INTRODUCTION TO CALCULUS

DERIVATIVES OF TRIG FUNCTIONS

OUTLINE

Find — Find the derivatives of the **sine** and **cosine** function.

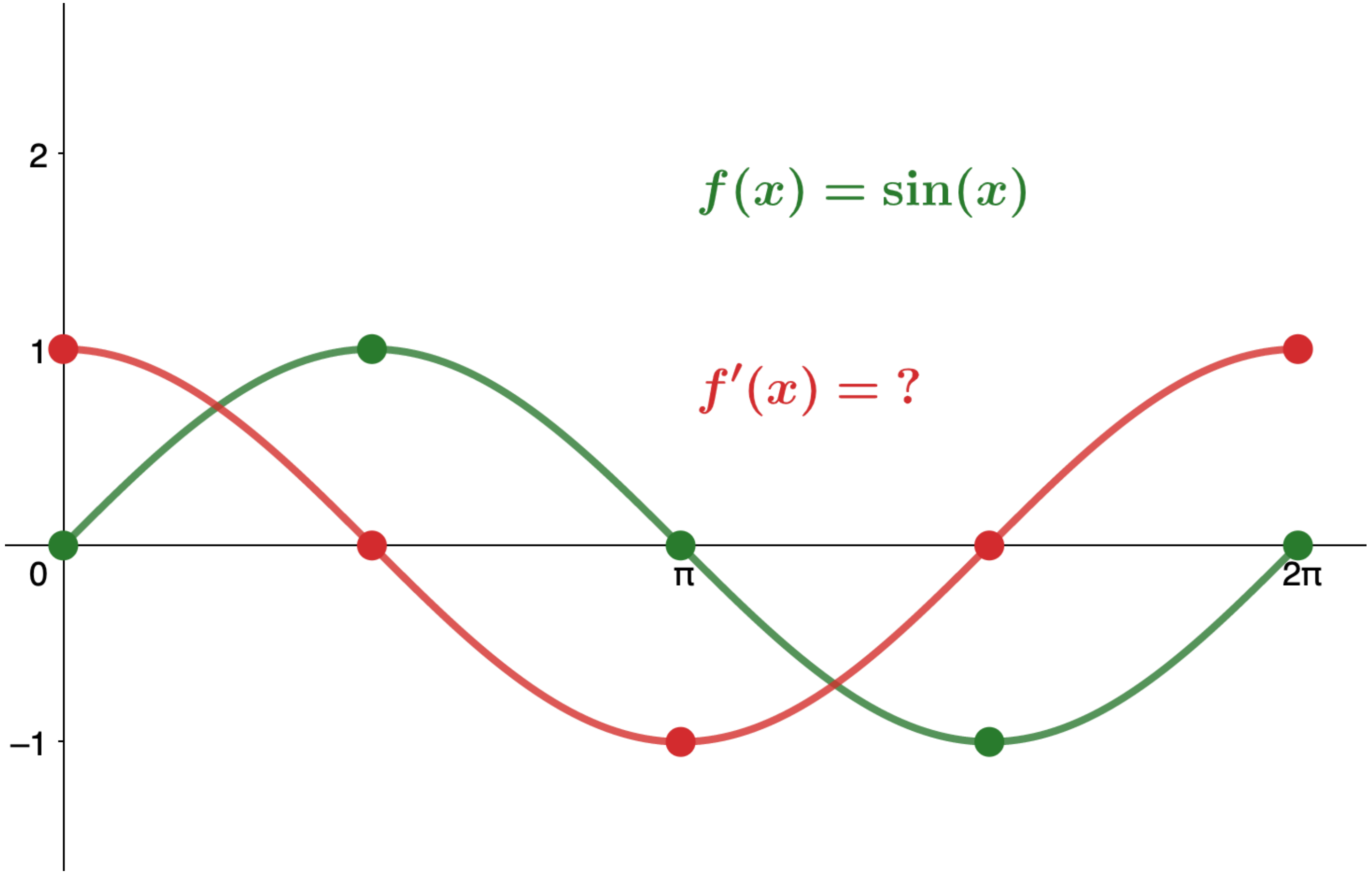
Find — Find the derivatives of the standard trigonometric functions.

Calculate — Calculate the **higher-order derivatives** of the sine and cosine.

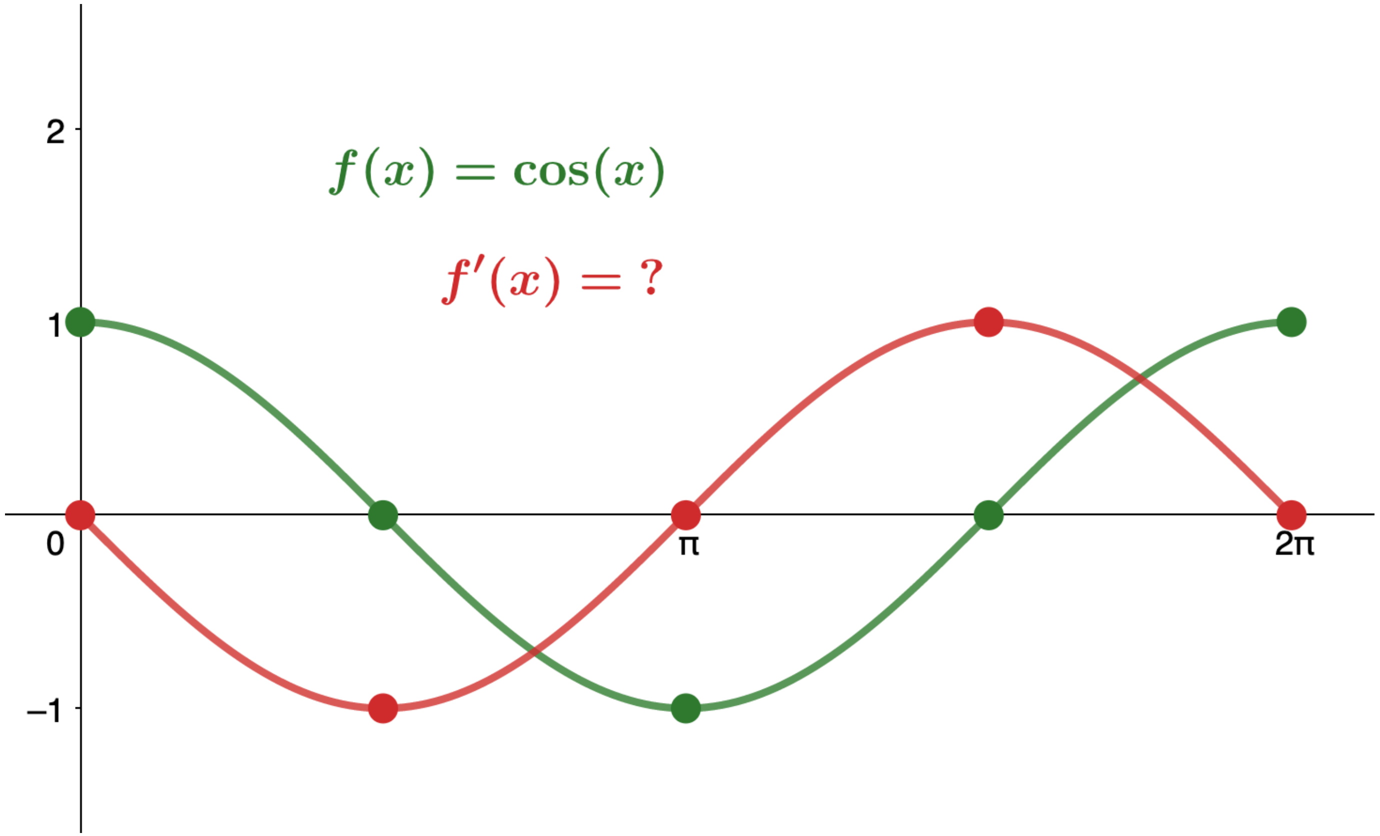
WHAT ARE WE AFTER?

- We only need to figure out the derivatives of … and ….

$$\sin(x) \text{ and } \cos(x)$$



GUESS



GUESS

THE DERIVATIVES

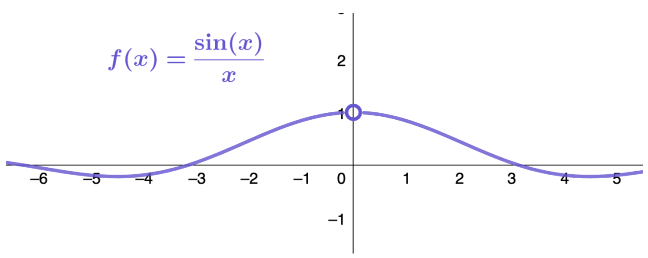
THEOREM 3.8

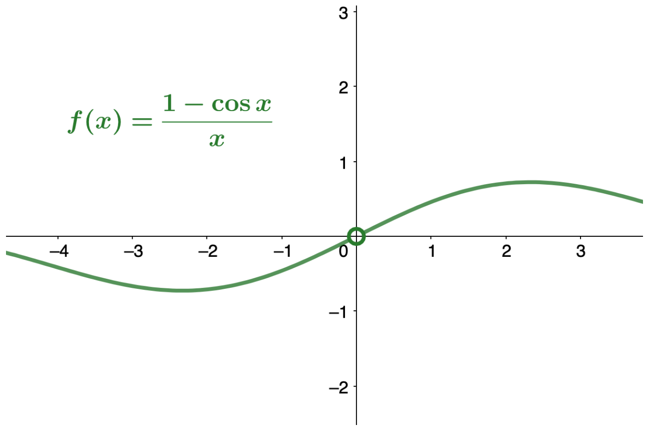
The Derivatives of sin *x* and cos *x*

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\boxed{\frac{d}{dx}(\sin x) = \cos x} \tag{3.11}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x} \tag{3.12}$$





PROOF: PREPARATION

- $$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$
- $$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$

PROOF

- Apply the definition of the derivative.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x+h) = \sin x\cos h + \cos x\sin h.$$

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\left(\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right) \\
&= \lim_{h\to 0}\left(\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right) \\
&= \sin x\cdot 0+\cos x\cdot 1 \\
&= \cos x
\end{aligned}
$$

PROOF

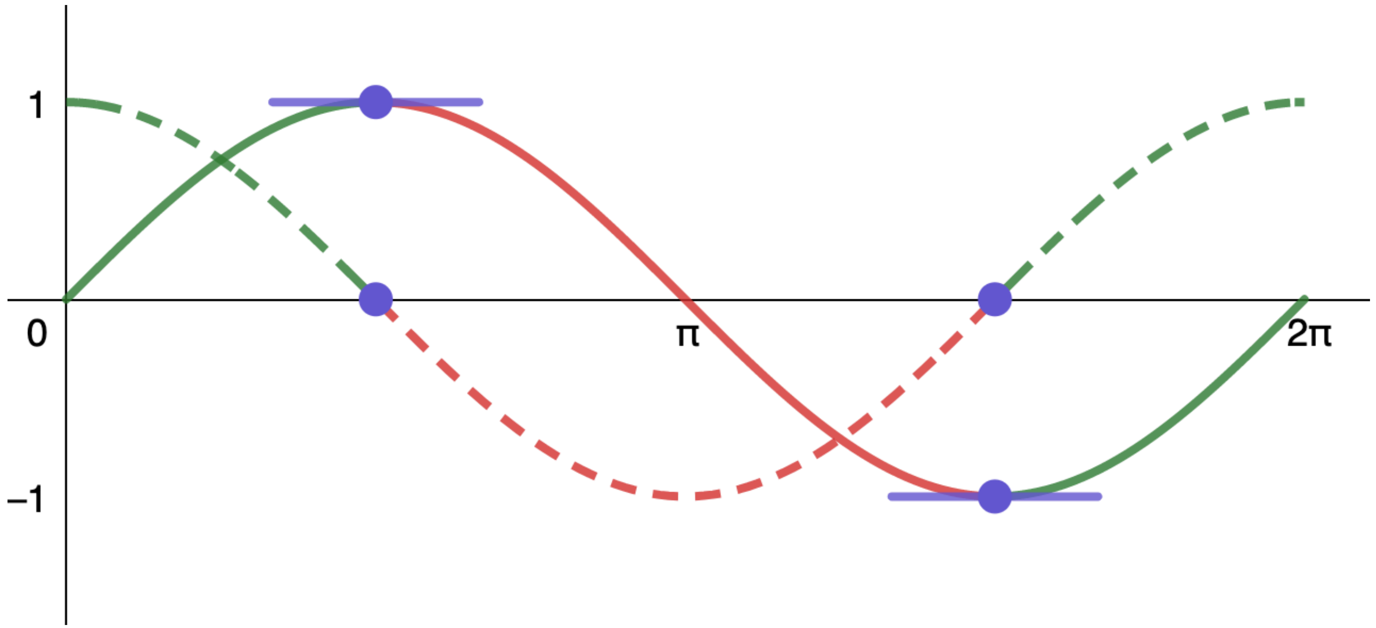
EXERCISE

PROOF

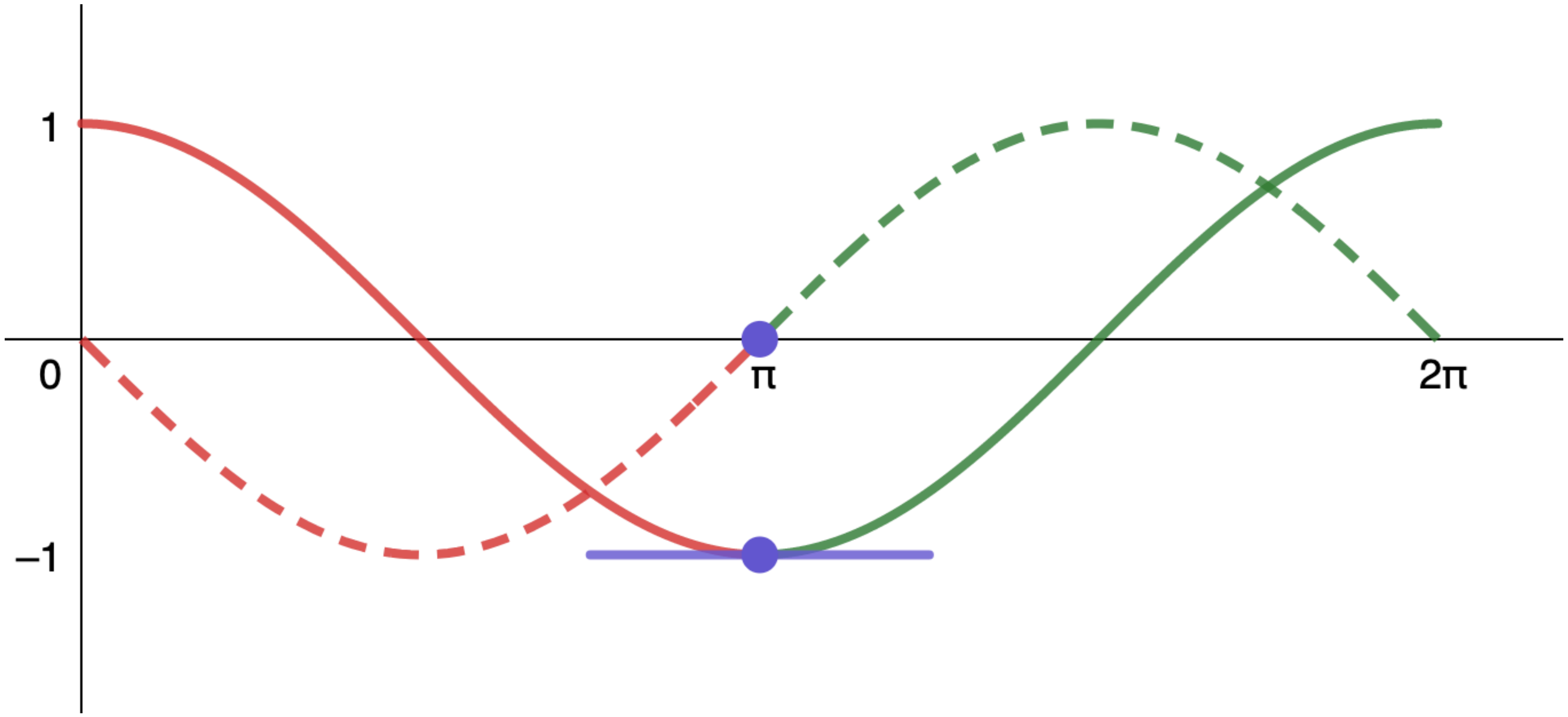
$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

SINE FUNCTION AND ITS DERIVATIVE

- At the points where $\sin(x)$ has a horizontal tangent, its derivative $\cos(x)$ takes on the value zero.
- Where $\sin(x)$ is increasing, $\cos(x) > 0$ and where $\sin(x)$ is decreasing, $\cos(x) < 0$.



COSINE FUNCTION AND ITS DERIVATIVE



EXERCISE ONE

Differentiating a Function Containing sin *x*

- Find the derivative of $2x^2 \sin(x)$.

EXERCISE TWO

Finding the Derivative of a Function Containing cos *x*

- $f(x) = \frac{\cos(x)}{x}$

EXERCISE THREE

- Find the derivative of $\sin(x)\cos(x)$.

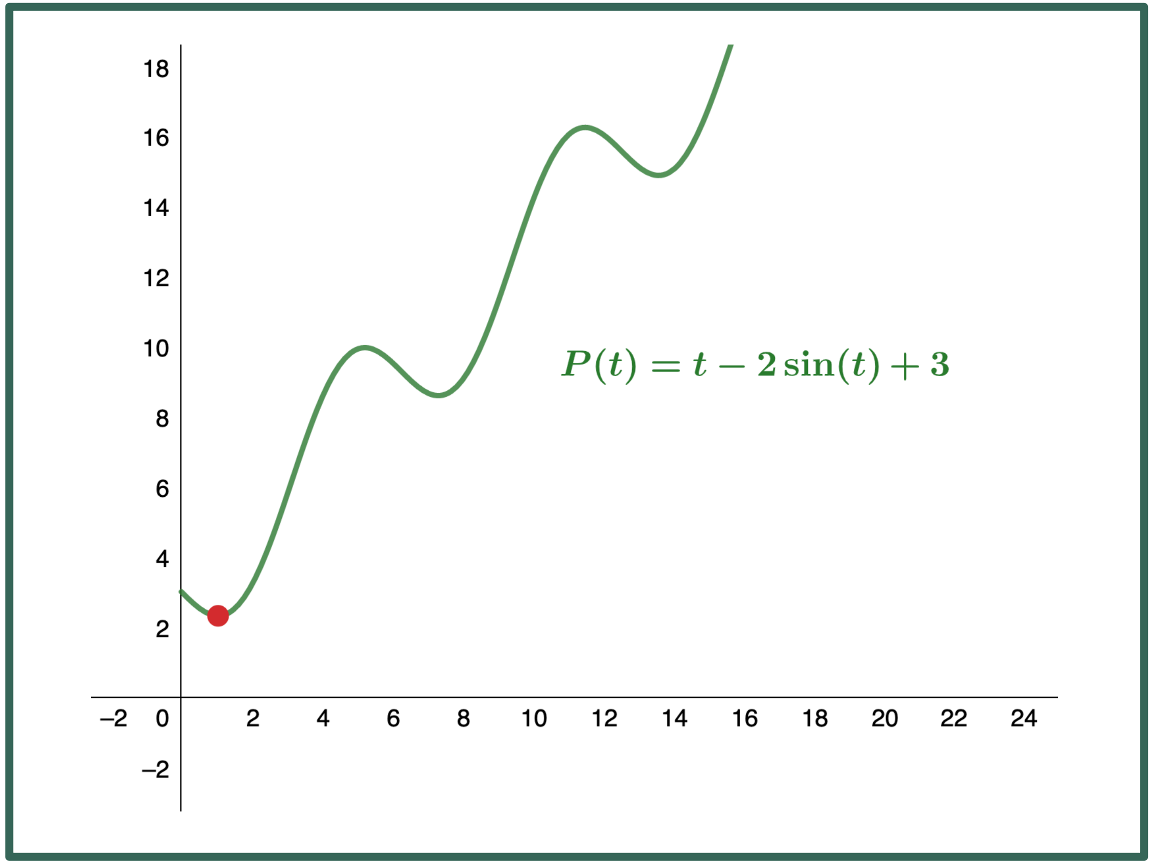
AN APPLICATION TO PHYSICS (VELOCITY)

- A dust moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2\cos(t) - t$ for $0 \leq t \leq 2\pi$.
- At what times is the dust at rest?

AN APPLICATION TO ECONOMY (STOCK PRICE)

- The price of the cryptocurrency dogecoin at time t is given by $P(t) = t - 2\sin(t) + 3$ for $t \geq 0$.
- At what time does the dogecoin hit its lowest price?

AN APPLICATION TO ECONOMY (STOCK PRICE)



- The price of the cryptocurrency dogecoin at time t is given by $P(t) = t - 2\sin(t) + 3$ for $t \geq 0$.
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DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS

- The power rule
- The Sum, Difference, and Constant Multiple Rules
- The product rule
- The quotient rule

EXAMPLE

The Derivative of the Tangent Function

- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

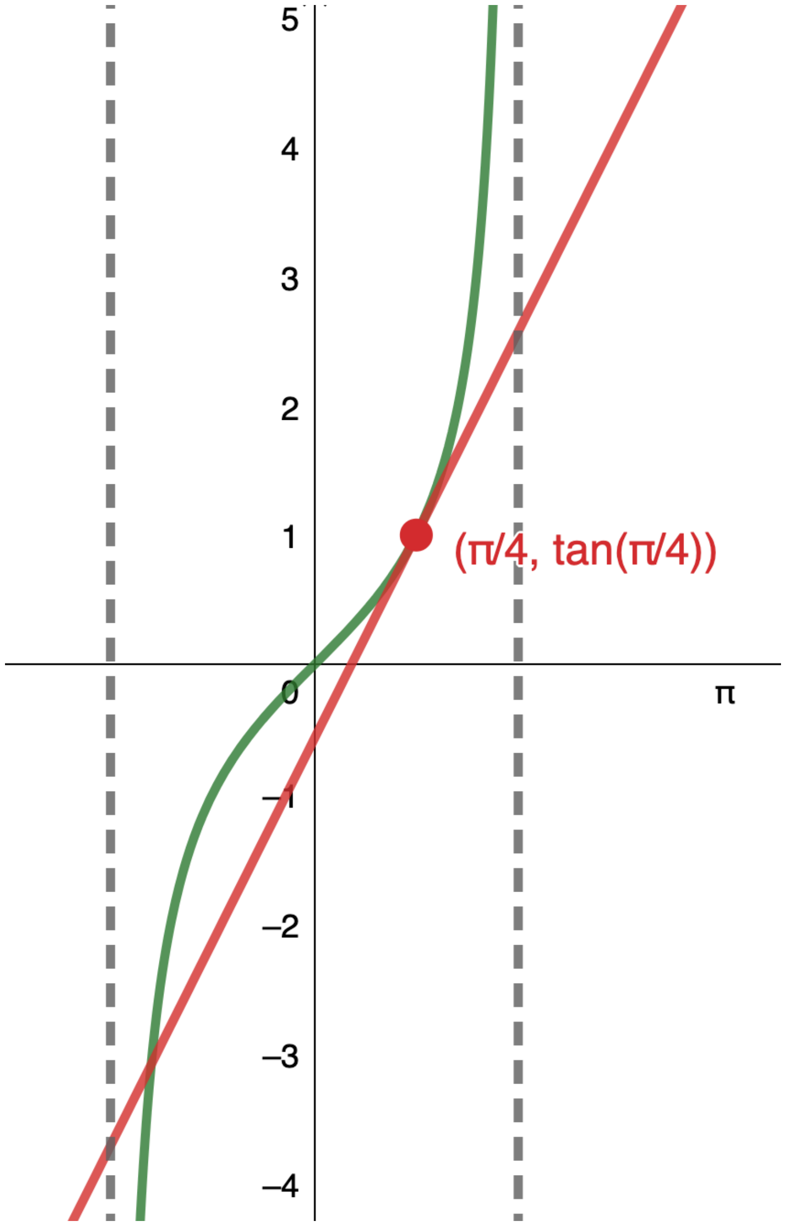
Use which rule?

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

EXERCISE ONE

Finding the Equation of a Tangent Line

- Find the equation of a line tangent to the graph of $f(x) = \tan(x)$ **at** $\boldsymbol{x = \frac{\pi}{4}}$.



EXERCISE

- The derivative of the Cotangent Function
- $\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

THEOREM 3.9

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x.$$

DERIVATIVES OF …

EXERCISE TWO

Finding the Derivative of Trigonometric Functions

- Find the derivative of $f(x) = \sec(x) + x\cot(x)$

EXERCISE THREE

Finding the Derivative of Squares of Trigonometric Functions

- Find the derivative of $f(x) = \csc^2(x)$.

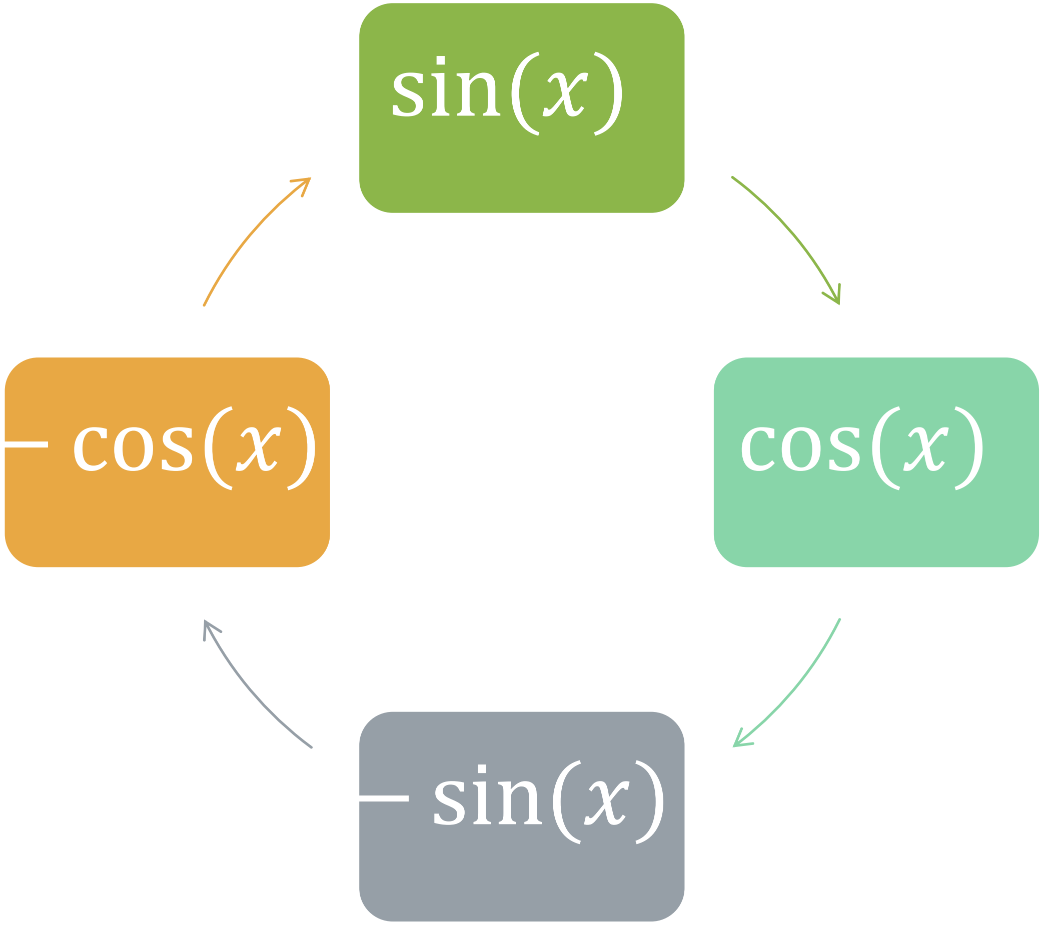
- The higher-order derivatives of $\sin(x)$ and $\cos(x)$ follow a repeating pattern.
- By following the pattern, we can find any higher-order derivative of $\sin(x)$ and $\cos(x)$.

HIGHER-ORDER DERIVATIVES

EXAMPLE ONE

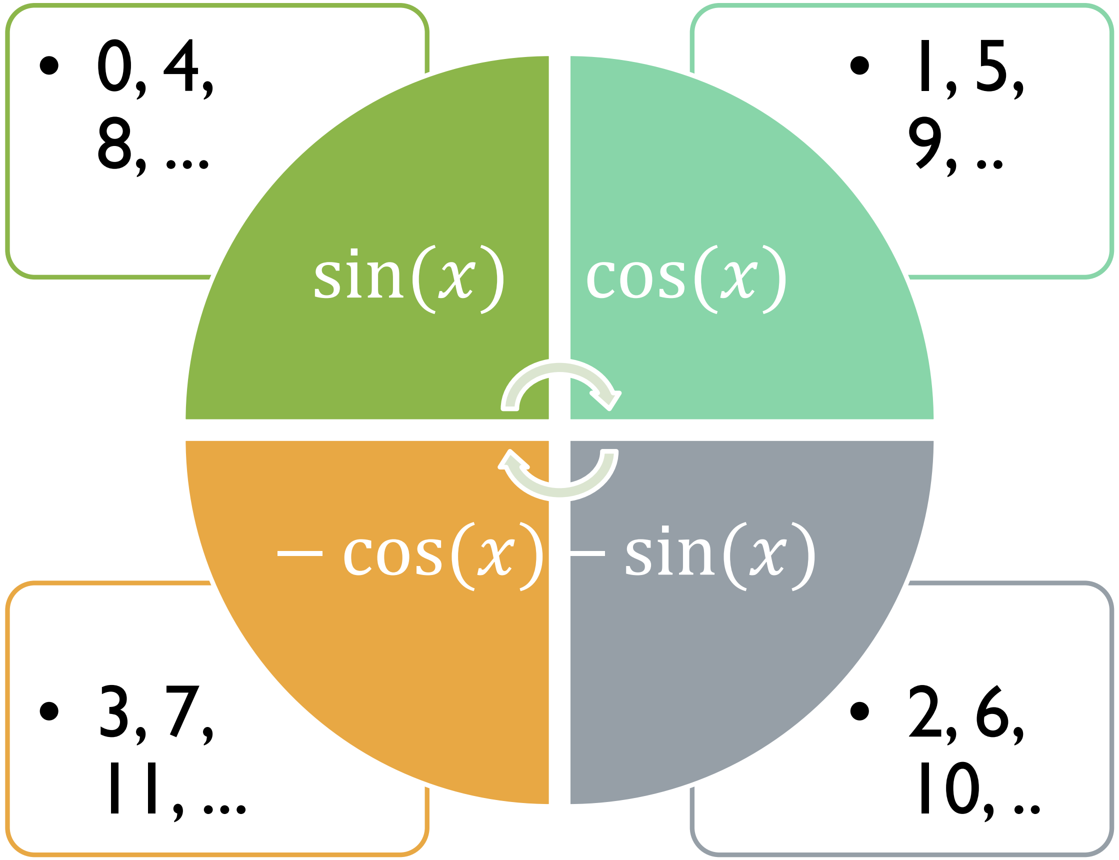
Finding Higher-Order Derivatives of $\boldsymbol{y} = \sin(\boldsymbol{x})$

- Find the first four derivatives of $\boldsymbol{y} = \sin(\boldsymbol{x})$
- $y = \sin(x)$
- $\frac{dy}{dx} = \frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d^2y}{dx^2} = \frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d^3y}{dx^3} = \frac{d}{dx}(-\sin(x)) = -\cos(x)$
- $\frac{d^4y}{dx^4} = \frac{d}{dx}(-\cos(x)) = \sin(x)$
- …



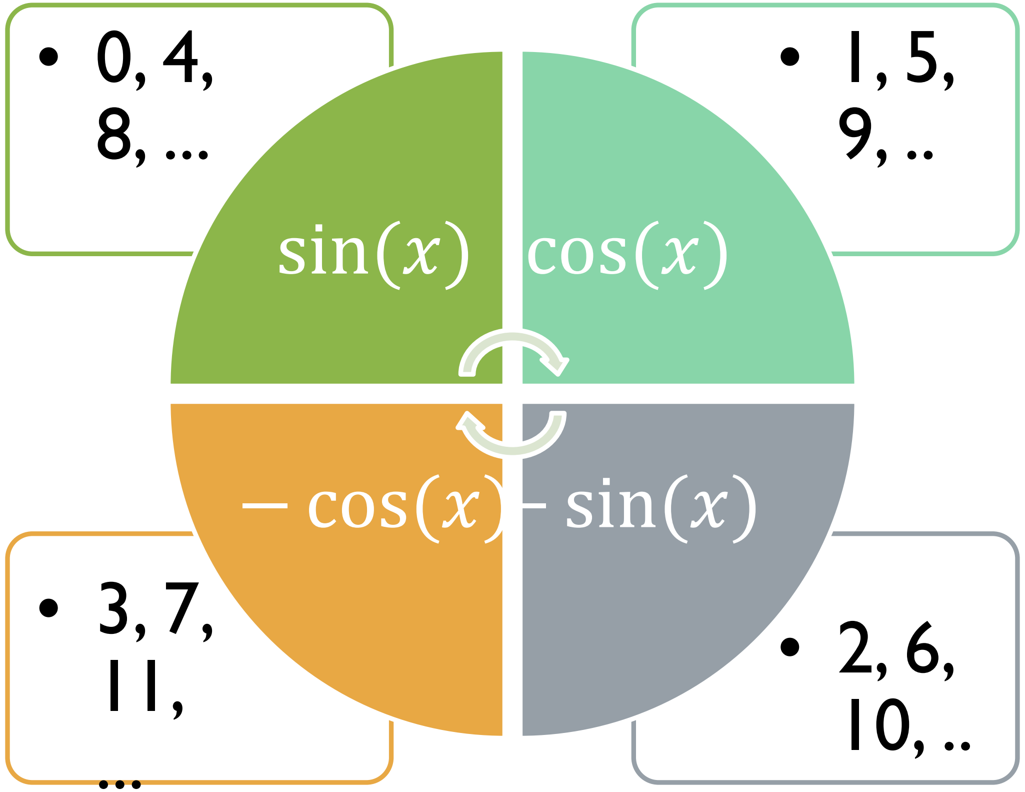
ANALYSIS

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the **step** in the pattern to which it corresponds.

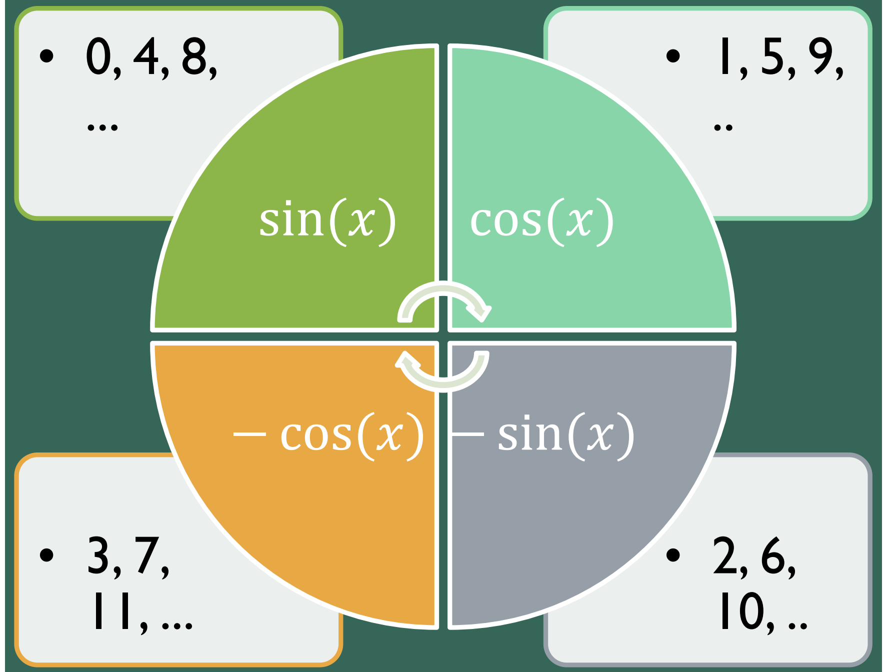


ANALYSIS

- $\frac{d}{dx}\sin(x) = \frac{d^5}{dx^5}\sin(x) = \frac{d^9}{dx^9}\sin(x) = \cdots = \cos(x)$
- $\frac{d^2}{dx^2}\sin(x) = \frac{d^6}{dx^6}\sin(x) = \frac{d^{10}}{dx^{10}}\sin(x) = \cdots = -\sin(x)$
- $\frac{d^3}{dx^3}\sin(x) = \frac{d^7}{dx^7}\sin(x) = \frac{d^{11}}{dx^{11}}\sin(x) = \cdots = -\cos(x)$
- $\frac{d^4}{dx^4}\sin(x) = \frac{d^8}{dx^8}\sin(x) = \frac{d^{12}}{dx^{12}}\sin(x) = \cdots = \sin(x)$



EXERCISE ONE



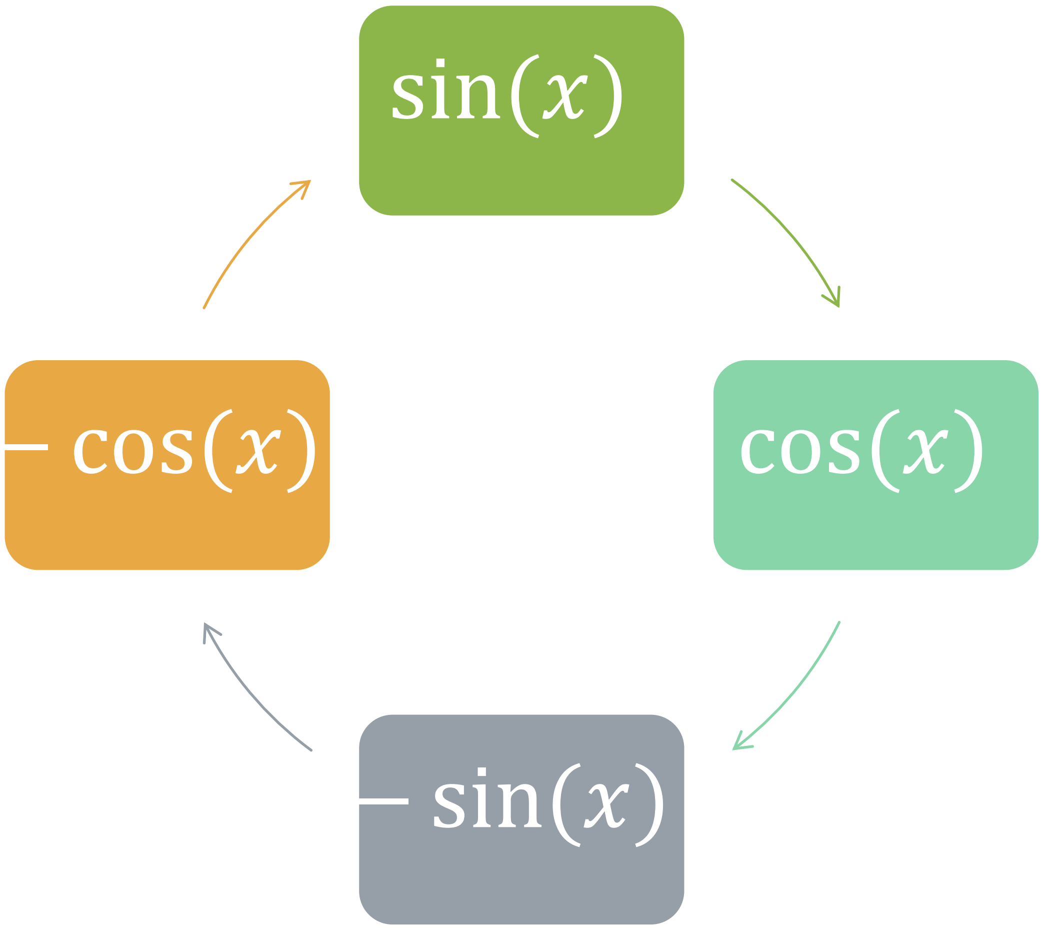
Using the Pattern for Higher-Order Derivatives of $\sin(x)$.

- Find $\frac{d^{2019}}{dx^{2019}} \sin(x)$.

EXAMPLE TWO

Finding Higher-Order Derivatives of $y = \cos(x)$.

- $y = \cos(x)$
- $\frac{dy}{dx} = \frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d^2y}{dx^2} = \frac{d}{dx}(-\sin(x)) = -\cos(x)$
- $\frac{d^3y}{dx^3} = \frac{d}{dx}(-\cos(x)) = \sin(x)$
- $\frac{d^4y}{dx^4} = \frac{d}{dx}(\sin(x)) = \cos(x)$
- …



ANALYSIS

- $\frac{d}{dx}\cos(\boldsymbol{x}) = \frac{d^5}{dx^5}\cos(\boldsymbol{x}) = \frac{d^9}{dx^9}\cos(\boldsymbol{x}) = \cdots = -\sin(x)$
- $\frac{d^2}{dx^2}\cos(\boldsymbol{x}) = \frac{d^6}{dx^6}\cos(\boldsymbol{x}) = \frac{d^{10}}{dx^{10}}\cos(\boldsymbol{x}) = \cdots = -\cos(x)$
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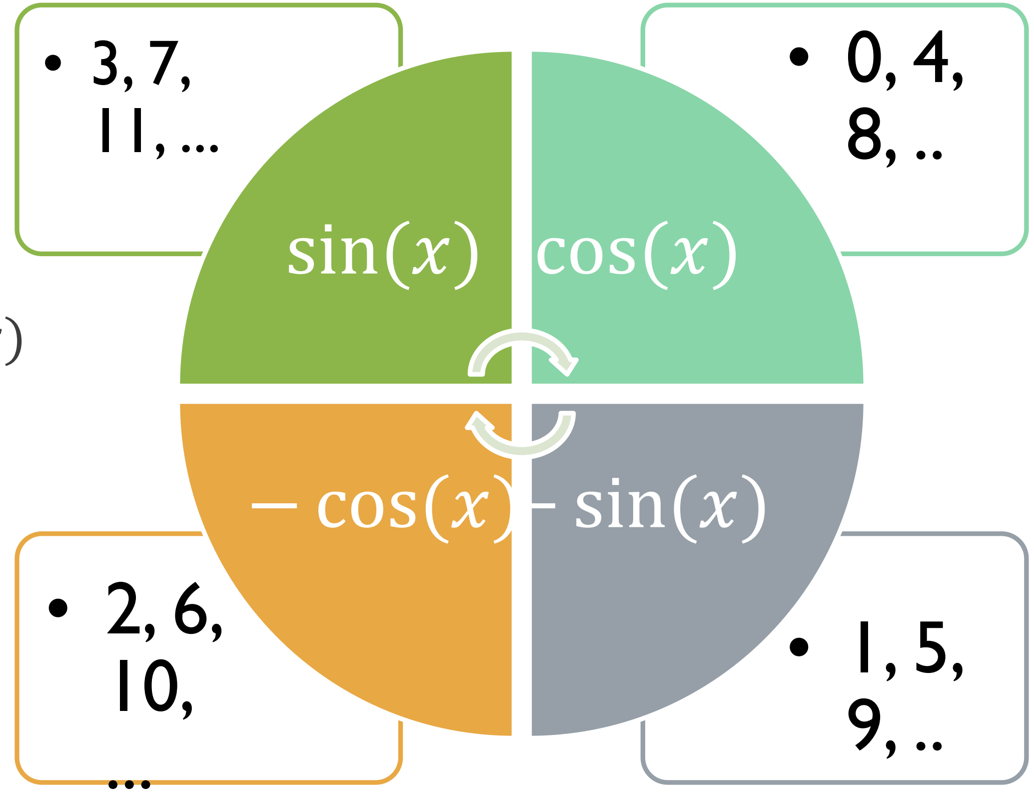
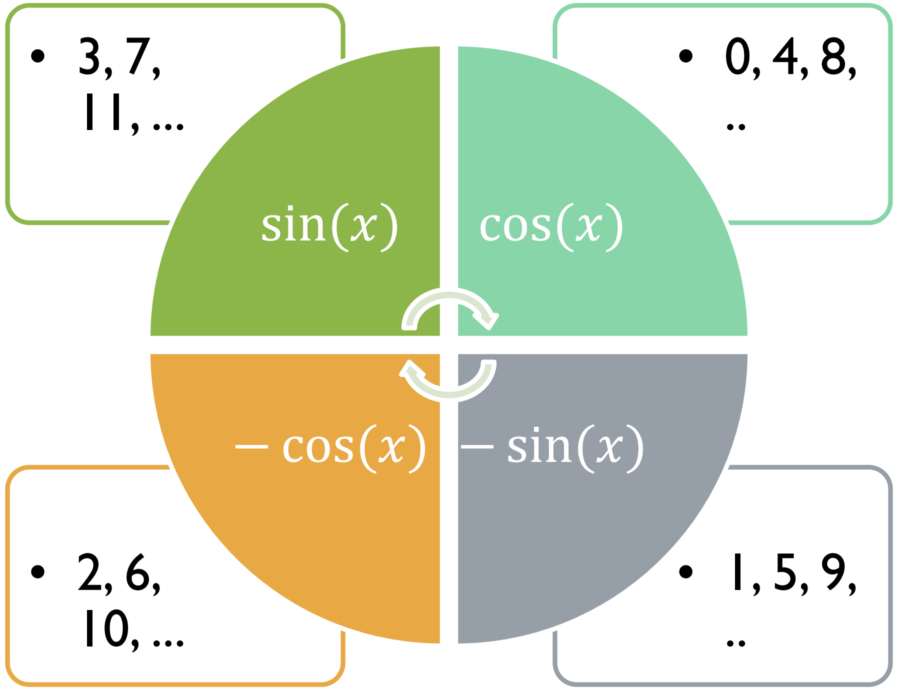


EXERCISE TWO



Using the Pattern for Higher-Order Derivatives of $\cos(x)$.

- Find $\frac{d^{1984}}{dx^{1984}}\cos(\boldsymbol{x})$.

ANOTHER WAY TO PROVE: Remember that $\frac{d}{dx}\sin(x) = \cos(x)$.

$$\frac{d^2}{dx^2}\sin(x) = \frac{d^6}{dx^6}\sin(x) = \frac{d^{10}}{dx^{10}}\sin(x) = \cdots = -\sin(x)$$

$$\frac{d}{dx}\cos(x) = \frac{d^5}{dx^5}\cos(x) = \frac{d^9}{dx^9}\cos(x) = \cdots = -\sin(x)$$

EXERCISE

An Application to Acceleration

- A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 + \cos(t)$.
- Find $v(\frac{\pi}{4})$ and $a(\frac{\pi}{4})$.
- Compare these values and decide whether the particle is speeding up or slowing down.