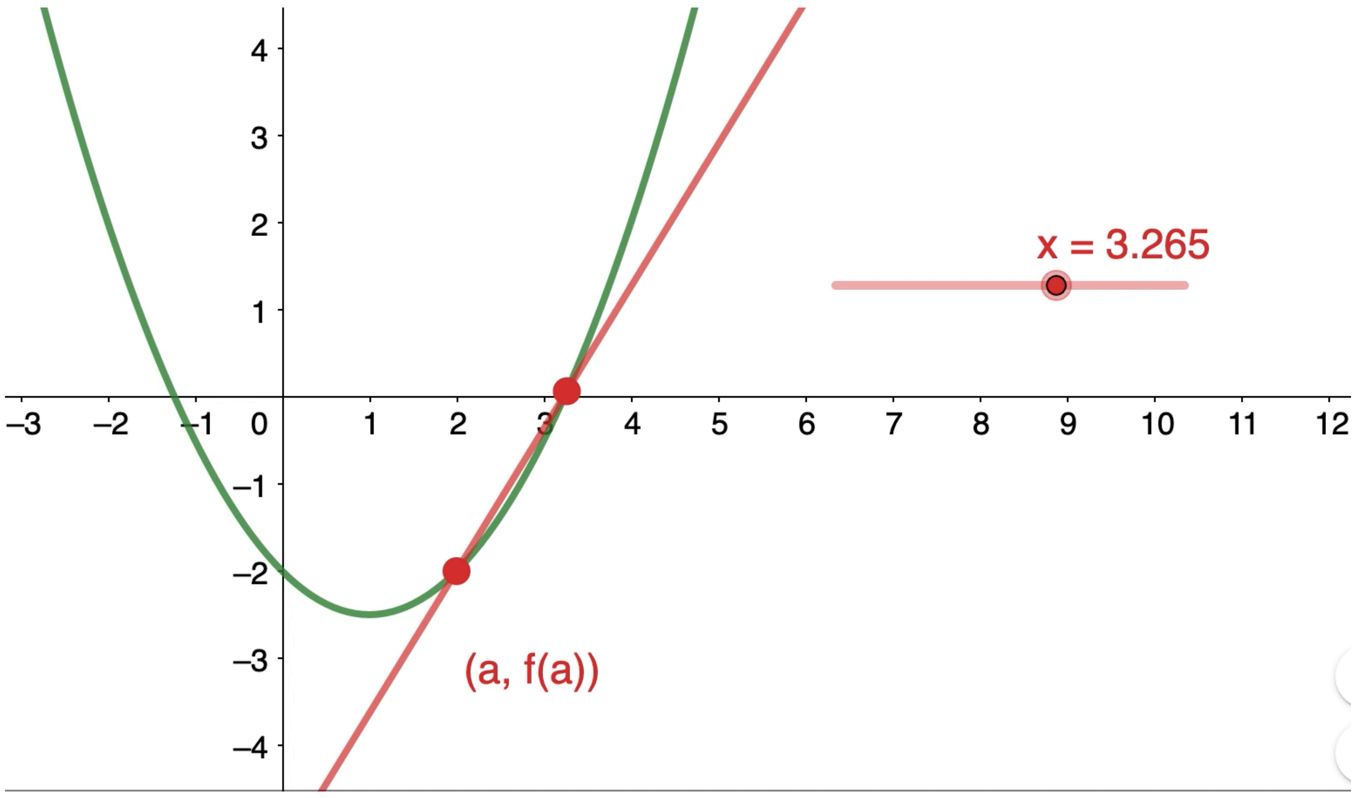
TWO DIFFERENT EXPRESSIONS

DEPENDING ON THE SETTING, WE CAN CHOOSE ONE OR THE OTHER.



FROM SECANT LINE TO TANGENT LINE

- as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a.
- Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines.

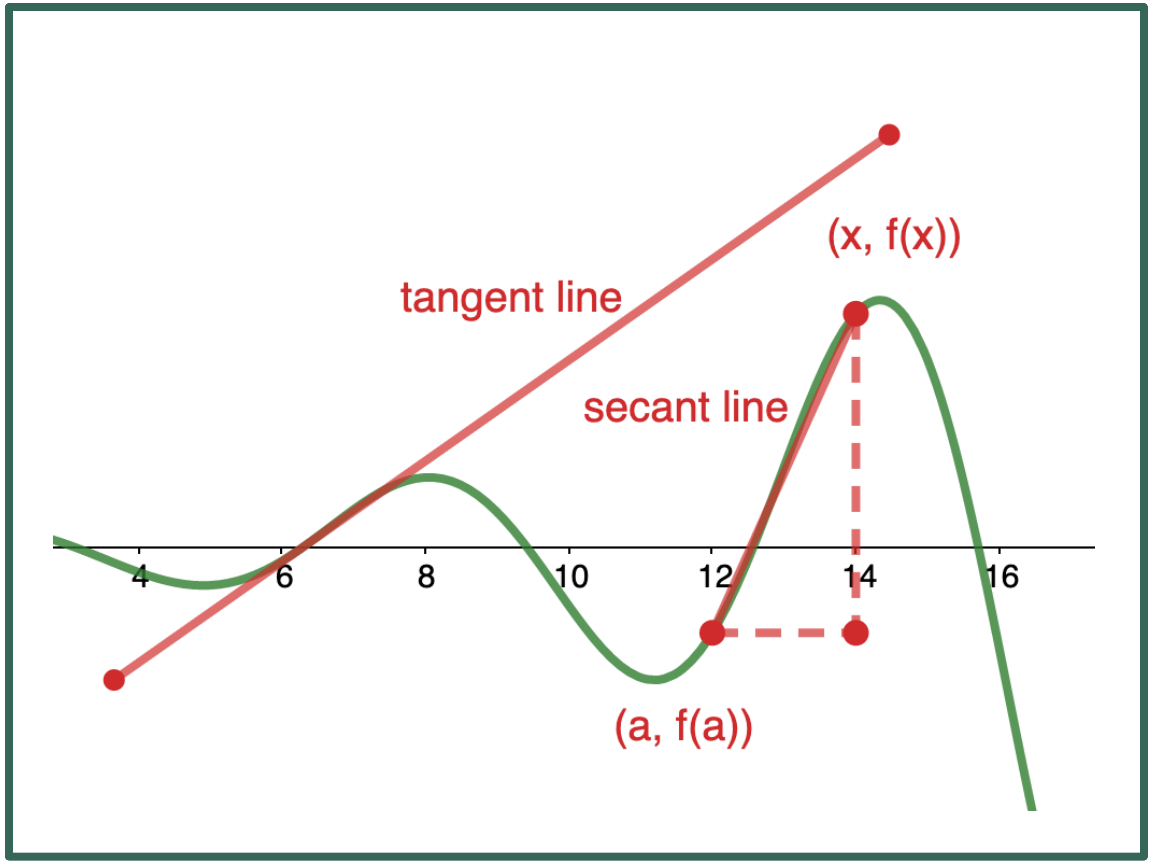


FROM SECANT LINE TO TANGENT LINE

$$m_{sec} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}.$$

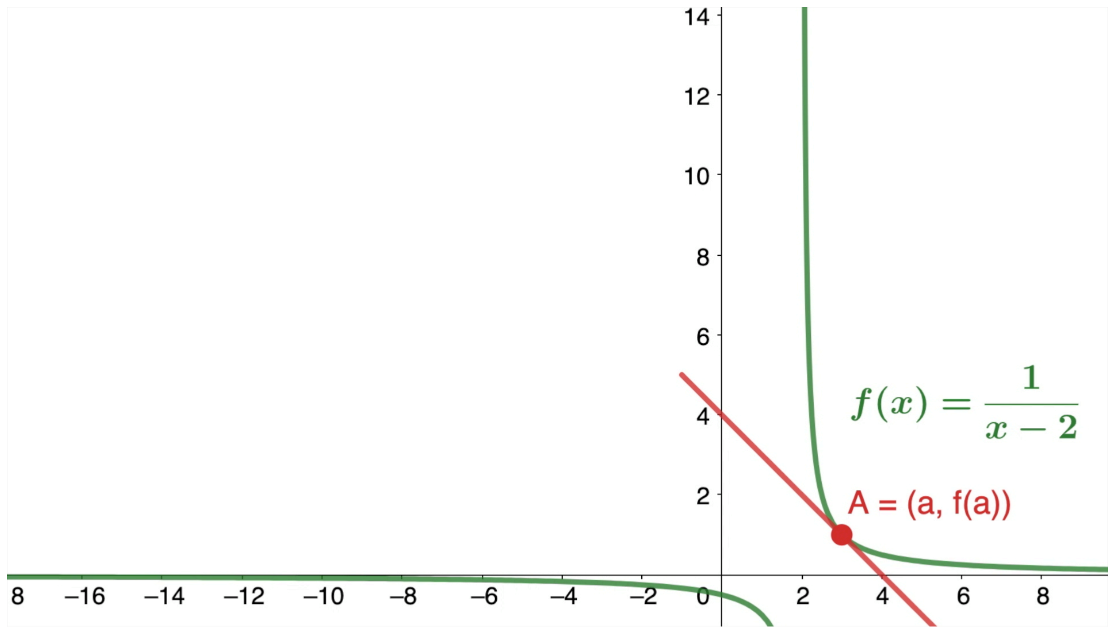
- Similarly, as the values of h get closer to 0, the secant lines also approach the tangent line.
- The slope of the tangent line at a is the rate of change of the function at a.

FROM SECANT LINE TO TANGENT LINE



- $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.
- $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

REMARKS



- We show the graph of $f(x)$ and its tangent line at $(a, f(a))$ in a series of tighter intervals about $x = a$.
- As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line **a good approximation** to the values of the function for choices of x close to a.

Let $f(x)$ be a function defined in an open interval containing a. The *tangent line* to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$\boxed{m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \tag{3.3}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$\boxed{m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}} \tag{3.4}$$

TANGENT LINE

EXERCISE ONE

Find the equation of the line tangent to the graph $f(x) = x^2$ at $x = 2$.

First find the slope of the tangent line.

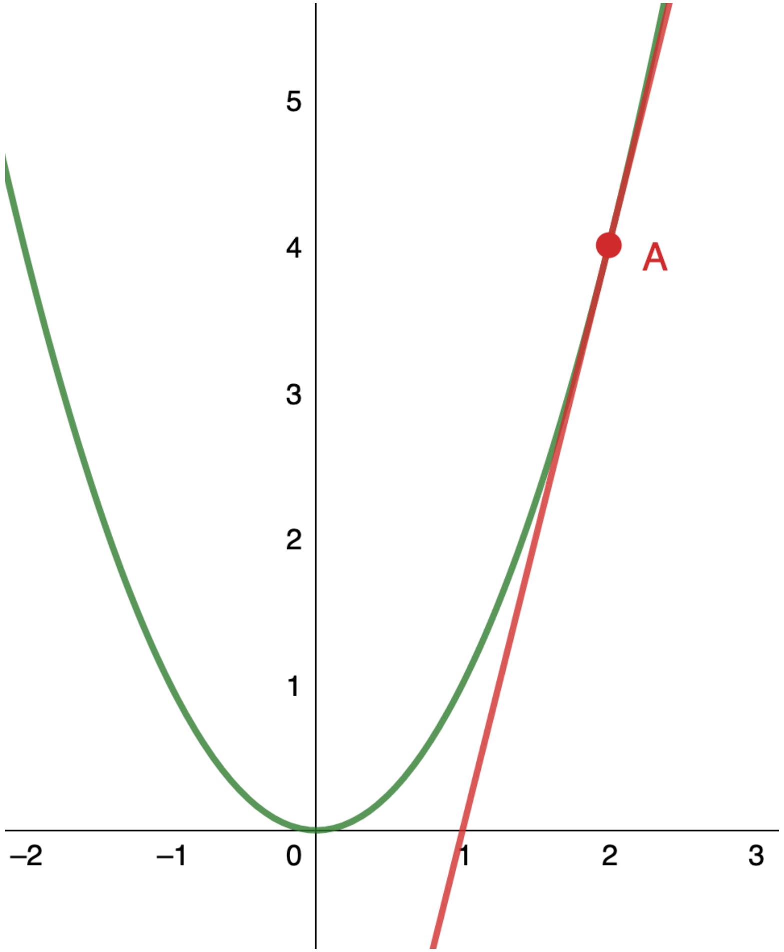
- $m_{tan} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to 2} \frac{x^2-2^2}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2}(x+2) = 4.$

Next, find a point on the tangent line.

- $\left(a, f(a)\right) = (2, 4)$

Using the point-slope equation of the line

- $y - 4 = 4(x - 2)$



EXERCISE ONE

Find the equation of the line tangent to the graph $f(x) = x^2$ at $x = 2$.

First find the slope of the tangent line.

- $m_{tan} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{(2+h)^2-2^2}{h} = \lim_{h \to 0} \frac{(4+h)h}{h} = \lim_{h \to 0}(4+h) = 4.$

Next, find a point on the tangent line.

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Using the point-slope equation of the line

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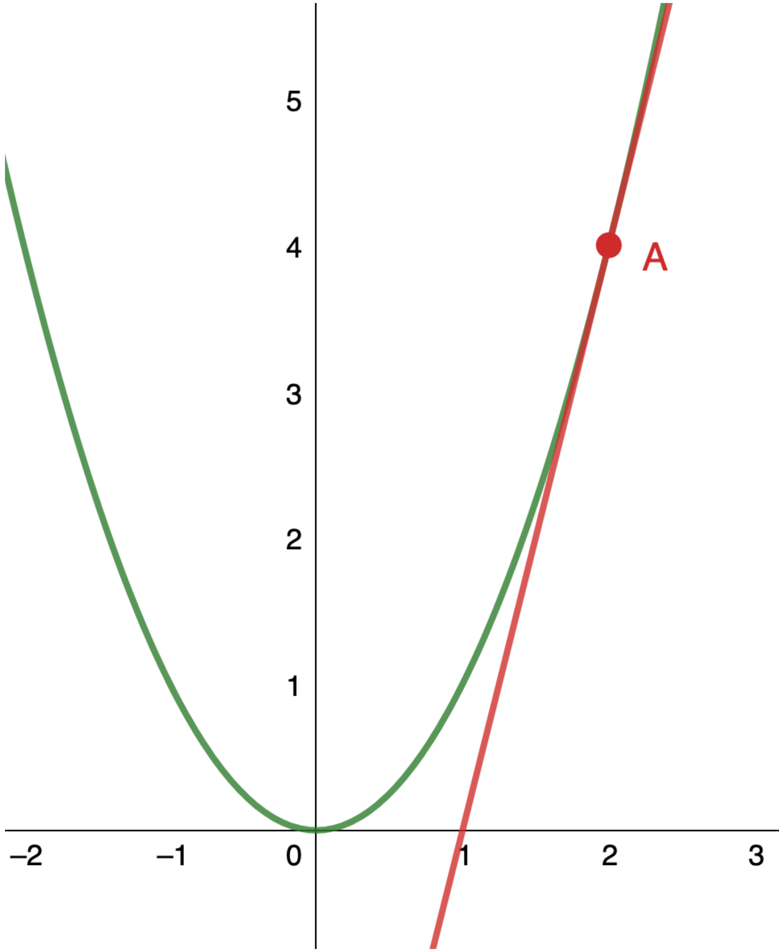
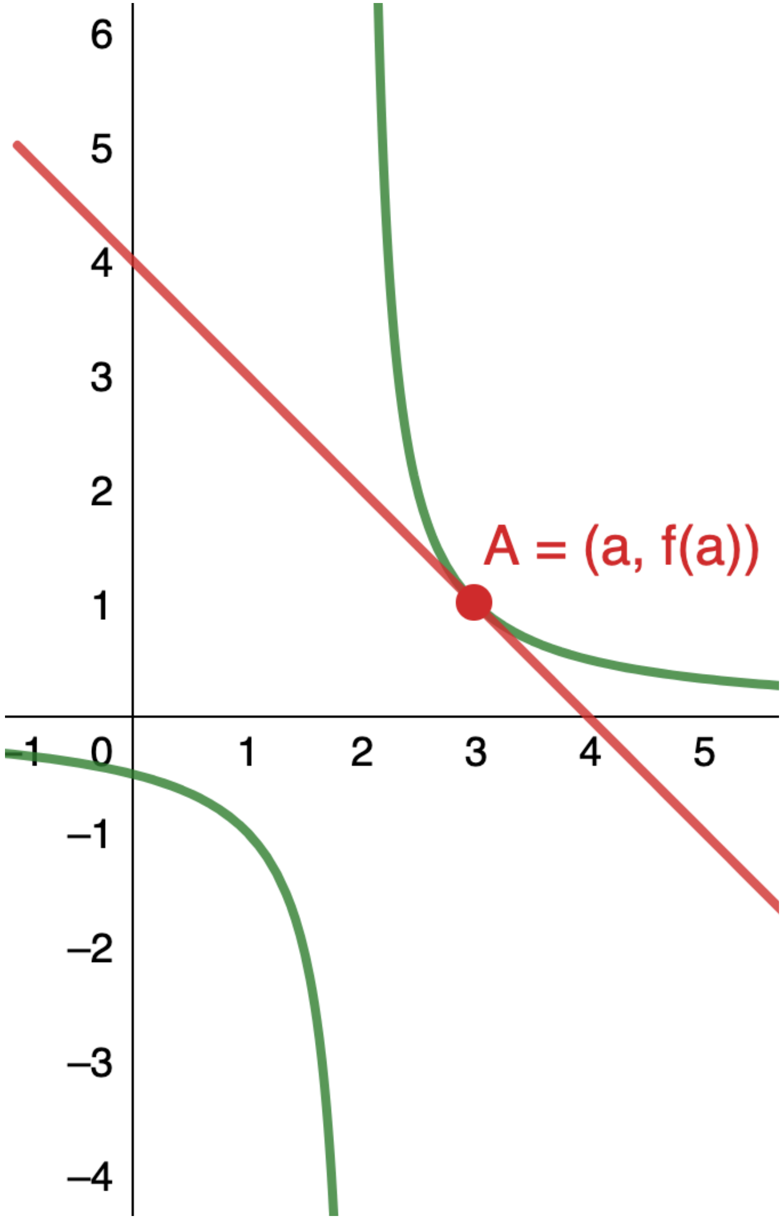




EXERCISE TWO

Find the equation of the line tangent to the graph $f(x) = \frac{1}{x-2}$ at $x = 3$.

First find the slope of the tangent line.

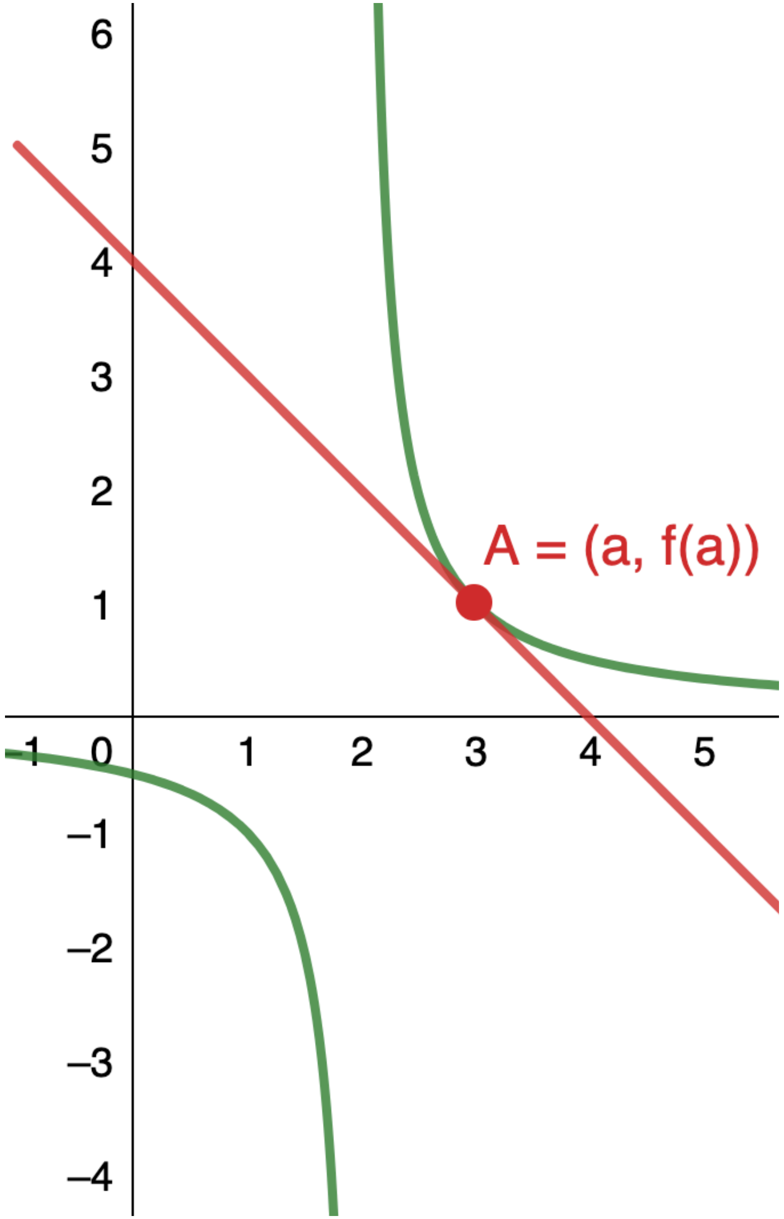
- $m_{tan} = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x\to 3} \frac{\frac{1}{x-2}-1}{x-3} = \lim\limits_{x\to 3} \frac{-(x-3)}{(x-2)(x-3)} = \lim\limits_{x\to 3}\left(-\frac{1}{x-2}\right) = -1.$

Next, find a point on the tangent line.

- $(a, f(a)) = (3, 1)$

Using the point-slope equation of the line

- $y - 1 = -(x - 3)$



EXERCISE TWO

Find the equation of the line tangent to the graph $f(x) = \frac{1}{x-2}$ at $x = 3$.

First find the slope of the tangent line.

- $m_{tan} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{1+h}-1}{h} = \lim_{h \to 0} \frac{-h}{h(1+h)} = \lim_{h \to 0} \left(-\frac{1}{1+h}\right) = -1.$

Next, find a point on the tangent line.

- $(a, f(a)) = (3, 1)$

Using the point-slope equation of the line

- $y - 1 = -(x - 3)$

DEFINE DERIVATIVE

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines.

These applications include **velocity and acceleration** in physics, marginal profit functions in business, and growth rates in biology.

This limit occurs so frequently that we give this value a special name: the **derivative**.

The process of finding a derivative is called **differentiation**.

DEFINITION

Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \qquad (3.5)$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.} \qquad (3.6)$$

DERIVATIVE

ESTIMATE A DERIVATIVE

x	$\frac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
2.001	4.001
2.01	4.01
2.1	4.1

For $f(x) = x^2$, use a table to estimate its derivative at $x = 2$.

- $f'(2)$

ESTIMATE DERIVATIVE EXERCISE ONE

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

For $f(x) = 2x^2 - 3x + 5$, find $f'(-1)$.

- $f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x+1} = \lim_{x \to -1} \frac{2x^2 - 3x - 5}{x+1} = \lim_{x \to -1} \frac{(x+1)(2x-5)}{x+1} = \lim_{x \to -1} (2x - 5) = -7.$

ESTIMATE DERIVATIVE EXERCISE TWO

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.}$$

For $f(x) = x^2 + x + 1$, find $f'(4)$.

- $f'(4) = \lim_{x \to 4} \frac{f(x)-f(4)}{x-4} = \lim_{x \to 4} \frac{x^2+x-20}{x-4} = \lim_{x \to 4} \frac{(x+5)(x-4)}{x-4} = \lim_{x \to 4}(x+5) = 9.$

VELOCITIES AND RATES OF CHANGE

- We have introduced a function $s(t)$, that gives the position of an object along a coordinate axis at any given time t.
- Velocity may be thought of as the rate of change of position.

VELOCITIES AND RATES OF CHANGE

DEFINITION

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \tag{2.2}$$

DEFINITION

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

INSTANTANEOUS VELOCITY

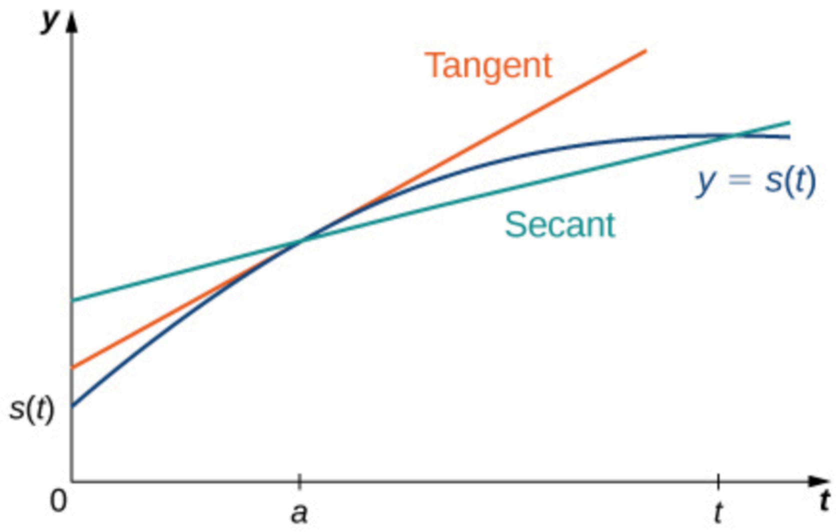


Figure 3.8 The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

- $v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$

ESTIMATE VELOCITY EXERCISE ONE

A cat is climbing up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin 2t$. Estimate $v(0)$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

ESTIMATE VELOCITY EXERCISE ONE

A cat is climbing up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin 2t$. Estimate $v(0)$.

- $v(0) = s'(0) = \lim_{t \to 0} \frac{s(t) - s(0)}{t} = \lim_{t \to 0} \frac{\sin 2t}{t} = \lim_{t \to 0} \frac{2 \sin t \cos t}{t} = \lim_{t \to 0} \frac{\sin t}{t} \lim_{t \to 0} 2 \cos t = 2$

ESTIMATE VELOCITY EXERCISE TWO

- Two zebras are wandering in the wood, seeking some little shelter from the wind and rain. They set off 10 days ago from Zoo Atlanta ($t = 0$) and happen to meet each other again in Hanover. Treating Washington DC as the origin, their positions are

$$S_1(t) = t^2 - 2t - 8 \text{ and } S_2(t) = 8t - 8.$$

- Use this information to decide which one of them are faster when they arrived at DC. What about now?

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$$S_1(t) = t^2 - 2t - 8 \text{ and } S_2(t) = 8t - 8.$$

- DC

$$S_1(a_1) = 0, S_2(a_2) = 0$$

$$a_1 = 4, a_2 = 1$$

- $v_1(4) = S_1'(4) = \lim_{t\to4}\frac{s_1(t)-s_1(4)}{t-4} = \lim_{t\to4}\frac{t^2-2t-8}{t-4} = \lim_{t\to4}\frac{(t+2)(t-4)}{t-4} = \lim_{t\to4}(t+2) = 6.$
- $v_2(1) = S_2'(1) = \lim_{t\to1}\frac{s_2(t)-s_2(1)}{t-1} = \lim_{t\to1}\frac{8t-8}{t-1} = \lim_{t\to1} 8 = 8.$

ESTIMATE VELOCITY EXERCISE TWO

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$$S_1(t) = t^2 - 2t - 8 \text{ and } S_2(t) = 8t - 8.$$

- Now

$$S_1(t) = S_2(t) \implies a = 10.$$

- $v_1(10) = S_1{}'(10) = \lim_{t \to 10} \frac{s_1(t) - s_1(10)}{t - 10} = \lim_{t \to 10} \frac{t^2 - 2t - 80}{t - 10} = \lim_{t \to 10} \frac{(t+8)(t-10)}{t-10} = \lim_{t \to 10} (t + 8) = 18.$
- $v_2(10) = 8.$

ESTIMATE **RATE OF CHANGE OF PROFIT EXERCISE**

- Morano Gelato determines that the daily profit on ice cream obtained by charging s dollars per cup is $P(s) = -50s^2 + 300s - 20$. The shop currently charges $\$4.00$ per cup. Find $P'(4.00)$ the rate of change of profit when the price is $\$4.00$ and decide whether or not the gelato shop should consider raising or lowering its prices.

ESTIMATE **RATE OF CHANGE OF PROFIT EXERCISE**

- $P(s) = -50s^2 + 300s - 20$. The shop currently charges $\$4.00$ per cup. Find $P'(4.00)$ the rate of change of profit when the price is $\$4.00$ and decide whether or not the gelato shop should consider raising or lowering its prices.
- $P'(4) = \lim_{h \to 0} \frac{P(4+h) - P(4)}{h} =$
$\lim_{h \to 0} \frac{-50(4+h)^2 + 300(4+h) + 504^2 - 300 \cdot 4}{h} =$
$\lim_{h \to 0} \frac{-400h - 50h^2 + 300h}{h} = \lim_{h \to 0}(-100 - 50h) =$
-100.