INTRODUCTION TO CALCULUS

BOUNDED AND CONVERGENT SEQUENCES

APART FROM THE SQUEEZE THEOREM

- One of the two most important theorems involving sequences: **the Monotone Convergence Theorem**.

BOUNDED AND UNBOUNDED SEQUENCES

Before stating the theorem, we need to introduce some terminology and motivation.

We begin by defining what it means for a sequence to be bounded.

BOUNDED AND UNBOUNDED SEQUENCES

Definition

A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$\boxed{a_n \leq M}$$

for all positive integers n.

A sequence $\{a_n\}$ is **bounded below** if there exists a real number M such that

$$\boxed{M \leq a_n}$$

for all positive integers n.

A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence**.

TESTS

$\left\{\frac{1}{n}\right\}$

$\{(-1)^n\}$

$\left\{(\frac{1}{3})^n\right\}$

$\{2^n\}$

$\left\{\sin\frac{n\pi}{2}\right\}$

Bounded below

Bounded above

A bounded sequence

A convergent sequence

THE RELATIONSHIP BETWEEN BOUNDEDNESS AND CONVERGENCE

- Suppose a sequence $\{a_n\}$ is unbounded.
- Then it is not bounded above, or not bounded below, or both.
- In either case, there are terms a_n that are arbitrarily large in magnitude as n gets larger.
- As a result, the sequence cannot converge.
- Therefore, **being bounded is a necessary condition for a sequence to converge**.

BEING BOUNDED IS A NECESSARY CONDITION FOR A SEQUENCE TO CONVERGE.

Theorem 5.5: Convergent Sequences Are Bounded

If a sequence $\{a_n\}$ converges, then it is bounded.

IS BEING BOUNDED A SUFFICIENT CONDITION FOR A SEQUENCE TO CONVERGE?

- Recall our tests!

A MONOTONE SEQUENCE

Definition

A sequence $\{a_n\}$ is increasing for all $n \geq n_0$ if

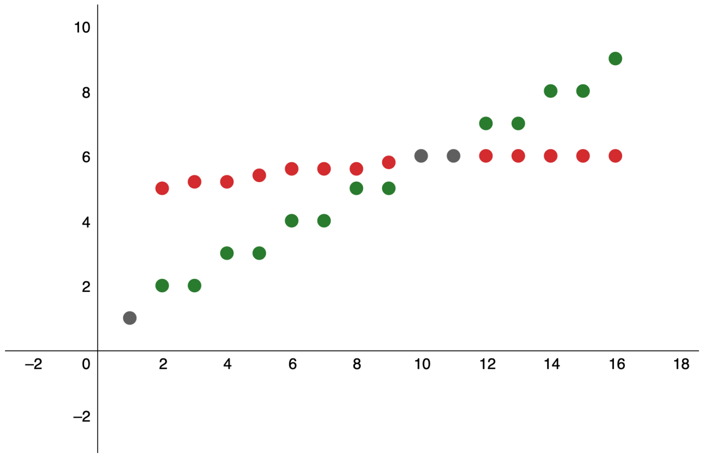
$$\boxed{a_n \leq a_{n+1} \text{ for all } n \geq n_0.}$$

A sequence $\{a_n\}$ is decreasing for all $n \geq n_0$ if

$$\boxed{a_n \geq a_{n+1} \text{ for all } n \geq n_0.}$$

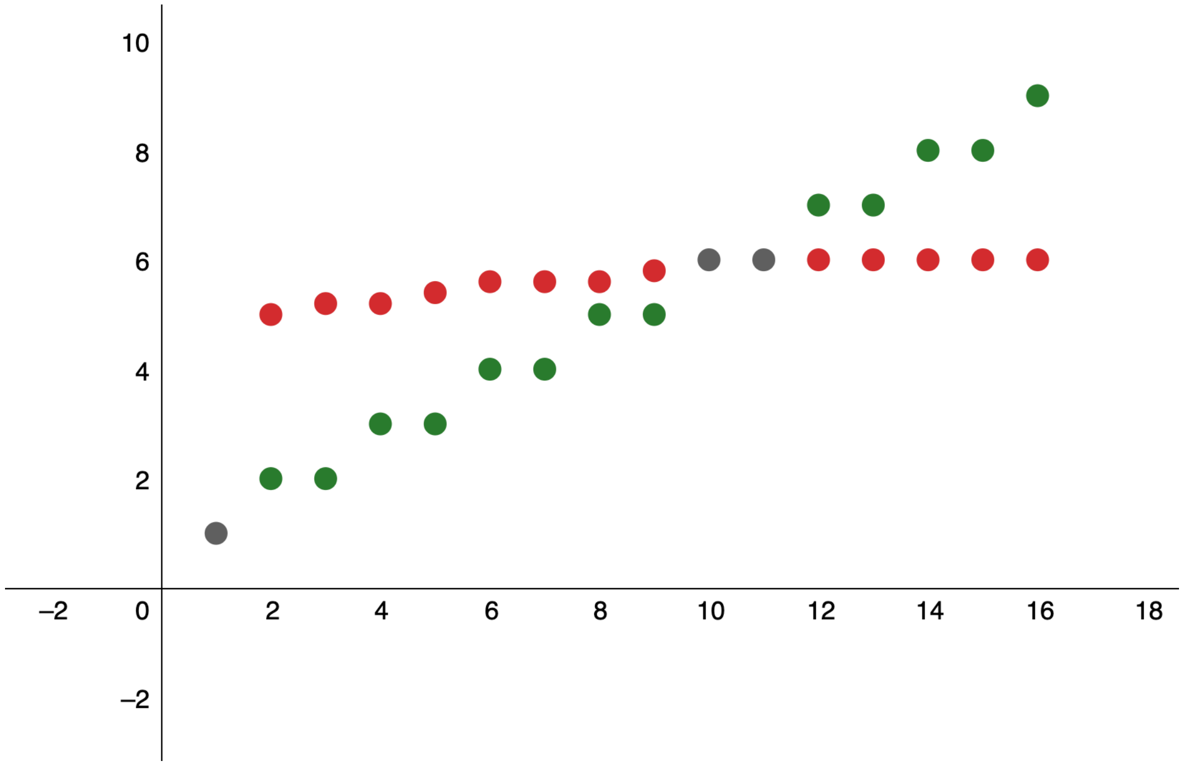
A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

AN INCREASING SEQUENCE

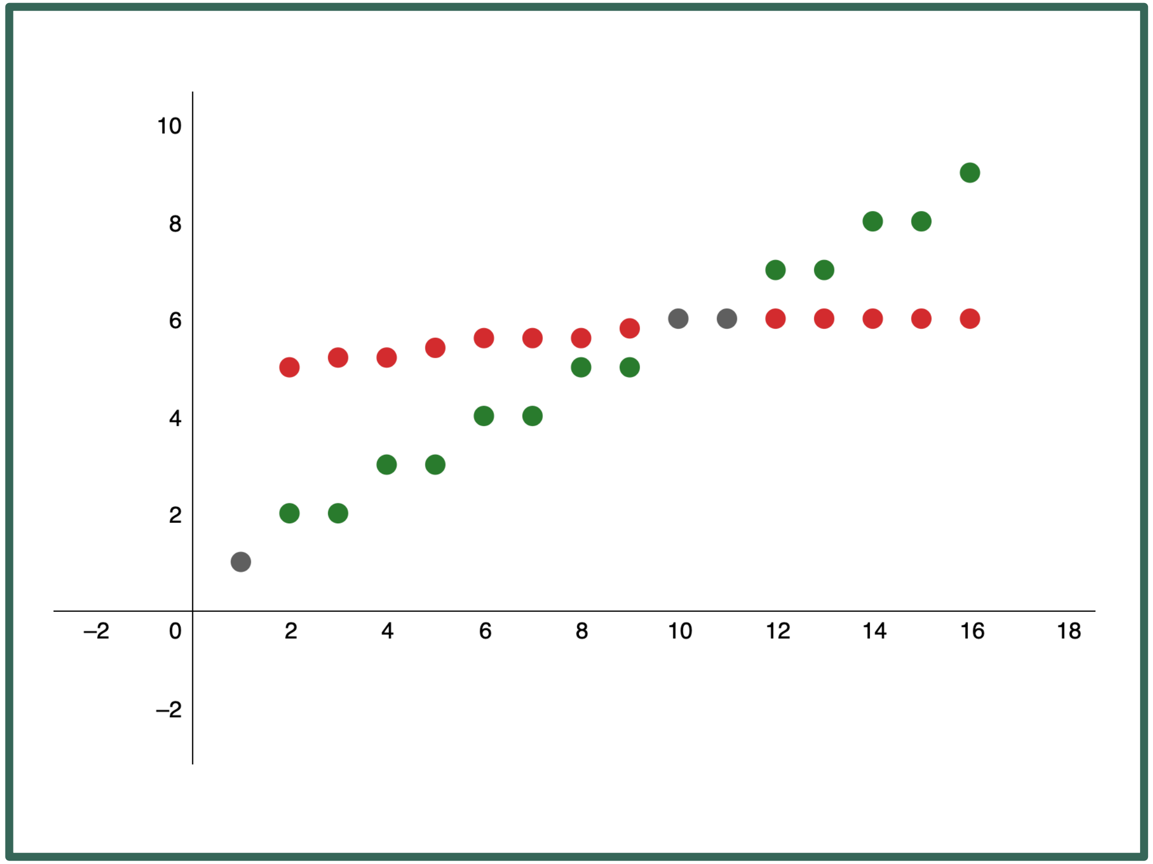


- Suppose the sequence $\{a_n\}$ is increasing.
- That is, $a_1 \leq a_2 \leq a_3 \cdots \cdots$
- Since the sequence is increasing, the terms are not oscillating.
- Therefore, there are two possibilities.
  - Diverge to infinity
  - Converge

HOW CAN AN INCREASING FUNCTION CONVERGE?



AN BOUNDED INCREASING SEQUENCE



- Since the (red) sequence is bounded, it is bounded above and the sequence cannot diverge to infinity.
- We conclude that it converges.

EXAMPLES

- $\left\{\frac{n-1}{n}\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\cdots\right\}$
- $\left\{1 - (\frac{1}{2})^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \cdots\cdots\right\}$

EVENTUALLY INCREASING SEQUENCES

- $\left\{100, 10, 1000, 1, 0, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}, \dots\dots\right\}$
- Even though the sequence is not increasing for all values of n we see that starting with the sixth term, $a_6 = -\frac{1}{2}$, the sequence is increasing.
- In this case, we say the sequence is eventually increasing.
- Since the sequence is bounded above, it converges.

ANALOGOUSLY

- If a sequence is decreasing (or eventually decreasing)
- How can it be convergent?
- Bounded below!!!

THE MONOTONE CONVERGENCE THEOREM (A SUFFICIENT CONDITION)

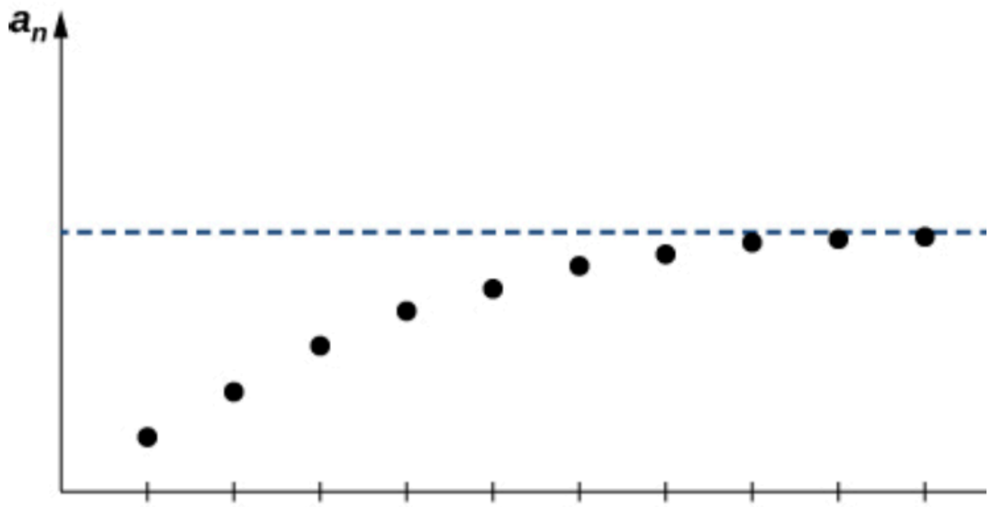


Figure 5.7 Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

If $\{a_n\}$ is a **bounded** sequence and there exists a positive integer n_0 such that $\{a_n\}$ is **monotone** for all $n \geq n_0$, then $\{a_n\}$ converges.

THE MONOTONE CONVERGENCE THEOREM (A SUFFICIENT CONDITION)

Increasing
Bounded above

Decreasing
Bounded below

EXAMPLE ONE

For the following sequences use the Monotone Convergence Theorem to show it converges and find its limit.

- $\left\{\frac{n}{2^n}\right\} = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \dots \dots\right\}$
- $a_{n+1} = \frac{n+1}{2n} a_n$

EXAMPLE ONE

- Decreasing
  - $a_{n+1} = \frac{n+1}{2n} a_n$
  - $a_{n+1} \leq a_n$
- Bounded below
  - $a_n = \frac{n}{2^n} \geq 0$
- What is the limit?

EXAMPLE ONE: WHAT IS THE LIMIT?

Recurrent relation

$$a_{n+1} = \frac{n+1}{2n} a_n$$

Take limits (both sides)

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{n+1}{2n} a_n$$

$$L = \frac{1}{2} \times L$$

Get the limit

$$L = 0$$

EXAMPLE TWO

For the following sequences use the Monotone Convergence Theorem to show it converges and find its limit.

- $\{a_n\}$ defines recursively.
  - $a_1 = 2$
  - $a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n})$, for all $n \geq 1$

EXAMPLE TWO

- Bounded below
  - $a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n}) \geq \frac{1}{2} \times 2 = 1$
- Decreasing
  - $a_{n+1} - a_n = \frac{1}{2}(a_n + \frac{1}{a_n}) - a_n = -\frac{1}{2}\left(a_n - \frac{1}{a_n}\right) = -\frac{1}{2}\frac{a_n^2 - 1}{a_n} \leq 0$
  - $a_{n+1} \leq a_n$
- What is the limit?

EXAMPLE TWO

Recurrent relation

$$a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n})$$

Take limits (both sides)

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + \frac{1}{a_n})$$

$$L = \frac{1}{2}(L + \frac{1}{L})$$

Get the limit

$$L = 1$$