## Problem 1.

Determine whether the following sequences are increasing, decreasing, or not monotonic, and explain. Is the sequence bounded? If yes, what is it bounded by?

(a) The sine function is bounded; it is always greater than or equal to -1 and less than or equal to 1. It is not monotonic, however, it is a wave, so it continually cycles up and down and never increases or decreases after some finite number of terms.

(b) This function is decreasing. We can back into this by noting that $2n+1 \geq 0$ for $n \geq 1$, so by adding $n^2+1$ to both sides, we get $n^2+2n+2 \geq n^2+1$, then by dividing both sides by both sides we get $\frac{1}{n^2+1} \geq \frac{1}{(n+1)^2+1}$, so $a_n \geq a_{n+1}$, and the sequence is decreasing. Thus it is bounded above by $a_1 = 1/2$, and it is bounded below by 0, because it is the ratio of two positive terms always.

(c) Again, solving the difference between consecutive terms gives us our answer. Starting with $3 \geq 0$, and adding $-n^2-2n$ to both sides gives us $3-2n-n^2 \geq -2n-n^2$, which we can rearrange to $\frac{1-n}{2+n} \geq \frac{1-(n+1)}{3+n}$, or $a_n \geq a_{n+1}$, so the sequence is decreasing. We can also calculate the limit is -1 by the algebraic limit laws, so it is bounded above by $a_1 = 0$ and bounded below by the limit -1.

(d) This sequence is not monotonic; it alternates in sign between positive and negative. It is bounded, because $(-1)^n/n \leq 1/n$, so it is bounded above by $1/2$, and $(-1)^n/n \geq -1/n$, so it is bounded below by $-1$.

## Problem 2.

Use the Monotone Convergence Theorem to show that each sequence converges.

(a) For $a_n = -(\frac{2}{3})^n$, we have that $a_n = 2/3 \cdot a_{n-1}$, so since $a_n$ is negative for all $n$, the sequence is increasing. It is bounded above by 0, since every term is negative, and bounded below by $a_1 = -2/3$, so the sequence is bounded and monotonic and thus by the Monotone Convergence Theorem, it converges.

(b) For $a_n = 1 + 1/n$, we have that the sequence is decreasing, because $a_n \geq a_{n+1}$, or $1+1/n \geq 1+1/(n+1)$, or $n+1 \geq n$. Thus it is bounded above by $a_1 = 3/2$, and since it is always 1 plus a positive number, it is bounded below by 1. Thus the sequence is bounded and monotonic, and by the Monotone Convergence Theorem, it converges.

(c) Since $(n+1)^2 \geq n^2$, we have that $a_n = 2/n^2 \geq 2/(n+1)^2 = a_{n+1}$, so the sequence is decreasing. Therefore it is bounded above by $a_1 = 2$, and bounded below by 0 since it is always a ratio of positive numbers, so positive. Thus the sequence is bounded and monotonic, and by the Monotone Convergence Theorem, it converges.

## Problem 3.

Give an example of a sequence that is bounded but not convergent.

There are plenty of these. One is $a_n = (-1)^n$, which is bounded above by 1 and below by -1, but which does not converge.

## Problem 4.

Give an example of a sequence that is not monotone but is convergent.

There are plenty of these. One is $a_n = \frac{(-1)^n}{n}$, which is not monotone because it alternates in sign, but converges because of the Squeeze Theorem with $\frac{-1}{n}$ and $\frac{1}{n}$.