IMPLICIT DIFFERENTIATION

INTRODUCTION TO CALCULUS

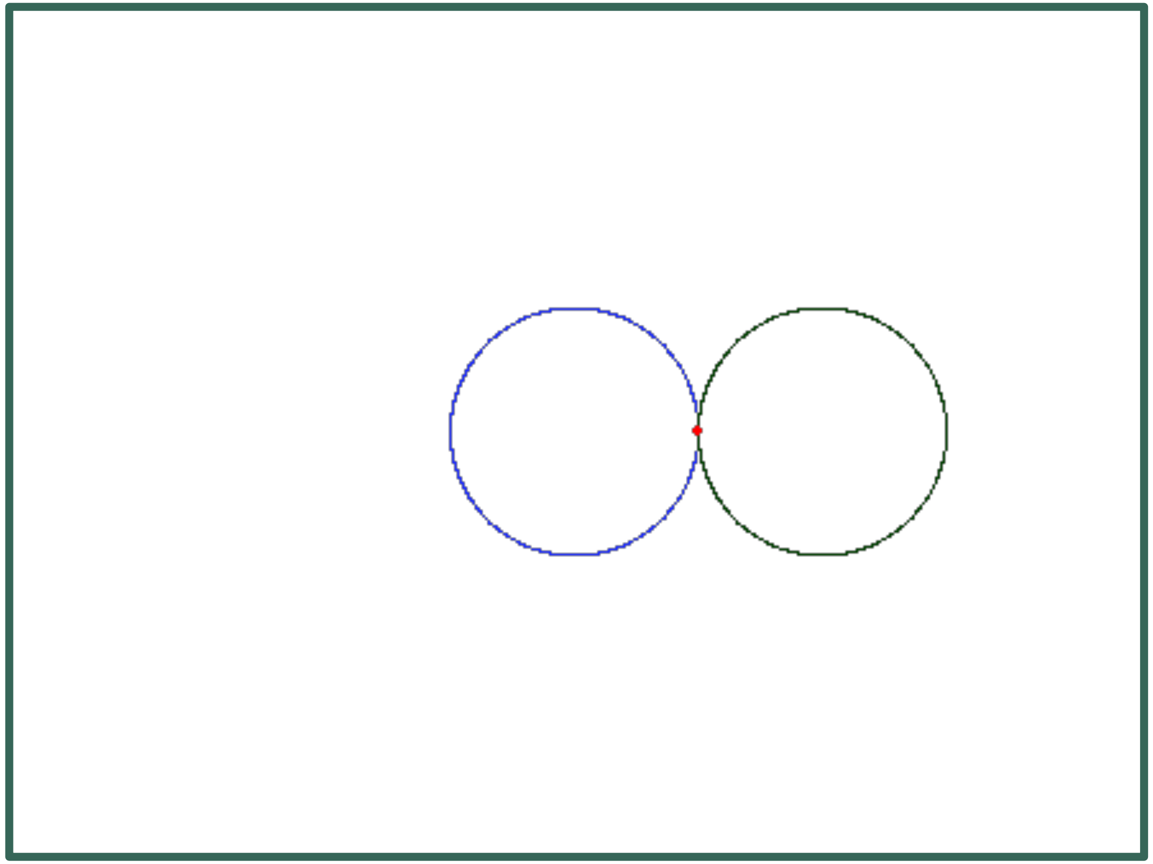
OUTLINE

- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

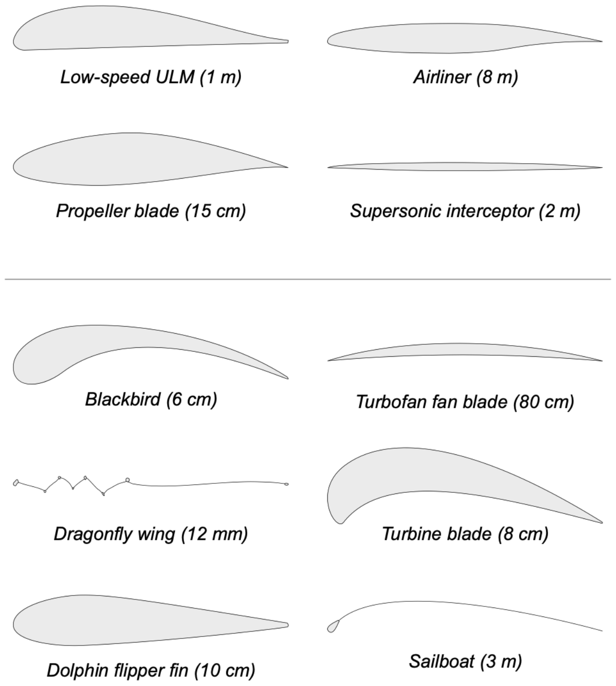
MOTIVATION

- How to find equations of tangent lines to functions and the rate of change of a function at a specific point?
- Given **the explicit equation**, we can differentiate these functions explicitly.
- How to find those equations if given **an arbitrary curve**?

MOTIVATION

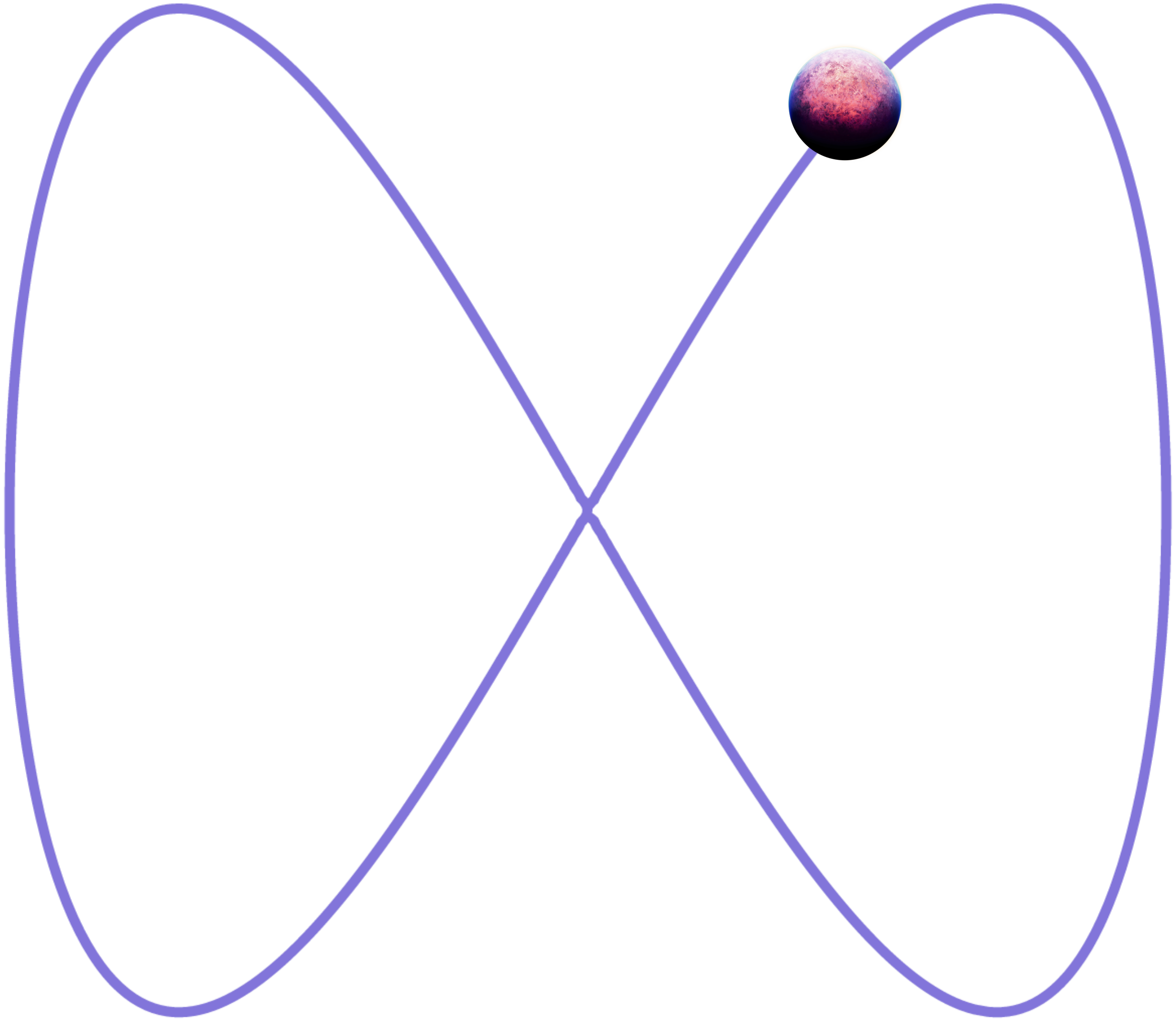


- How to find those equations if given an arbitrary curve?
- We solve these problems by finding the derivatives of functions that define y implicitly in terms of x.



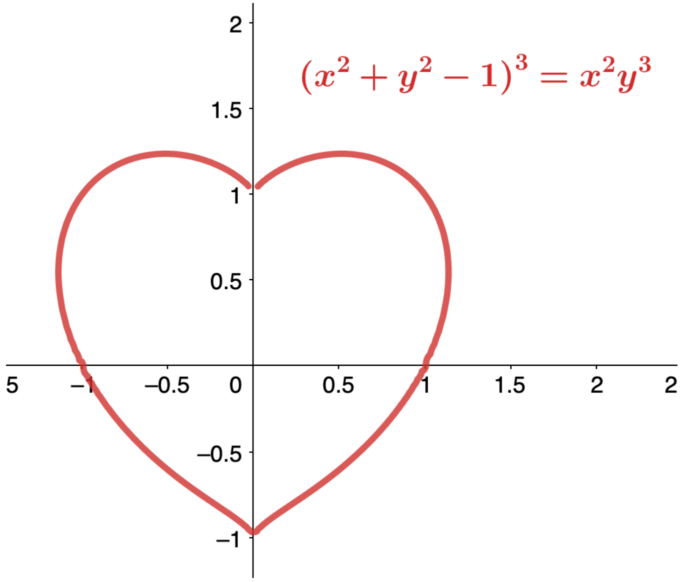
MOTIVATION

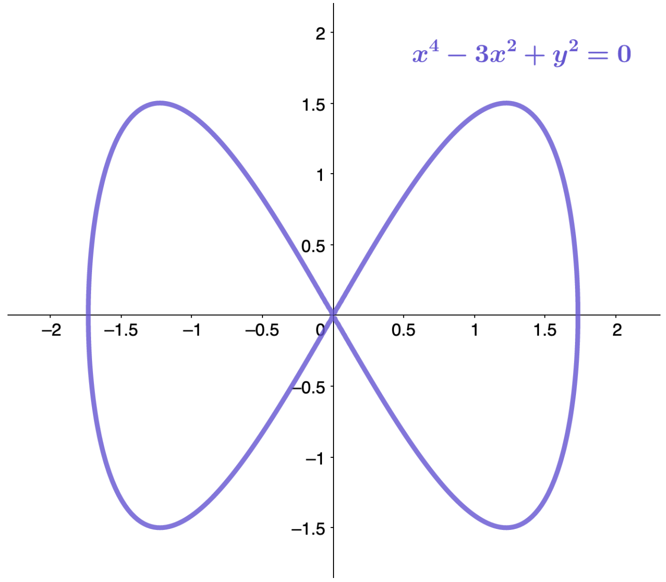
AIR FOILS



MOTIVATION

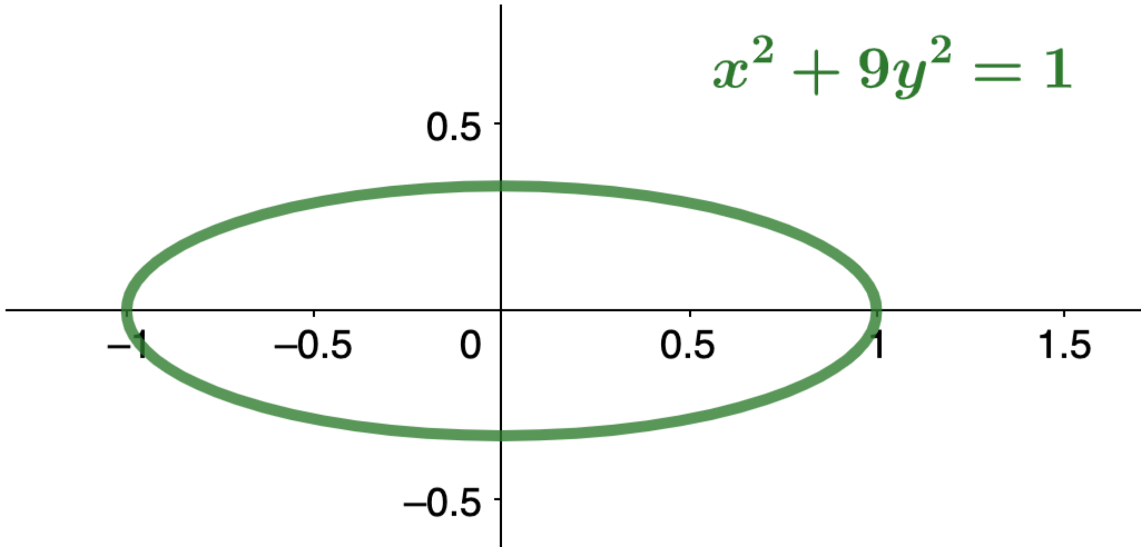
SATELLITE MOVEMENT

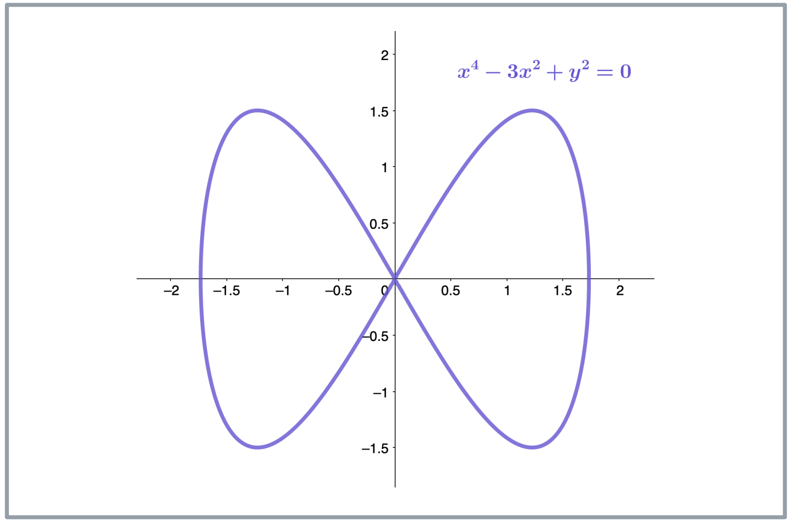


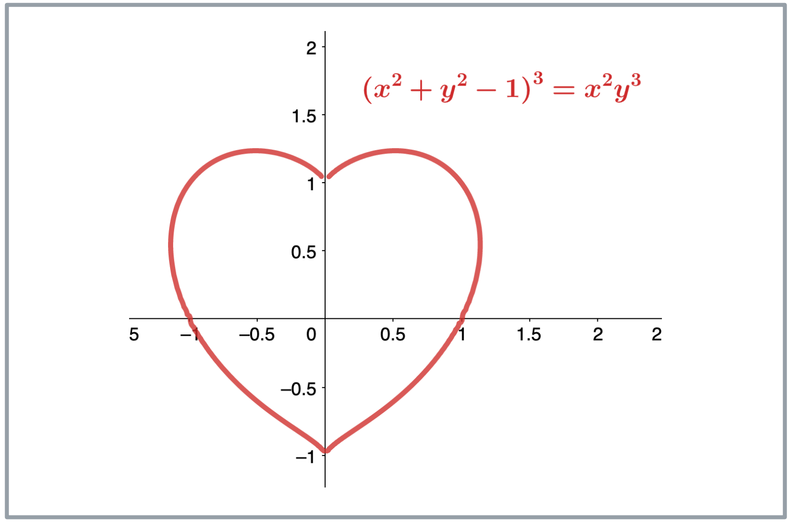


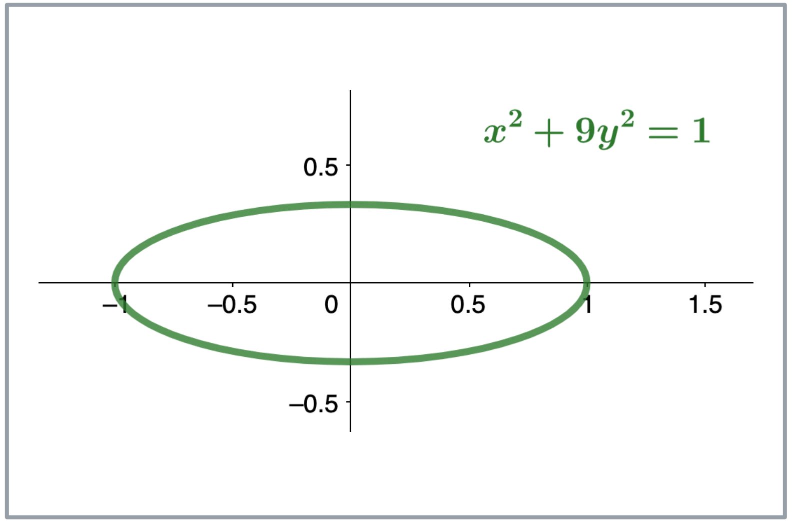
CURVES THAT ARE CLEARLY NOT FUNCTIONS

- The equation defines y implicitly in terms of x.
- It fail the vertical line test!
- An equation may define many different functions implicitly.









FORTUNATELY…

- The technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly.

STRATEGY

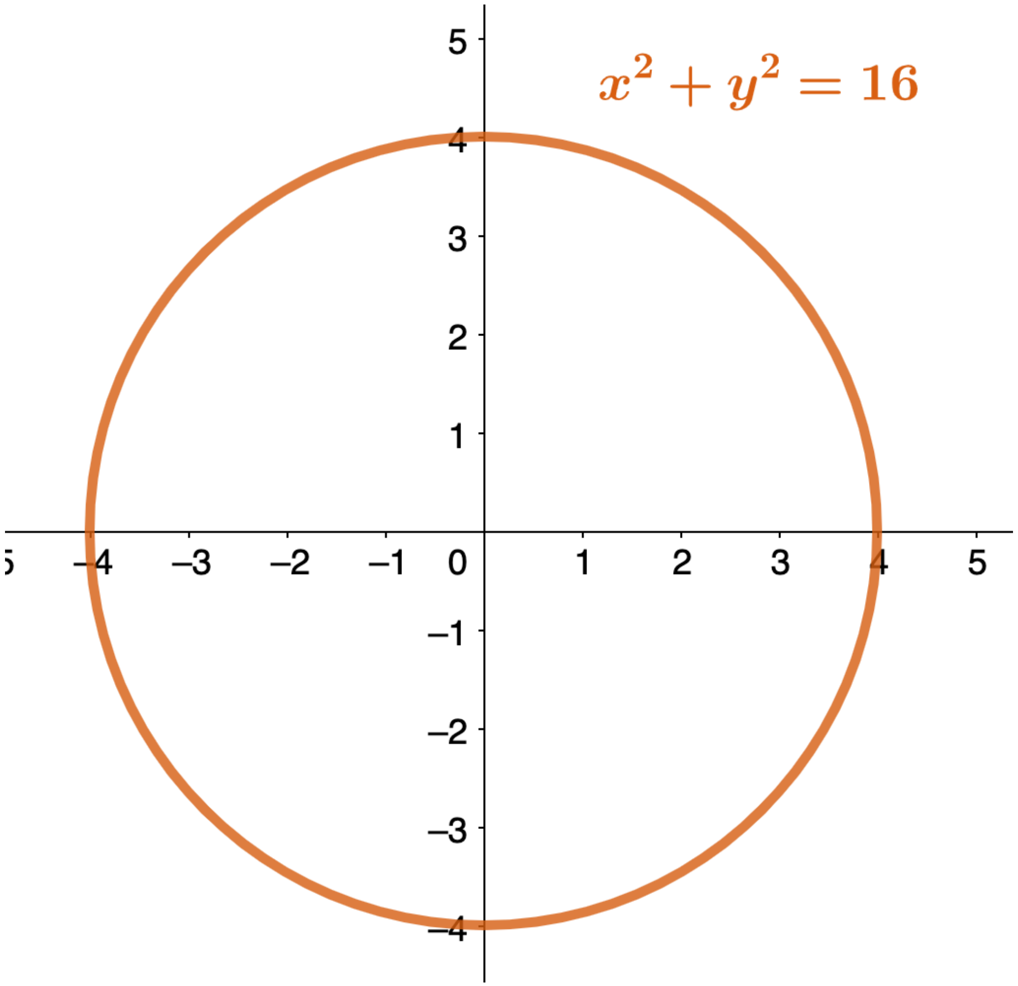
PROBLEM-SOLVING STRATEGY: IMPLICIT DIFFERENTIATION

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

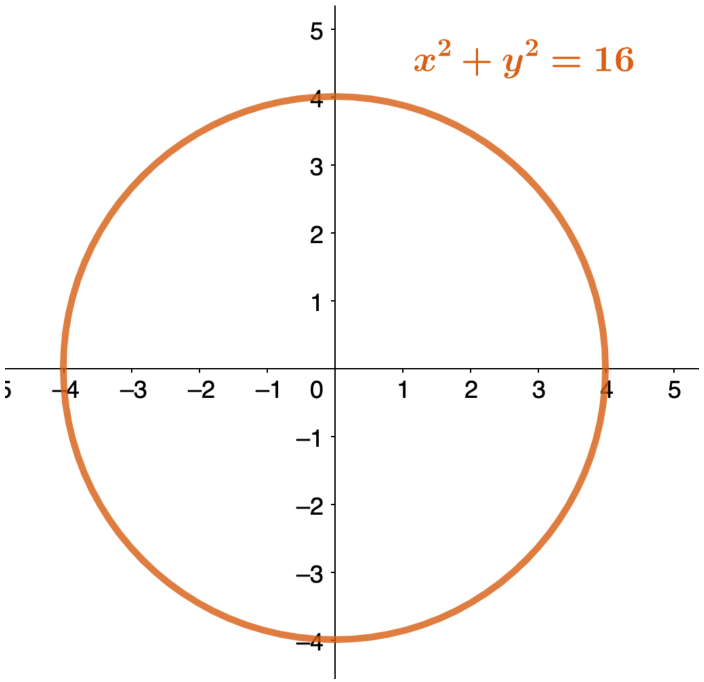
1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin y$ with respect to x.
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

EXAMPLE

- Assuming that y is defined implicitly by the equation $x^2 + y^2 = 16$, find $\frac{dy}{dx}$.



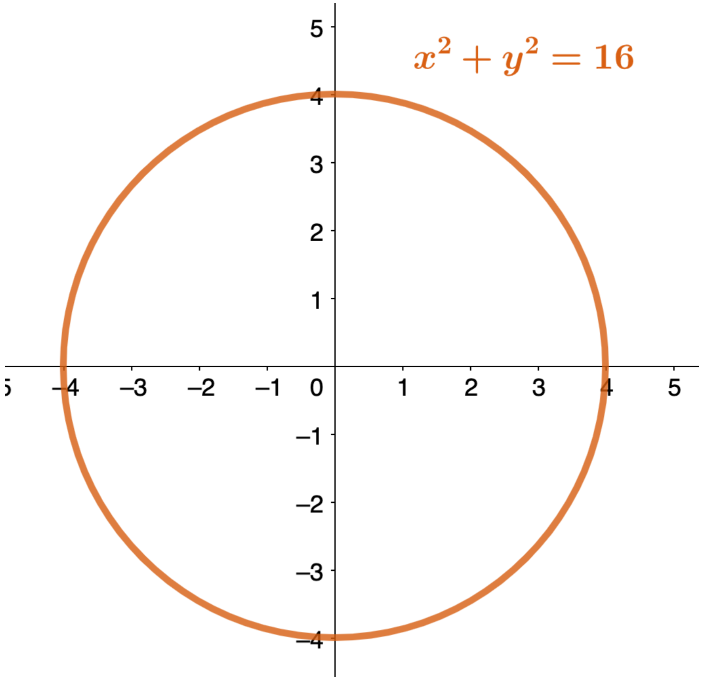
EXAMPLE ONE: STEP ONE



Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\sin(y) = \cos(y)\frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin(y)$ with respect to x.

- $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(16)$
- $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$
- $2x + 2y\frac{dy}{dx} = 0$

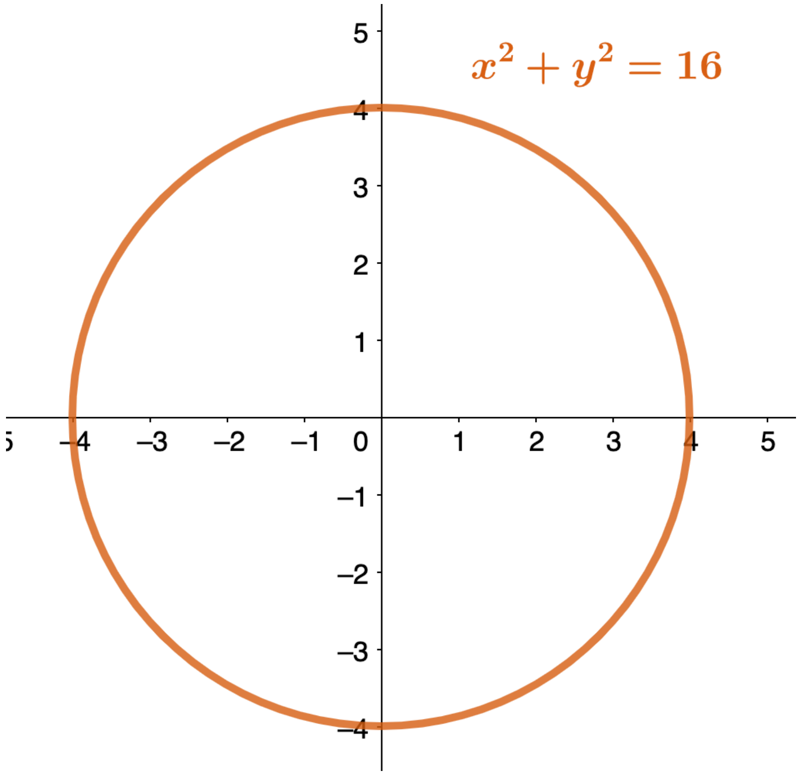
EXAMPLE ONE: STEP TWO



Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.

- $2x + 2y\frac{dy}{dx} = 0$
- $2y\frac{dy}{dx} = -2x$

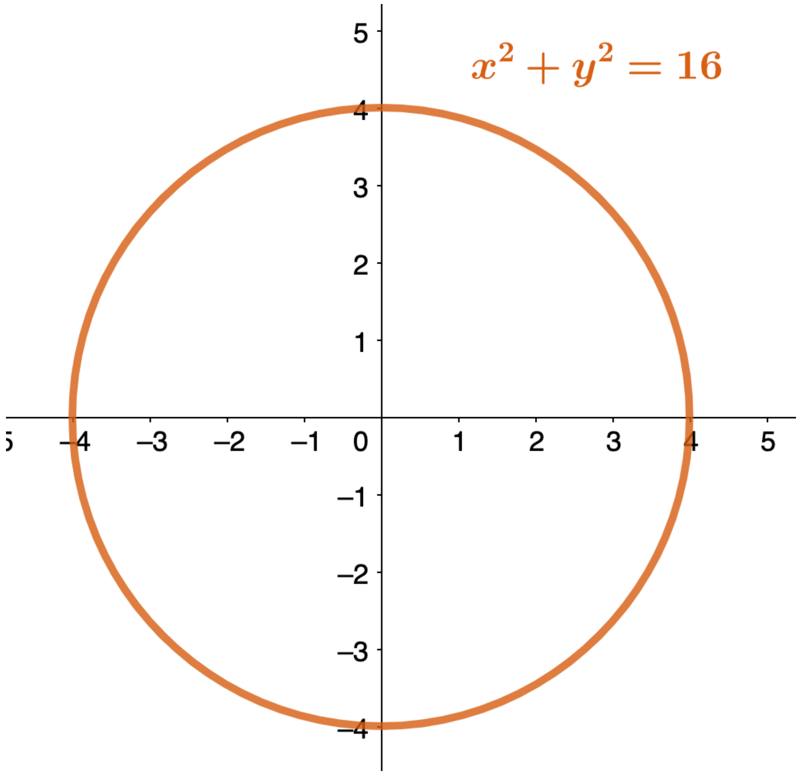
EXAMPLE ONE: STEP THREE



Factor out $\frac{dy}{dx}$ on the left.

- $2y\frac{dy}{dx} = -2x$

EXAMPLE ONE: STEP FOUR

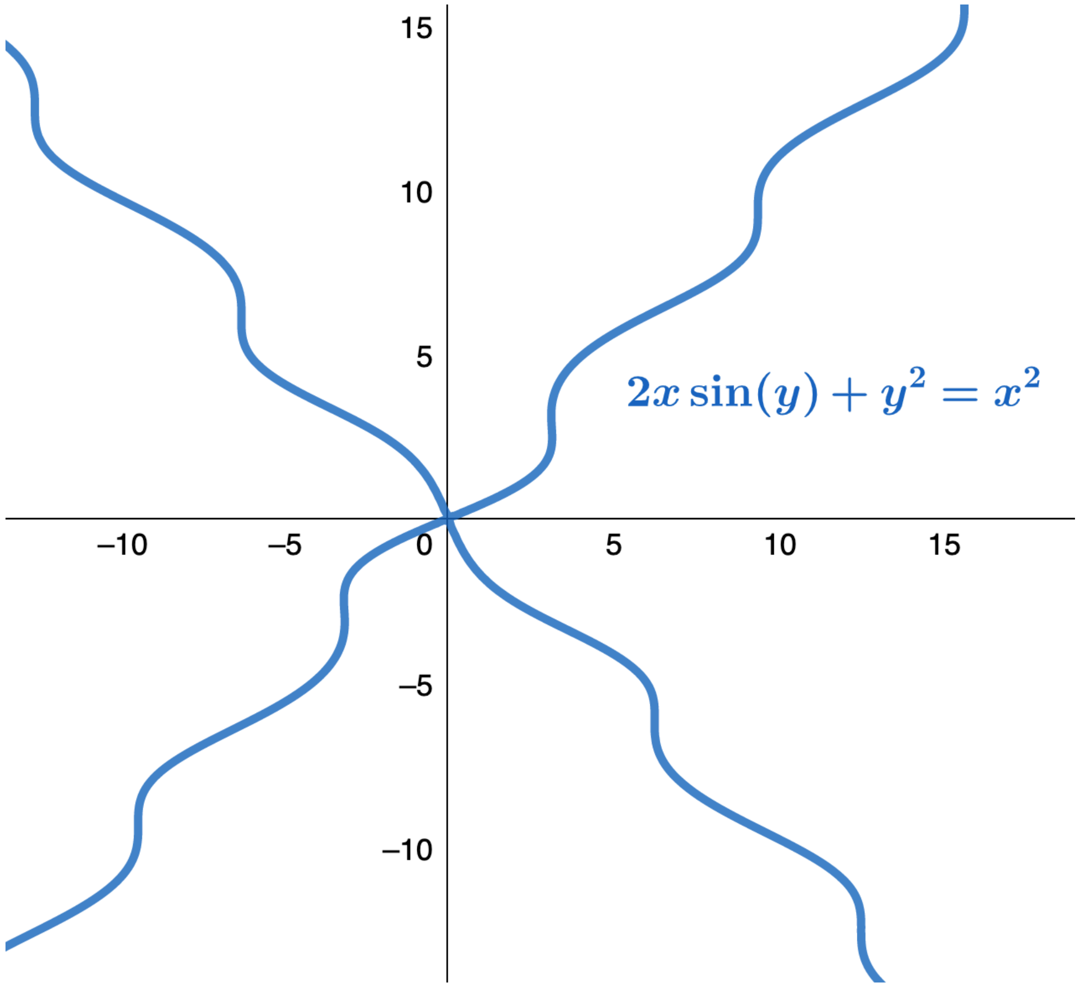


Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

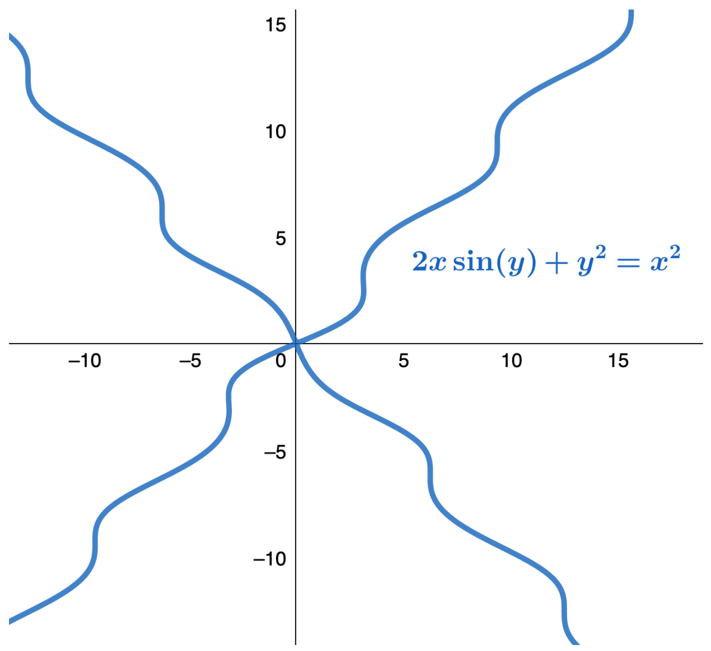
- $2y\frac{dy}{dx} = -2x$
- $\frac{dy}{dx} = -\frac{x}{y}$

EXAMPLE TWO

- Assuming that y is defined implicitly by the equation $2x\sin(y) + y^2 = x^2$, find $\frac{dy}{dx}$.



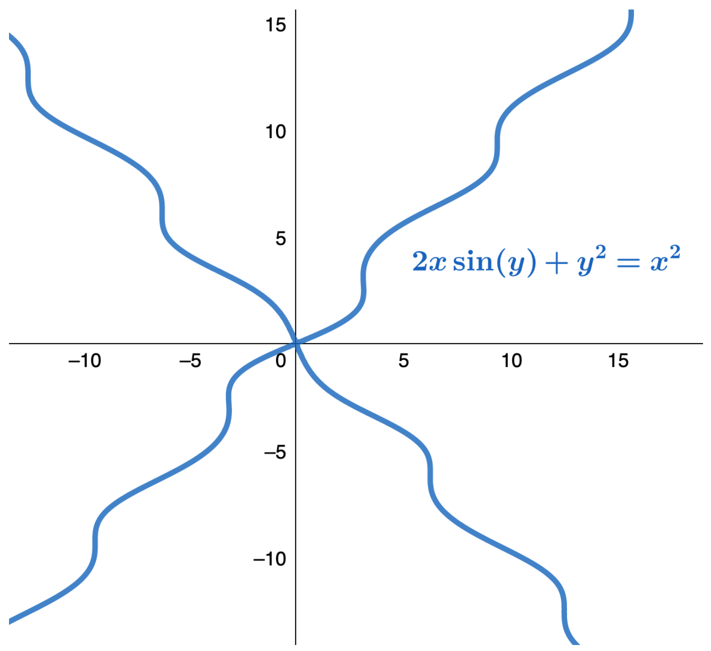
EXAMPLE TWO: STEP ONE



Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\sin(y) = \cos(y)\frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin(y)$ with respect to x.

- $\frac{d}{dx}(2x\sin(y) + y^2) = \frac{d}{dx}(x^2)$
- $2\frac{d}{dx}(x\sin(y)) + \frac{d}{dx}(y^2) = 2x$
- $2(\frac{dx}{dx} \cdot \sin(y) + x \cdot \frac{d\sin(y)}{dx}) + 2y\frac{dy}{dx} = 2x$
- $2(\sin(y) + x\cos(y)\frac{dy}{dx}) + 2y\frac{dy}{dx} = 2x$
- $\sin(y) + x\cos(y)\frac{dy}{dx} + y\frac{dy}{dx} = x$

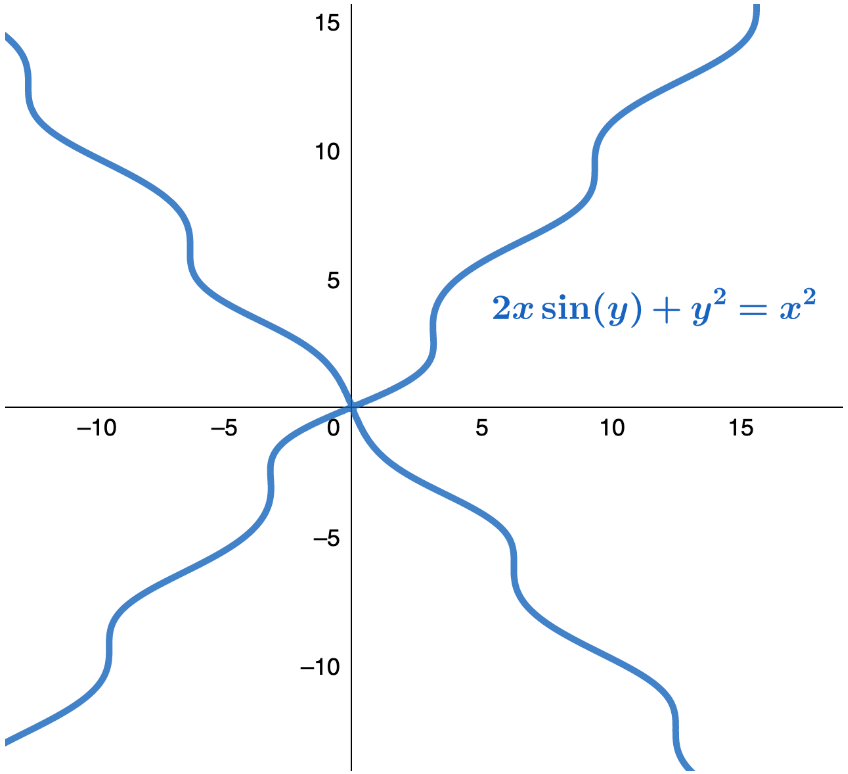
EXAMPLE TWO: STEP TWO



Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.

- $\sin(y) + x\cos(y)\frac{dy}{dx} + y\frac{dy}{dx} = x$
- $x\cos(y)\frac{dy}{dx} + y\frac{dy}{dx} = x - \sin(y)$

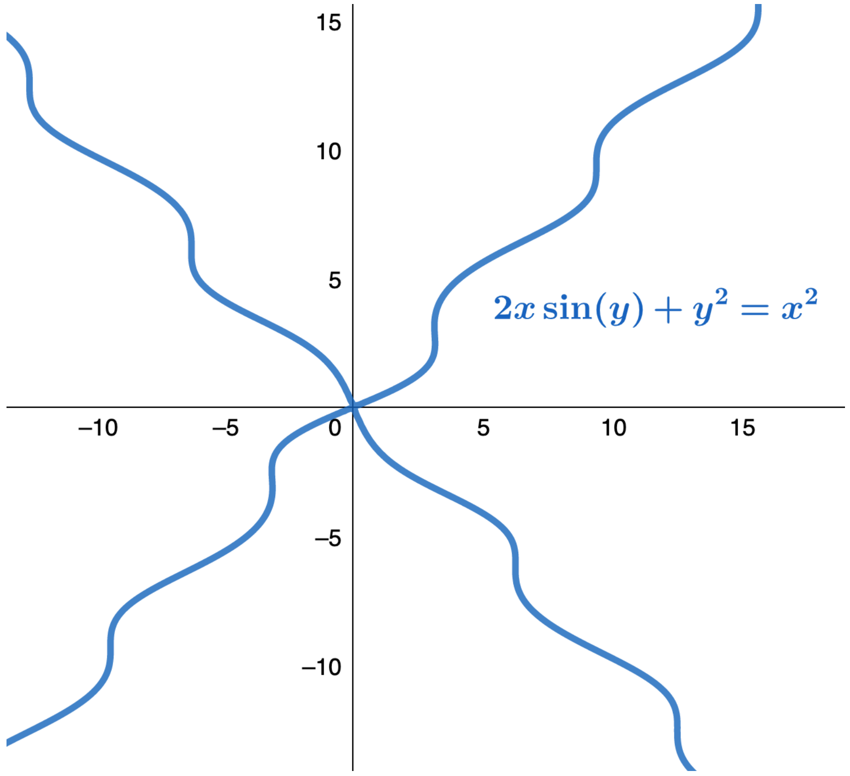
EXAMPLE TWO: STEP THREE



Factor out $\frac{dy}{dx}$ on the left.

- $x\cos(y)\frac{dy}{dx} + y\frac{dy}{dx} = x - \sin(y)$
- $(x\cos(y) + y)\frac{dy}{dx} = x - \sin(y)$

EXAMPLE TWO: STEP FOUR

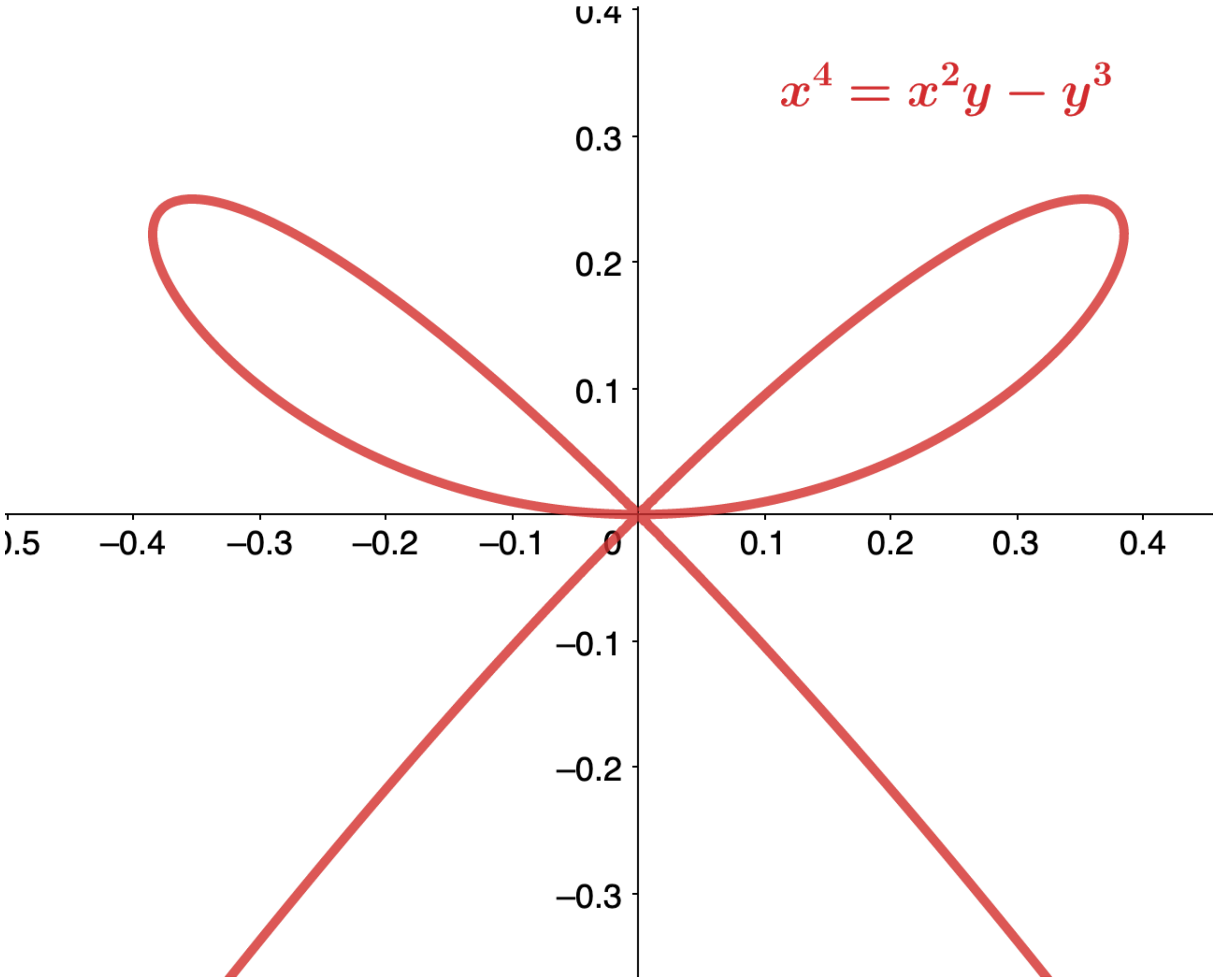


Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

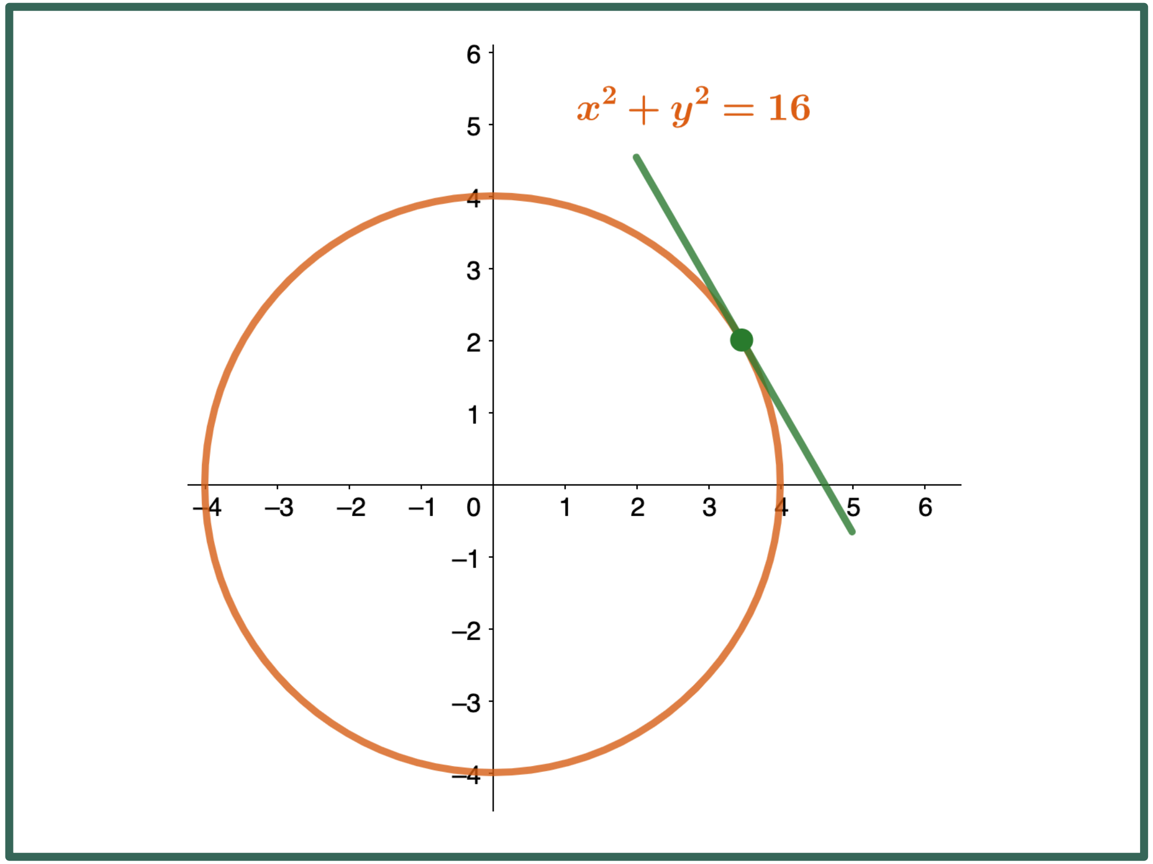
- $(x\cos(y) + y)\frac{dy}{dx} = x - \sin(y)$
- $\frac{dy}{dx} = \frac{x-\sin(y)}{x\cos(y)+y}$

EXERCISE ONE

- Assuming that y is defined implicitly by the equation $x^4 = x^2y - y^3$, find $\frac{dy}{dx}$.

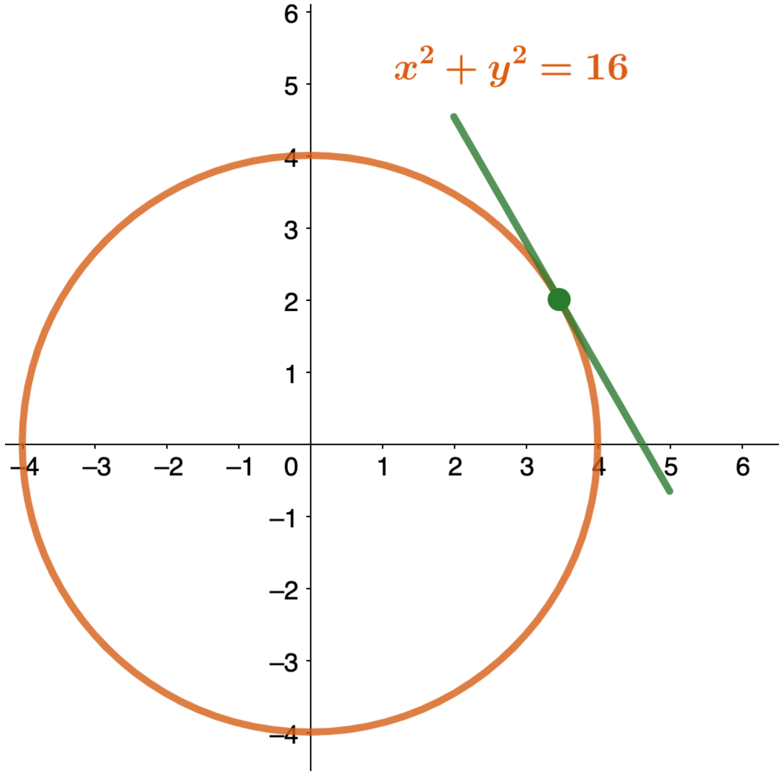


EXERCISE TWO



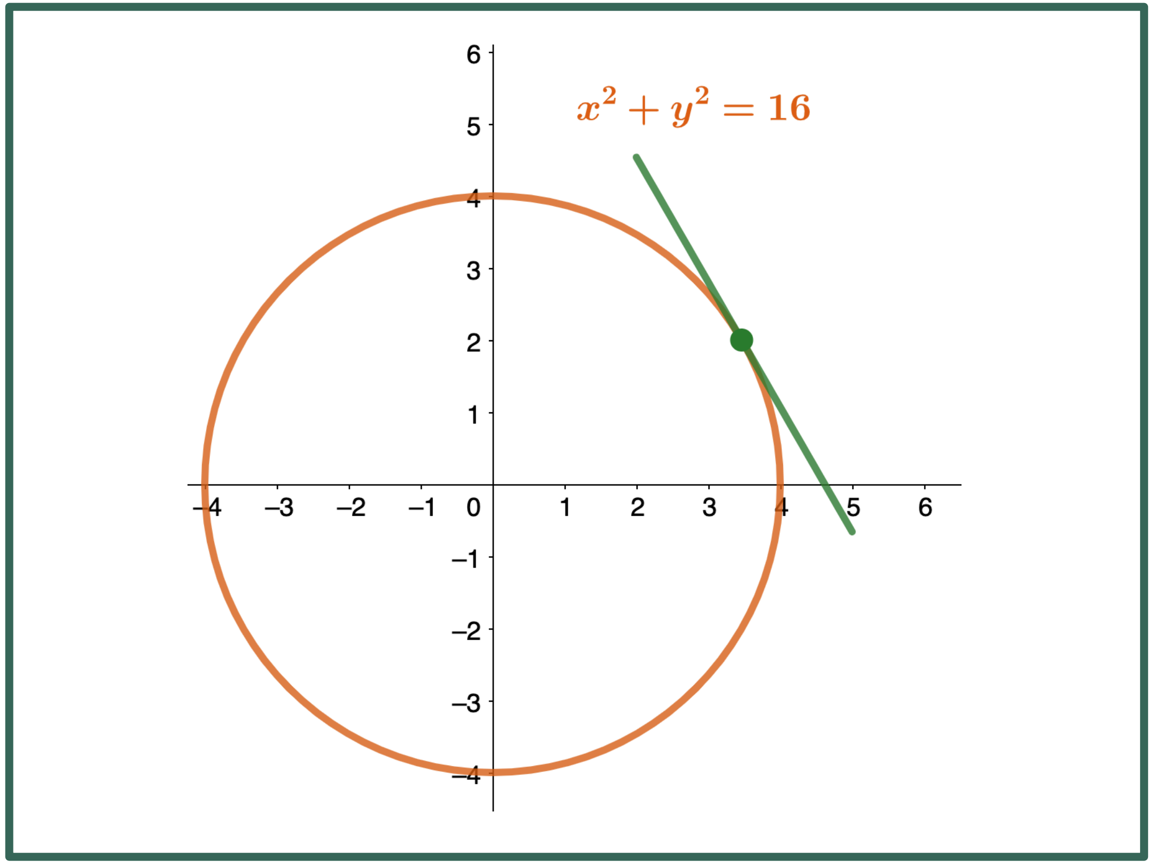
- **Using Implicit Differentiation to Find a Second Derivative**
- Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 16$.
- Hint: from the first example we know that $\frac{dy}{dx} = -\frac{x}{y}$.

EXERCISE TWO



- Hint: from the first example we know that $\frac{dy}{dx} = -\frac{x}{y}$.
- We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

EXERCISE TWO



Use the quotient rule

- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x}{y}\right) = -\frac{d}{dx}\left(\frac{x}{y}\right) = -\frac{\frac{dx}{dx}\cdot y - x\frac{dy}{dx}}{y^2}$

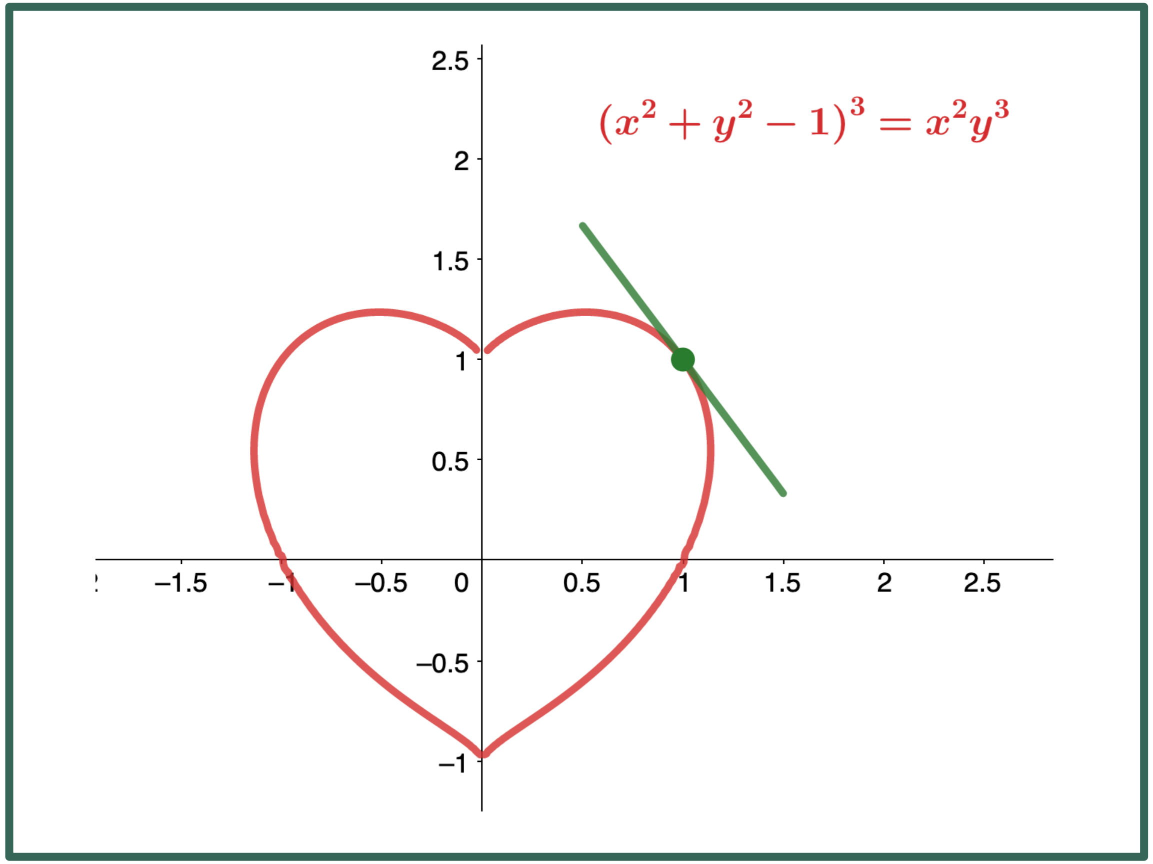
Use the fact that $\frac{dy}{dx} = -\frac{x}{y}$

- $\frac{d^2y}{dx^2} = -\frac{\frac{dx}{dx}\cdot y - x\frac{dy}{dx}}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2+x^2}{y^3}$

Use the fact that $x^2 + y^2 = 16$

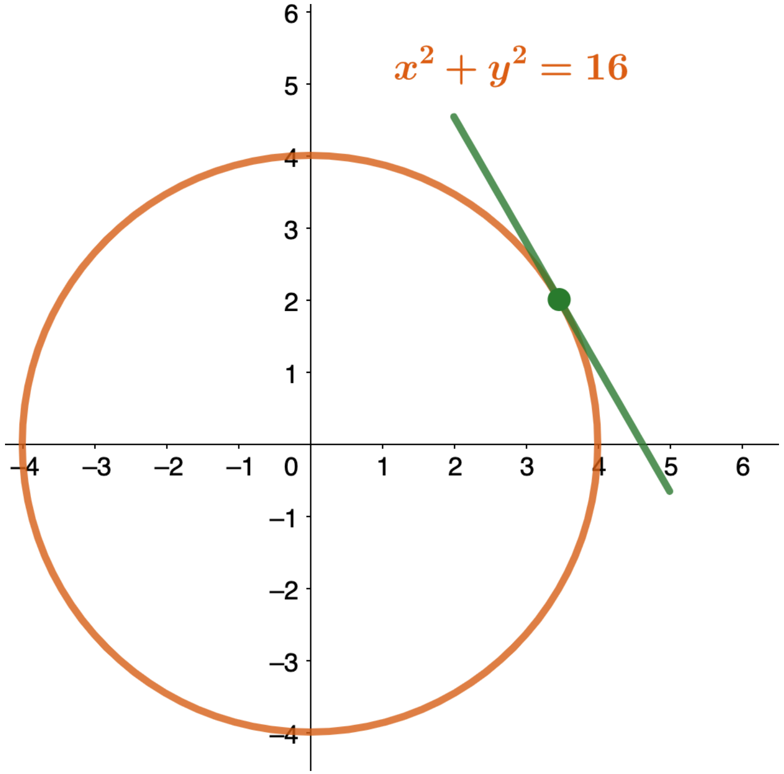
- $\frac{d^2y}{dx^2} = -\frac{y^2+x^2}{y^3} = -\frac{16}{y^3}$

FINDING TANGENT LINES IMPLICITLY



- Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

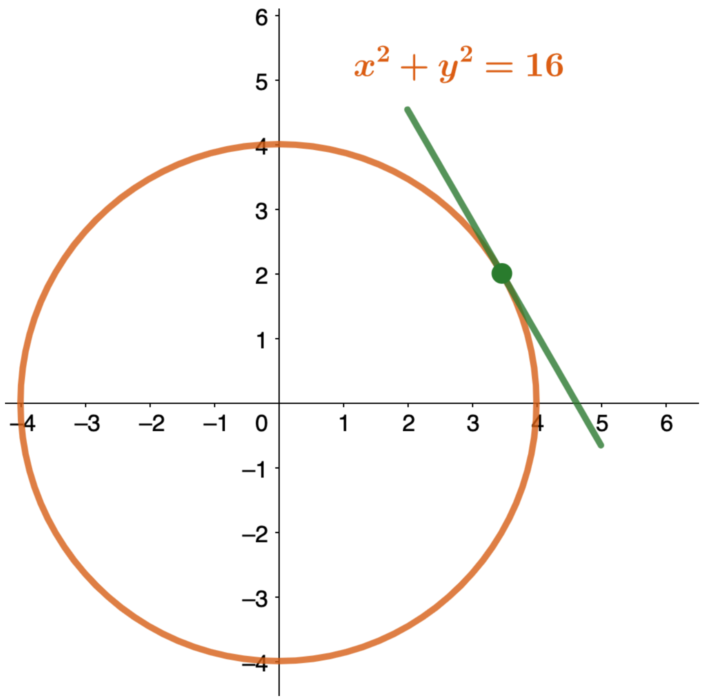
EXAMPLE



Finding a Tangent Line to a Circle

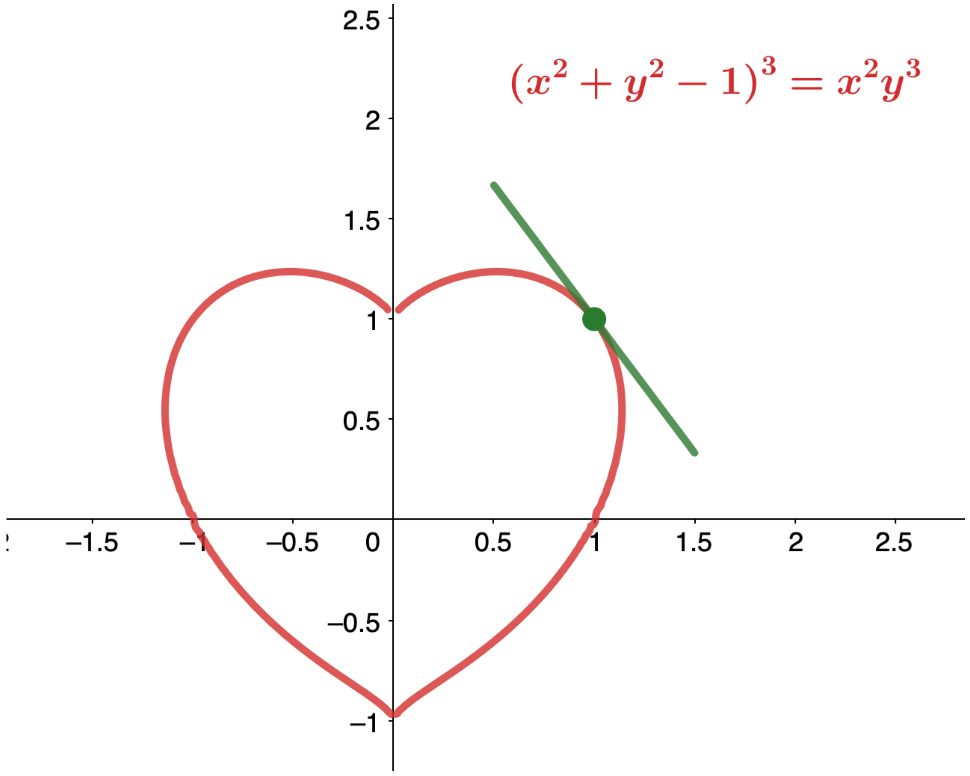
- Find the equation of the line tangent to the curve $x^2 + y^2 = 16$ at the point $(2\sqrt{3}, 2)$.

EXAMPLE



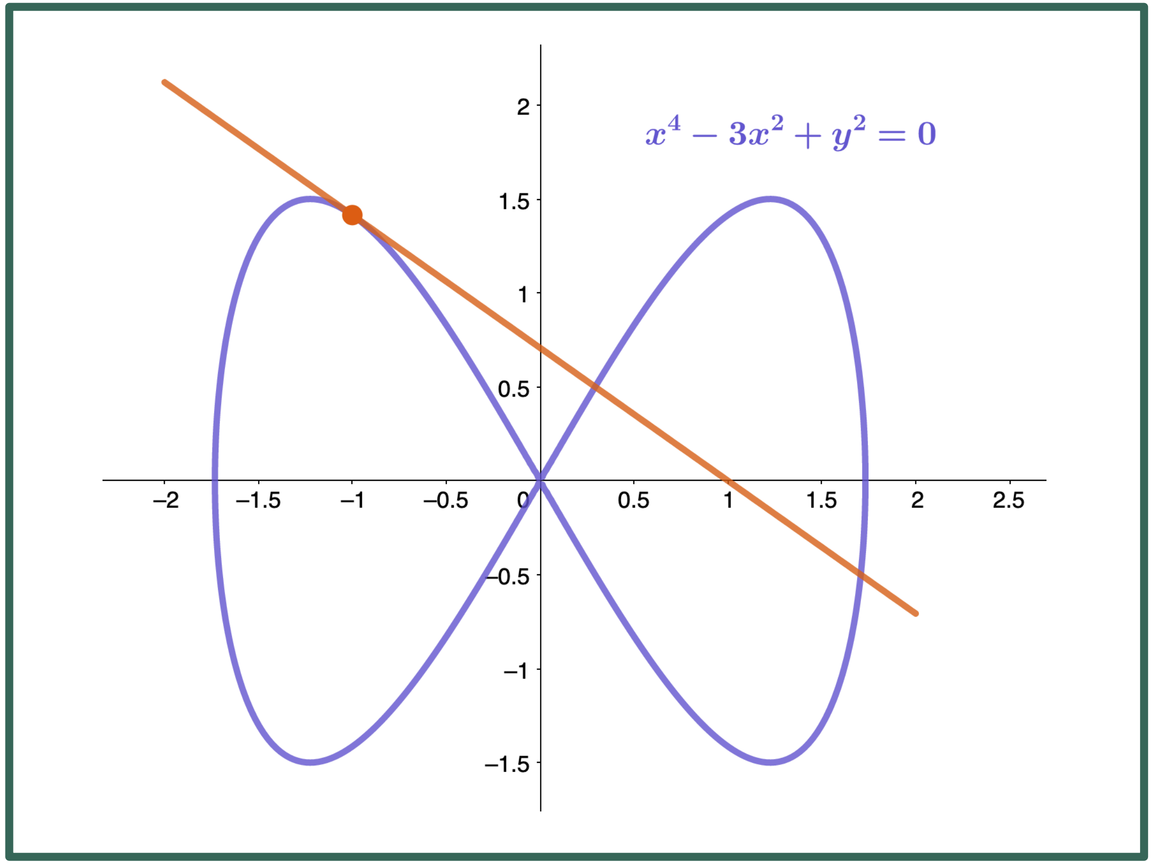
- Previously, we found $\frac{dy}{dx} = -\frac{x}{y}$.
- The slope of the tangent line is found by substituting $(2\sqrt{3}, 2)$ into this expression. Consequently, the slope of the tangent line is $\frac{dy}{dx}\Big|_{(2\sqrt{3},2)} = -\frac{2\sqrt{3}}{2} = -\sqrt{3}$.
- Using the point $(2\sqrt{3}, 2)$ and the slope $-\sqrt{3}$ in the point-slope equation of the line, we obtain the equation $y - 2 = -\sqrt{3}(x - 2\sqrt{3})$.

EXERCISE ONE



- Find the equation of the line tangent to the curve $(x^2 + y^2 - 1)^3 = x^2y^3$ at the point $(1, 1)$.

EXERCISE TWO



- In a simple video game, a rocket travels in an orbit whose path is described by the equation $x^4 - 3x^2 + y^2 = 0$.
- The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0, 0)$.
- If the rocket fires a missile when it is located at $(-1, \sqrt{2})$, where will it intersect the x-axis?