# MATH 1: INTRODUCTION TO CALCULUS
# FINAL EXAM

Name: ______________________________

Section (please circle): 10 Winkeler 11 Tripp 12 Chen 2 Xiao

(1) Write your name *legibly* and circle your section above.
(2) This exam is closed book. You may not use notes, calculators, or any other external resource. It is a violation of the honor code to give or receive help on this exam.
(3) You must explain what you are doing, justify your answer, and show your work. You will be *graded on your work*, not just on your answer. Please write clearly.
(4) It is fine to leave your answer in a form such as $14\sqrt{239}$. However, if an expression can be easily simplified (such as $\cos(\pi/2)$ or $2+3$), please simplify it.
(5) If you use the backside of a page or the scratch paper at the end of the exam, and you want us to look at it, please write on a problem "Continued on back/page…".

| **Problem** | **Score** | **Problem** | **Score** |
|---|---|---|---|
| 1 | /8 | 6 | /9 |
| 2 | /11 | 7 | /11 |
| 3 | /12 | 8 | /9 |
| 4 | /12 | 9 | /7 |
| 5 | /9 | 10 | /5 |
| | | Total | /93 |

*Date*: Friday, 22 November 2019.

**Problem 1**. [8 points]

(a) Let $h(x) = f(x) \cdot g(x)$. Using the Product Rule, write $h'(x)$ in terms of $f(x)$, $g(x)$, and their derivatives.

(b) Let $j(x) = \frac{f(x)}{g(x)}$. Using the Quotient Rule, write $j'(x)$ in terms of $f(x)$, $g(x)$, and their derivatives.

(c) Let $p(x) = (f \circ g)(x)$. Using the Chain Rule, write $p'(x)$ in terms of $f(x)$, $g(x)$, and their derivatives.

(d) Using the Inverse Function Theorem, write $(f^{-1})'(x)$ in terms of $f(x)$, $f'(x)$, and $f^{-1}(x)$.

**Problem 2**. [11 points] Mark the following statements as true or false.

__________ If a function $f(x)$ is differentiable at a point $c$, then $f(x)$ is continuous at $c$.

__________ According to the Power Rule, the derivative of $2^x$ is $x \cdot 2^{x-1}$.

__________ If $f(x) = x^2 \cos(x)$, then $f'(x) = 2x(-\sin(x))$.

__________ If $g(x) = \pi^\pi$, then $g'(x) = 0$.

__________ If $h(x) = 7^x$, then $h'(x) = 7^x \log_e 7 = 7^x \ln 7$.

__________ If a sequence is convergent, then it must be bounded.

__________ If $\lim\limits_{x \to a} f(x)$ exists, then $f$ must be continuous at $a$.

__________ If $f(x)$ and $g(x)$ are continuous on $(-\infty, \infty)$, then $\dfrac{f(x)}{g(x)}$ is also continuous on $(-\infty, \infty)$.

__________ For all $y$ in the domain of $f^{-1}$, $f(f^{-1}(y)) = y$.

__________ If a graph fails the horizontal line test, the graph does not correspond to a function.

__________ $\log_b(x + y) = \log_b(x) + \log_b(y)$.

**Problem 3**. [12 points]

(a) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\ln(x^4)+1\big)$.

(b) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x}{\sin x\cos x}\right)$.

(c) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\sin^{-1}(2x^2)\big)$.

(d) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2} + e^x\right)$.

(e) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{x^2+3}\right)$.

(f) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left((\sin 2^x)^2\right)$.

**Problem 4**. [12 points]

(a) Give an example of an explicit formula for a bounded sequence.

(b) What does it mean for a sequence to be monotone?

(c) Classify this sequence as arithmetic, geometric, both, or neither, **and** write an explicit formula for it: $\{a_n\}_{n=1}^{\infty} = \{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \cdots\}$.

(d) Find $\lim_{n\to\infty} \dfrac{5n^3+4}{4+n^2+2n^3}$.

(e) Find $\lim_{n\to\infty} \dfrac{\sin(n)\cos(n)}{n}$.

(f) Find $\lim_{n\to\infty} 1^n$.

**Problem 5**. [9 points] Imagine a population of deer live in the forest of Hanover. Where $t$ is the number of years since 1900, the number of deer can be modeled by the function:

$$P(t) = \frac{50}{1 + 9e^{-t/2}}.$$

(a) What was the population of deer in 1900?

(b) After how many years did the population of deer reach 25? [No need to simplify.]

(c) Find the derivative of $P(t)$, which is the rate of change of the deer population at time $t$.

(d) How fast is the deer population growing at time $t = 10$ years?

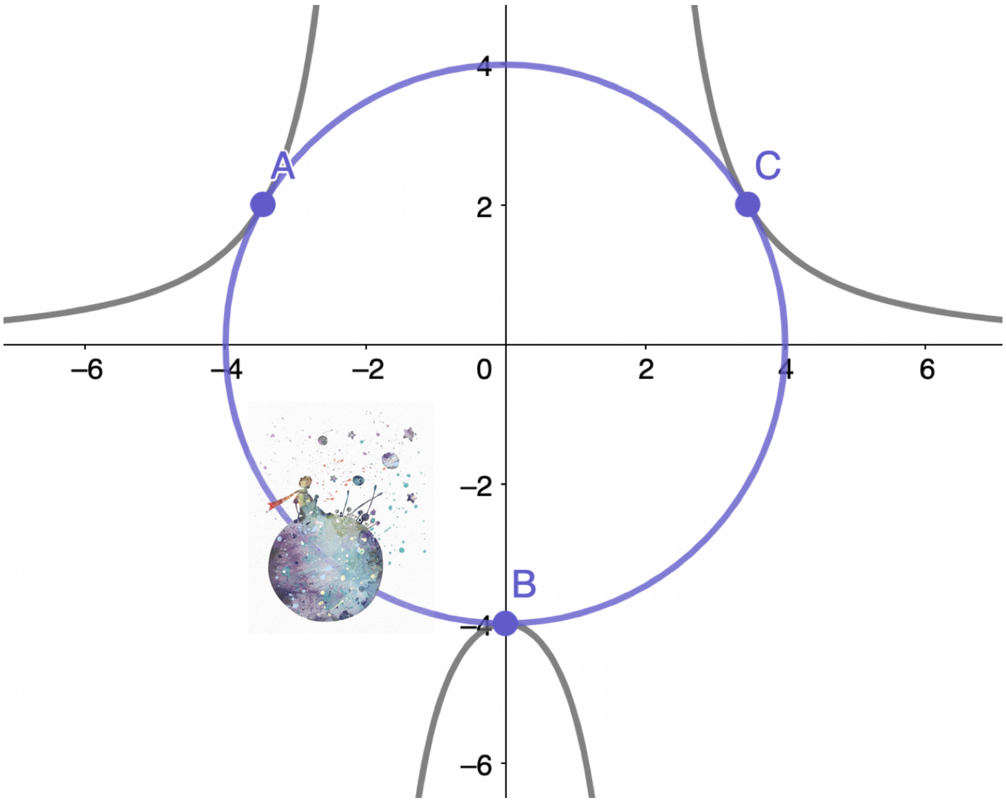
**Problem 6**. [9 points] The little prince lives in his tiny home planet, known as "B 612" on earth. The planet travels in a purple orbit whose path is described by the equation

$$x^2 + y^2 = 16.$$

One day, the little prince decided to leave the planet to explore the rest of the universe. To start the journey, he first needed to jump to the grey orbit, whose path is described by the equation

$$x^2y - 4y = 16.$$

He has since visited six other planets and finally landed in a desert on earth.




(a) The equation for the purple orbit of his home planet, $x^2 + y^2 = 16$, implicitly defines $y$ as a function of $x$. Find $\frac{dy}{dx}$.

(b) The equation for the gray orbit, $x^2y - 4y = 16$, implicitly defines $y$ as a function of $x$. Find $\frac{dy}{dx}$.

(c) As can be observed from the figure, the two orbits intersect in three different points: $A = (-2\sqrt{3}, 2)$, $B = (0, -4)$ and $C = (2\sqrt{3}, 2)$. To guarantee a safe travel,

- the little prince needs to jump when the two orbits intersect, and
- the tangent lines to the two orbits at the point where he jumps should be the same.

For the three points $A$, $B$, and $C$ above, verify that the slopes of the tangent line to the two orbits are the same.

**Problem 7**. [11 points] Let $f(x) = \dfrac{x-2}{x^2-5x+6}$.

(a) Find the domain and range of $f$.

(b) Find and classify all discontinuities of $f$.

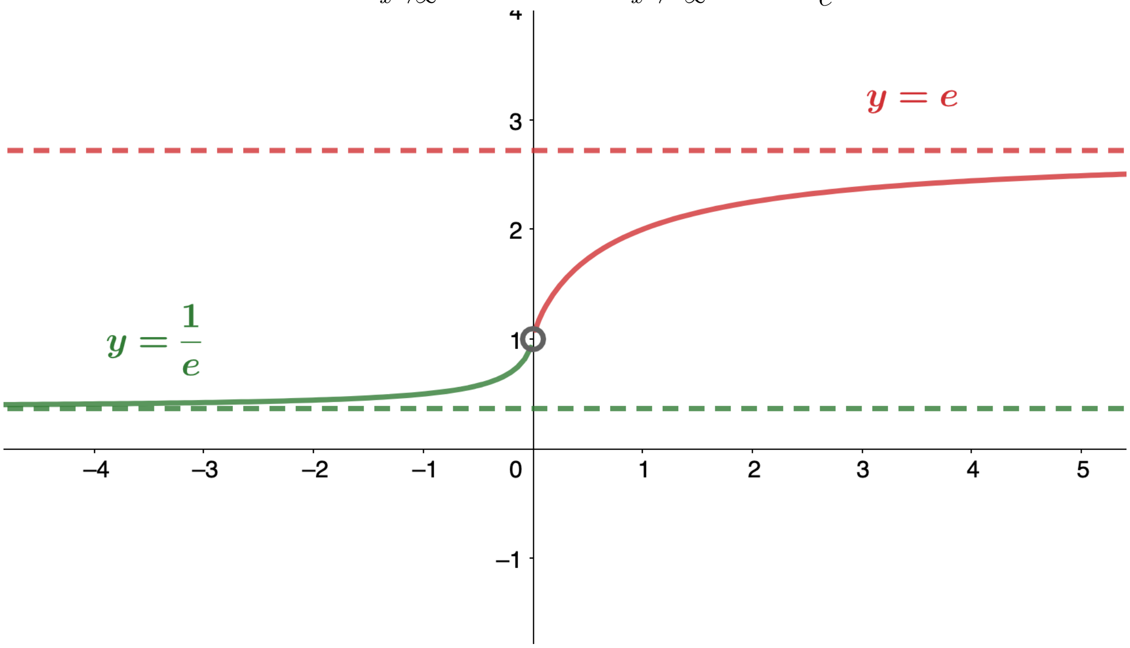
(c) Find $f'(x)$.

(d) On which intervals is $f$ increasing? On which intervals is $f$ decreasing? [Hint: sketch a graph of $f(x)$, or use $f'(x)$. ]

**Problem 8**. [9 points] Consider the following piecewise function

$$f(x) = \begin{cases} (1-\frac{1}{x})^x, & x < 0 \\ (1+\frac{1}{x})^x. & x > 0 \end{cases} \tag{1}$$

The graph of $f(x)$ is given. Notice that the function has horizontal asymptotes $y = e$ and $y = \frac{1}{e}$. To be more precise, $\lim_{x\to\infty} f(x) = e$ and $\lim_{x\to-\infty} f(x) = \dfrac{1}{e}$.




(a) Decide whether $f(x)$ is continuous at $x = 0$. If the function has a discontinuity there, state which type of discontinuity it is. [Hint: use the graph.]

(b) Find the derivative of $f(x)$. [Hint: Use logarithmic differentiation. Your derivatives should be functions just of $x$, not of $y$ at the end. ]

- $f'(x)$ when $x > 0$;

- $f'(x)$ when $x < 0$.

- Write down $f'(x)$ as a piecewise function.

**Problem 9**. [7 points]

(a) Find the equation of the tangent line to $x^2 + y^2 = -2xy$ at $x = 1$.

(b) For $f(x) = x - \frac{4}{x}$ defined on $x < 0$, and the inverse function $f^{-1}(x)$, find $(f^{-1})'(-3)$.

**Problem 10**. [5 points] Apply the Intermediate Value Theorem to the following problems.

(a) For $f(x) = \frac{1}{x}$, $f(-1) < 0$, $f(1) > 0$, can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

(b) Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

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