



# MATH 20: PROBABILITY

Combinations





**SMALL PLATES**

**LARGE PLATES**

**SWEET BITES**

**3 choose 2**

How many possible choices are there in total?

Is it 6?

?



**SMALL PLATES = 1**

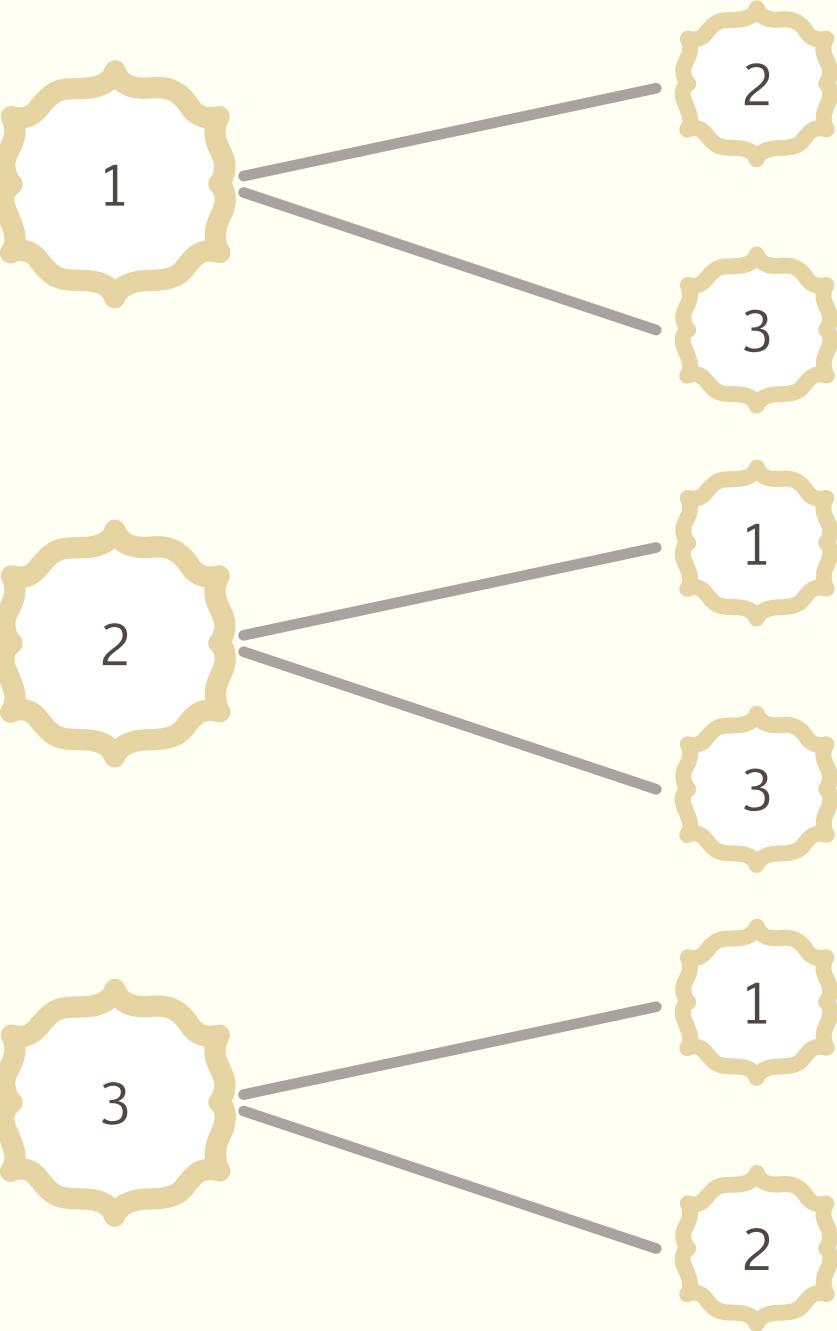
**LARGE PLATES = 2**

**SWEET BITES = 3**

## 3 choose 2

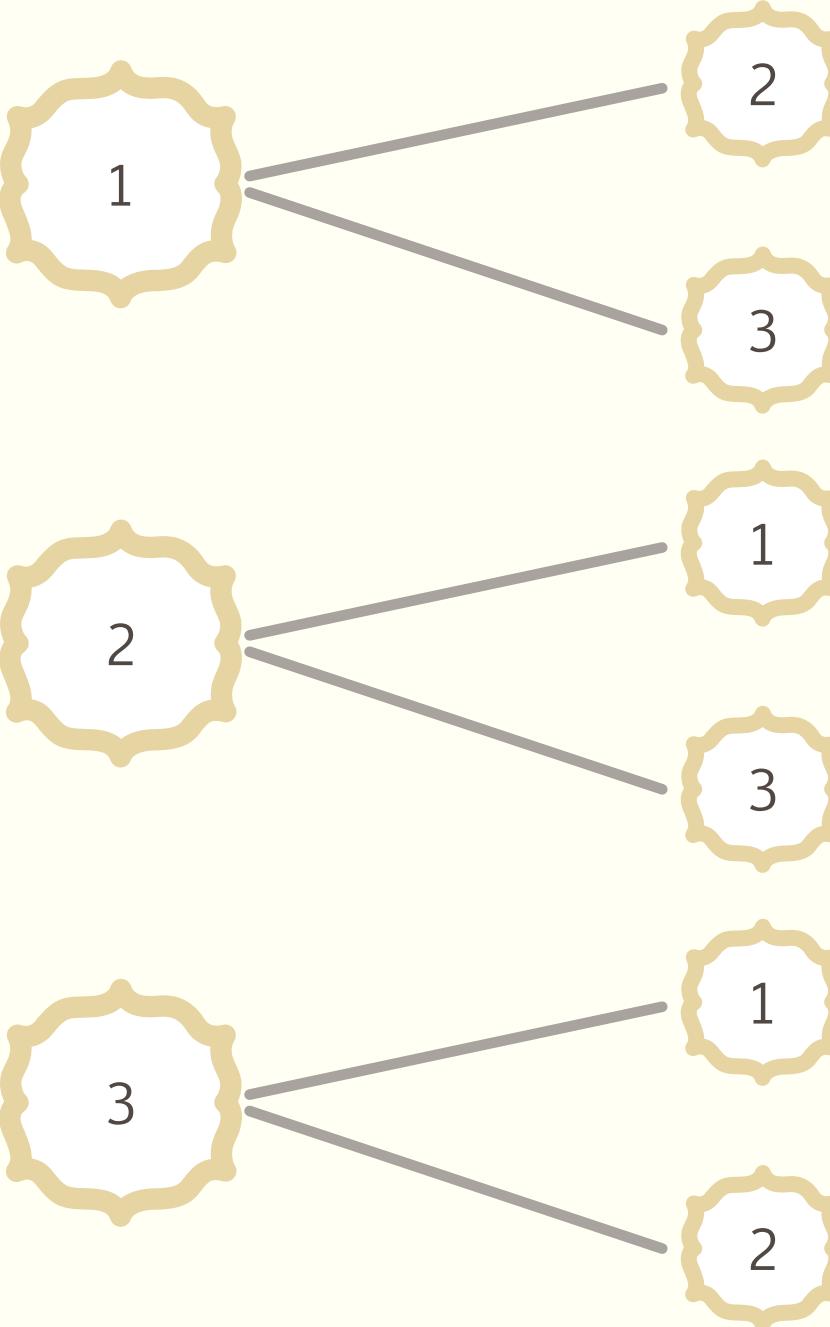
How many possible choices are there in total?

subset	subset
{sp, lp}	{1, 2}
{sp, sb}	{1, 3}
{lp, sb}	{2, 3}



**3 choose 2**

How many possible choices are there in total?



$(3)_2$

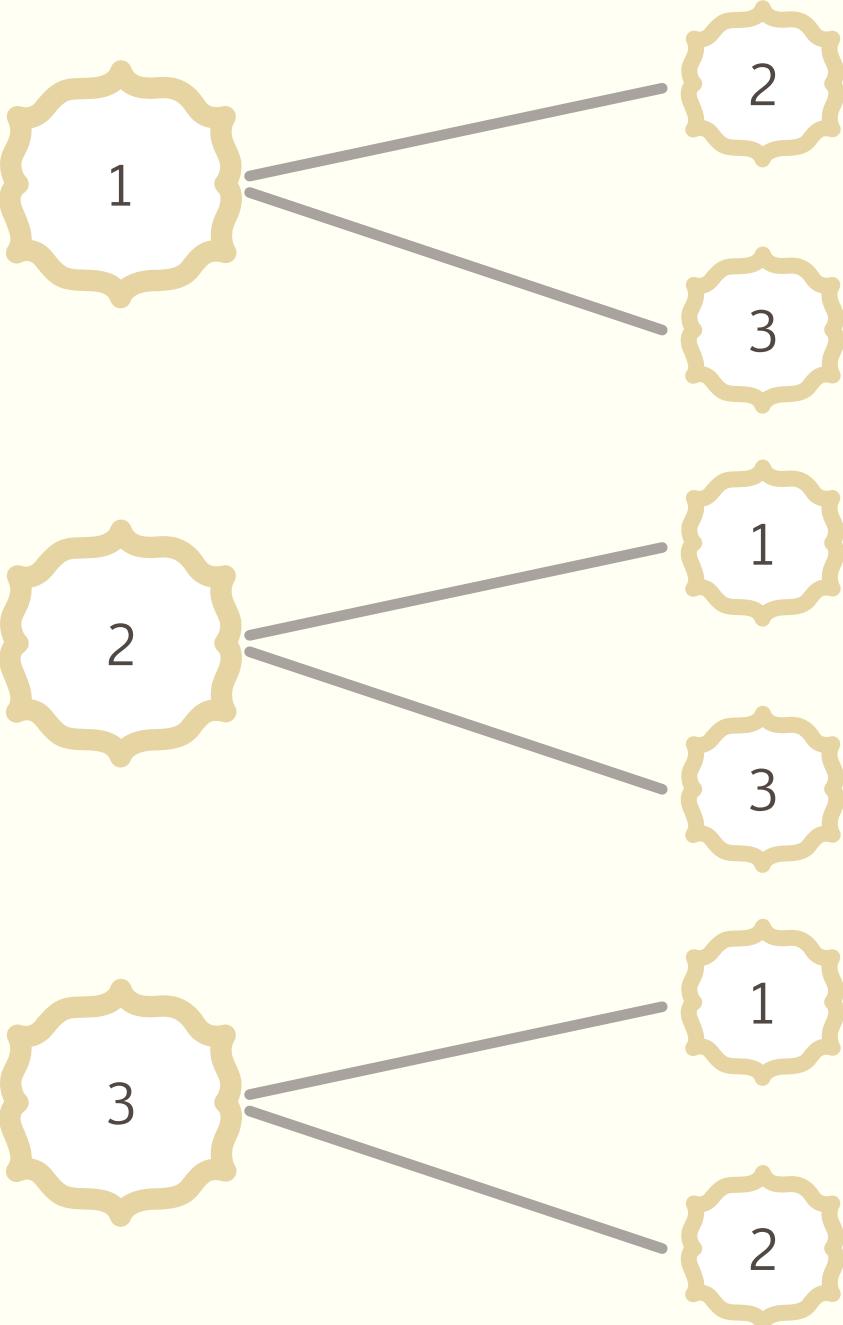
Do we care about the order?

?

The number of permutations of  
 $n$  elements is given by  $n!$

!

$2!$



Do we care about the order?

?

The number of permutations of  $n$  elements is given by  $n!$

!

$$3 = \frac{3 \times 2}{2 \times 1} = \frac{(3)_2}{2!}$$

!

$(3)_2$

$2!$

3 choose 2

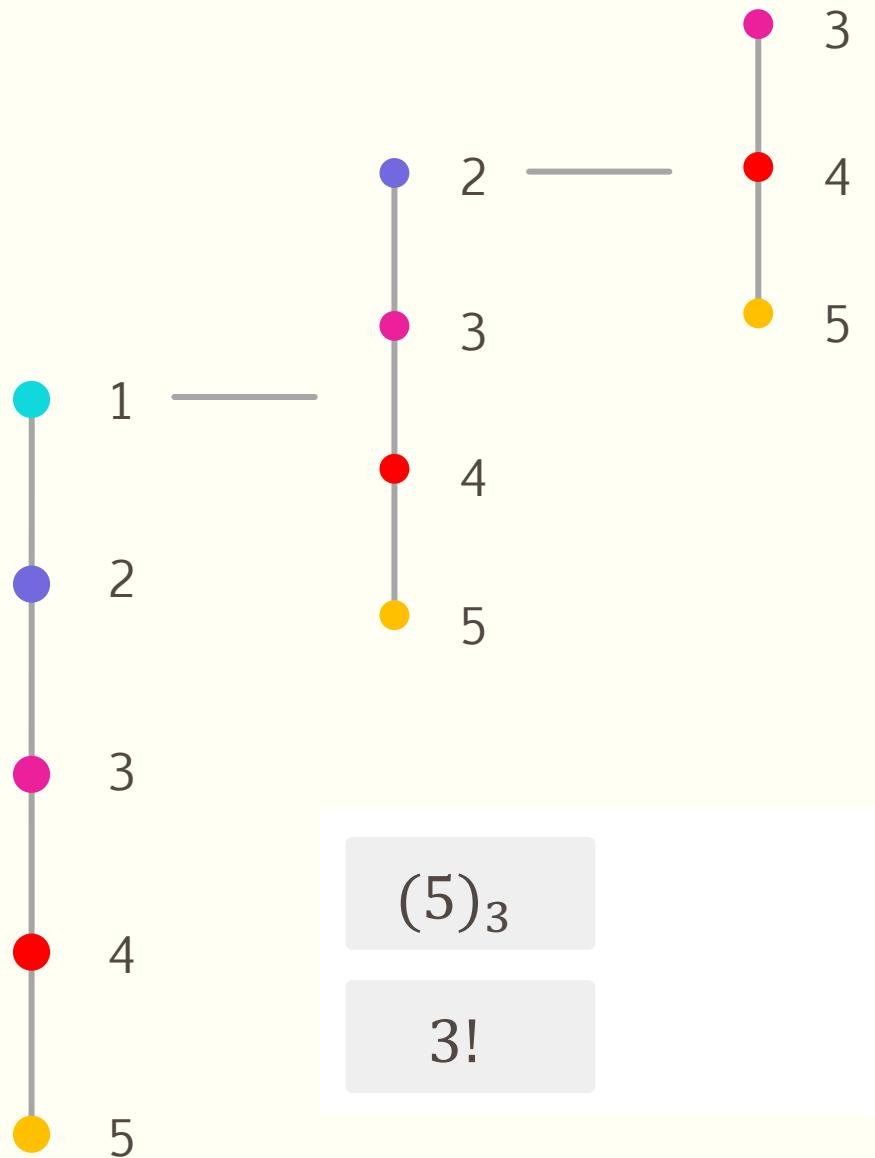
# 5 choose 3



- United States
- Brazil
- Egypt
- China
- Australia

How many possible choices are there  
in total?

## 5 choose 3



Do we care about the order?

?

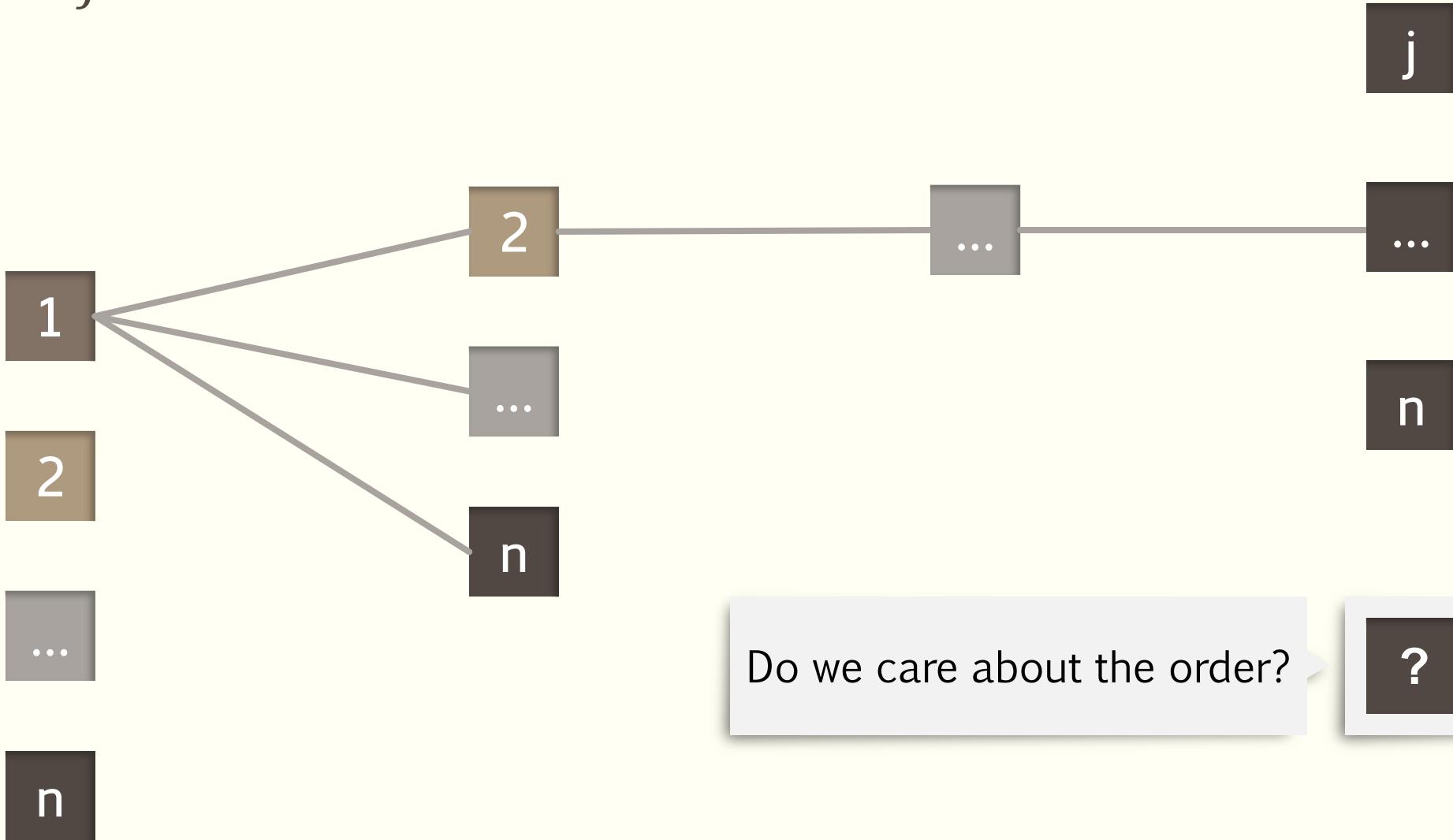
The number of permutations of  $n$  elements is given by  $n!$

!

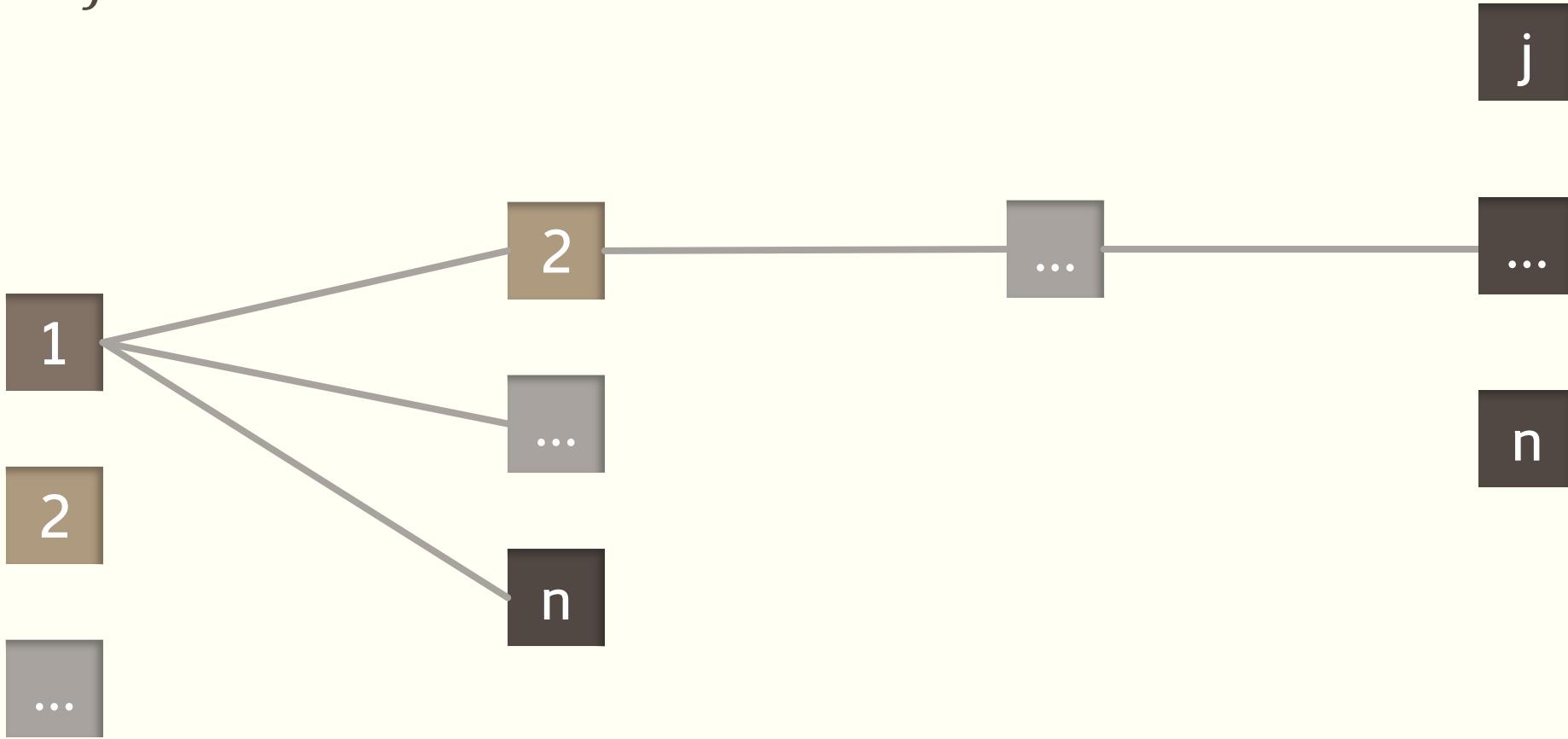
$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = \frac{(5)_3}{3!} = 10.$$

=

$n$  choose  $j$



$n$  choose  $j$



$$\frac{(n)_j}{j!}$$

$$\frac{n \times (n-1) \times \cdots \times (n-j+1)}{j \times (j-1) \times \cdots \times 1} = \frac{(n)_j}{j!}.$$

!

# Binomial Coefficients

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- The number of distinct subsets with  $j$  elements that can be chosen from a set with  $n$  elements is denoted by  $\binom{n}{j}$ .
- The number  $\binom{n}{j}$  is called a binomial coefficient.

$$\binom{n}{0} = \binom{n}{n} = 1.$$

1

$$\binom{n}{j} = \frac{(n)_j}{j!} = \frac{n!}{j!(n-j)!} = \binom{n}{n-j}.$$

2

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

3

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

3



$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

3



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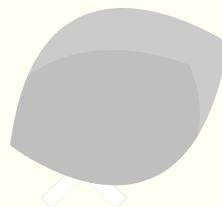
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## Recurrence Relation: Pascal's Triangle

For integers  $n$  and  $j$ , with  $0 < j < n$ , the binomial coefficients satisfy:

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4



do not include

include

## Recurrence Relation: Pascal's Triangle

For integers  $n$  and  $j$ , with  $0 < j < n$ , the binomial coefficients satisfy:

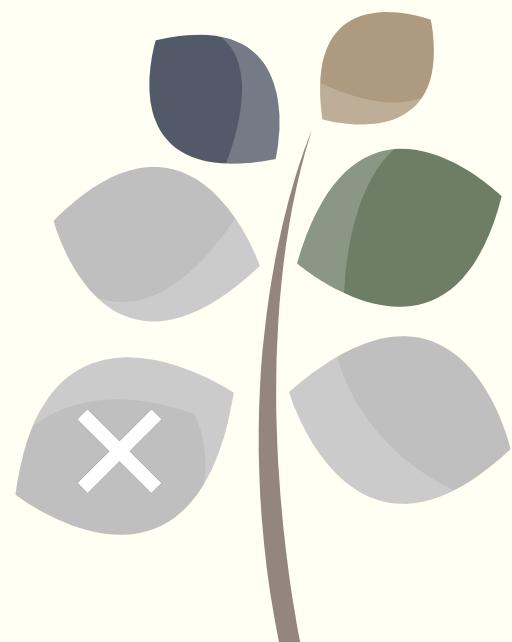
$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4



do not include

include



## Recurrence Relation: Pascal's Triangle

For integers  $n$  and  $j$ , with  $0 < j < n$ , the binomial coefficients satisfy:

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

- **Pascal's Triangle**

```
from scipy.special import comb  
pascal(n = 10, j = 5)
```



```
# Pascal's Triangle  
def pascal(n, j):  
    n_choose_j = int(comb(n, j))  
    sum_choose = int(comb(n-1, j) + comb(n-1, j-1))  
    return n_choose_j, sum_choose
```

# Pascal's Triangle

n/j	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

$\binom{n}{0} = \binom{n}{n} = 1.$  1

a column and the diagonal

$\binom{n}{j} = \frac{(n)_j}{j!} = \frac{n!}{j!(n-j)!} = \binom{n}{n-j}.$  2

symmetry in every row

# Pascal's Triangle

n/j	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$ .

**Pascal's Triangle**

## Example

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Find integers  $n$  and  $r$  such that the following equation is true:

$$\binom{13}{5} + 2\binom{13}{6} + \binom{13}{7} = \binom{n}{r}.$$

=

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4

Find integers n and r such that the following equation is true:

$$\binom{13}{5} + 2\binom{13}{6} + \binom{13}{7} = \binom{n}{r}.$$

=

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

4

$$\binom{13}{6} + \binom{13}{5} = \binom{14}{6}.$$

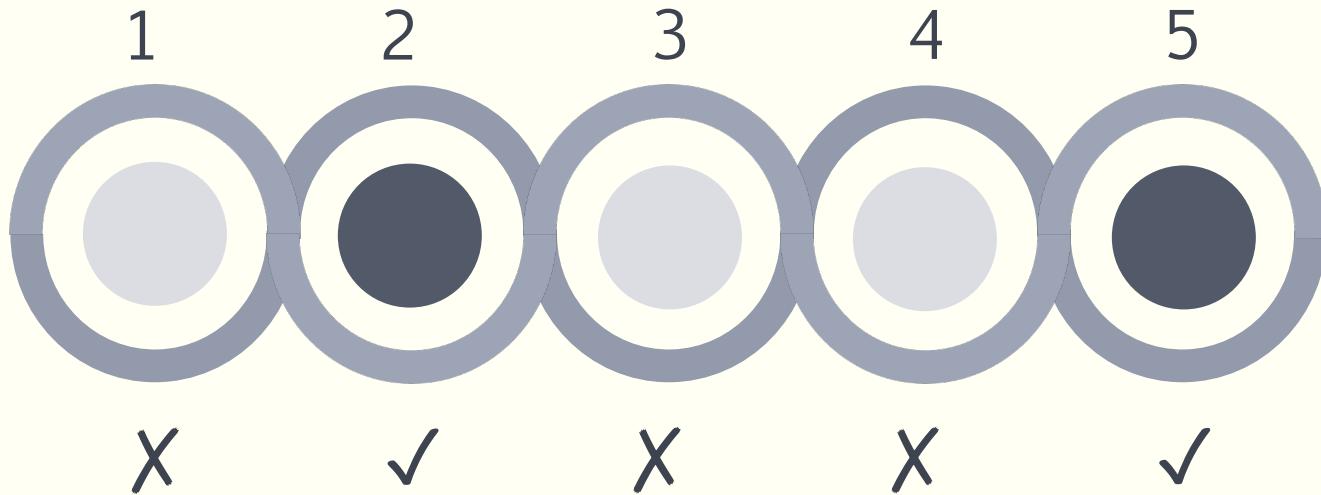
$$\binom{13}{7} + \binom{13}{6} = \binom{14}{7}.$$

$$\binom{14}{7} + \binom{14}{6} = \binom{15}{7}.$$

# TOSS A COIN

Toss a coin 5 times. What is the probability that there are 2 flips that land heads?



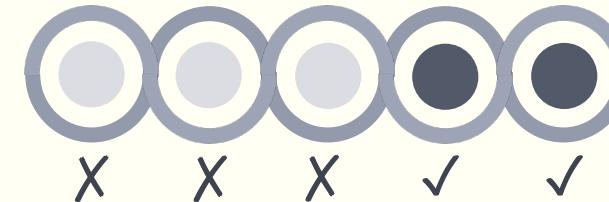
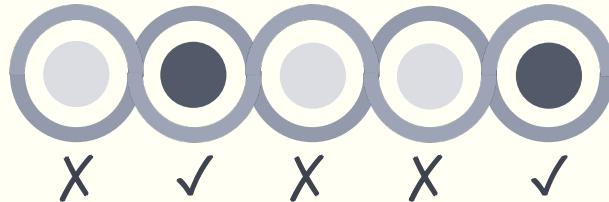
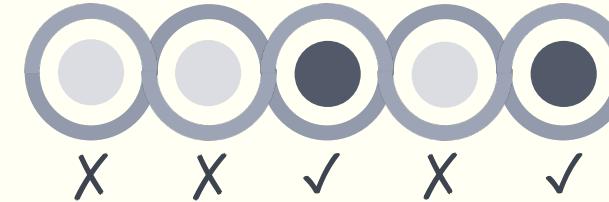
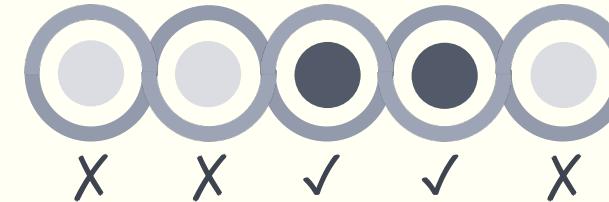
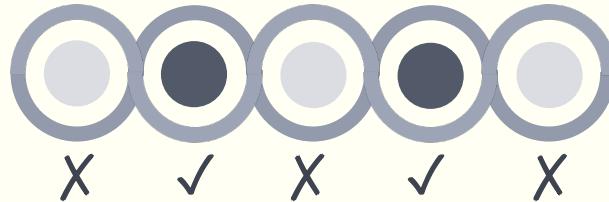
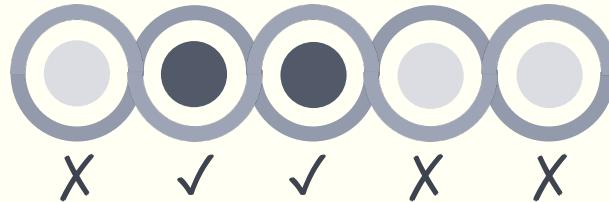
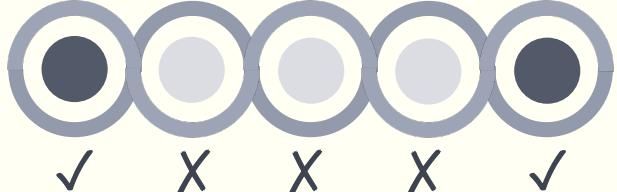
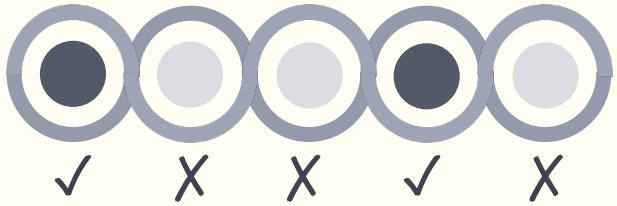
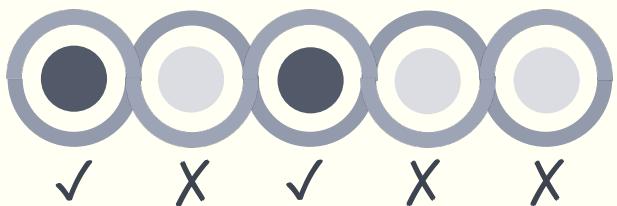
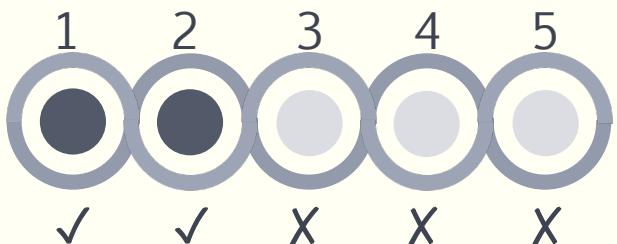


The 2<sup>nd</sup> and the 5<sup>th</sup> trials land heads:

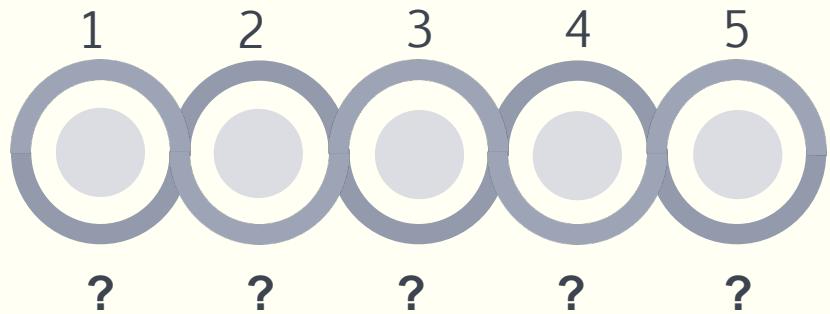
$$p = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

=

# 5 choose 2



5 choose 2



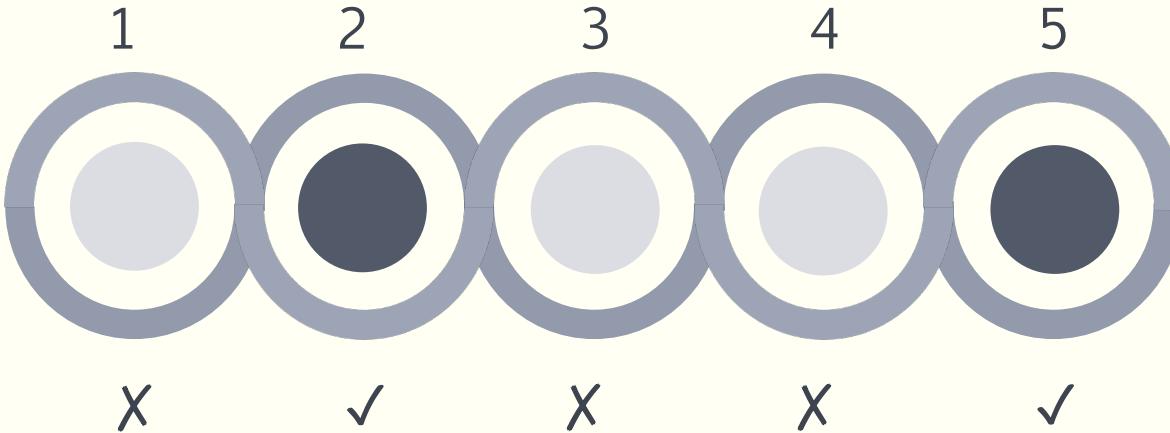
How many possible choices?

?

$$\binom{5}{2} = \frac{(5)_2}{2!} = \frac{5 \times 4}{2 \times 1} = 10.$$

=

# Probability that there are 2 flips that land heads



5 choose 2

!

2 heads & 3 tails

!

$$\binom{5}{2} \times \left(\frac{1}{2}\right)^5 = 10 \times \frac{1}{32} = \frac{5}{16}.$$

X

# Weather Forecast

Hanover  
12:25 am



☀ WEATHER

33°

Monday 16<sup>th</sup>



4mph / 67°

Tue



31°

Wed



30°

Thu



33°

Fri



32°

Sat



33°

Sun



31°

# Weather Forecast

Hanover  
12:25 pm



☀ WEATHER

33°



Monday

Tue



31°

Wed



30°

Thu



33°

Fri



32°

Sat



33°

Sun



31°

- From Tuesday to Friday, it has 40% chance of raining everyday.
- What is the probability that on three of the four days it does not rain?

# Weather Forecast

Hanover  
12:25 pm



- From Tuesday to Friday, it has 40% chance of raining everyday.
- What is the probability that on three of the four days it does not rain?

4 choose 3

!

3 not rainy & 1 rainy

!

$$\binom{4}{3} \times \left(\frac{3}{5}\right)^3 \times \left(\frac{2}{5}\right)^1 = 4 \times \frac{54}{625} = \frac{216}{625}$$

×

# Weather Forecast

Hanover  
12:25 pm



- From Tuesday to Friday, it has 40% chance of raining everyday.
- What is the probability that on three of the four days it does not rain?

How do we define success and failure?

?

not rainy

!

rainy

!

# Weather Forecast

Hanover  
12:25 pm



- What is the probability that on three of the four days it does not rain?
- What is the probability that on one of the four days it rains?

4 choose 3

!

4 choose 1

!

3 not rainy & 1 rainy

!

1 rainy & 3 not rainy

!

$$\binom{4}{3} \times \left(\frac{3}{5}\right)^3 \times \left(\frac{2}{5}\right)^1 = 4 \times \frac{54}{625} = \frac{216}{625}$$

×

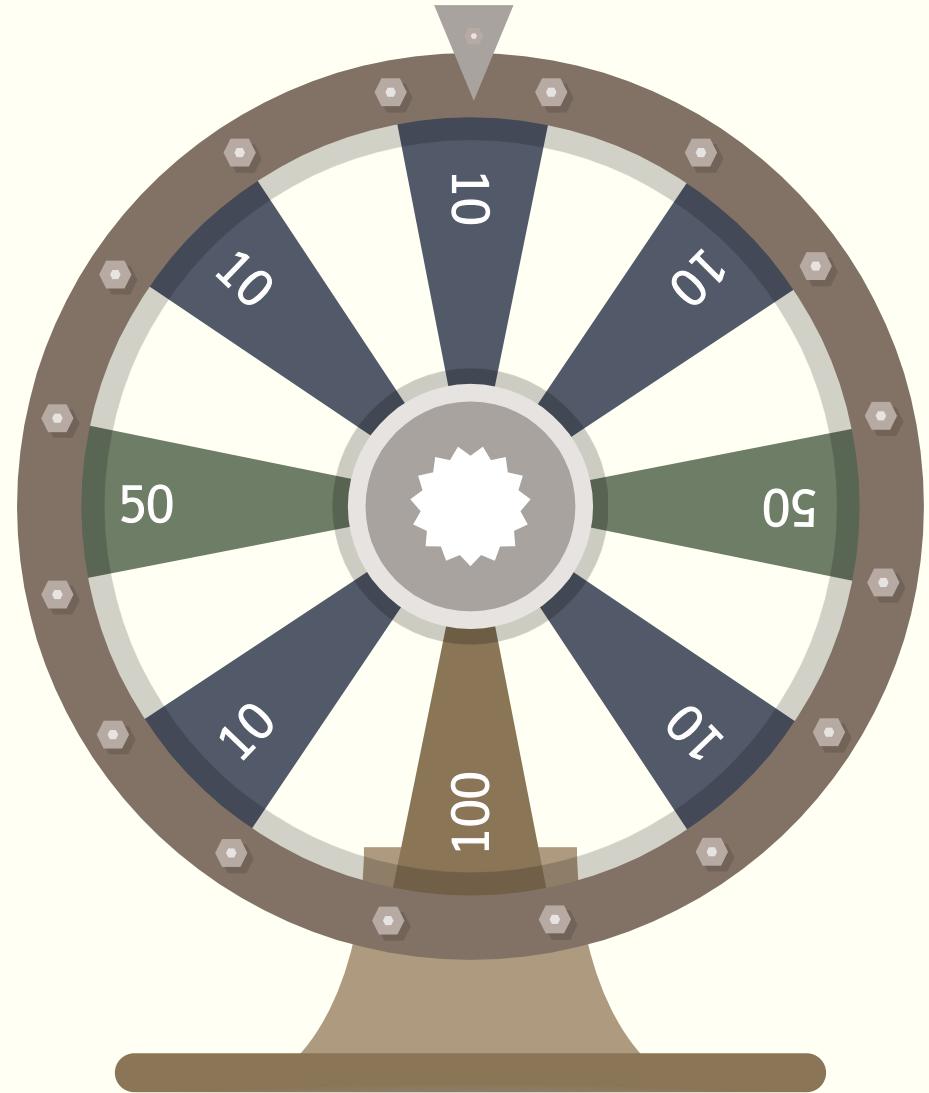
$$\binom{4}{1} \times \left(\frac{2}{5}\right)^1 \times \left(\frac{3}{5}\right)^3 = 4 \times \frac{54}{625} = \frac{216}{625}$$

×

# Wheel of Fortune

- Turn the wheel 10 times.
- What is the probability of getting 50 points twice?

Points	Probability
0	$1/2$
10	$5/16$
50	$1/8$
100	$1/16$



# Wheel of Fortune

- Turn the wheel 10 times.
- What is the probability of getting 50 points twice?

Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16

10 choose 2

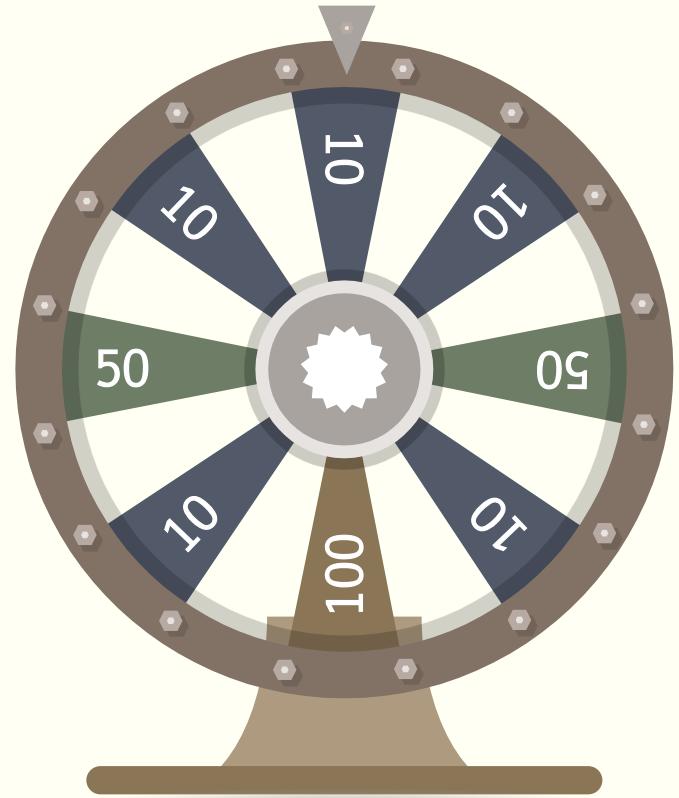
!

2 50 points & 8 others

!

$$\binom{10}{2} \times \left(\frac{1}{8}\right)^2 \left(\frac{7}{8}\right)^8 = 45 \times \frac{7^8}{8^{10}}.$$

×



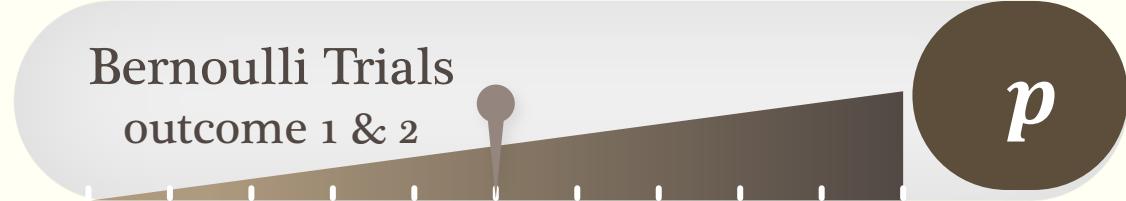
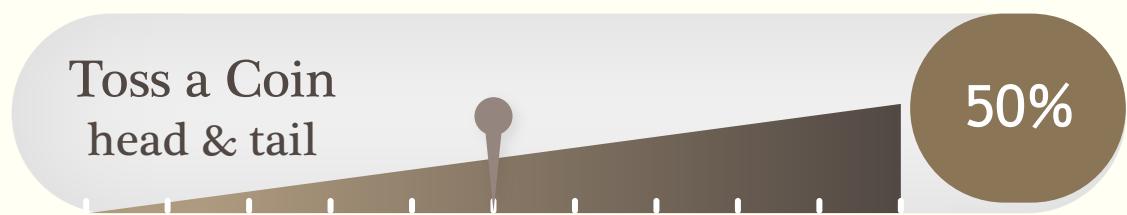
# Bernoulli Trials

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A Bernoulli trials process is a sequence of  $n$  chance experiments such that

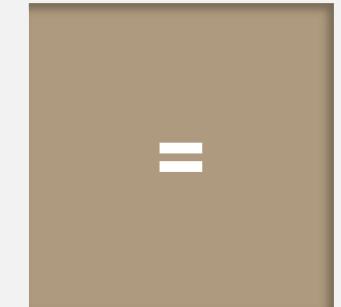
- Each experiment has two possible outcomes, which we may call success and failure.
- The probability  $p$  of success on each experiment is the same for each experiment, and this probability is not affected by any knowledge of previous outcomes. The probability  $q$  of failure is given by  $q = 1 - p$ .



## Bernoulli Probabilities

$B(n, p, j)$ , the probability that in  $n$  Bernoulli trials there are exactly  $j$  successes. We have:

$$B(n, p, j) = \binom{n}{j} p^j q^{n-j}.$$

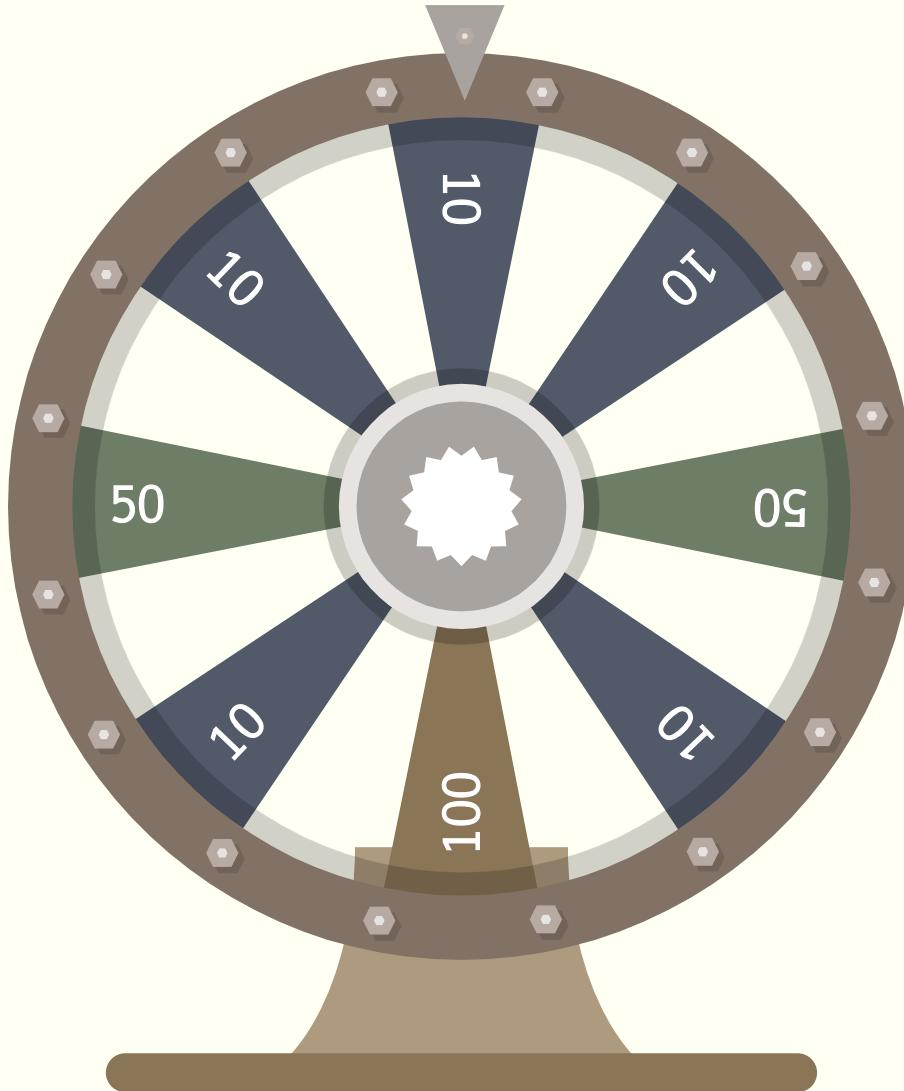


Bernoulli Trials	$n$	$p$	$j$	Probability
Toss a Coin	5	50%	2	$\binom{5}{2} \times (\frac{1}{2})^5$
Weather Forecast	4	60%	3	$\binom{4}{3} \times (\frac{3}{5})^3 \times (\frac{2}{5})^1$
Wheel Fortune	10	12.5%	2	$\binom{10}{2} \times (\frac{1}{8})^2 (\frac{7}{8})^8$

# Wheel of Fortune revisit

- Turn the wheel 5 times.
- What is the probability of getting 100 points in total?

Points	Probability
0	$1/2$
10	$5/16$
50	$1/8$
100	$1/16$



# Wheel of Fortune revisit

- Turn the wheel 5 times.
- What is the probability of getting 100 points in total?

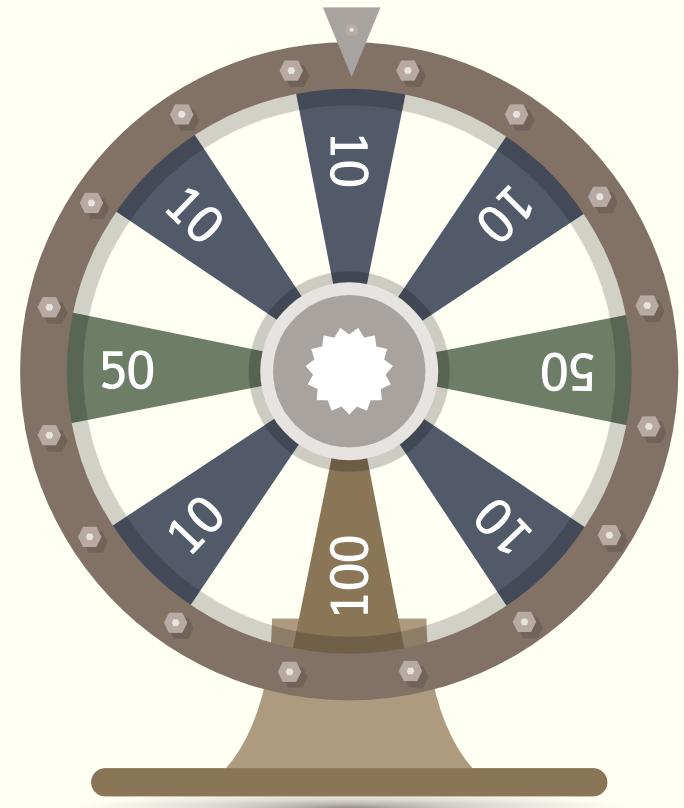
Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16

$$50 \times 2 + 0 \times 3 = 100$$

A

$$100 \times 1 + 0 \times 4 = 100$$

B



# Wheel of Fortune revisit

Points	Probability
0	1/2
10	5/16
50	1/8
100	1/16

$$50 \times 2 + 0 \times 3 = 100$$

A

$$100 \times 1 + 0 \times 4 = 100$$

B

$$\binom{5}{2} \times \left(\frac{1}{8}\right)^2 \left(\frac{1}{2}\right)^3$$

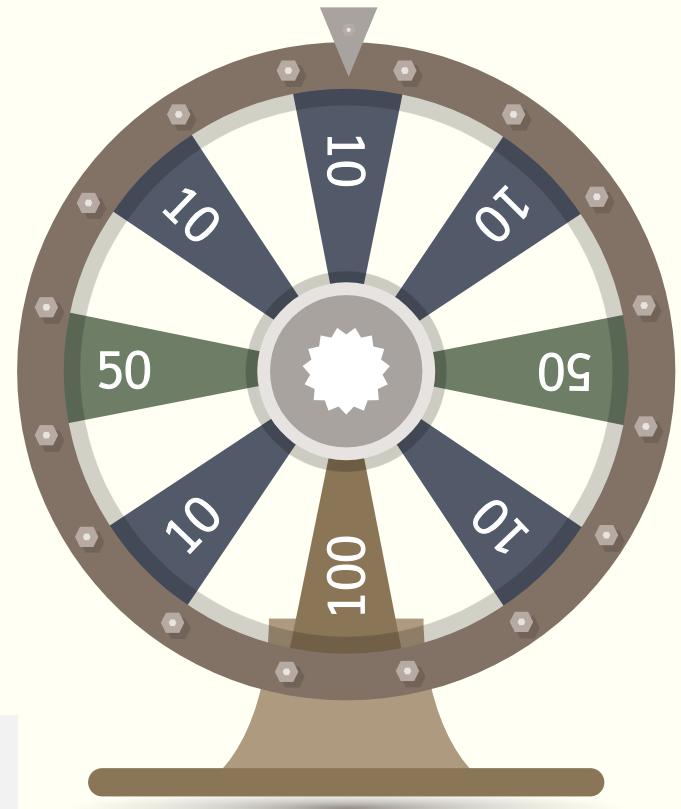
$p_A$

+

$$p = p_A + p_B$$

$$\binom{5}{1} \times \left(\frac{1}{16}\right)^1 \left(\frac{1}{2}\right)^4$$

$p_B$



# Binomial Distributions

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- Let  $n$  be a positive integer and let  $p$  be a real number between 0 and 1.
- Let  $B$  be the random variable which counts the number of successes in a Bernoulli trials process with parameters  $n$  and  $p$ .
- Then the distribution  $b(n, p, k)$  of  $B$  is called the binomial distribution.

$$\binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots + \binom{n}{n} p^n q^0 = \dots$$

?

# Binomial Expansion

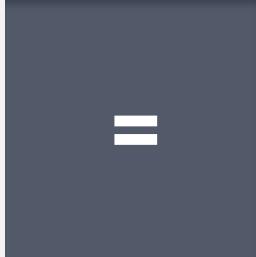
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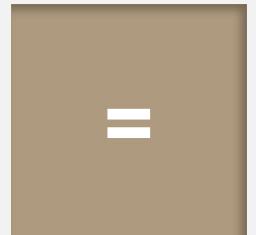
## Binomial Theorem

The quantity  $(a + b)^n$  can be expressed in the form:

$$(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^j b^{n-j}.$$



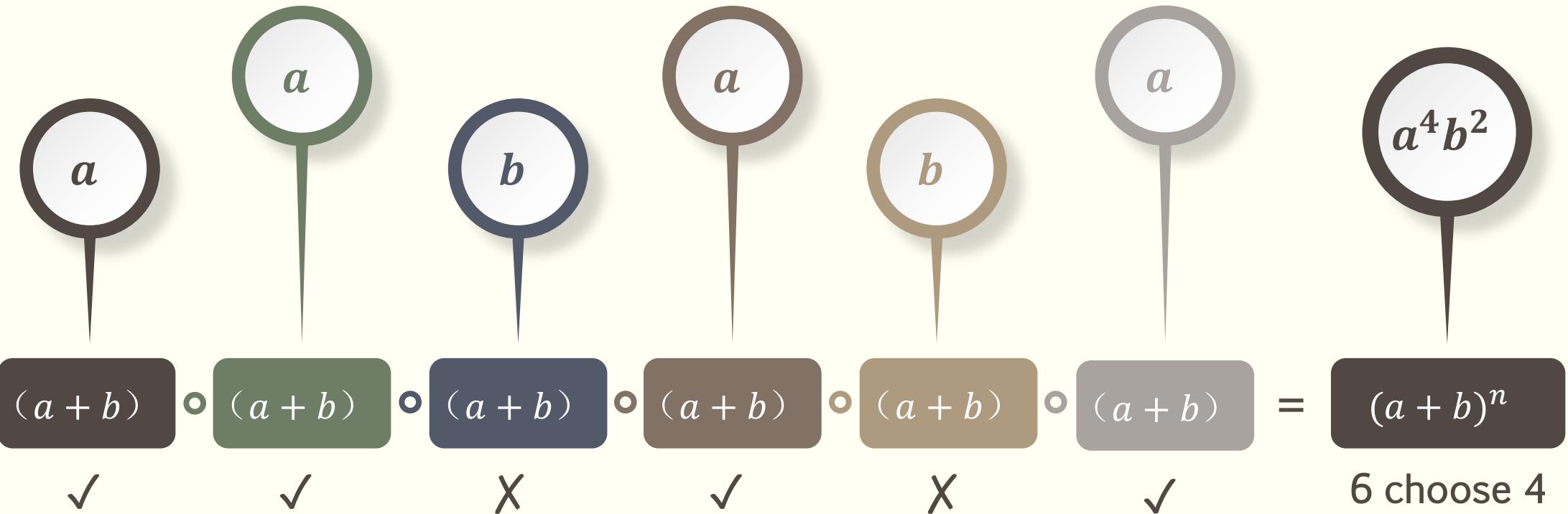
$$(p + q)^n = \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots + \binom{n}{n} p^n q^0 = 1.$$



## Binomial Theorem

$$(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^j b^{n-j}.$$

=



## Binomial Theorem

$$(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^j b^{n-j}.$$

=

Let  $n = 2$ , we have

$$(a + b)^2 = \binom{2}{0}b^2 + \binom{2}{1}ab + \binom{2}{2}a^2 = a^2 + 2ab + b^2.$$

=

Let  $n = 3$ , we have

$$(a + b)^3 = \binom{3}{0}b^3 + \binom{3}{1}ab^2 + \binom{3}{2}a^2b + \binom{3}{3}a^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

=

## Binomial Theorem

$$(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^j b^{n-j}.$$

=

Let  $a = b = 1$ , we have

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

=

Let  $a = -1, b = 1$ , we have

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}.$$

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## Inclusion-Exclusion Principle

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- Let  $P$  be a probability measure on a sample space  $\Omega$ , and let  $\{A_1, A_2, \dots, A_n\}$  be a finite set of events. Then

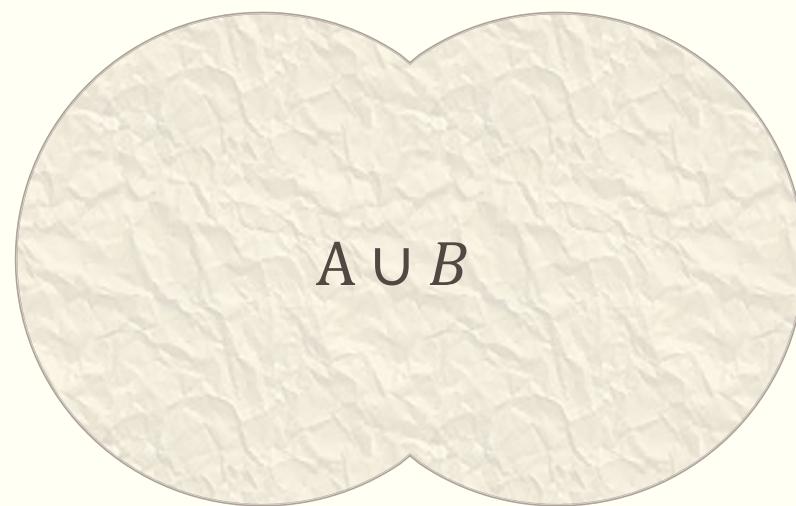
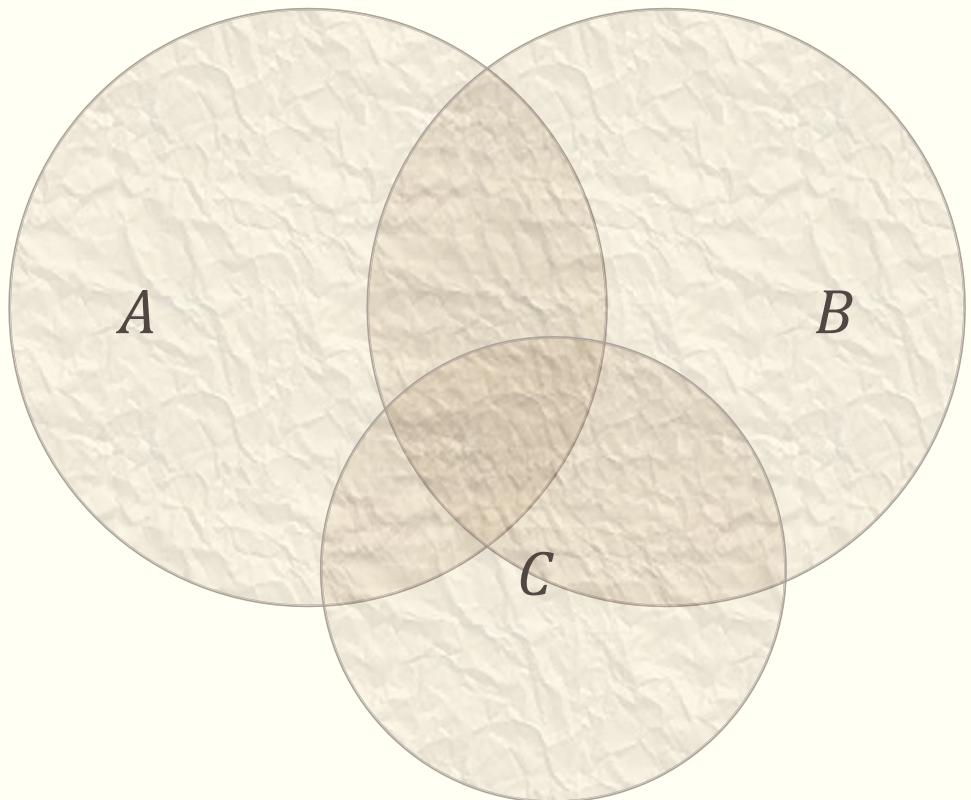
$$\begin{aligned} P(A_1 \cup A_2 \cup \dots \cup A_n) &= \\ \sum_{i=1}^n P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) - \dots \end{aligned}$$

- If  $A$  and  $B$  are subsets of  $\Omega$ , then

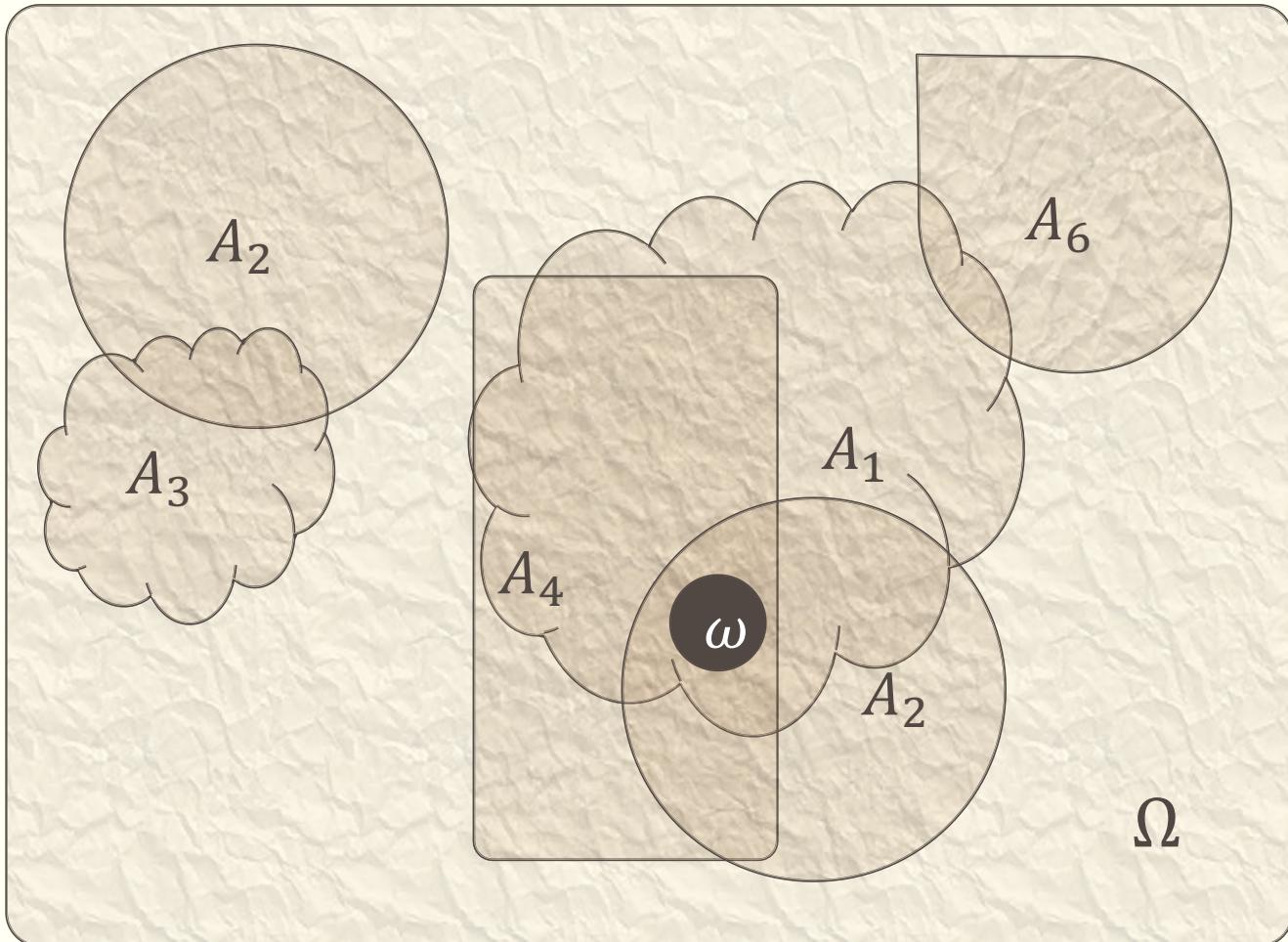
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

- If  $A, B$  and  $C$  are subsets of  $\Omega$ , then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

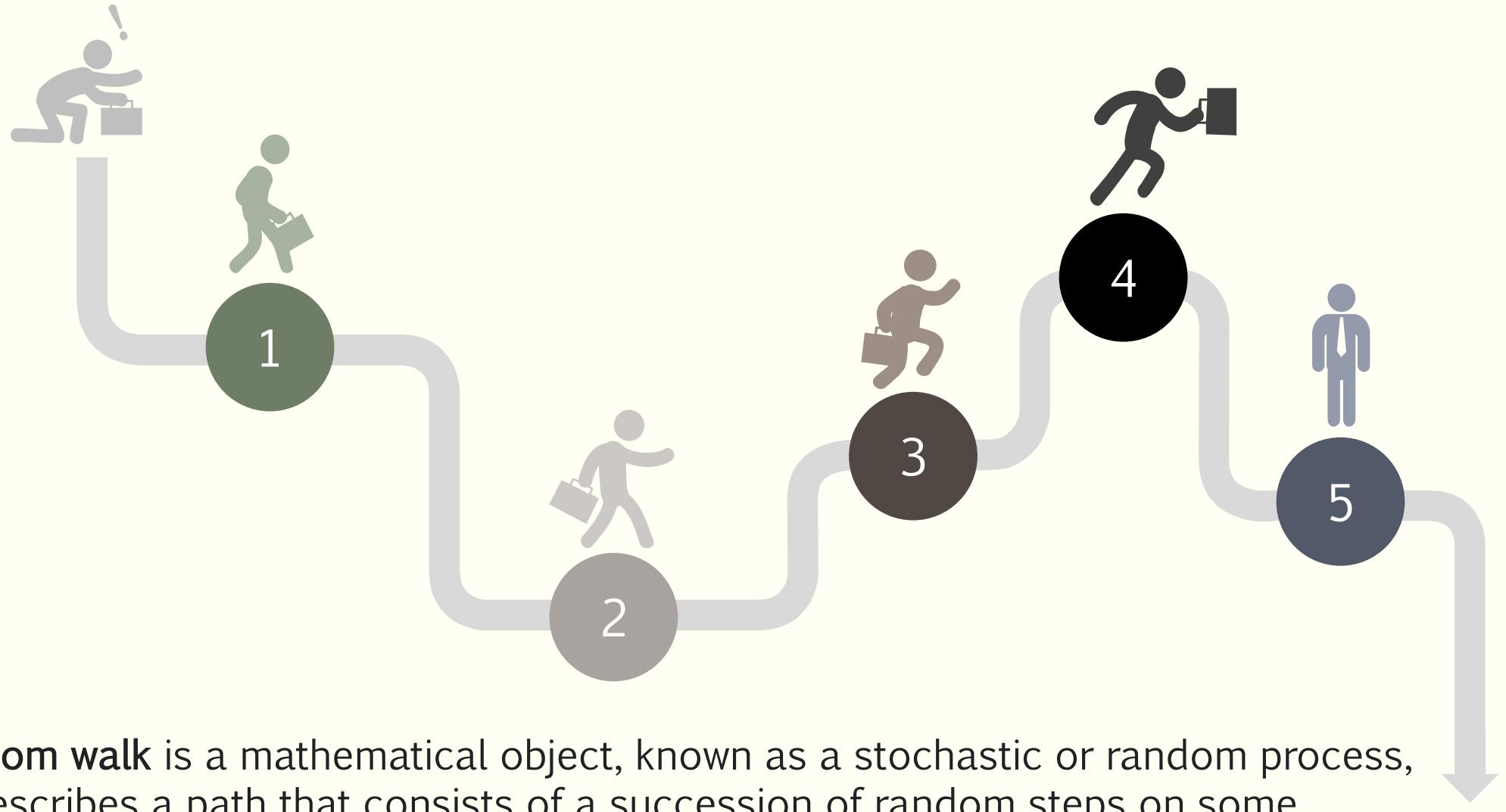


# Proof



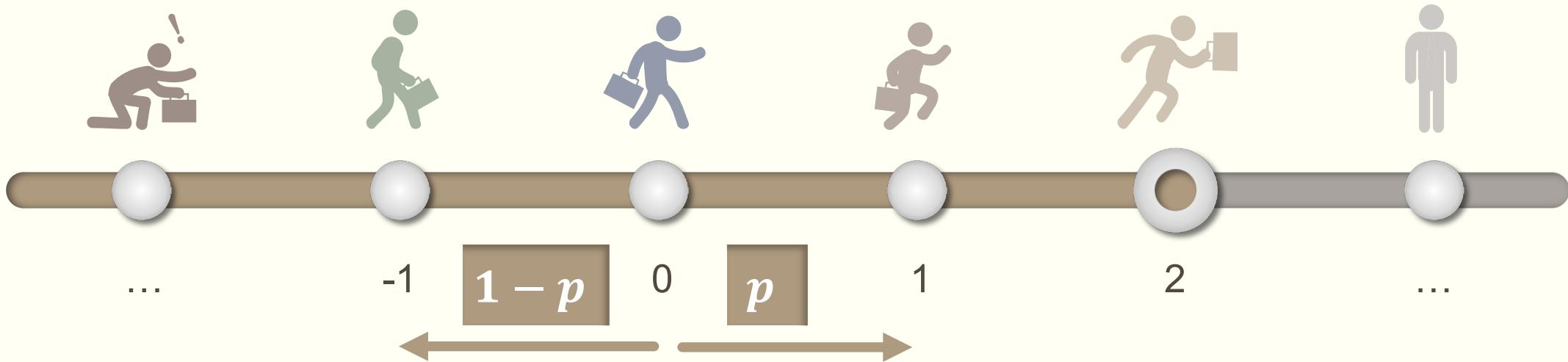
- If the outcome  $\omega$  occurs in at least one of the events  $A_i$ , its probability is added exactly once by the left side.
- We must show that it is also added exactly once by the right side.
- Assume that  $\omega$  is in exactly  $k$  of the sets. Then its probability is added  $k$  times in the first term, subtracted  $\binom{k}{2}$  times in the second, added  $\binom{k}{3}$  times in the third term, and so forth.

# Random Walk



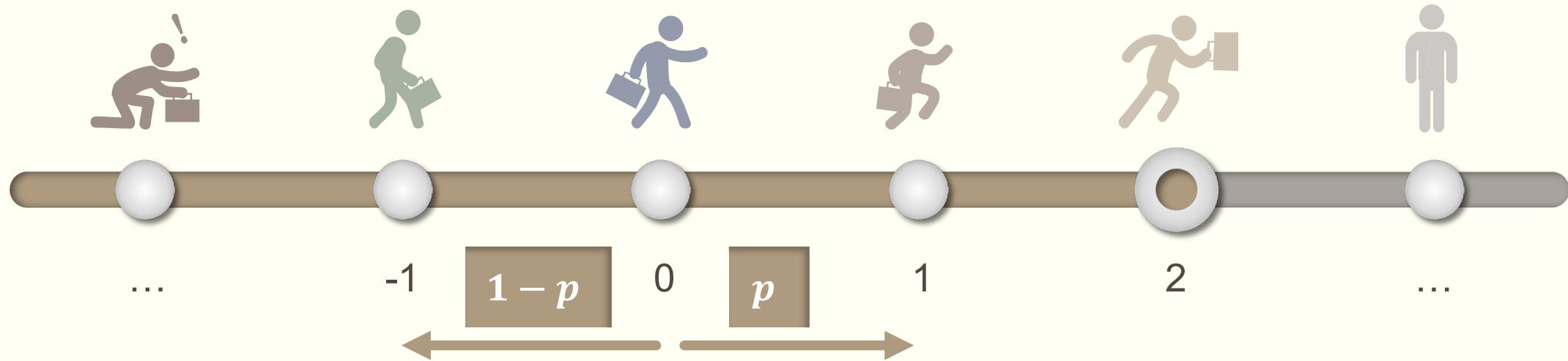
A **random walk** is a mathematical object, known as a stochastic or random process, that describes a path that consists of a succession of random steps on some mathematical space such as the integers.

# Random Walk (1 dimensional)



- An elementary example of a random walk is the random walk on the integer number line.
- It starts at 0 and at each step moves +1 or -1 with probability  $p$  and  $1 - p$ .

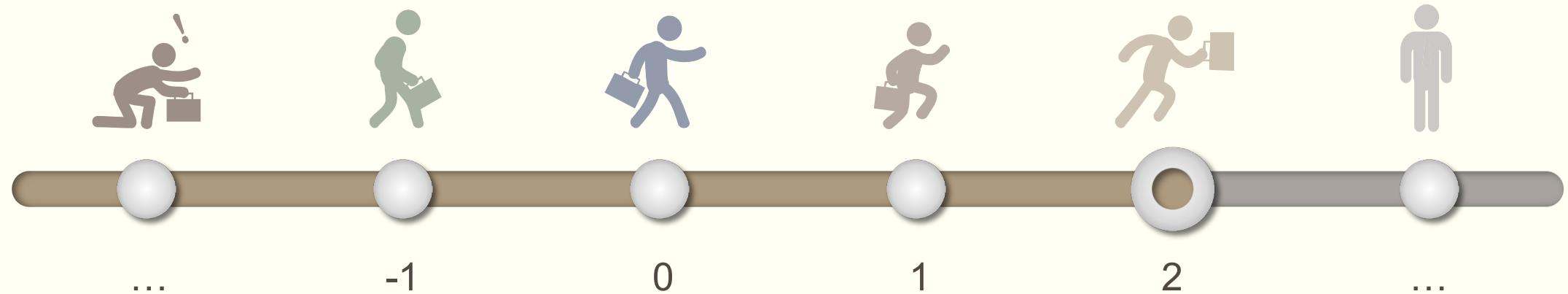
# Random Walk (1 d)



After 10 steps, what is the probability of landing on

- 0
- 1
- 2
- -1
- -2
- ...

# Random Walk



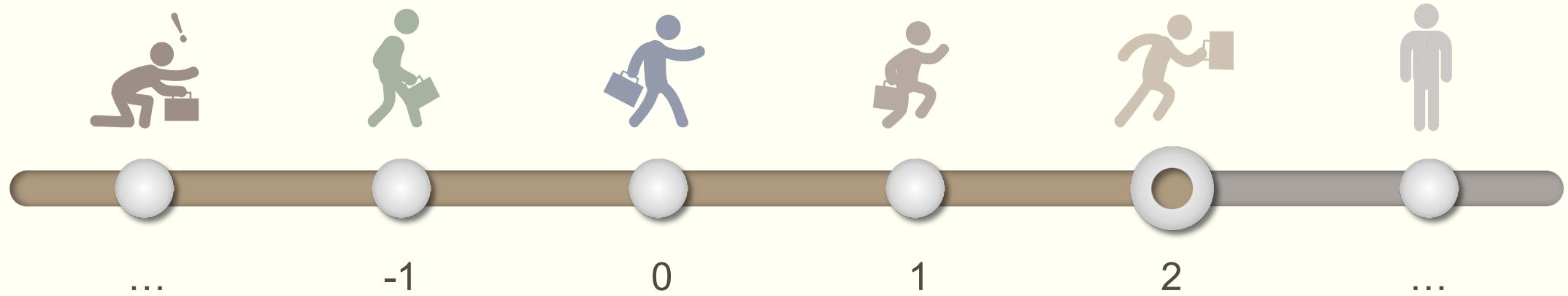
coordinate	...	-1	0	1	2	...
left	...		5		4	
right	...		5		6	
probability	...	0		0		...

$$\binom{10}{5} \times p^5(1-p)^5$$

0

$$\binom{10}{6} \times p^6(1-p)^4$$

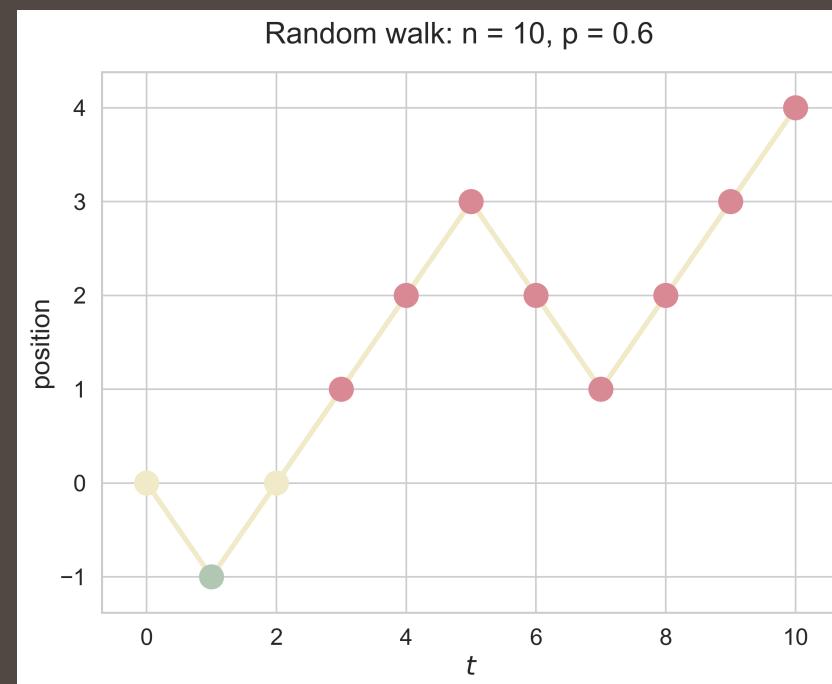
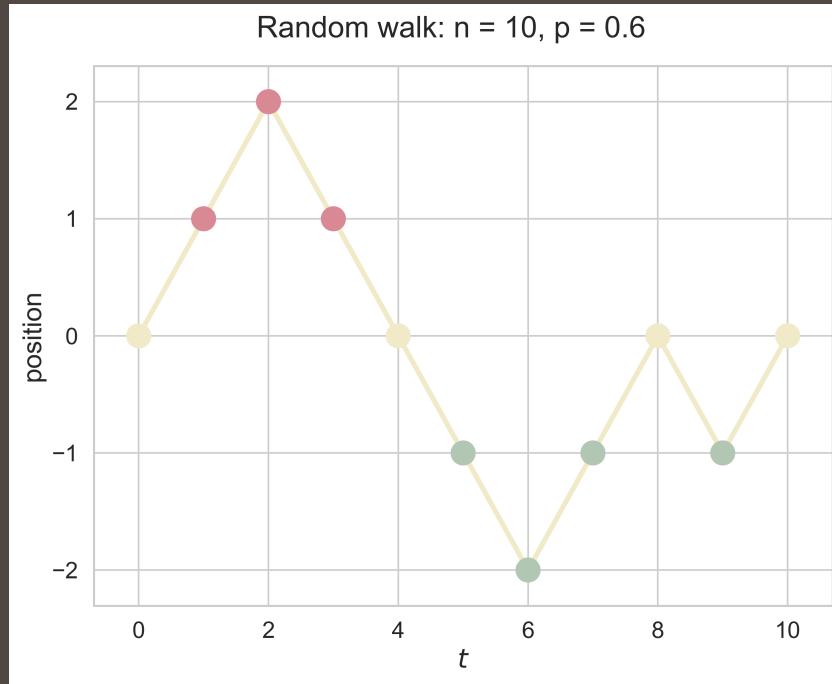
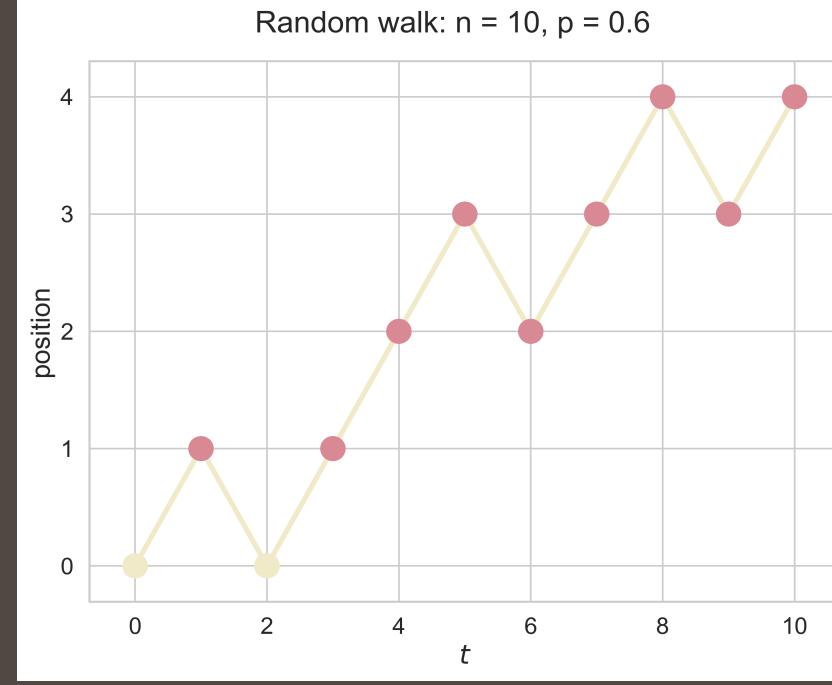
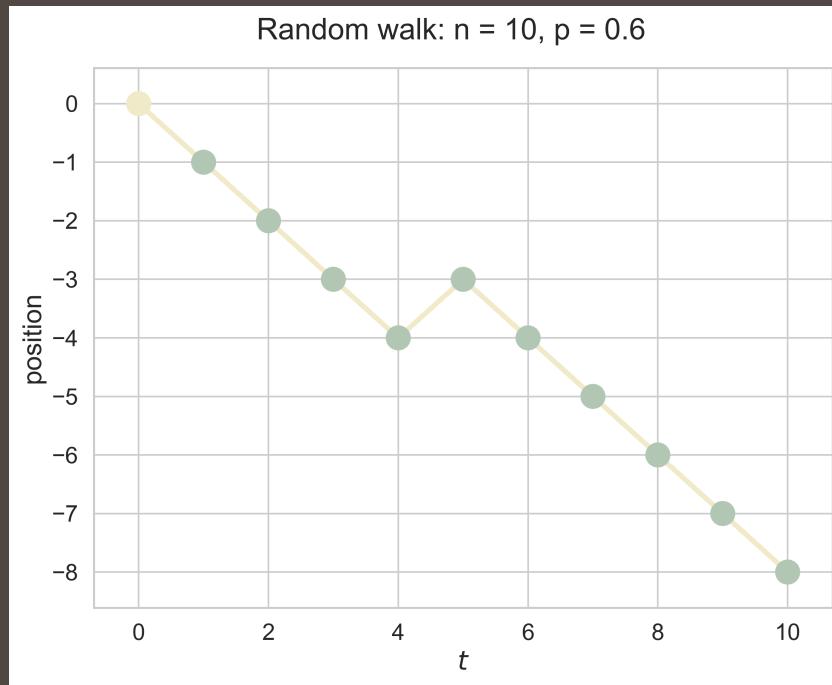
2



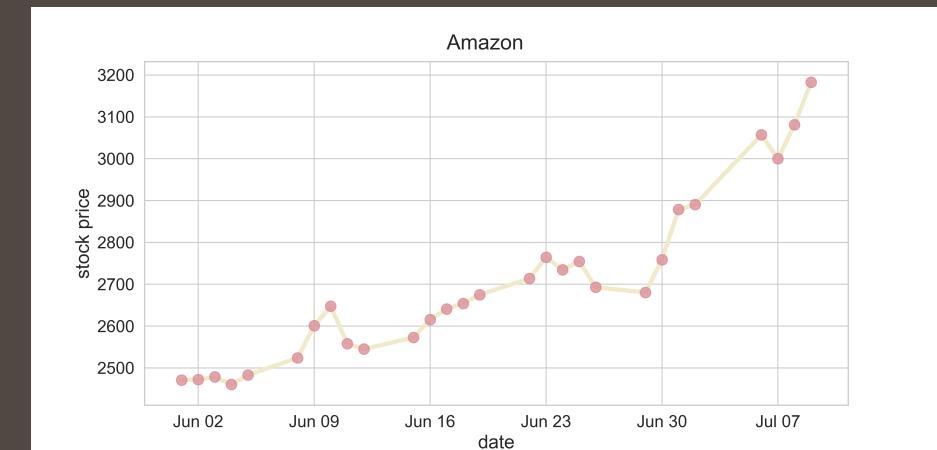
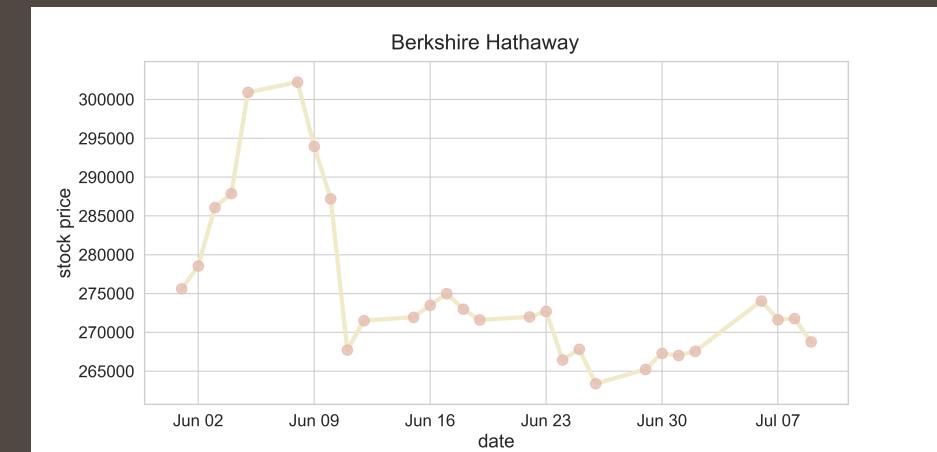
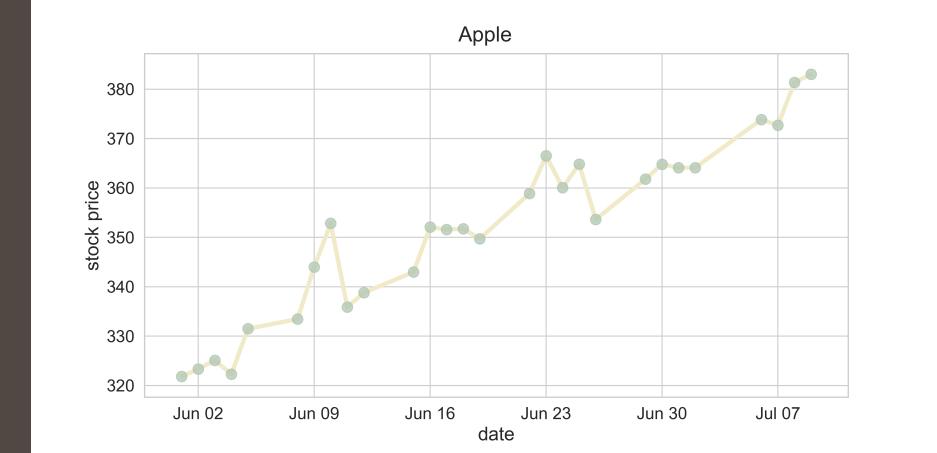
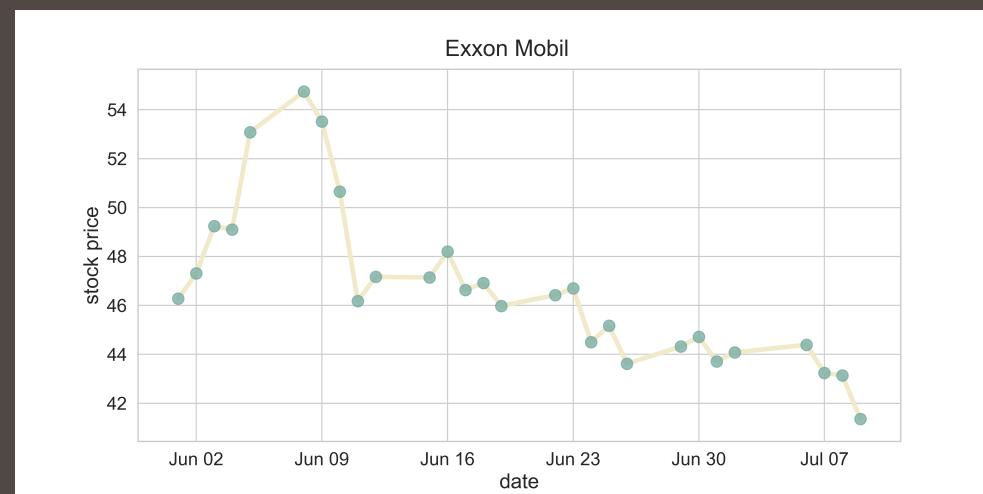
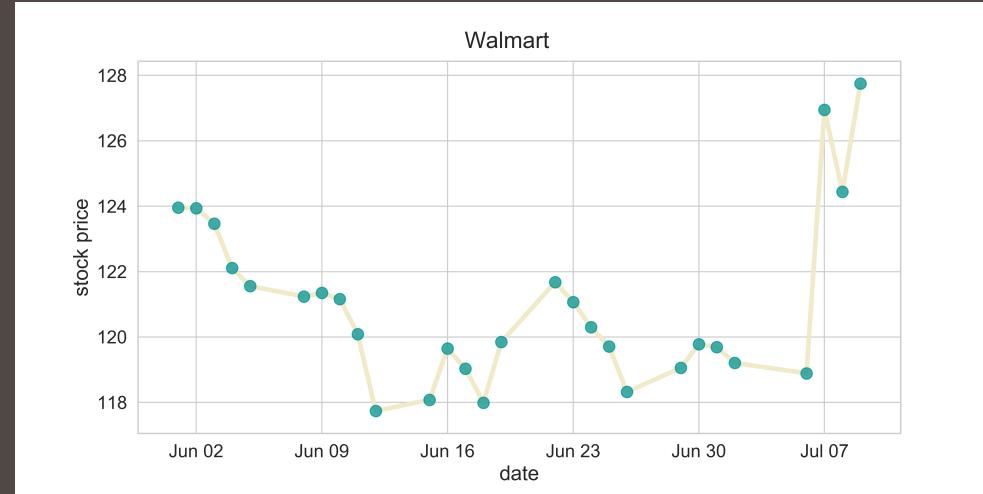
- One Dimensional Random Walk

```
random_walk_1D(n = 10, p = 0.6)
path_rw_2D(n = 10, p = 0.6, fsize = (8, 6), fs = 18, index = 1)
```

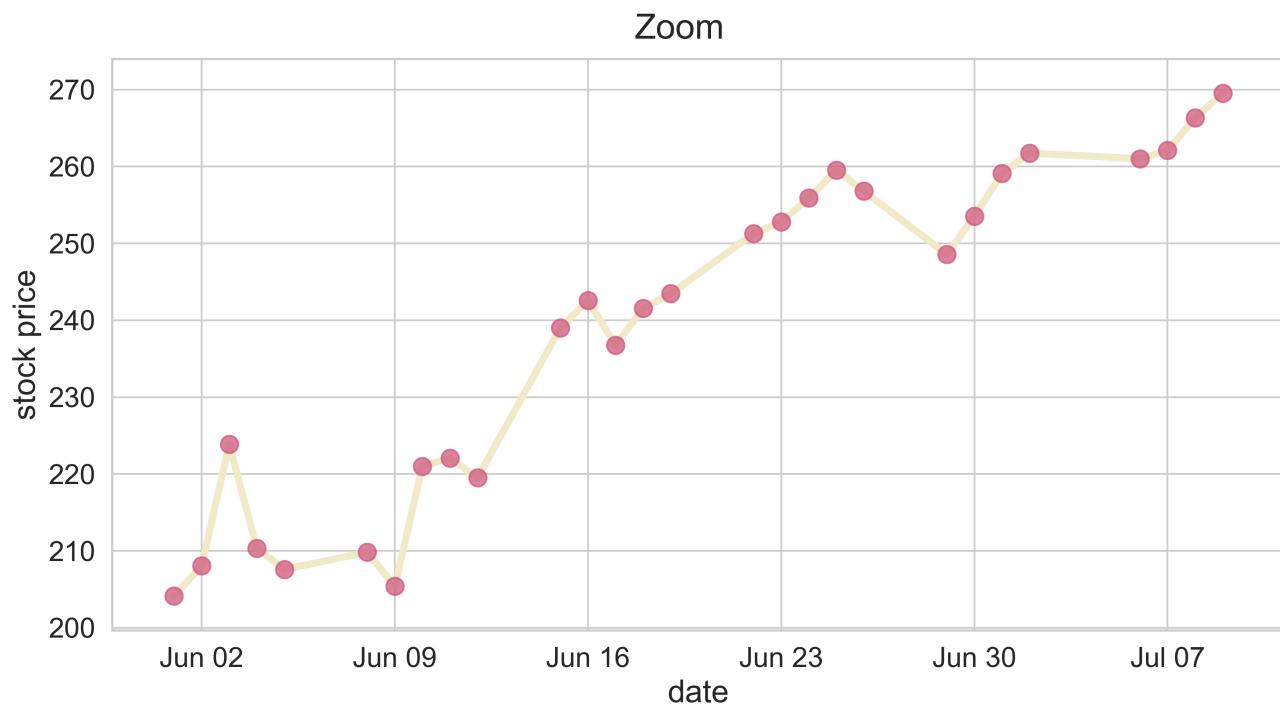
```
# Take n steps
for i in range(n):
    # Generate a random number between 0 and 1
    u = random.uniform(0, 1)
    if u <= p:
        pos += 1 # go right
    else:
        pos -= 1 # go left
    positions.append(pos)
return positions
```



# Stock Market



# Binomial lattice model for stock prices



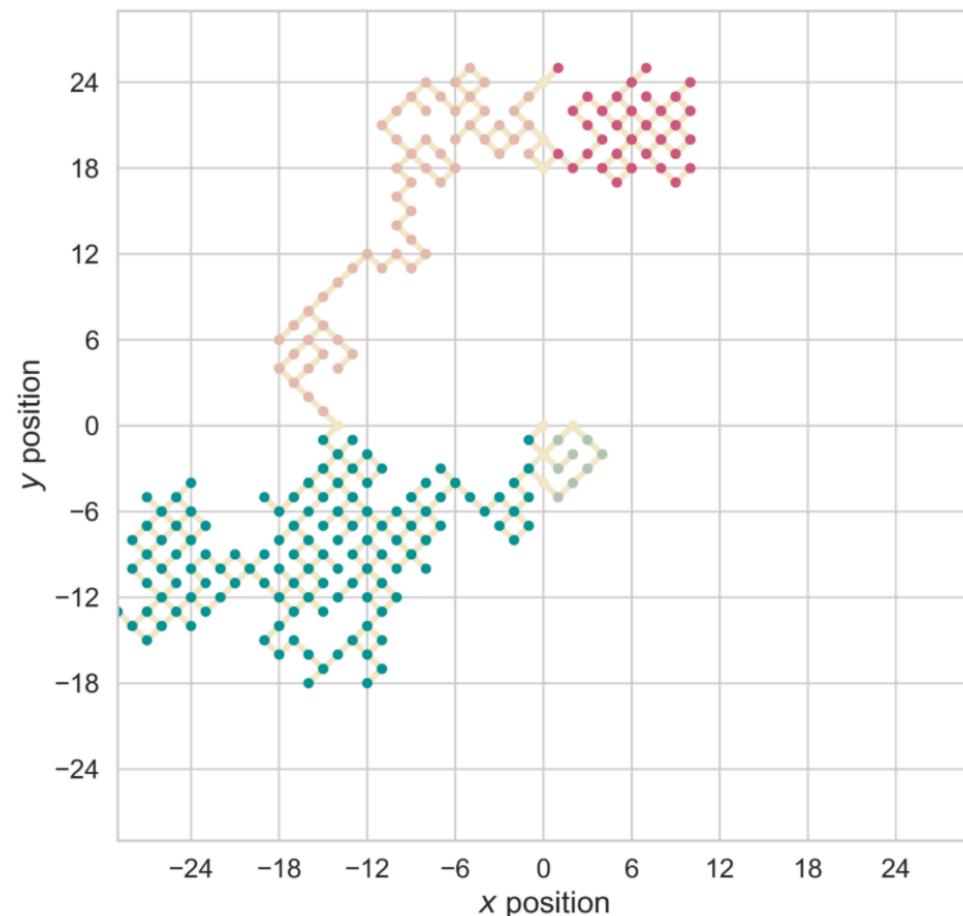
$$S(t+1) = \begin{cases} uS(t), & \text{with prob } p \\ dS(t), & \text{with prob } 1-p \end{cases}$$

```
import yfinance as yf
load_STOCK_raw(company_index = 'AAPL')
figure_stock_price(company_index = 'AMZN', date_initial = datetime.date(int(2020),int(6),int(1)),
                   fsize = (12, 6), fs = 20)
```

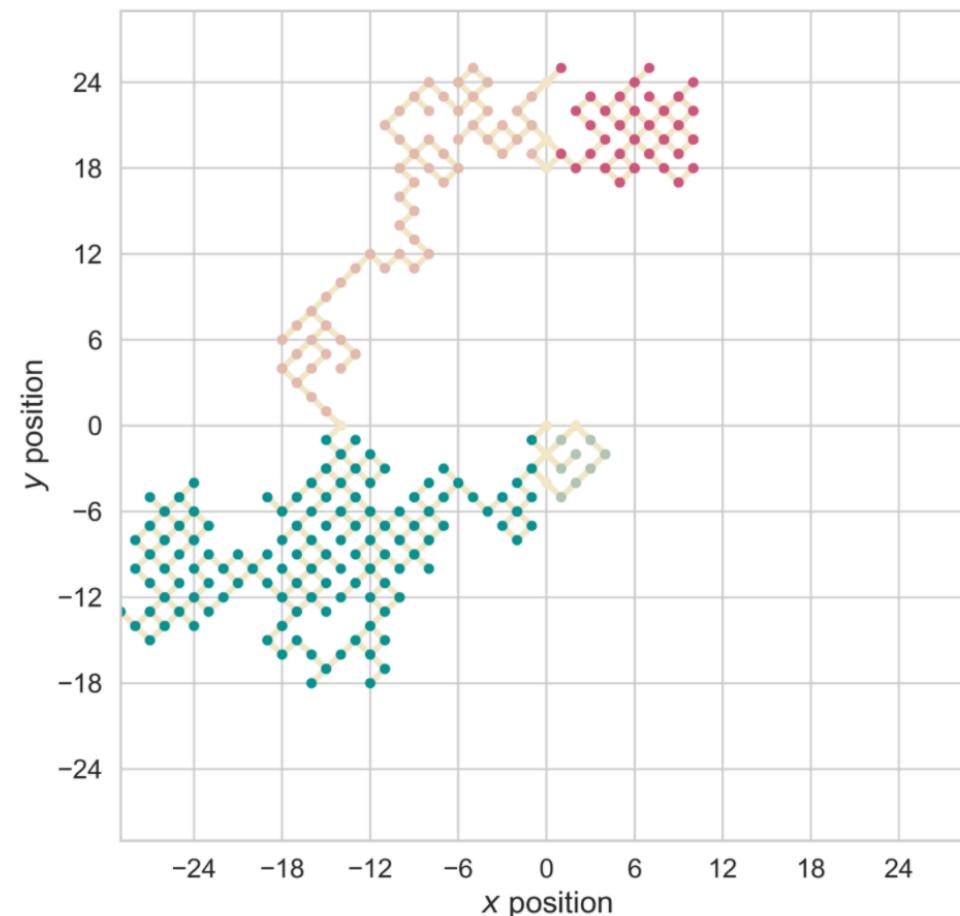
- Two Dimensional Random Walk (Take-Home Problem)

```
random_walk_2D(n = 500, p_x = 0.5, p_y = 0.5)
path_rw_2D(n = 500, p_x = 0.5, p_y = 0.5, fsize = (8, 8), fs = 18, index = 1)
```

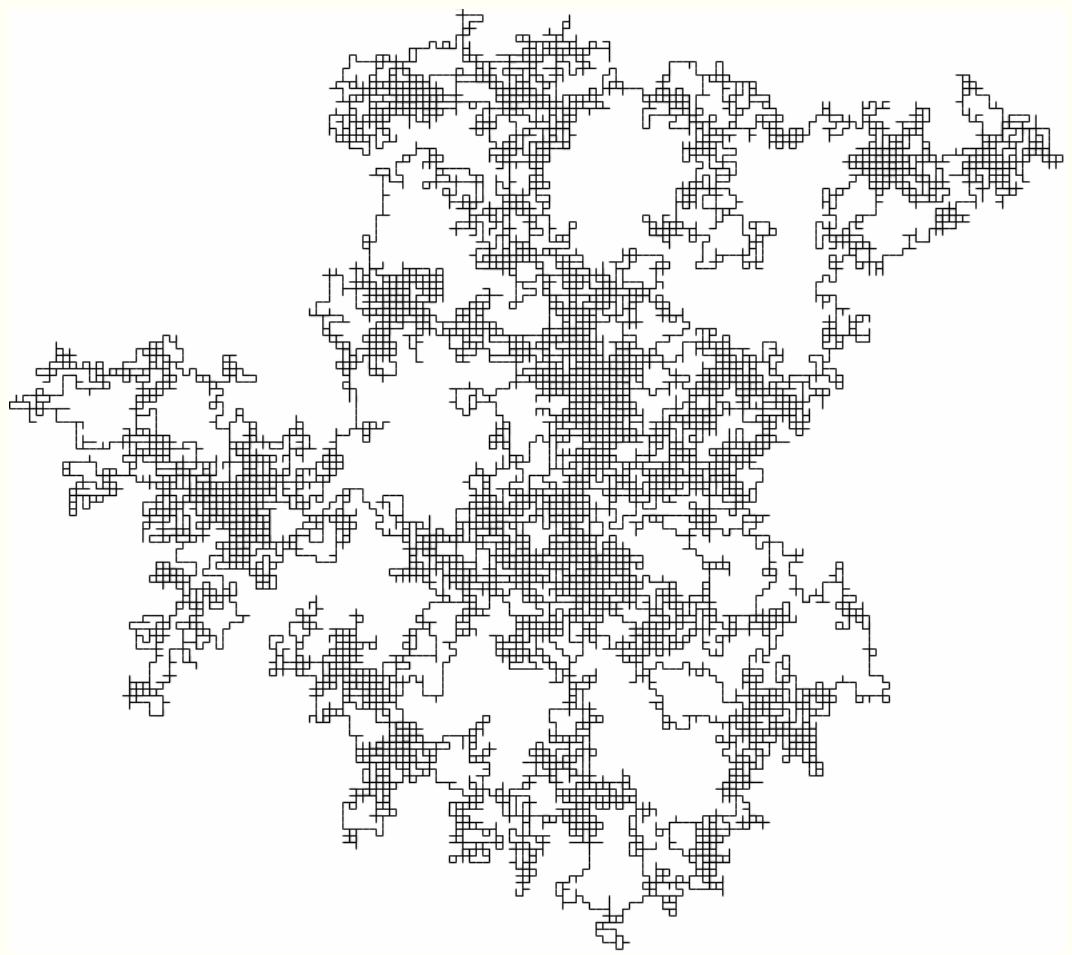
2D Random walk:  $n = 500, p = (0.5, 0.5)$



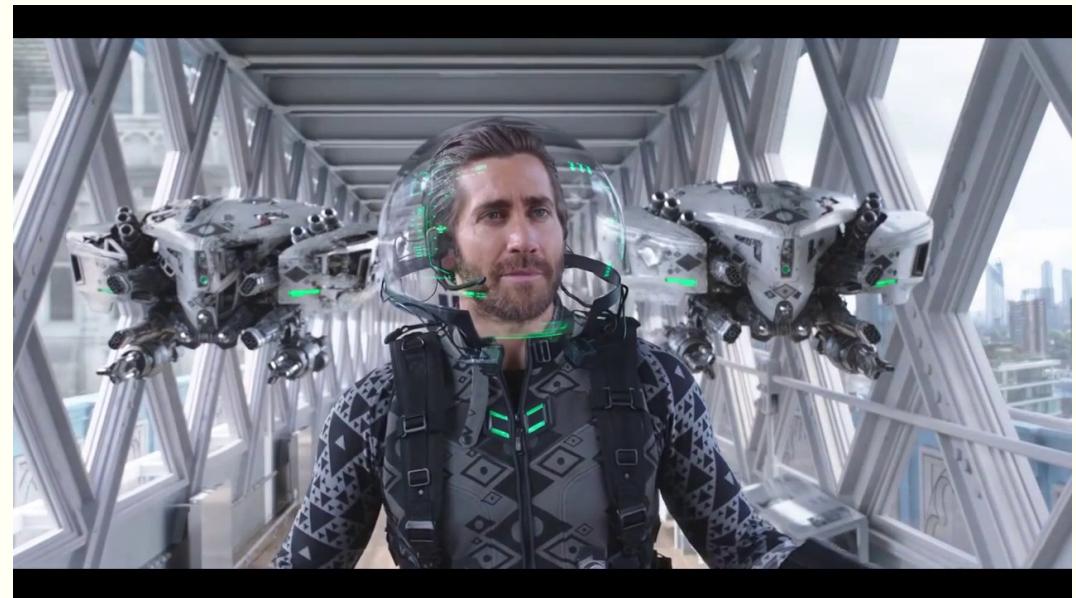
2D Random walk:  $n = 500, p = (0.5, 0.5)$

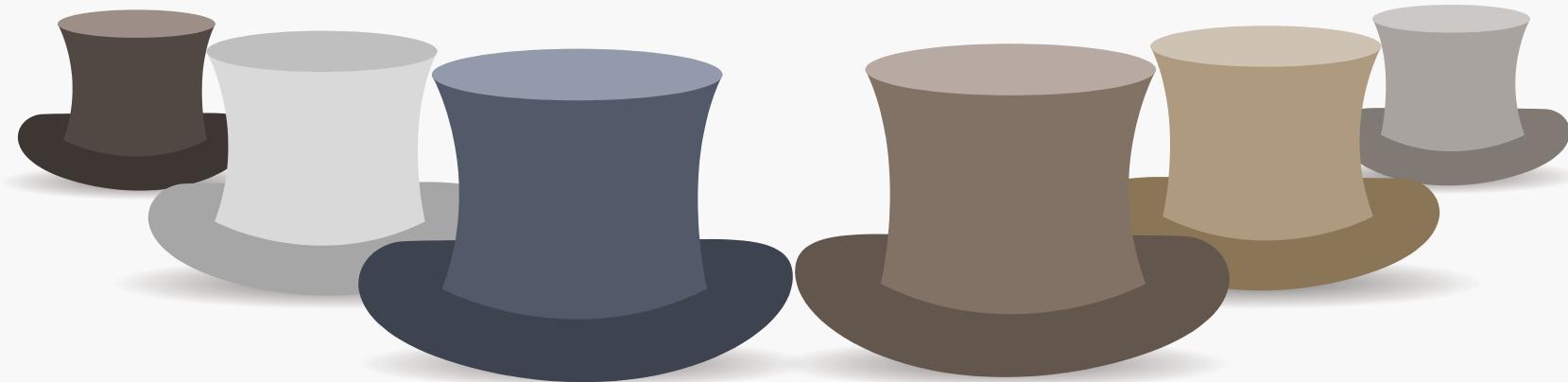


# Random Walk and AI (Robots)



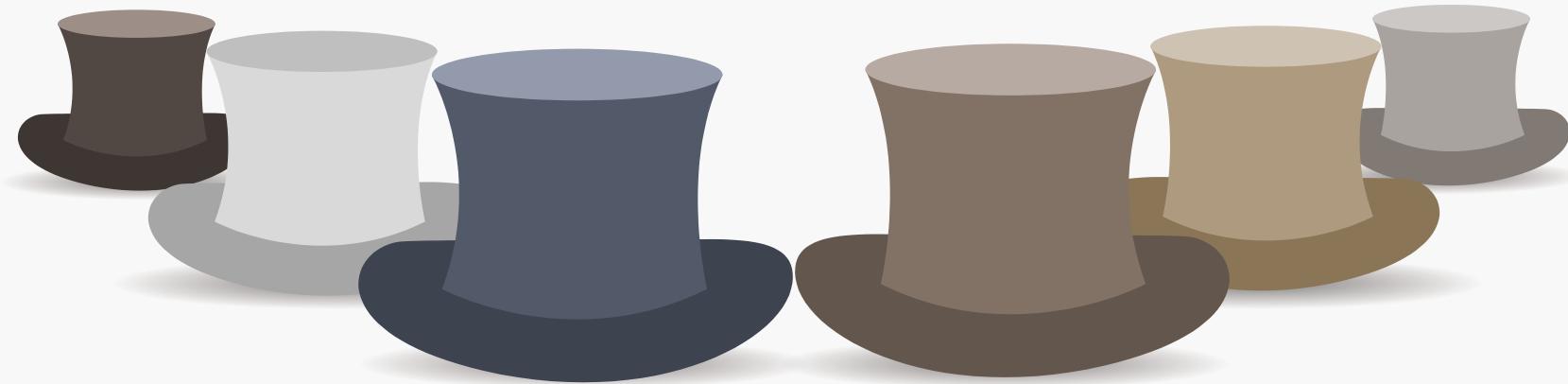
micro drones  
swarm robotics





# Hat Check Problem

In a restaurant  $n$  hats are checked and they are hopelessly scrambled.  
What is the probability that no one gets his own hat back?



# Hat Check Problem

$$p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!}.$$

# Hat Check Problem

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a random permutation

- Find the probability that it contains no fixed point.
- Recall that it is a one-to-one map of a set  $A = \{a_1, a_2, \dots, a_n\}$  onto itself.
- Let  $A_i$  be the event that the  $i$ th element  $a_i$  remains fixed under this map.

$$p_0(n) = 1 - P(A_1 \cup A_2 \cup \dots \cup A_n)$$

# Hat Check Problem

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$$P(A_i)$$

- Let  $A_i$  be the event that the  $i$ th element  $a_i$  remains fixed under this map.
- If we require that  $a_i$  is fixed, then the map of the remaining  $n - 1$  elements provides an arbitrary permutation of  $(n - 1)$  objects.
- Since there are  $(n - 1)!$  such permutations,  $P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$ .

$$\sum_{i=1}^n P(A_i) = n \times \frac{1}{n} = 1.$$

# Hat Check Problem

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$$P(A_i \cap A_j)$$

- To have a particular pair  $(a_i, a_j)$  fixed, we can choose any permutation of the remaining  $n - 2$  elements.
- There are  $(n - 2)!$  such choices and thus  $P(A_i \cap A_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$

$$\sum_{1 \leq i < j \leq n} P(A_i \cap A_j) = \binom{n}{2} \times \frac{1}{n(n-1)} = \frac{1}{2!}$$

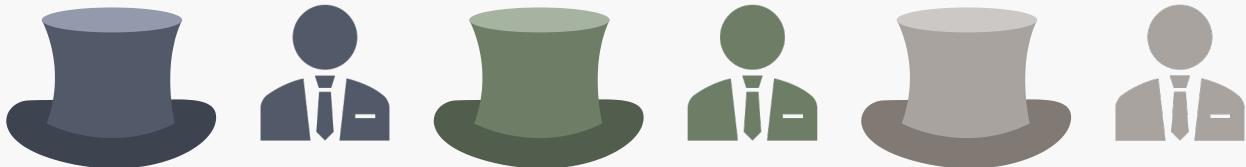


# Hat Check Problem

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- $P(A_i) = \frac{(n-1)!}{n!}$
- $P(A_i \cap A_j) = \frac{(n-2)!}{n!}$
- $P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!}$
- ...
- $\sum_{i=1}^n P(A_i) = n \times \frac{(n-1)!}{n!} = 1$
- $\sum_{1 \leq i < j \leq n} P(A_i \cap A_j) = \binom{n}{2} \times \frac{1}{n(n-1)} = \frac{1}{2!}$
- $\sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) = \binom{n}{3} \times \frac{(n-3)!}{n!} = \frac{1}{3!}$
- ...



# Hat Check Problem

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$$p_0(n) = 1 - P(A_1 \cup A_2 \cup \dots \cup A_n)$$

- $\sum_{i=1}^n P(A_i) = n \times \frac{(n-1)!}{n!} = 1$
- $\sum_{1 \leq i < j \leq n} P(A_i \cap A_j) = \binom{n}{2} \times \frac{1}{n(n-1)} = \frac{1}{2!}$
- $\sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) = \binom{n}{3} \times \frac{(n-3)!}{n!} = \frac{1}{3!}$
- ...

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^n P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) - \dots.$$

$$p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots + \frac{(-1)^n}{n!}.$$

