BAYES' FORMULA

a two-stage experiment

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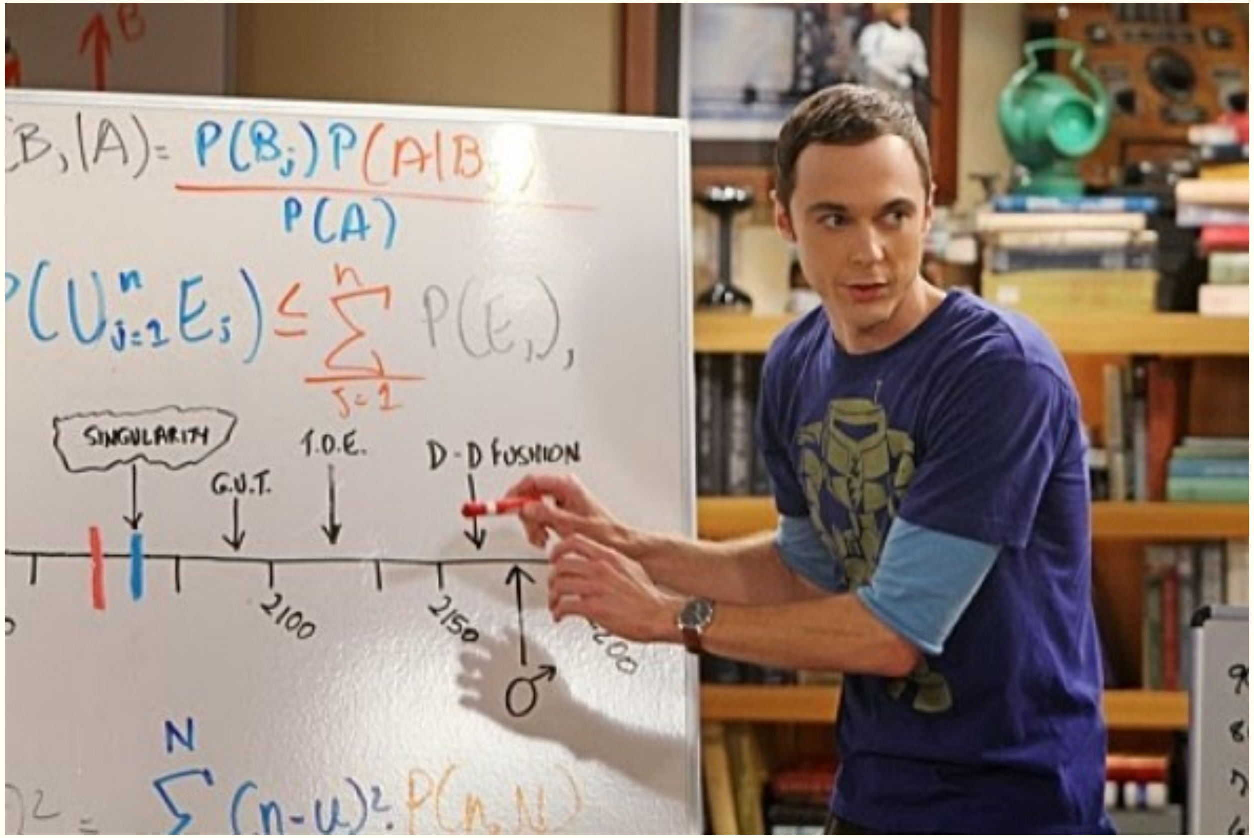
Simplest Bayes' Formula



$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}.$$

B

Bayes probabilities

Bayes' theorem links the degree of belief in a proposition **before and after** accounting for **evidence**.

B

Simplest Bayes' Formula

$$P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

=

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

=

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}.$$

B

Weather Forecast

Event-1: rain

$$P(\text{rain}) = 0.6$$

Prior probability

The **prior probability** of an event (often simply called **the prior**) is its probability obtained from some prior information.

Event-2: windy & cloudy

$$P(\text{windy \& cloudy}) = 0.48$$

Evidence

The **evidence** term in Bayes' theorem refers to the **overall probability** of this new piece of information.

Event-1: rain

Prior probability

$$P(\text{rain}) = 0.6$$

Event-2: windy & cloudy

Evidence

$$P(\text{windy \& cloudy}) = 0.48$$

$$P(\text{rain}|\text{windy \& cloudy}) = \frac{P(\text{rain})P(\text{windy \& cloudy}|\text{rain})}{P(\text{windy \& cloudy})}.$$

B

Event-1: rain

Prior probability

$$P(\text{rain}) = 0.6$$

Event-2: windy & cloudy

Evidence

$$P(\text{windy \& cloudy}) = 0.48$$

windy & cloudy | rain

$$P(\text{windy \& cloudy} \mid \text{rain}) = 0.64$$

Likelihood

The **likelihood** represents a conditional probability. It is the degree to which the first event is consistent with the second event.

$$P(\text{rain}|\text{windy \& cloudy}) = \frac{P(\text{rain})P(\text{windy \& cloudy}|\text{rain})}{P(\text{windy \& cloudy})}.$$

B

Event-1: rain

Prior probability

$$P(\text{rain}) = 0.6$$

Event-2: windy & cloudy

Evidence

$$P(\text{windy \& cloudy}) = 0.48$$

rain | windy & cloudy

Posterior probability

$$P(\text{rain}|\text{windy \& cloudy}) = \cdots$$

windy & cloudy | rain

Likelihood

$$P(\text{windy \& cloudy} \mid \text{rain}) = 0.64$$

$$P(\text{rain}|\text{windy \& cloudy}) = \frac{P(\text{rain})P(\text{windy \& cloudy}|\text{rain})}{P(\text{windy \& cloudy})}.$$

B

rain | windy & cloudy

Posterior probability

$$P(\text{rain}|\text{windy \& cloudy}) = \frac{0.6\times0.64}{0.48} = 0.8$$

Prior probability
rain

The **prior probability** of an event (often simply called **the prior**) is its probability obtained from some prior information.

Posterior probability
rain | windy & cloudy

The **posterior probability** represents the **updated prior probability** after taking into account some new piece of information.

Evidence
windy & cloudy

The **evidence** term in Bayes' theorem refers to the **overall probability** of this new piece of information.

Likelihood
windy & cloudy | rain

The **likelihood** represents a conditional probability. It is the degree to which the first event is consistent with the second event.

$$\text{Posterior probability} = \frac{\text{Prior probability} \times \text{Likelihood}}{\text{Evidence}}.$$

B

Bayes probabilities

Bayes probabilities are particularly appropriate for medical diagnosis.

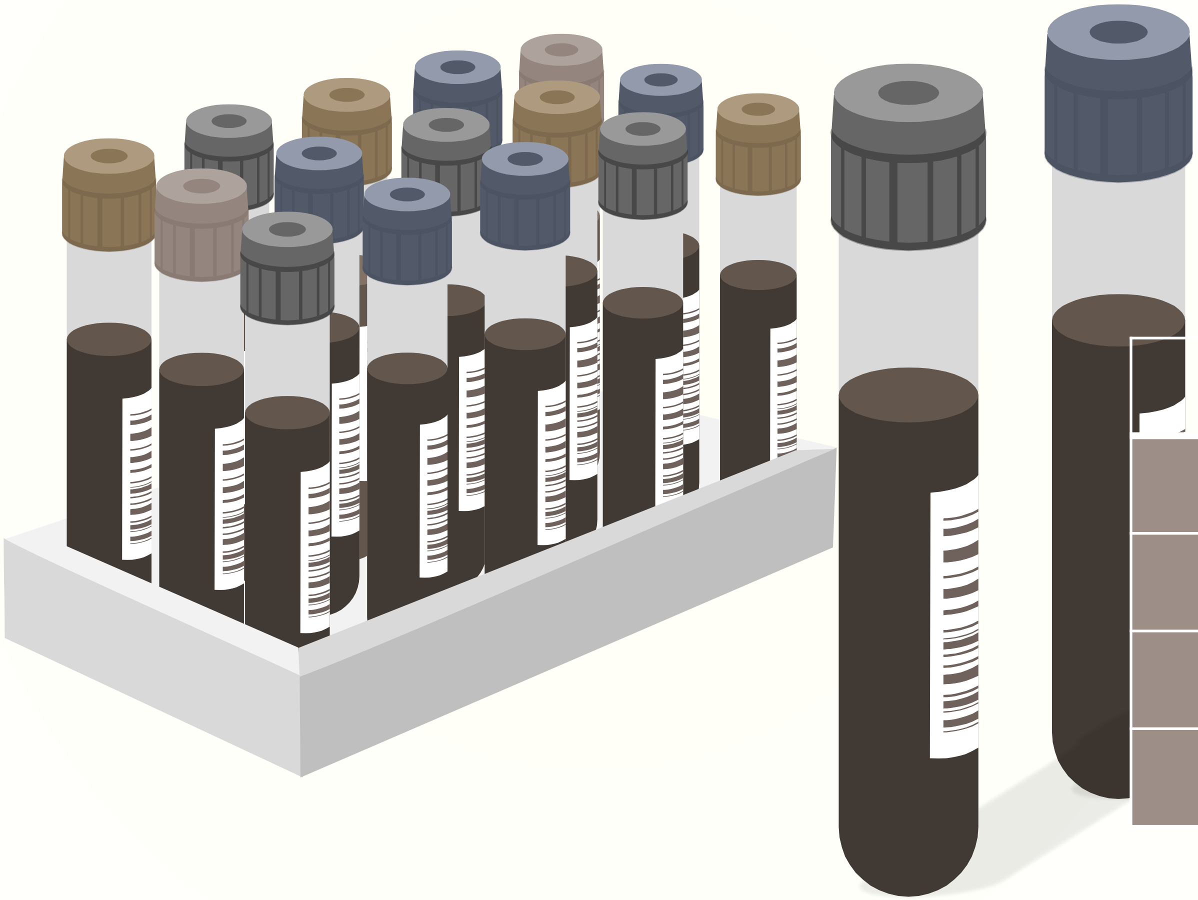
Solving Inverse Problems Using Bayes Probabilities

- A doctor is trying to decide if a patient has one of three diseases d_1, d_2, or d_3. Two tests are to be carried out, each of which results in a positive (+) or a negative (–) outcome. There are four possible test patterns ++, +−, −+, and −−.
- National records have indicated that, for 10,000 people having one of these three diseases, the distribution of diseases and test results are as in Table below.

Disease	Number having this disease	Number having this disease			
		+ +	+ −	− +	− −
d_1	3215	2110	301	704	100
d_2	2125	396	132	1187	410
d_3	4660	510	3568	73	509
Total	10000	3016	4001	1964	1019

Use Bayes' formula

to compute various posterior probabilities



	d_1	d_2	d_3
+ +			
+ −			
− +			
− −			

Use Bayes' formula to compute various posterior probabilities

	d_1	d_2	d_3
$+\ +$			
$+\ -$			
$-\ +$			
$-\ -$			

Posterior probability: d_i | $+\ +$

$$P(d_i|+\ +) = \frac{P(+\ +|d_i)P(d_i)}{P(+\ +)}$$

Posterior probability: d_i | $+\ -$

$$P(d_i|+\ -) = \frac{P(+\ -|d_i)P(d_i)}{P(+\ -)}$$

Posterior probability: d_i | $-\ +$

$$P(d_i|-\ +) = \frac{P(-\ +|d_i)P(d_i)}{P(-\ +)}$$

Posterior probability: d_i | $-\ -$

$$P(d_i|-\ -) = \frac{P(-\ -|d_i)P(d_i)}{P(-\ -)}$$

Prior Probability

Disease	Number having this disease	Number having this disease			
		+ +	+ −	− +	− −
d_1	3215	2110	301	704	100
d_2	2125	396	132	1187	410
d_3	4660	510	3568	73	509
Total	10000	3016	4001	1964	1019

Prior probability: disease 1

$$P(d_1) = \frac{3215}{10000} = 0.3215$$

Prior probability: disease 2

$$P(d_2) = \frac{2125}{10000} = 0.2125$$

Prior probability: disease 3

$$P(d_3) = \frac{4660}{10000} = 0.466$$

Evidence

Disease	Number having this disease	Number having this disease			
		+ +	+ −	− +	− −
d_1	3215	2110	301	704	100
d_2	2125	396	132	1187	410
d_3	4660	510	3568	73	509
Total	10000	3016	4001	1964	1019

Evidence: + +

$$P(+\ +) = \frac{3016}{10000} = 0.3016$$

Evidence: + −

$$P(+\ -) = \frac{4001}{10000} = 0.4001$$

Evidence: − +

$$P(-\ +) = \frac{1964}{10000} = 0.1964$$

Evidence − −

$$P(-\ -) = \frac{1019}{10000} = 0.1019$$

Likelihood

Disease	Number having this disease	Number having this disease			
		+ +	+ −	− +	− −
d_1	3215	2110	301	704	100
d_2	2125	396	132	1187	410
d_3	4660	510	3568	73	509
Total	10000	3016	4001	1964	1019

Likelihood: + + | d_1

$$P(+\ +|d_1) = \frac{2110}{3215}$$

Likelihood: + − | d_1

$$P(+\ -|d_1) = \frac{301}{3215}$$

Likelihood: − + | d_1

$$P(-\ +) = \frac{704}{3215}$$

Likelihood: − − | d_1

$$P(-\ -) = \frac{100}{3215}$$

Posterior Probability

Disease	Number having this disease	Number having this disease			
		+ +	+ −	− +	− −
d_1	3215	2110	301	704	100
d_2	2125	396	132	1187	410
d_3	4660	510	3568	73	509
Total	10000	3016	4001	1964	1019

Prior probability: disease 1

$$P(d_1) = \frac{3215}{10000} = 0.3215$$

Evidence: + +

$$P(+\ +) = \frac{3016}{10000} = 0.3016$$

Likelihood: + + | d_1

$$P(+\ +|d_1) = \frac{2110}{3215}$$

Posterior probability: d_1 | + +

$$\mathbf{P}(\boldsymbol{d_1}|+\ +) = \frac{\boldsymbol{P}(+\ +|\boldsymbol{d_1})\boldsymbol{P}(\boldsymbol{d_1})}{\boldsymbol{P}(+\ +)}$$

$$= \frac{\frac{2110}{3215}\frac{3215}{10000}}{\frac{3016}{10000}} = \frac{2110}{3016} = \mathbf{0.700}$$

Posterior Probability

	d_1	d_2	d_3
+ +	0.700	0.131	0.169
+ −	0.075	0.033	0.892
− +	0.358	0.604	0.038
− −	0.098	0.403	0.499

Posterior probability: d_i | + +

$$\mathrm{P}(d_i|+\ +) = \frac{P(+\ +|d_i)P(d_i)}{P(+\ +)}$$

Posterior probability: d_i | + −

$$\mathrm{P}(d_i|+\ -) = \frac{P(+\ -|d_i)P(d_i)}{P(+\ -)}$$

Posterior probability: d_i | − +

$$\mathrm{P}(d_i|-\ +) = \frac{P(-\ +|d_i)P(d_i)}{P(-\ +)}$$

Posterior probability: d_i | − −

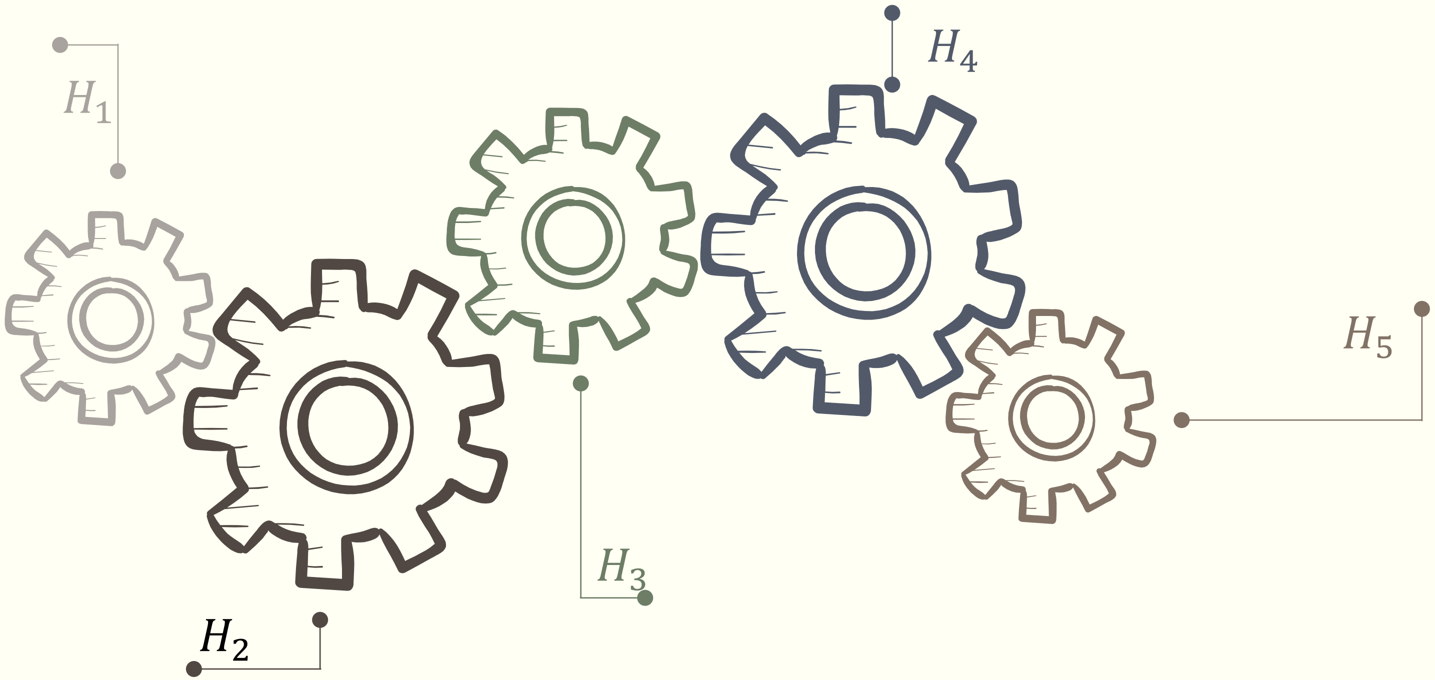
$$\mathrm{P}(d_i|-\ -) = \frac{P(-\ -|d_i)P(d_i)}{P(-\ -)}$$

HOW TO GET THE EVIDENCE?

Sometimes the evidence is not directly given to us…

Bayes' Formula

- **Bayes probabilities**: given the outcome of the second stage of a two-stage experiment, the probability for an outcome at the first stage.
- Suppose we have a set of hypotheses $H_1, H_2, \cdots, H_m$, which are pairwise disjoint and such that $\Omega = H_1 \cup H_2 \cup \cdots \cup H_m$. We have a set of prior probabilities $P(H_1), P(H_2), \cdots, P(H_m)$ for the hypotheses.



Bayes' Formula

- **Bayes probabilities**: given the outcome of the second stage of a two-stage experiment, the probability for an outcome at the first stage.
- Suppose we have a set of hypotheses $H_1, H_2, \cdots, H_m$, which are pairwise disjoint and such that $\Omega = H_1 \cup H_2 \cup \cdots \cup H_m$. We have a set of prior probabilities $P(H_1), P(H_2), \cdots, P(H_m)$ for the hypotheses.
- We also have an event **evidence** E, which can tell us further information about which hypothesis is correct. Suppose we know $P(E|H_i)$ for all i: if we know the correct hypothesis, we know the probability for the evidence E. The conditional probability $P(H_i|E)$ is the probability for the hypothesis given the evidence E, and is called the **posterior probability**.

Bayes' formula

$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{P(E)}$$

$$P(E|H_i) = \frac{P(E \cap H_i)}{P(H_i)}.$$

=

Important Properties

- If $H_1, \ldots, H_n$ are pairwise disjoint subsets of Ω (i.e., no two of the H_i have an element in common), then

$$P(H_1 \cup \cdots \cup H_n) = \sum_{i=1}^{n} P(H_i).$$

- If $H_1, \ldots, H_n$ are pairwise disjoint subsets with $\Omega = H_1 \cup \cdots \cup H_n$, and let E be any event. Then

$$P(E) = \sum_{i=1}^{n} P(E \cap H_i).$$

Bayes' formula

$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{P(E)}$$

$$P(E \cap H_i) = P(E|H_i)P(H_i).$$

=

$$P(E) = \sum_{i=1}^{m} P(E \cap H_i).$$

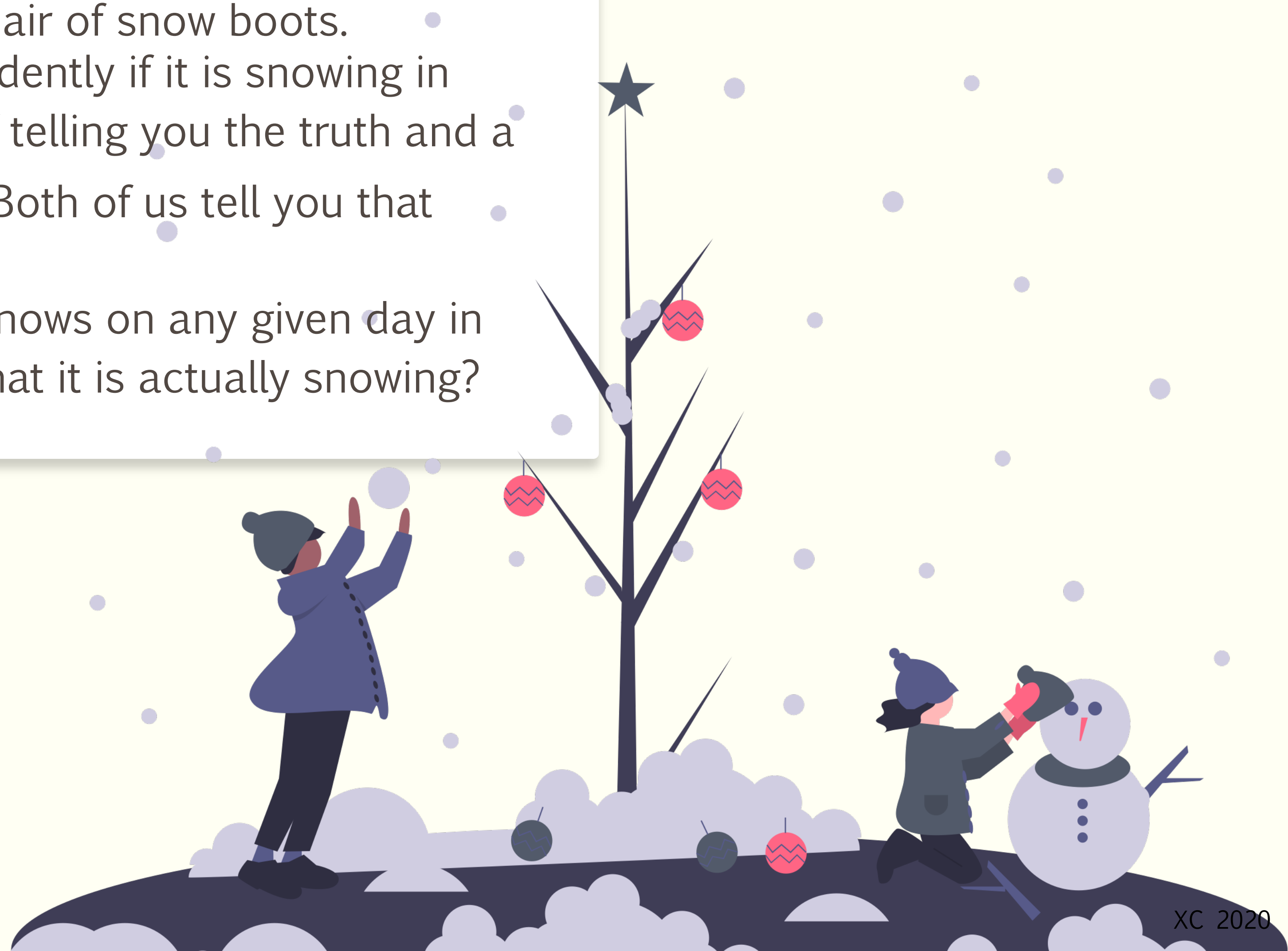
$$P(E) = \sum_{i=1}^{m} P(E|H_i)P(H_i).$$

Bayes' formula

$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{\sum_{i=1}^{m} P(E|H_i)P(H_i)}$$

Snow in Hanover

- Four months later… xue hua piao piao
- You are about to get on the Dartmouth Coach and back to school. You want to know if you should wear a pair of snow boots.
- You ask both the driver and me independently if it is snowing in Hanover. Both of us have a $\frac{3}{4}$ chance of telling you the truth and a $\frac{1}{4}$ chance of messing with you by lying. Both of us tell you that "YES" it is snowing.
- Here in Hanover, the probability that it snows on any given day in November is $\frac{1}{2}$. What is the probability that it is actually snowing?



Prior probability

The **prior probability** of an event (often simply called **the prior**) is its probability obtained from some prior information.

Posterior probability

The **posterior probability** represents the **updated prior probability** after taking into account some new piece of information.

Evidence

The **evidence** term in Bayes' theorem refers to the **overall probability** of this new piece of information.

Likelihood

The **likelihood** represents a conditional probability. It is the degree to which the first event is consistent with the second event.

$$\text{Posterior probability} = \frac{\text{Prior probability} \times \text{Likelihood}}{\text{Evidence}}.$$

B

Event-1: snow

Prior probability

$$P(\text{snow}) = \frac{1}{2}$$

snow | yes, yes

Posterior probability

$$P(\text{snow}|\text{yes} \,\&\, \text{yes}) = \cdots$$

Event-2: yes & yes

Evidence

$$P(\text{yes} \,\&\, \text{yes}) = \cdots$$

Yes, Yes | snow

Likelihood

$$P(\text{yes} \,\&\, \text{yes}|\text{snow}) = \cdots$$

$$P(\text{snow}|\text{yes} \,\&\, \text{yes}) = \frac{P(\text{snow})P(\text{yes} \,\&\, \text{yes}|\text{snow})}{P(\text{yes} \,\&\, \text{yes})}.$$

B

Bayes' formula

$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{\sum_{i=1}^{m} P(E|H_i)P(H_i)}$$

H_i: snow, not snow

E: yes & yes

Snow in Hanover

$$P(\text{snow}|\text{yes \& yes}) = \frac{P(\text{snow})P(\text{yes \& yes}|\text{snow})}{P(\text{yes \& yes})}.$$

B

Bayes' formula

$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{\sum_{i=1}^{m} P(E|H_i)P(H_i)}$$

H_i: snow, not snow
E: yes & yes

$$\begin{aligned} &P(\text{yes \& yes}) \\ &= P(\text{snow})P(\text{yes \& yes}|\text{snow}) + P(\text{not snow})P(\text{yes \& yes}|\text{not snow}) \\ &= \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{2}\left(\frac{1}{4}\right)^2 \end{aligned}$$

$$P(\text{yes \& yes})$$
$$= P(\text{snow})P(\text{yes \& yes}|\text{snow}) + P(\text{not snow})P(\text{yes \& yes}|\text{not snow})$$
$$= \frac{1}{2}(\frac{3}{4})^2 + \frac{1}{2}(\frac{1}{4})^2$$

$$P(\text{snow}|\text{yes \& yes}) = \frac{P(\text{snow})P(\text{yes \& yes}|\text{snow})}{P(\text{yes \& yes})}$$
$$= \frac{\frac{1}{2}(\frac{3}{4})^2}{\frac{1}{2}(\frac{3}{4})^2 + \frac{1}{2}(\frac{1}{4})^2}$$
$$= \frac{9}{10}$$

Chocolate Box

- Forrest Gump has a box of assorted chocolate. It includes 10 pieces of different flavors.
- A piece of chocolate is taken out at random. Let X be the chosen piece.
- If Jenny does not buy him a new box of chocolate and the second piece of chocolate Y is chosen from the original box at random.
- Consider the distributions of X and Y. Are the two distributions the same?

Chocolate:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Random variable X

$$m(x) = \frac{1}{10}$$

Random variable Y

$$P(Y = 1)$$
$$= P(Y = 1|X = 1)P(X = 1) + P(Y = 1|X \neq 1)P(X \neq 1)$$
$$= 0 \times \frac{1}{10} + \frac{1}{9} \times \frac{9}{10}$$
$$= \frac{1}{10}$$

July 2020

Sun	Mon	Tue	Wed	Thu	Fri	Sat
28	29	30	01	02	03	04
05	06	07	08	09	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	01

Midterm 1

Open book

Scope: Chapters 1, 2, 3, and 4

- discrete probability distributions
- continuous probability densities
- permutations and combinations
- conditional probability

Materials: Slides, homework, quizzes, textbook

Date & Time: July 20, 3 hours, 24 hours

Office hours: July 20, July 21