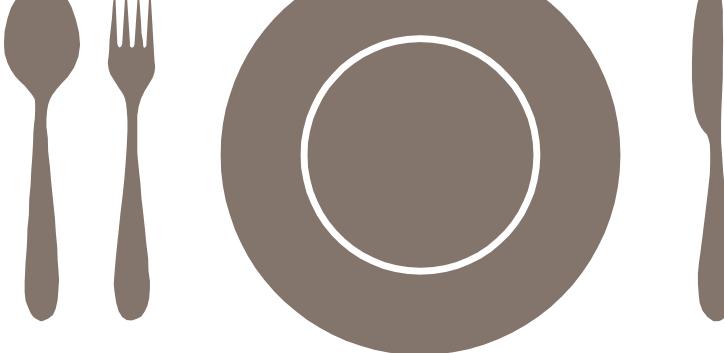


# MATH 20: PROBABILITY

Lecture 3: Permutations

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PINE

2 East Wheelock Street, Hanover, NH

## SMALL PLATES

- Buffalo Cauliflower
- Pine Fries

**2 options**

## LARGE PLATES

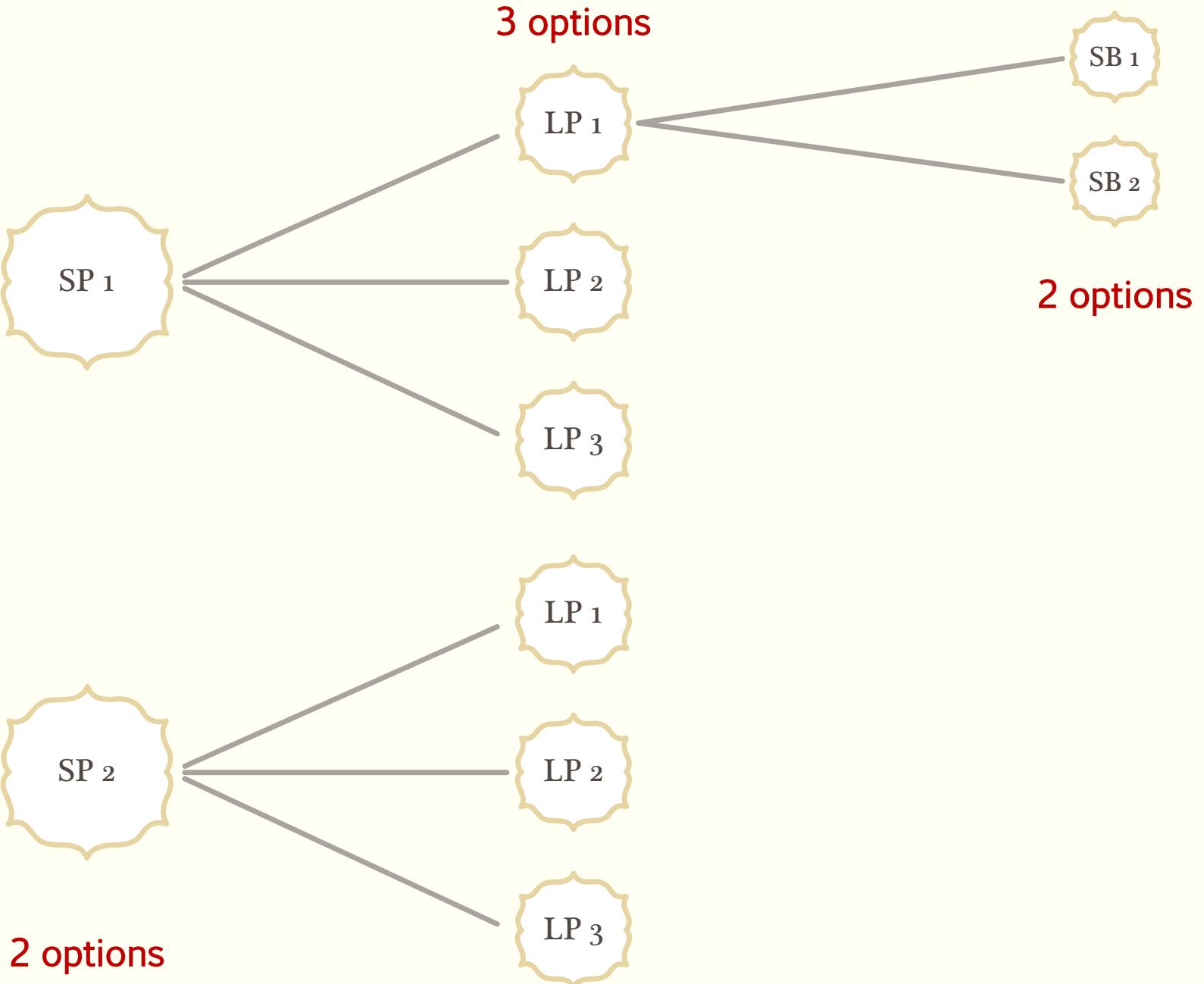
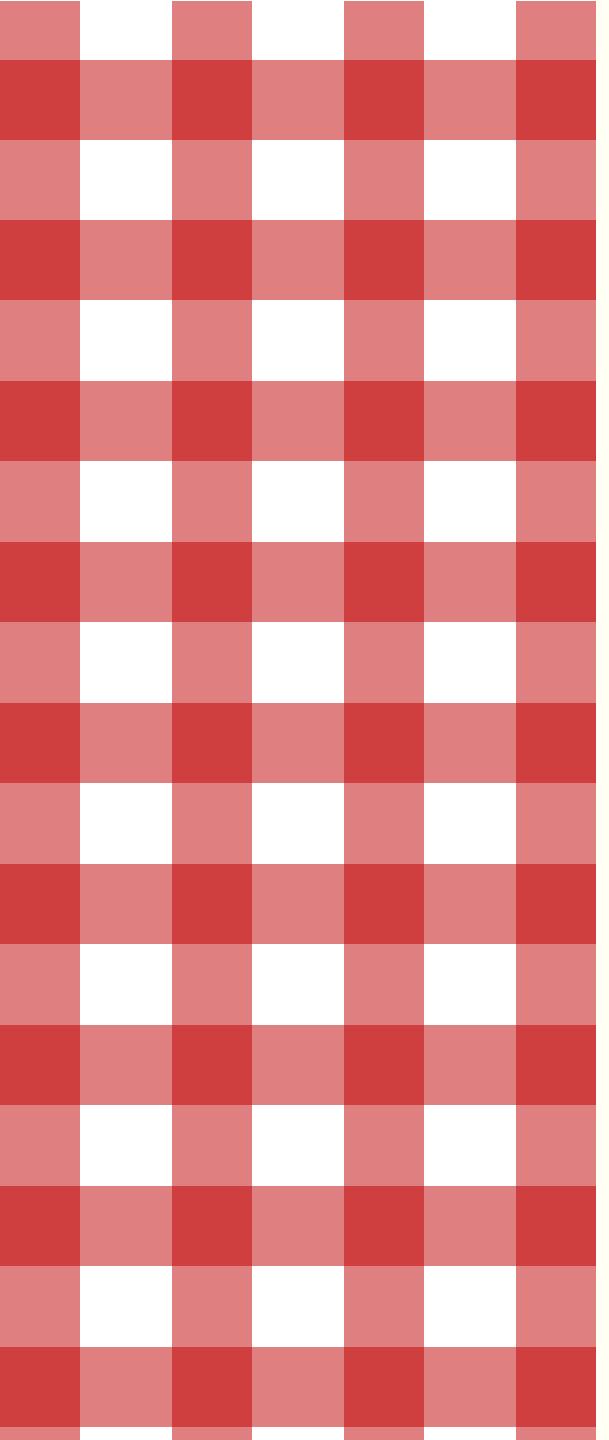
- Crispy Fish Taco
- Maine Lobster Roll
- Hanover Inn Burger

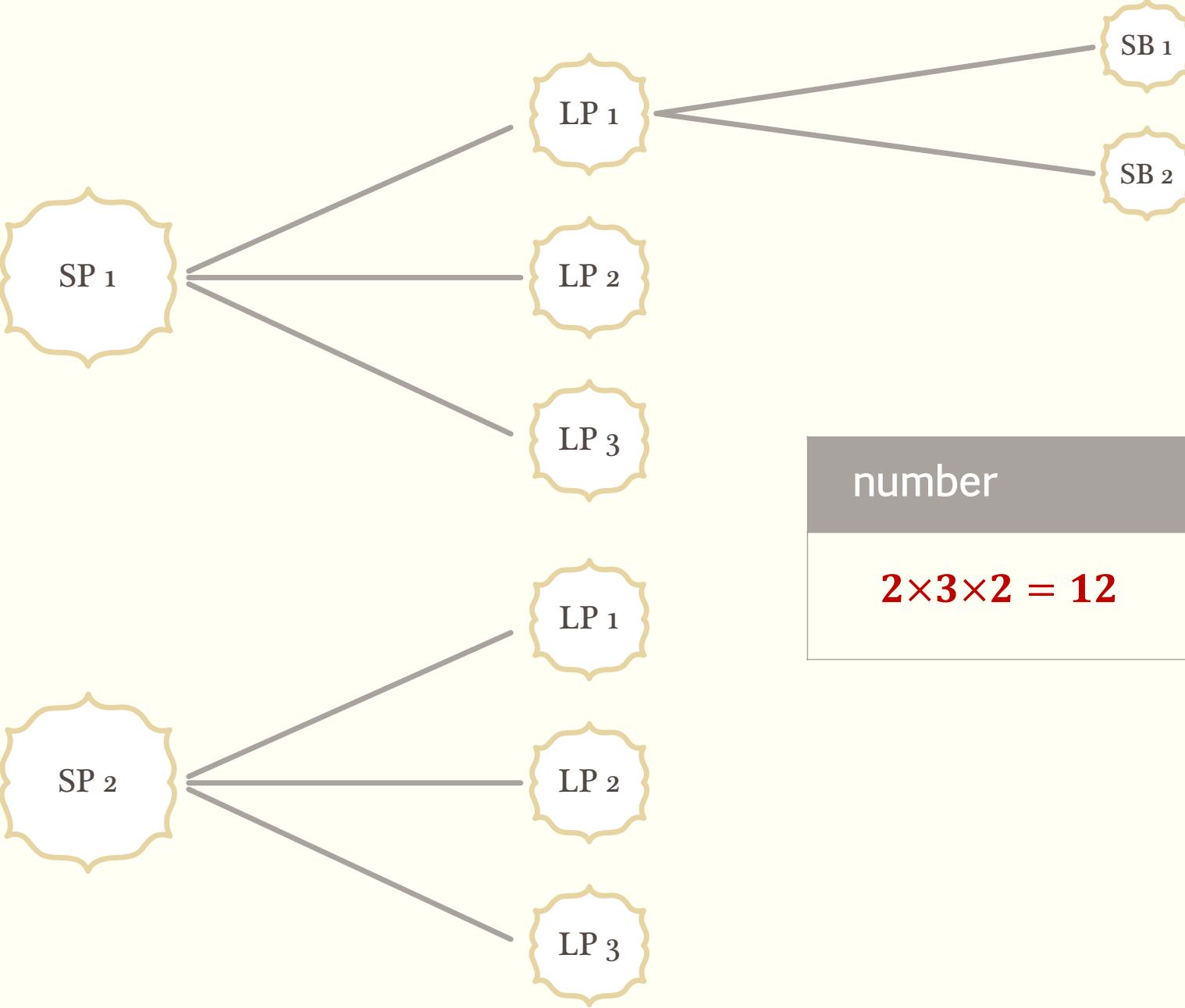
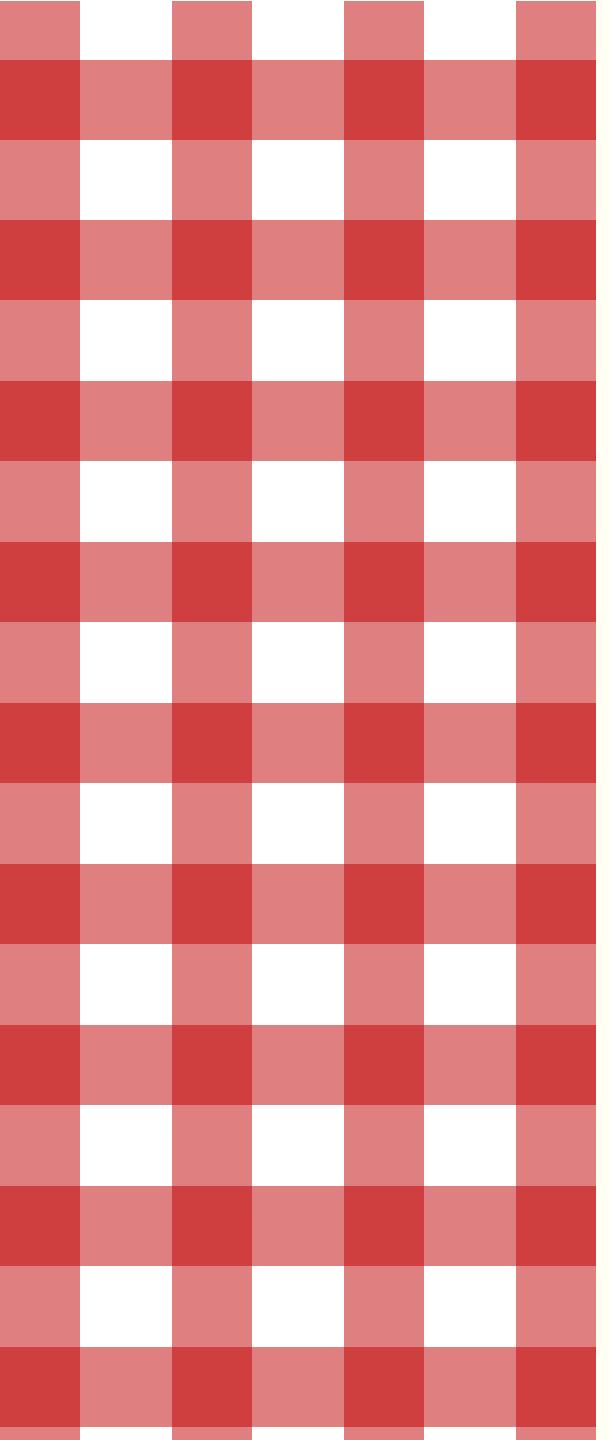
**3 options**

## SWEET BITES

- Citrus Vanilla Cheesecake
- House-Made Ice Creams

**2 options**





# Le tour du monde en quatre-vingts jours



## Options

- Airplane
- Ferry
- Train

# Le tour du monde en quatre-vingts jours



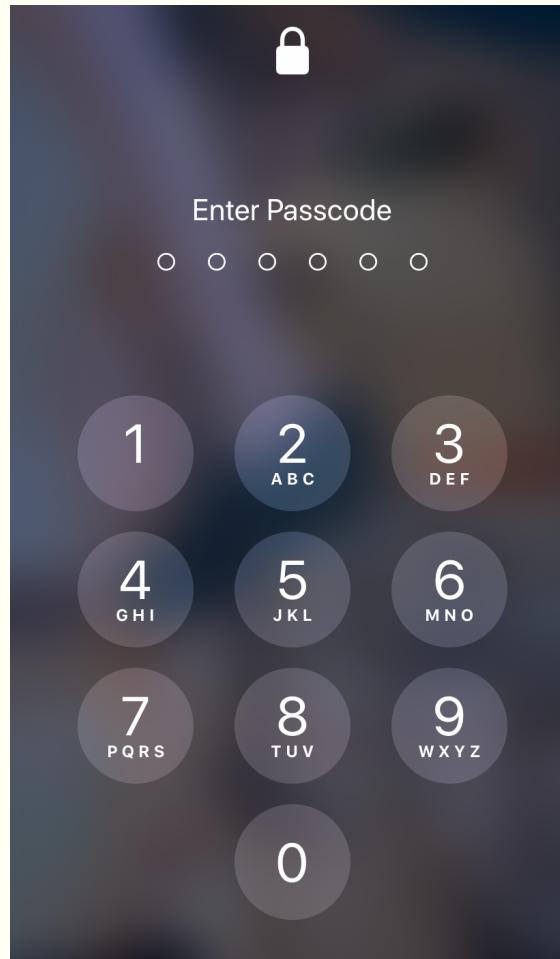
# Lockscreen Password

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Forgot your password?

Maximum number of attempts  
required is ...



# Lockscreen Password

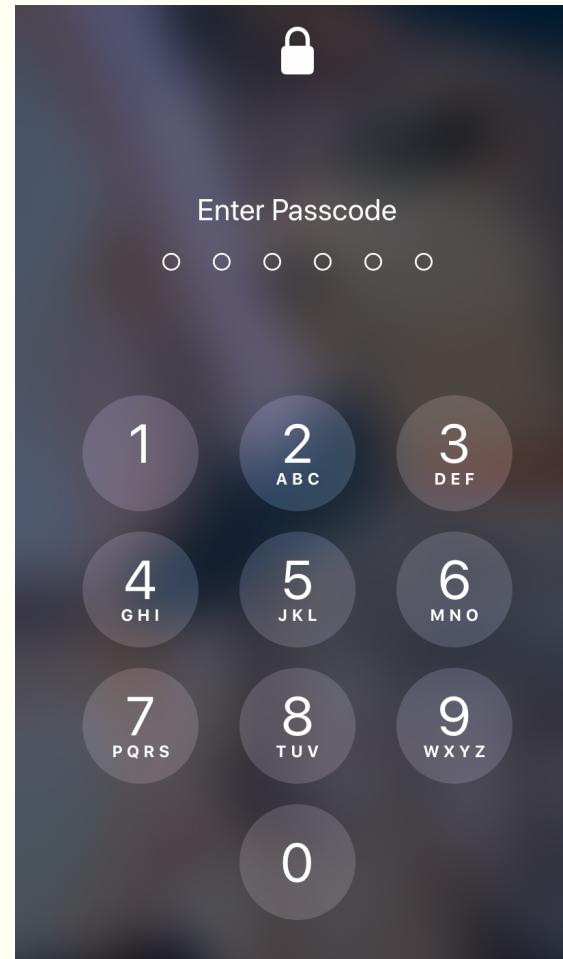
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Forgot your password?

Maximum number of attempts required is ...

number  
 $10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$



# Counting Problems

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- Experiment is composed of multiple **independent** stages.
  - The numbers of outcomes may be different for each stage.
- 
- Examples: restaurant menu, multi-destination travel, password, name initials...
  - Counting technique: **tree diagram**.

What does independent mean here?

?



# Birthday Problem

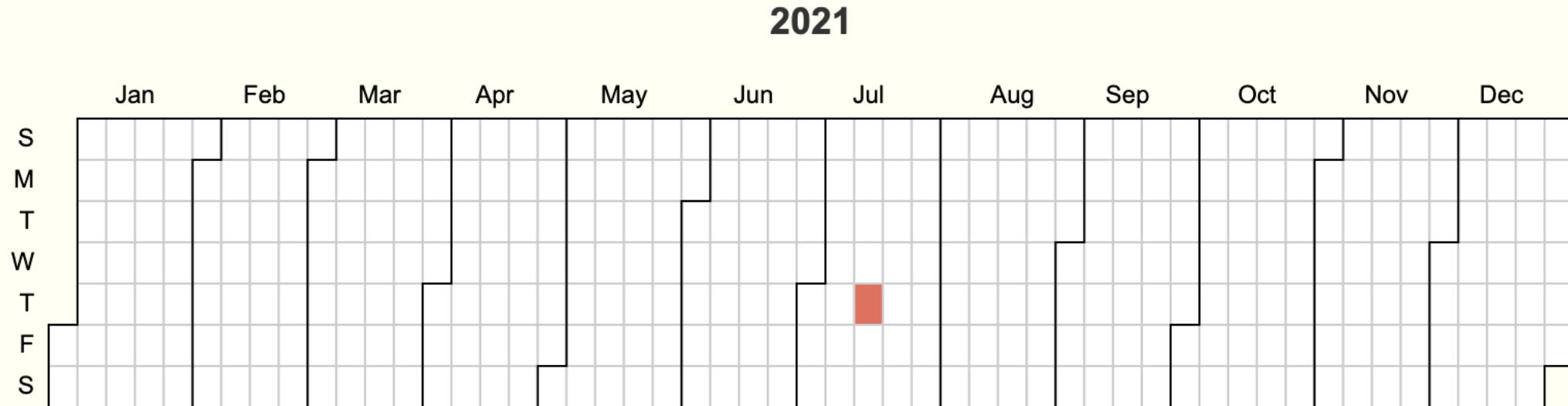
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How many students do we need to have in our hours section to make it favorable bet (that is, probability of success greater than  $\frac{1}{2}$ ) that two people in the classroom will have the same birthday?

# Birthday Problem

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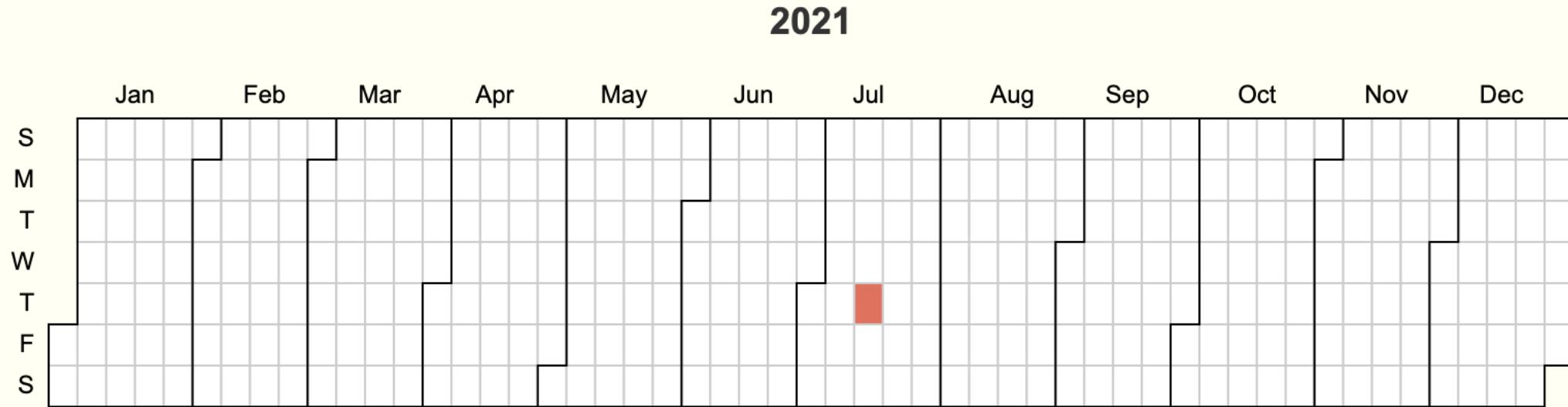
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- Ignore leap years in our calculation ( $1y = 365\ d$ ). Assume birthdays are equally likely to fall on any particular day.
- Order the students from 1 to  $n$ . There are  $\clubsuit$  possibilities for the birthday of a student. There are  $\clubsuit$  possibilities altogether.

# Birthday Problem

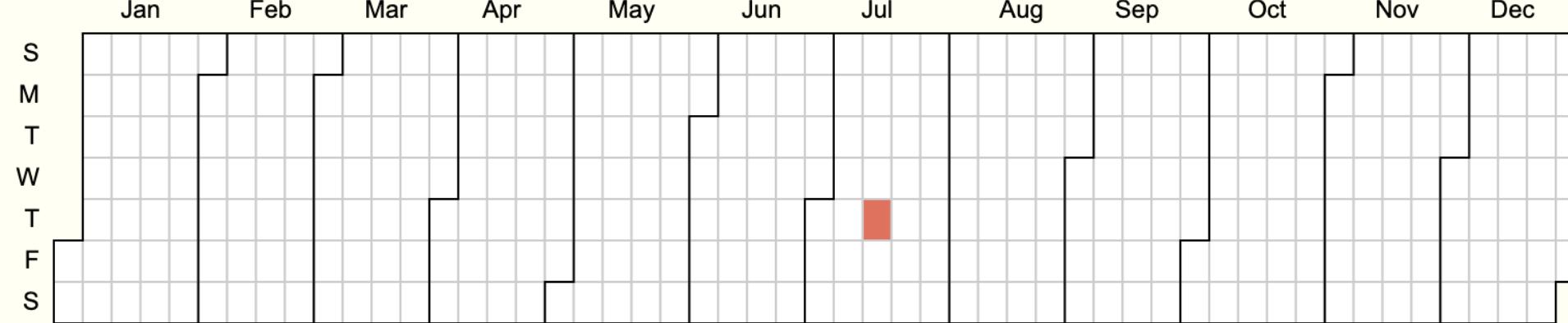
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- Ignore leap years in our calculation ( $1y = 365\ d$ ). Assume birthdays are equally likely to fall on any particular day.
- Order the students from 1 to  $n$ . There are 365 possibilities for the birthday of a student. There are  $365^n$  possibilities altogether.

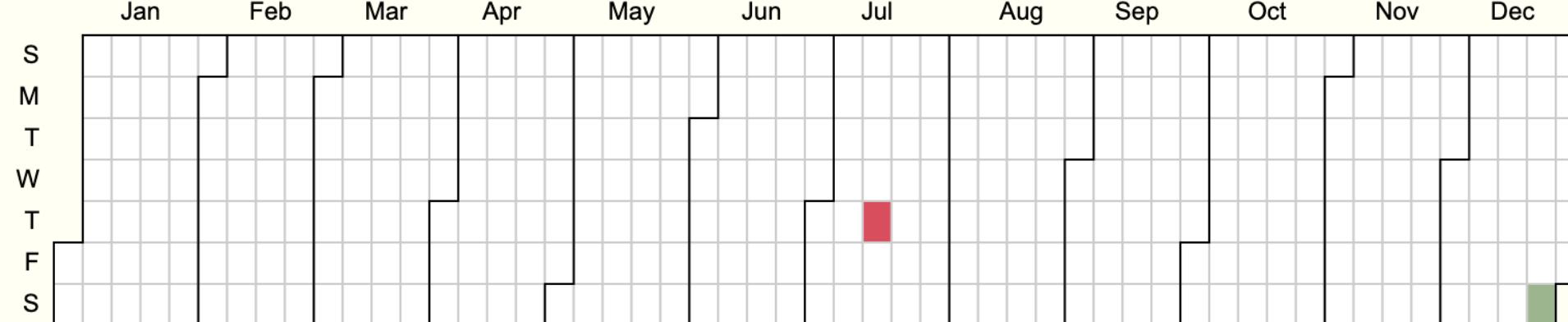
# Birthdays are different

2021



365 possibilities

2021



365×364 possibilities

# Birthday Problem

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- The probability that all birthdays are different for  $n$  students:

$$\frac{(365)_n}{365^n}.$$

We denote the product  $k \times (k - 1) \times \cdots \times (k - r + 1)$  by  $(k)_r$  (read ‘ $k$  down  $r$ ’ or ‘ $k$  lower  $r$ ’).

# Birthday Problem

---

How many students do we need to have in our hours section to make it favorable bet (that is, probability of success greater than  $\frac{1}{2}$ ) that two people in the classroom will have the same birthday?

probability

$$P = 1 - \frac{(365)_n}{365^n} > \frac{1}{2}$$

$$n = 23, 24, \dots$$





**SMALL PLATES**

**LARGE PLATES**

**SWEET BITES**

## Serving Orders

The waiter serve one course at a time. How many possible serving orders are there in total?

Order	Course
1	SP
2	LP
3	SB



**SMALL PLATES**

**LARGE PLATES**

**SWEET BITES**

## Serving Orders

Order	1	2	3
Course	SP	LP	SB
Course	SP	SB	LP
Course	LP	SP	SB
Course	LP	SB	SP
Course	SB	SP	LP
Course	SB	LP	SP

$6 = 3 \times 2 \times 1.$

!

**SMALL PLATES = 1**

**LARGE PLATES = 2**

**SWEET BITES = 3**

## Serving Orders

Order	1	2	3
Course	1	2	3
Course	1	3	2
Course	2	1	3
Course	2	3	1
Course	3	1	2
Course	3	2	1

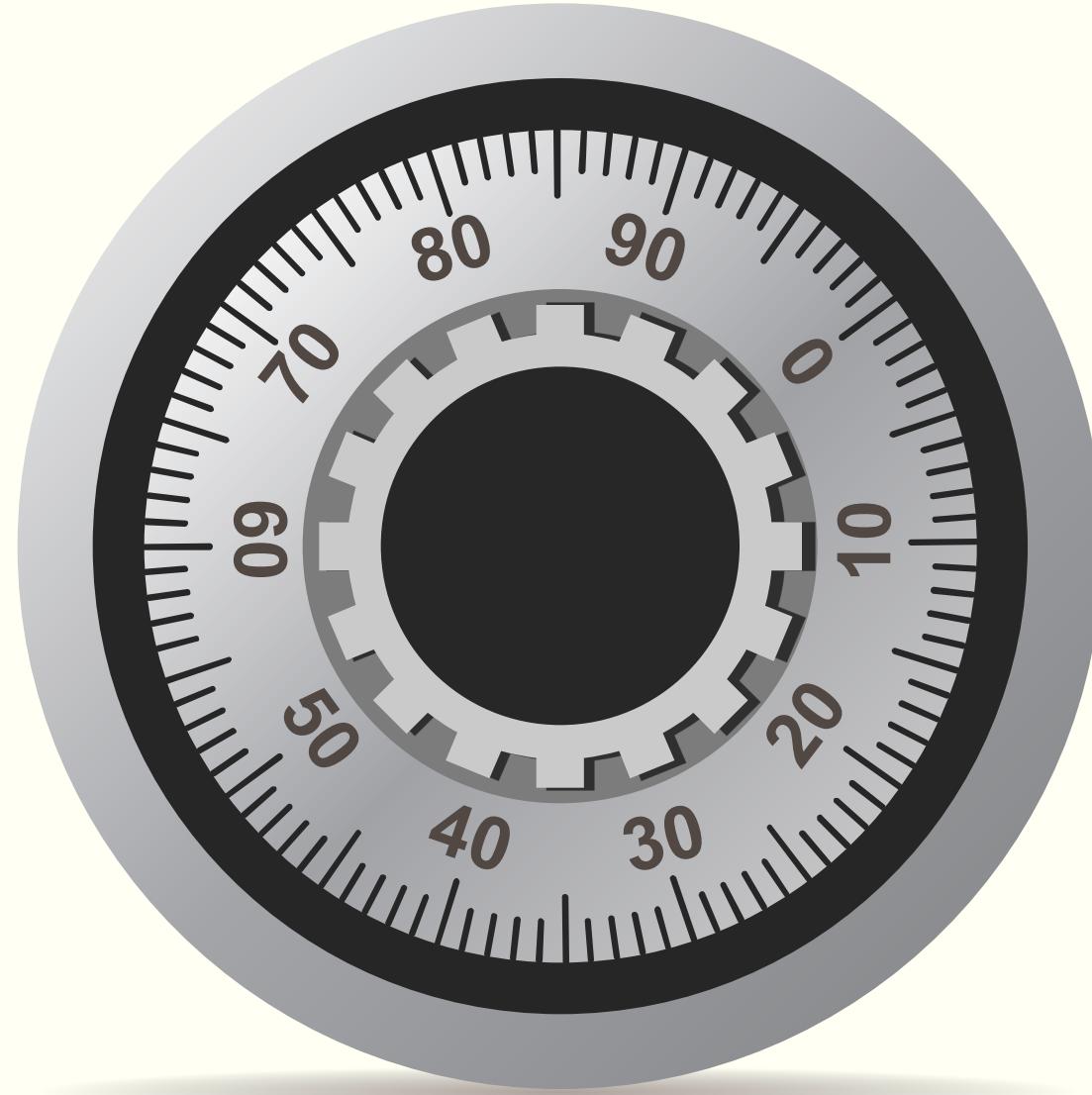
$$6 = 3 \times 2 \times 1.$$

!

# Combination Lock

## What is known

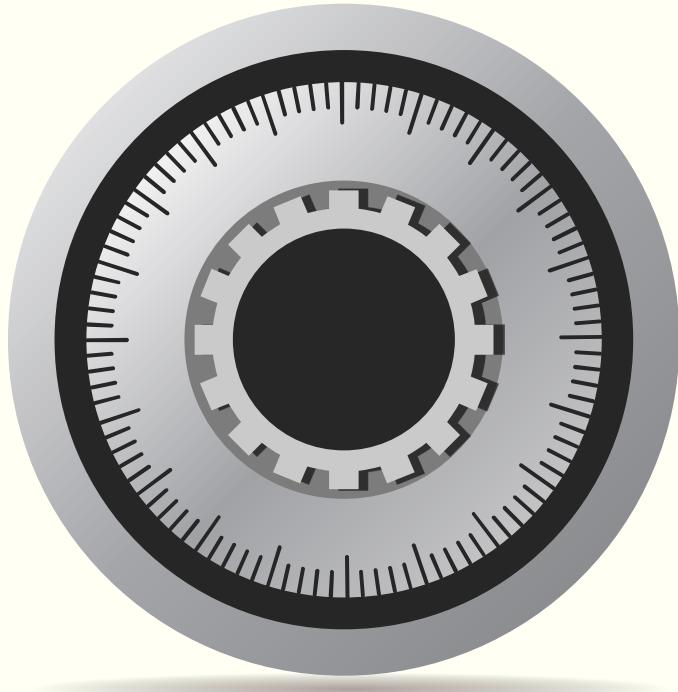
- The password is a combination of four numbers.
- The four numbers are 12, 25, 33, and 77.



## What is unknown

The order of these four numbers is unknown.

# Password Number Orders



What's the maximum number of tries to open the lock?

For ease of notation:

- 12 – 1<sup>st</sup>, 25 – 2<sup>nd</sup>, 33 – 3<sup>rd</sup>, 77 – 4<sup>st</sup>.

Further:

- 12 – 1, 25 – 2, 33 – 3, 77 – 4.

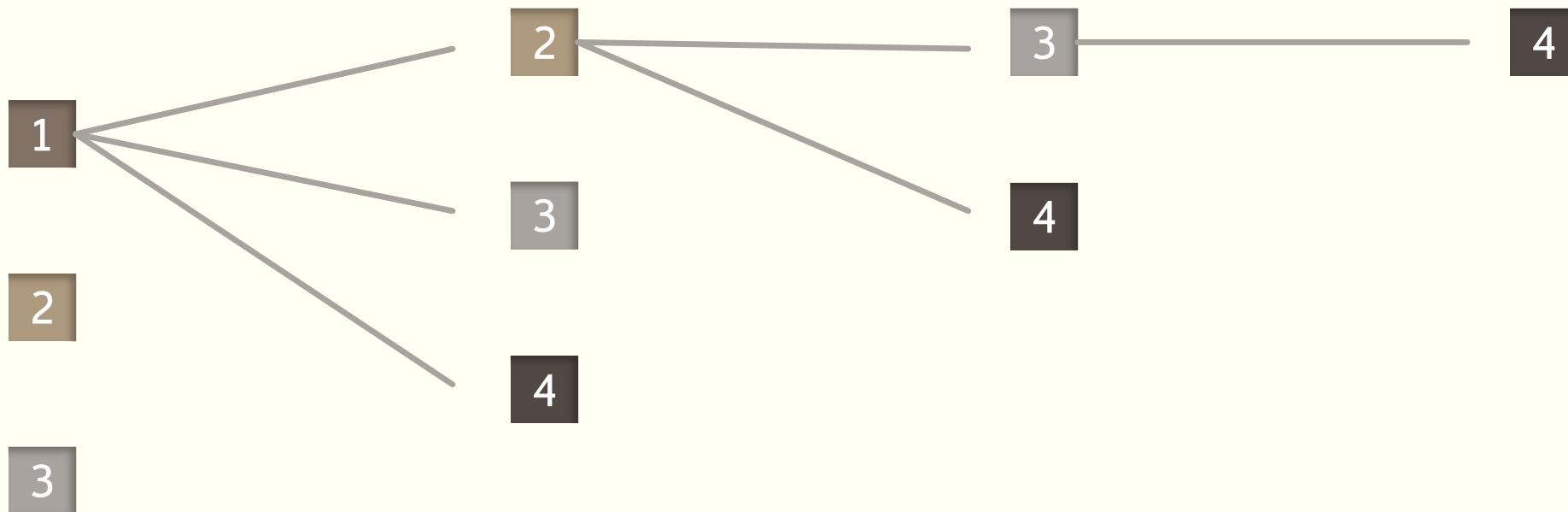
1	2	3	4
1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
...	...	...	...

1<sup>st</sup>

2<sup>nd</sup>

3<sup>rd</sup>

4<sup>st</sup>



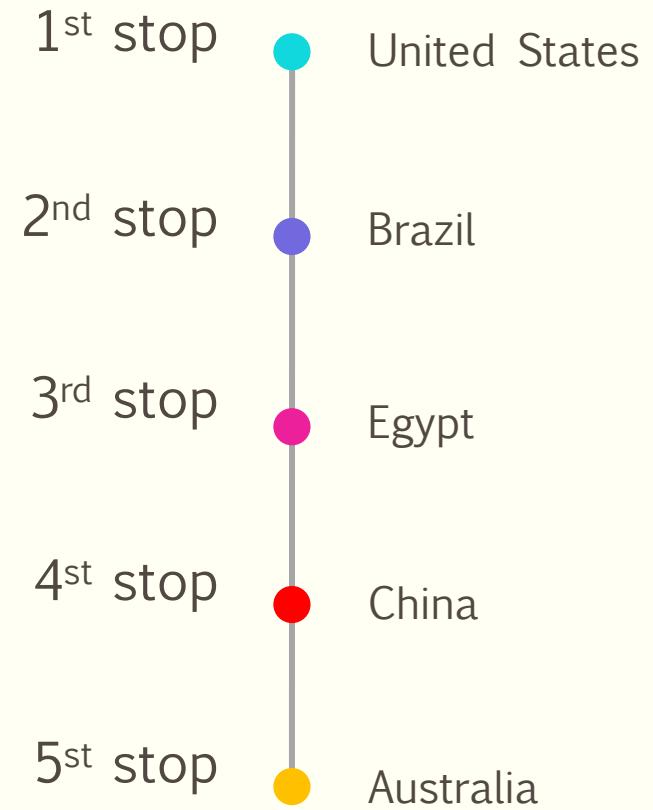
$$24 = 4 \times 3 \times 2 \times 1.$$

!

# Travel orders



Mr. Fogg visit one country at a time. How many possible travel orders are there in total?

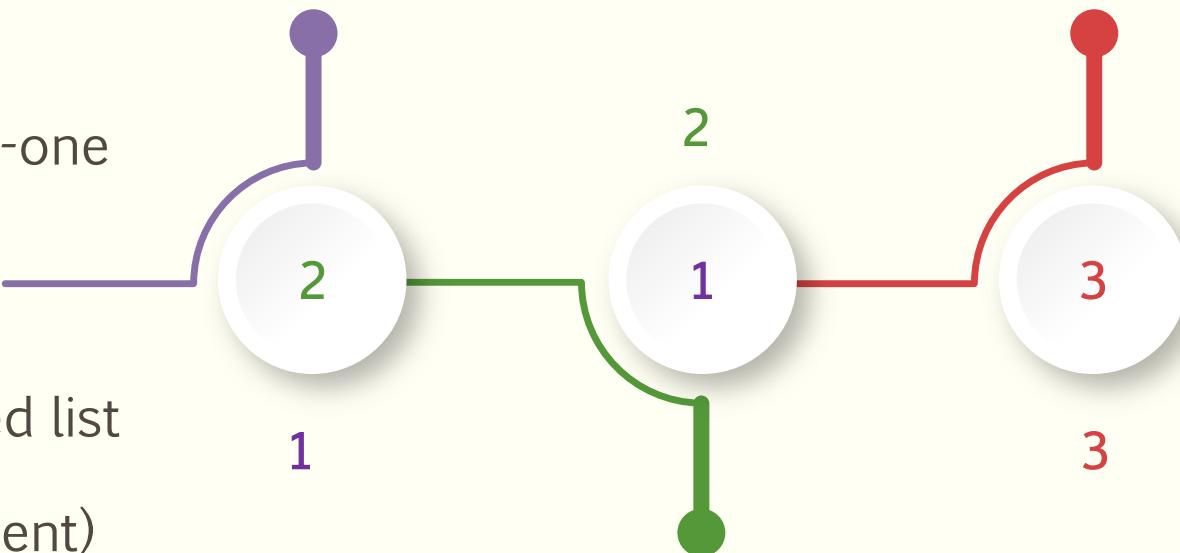


# Permutations

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- Let  $A$  be any finite set.
- A permutation of  $A$ ,  $\sigma$ , is a one-to-one mapping of  $A$  onto itself.
- For example, if we have an ordered list of elements  $A = \{a_1, a_2, a_3\}$ , a possible permutation (re-arrangement) can be prescribed by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$



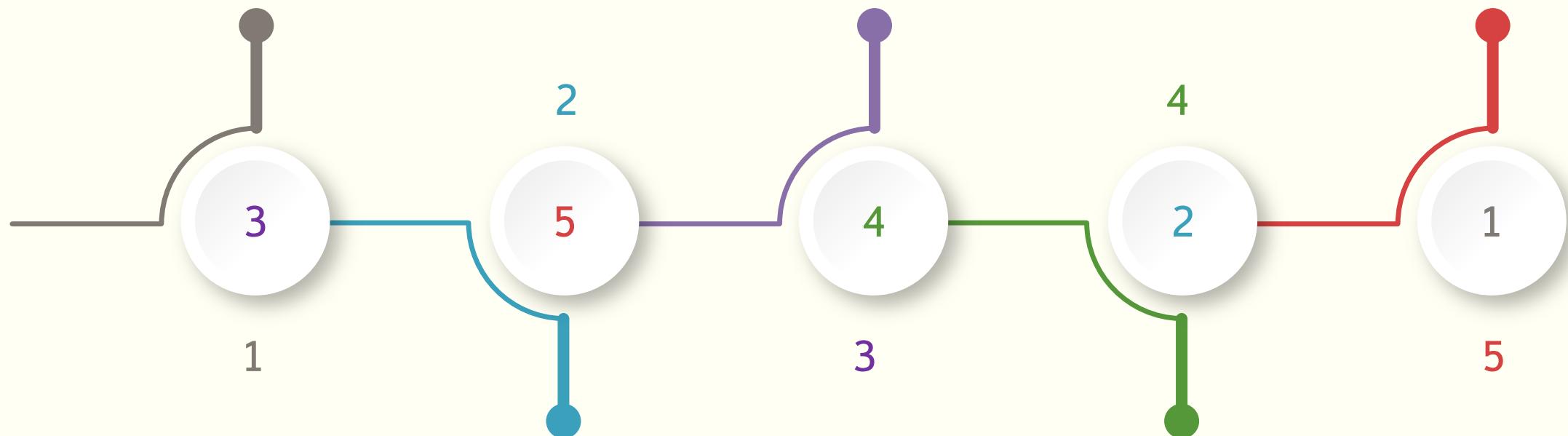
# Permutations

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## Theorem

The total number of permutations of a set  $A$  of  $n$  elements is given by  $n! = n \times (n - 1) \times (n - 2) \times \dots \times 1$ .

!



# $n$ factorial $n!$

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## Stirling's Formula

The sequence  $n!$  is asymptotically equal to  
$$n^n e^{-n} \sqrt{2\pi n}.$$

!

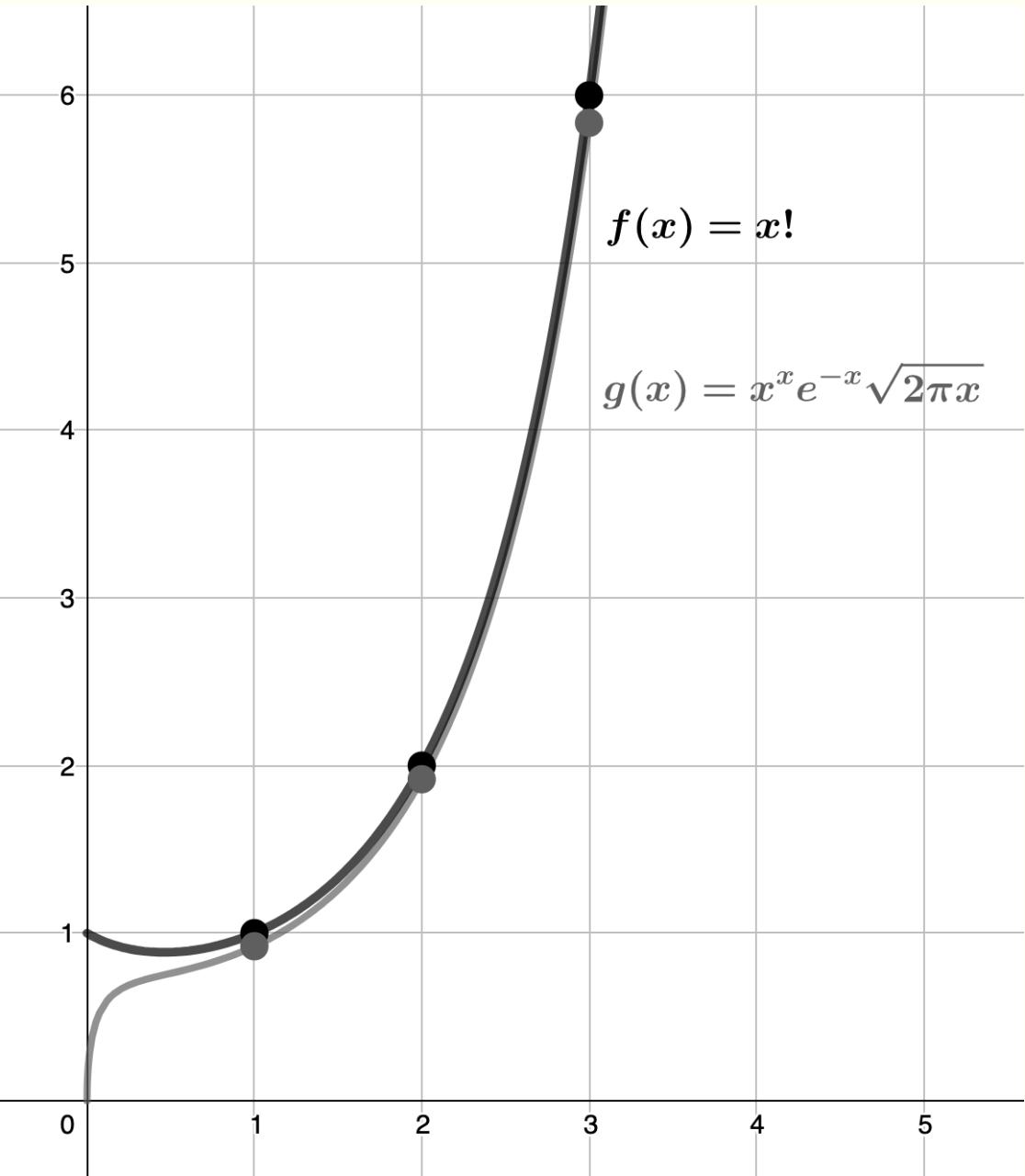
$n$	$n$ factorial	Stirling's Formula	ratio
1	1	0.9221	1.0844
2	2	1.919	1.0422
3	6	5.8362	1.0281
4	24	23.5062	1.021
5	120	118.0192	1.0168
6	720	710.0782	1.014
...	...	...	...

## Stirling's Formula

The sequence  $n!$  is asymptotically equal to  $n^n e^{-n} \sqrt{2\pi n}$ .

## Birthday Problem

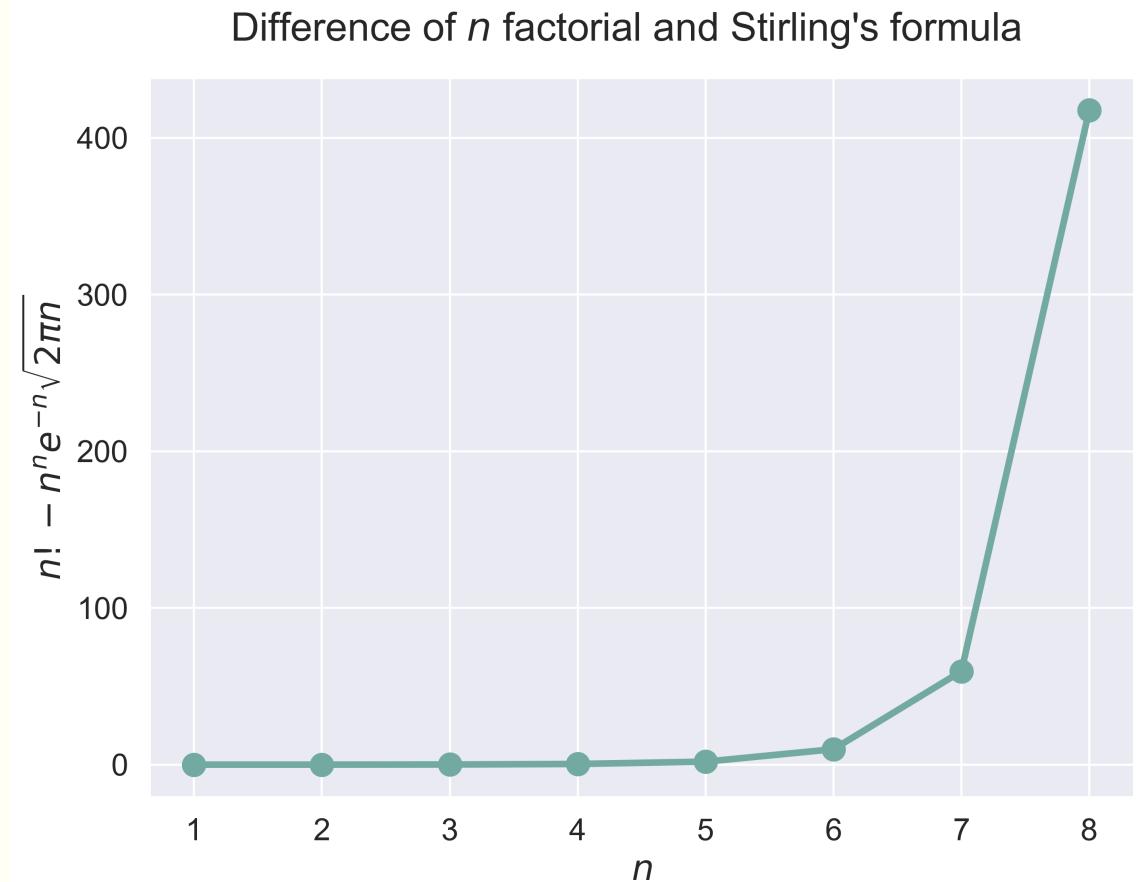
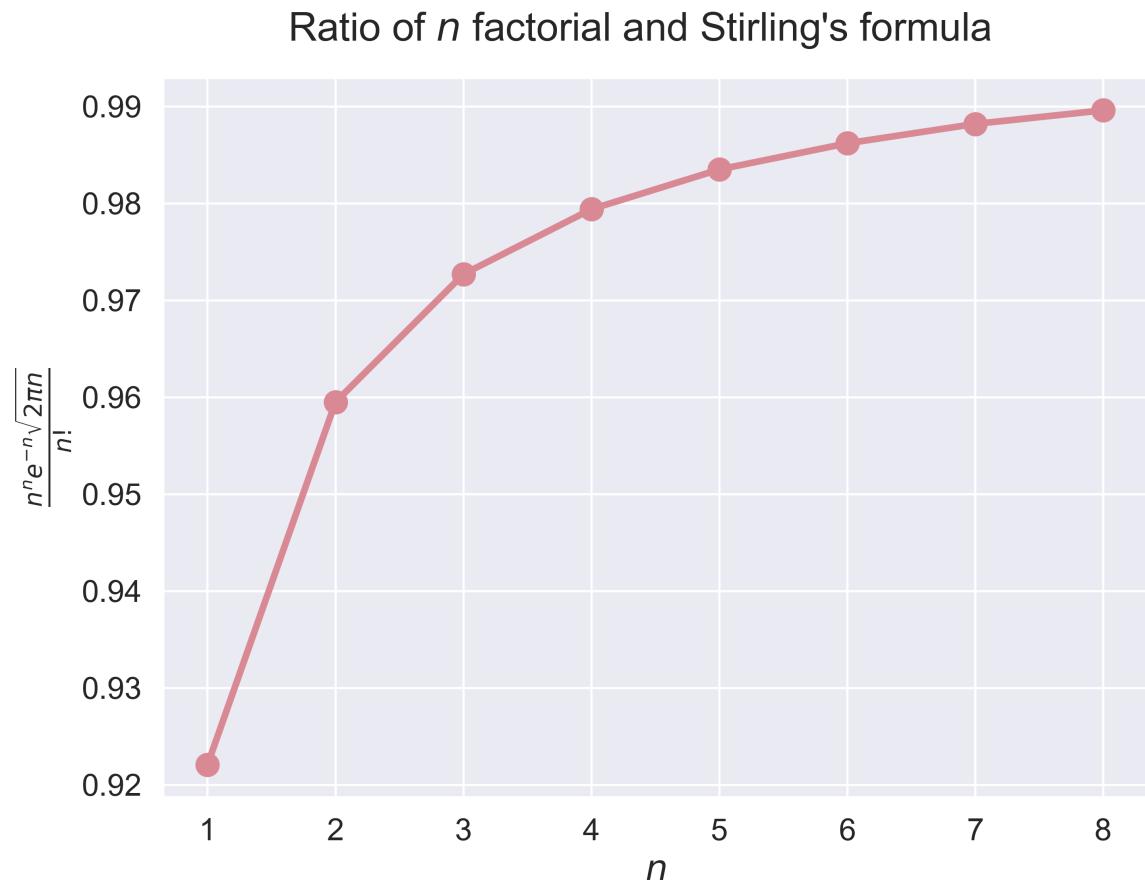
Apply Stirling's formula to estimate the number of students needed such that  $p_n = \frac{(365)_n}{365^n} = \frac{1}{2}$ .  
 $n = 23$ .



# Stirling's formula

$n$	$n!$	$n^n e^{-n} \sqrt{2\pi n}$	ratio	difference
1	1	0.9221	0.9221	0.0779
2	2	1.919	0.9595	0.081
3	6	5.8362	0.9727	0.1638
4	24	23.5062	0.9794	0.4938
5	120	118.0192	0.9835	1.9808
6	720	710.0782	0.9862	9.9218
7	5040	4980.3958	0.9882	59.6042
8	40320	39902.3955	0.9896	417.6045
...	...	...	...	...

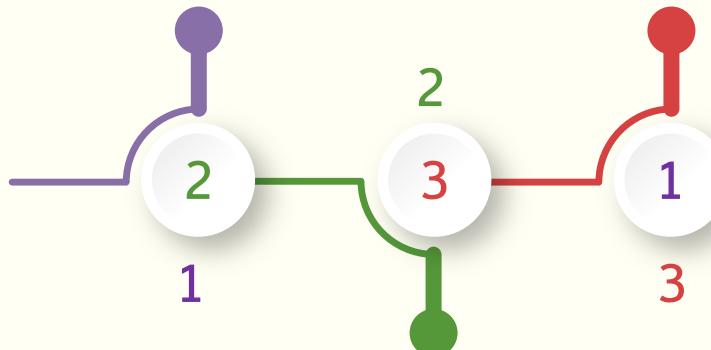
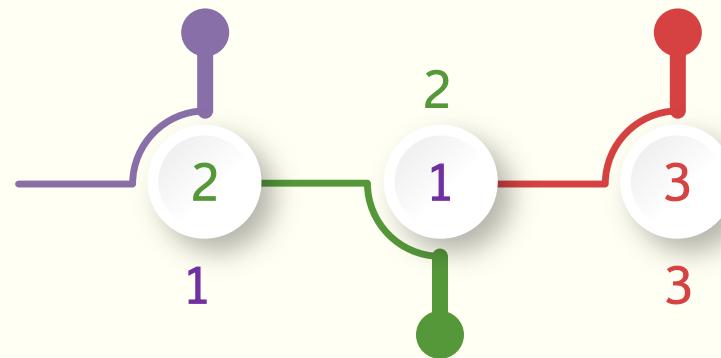
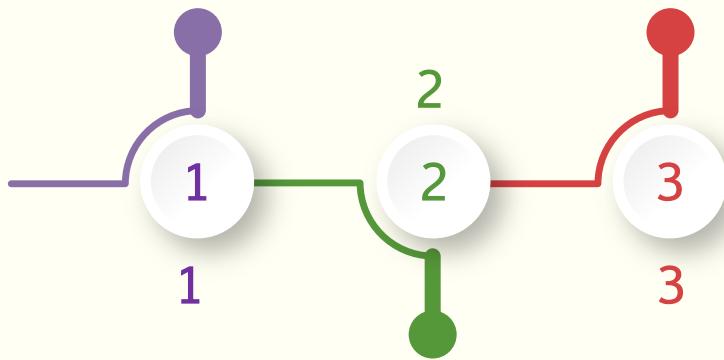
# Stirling's formula



# Fixed Points

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- Since a permutation is a one-to-one mapping of the set onto itself, it is of interest to ask how many points (elements) are mapped onto themselves. Such points are called **fixed points** of the mapping.



# Fixed Points

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- Let  $p_k(n)$  denote the probability that a random permutation of the set  $\{1, 2, \dots, n\}$  has exactly  $k$  fixed points.
- What is the probability of no fixed points for a permutation of a set of  $n$  elements,  $p_0(n)$ ? This is the famous **hat check** problem.

$$p_0(3) = \dots$$

?

1	2	3
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

# Fixed Points

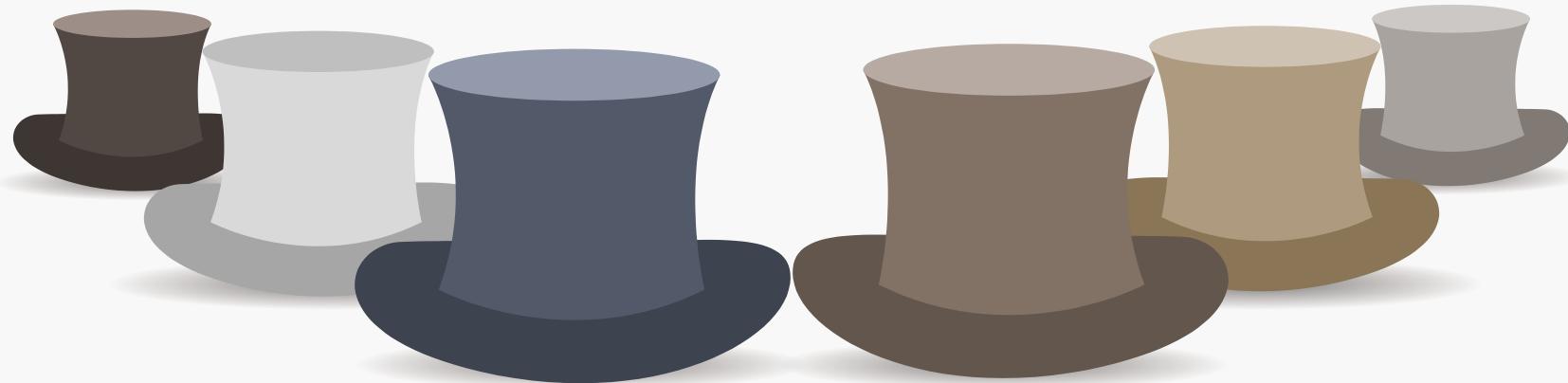
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- Let  $p_k(n)$  denote the probability that a random permutation of the set  $\{1, 2, \dots, n\}$  has exactly  $k$  fixed points.
- What is the probability of no fixed points for a permutation of a set of  $n$  elements,  $p_0(n)$ ? This is the famous hat check problem.

$$p_0(3) = \frac{2}{3!} = \frac{1}{3}.$$

?

1	2	3
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1



# Hat Check Problem

In a restaurant  $n$  hats are checked and they are hopelessly scrambled.  
What is the probability that no one gets his own hat back?

# Hat Check Problem

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- Number of derangements  $D_0(n)$
- If there is a derangement, then man #1 will not have his correct hat.
- We begin by looking at the case where man #1 gets hat #2. Note that this case can be broken down into two subcases:
  - a) man #2 gets hat #1, or
  - b) man #2 does not get hat #1



# Hat Check Problem

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- Number of derangements  $D_0(n)$ 
  - In case (a), for a derangement to occur, we need the remaining  $n - 2$  men to get the wrong hats. Therefore, the total number of derangements in this subcase is simply  $D_0(n - 2)$ .
  - In case (b), for a derangement to occur, man #2 cannot get hat #1 (that's case a), man #3 cannot get hat #3, man # $i$  cannot get hat # $i$ , etc. In this subcase, the number of derangements is  $D_0(n - 1)$ .

# Hat Check Problem

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- Number of derangements  $D_0(n)$
- We can treat the cases where man #1 receives hat #3, or hat #4, or hat  $#i$ , in exactly the same way.
- Therefore, to account for all possible derangements, there are  $n - 1$  such possibilities for all the different incorrect hats which man #1 can get.

$$D_0(n) = (n - 1)[D_0(n - 1) + D_0(n - 2)].$$

# Hat Check Problem

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- $D_0(n) = (n - 1)[D_0(n - 1) + D_0(n - 2)].$
- $p_0(n) = \frac{D_0(n)}{n!}.$
- $p_0(n) = p_0(n - 1) - \frac{1}{n}[p_0(n - 1) - p_0(n - 2)].$



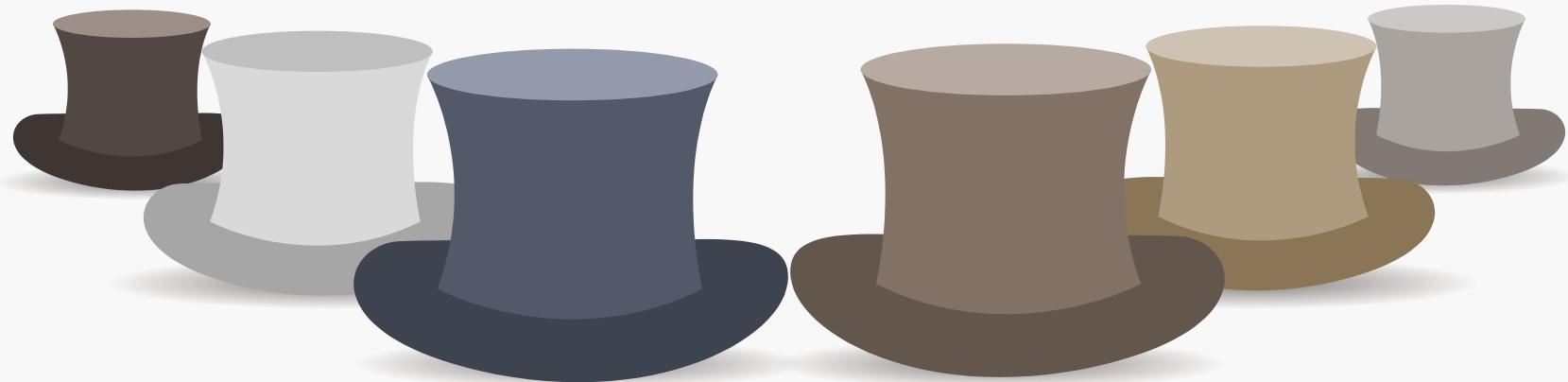
# Hat Check Problem

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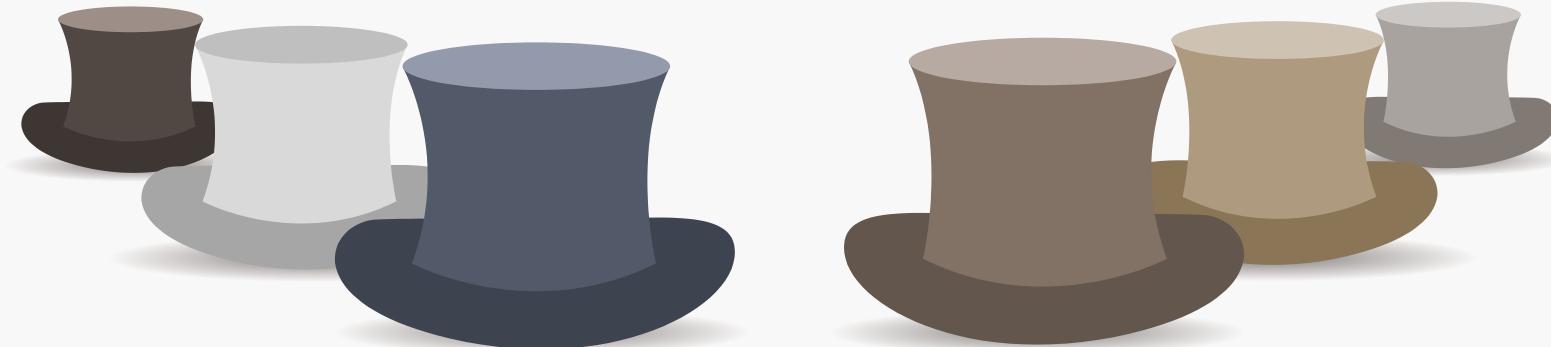
- $p_0(n) = p_0(n - 1) - \frac{1}{n} [p_0(n - 1) - p_o(n - 2)].$
- $p_0(1) = 0, p_0(2) = \frac{1}{2}.$
- $p_0(3) = p_0(2) - \frac{1}{3} [p_0(2) - p_o(1)] = \frac{1}{2} - \frac{1}{6}.$
- $p_0(4) = p_0(3) - \frac{1}{4} [p_0(3) - p_o(2)] = \frac{1}{2} - \frac{1}{6} + \frac{1}{24}.$
- ...





# Hat Check Problem

$$p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!}.$$



## Hat Check Problem

$$p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots + \frac{(-1)^n}{n!}.$$

Taylor Expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n + \dots.$$

$e^x$

## Taylor Expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots.$$

$e^x$

$p_0(n)$

## Hat Check Problem

$$e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!} + \cdots \approx .3679.$$

- Hat Check Problem (Fixed Point)

```
hat_check(n = 10)
figure_hat_check(n = 15, fsize = (12, 6), fs = 20)
```

```
# Calculate the probabilities of p_0(n)
def hat_check(n):
    p_1 = 0
    p_2 = 0.5
    p_list = [p_1, p_2]
    while len(p_list) < n:
        # recursive relation
        p_list.append(p_list[-1] - 1/(len(p_list)+1)*(p_list[-1] - p_list[-2]))
    return p_list
```

## Hat Check Problem

