MATH 20: PROBABILITY

Combinations

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SMALL PLATES

LARGE PLATES

SWEET BITES

3 choose 2

How many possible choices are there in total?

Is it 6?

?

SMALL PLATES = 1

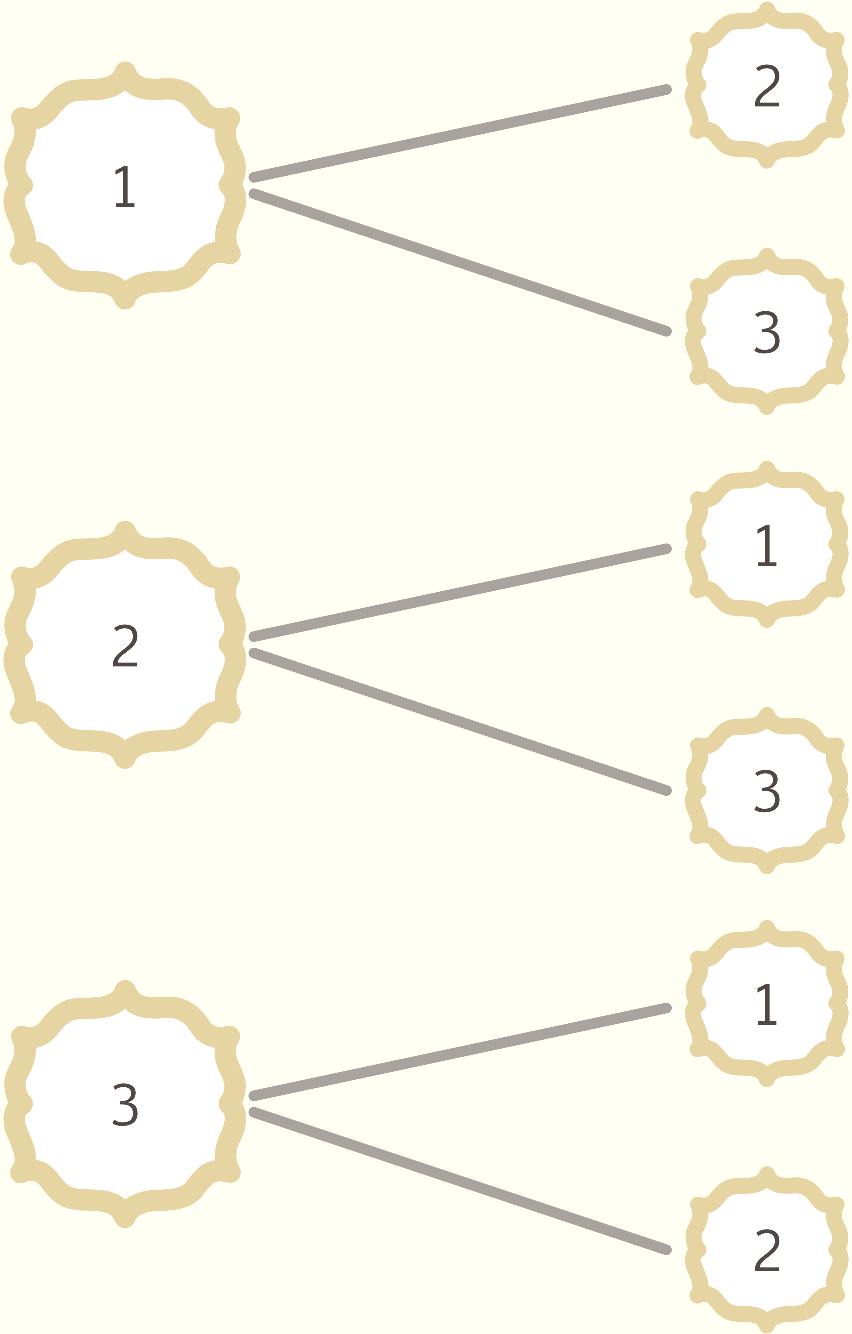
LARGE PLATES = 2

SWEET BITES = 3

3 choose 2

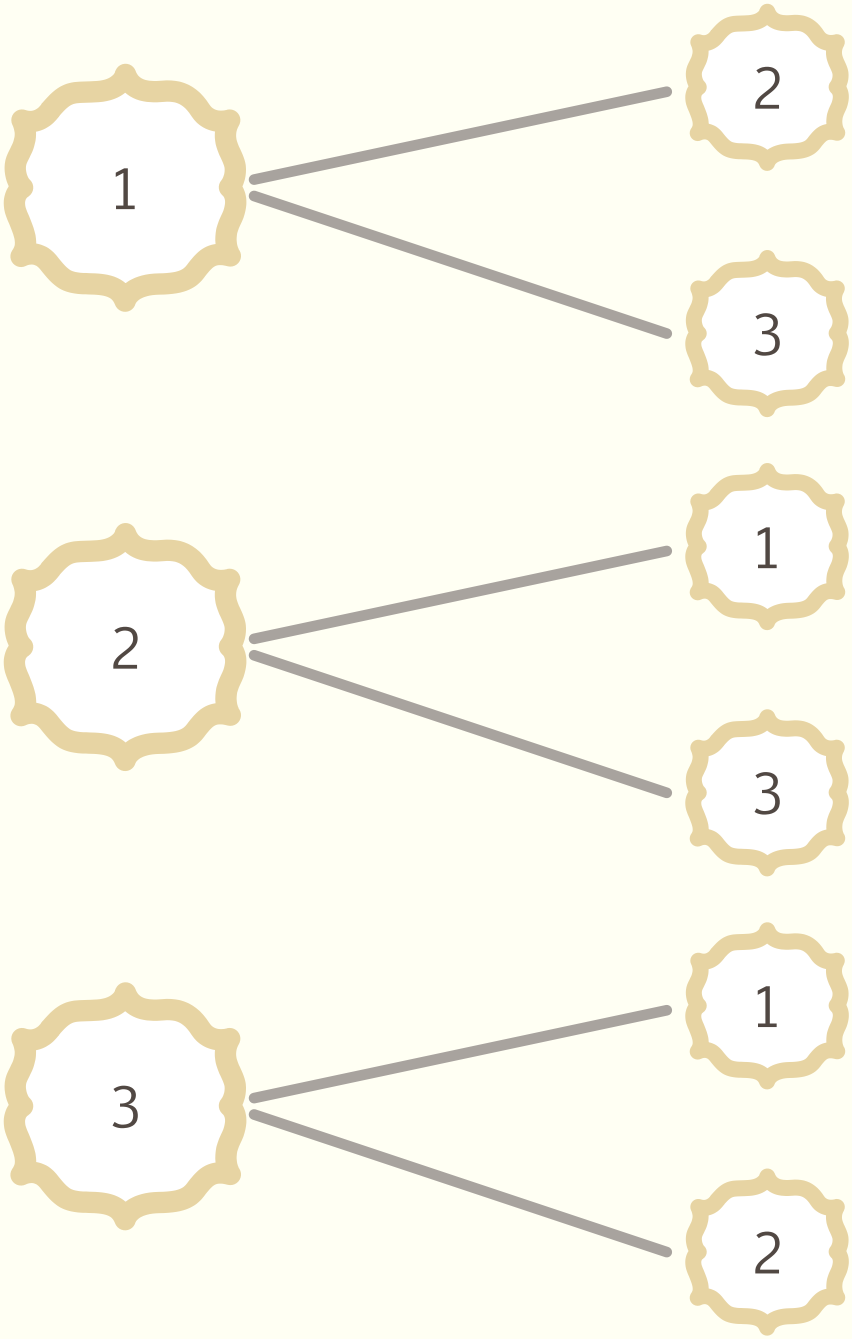
How many possible choices are there in total?

subset	subset
{sp, lp}	{1, 2}
{sp, sb}	{1, 3}
{lp, sb}	{2, 3}



3 choose 2

How many possible choices are there in total?



$(3)_2$

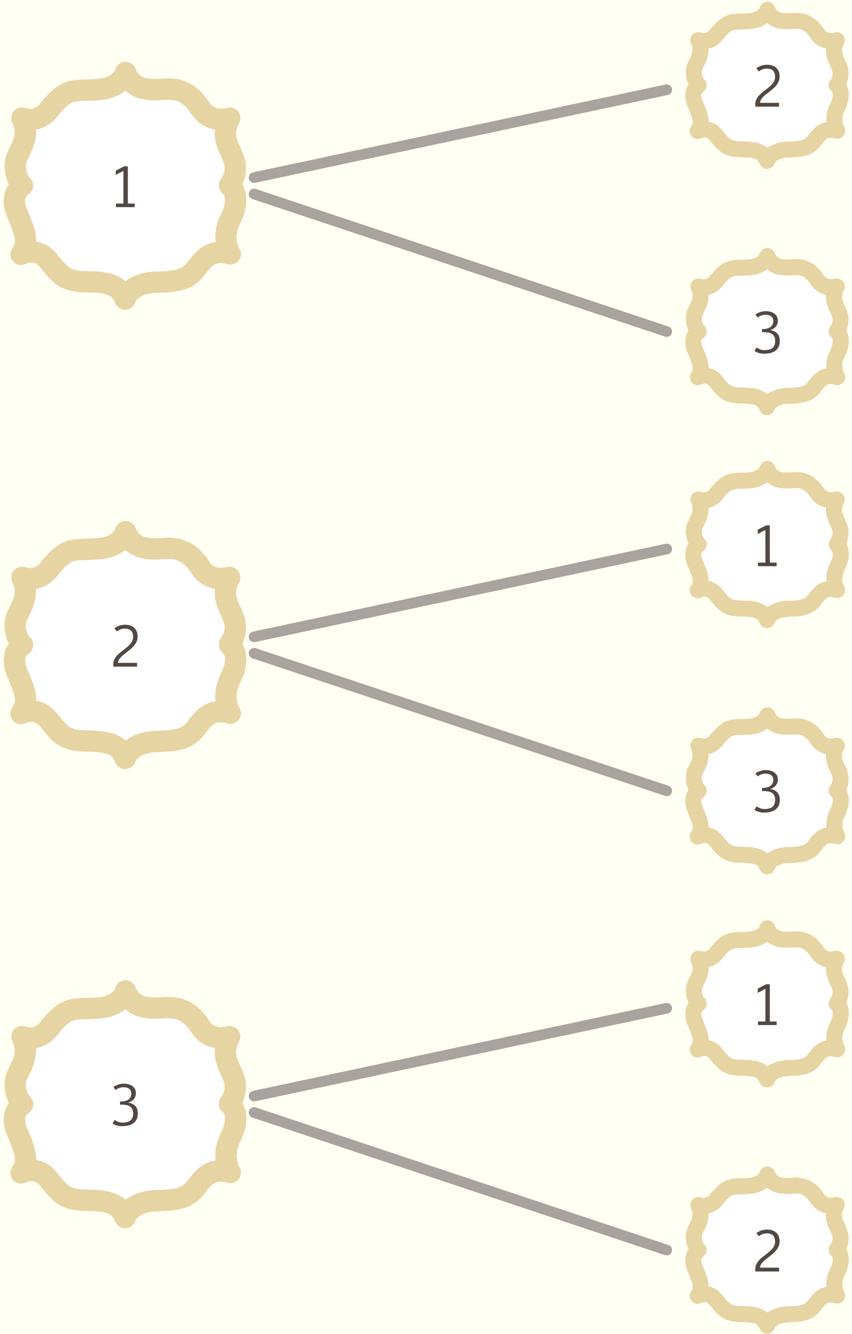
Do we care about the order?

?

The number of permutations of n elements is given by $n!$

!

$2!$



Do we care about the order? ?

The number of permutations of n elements is given by $n!$!

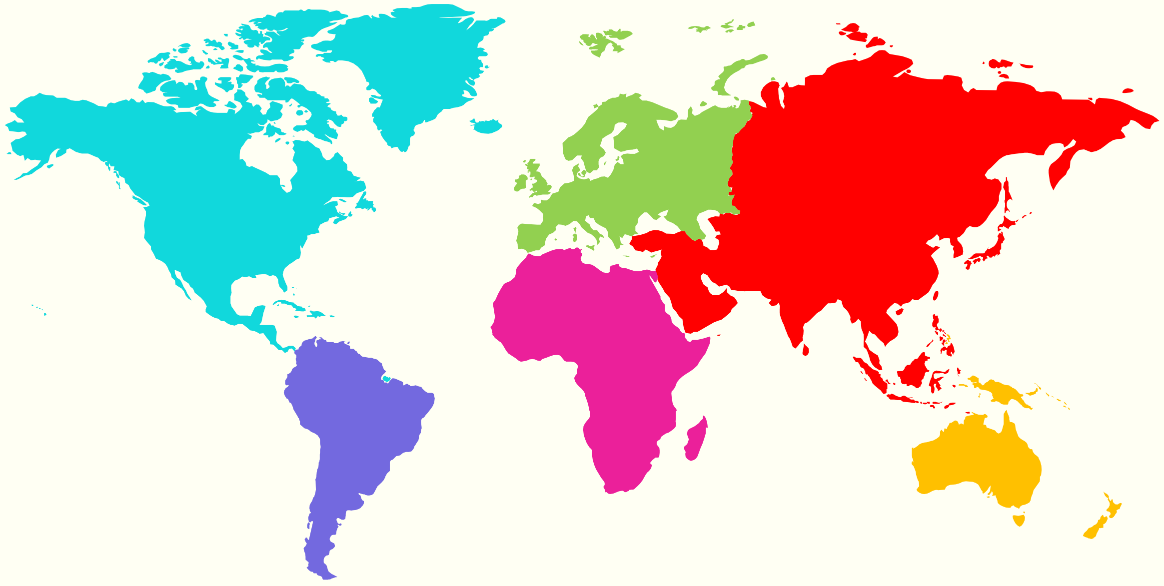
$$3 = \frac{3\times 2}{2\times 1} = \frac{(3)_2}{2!}$$!

$(3)_2$

$2!$

3 choose 2

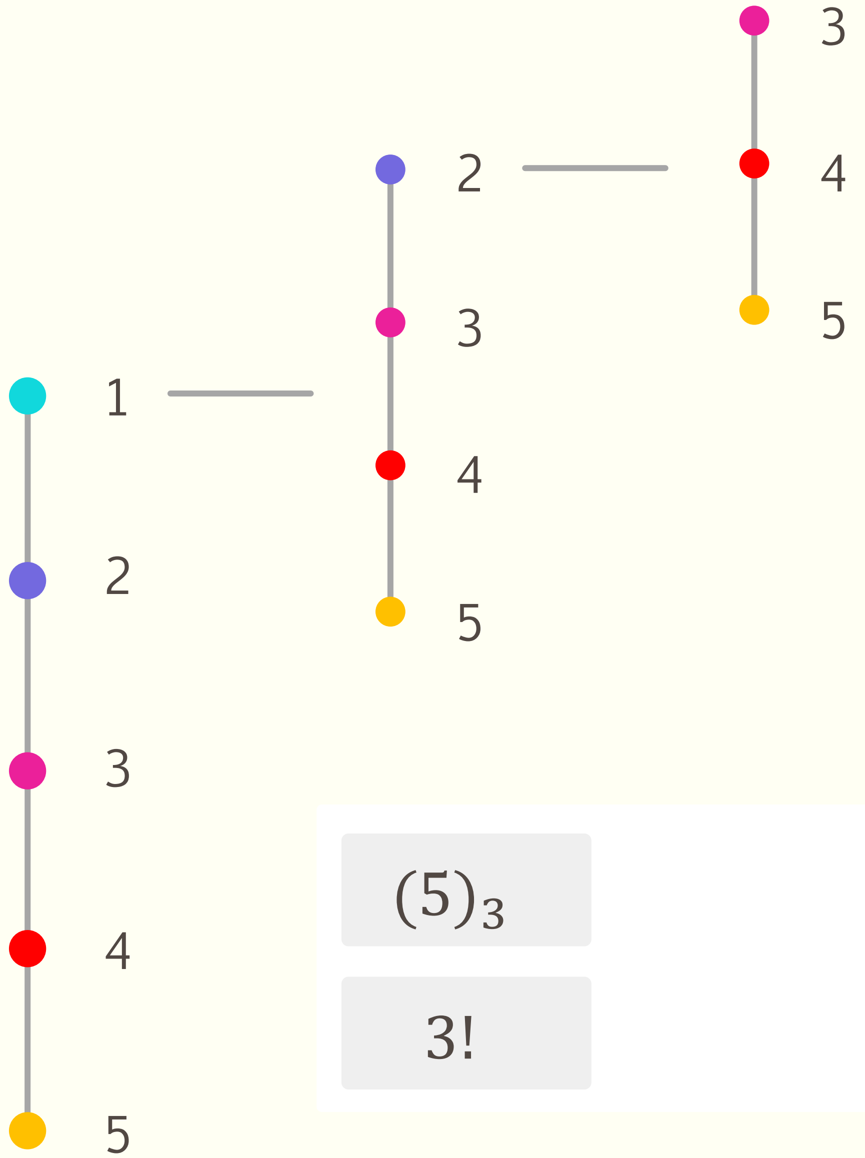
5 choose 3



- United States
- Brazil
- Egypt
- China
- Australia

How many possible choices are there in total?

5 choose 3



$(5)_3$

$3!$

Do we care about the order?

?

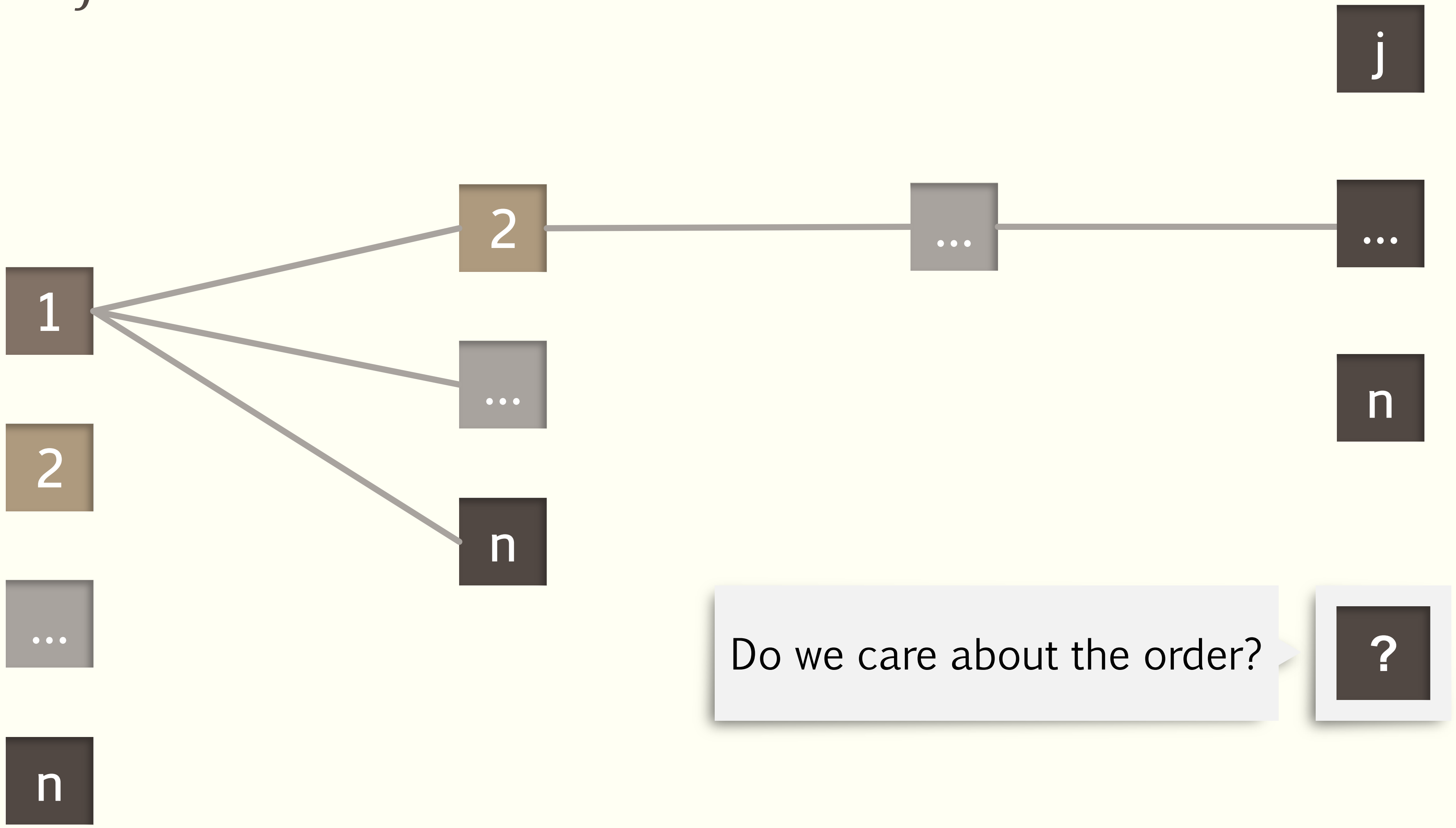
The number of permutations of n elements is given by $n!$

!

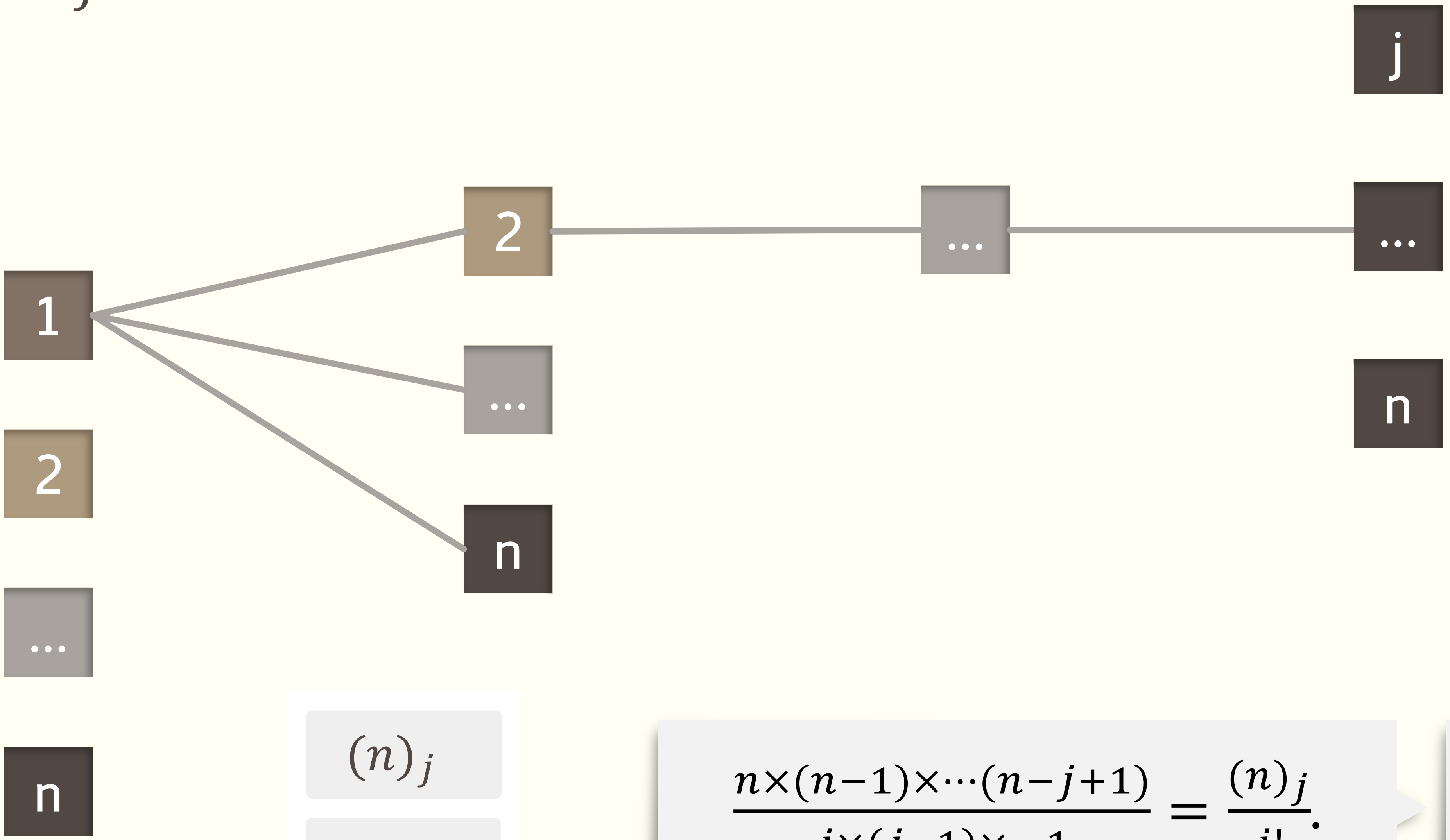
$$\frac{5\times4\times3}{3\times2\times1} = \frac{(5)_3}{3!} = 10.$$

=

n choose j



n choose j



$(n)_j$

j!

$$\frac{n\times(n-1)\times\cdots(n-j+1)}{j\times(j-1)\times\cdots 1}=\frac{(n)_j}{j!}.$$

!

Binomial Coefficients

- The number of distinct subsets with j elements that can be chosen from a set with n elements is denoted by $\binom{n}{j}$.
- The number $\binom{n}{j}$ is called a binomial coefficient.

$$\binom{n}{0} = \binom{n}{n} = 1.$$

1

$$\binom{n}{j} = \frac{(n)_j}{j!} = \frac{n!}{j!(n-j)!} = \binom{n}{n-j}.$$

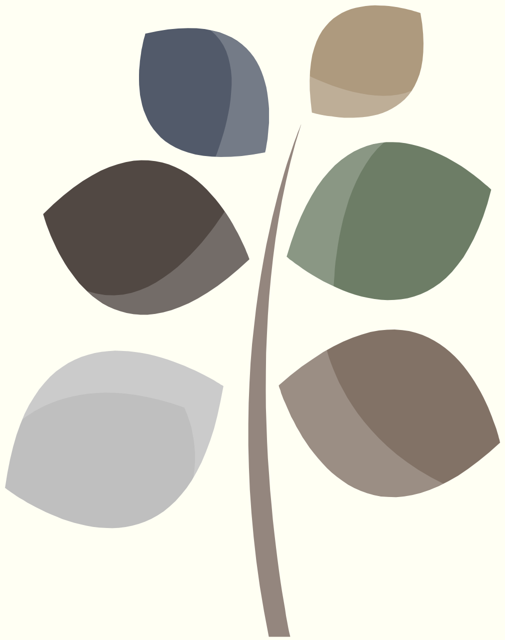
2

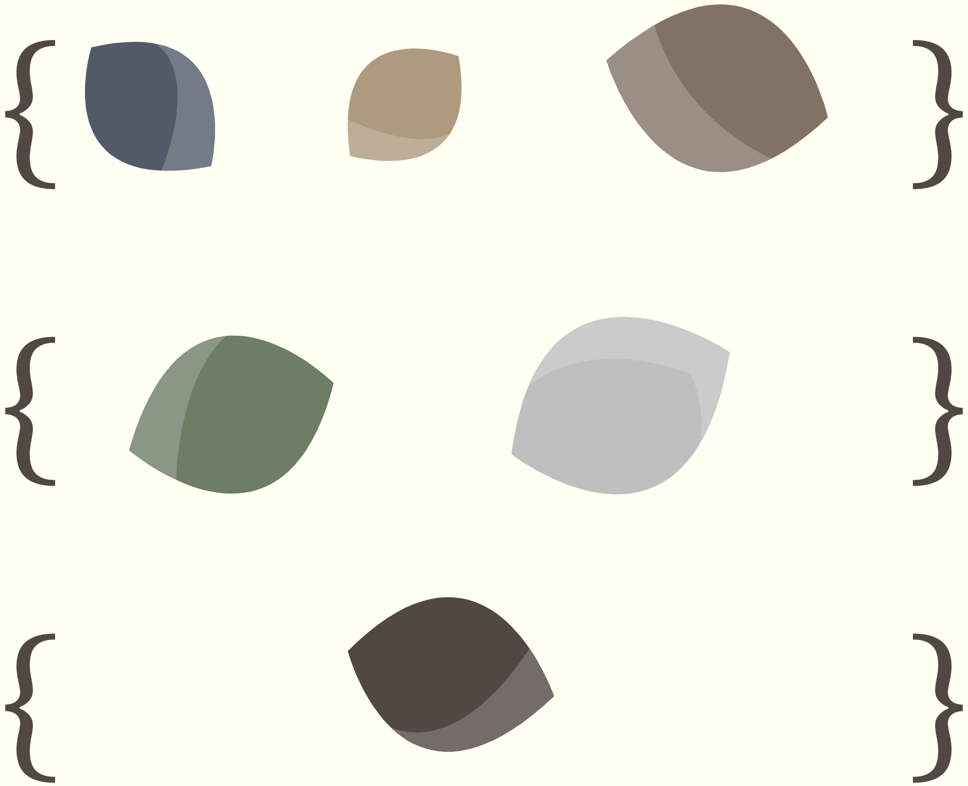
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

3

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

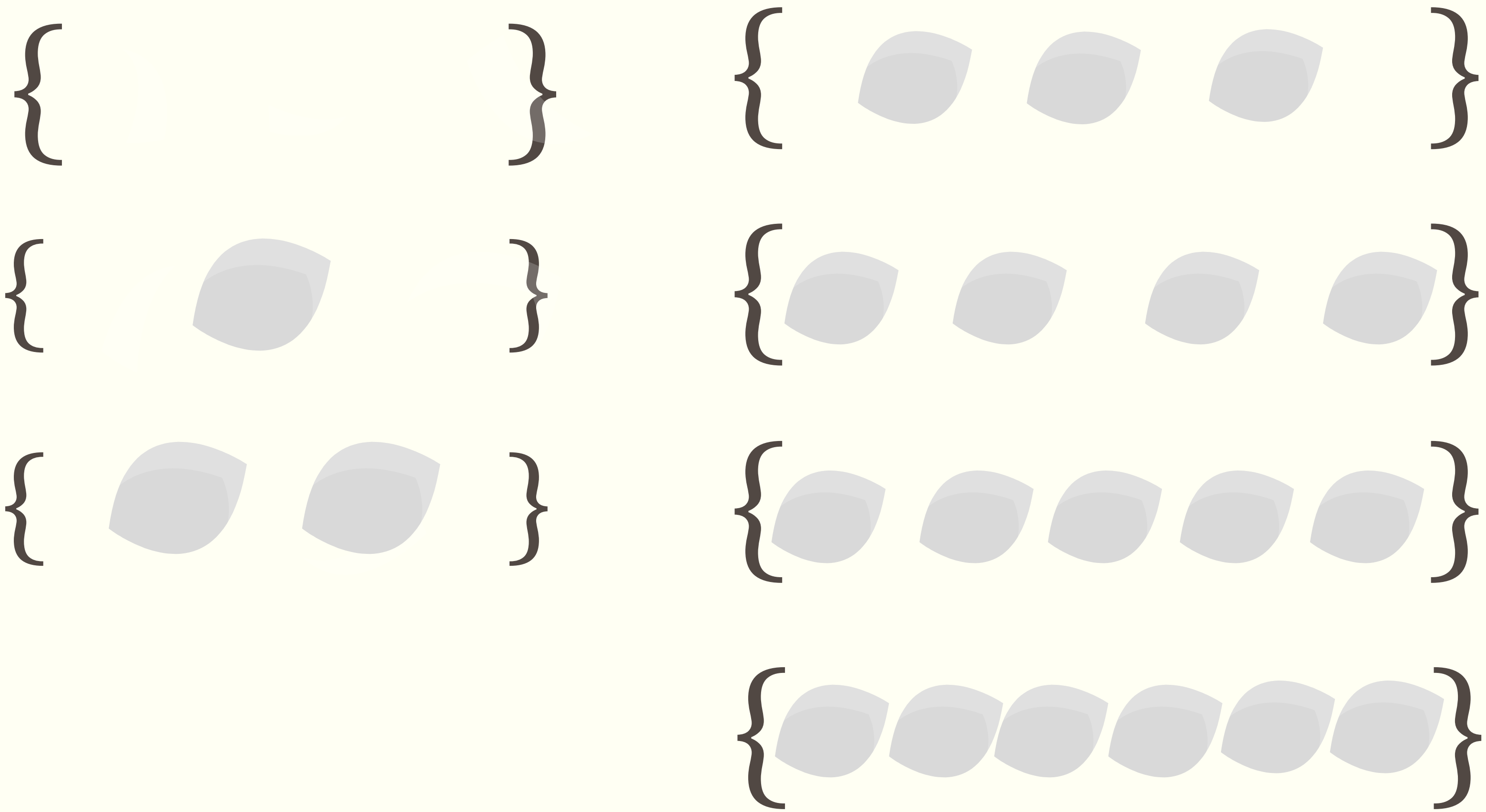
3





$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

3

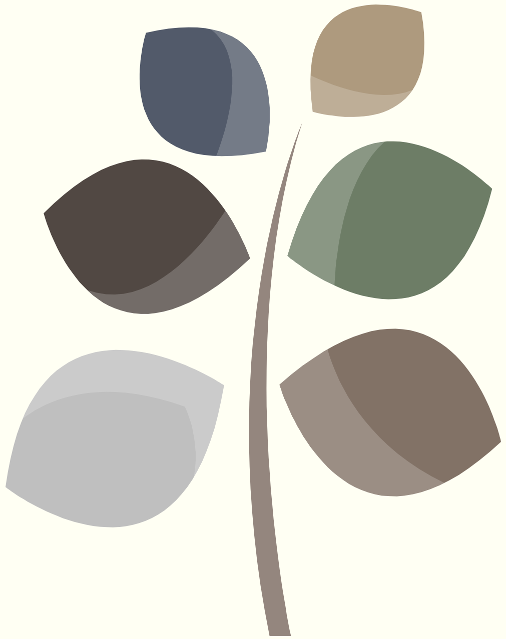


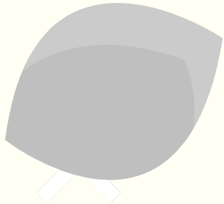
Recurrence Relation: Pascal's Triangle

For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4





do not include

include

Recurrence Relation: Pascal's Triangle

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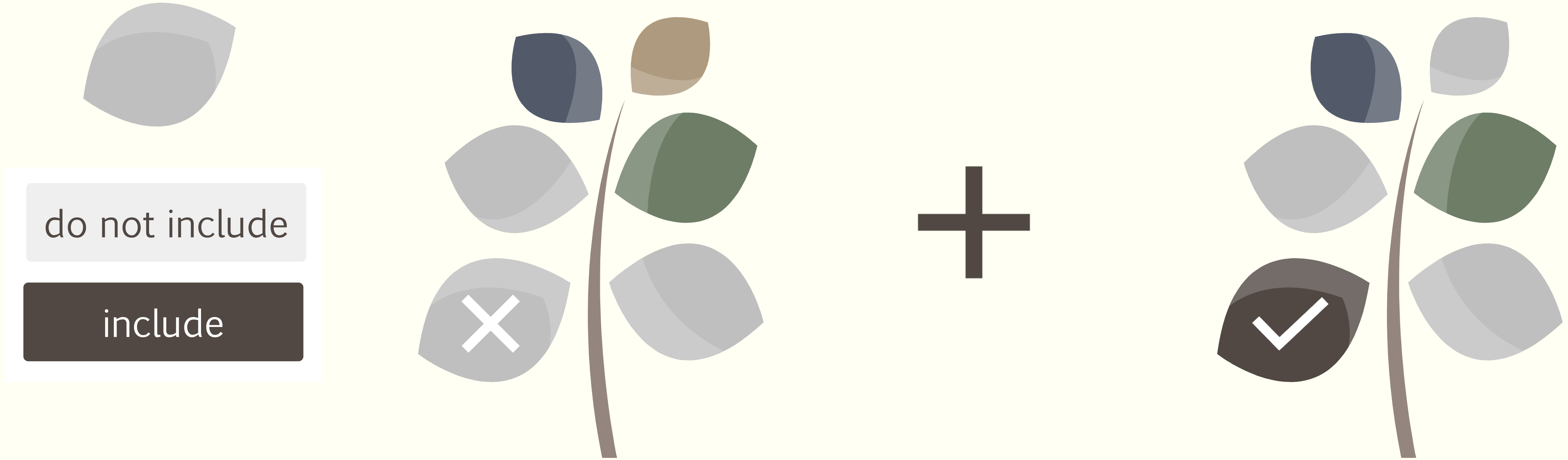


Recurrence Relation: Pascal's Triangle

For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4

- **Pascal's Triangle**

```
from scipy.special import comb
pascal(n = 10, j = 5)
```

```
# Pascal's Triangle
def pascal(n, j):
    n_choose_j = int(comb(n, j))
    sum_choose = int(comb(n-1, j) + comb(n-1, j-1))
    return n_choose_j, sum_choose
```

Pascal's Triangle

n/j	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

$$\binom{n}{0} = \binom{n}{n} = 1.$$

1

a column and the diagonal

$$\binom{n}{j} = \frac{(n)_j}{j!} = \frac{n!}{j!(n-j)!} = \binom{n}{n-j}.$$

2

symmetry in every row

Pascal's Triangle

n/j	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4

Pascal's Triangle

Example

Find integers n and r such that the following equation is true:

$$\binom{13}{5} + 2\binom{13}{6} + \binom{13}{7} = \binom{n}{r}.$$

=

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4

Find integers n and r such that the following equation is true:

$$\binom{13}{5} + 2\binom{13}{6} + \binom{13}{7} = \binom{n}{r}.$$

=

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4

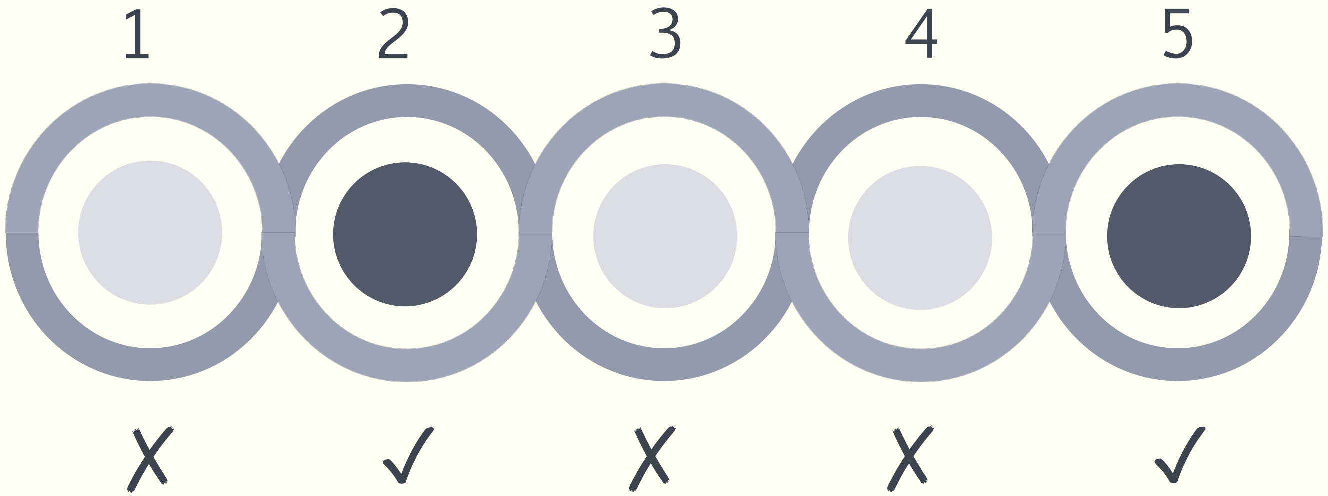
$$\binom{13}{6} + \binom{13}{5} = \binom{14}{6}.$$

$$\binom{13}{7} + \binom{13}{6} = \binom{14}{7}.$$

$$\binom{14}{7} + \binom{14}{6} = \binom{15}{7}.$$

TOSS A COIN

Toss a coin 5 times. What is the probability that there are 2 flips that land heads?

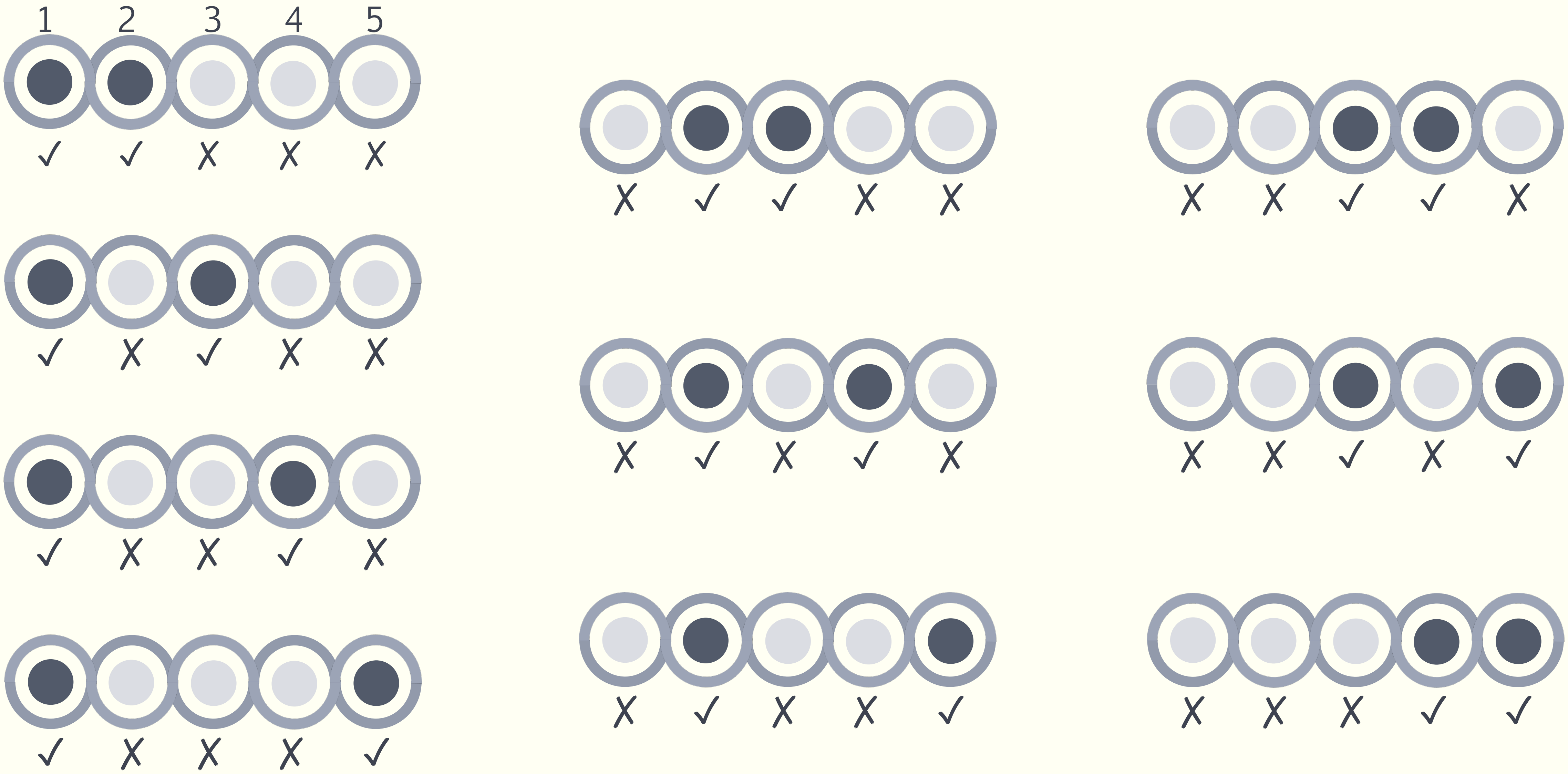


The 2nd and the 5th trials land heads:

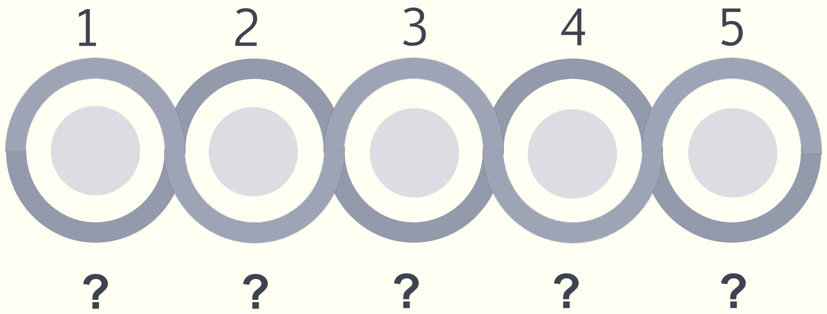
$$p = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

=

5 choose 2



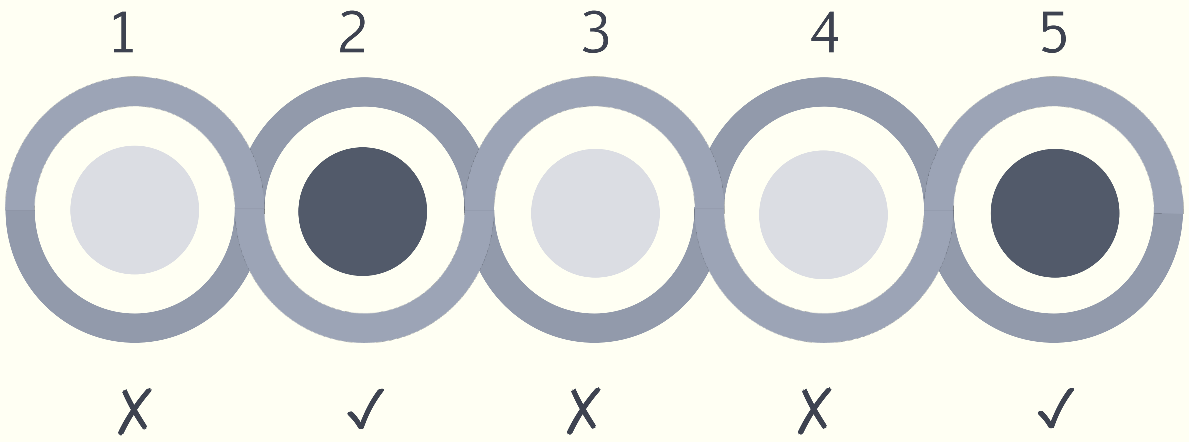
5 choose 2



How many possible choices?

$$\binom{5}{2} = \frac{(5)_2}{2!} = \frac{5\times 4}{2\times 1} = 10.$$

Probability that there are 2 flips that land heads



5 choose 2

!

2 heads & 3 tails

!

$$\binom{5}{2}\times\left(\frac{1}{2}\right)^5 = 10\times\frac{1}{32} = \frac{5}{16}.$$

×

Weather Forecast

Hanover
12:25 am

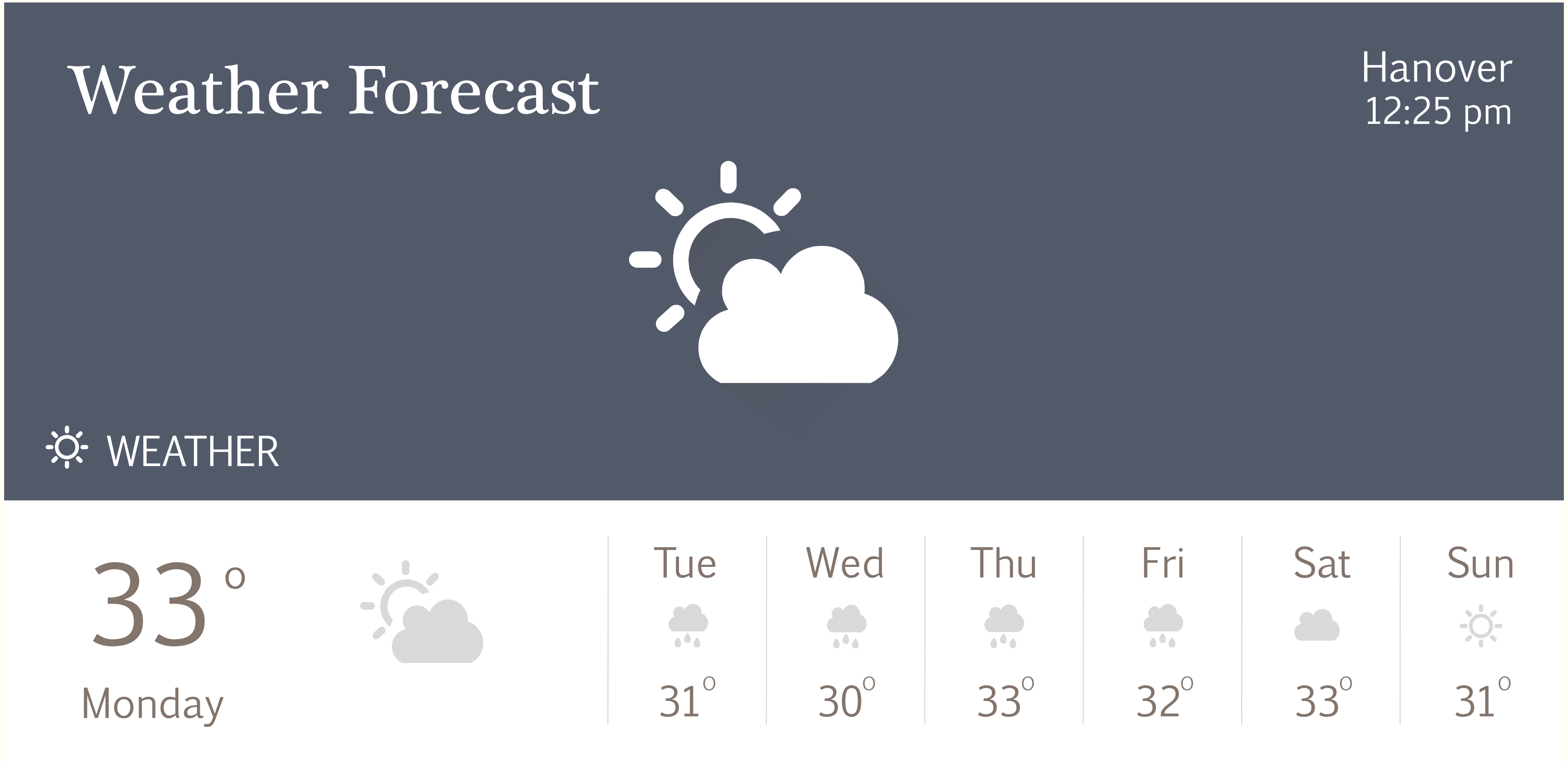
WEATHER

33°

Monday 16^{th}

4mph / 67°

Tue	Wed	Thu	Fri	Sat	Sun
31°	30°	33°	32°	33°	31°



- From Tuesday to Friday, it has 40% chance of raining everyday.
- What is the probability that on three of the four days it does not rain?

Weather Forecast

Hanover
12:25 pm

- From Tuesday to Friday, it has 40% chance of raining everyday.
- What is the probability that on three of the four days it does not rain?

4 choose 3 | !

3 not rainy & 1 rainy | !

$$\binom{4}{3}\times\left(\frac{3}{5}\right)^{3}\times\left(\frac{2}{5}\right)^{1}= 4\times\frac{54}{625} = \frac{216}{625}.$$

×

Weather Forecast

Hanover
12:25 pm

- From Tuesday to Friday, it has 40% chance of raining everyday.
- What is the probability that on three of the four days it does not rain?

How do we define success and failure?

?

not rainy

!

rainy

!

Weather Forecast

Hanover
12:25 pm

- What is the probability that on three of the four days it does not rain?
- What is the probability that on one of the four days it rains?

4 choose 3 | !

3 not rainy & 1 rainy | !

$\binom{4}{3}\times\left(\frac{3}{5}\right)^3\times\left(\frac{2}{5}\right)^1 = 4\times\frac{54}{625} = \frac{216}{625}.$

4 choose 1 | !

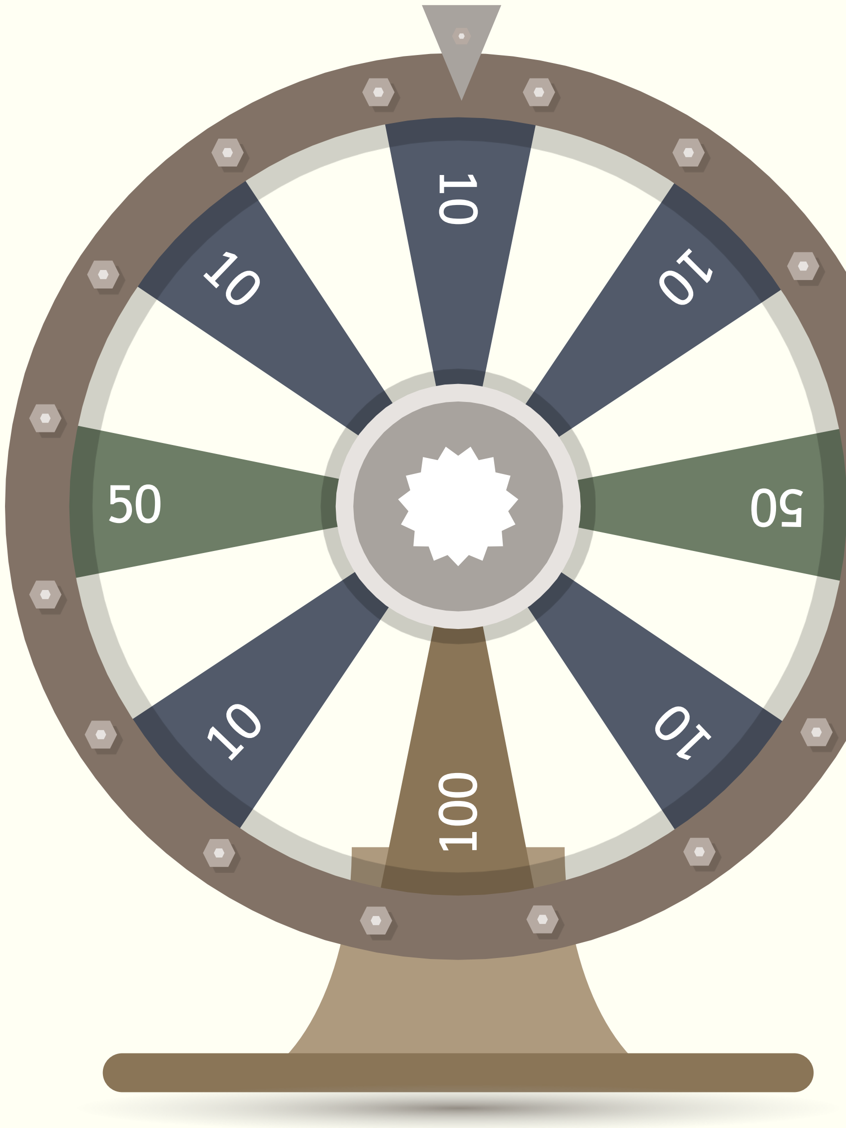
1 rainy & 3 not rainy | !

$\binom{4}{1}\times\left(\frac{2}{5}\right)^1\times\left(\frac{3}{5}\right)^3 = 4\times\frac{54}{625} = \frac{216}{625}.$

Wheel of Fortune

- Turn the wheel 10 times.
- What is the probability of getting 50 points twice?

Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16



Wheel of Fortune

- Turn the wheel 10 times.
- What is the probability of getting 50 points twice?

Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16

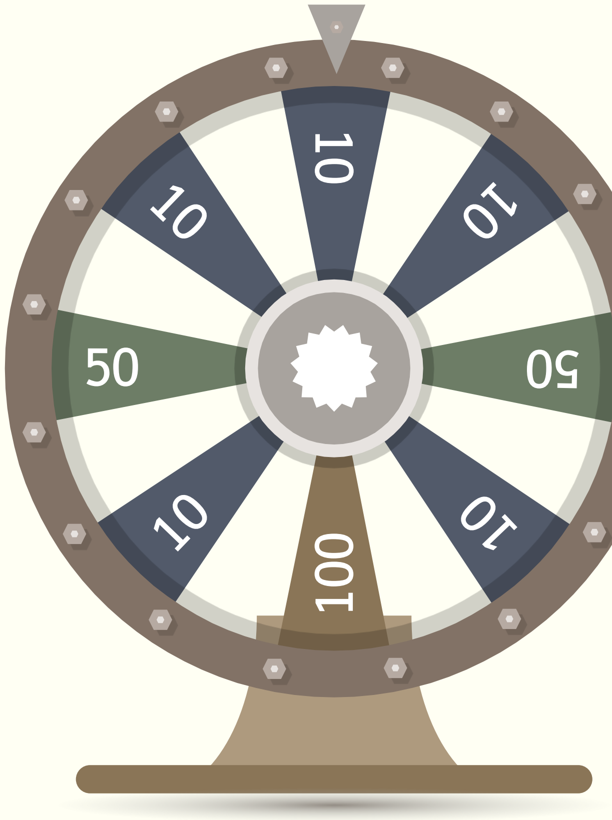
10 choose 2

!

2 50 points & 8 others

!

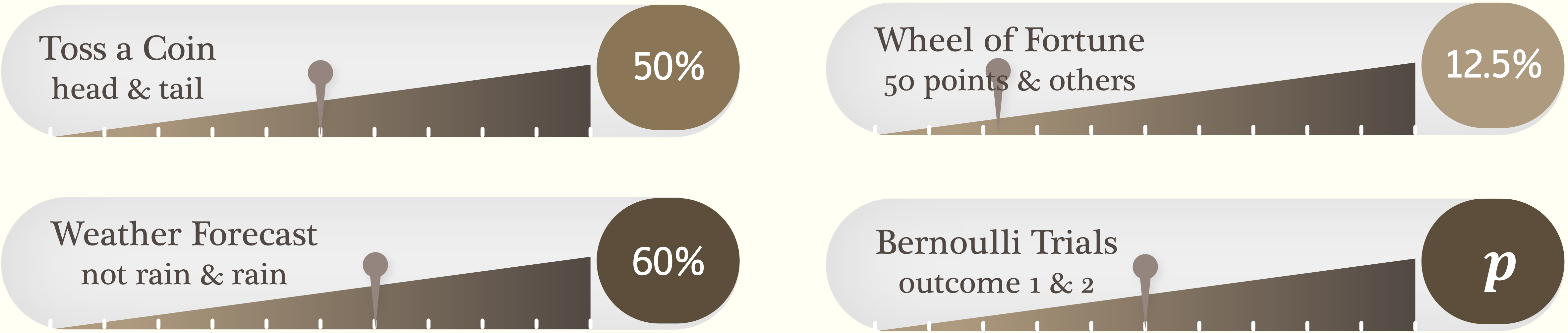
$$\binom{10}{2}\times\left(\frac{1}{8}\right)^2\left(\frac{7}{8}\right)^8 = 45\times\frac{7^8}{8^{10}}.$$



Bernoulli Trials

A **Bernoulli trials process** is a sequence of n chance experiments such that

- Each experiment has **two possible outcomes**, which we may call success and failure.
- The probability p of success on each experiment is the same for each experiment, and this probability is not affected by any knowledge of previous outcomes. The probability q of failure is given by $q = 1 - p$.



Bernoulli Probabilities

$B(n,p,j)$, the probability that in n Bernoulli trials there are exactly j successes. We have:

$$B(n,p,j) = \binom{n}{j} p^j q^{n-j}.$$

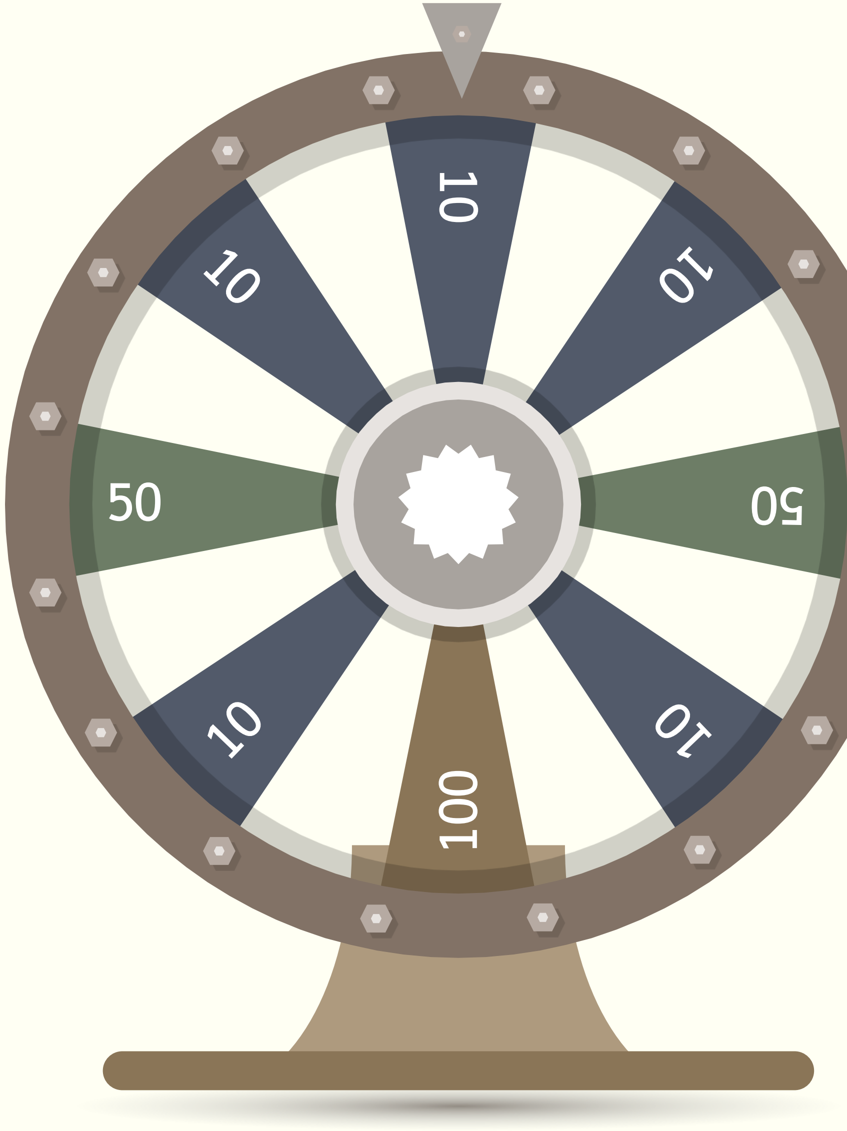
=

Bernoulli Trials	n	p	j	Probability
Toss a Coin	5	50%	2	$\binom{5}{2} \times (\frac{1}{2})^5$
Weather Forecast	4	60%	3	$\binom{4}{3} \times (\frac{3}{5})^3 \times (\frac{2}{5})^1$
Wheel Fortune	10	12.5%	2	$\binom{10}{2} \times (\frac{1}{8})^2 (\frac{7}{8})^8$

Wheel of Fortune revisit

- Turn the wheel 5 times.
- What is the probability of getting 100 points in total?

Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16



Wheel of Fortune revisit

- Turn the wheel 5 times.
- What is the probability of getting 100 points in total?

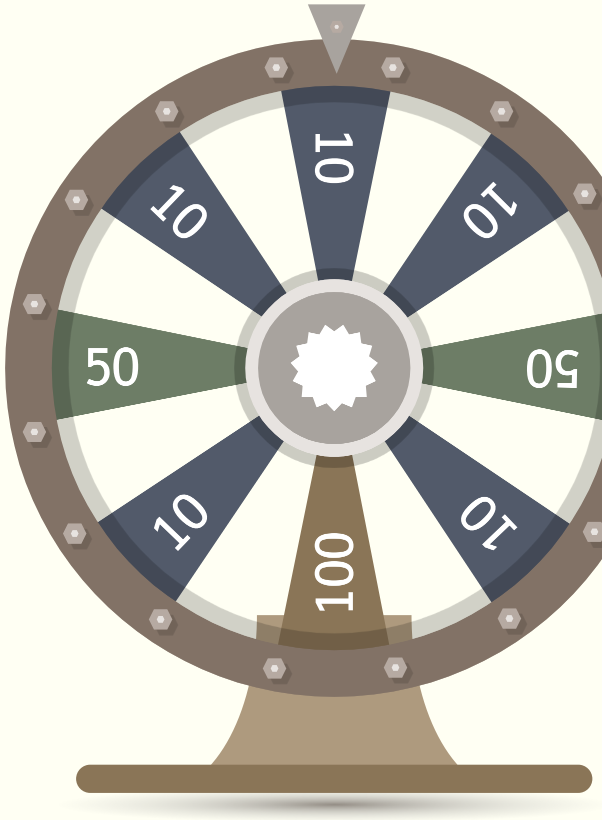
Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16

$$50\times 2 + 0\times 3 = 100$$

A

$$100\times 1 + 0\times 4 = 100$$

B



Wheel of Fortune revisit

Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16

$50\times2 + 0\times3 = 100$ **A**

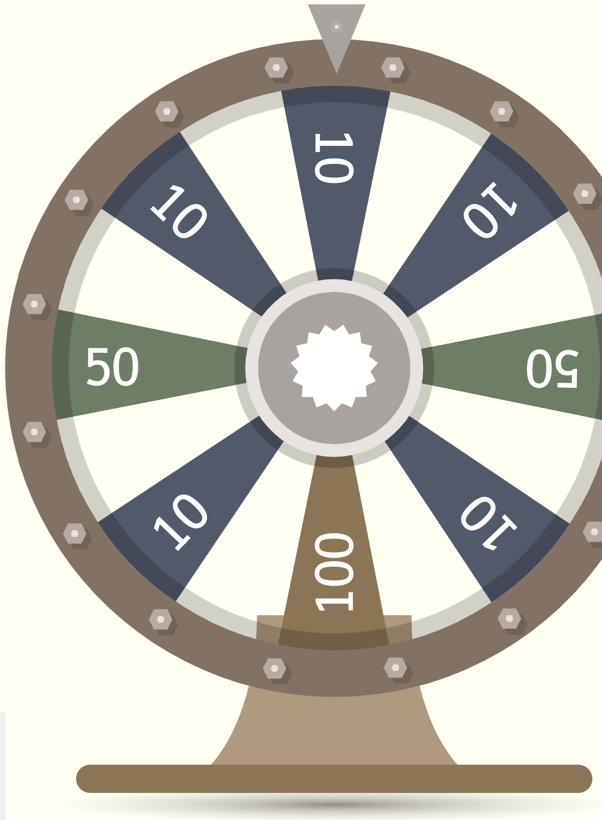
$100\times1 + 0\times4 = 100$ **B**

$\binom{5}{2}\times(\frac{1}{8})^2(\frac{1}{2})^3$ p_A

$\binom{5}{1}\times(\frac{1}{16})^1(\frac{1}{2})^4$ p_B

$+$

$p = p_A + p_B$



Binomial Distributions

- Let n be a positive integer and let p be a real number between 0 and 1.
- Let B be the random variable which counts the number of successes in a Bernoulli trials process with parameters n and p.
- Then the distribution $b(n, p, k)$ of B is called the binomial distribution.

$$\binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots + \binom{n}{n} p^n q^0 = \cdots$$

?

Binomial Expansion

Binomial Theorem

The quantity $(a+b)^n$ can be expressed in the form:

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

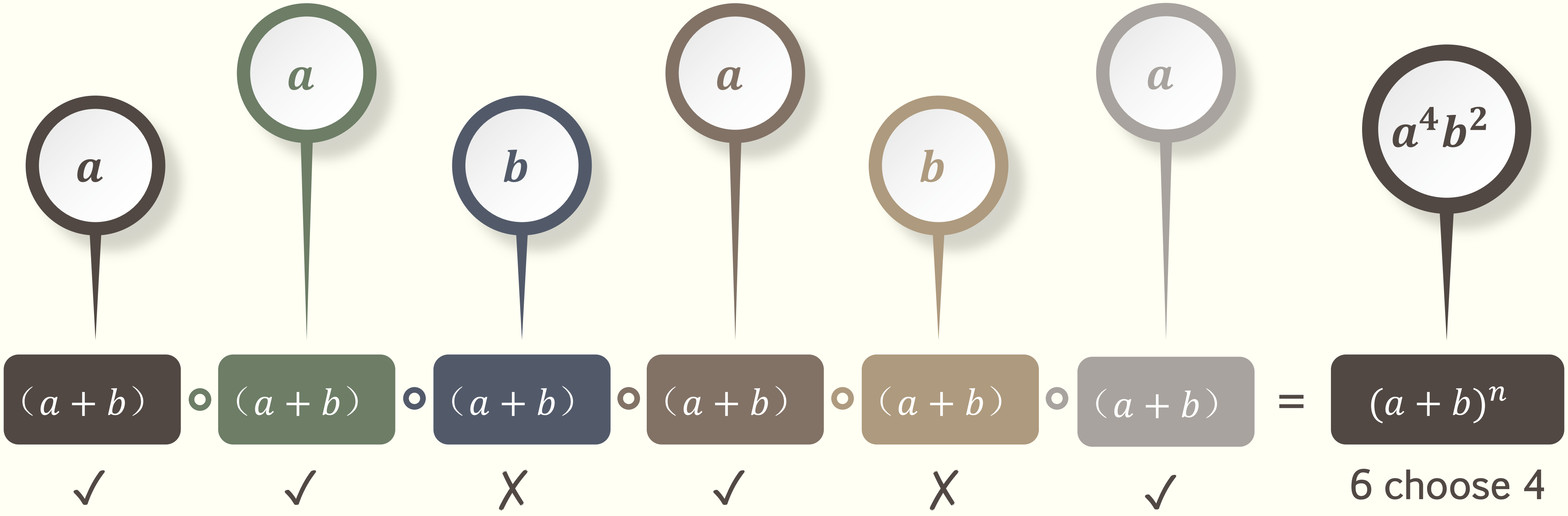
=

$$(p+q)^n = \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots + \binom{n}{n} p^n q^0 = 1.$$

=

Binomial Theorem

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$



Binomial Theorem

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

=

Let $n = 2$, we have

$$(a+b)^2 = \binom{2}{0} b^2 + \binom{2}{1} ab + \binom{2}{2} a^2 = a^2 + 2ab + b^2.$$

=

Let $n = 3$, we have

$$(a+b)^3 = \binom{3}{0} b^3 + \binom{3}{1} ab^2 + \binom{3}{2} a^2 b + \binom{3}{3} a^3 = a^3 + 3a^2 b + 3ab^2 + b^3.$$

=

Binomial Theorem

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

=

Let $a = b = 1$, we have

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

=

Let $a = -1$, $b = 1$, we have

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}.$$

=

Inclusion-Exclusion Principle

- Let P be a probability measure on a sample space Ω, and let $\{A_1, A_2, \cdots, A_n\}$ be a finite set of events. Then

$$P(A_1 \cup A_2 \cup \cdots A_n) =$$

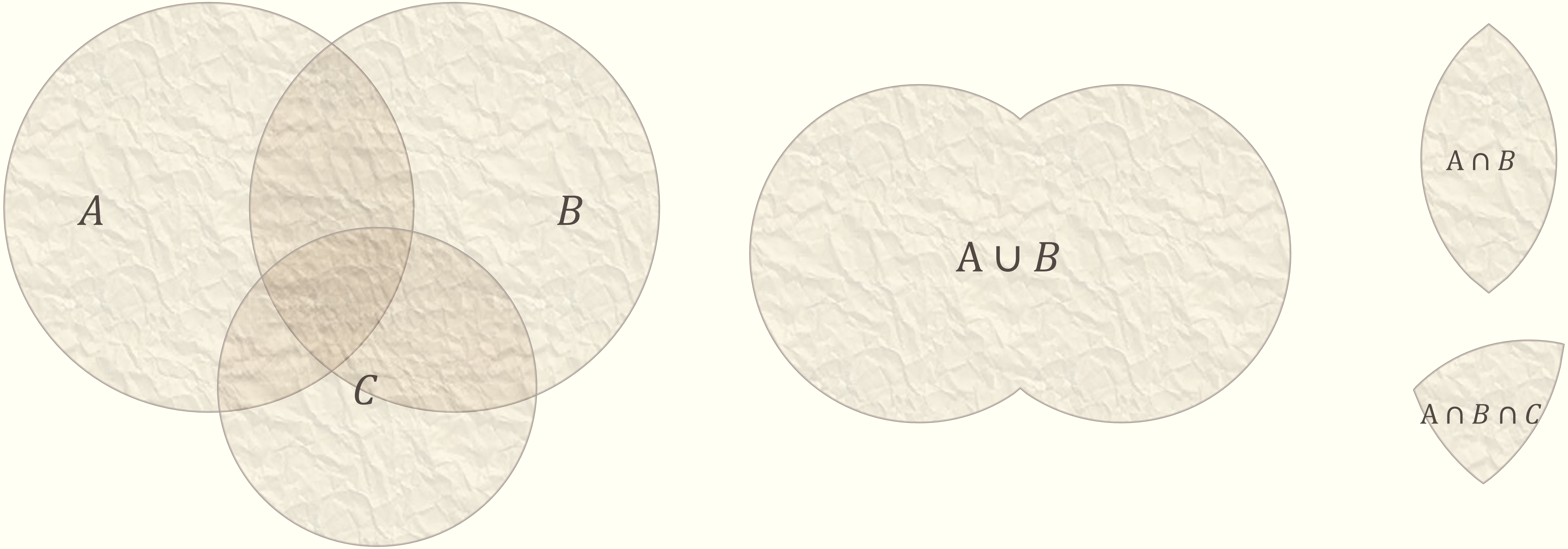
$$\sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots.$$

- If A and B are subsets of Ω, then

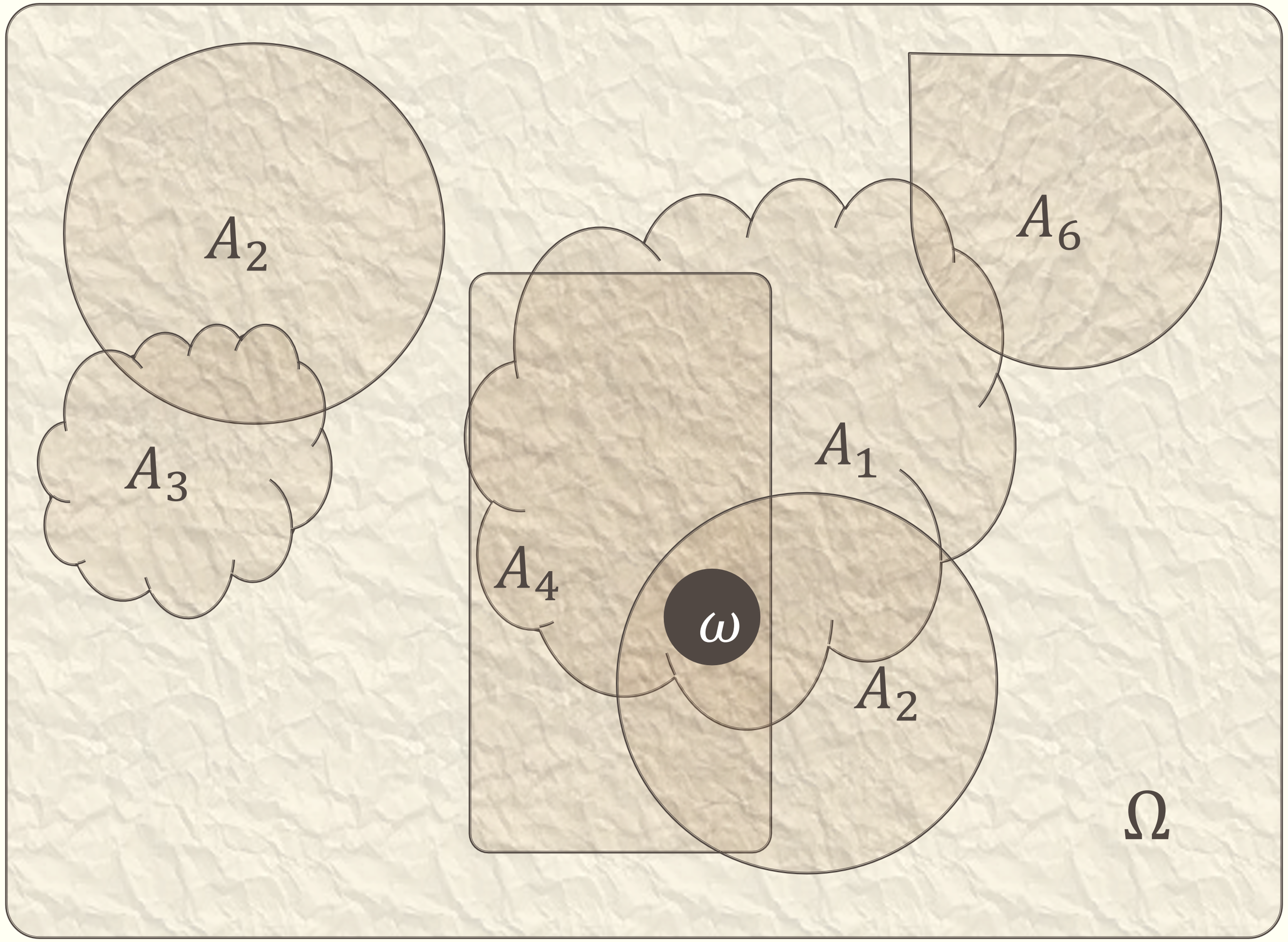
$$P(\mathrm{A} \cup B) = P(A) + P(B) - P(\mathrm{A} \cap B).$$

- If A, B and C are subsets of Ω, then

$$P(\mathrm{A} \cup B \cup C) = P(A) + P(B) + P(C) - P(\mathrm{A} \cap B) - \mathrm{P}(\mathrm{B} \cap C) - \mathrm{P}(\mathrm{A} \cap C) + \mathrm{P}(\mathrm{A} \cap B \cap C).$$

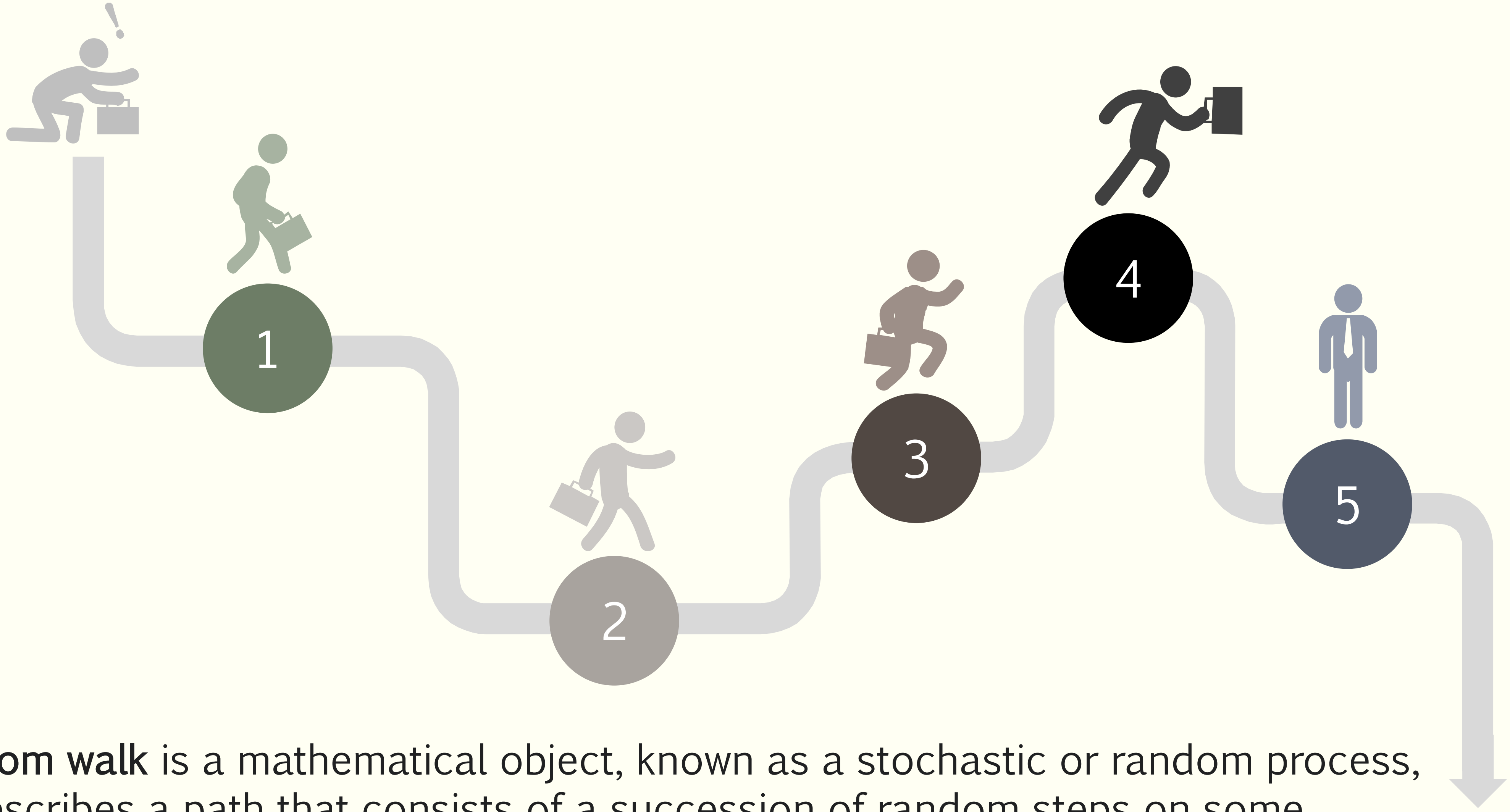


Proof



- If the outcome ω occurs in at least one of the events A_i, its probability is added exactly once by the left side.
- We must show that it is also added exactly once by the right side.
- Assume that ω is in exactly k of the sets. Then its probability is added k times in the first term, subtracted $\binom{k}{2}$ times in the second, added $\binom{k}{3}$ times in the third term, and so forth.

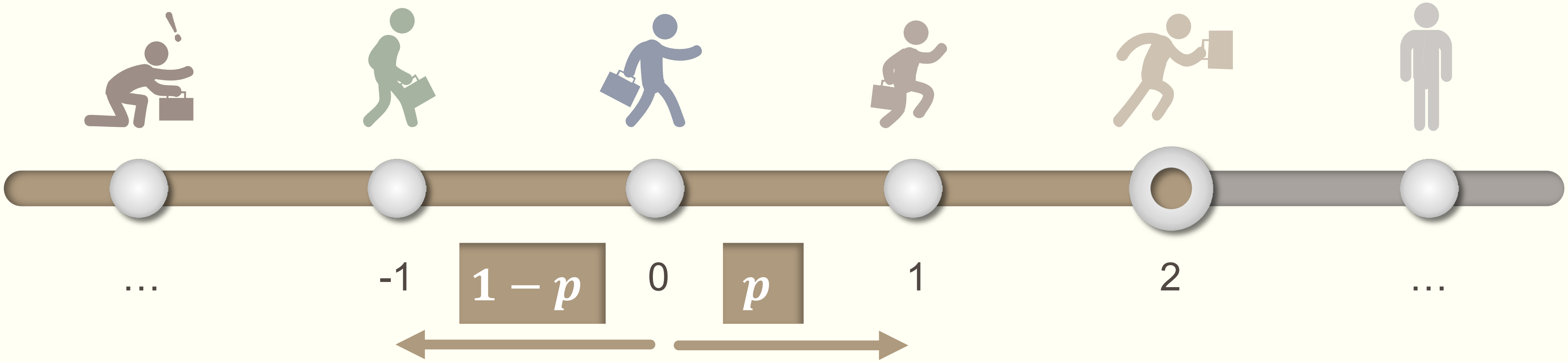
Random Walk



A **random walk** is a mathematical object, known as a stochastic or random process, that describes a path that consists of a succession of random steps on some mathematical space such as the integers.

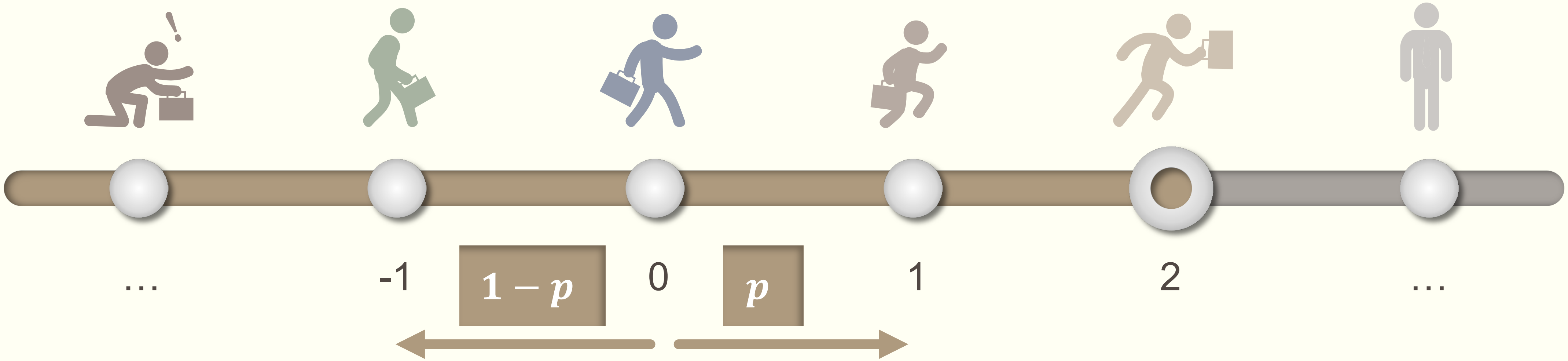
…

Random Walk (1 dimensional)



- An elementary example of a random walk is the random walk on the integer number line.
- It starts at 0 and at each step moves +1 or −1 with probability p and $1-p$.

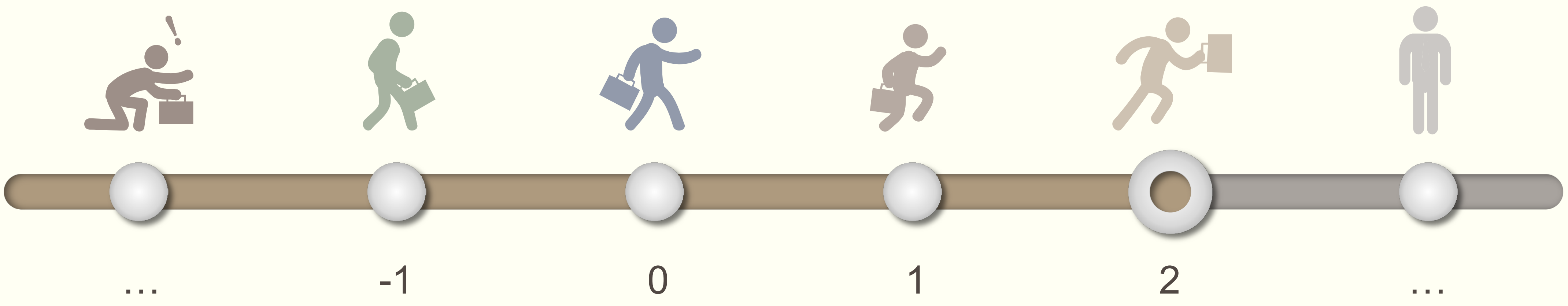
Random Walk (1 d)



After 10 steps, what is the probability of landing on

- 0
- 1
- 2
- -1
- -2
- …

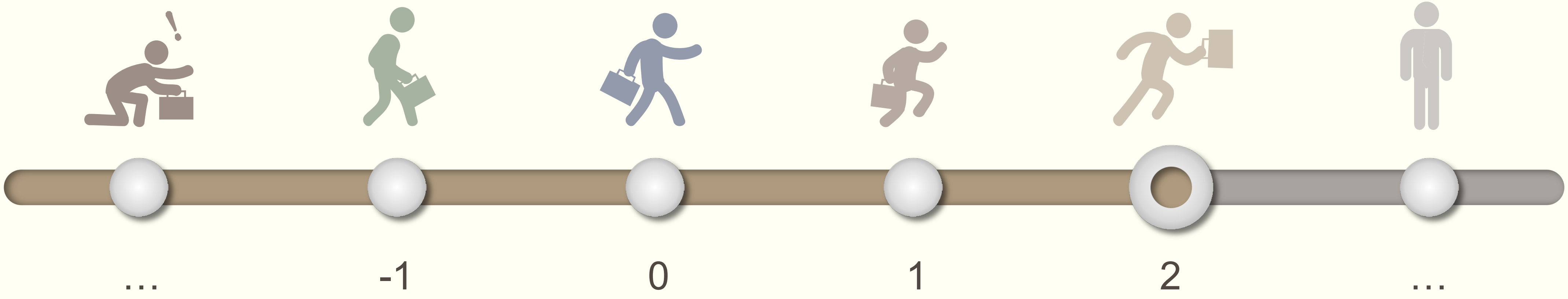
Random Walk



coordinate	...	-1	0	1	2	...
left	...		5		4	
right	...		5		6	
probability	...	0		0		...

$\binom{10}{5}\times p^5(1-p)^5$ **0**

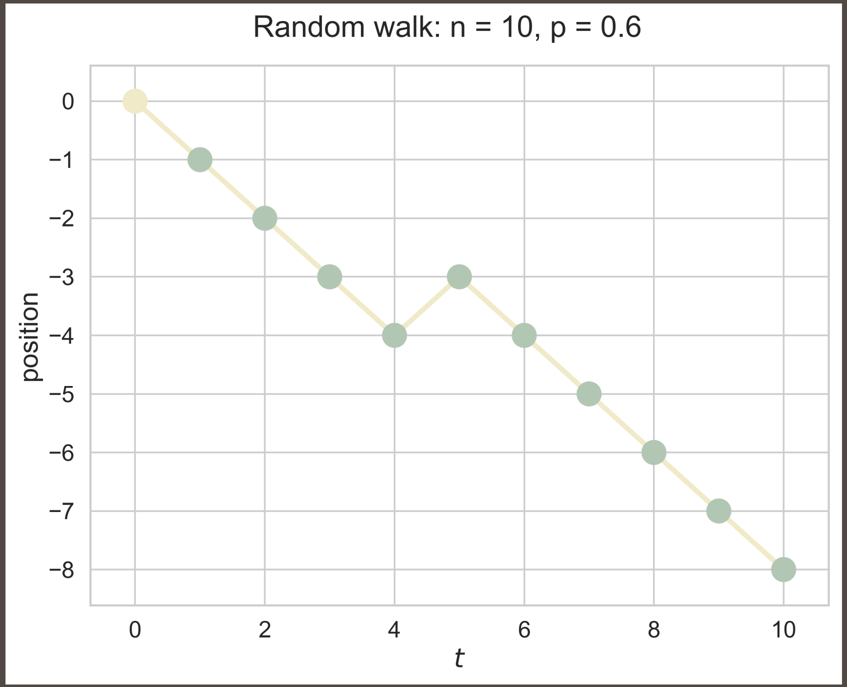
$\binom{10}{6}\times p^6(1-p)^4$ **2**

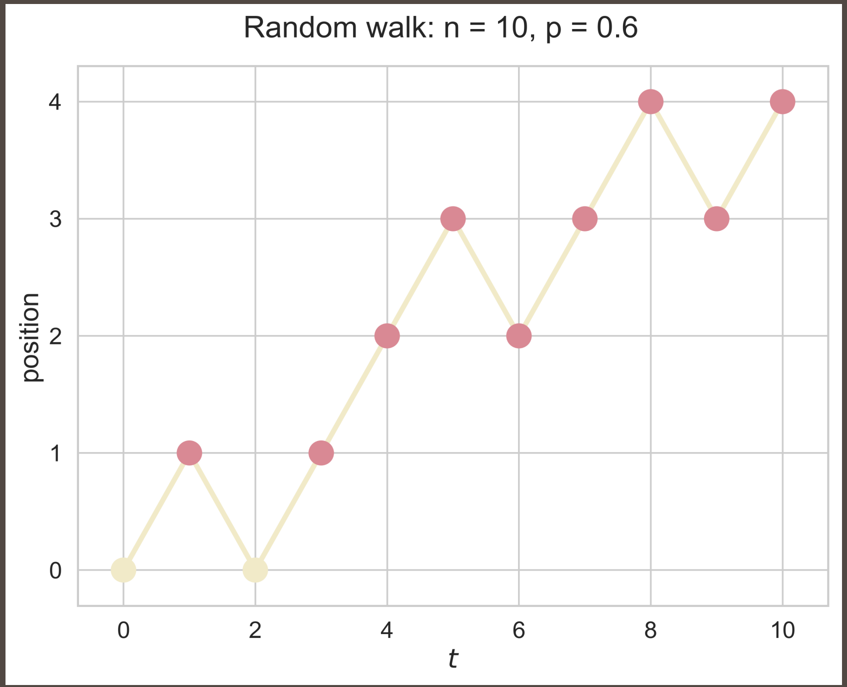


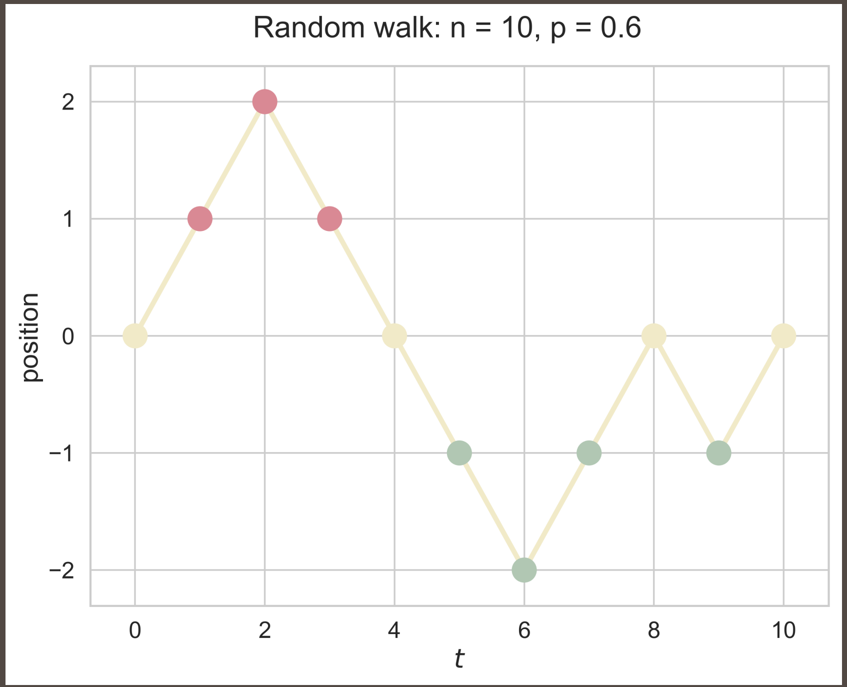
- **One Dimensional Random Walk**

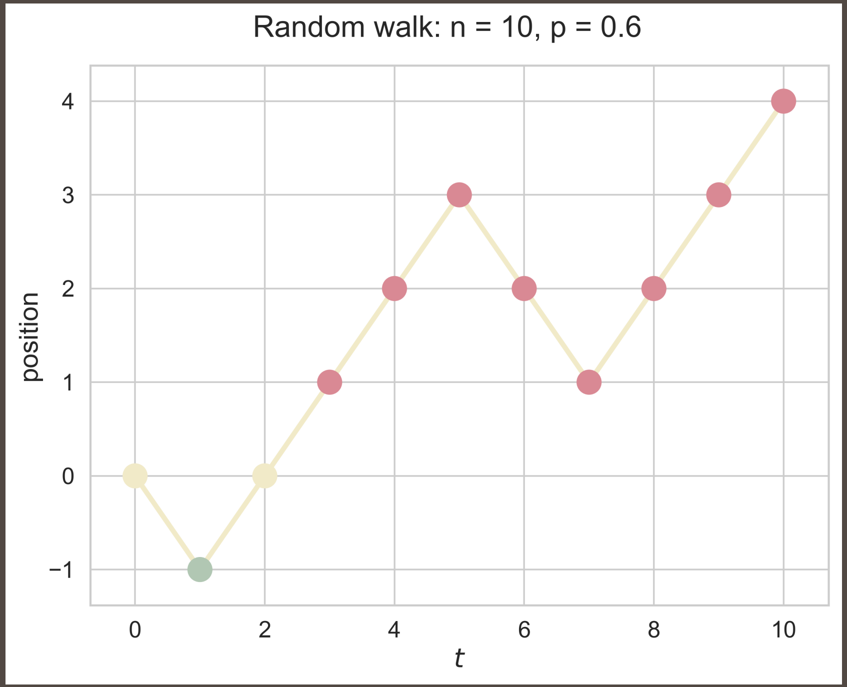
```
random_walk_1D(n = 10, p = 0.6)
path_rw_2D(n = 10, p = 0.6, fsize = (8, 6), fs = 18, index = 1)
```

```
# Take n steps
for i in range(n):
    # Generate a random number between 0 and 1
    u = random.uniform(0, 1)
    if u <= p:
        pos += 1 # go right
    else:
        pos -= 1 # go left
    positions.append(pos)
return positions
```

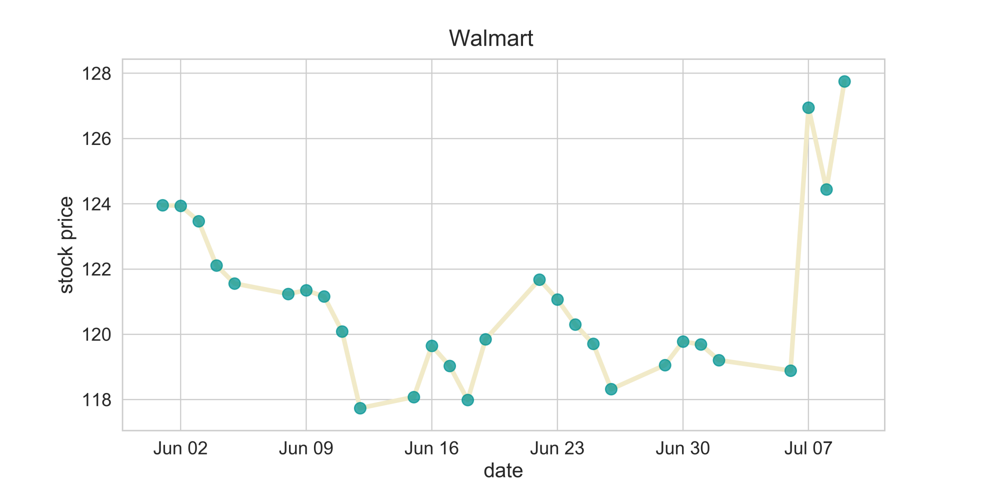

Random walk: n = 10, p = 0.6
position
t
0
−1
−2
−3
−4
−5
−6
−7
−8
0
2
4
6
8
10

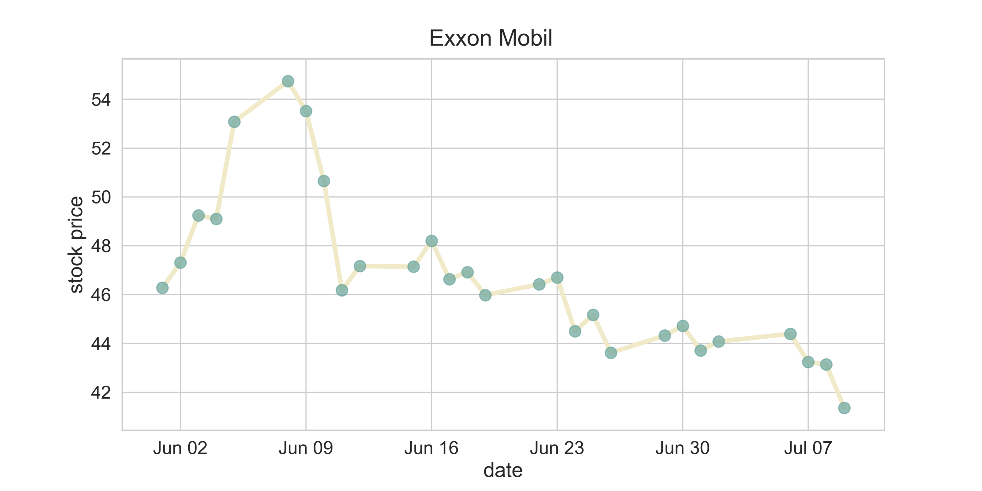

Random walk: n = 10, p = 0.6
position
t
4
3
2
1
0
0
2
4
6
8
10

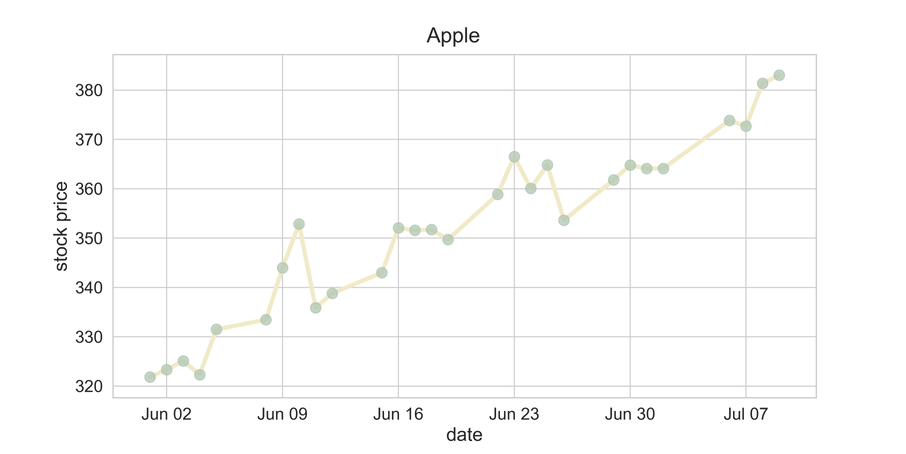

Random walk: n = 10, p = 0.6
position
t
2
1
0
−1
−2
0
2
4
6
8
10

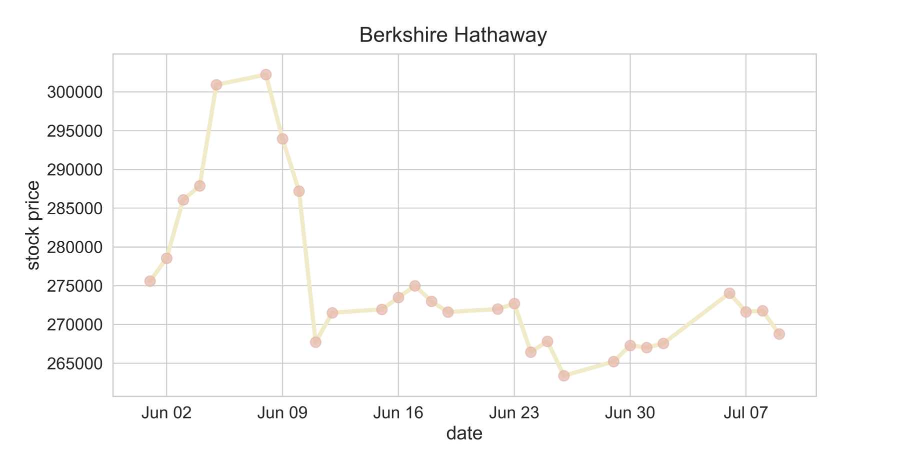

Random walk: n = 10, p = 0.6
position
t
4
3
2
1
0
−1
0
2
4
6
8
10

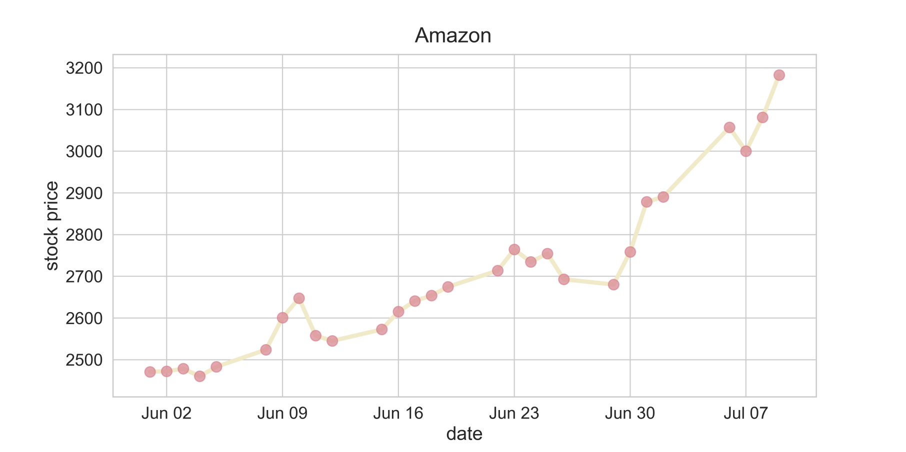
Stock Market



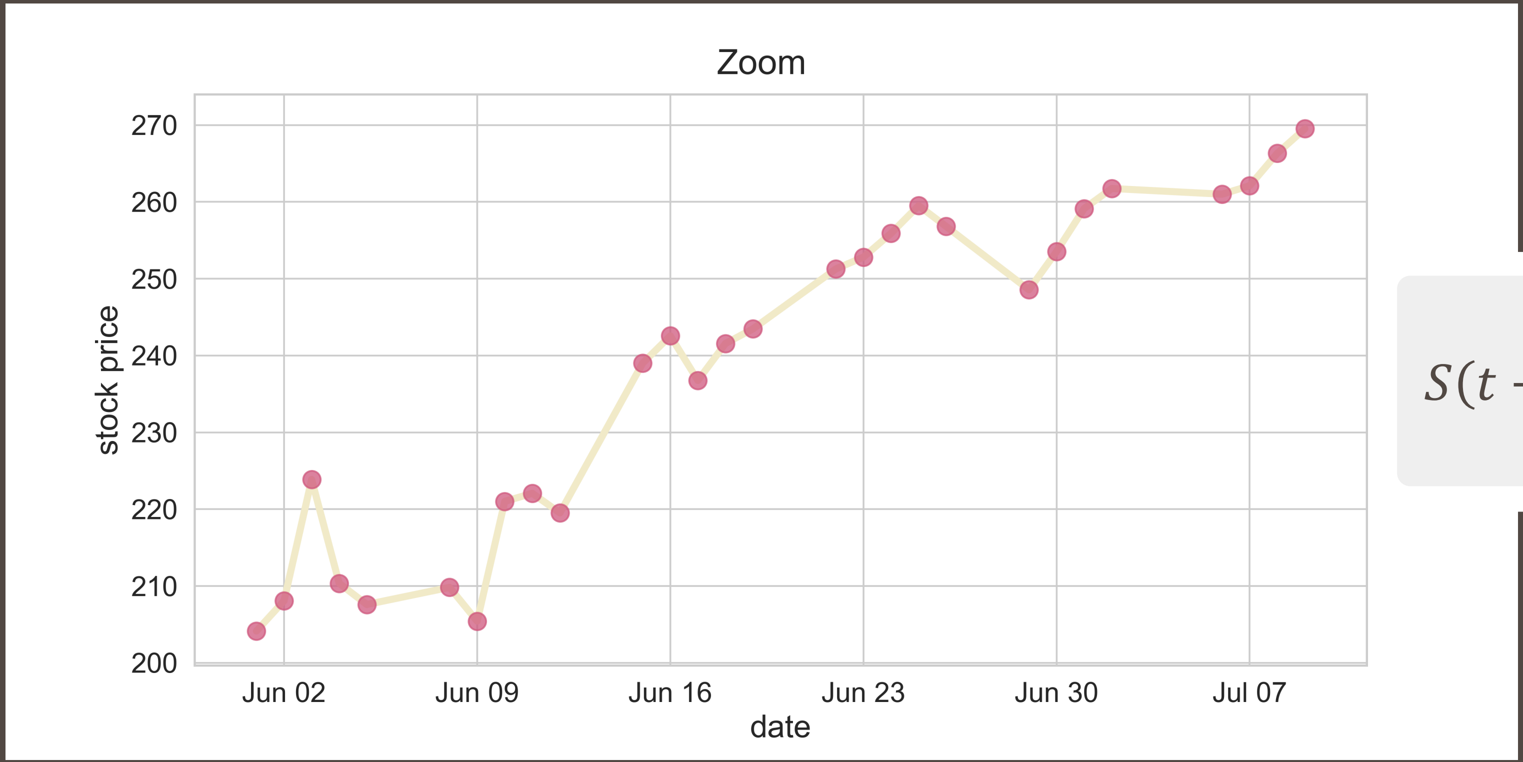








Binomial lattice model for stock prices

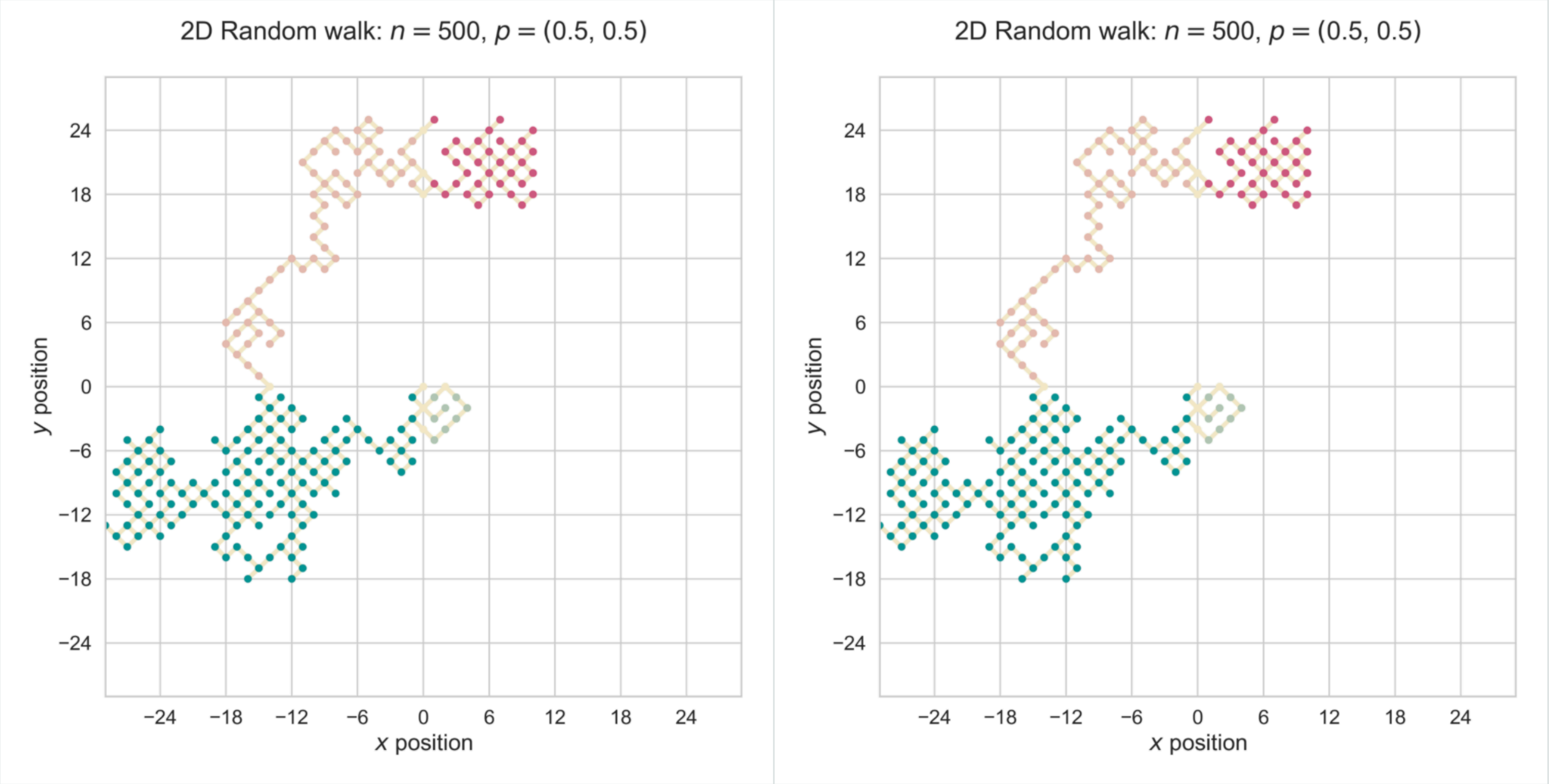


$$S(t+1) = \begin{cases} uS(t), & \text{with prob } p \\ dS(t), & \text{with prob } 1-p \end{cases}$$

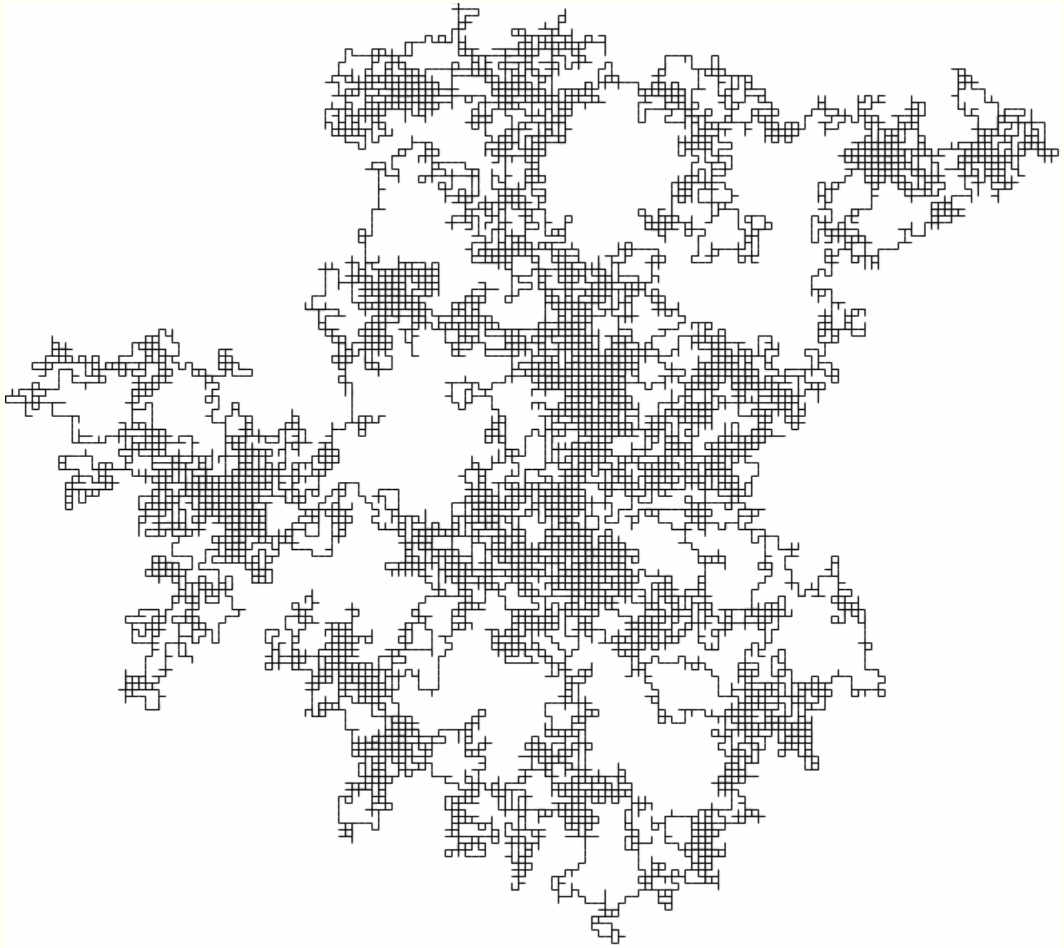
```
import yfinance as yf
load_STOCK_raw(company_index = 'AAPL')
figure_stock_price(company_index = 'AMZN', date_initial = datetime.date(int(2020),int(6),int(1)),
                   fsize = (12, 6), fs = 20)
```

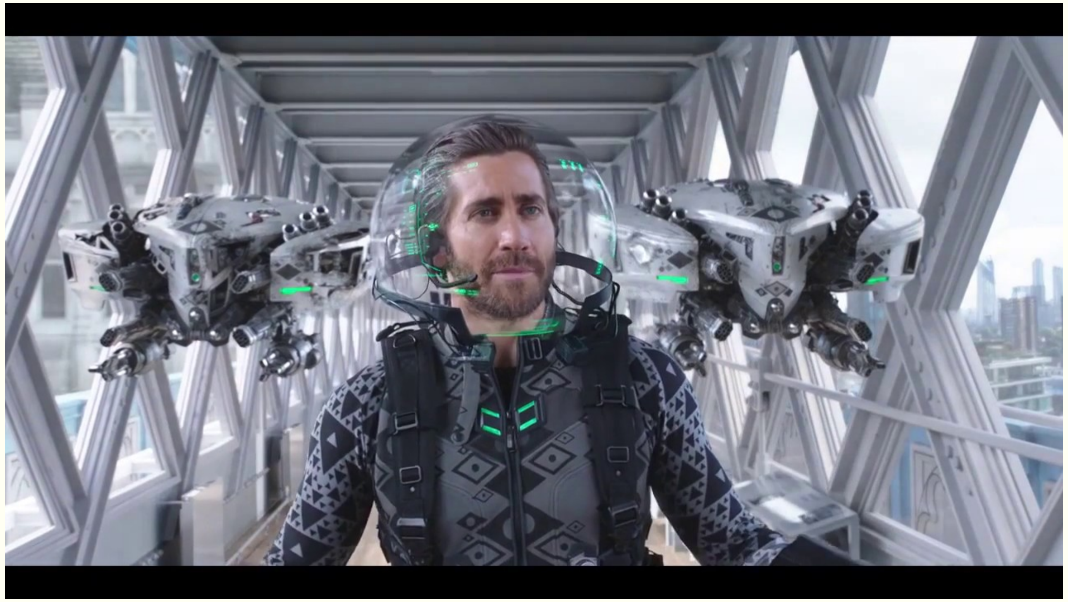
- **Two Dimensional Random Walk (Take-Home Problem)**

```
random_walk_2D(n = 500, p_x = 0.5, p_y = 0.5)
path_rw_2D(n = 500, p_x = 0.5, p_y = 0.5, fsize = (8, 8), fs = 18, index = 1)
```

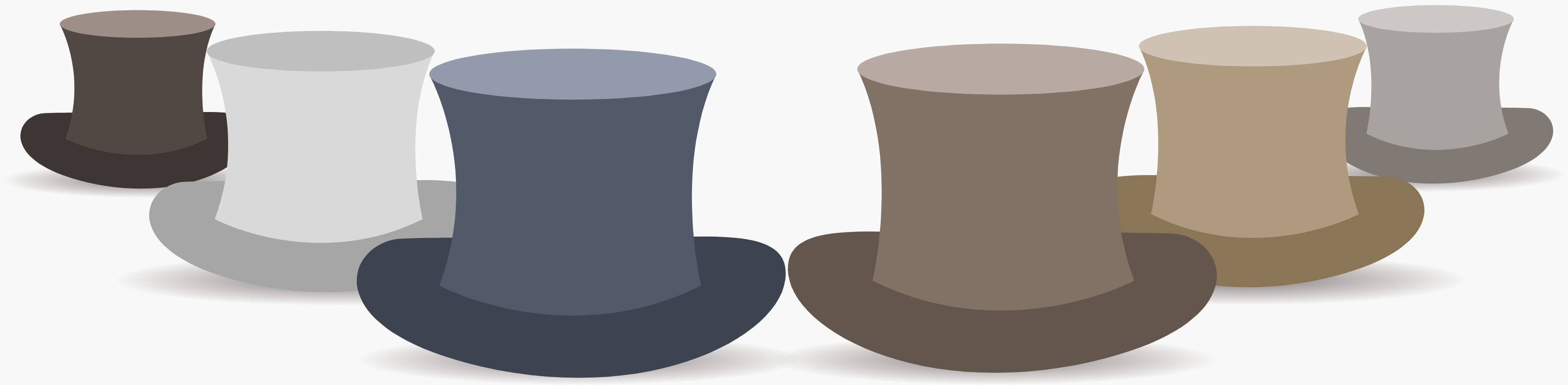


Random Walk and AI (Robots)



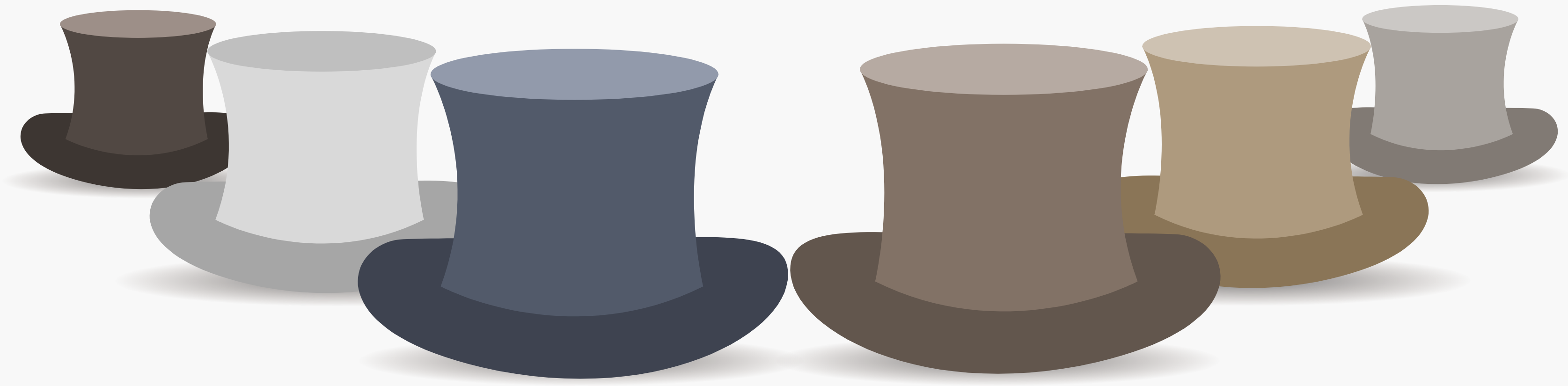


micro drones
swarm robotics



Hat Check Problem

In a restaurant n hats are checked and they are hopelessly scrambled. What is the probability that no one gets his own hat back?



Hat Check Problem

$$p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!}.$$

Hat Check Problem

a random permutation

- Find the probability that it contains no fixed point.
- Recall that it is a one-to-one map of a set $A = \{a_1, a_2, \cdots, a_n\}$ onto itself.
- Let A_i be the event that the ith element a_i remains fixed under this map.

$$\boldsymbol{p_0(n) = 1 - P(A_1 \cup A_2 \cup \cdots A_n)}$$

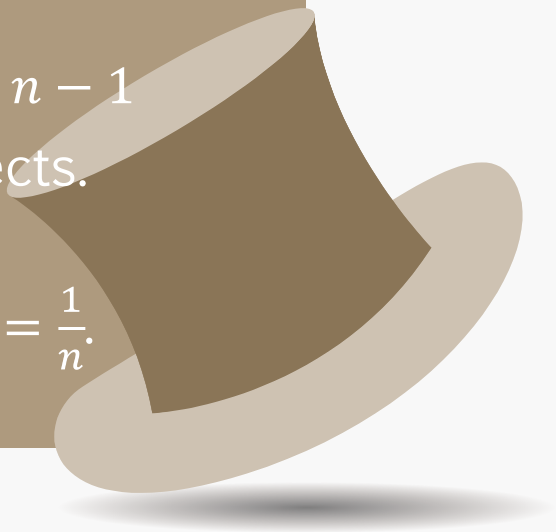
Hat Check Problem

$P(A_i)$

- Let A_i be the event that the ith element a_i remains fixed under this map.

- If we require that a_i is fixed, then the map of the remaining $n-1$ elements provides an arbitrary permutation of $(n-1)$ objects.

- Since there are $(n-1)!$ such permutations, $P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$.



$$\sum_{i=1}^{n} P(A_i) = n \times \frac{1}{n} = 1.$$

Hat Check Problem

$P(A_i \cap A_j)$

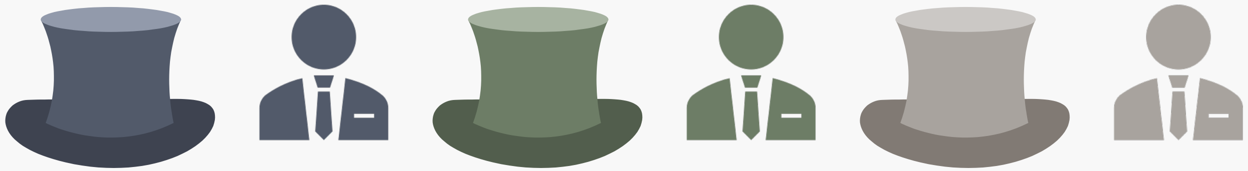
- To have a particular pair (a_i, a_j) fixed, we can choose any permutation of the remaining $n-2$ elements.
- There are $(n-2)!$ such choices and thus $P\left(A_i \cap A_j\right) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$.

$$\sum_{1 \le i < j \le n} P\left(A_i \cap A_j\right) = \binom{n}{2} \times \frac{1}{n(n-1)} = \frac{1}{2!}$$

Hat Check Problem

- $P(A_i) = \frac{(n-1)!}{n!}$
- $P\left(A_i \cap A_j\right) = \frac{(n-2)!}{n!}$
- $P\left(A_i \cap A_j \cap A_k\right) = \frac{(n-3)!}{n!}$
- …

- $\sum_{i=1}^{n} P(A_i) = n \times \frac{(n-1)!}{n!} = 1$
- $\sum_{1 \le i < j \le n} P\left(A_i \cap A_j\right) = \binom{n}{2} \times \frac{1}{n(n-1)} = \frac{1}{2!}$
- $\sum_{1 \le i < j < k \le n} P\left(A_i \cap A_j \cap A_k\right) = \binom{n}{3} \times \frac{(n-3)!}{n!} = \frac{1}{3!}$
- …



Hat Check Problem

$$\boldsymbol{p_0(n) = 1 - P(A_1 \cup A_2 \cup \cdots A_n)}$$

- $\sum_{i=1}^{n} P(A_i) = n \times \frac{(n-1)!}{n!} = 1$
- $\sum_{1 \le i < j \le n} P\left(A_i \cap A_j\right) = \binom{n}{2} \times \frac{1}{n(n-1)} = \frac{1}{2!}$
- $\sum_{1 \le i < j < k \le n} P\left(A_i \cap A_j \cap A_k\right) = \binom{n}{3} \times \frac{(n-3)!}{n!} = \frac{1}{3!}$
- …

$$\boldsymbol{P(A_1 \cup A_2 \cup \cdots A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P\left(A_i \cap A_j\right) + \sum_{1 \le i < j < k \le n} P\left(A_i \cap A_j \cap A_k\right) - \cdots.}$$

$$\boldsymbol{p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!}.}$$