EXPONENTIAL AND LOGARITHMIC FUNCTIONS

INTRODUCTION TO CALCULUS

THE RICE AND CHESSBOARD STORY

There's a famous legend about the origin of chess. When the inventor of the game showed it to the emperor, the emperor was so impressed by the new game, that he said to the man

"*Name your reward!*"

The man responded,

"*Oh emperor, my wishes are simple. I only wish for this. Give me one grain of rice for the first square of the chessboard, two grains for the next square, four for the next, eight for the next and so on for all 64 squares, with each square having double the number of grains as the square before.*"

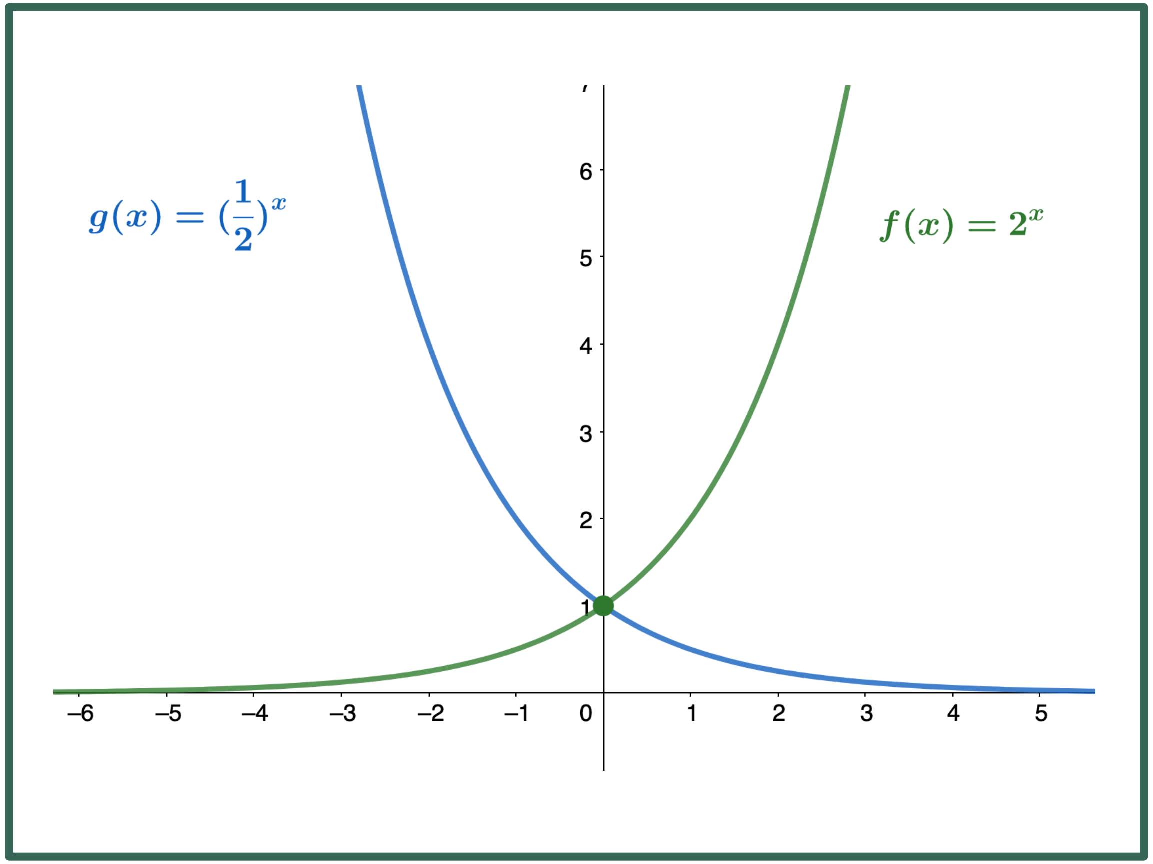
The emperor agreed, amazed that the man had asked for such a small reward - or so he thought.

After a week, his treasurer came back and informed him that the reward would add up to an astronomical sum, far greater than all the rice that could conceivably be produced in many many centuries!

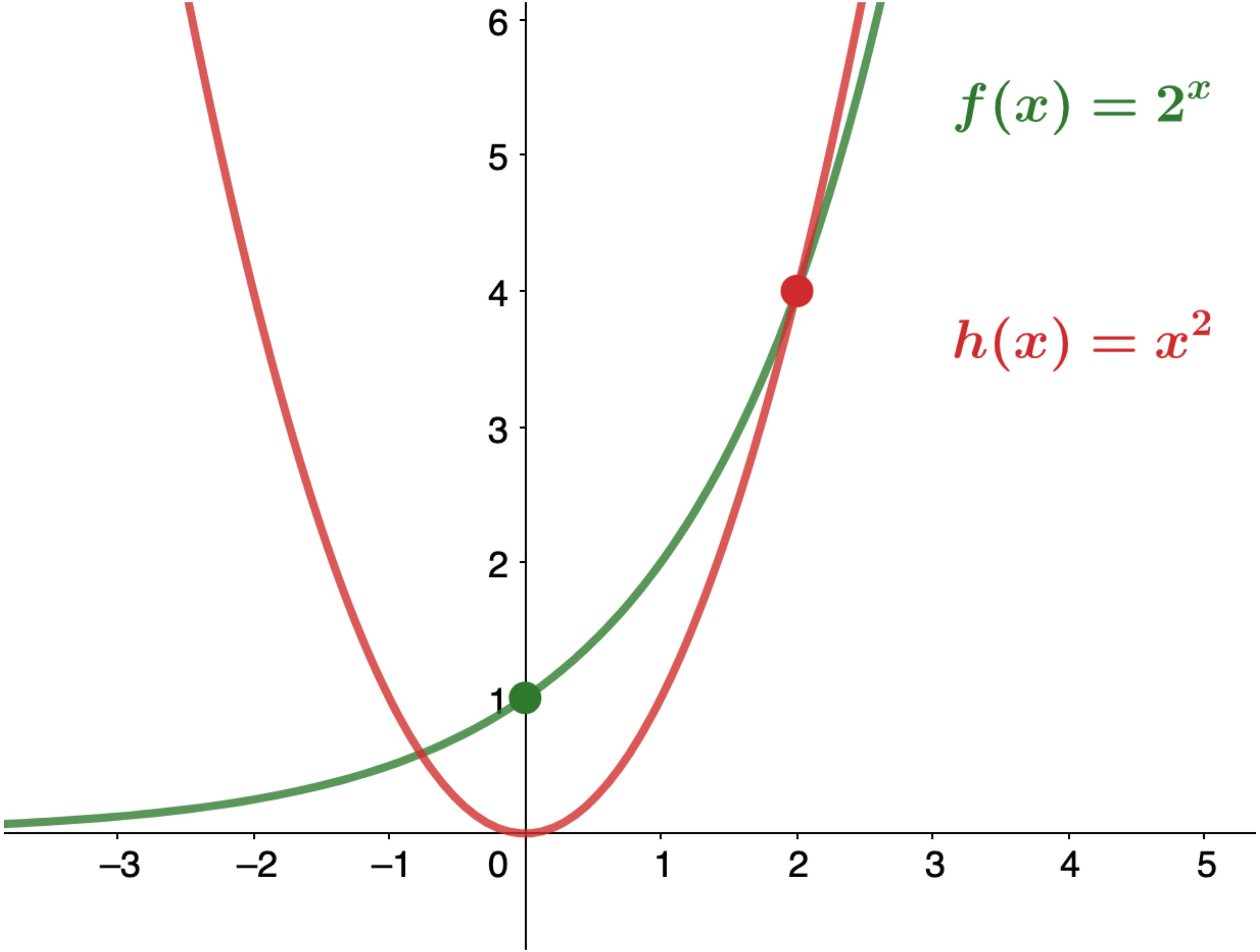
THE RICE AND CHESSBOARD STORY

- In general, the number of grains on the nth square is
- $1 \times 2 \times 2 \times \cdots \times 2 = 2^{n-1}$.

EXPONENTIAL FUNCTIONS



- Any function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, is an exponential function
  with **base** b and **exponent** x.
- Exponential functions have constant bases and variable exponents.
- **Is $f(x) = x^b$ for some constant b an exponential function?**



THE DIFFERENCE BETWEEN AN EXPONENTIAL FUNCTION AND A POWER FUNCTION

EVALUATE EXPONENTIAL FUNCTIONS

x	-3	0	$\frac{3}{4}$	1	$\sqrt{2}$	2
2^x	$\frac{1}{8}$	1	$\sqrt[4]{8}$	2		4

- $f(x) = b^x$
- x is a positive integer
  - $b \times b \times \cdots \times b$
- x is a negative integer
  - $\frac{1}{b^{(-x)}}$
- $x = \frac{p}{q}$ is a rational number
  - $\sqrt[q]{b^p}$
- x is an irrational number
  - ?

LAWS OF EXPONENTS

EXPONENTIAL FUNCTIONS SATISFY THE GENERAL LAWS OF EXPONENTS.

RULE: LAWS OF EXPONENTS

For any constants $a > 0, b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\frac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

EXERCISE

- $2^5 \cdot 2^3$
- $\dfrac{2^{2019}}{2^{2001}}$
- $(3^3)^5$
- $(4 \cdot x)^3$
- $\dfrac{5^{2019}}{6^{2019}}$

GRAPH EXPONENTIAL FUNCTIONS

Domain

- $\mathbb{R}$

Range

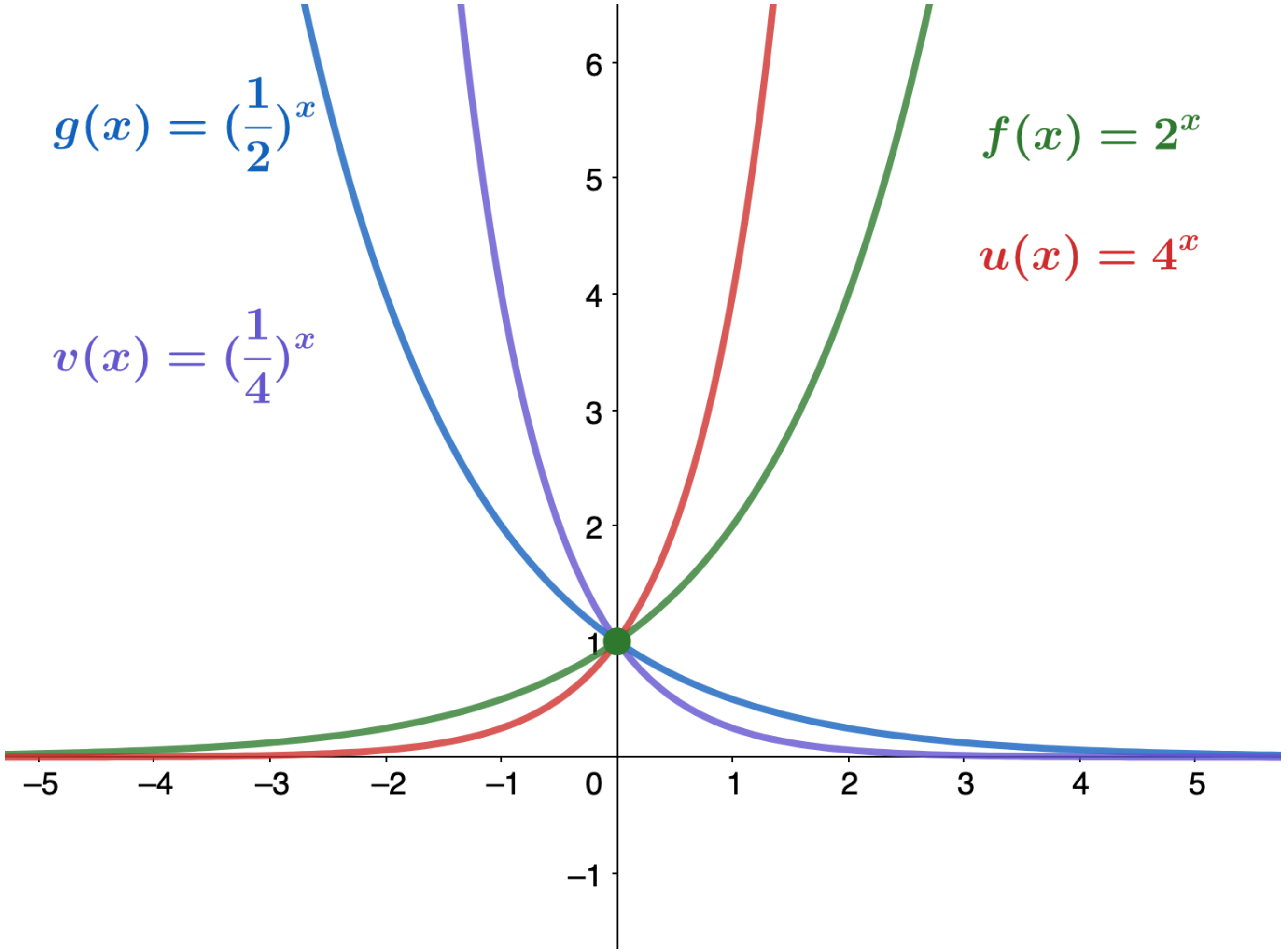
- $(0, +\infty)$

Monotonicity

- $b > 1$ increasing, $0 < b < 1$ decreasing

Asymptote

- $y = 0$ (the x-axis)



THE COMPOUND INTEREST PROBLEM

- Imagine a wonderful bank in Hanover that pays 100% interest.
- In one year you could turn $1000 into $2000.
- Now imagine the bank pays twice a year, that is 50% and 50%.
- Half-way through the year you have $1500, you reinvest for the rest of the year and your $1500 grows to $2250.
- You got **more money**, because you reinvested half way through.
- That is called **compound interest**.

THE COMPOUND INTEREST PROBLEM

- Could we get even *more* if we broke the year up into months?
- Weeks?
- Days?
- Hours?
- Minutes?
- Seconds?
- …

THE COMPOUND INTEREST PROBLEM

We can use the formula to calculate the amount of money in the account after a year

- $P(1+\frac{r}{n})^n$

Here, P is the principle ($P = \$1000$), r is the annual interest rate ($r = 1$) and n is the number of periods within a year.

Our half yearly example is:

- $(1+\frac{1}{\mathbf{2}})^{\mathbf{2}} = 2.25$

Lets try it monthly:

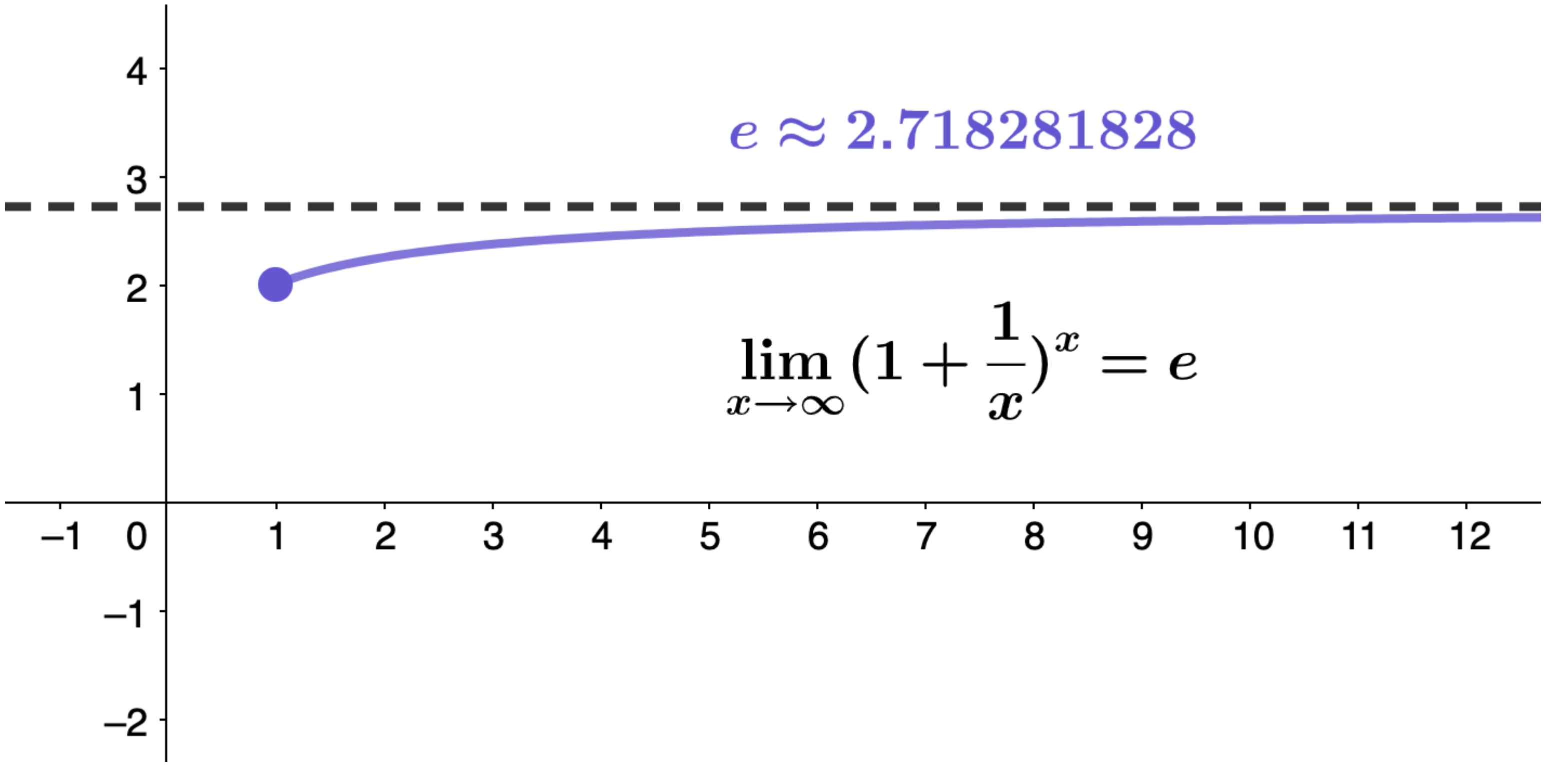
- $(1+\frac{1}{\mathbf{12}})^{\mathbf{12}} \approx 2.613$

Lets try it daily:

- $(1+\frac{1}{\mathbf{356}})^{\mathbf{365}} \approx 2.715$

it is heading towards a special number in mathematics!!!

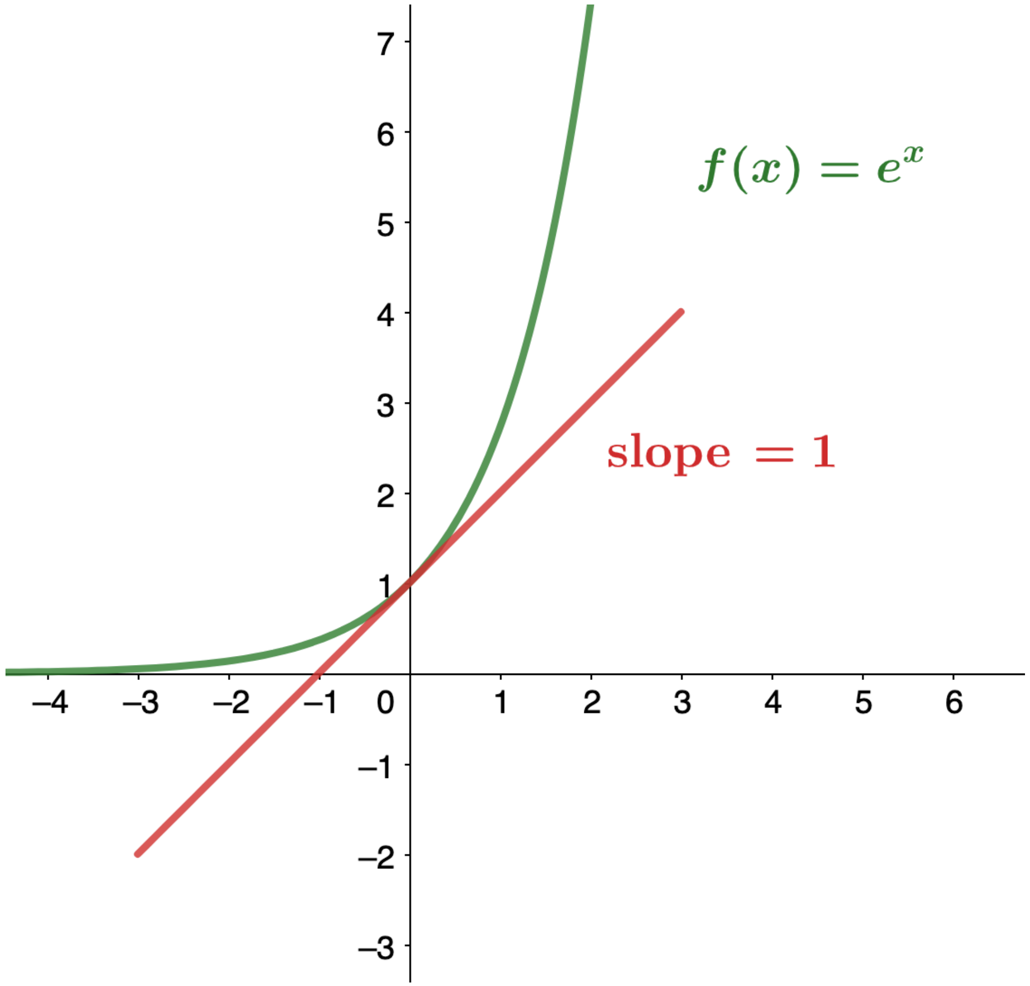
THE NUMBER *e*



$e \approx 2.718281828$
$\lim_{x \to \infty} (1 + \frac{1}{x})^x = e$
4
3
2
1
–1
0
1
2
3
4
5
6
7
8
9
10
11
12
–1
–2

THE NATURAL EXPONENTIAL FUNCTION

- The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1.
- As we see later in the text, having this property makes the natural exponential function the simplest exponential function to use in many instances.



LOGARITHMIC FUNCTIONS

- Does the exponential function $f(x) = b^x$ has an inverse function?
  - It is one-to-one, with domain $\mathbb{R}$ and range $(0, +\infty)$.
- Therefore, it has an inverse function, called the *logarithmic function with base b*.
- For any $b > 0$ and $b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, +\infty)$ and range $\mathbb{R}$ and satisfies
  - $\log_b(x) = y$ if and only if $b^y = x$.

LOGARITHMIC FUNCTIONS

- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$

$\log_2(4)$

$\log_2(\frac{1}{8})$

$\log_3(1)$

$\log_3(27)$

-3

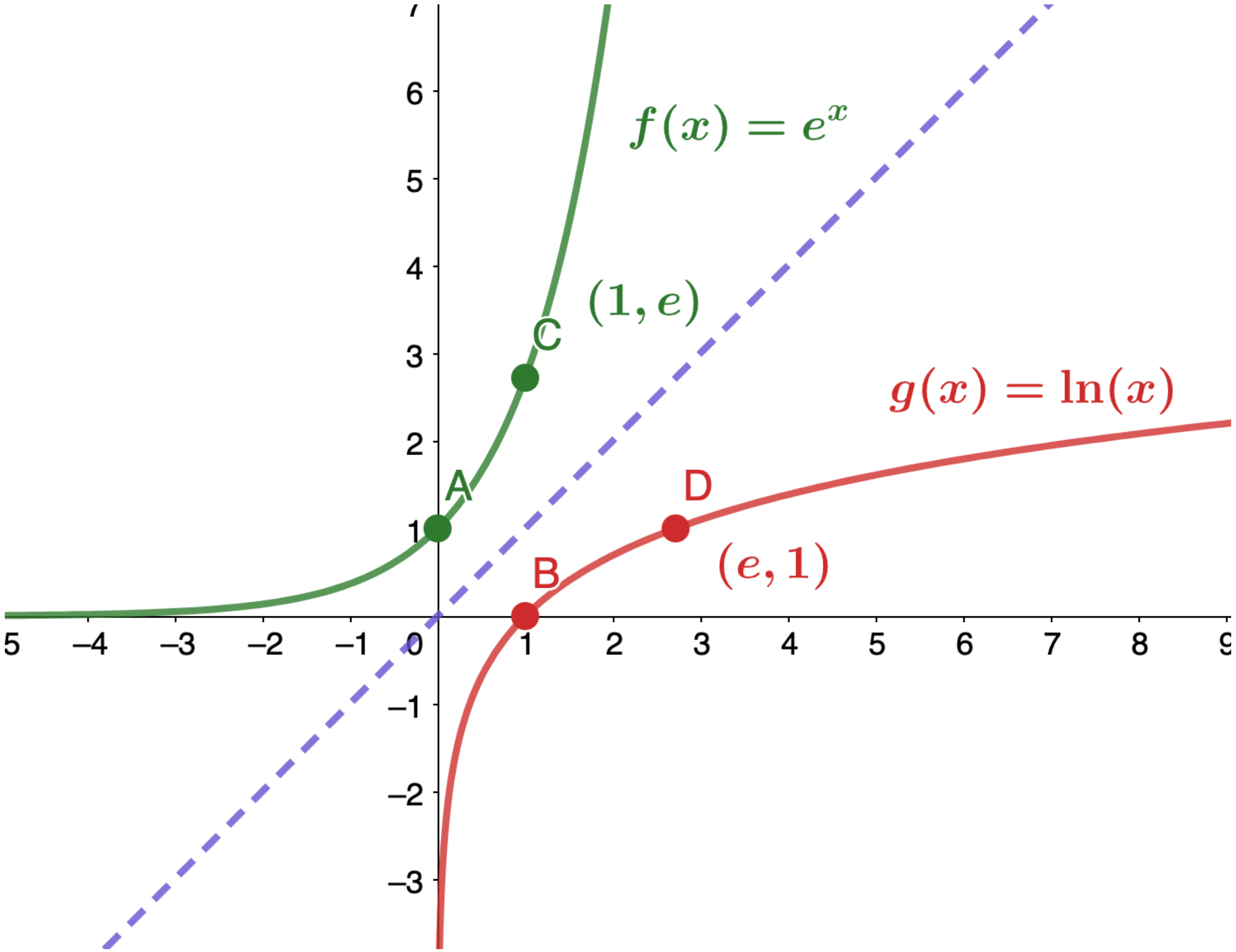
0

2

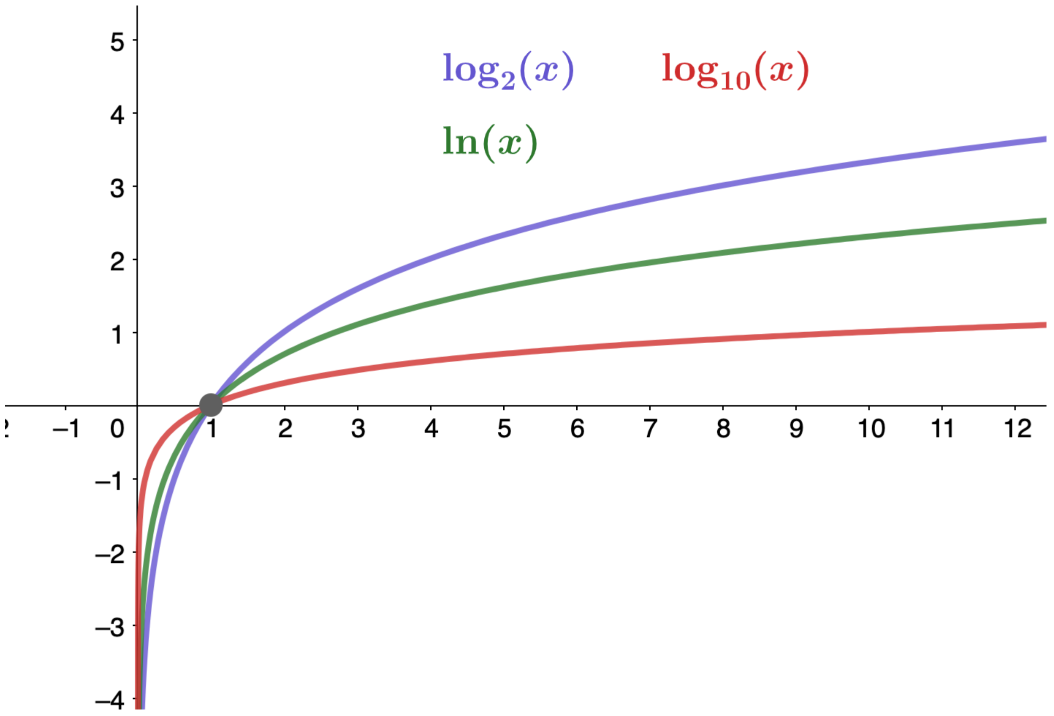
3

THE NATURAL LOGARITHM

- We use the notation $\mathbf{ln}(\boldsymbol{x})$ or $\mathbf{ln}\,\boldsymbol{x}$ to denote $\log_e(x)$.
- $\ln 1 = ?$
- $\ln e = ?$
- $\ln e^x = x$
- $e^{\ln x} = x$



GRAPHS OF LOGARITHMIC FUNCTIONS



- In general, for any base $b > 0$ and $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$.
- Where is the x-intercept?
- Where is the asymptote?

RULE: PROPERTIES OF LOGARITHMS

If $a, b, c > 0, b \neq 1$, and r is any real number, then

1. $\log_b(ac) = \log_b(a) + \log_b(c)$ (Product property)
2. $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ (Quotient property)
3. $\log_b(a^r) = r\log_b(a)$ (Power property)

THE BASIC PROPERTIES OF LOGARITHMS

EXERCISE

- Solve the following equations involving exponential and logarithmic functions.

- $\ln \frac{x+1}{x+3} = -1$
- $\ln x^2 + 3 \ln x = \frac{5}{2}$
- $e^{2x} - 4e^{-2x} = 3$

EVALUATE A LOGARITHMIC EXPRESSION

The common logarithm

- $\log_{10} x$ or $\log x$

The natural logarithm

- $\ln x$

A general logarithm

- $\log_b x$

How to evaluate an expression with a different base?

CHANGE-OF-BASE FORMULAS

USING THIS CHANGE OF BASE, WE TYPICALLY WRITE A GIVEN EXPONENTIAL OR LOGARITHMIC FUNCTION IN TERMS OF THE NATURAL EXPONENTIAL AND NATURAL LOGARITHMIC FUNCTIONS.

RULE: CHANGE-OF-BASE FORMULAS

Let $a > 0, b > 0$, and $a \neq 1, b \neq 1$.

1. $a^x = b^{x \log_b a}$ for any real number x.
   If $b = e$, this equation reduces to $a^x = e^{x \log_e a} = e^{x \ln a}$.
2. $\log_a x = \frac{\log_b x}{\log_b a}$ for any real number $x > 0$.
   If $b = e$, this equation reduces to $\log_a x = \frac{\ln x}{\ln a}$.

PROOF

$$a^x = b^{x \log_b a}$$

$$x \log_b a = \log_b(a^x)$$

$$b^{\log_b(a^x)} = a^x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$\log_a x \cdot \log_b a = \log_b x$$

$$b^{\log_a x \cdot \log_b a} = a^{\log_a x} = x$$

$$b^{\log_b x} = x$$

EXERCISE

For the following exercises, use the change-of-base formula and either base 10 or base e to evaluate the given expressions.

- $\log_5 36$
- $\log_{0.2} 2$
- $\log_3 e$

SECRETARY PROBLEM (LARGEST DIAMOND PROBLEM)

- In a building of n floors, a diamond has been placed outside the elevator door on every floor in random order.
- A girl gets into the elevator on the entry floor. It will stop at every floor and she needs to decide whether to pick up the diamond or not.
- If she does that on a certain floor, she cannot take another diamond after the elevator starts rising again.
- During the journey, the girl gains information sufficient to rank the diamond among all diamonds that have appeared so far, but is unaware of the size of yet unseen diamonds.
- The question is about the optimal strategy (stopping rule) to maximize the probability of selecting the largest diamond.

SECRETARY PROBLEM (LARGEST DIAMOND PROBLEM)

- The $\frac{1}{e}$ stopping rule: the optimal stopping rule prescribes always rejecting the first $\approx \frac{n}{e}$ diamonds, and then choosing the first diamond which is better than every diamond seen so far (or continuing to the top floor if this never occurs).