MATH 20: PROBABILITY

Expected Value & Variance of Continuous Random Variables

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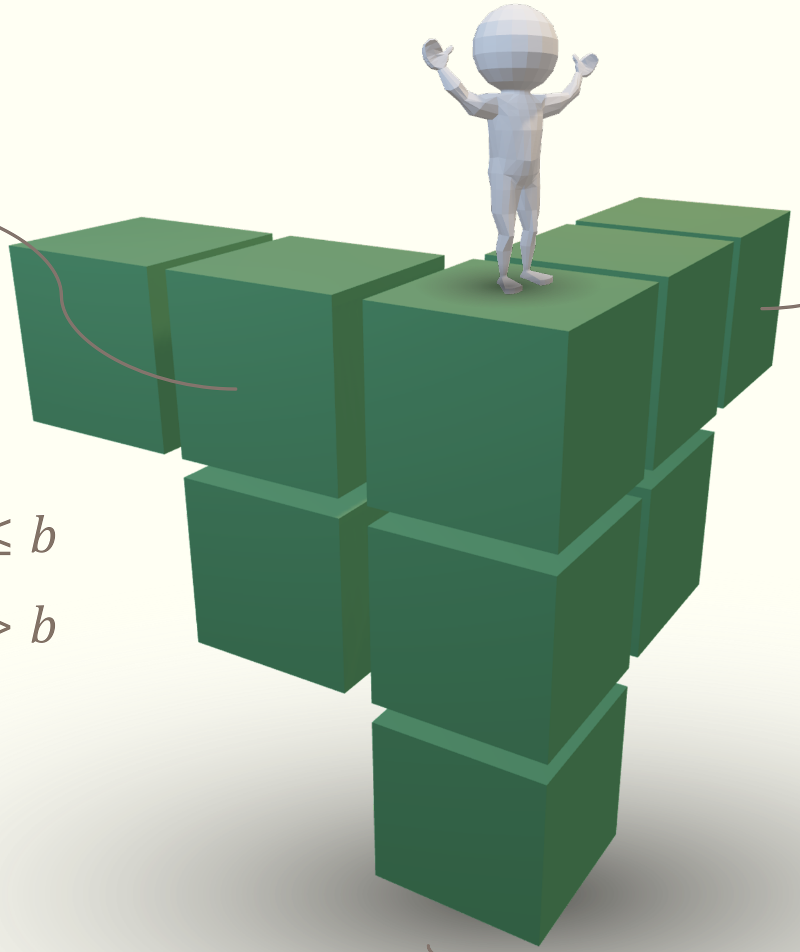
Important Densities

Continuous uniform density

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & x < a \text{ or } x > b \end{cases}$$



Exponential density

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}.$$

Normal density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Continuous uniform

$$f(x) = \frac{1}{b-a} \qquad E(X) = \frac{1}{2}(a+b) \qquad V(X) = \frac{1}{12}(b-a)^2$$

Exponential

$$f(x) = \lambda e^{-\lambda x} \qquad E(X) = \frac{1}{\lambda} \qquad V(X) = \frac{1}{\lambda^2}$$

Normal

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad E(X) = \mu \qquad V(X) = \sigma^2$$

Expected Value

- Let X be a **real-valued** random variable with density function $f(x)$.
- The expected value $\mu = E(X)$ is defined by

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx,$$

provided $\int_{-\infty}^{+\infty} |x| f(x) dx$ is finite.

Discrete expected value $E(X)$
$\sum_{x \in \Omega} x m(x)$

Continuous expected value $E(X)$
$\int_{-\infty}^{+\infty} x f(x) dx$

Expectation of Functions of Random Variables

- If X is a real-valued random variable and if $\phi\colon R \to R$ is a continuous real-valued function with domain $[a, b]$, then

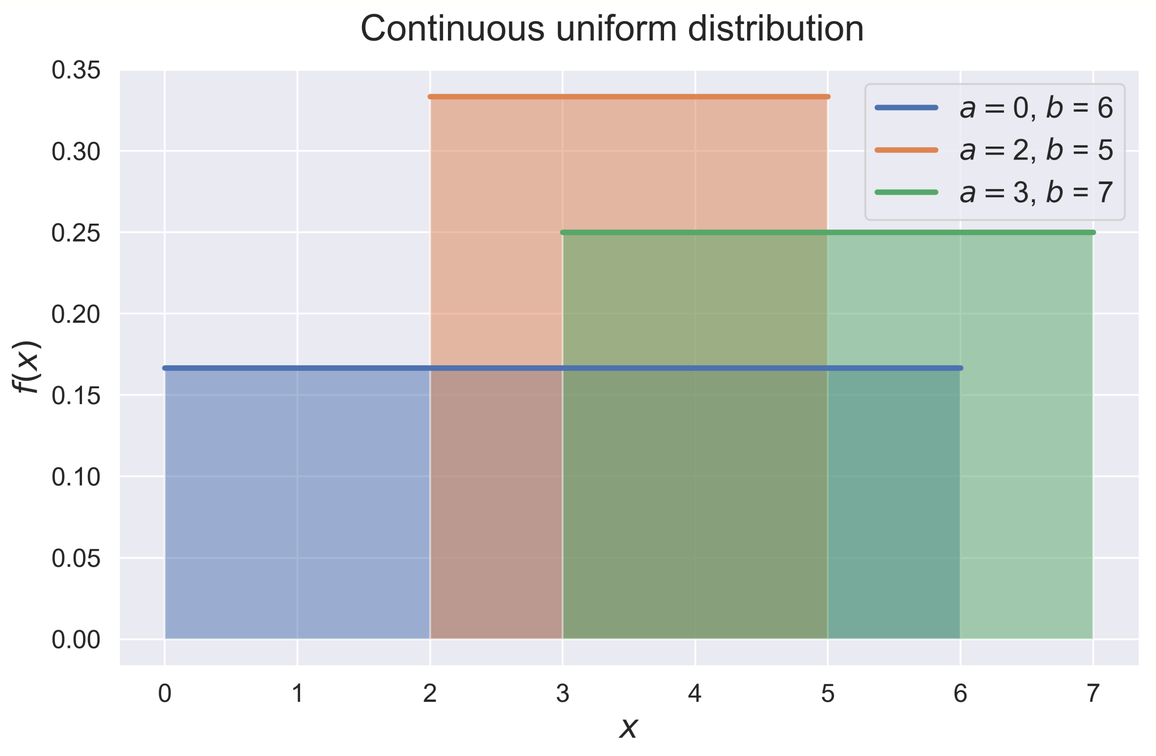
$$E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x) f(x) dx,$$

provided the integral exists.

Discrete expected value $\boldsymbol{E(\phi(X))}$	Continuous expected value $\boldsymbol{E(\phi(X))}$
$\sum_{x \in \Omega} \phi(x) m(x)$	$\int_{-\infty}^{+\infty} \phi(x) f(x) dx$

Example 1

Continuous uniform distribution
$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b$$
$$E(X) = \cdots$$



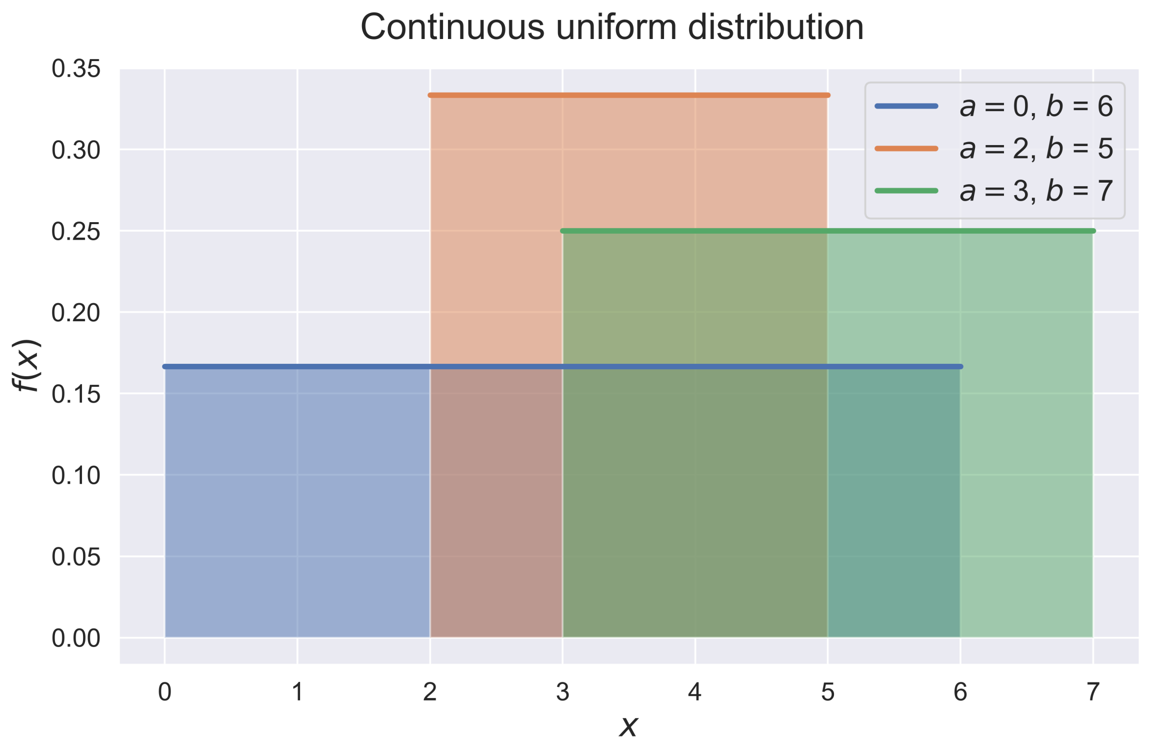
Continuous uniform distribution

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \frac{1}{2} x^2 |_a^b = \frac{1}{2} \frac{b^2 - a^2}{b-a}$$

$$= \frac{1}{2}(a+b)$$

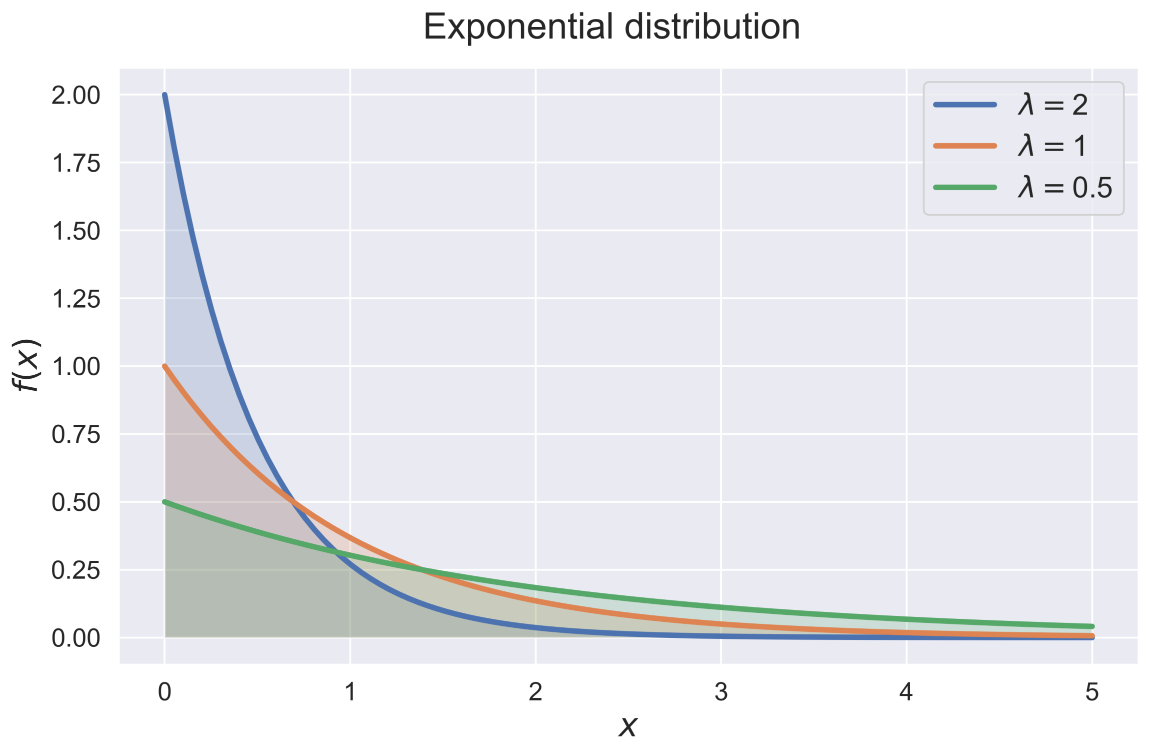


Example 2

Exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

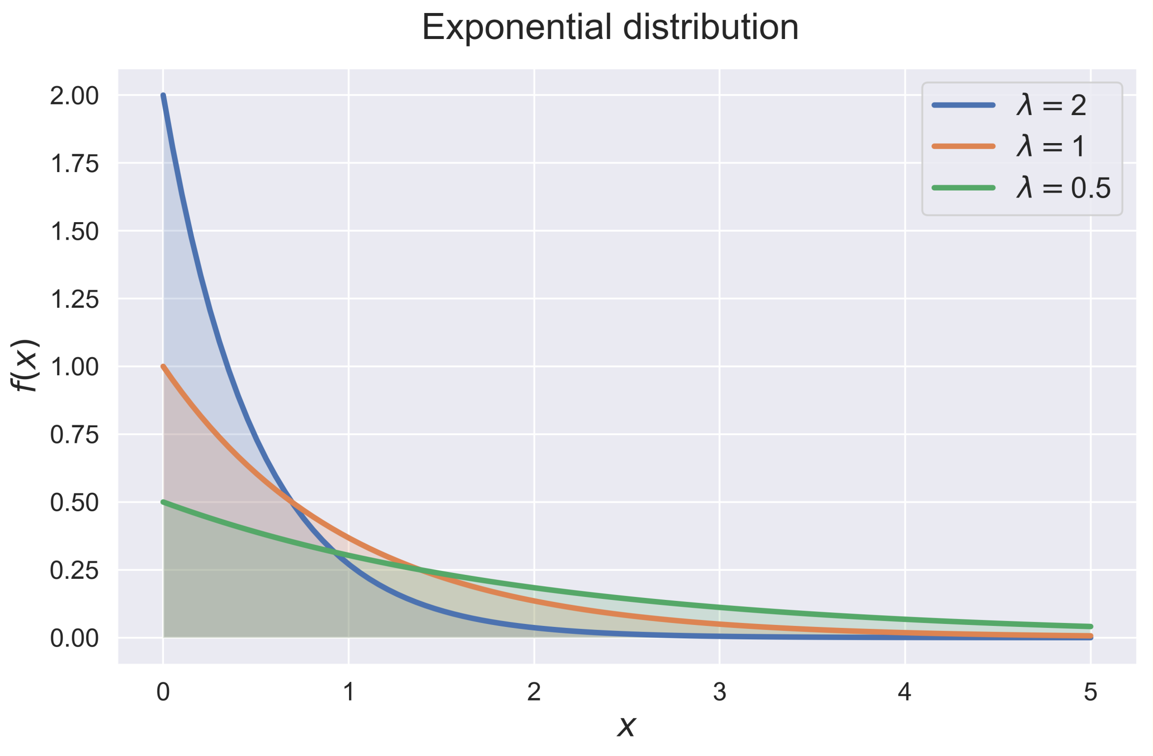
$$E(X) = \cdots$$



Exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_{0}^{+\infty} x \lambda e^{-\lambda x} dx$$
$$= \int_{0}^{+\infty} -x d e^{-\lambda x} = -x e^{-\lambda x}|_{0}^{+\infty} + \int_{0}^{+\infty} e^{-\lambda x} dx$$
$$= \int_{0}^{+\infty} e^{-\lambda x} dx = -\frac{1}{\lambda} e^{-\lambda x}|_{0}^{+\infty}$$
$$= \frac{1}{\lambda}$$



Example 3

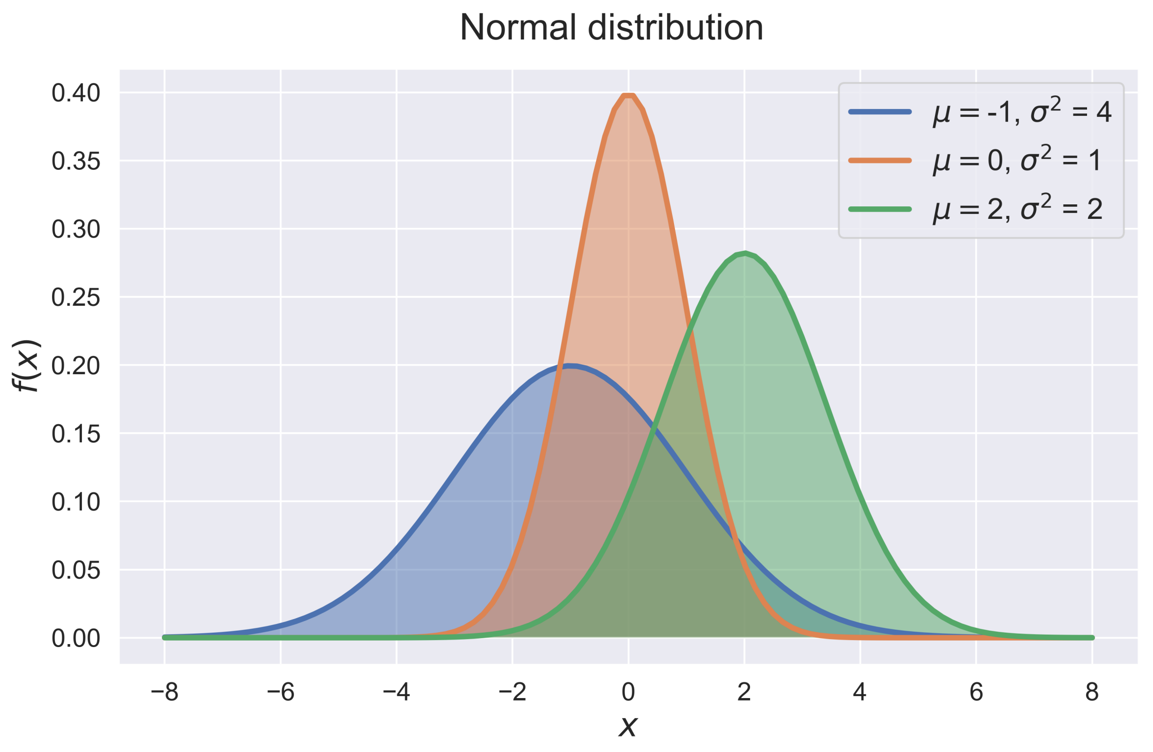
Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

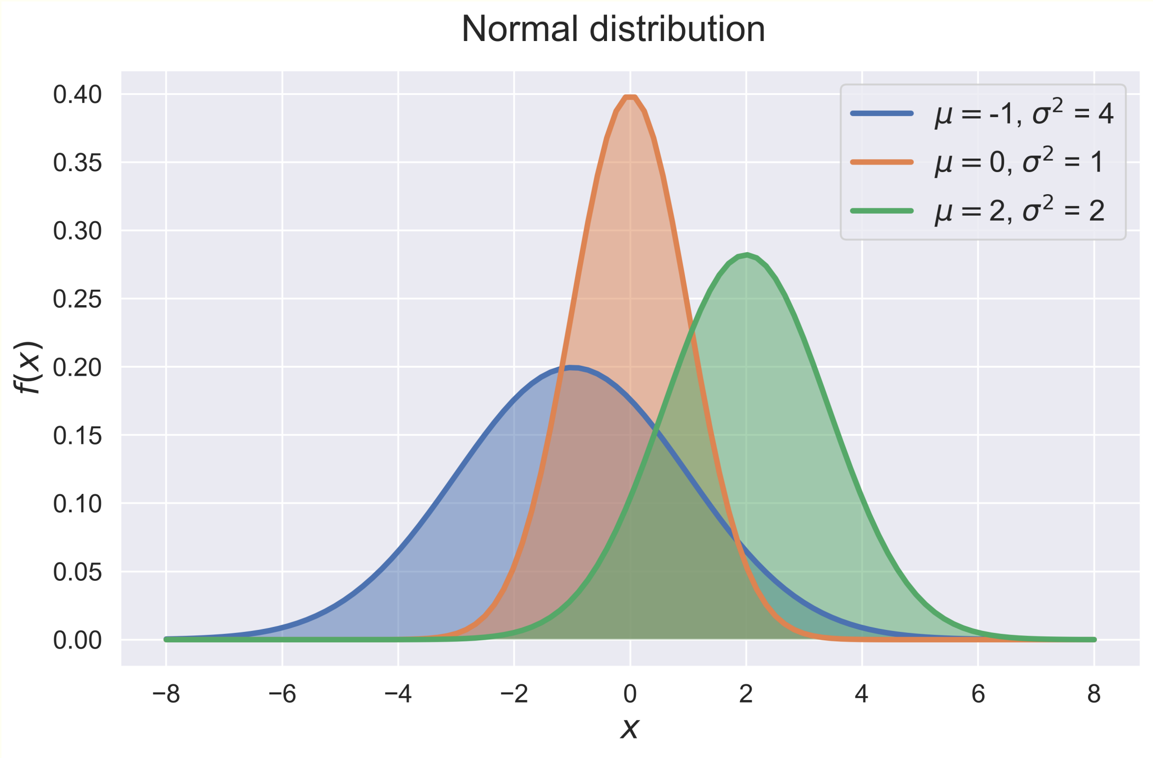
$$E(X) = \cdots$$



Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{1}{2}x^2} dx = 0$$



Sum of Random Variables

- If X and Y are real-valued random variables with finite expected values. Then

$$E(X + Y) = E(X) + E(Y),$$

  and if c is any constant, then

$$E(cX) = cE(X).$$

- In general, for a linear combination of n real-valued random variables X_i with constants c_i, we have

$$E(c_1X_1 + c_2X_2 + \cdots + c_nX_n) = c_1E(X_1) + c_2E(X_2) + \cdots + c_nE(X_n).$$

Do we need mutual independence of the summands?

?

The Product of Two Random Variables

- Let X and Y be independent real-valued continuous random variables with finite expected values. Then we have

$$E(XY) = E(X)E(Y).$$

- More generally, for n mutually independent random variables X_i, we have

$$E(X_1X_2\cdots X_n) = E(X_1)E(X_2)\cdots E(X_n).$$

Do we need mutual independence of the factors?

?

Example 3 revisited

Standard normal distribution
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
$E(X) = 0$

Normal distribution
$Z = \sigma X + \mu$
$f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{z-\mu}{\sigma})^2}$
$E(Z) = E(\sigma X + \mu) = \mu$

Conditional Expectation

- Conditional density

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

joint density

marginal density

- Conditional expected value

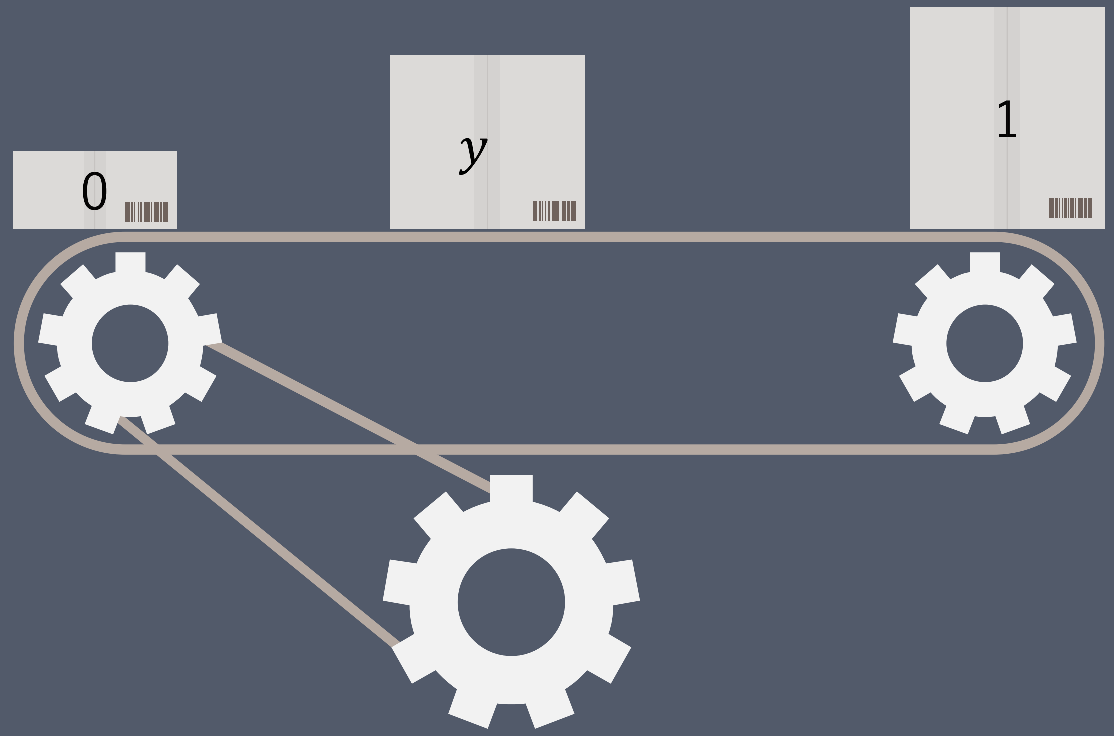
$$E(X|Y=y) = \int x f_{X|Y}(x|y)dx.$$

Example

- A point Y is chosen at random from $[0, 1]$ uniformly. A second point X is then uniformly and randomly chosen from the interval $[0, Y]$. Find the density and conditional expectation for X.

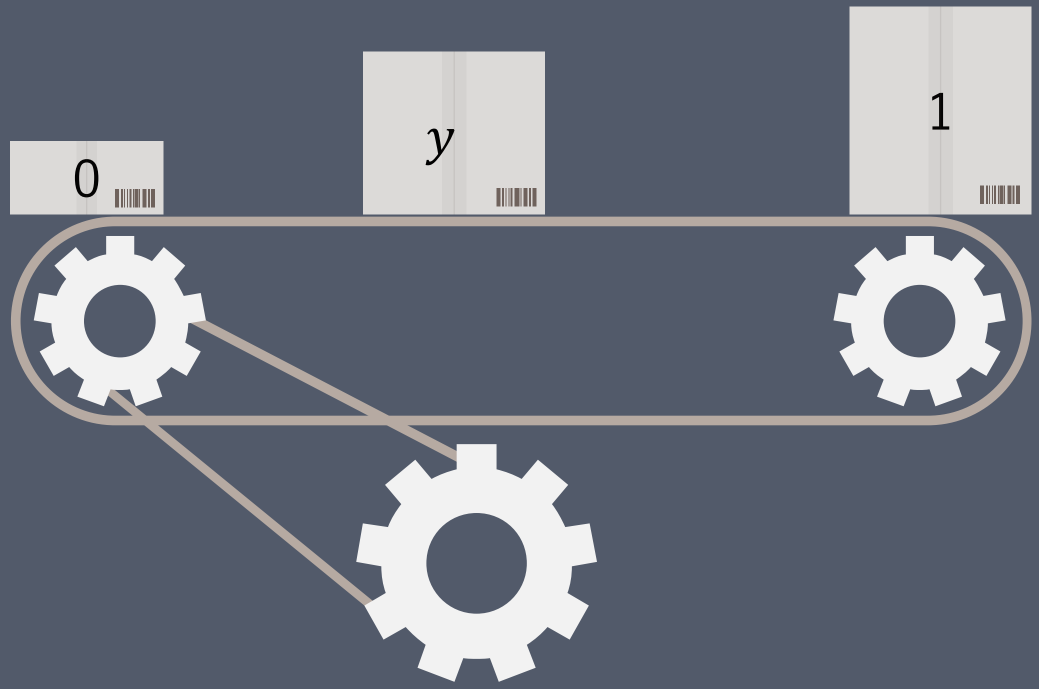
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,\ y)}{f_Y(y)}$$

$$E(X|Y = y) = \int x f_{X|Y}(x|y)dx$$



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,\ y)}{f_Y(y)}$$

$$E(X|Y=y) = \int x f_{X|Y}(x|y)dx$$



$$f_{X,Y}(x,y) = \frac{1}{y}$$

$$f_Y(y) = 1$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,\ y)}{f_Y(y)} = \frac{1}{y}$$

$$E(X|Y=y) = \int_0^y \frac{x}{y}dx = \frac{1}{2}y$$

Conditional Expectation

- Conditional density

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

joint density

marginal density

- Conditional expected value

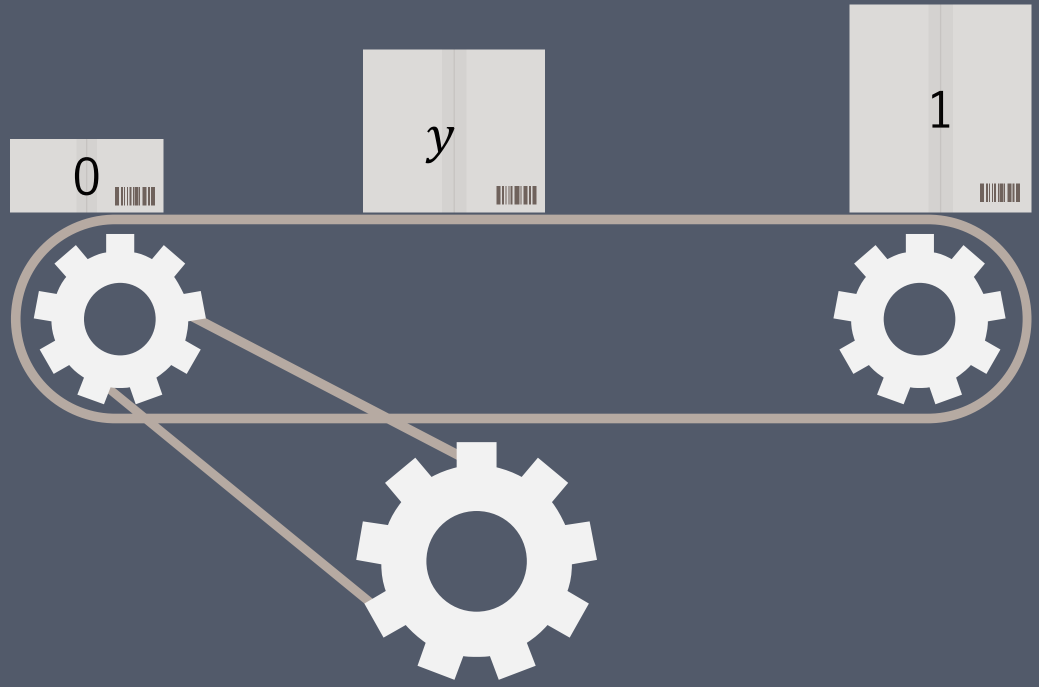
$$E(X|Y=y) = \int x f_{X|Y}(x|y)dx.$$

- Expected value

$$E(X) = \int E(X|Y=y) f_Y(y)dy.$$

$$E(X) = \int E(X|Y=y) f_Y(y) dy$$

!



$$E(X|Y=y) = \int_0^y \frac{x}{y} dx = \frac{1}{2} y$$

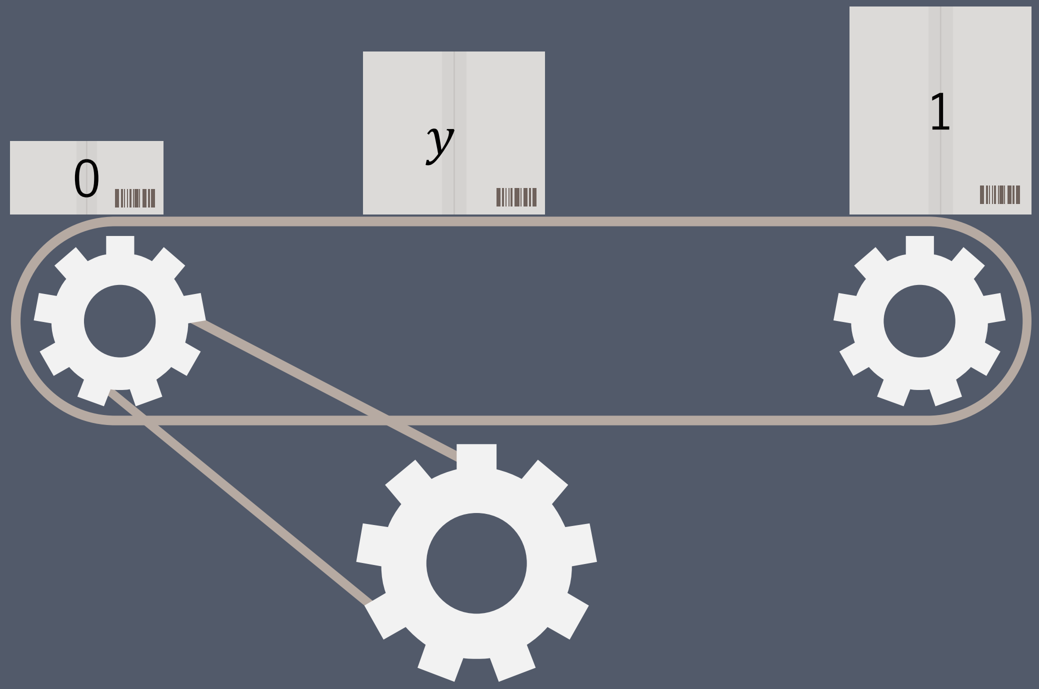
$$E(X) = \int_0^1 \frac{1}{2} y dy = \frac{1}{4}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{y}$$

marginal density: $f_Y(y) = 1$

joint density: $f_{X,Y}(x,y) = \frac{1}{y}$

marginal density: $f_X(x) = \cdots$



$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy$$

!

marginal density: $f_Y(y) = 1$

joint density: $f_{X,Y}(x,y) = \frac{1}{y}$

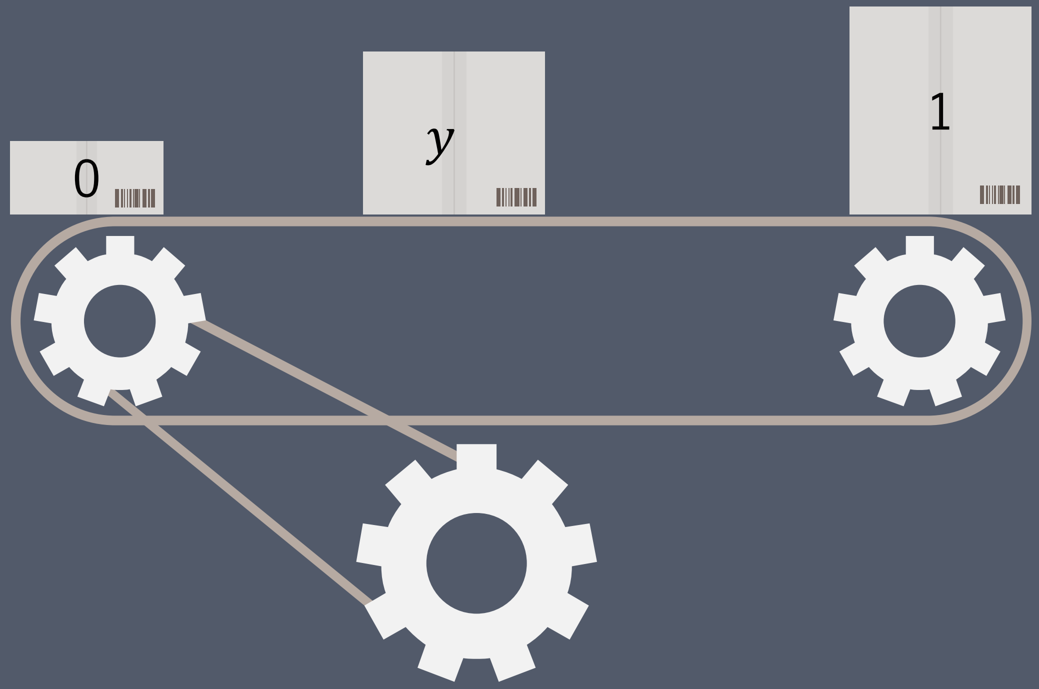
$0 \le x \le y$

marginal density: $f_X(x)$

$$= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \int_0^1 f_{X,Y}(x,y)dy$$

$$= \int_x^1 \frac{1}{y} dy$$

$$= -\ln(x)$$



$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy$$

!

Continuous uniform

$$f(x) = \frac{1}{b-a} \qquad E(X) = \frac{1}{2}(a+b) \qquad V(X) = \frac{1}{12}(b-a)^2$$

Exponential

$$f(x) = \lambda e^{-\lambda x} \qquad E(X) = \frac{1}{\lambda} \qquad V(X) = \frac{1}{\lambda^2}$$

Normal

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad E(X) = \mu \qquad V(X) = \sigma^2$$

Variance

- Let X be real-valued random variable with density function $f(x)$ and with the expected value $E(X) = \mu$. The variance $\sigma^2 = V(X)$ is defined by

$$\sigma^2 = V(X) = E\left((X-\mu)^2\right) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx\,.$$

Discrete variance $V(X)$

$$\sum_{x\in\Omega}(x-\mu)^2 m(x)$$

Continuous variance $V(X)$

$$\int_{-\infty}^{+\infty}(x-\mu)^2 f(x)dx$$

$$V(X) = E(X^2) - \mu^2$$

Properties of Variance

- If X is any random variable and c is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

- If X and Y are independent real-valued random variables, then

$$V(X + Y) = V(X) + V(Y).$$

When do we need independence?

$X + Y$

XY

Do not need

$E(X + Y) = E(X) + E(Y)$

...

...

Need

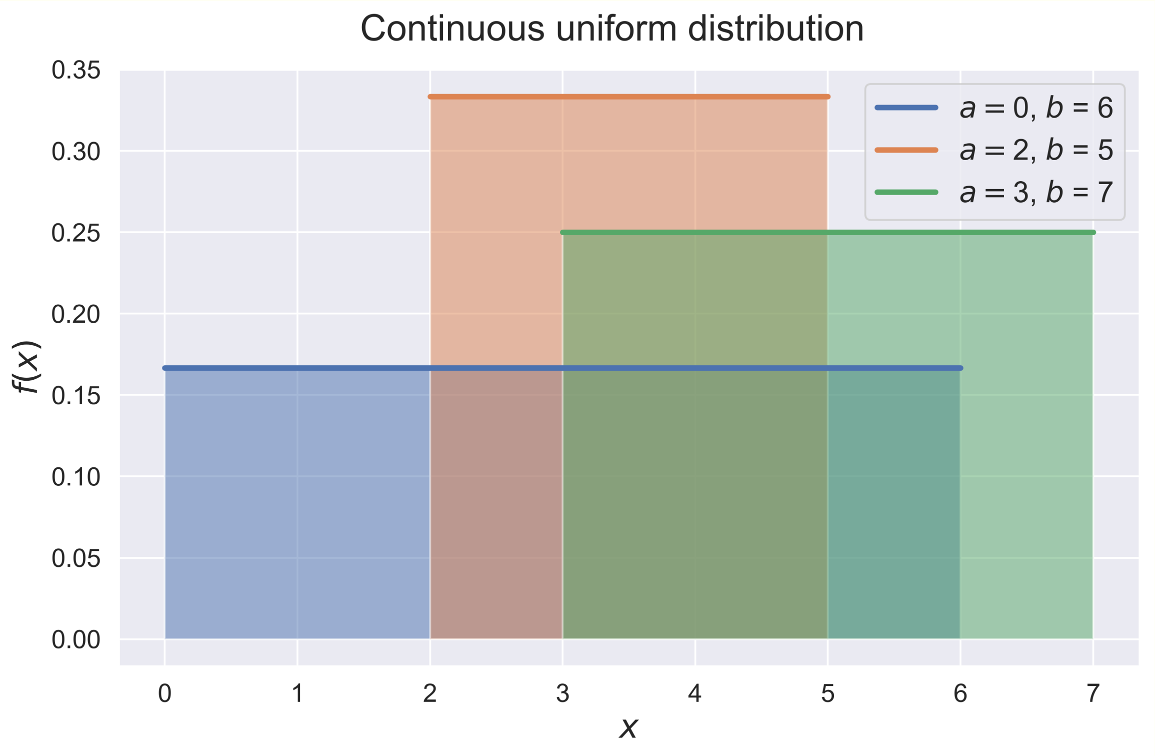
$E(XY) = E(X)E(Y)$

$V(X + Y) = V(X) + V(Y)$

...

Example 1

Continuous uniform distribution
$f(x) = \dfrac{1}{b-a}, \qquad a \le x \le b$
$E(X) = \dfrac{1}{2}(a+b)$
$V(X) = \cdots$

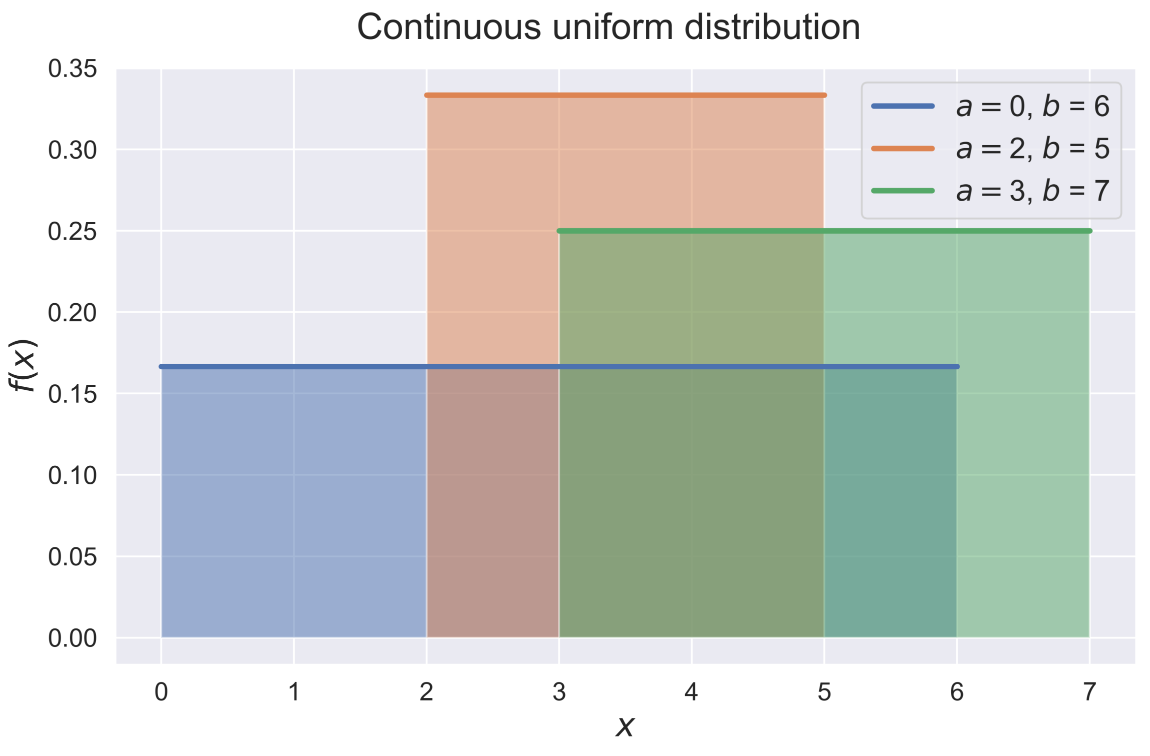


Continuous uniform distribution

$$f(x) = \frac{1}{b-a}, \qquad a \leq x \leq b$$

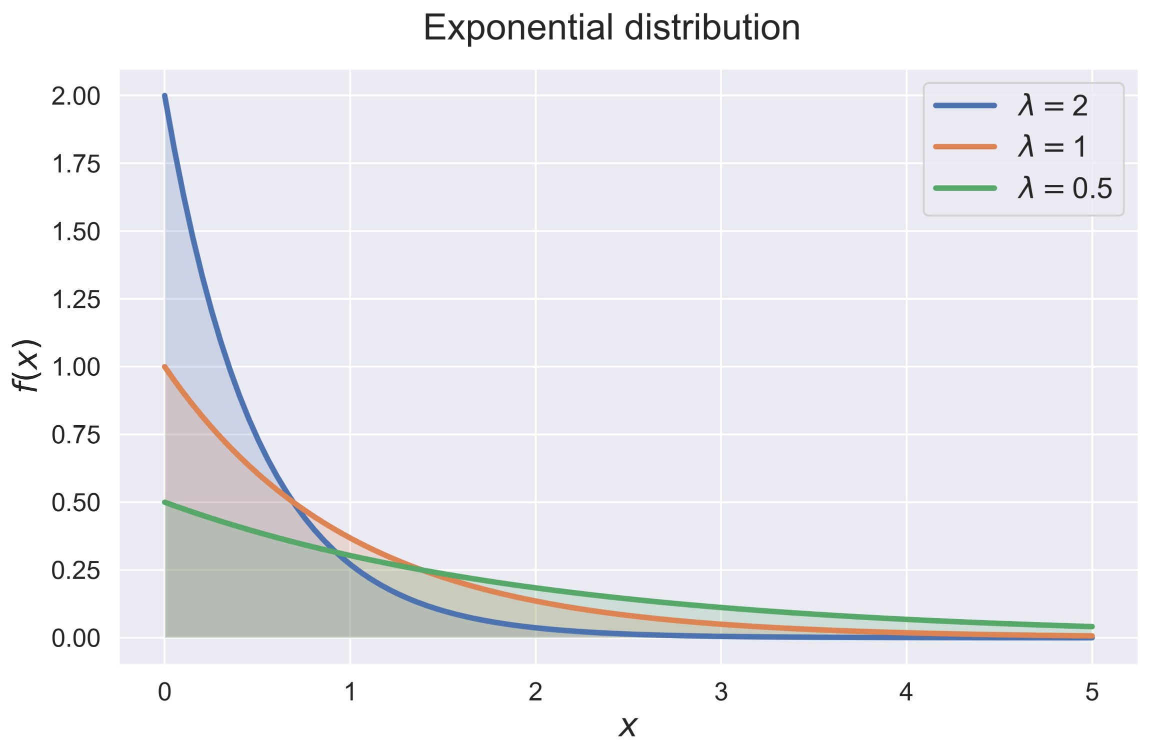
$$E(X) = \frac{1}{2}(a+b)$$

$$V(X) = E(X^2) - \mu^2$$
$$= \int_a^b x^2 \frac{1}{b-a} dx - \frac{1}{4}(a+b)^2$$
$$= \frac{1}{b-a}\frac{1}{3}x^3|_a^b - \frac{1}{4}(a+b)^2$$
$$= \frac{1}{3}\frac{b^3-a^3}{b-a} - \frac{1}{4}(a+b)^2$$
$$= \frac{1}{12}(b-a)^2$$



Example 2

Exponential distribution
$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$
$$E(X) = \frac{1}{\lambda}$$

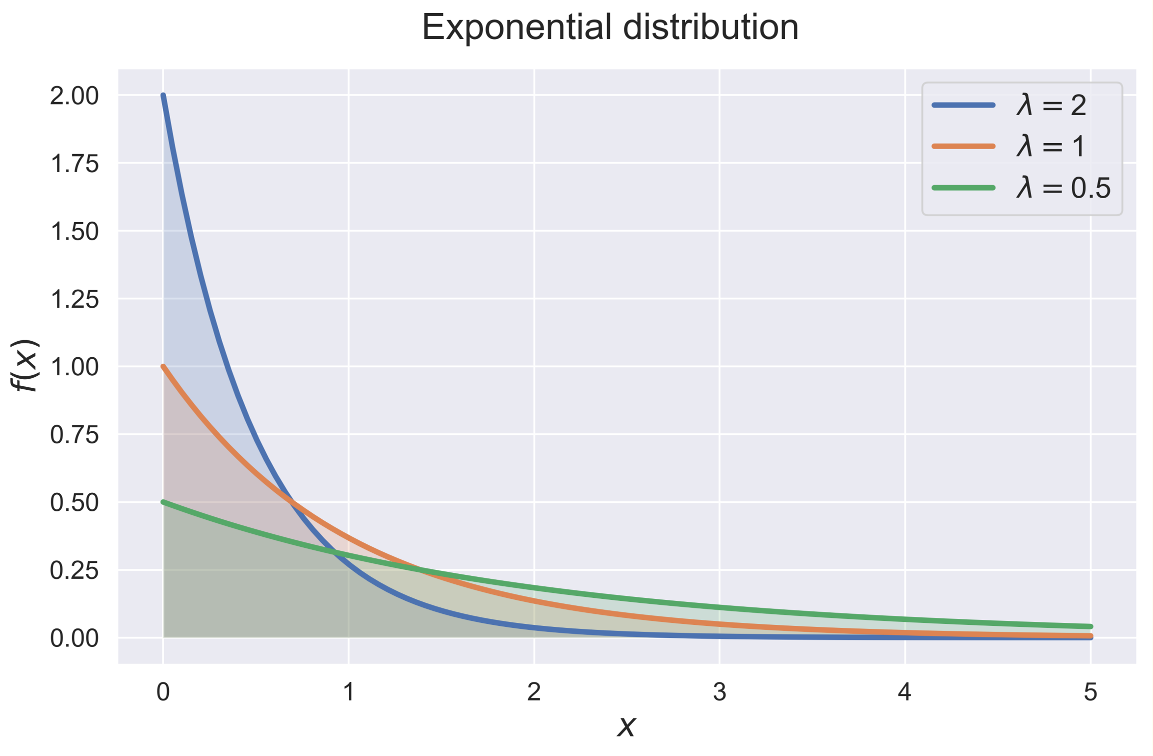


Exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

$$E(X) = \frac{1}{\lambda}$$

$$V(X) = E(X^2) - \mu^2 = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx - \frac{1}{\lambda^2}$$
$$= \int_0^{+\infty} -x^2 d e^{-\lambda x} - \frac{1}{\lambda^2}$$
$$= -x^2 e^{-\lambda x}|_0^{+\infty} + \int_0^{+\infty} 2x e^{-\lambda x} dx - \frac{1}{\lambda^2}$$
$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$



Example 3

Normal distribution

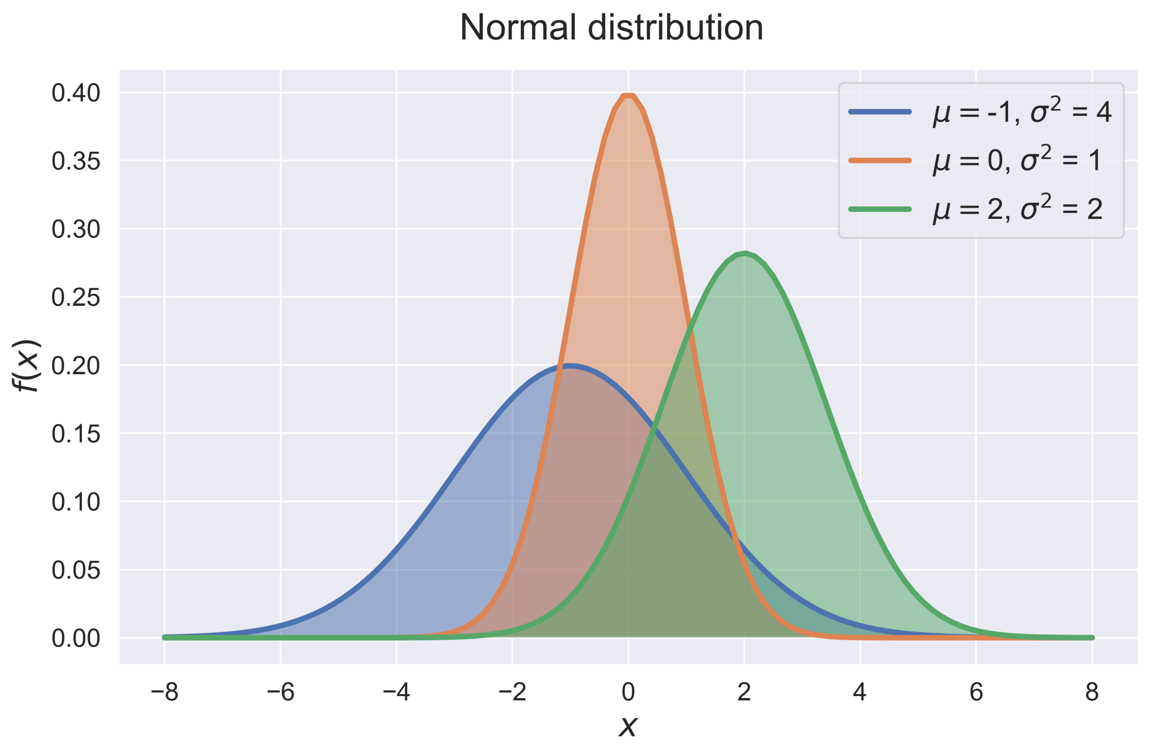
$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$E(X) = 0$$

$$V(X) = \cdots$$



Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$E(X) = 0$$

$$V(X) = E(X^2) - \mu^2 = \int_{-\infty}^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

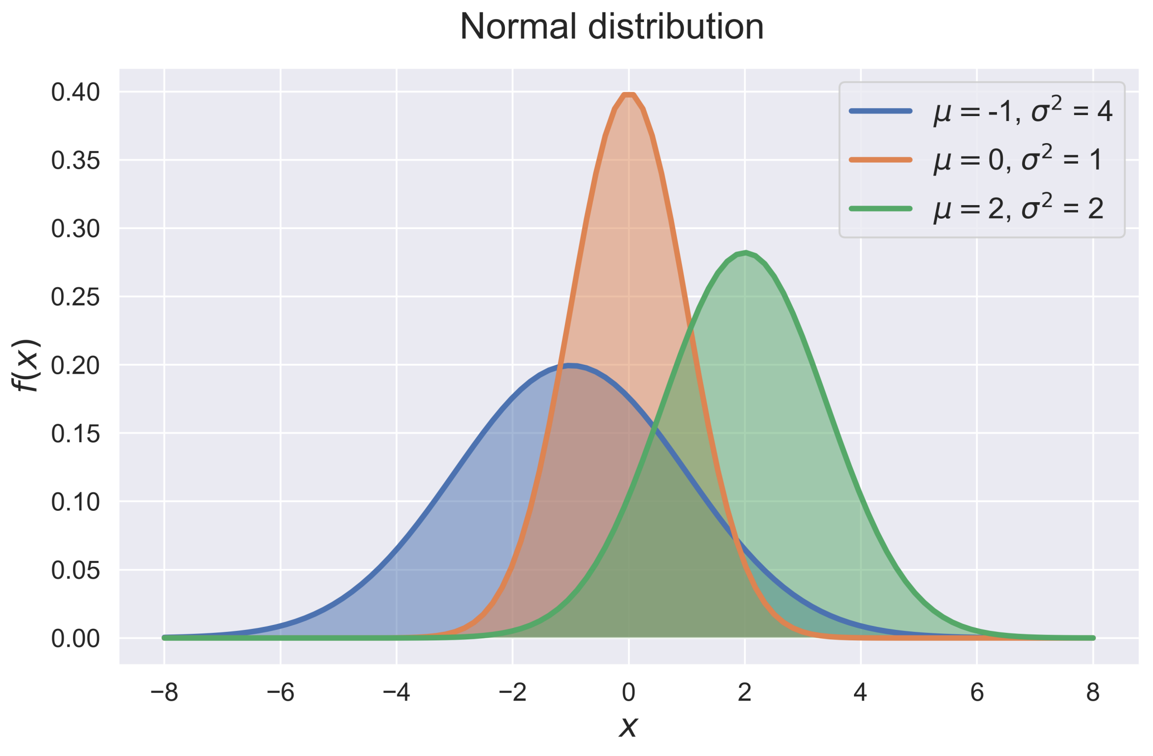
$$= 2\int_0^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 2\int_0^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} d\frac{1}{2}x^2$$

$$= 2\int_0^{+\infty} -x \frac{1}{\sqrt{2\pi}} de^{-\frac{1}{2}x^2}$$

$$= -2x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} |_0^{+\infty} + 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= 1$$



Example 3 continued

Standard normal distribution
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
$E(X) = 0$
$V(X) = 1$

Normal distribution
$Z = \sigma X + \mu$
$f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{z-\mu}{\sigma})^2}$
$E(Z) = E(\sigma X + \mu) = \mu$
$V(X) = \sigma^2$

Quiz 10: Question 5

Let X be a random variable with range $[-1, 1]$ and let $f_X(x)$ be the density function of X.

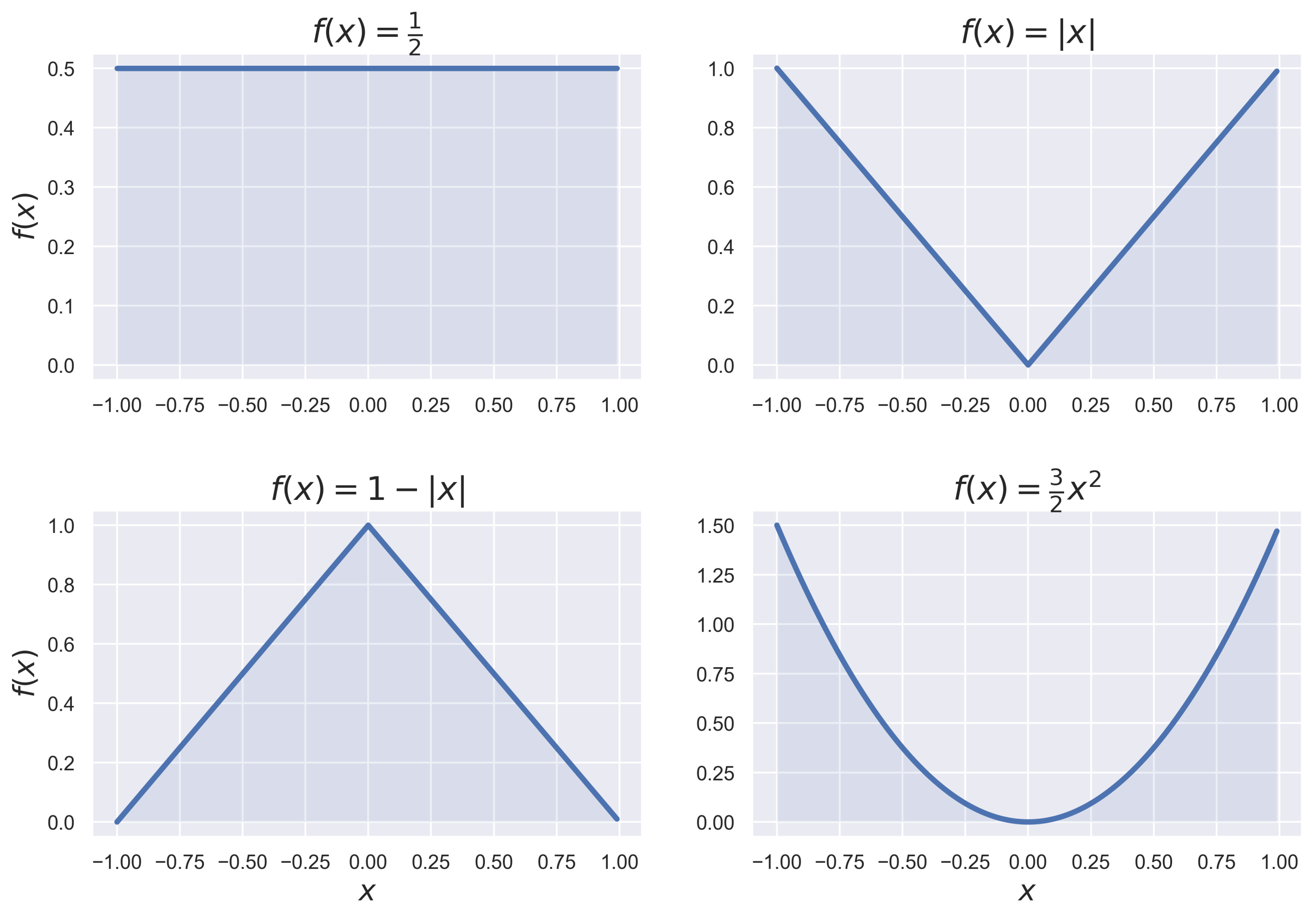
Find $\mu(X)$ and $\sigma^2(X)$ if, for $|X| < 1$,

- $f_X(x) = \frac{1}{2}$
- $f_X(x) = |x|$
- $f_X(x) = 1 - |x|$
- $f_X(x) = \frac{3}{2}x^2$

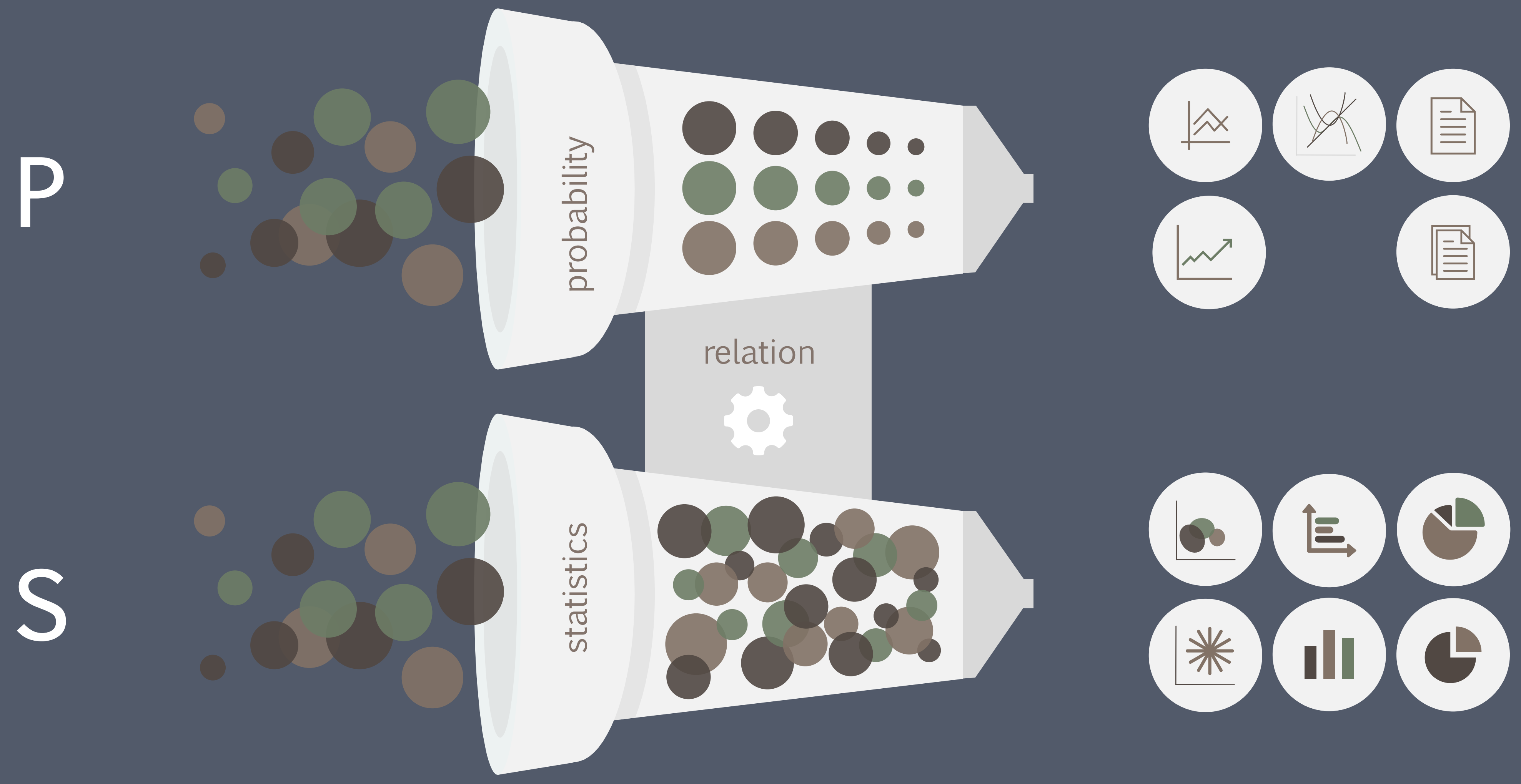
$\mu = 0$

- $\sigma^2 = \frac{1}{3}$
- $\sigma^2 = \frac{1}{2}$
- $\sigma^2 = \frac{1}{6}$
- $\sigma^2 = \frac{3}{5}$

How to compare the variances without calculating them?



Probability and Statistics



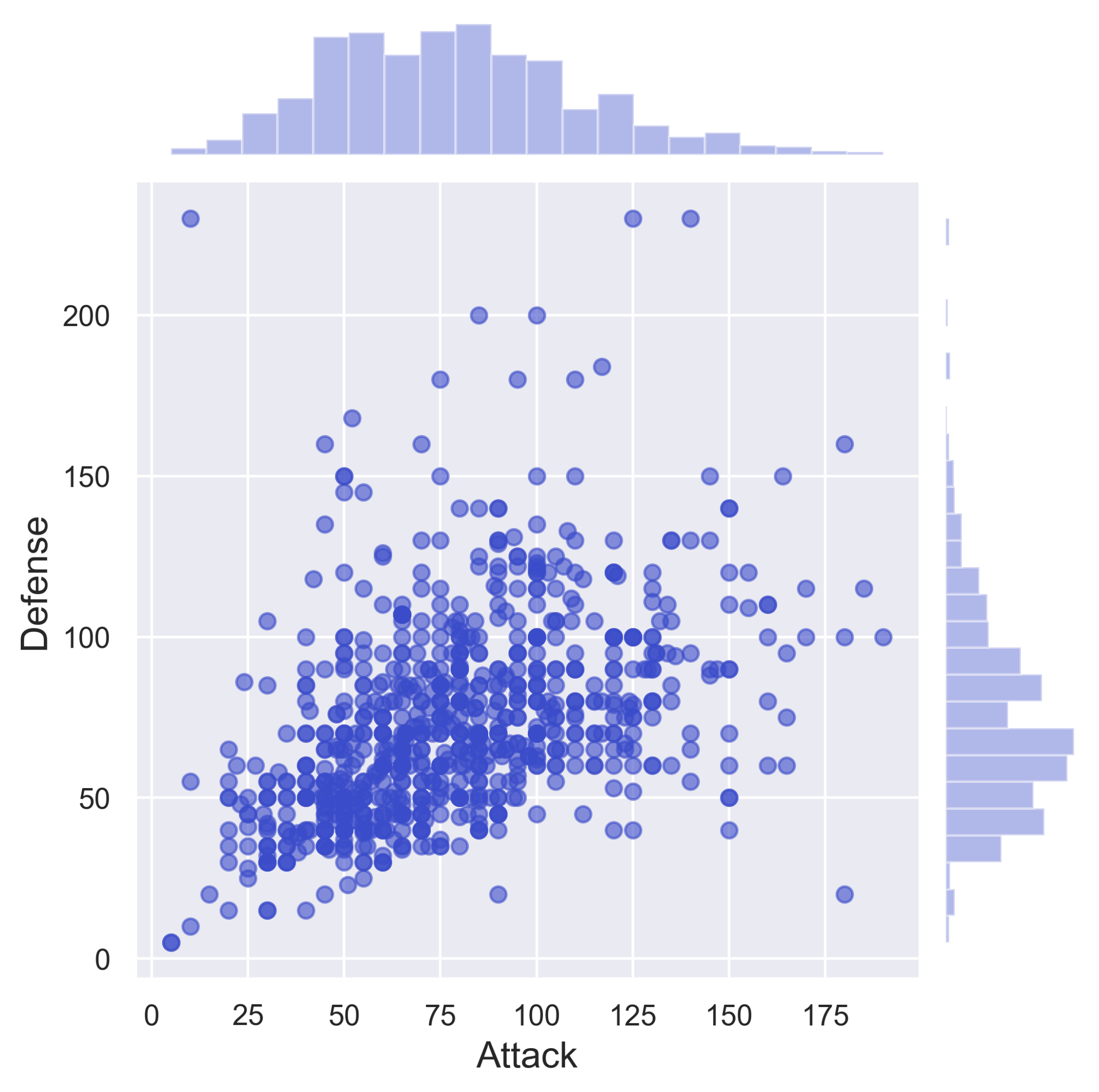
POKEMON

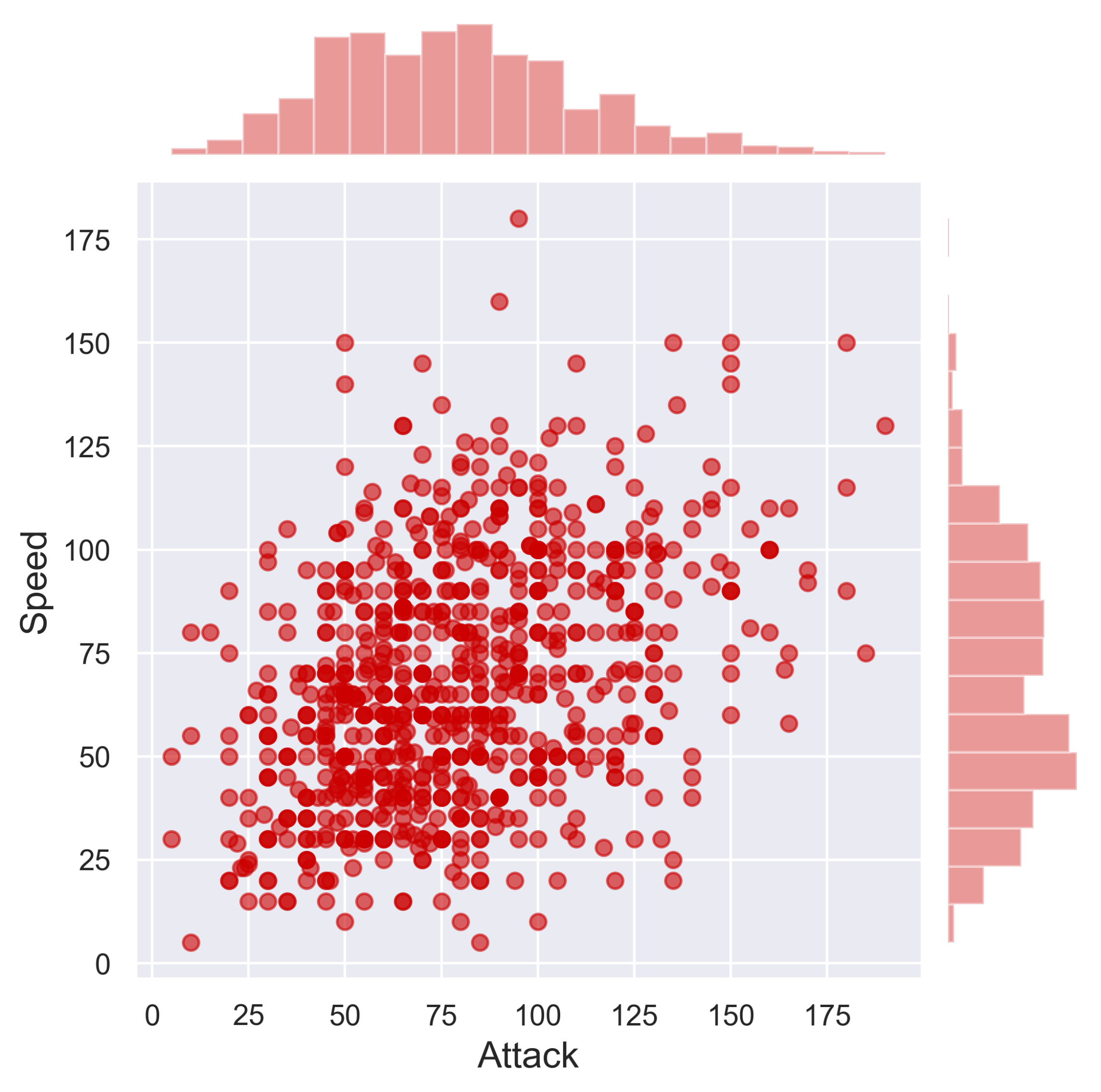
Dataset

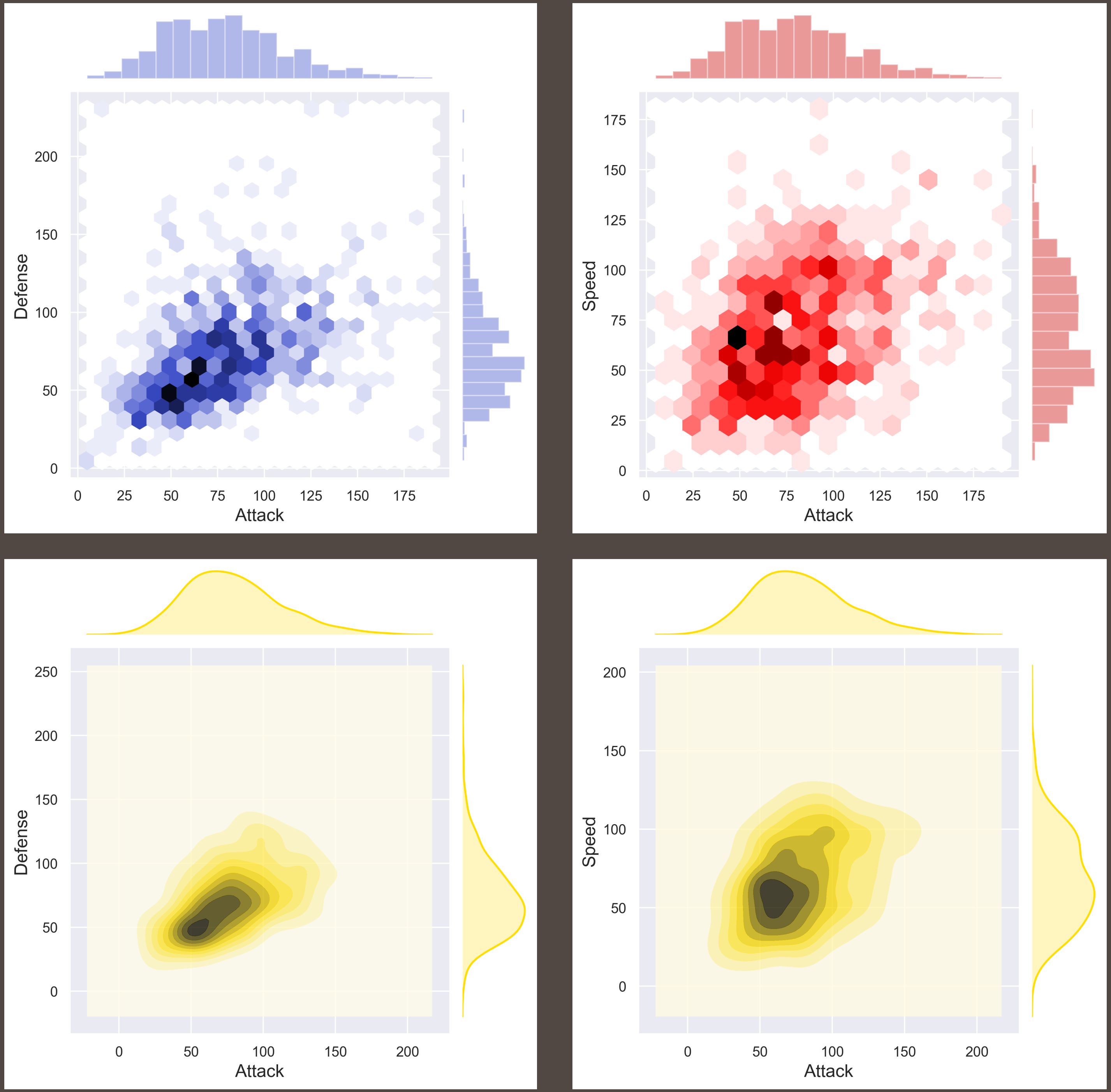
- The data set includes 721 Pokemon, including their number, name, first and second type, and basic stats: HP, Attack, Defense, Special Attack, Special Defense, and Speed.
- This are the raw attributes that are used for calculating how much damage an attack will do in the games.

	Name	Type 1	Type 2	Total	HP	Attack	Defense	Sp. Atk	Sp. Def	Speed	Generation	Legendary
0	Bulbasaur	Grass	Poison	318	45	49	49	65	65	45	1	False
1	Ivysaur	Grass	Poison	405	60	62	63	80	80	60	1	False
2	Venusaur	Grass	Poison	525	80	82	83	100	100	80	1	False
3	VenusaurMega Venusaur	Grass	Poison	625	80	100	123	122	120	80	1	False
4	Charmander	Fire	NaN	309	39	52	43	60	50	65	1	False

Attribute
▪ Attack: the base modifier for normal attacks (eg. Scratch, Punch)
▪ Defense: the base damage resistance against normal attacks
▪ Speed: determines which pokemon attacks first each round


Defense
Attack
0
25
50
75
100
125
150
175
200


Speed
Attack
0
25
50
75
100
125
150
175



- **Pokemon**

```
figure_pokemon(df = data_pokemon,
               column_A = 'Attack',
               column_B = 'Defense',
               job = 'scatter',
               fs = 16)

               job = 'hex'

               job = 'kde'
```

Covariance

- Let X and Y be real-valued random variables. The covariance $\mathrm{cov}(X, Y)$ is defined as

$$\mathrm{cov}(X, Y) = E\Big(\big(X - \mu(X)\big)\big(Y - \mu(Y)\big)\Big).$$

- If X and Y are independent, $\mathrm{cov}(X, Y) = 0$. The reverse is not necessarily true.

$$\mathrm{cov}(X, Y) = E(XY) - E(X)E(Y).$$

!

$$V(X + Y) = V(X) + V(Y) + 2\mathrm{cov}(X, Y).$$

!

Proof

Let X and Y be real-valued random variables with expected values $\mu(X) = E(X)$ and $\mu(Y) = E(Y)$.

$$\operatorname{cov}(X, Y) = E\Big(\big(X - \mu(X)\big)\big(Y - \mu(Y)\big)\Big).$$

$$E(aX + b) = aE(X) + b$$

$$\begin{aligned} \operatorname{cov}(X, Y) &= E\big(XY - X\mu(Y) - Y\mu(X) + \mu(X)\mu(Y)\big) \\ &= E(XY) - \mu(Y)E(X) - \mu(X)E(Y) + \mu(X)\mu(Y) \\ &= E(XY) - E(X)E(Y). \end{aligned}$$

$$V(X) = E(X^2) - \mu^2$$

$$\begin{aligned} V(X + Y) &= E\big((X + Y)^2\big) - \big(\mu(X) + \mu(Y)\big)^2 \\ &= E(X^2 + 2XY + Y^2) - \big(\mu(X)\big)^2 - 2\mu(X)\mu(Y) - \big(\mu(Y)\big)^2 \\ &= E(X^2) - \big(\mu(X)\big)^2 + E(Y^2) - \big(\mu(Y)\big)^2 + 2E(XY) - 2E(X)E(Y) \\ &= V(X) + V(Y) + 2\operatorname{cov}(X, Y). \end{aligned}$$

When do we need independence?

$X + Y$

XY

$\mathrm{COV}(X, Y)$

Do not need

$E(X+Y) = E(X) + E(Y)$

$\mathrm{COV}(X, Y) = 0$

...

$X \sim U(-1, 1)$

$Y = X^2$

$\mathrm{COV}(X, Y) = \cdots$

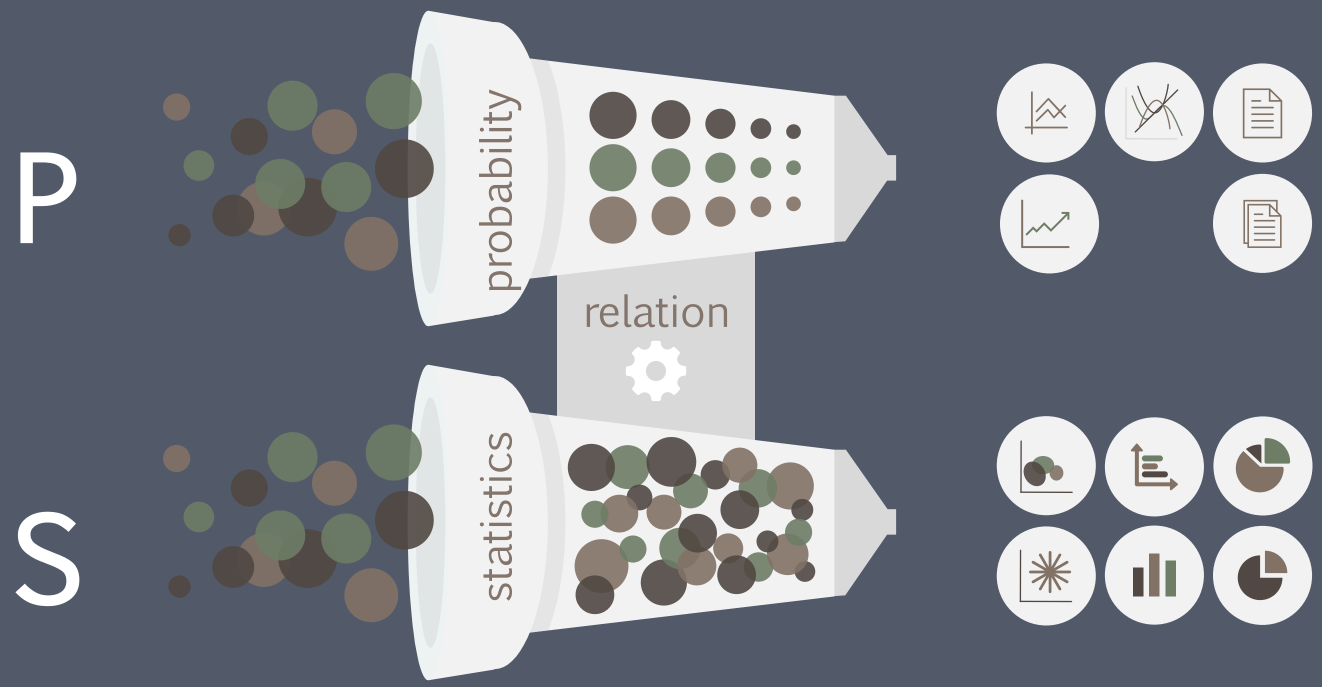
Need

$E(XY) = E(X)E(Y)$

$V(X+Y) = V(X) + V(Y)$

...

Probability and Statistics



$\mathrm{COV}(X, Y)$

Pearson's r

$$\rho = \frac{\mathrm{COV}(X, Y)}{\sigma_X \sigma_Y}$$

RED WINE

dataset

- The dataset is related to red variants of the Portuguese "Vinho Verde" wine.
- Input variables (based on physicochemical tests):
  - fixed acidity, volatile acidity, citric acid, residual sugar, chlorides, free sulfur dioxide, total sulfur dioxide, density, pH, sulphates and alcohol.
- Output variable (based on sensory data):
  - quality (score between 0 and 10).

	fixed acidity	volatile acidity	citric acid	residual sugar	chlorides	free sulfur dioxide	total sulfur dioxide	density	pH	sulphates	alcohol	quality
0	7.4	0.70	0.00	1.9	0.076	11.0	34.0	0.9978	3.51	0.56	9.4	5
1	7.8	0.88	0.00	2.6	0.098	25.0	67.0	0.9968	3.20	0.68	9.8	5
2	7.8	0.76	0.04	2.3	0.092	15.0	54.0	0.9970	3.26	0.65	9.8	5
3	11.2	0.28	0.56	1.9	0.075	17.0	60.0	0.9980	3.16	0.58	9.8	6
4	7.4	0.70	0.00	1.9	0.076	11.0	34.0	0.9978	3.51	0.56	9.4	5

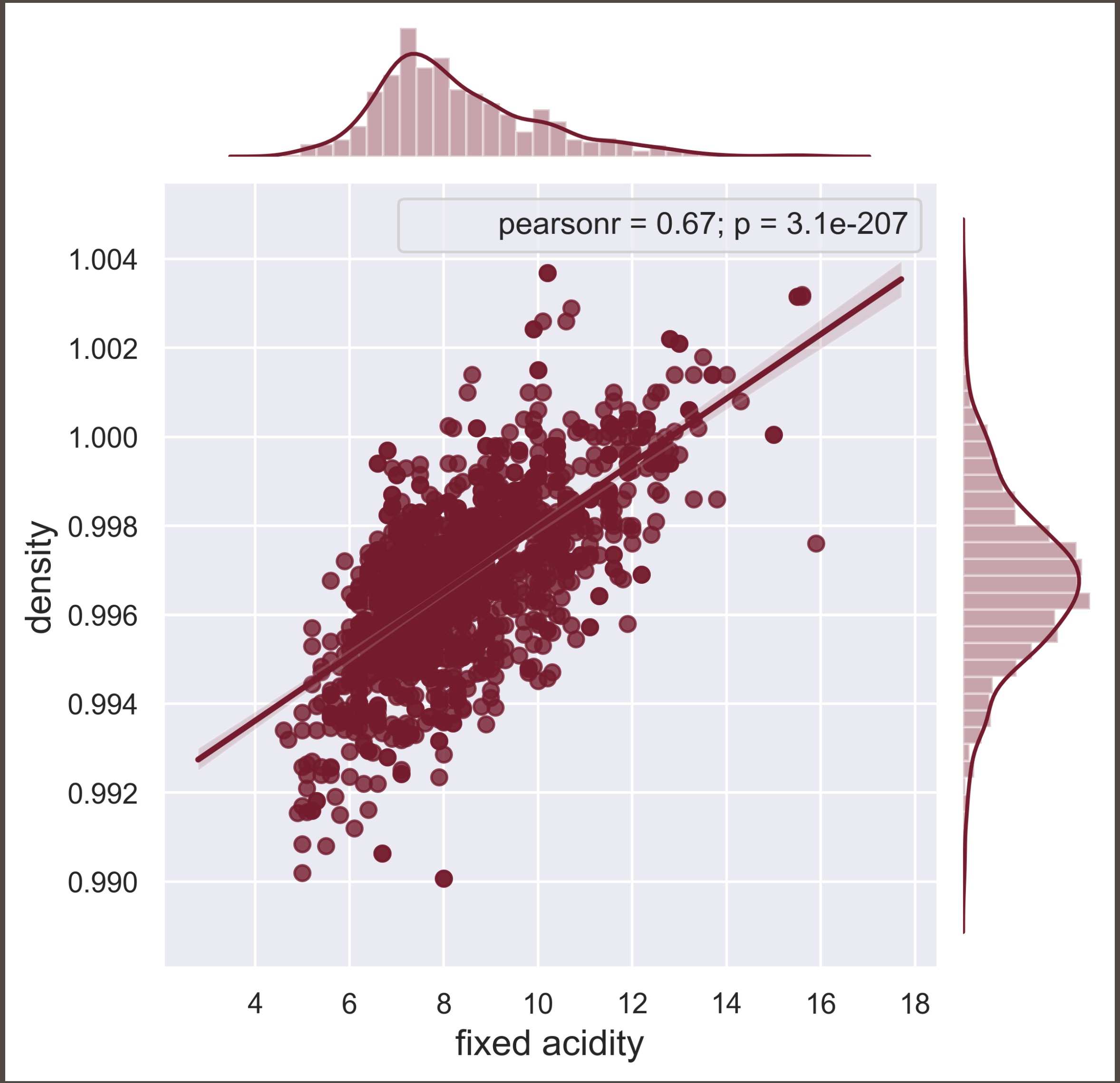
fixed acidity

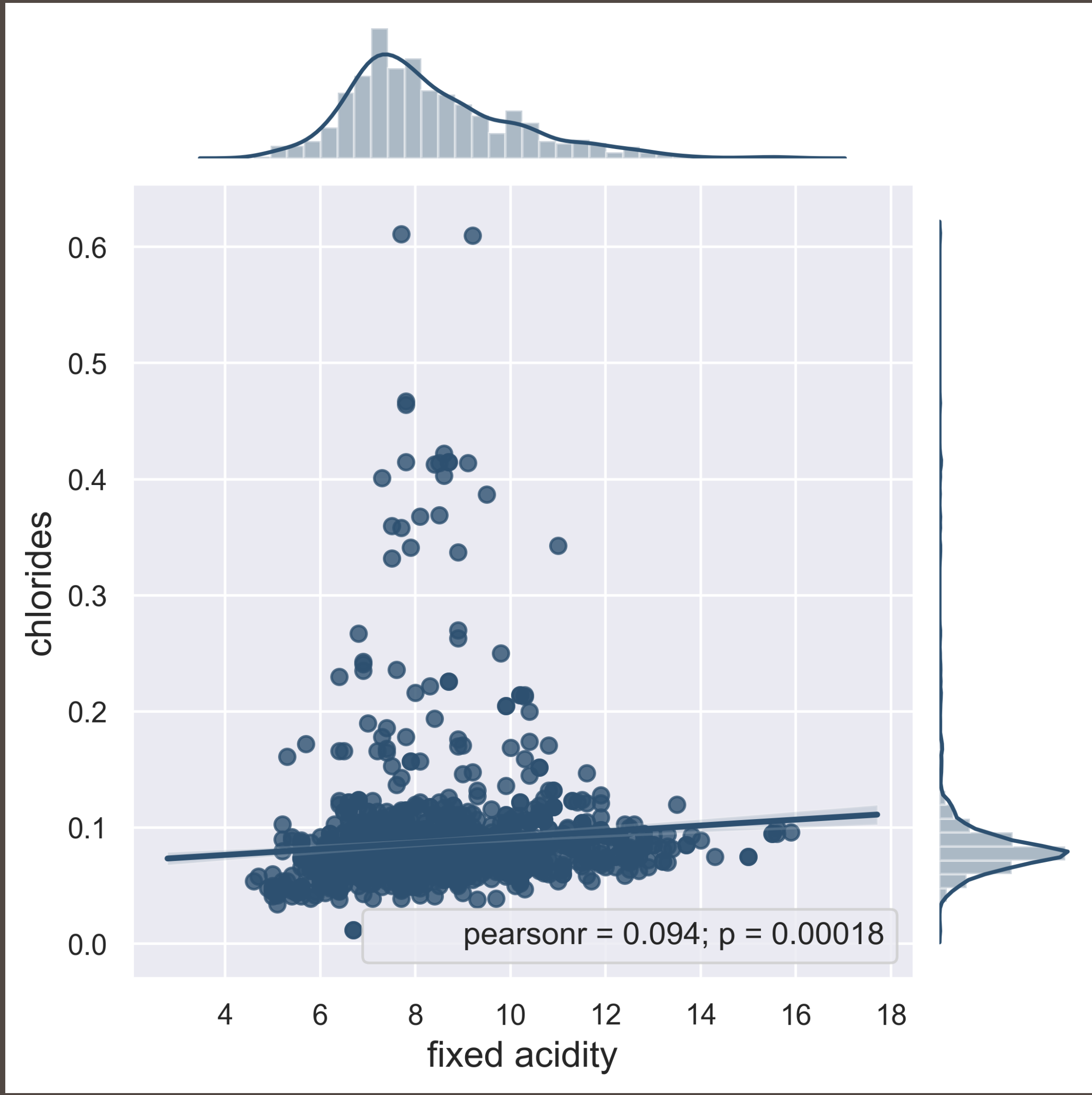
chlorides

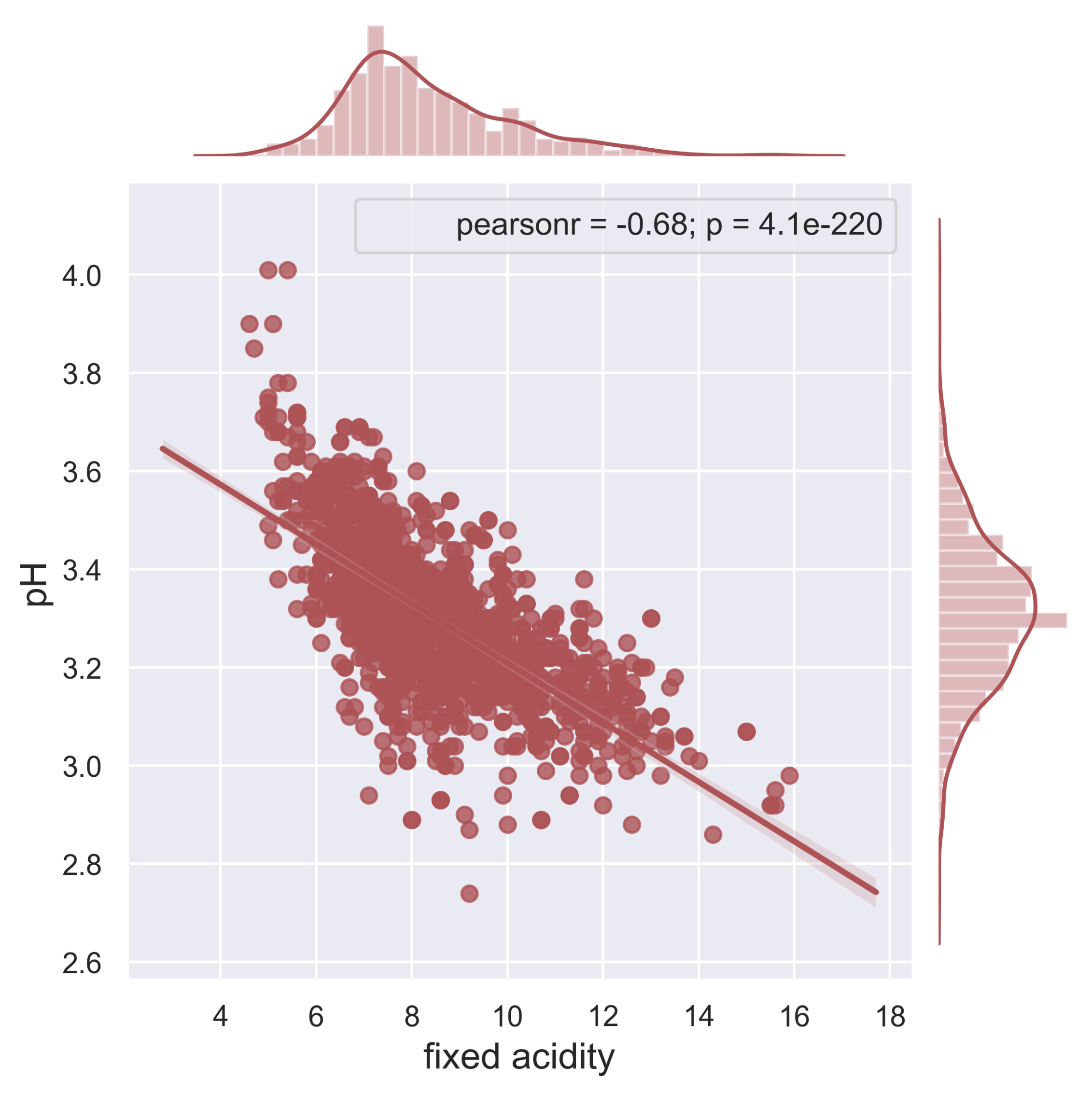
density

pH

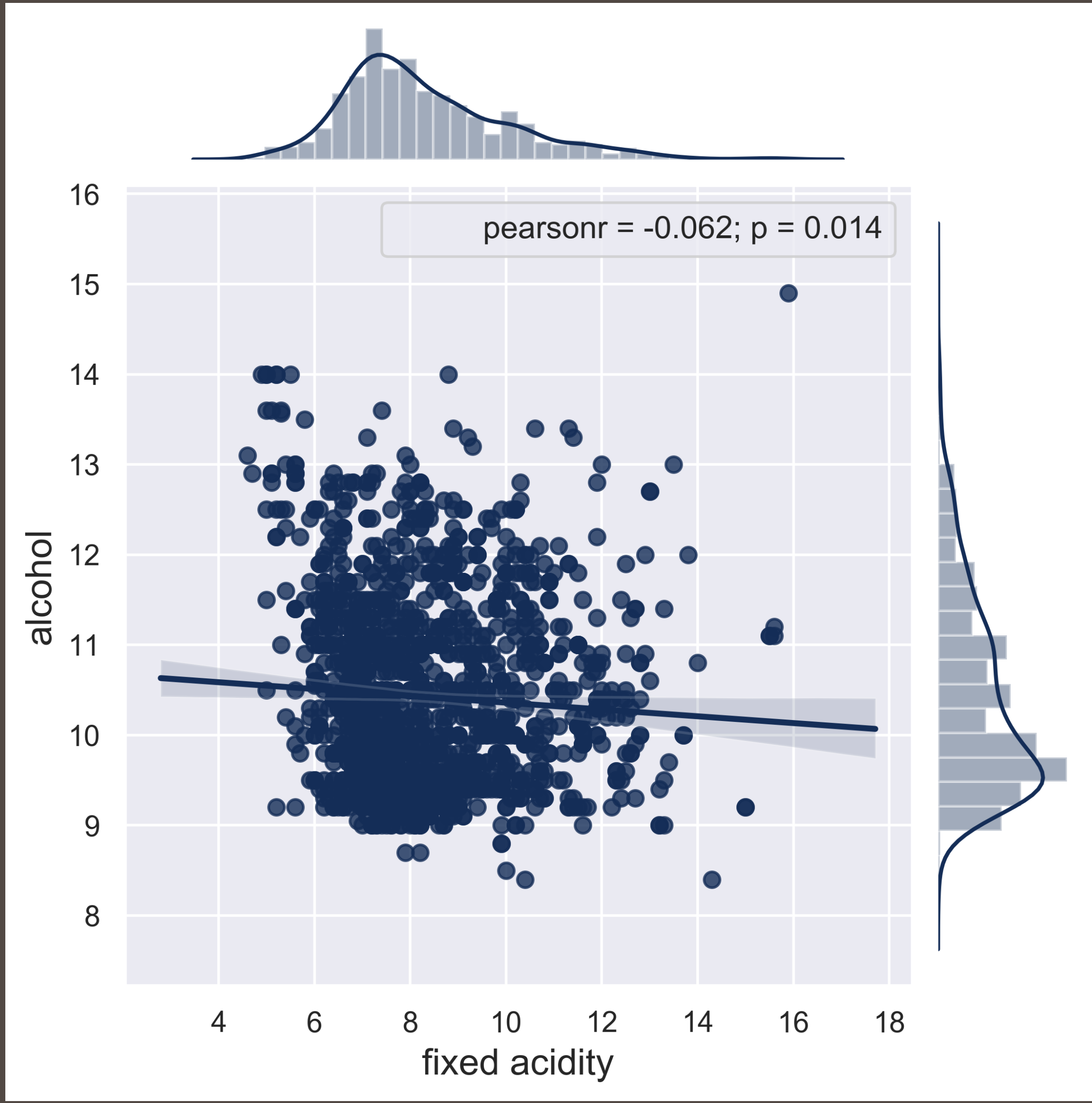
alcohol


pearsonr = 0.67; p = 3.1e-207
density
1.004
1.002
1.000
0.998
0.996
0.994
0.992
0.990
4
6
8
10
12
14
16
18
fixed acidity


chlorides
0.6
0.5
0.4
0.3
0.2
0.1
0.0
pearsonr = 0.094; p = 0.00018
4
6
8
10
12
14
16
18
fixed acidity



pearsonr = -0.68; p = 4.1e-220
pH
fixed acidity



pearsonr = -0.062; p = 0.014
alcohol
fixed acidity

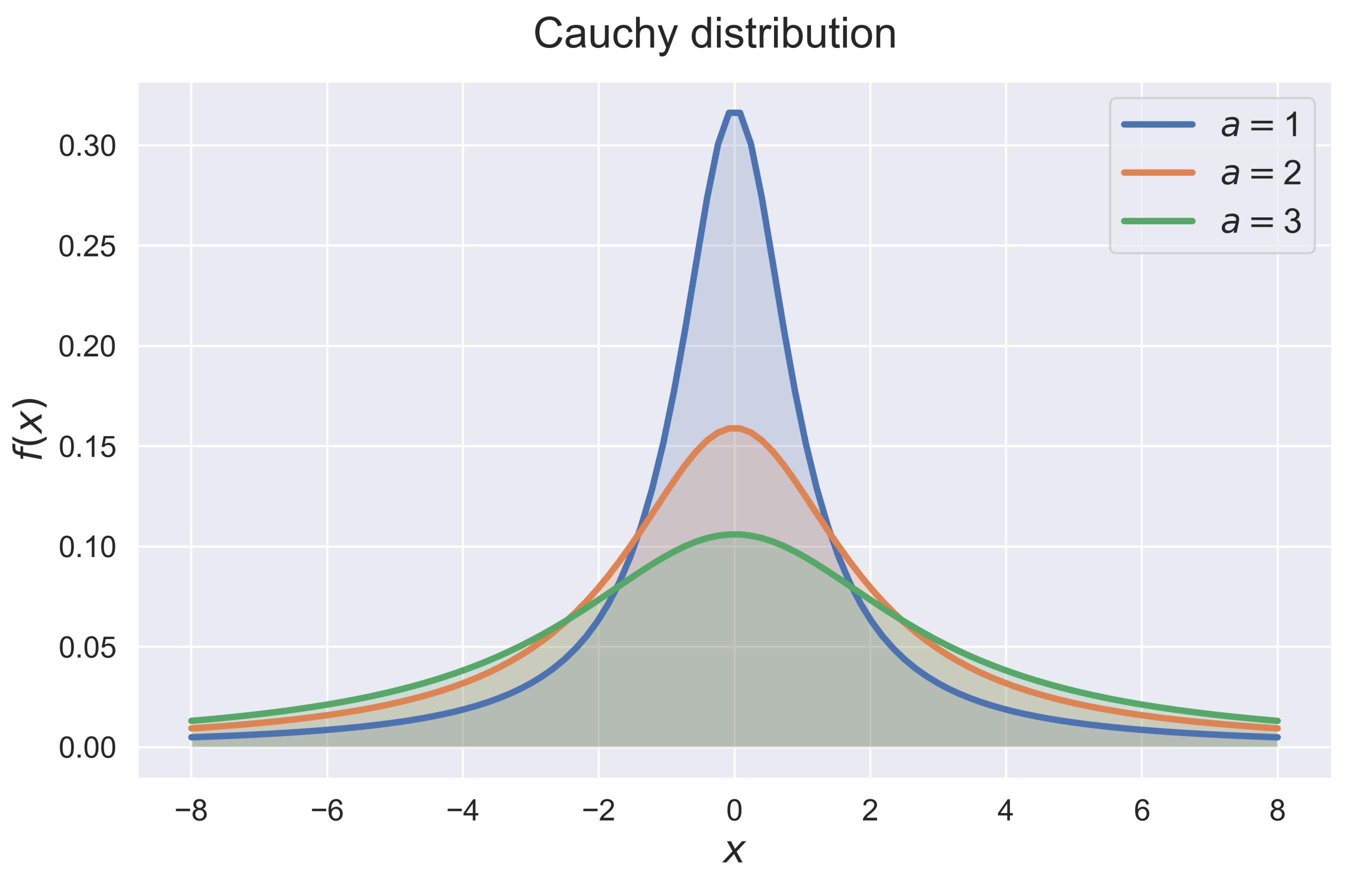
Special example: Cauchy density

Let X be continuous random variable with the Cauchy density function

$f_X(x) = \frac{a}{\pi}\frac{1}{a^2+x^2}.$

Does the expectation of X exist?

And its variance?



Special example

Cauchy distribution

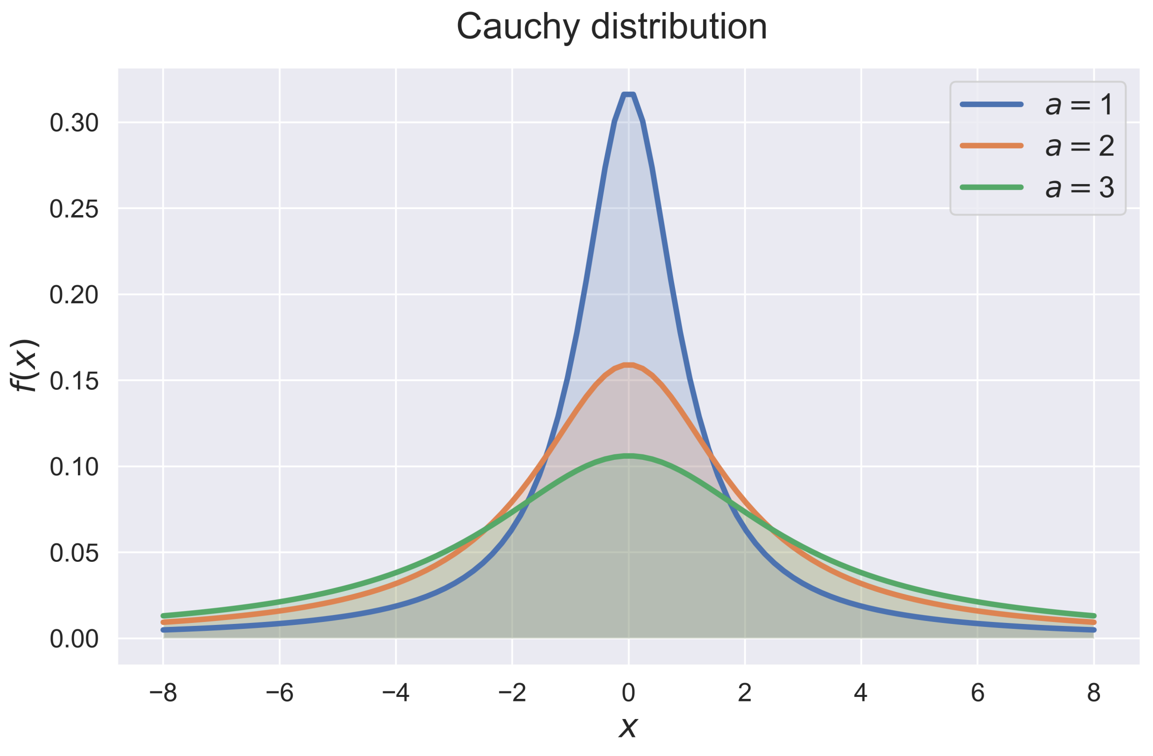
$a = 1$

$$f_X(x) = \frac{a}{\pi}\frac{1}{a^2 + x^2} = \frac{1}{\pi(1 + x^2)}$$

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\pi(1 + x^2)} dx$$

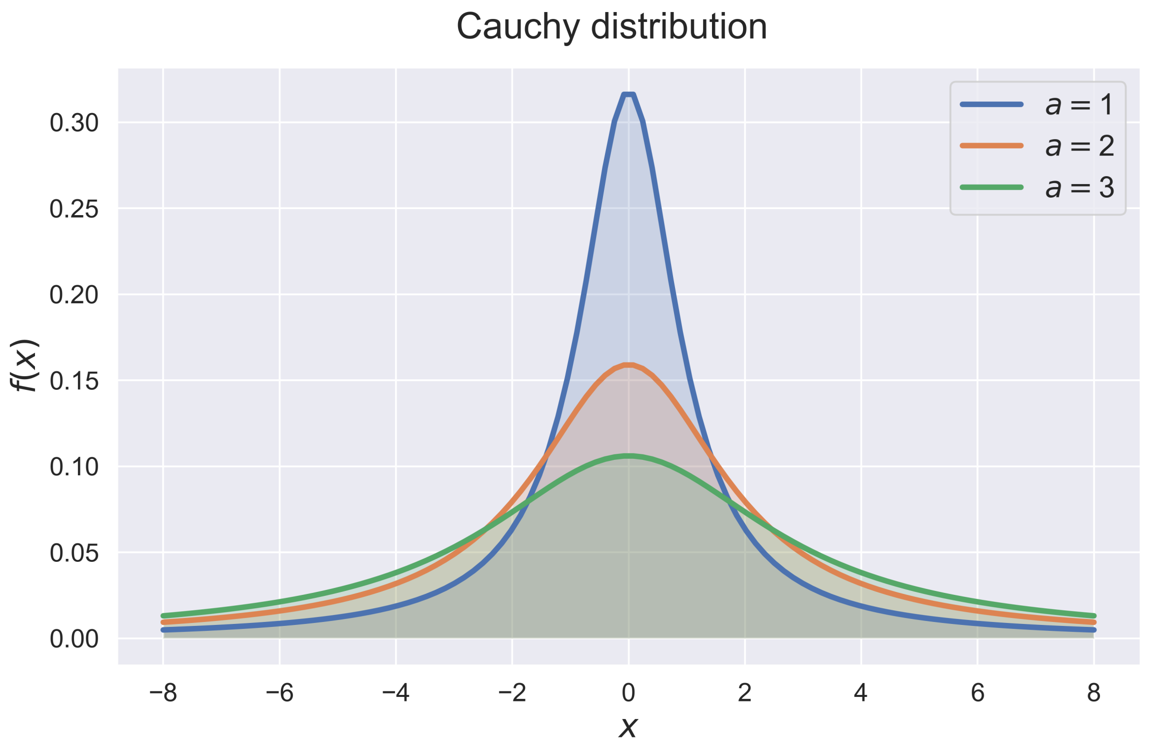
$$\lim_{c \to +\infty} \int_{-c}^{+c} \frac{x}{\pi(1 + x^2)} dx$$

$$\lim_{c \to +\infty} \int_{c}^{2c} \frac{x}{\pi(1 + x^2)} dx$$



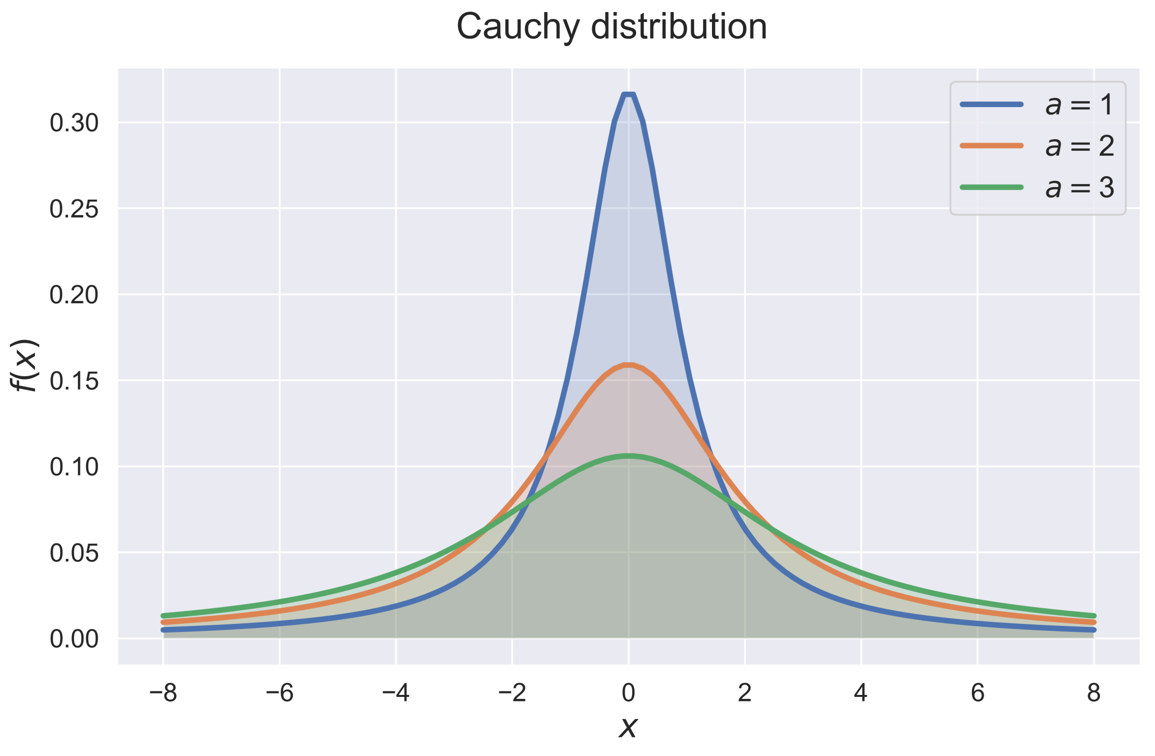
Special example

Cauchy distribution
$\boldsymbol{a = 1}$
$f_X(x) = \frac{a}{\pi}\frac{1}{a^2 + x^2} = \frac{1}{\pi(1 + x^2)}$
$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\pi(1 + x^2)} dx$
$\lim_{c \to +\infty} \int_{-c}^{+c} \frac{x}{\pi(1 + x^2)} dx = 0$ $\lim_{c \to +\infty} \int_{c}^{2c} \frac{x}{\pi(1 + x^2)} dx = +\infty$



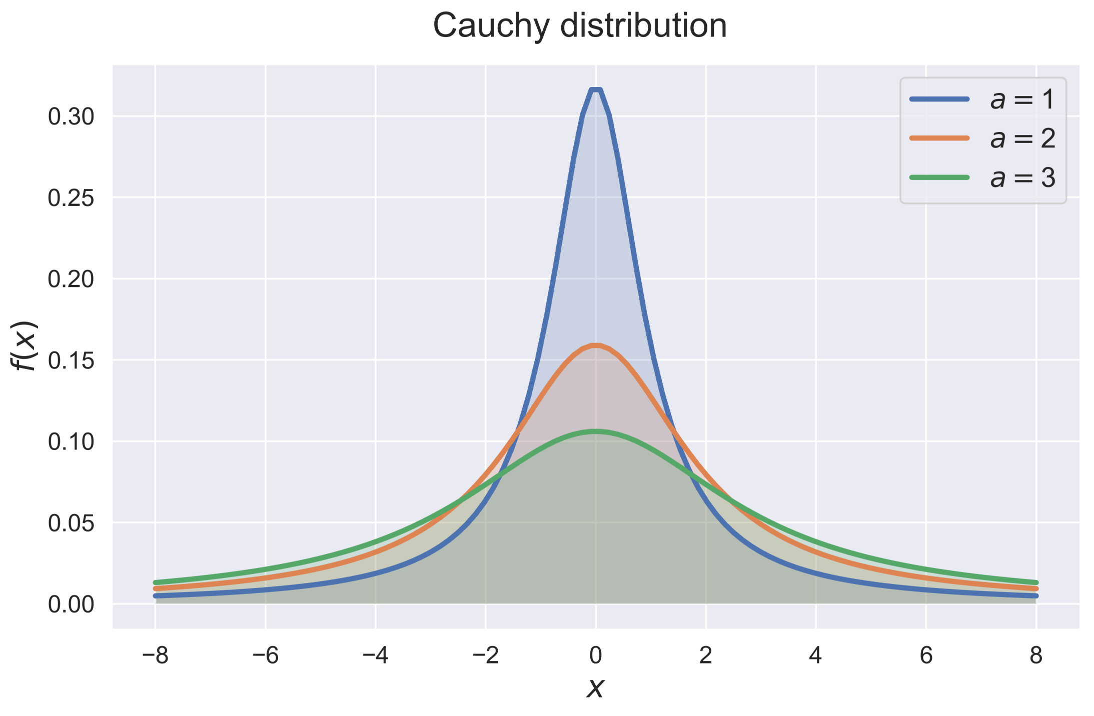
Special example

Cauchy distribution
$\boldsymbol{a = 1}$
$f_X(x) = \dfrac{a}{\pi}\dfrac{1}{a^2 + x^2} = \dfrac{1}{\pi(1 + x^2)}$
$E(X^2) = \int_{-\infty}^{+\infty} \dfrac{x^2}{\pi(1 + x^2)} dx = \dfrac{1}{\pi}\int_{-\infty}^{+\infty} 1 - \dfrac{1}{1 + x^2} dx = \dfrac{1}{\pi}\left(\int_{-\infty}^{+\infty} 1 dx - \pi\right) = +\infty$



CAUCHY DENSITY

The Cauchy distribution is often used in statistics as the canonical example of a ”pathological" distribution since both its expected value and its variance are undefined.



Final example

- A game is played as follows: a random number X is chosen uniformly from $[0, 1]$.
- Then a sequence $Y_1, Y_2, \cdots$ of random numbers is chosen independently and uniformly from $[0, 1]$.
- The game ends the first time that $Y_i > X$. You are paid $(i - 1)$ dollars.
- What is a fair entrance fee for this game?