INTRODUCTION TO CALCULUS

DERIVATIVES OF INVERSE (TRIG) FUNCTIONS

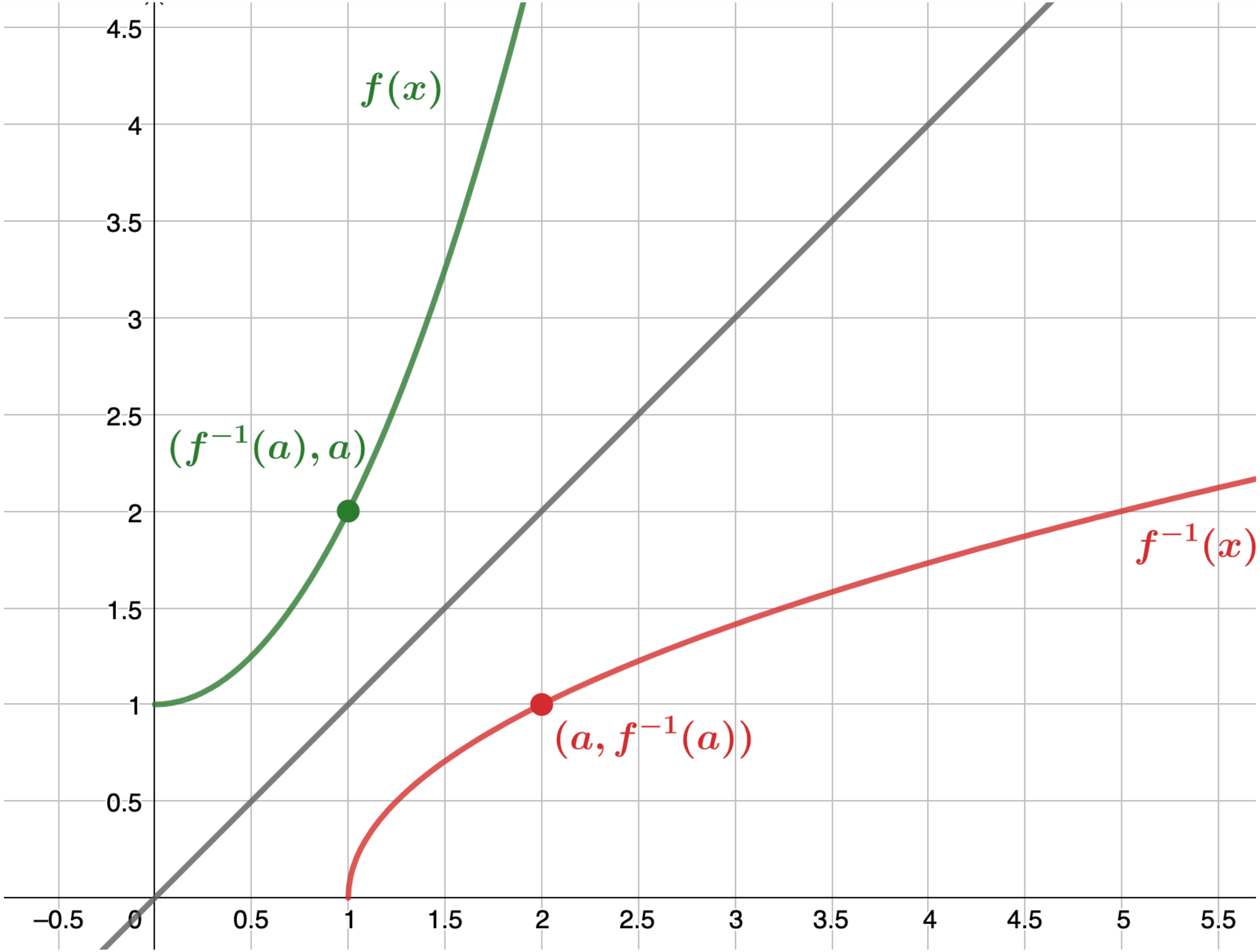
Calculate the derivative of an inverse function.

Recognize the derivatives of the standard **inverse trigonometric functions**.

OUTLINE

THE RELATIONSHIP BETWEEN THE DERIVATIVE OF A FUNCTION AND THE DERIVATIVE OF ITS INVERSE

- For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative.
- In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions.



HOW TO FIND THE RELATION?

METHOD ONE

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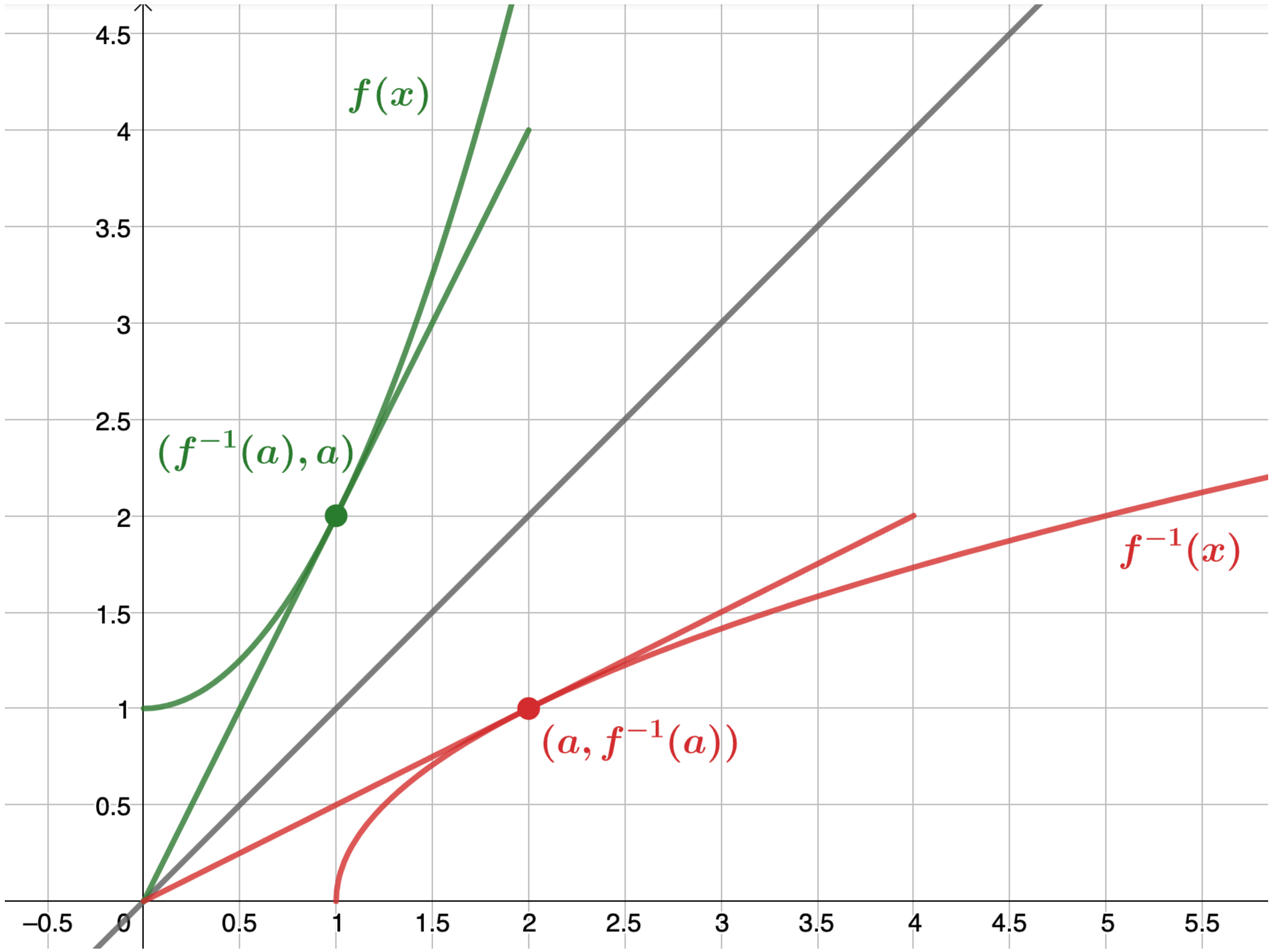
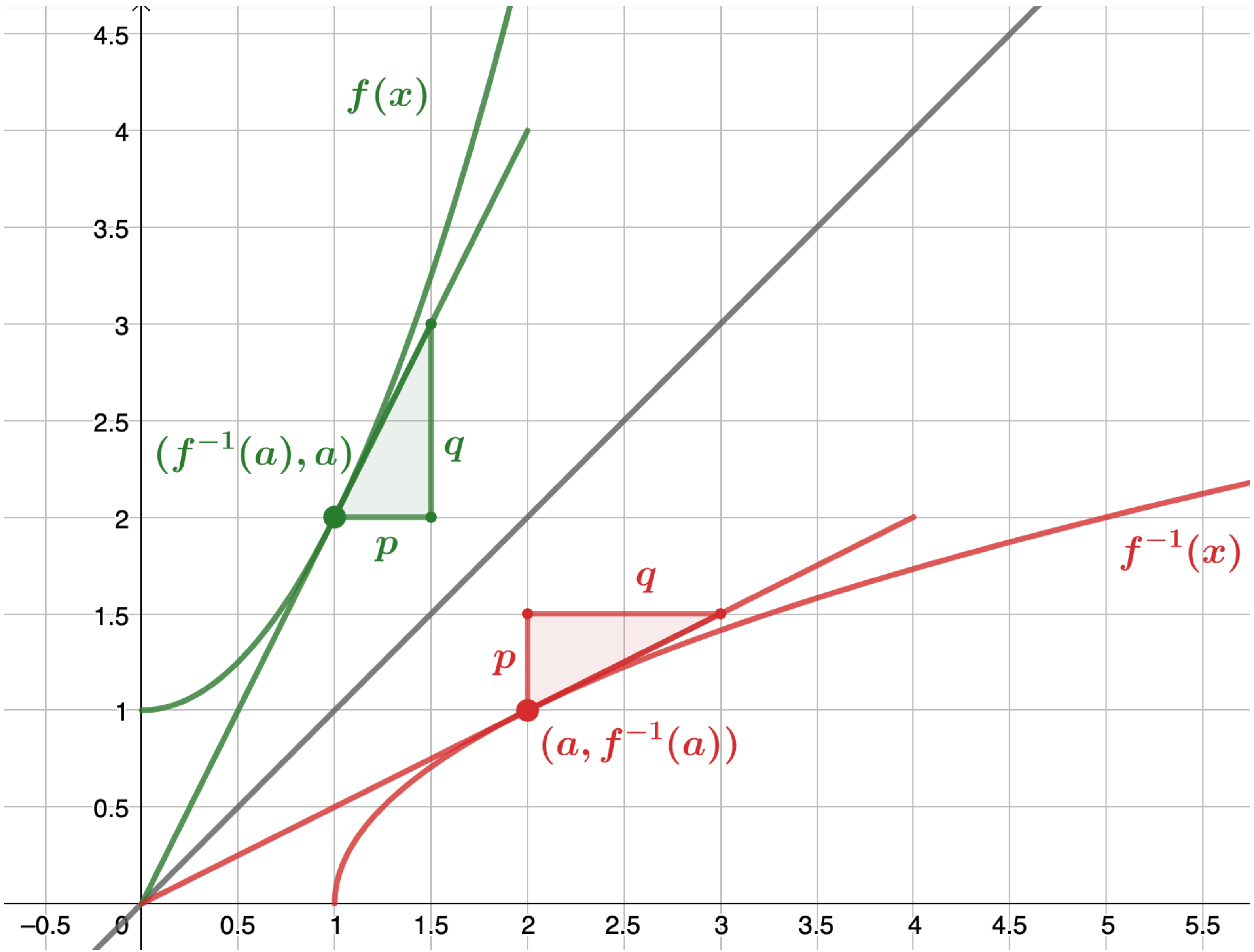
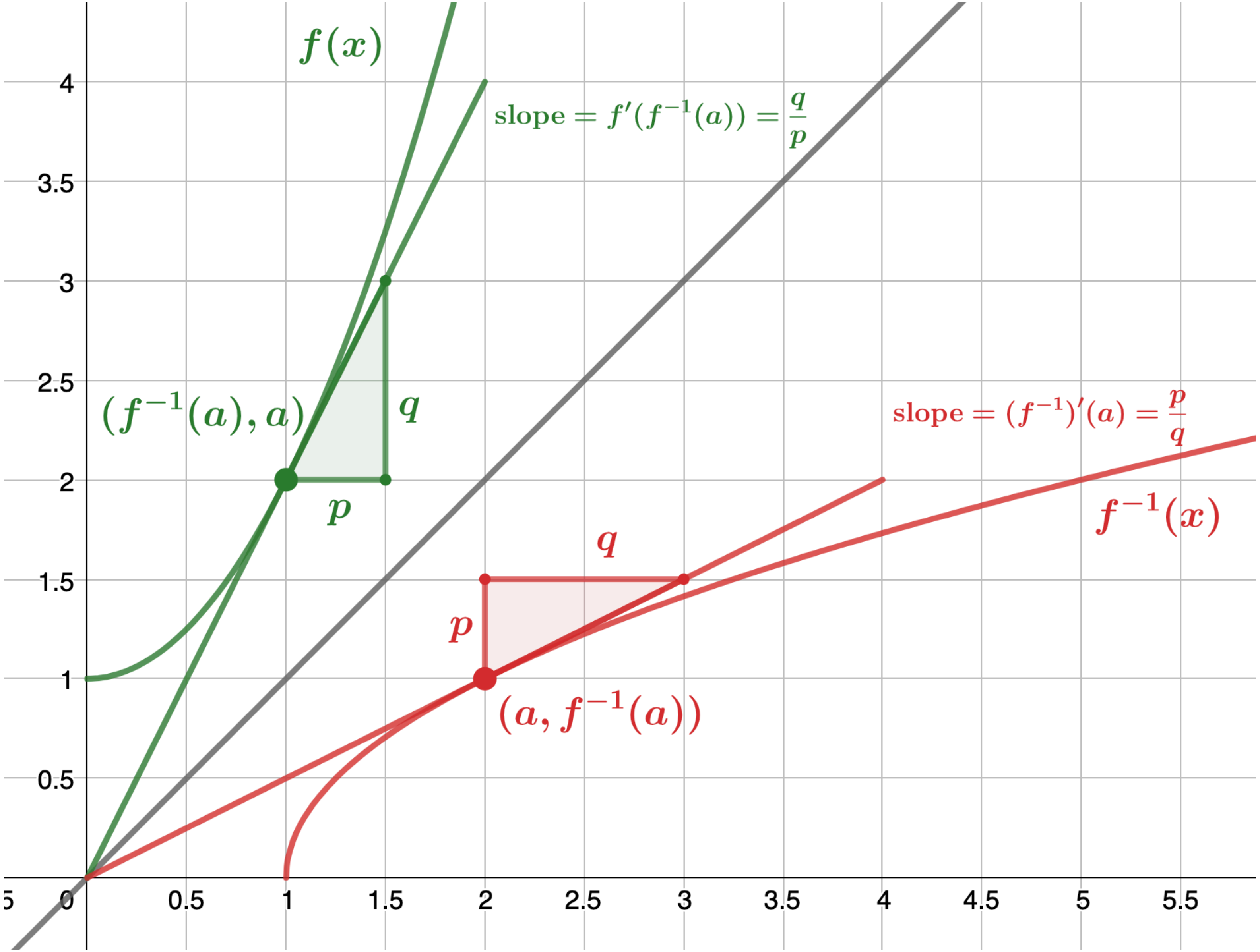
- METHOD ONE





HOW TO FIND THE RELATION

METHOD ONE

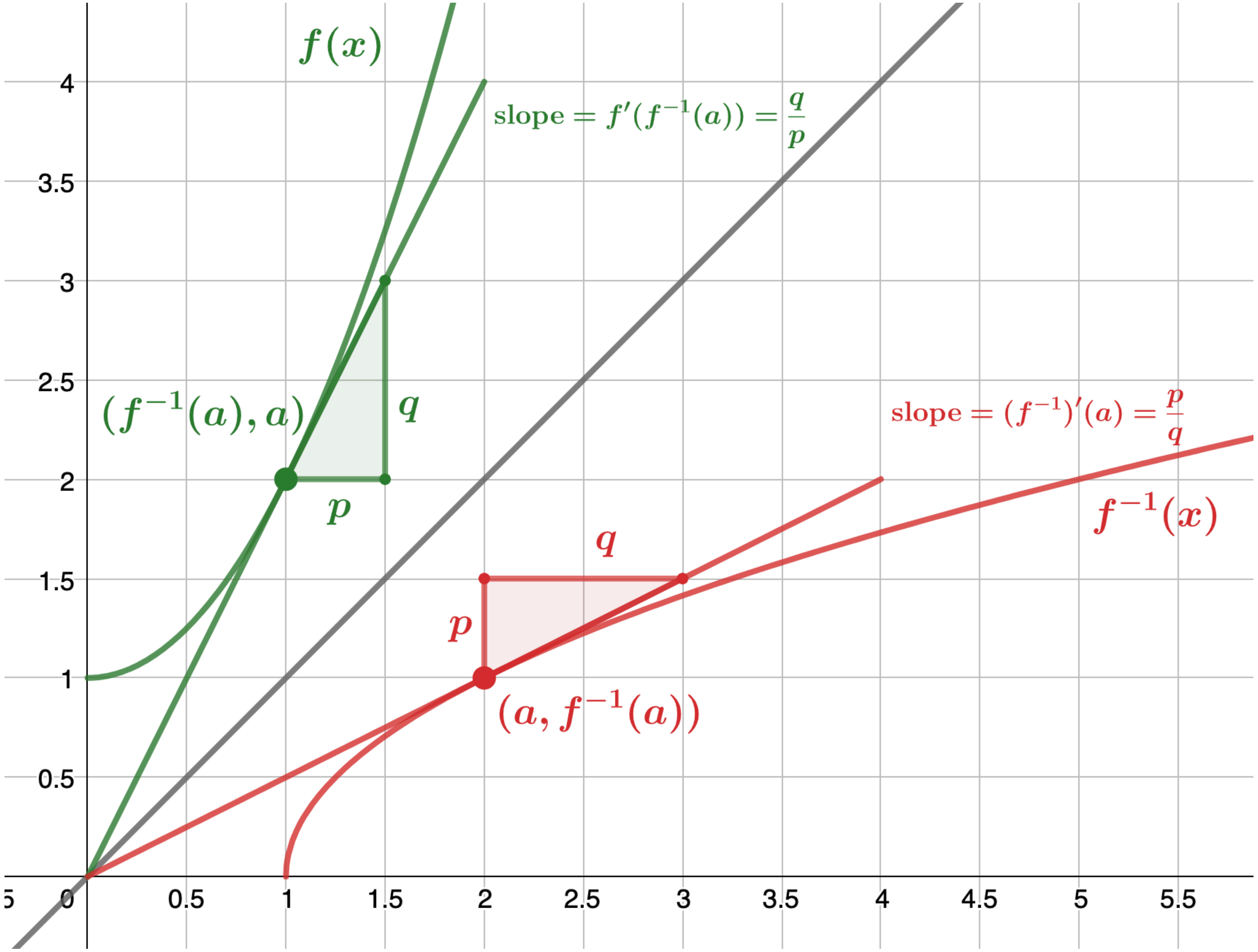


HOW TO FIND THE RELATION

METHOD ONE

HOW TO FIND THE RELATION

- $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$



HOW TO FIND THE RELATION

Method two

Recall that

- $x = f(f^{-1}(x))$.

Differentiating both sides of this equation (using the chain rule on the right), we obtain

- $1 = f'(f^{-1}(x))(f^{-1})'(x)$.

Solving for $(f^{-1})'(x)$, we obtain

- $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

SUMMARY

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

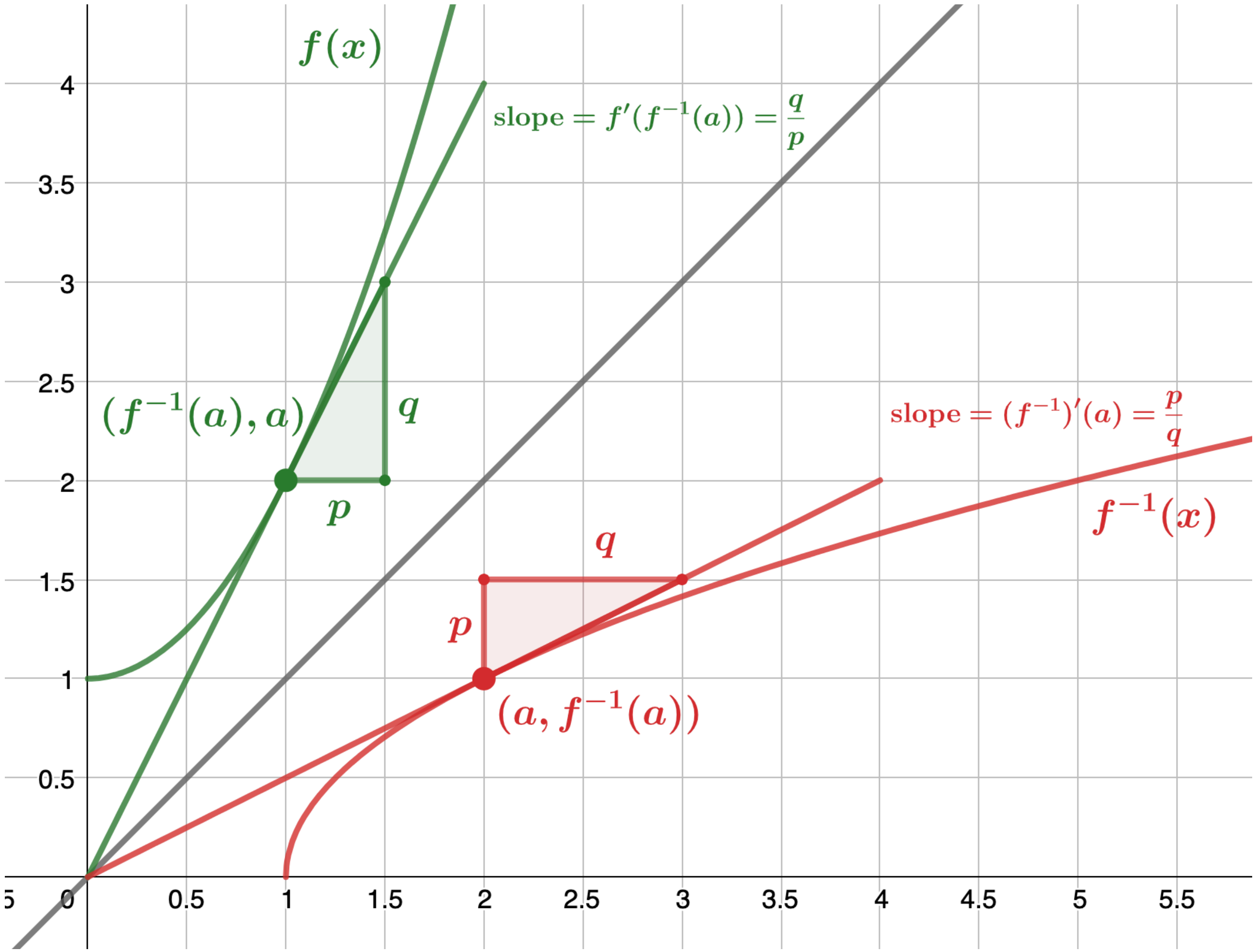
$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \boxed{\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.}$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

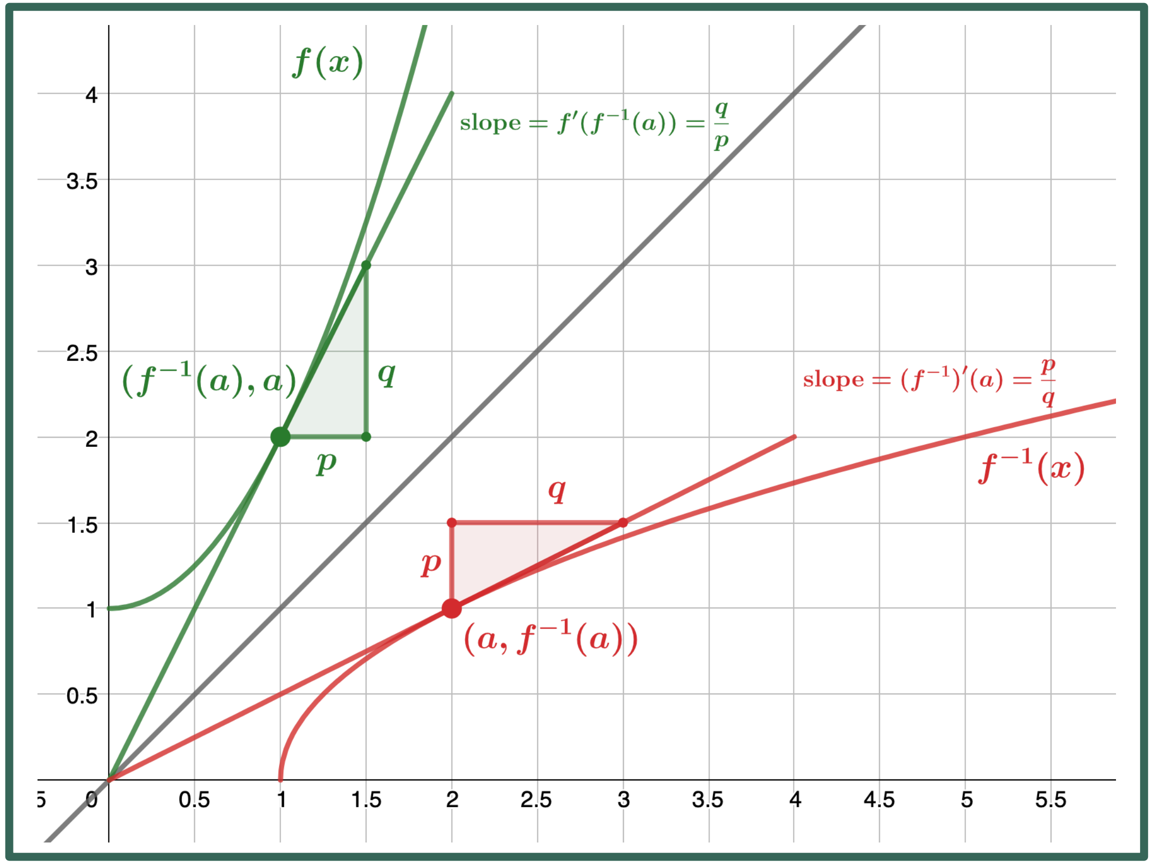
$$\boxed{g'(x) = \frac{1}{f'(g(x))}.}$$

EXAMPLE

- Use the inverse function theorem to find the derivative of $g(x) = \sqrt{x-1}$. Compare the result obtained by differentiating $g(x)$ dir ectly.



EXAMPLE



- $f(x) = x^2 + 1, x \geq 0$
- $g(x) = f^{-1}(x) = \sqrt{x-1}$
- $f'(x) = 2x, x \geq 0$
- $g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{2\sqrt{x-1}}$

TWO IMPORTANT APPLICATIONS

- **Extend the Power Rule to Rational Exponents**
- **Derive derivatives of Inverse Trigonometric Functions**

EXTEND THE POWER RULE TO RATIONAL EXPONENTS

Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\boxed{\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}.}$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\boxed{\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.}$$

THE GENERAL POWER RULE

- $\frac{d}{dx} x^b = b x^{b-1}$

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n} x^{(m/n)-1}.$$

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n} x^{(1/n)-1}.$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

PROOF

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}.$$

We first try $g(x) = x^{1/n}$.

- $f(x) = x^n$
- $g(x) = f^{-1}(x) = x^{1/n}$
- $f'(x) = nx^{n-1}$
- $g'(x) = \dfrac{1}{nx^{(n-1)/n}} = \dfrac{1}{n}x^{-(n-1)/n} = \dfrac{1}{n}x^{(1/n)-1}$

PROOF

$$\boxed{\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.}$$

We then try $g(x) = x^{m/n}$.

- $g(x) = x^{m/n} = (x^{1/n})^m$

Apply the chain rule!

- $g'(x) = m(x^{1/n})^{m-1} \cdot \frac{1}{n} x^{(1/n)-1} = \frac{m}{n} x^{(m-1)/n} \cdot x^{(1/n)-1} = \frac{m}{n} x^{(m/n)-1}$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

- The derivatives of inverse trigonometric functions are actually **algebraic functions**.
- Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions.
- Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

EXAMPLE ONE

$$\boxed{g'(x) = \frac{1}{f'(g(x))}.}$$

Derivative of the Inverse Sine Function

- Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1}(x)$.

EXAMPLE ONE

$$\boxed{g'(x) = \frac{1}{f'(g(x))}.}$$

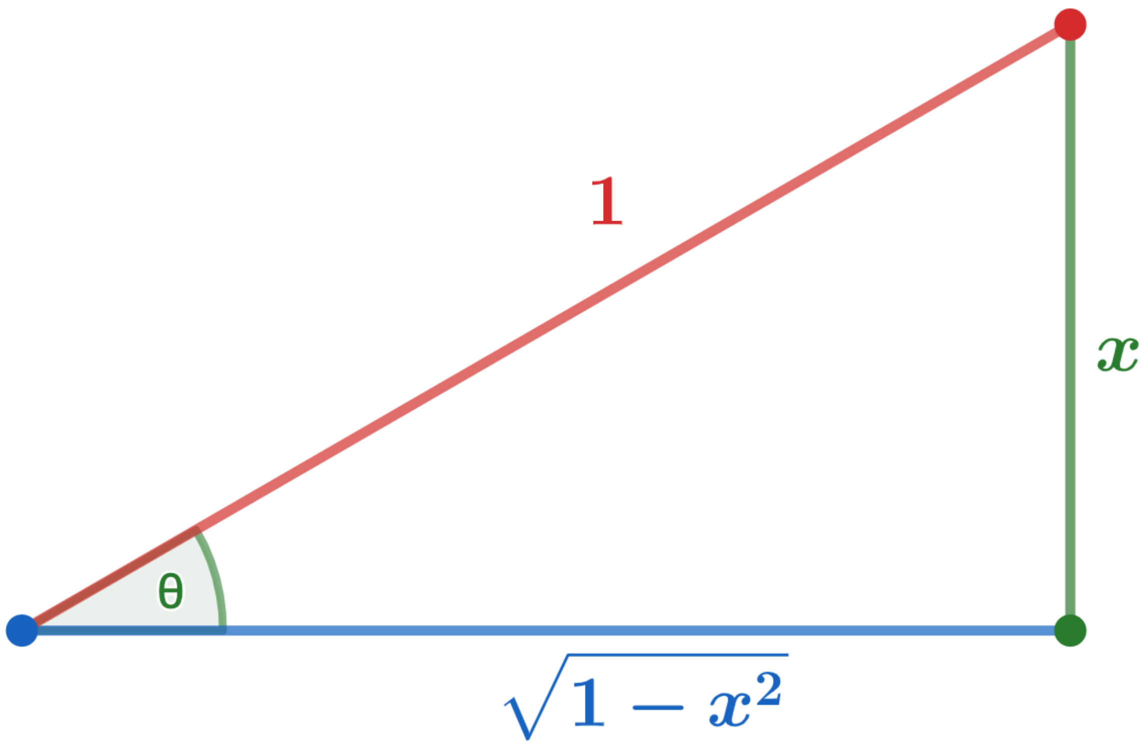
- Since for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $f(x) = \sin(x)$ is in the inverse of $g(x) = \sin^{-1}(x)$, begin by finding $f'(x)$.
- Since $f'(x) = \cos(x)$, it follows that $f'\big(g(x)\big) = \cos(\sin^{-1}(x))$ and $g'(x) = \frac{1}{\cos(\sin^{-1}(x))}$.
- But what on earth is $\cos(\sin^{-1}(x))$?

EXAMPLE ONE

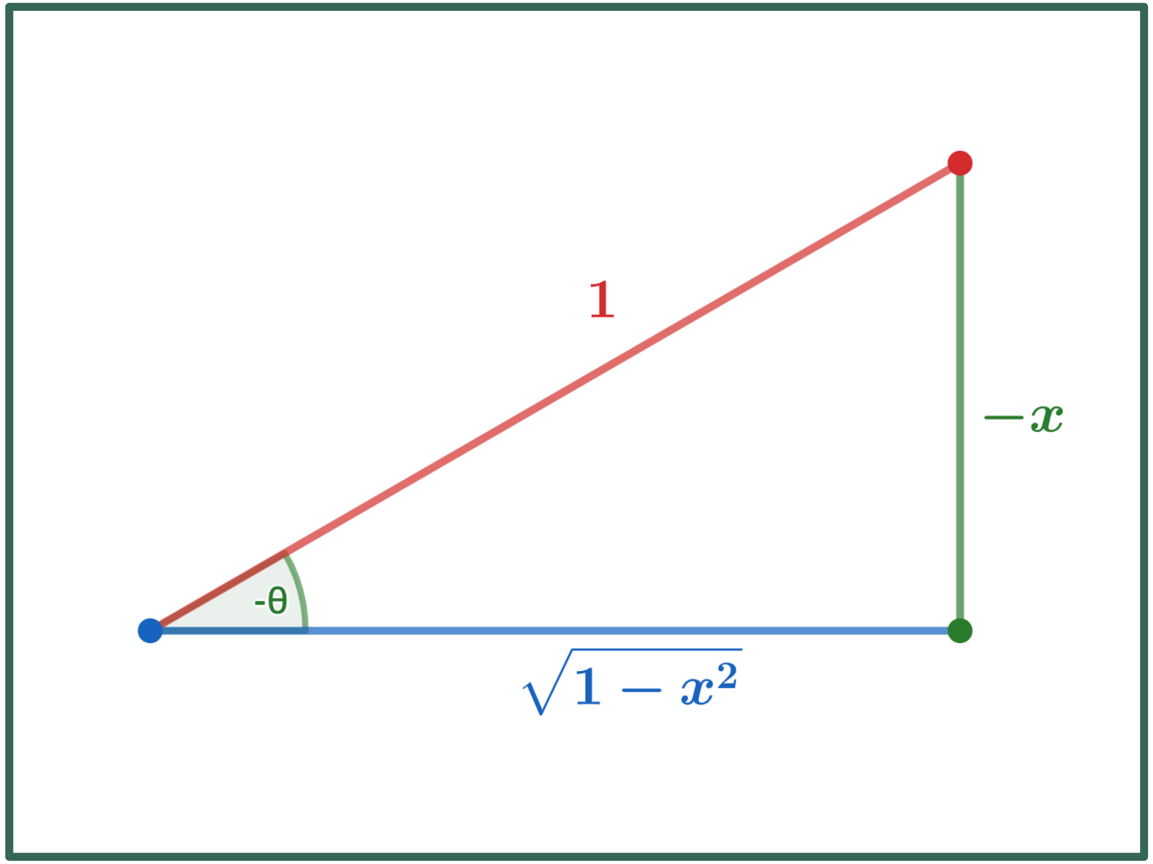
- What is $\cos(\sin^{-1}(x))$?
- Set $\sin^{-1}(x) = \theta$. Therefore, $\sin(\theta) = x$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- We first consider the case where $\mathbf{0 < \theta < \frac{\pi}{2}}$.

EXAMPLE ONE

- $\cos(\sin^{-1}(x)) = \cos(\theta) = \sqrt{1 - x^2}$.



EXAMPLE ONE



- In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence
- $\cos(\sin^{-1}(x)) = \cos(\theta) = \cos(-\theta) = \sqrt{1 - x^2}$.

EXAMPLE ONE

- Now if $\boldsymbol{\theta = \frac{\pi}{2}}$ **or** $\boldsymbol{\theta = -\frac{\pi}{2}}$, $x = 1$ or $x = -1$. And since in either case $\cos(\theta) = 0$ and $\sqrt{1 - x^2} = 0$, we have
- $\cos(\sin^{-1}(x)) = \cos(\theta) = \sqrt{1 - x^2} = 0.$

EXAMPLE ONE

- Consequently, in all cases, $\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}$.
- $g(x) = \sin^{-1}(x)$.
- $g'(x) = \dfrac{1}{\cos(\sin^{-1}(x))} = \dfrac{1}{\sqrt{1 - x^2}}$.

$$g'(x) = \frac{1}{f'(g(x))}.$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

EXERCISE

- Use the inverse function theorem to find the derivative of $\tan^{-1}(x)$.

SUMMARY

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-(x)^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+(x)^2}$$

EXERCISE ONE

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}}$$

Applying the Chain Rule to the Inverse Sine Function

- Find the derivative of $h(x) = \sin^{-1}(g(x))$ and use the result to find the derivative of $h(x) = \sin^{-1}(2x^3)$.

EXERCISE TWO

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}}$$

Applying Differentiation Formulas to an Inverse Sine Function

- Find the derivative of $h(x) = x^3 \sin^{-1}(x)$.

EXERCISE THREE

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+(x)^2}$$

Applying Differentiation Formulas to an Inverse Tangent Function

- Find the derivative of $h(x) = \tan^{-1}(x^2 + x + 1)$.