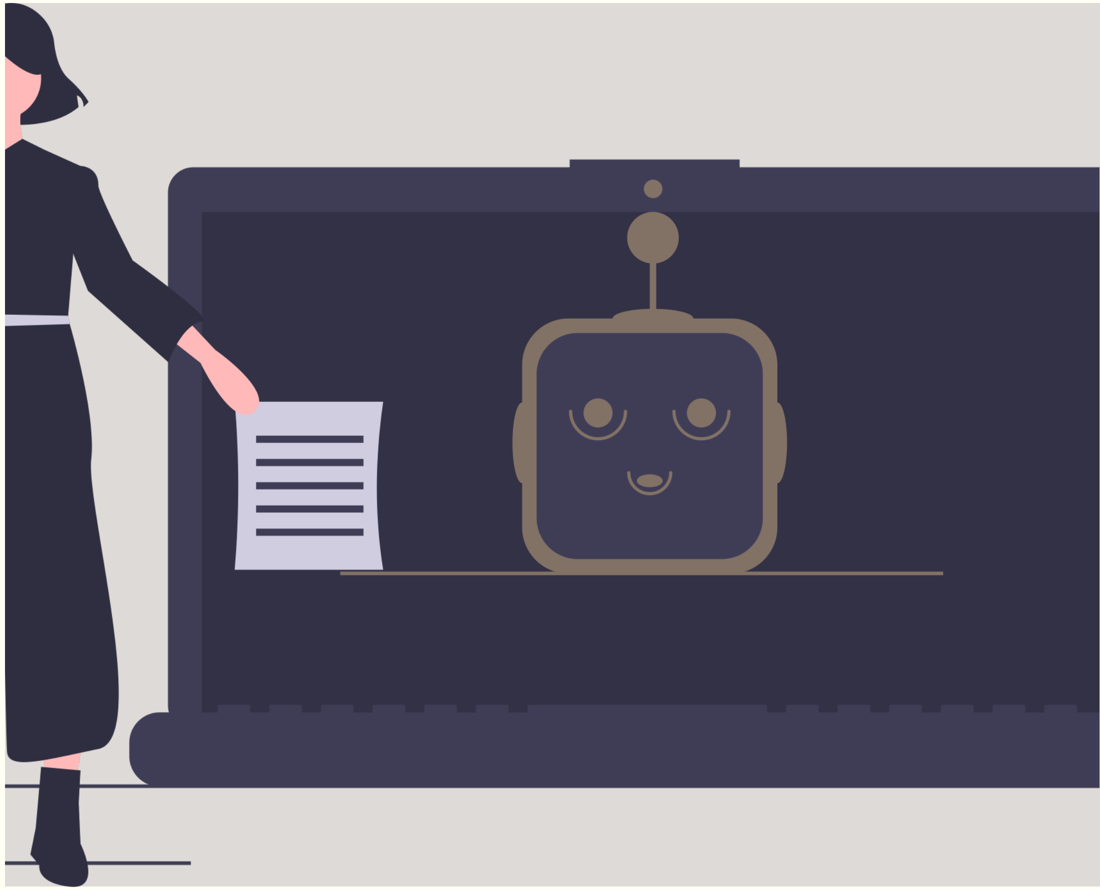
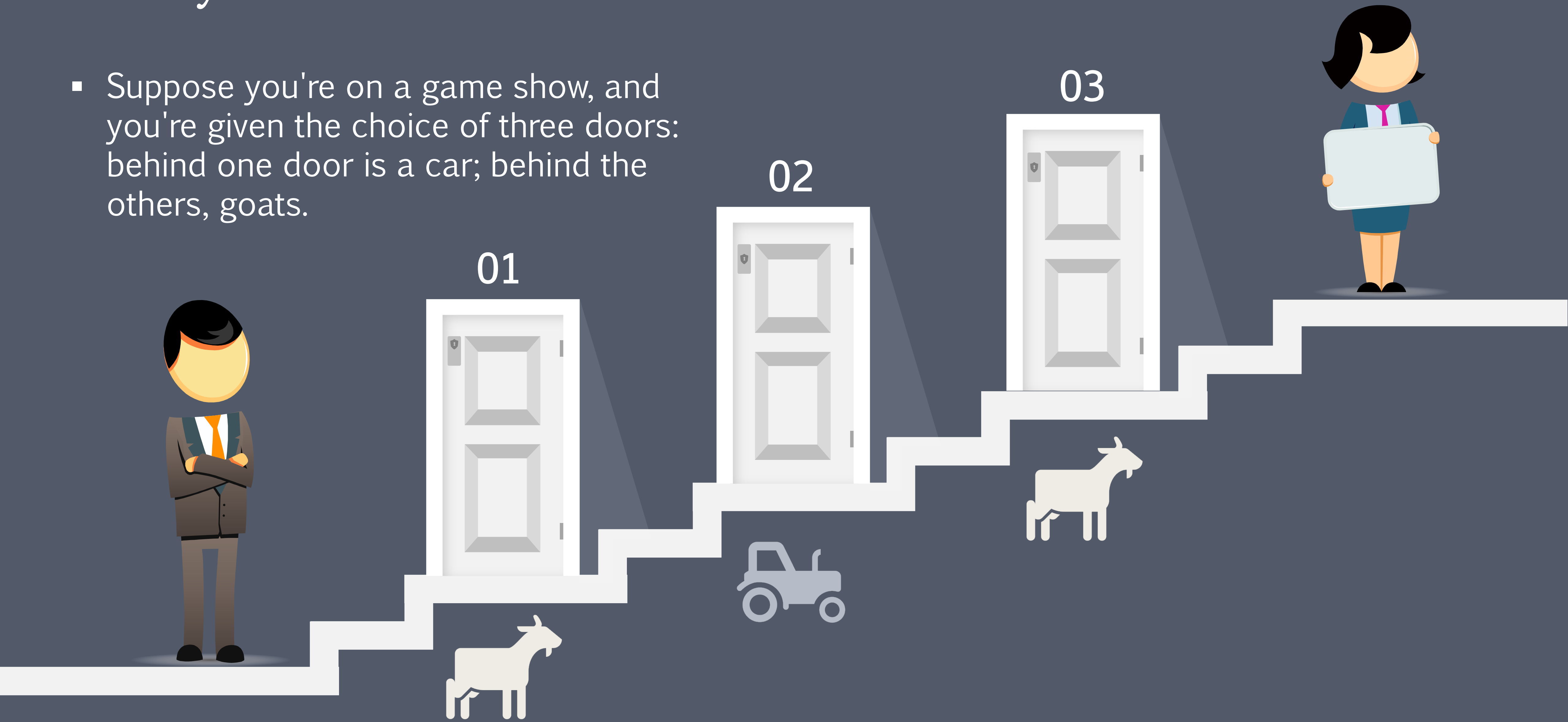
MATH 20: PROBABILITY

Discrete **Conditional** Probability



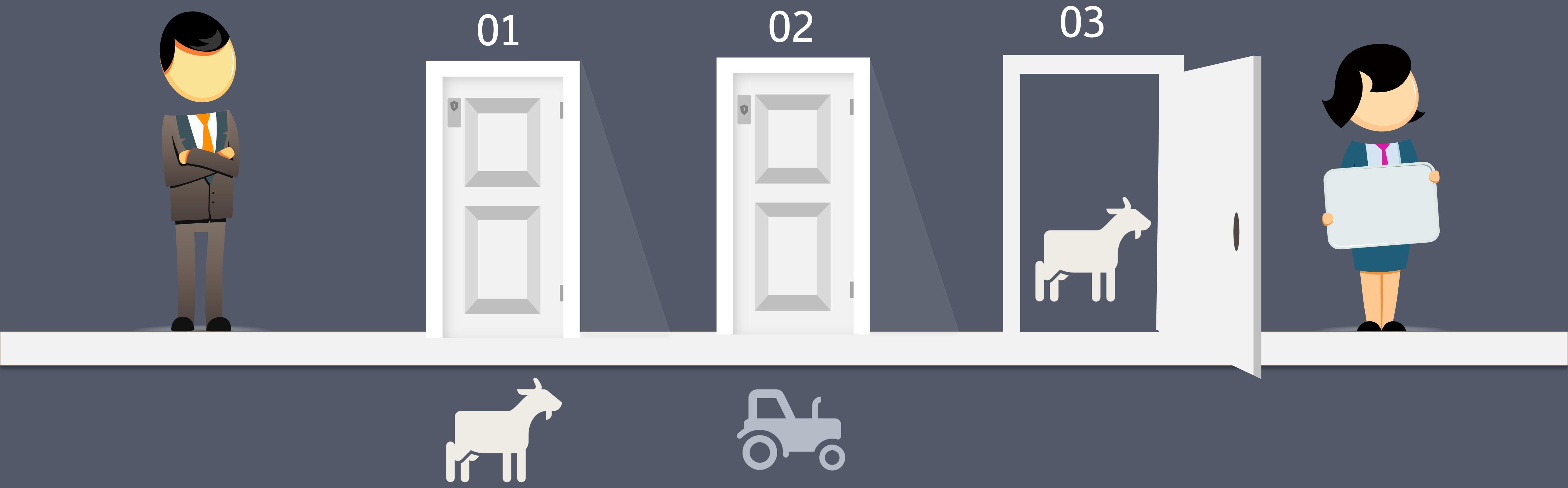
Monty Hall Problem

- Suppose you're on a game show, and you're given the choice of three doors: behind one door is a car; behind the others, goats.



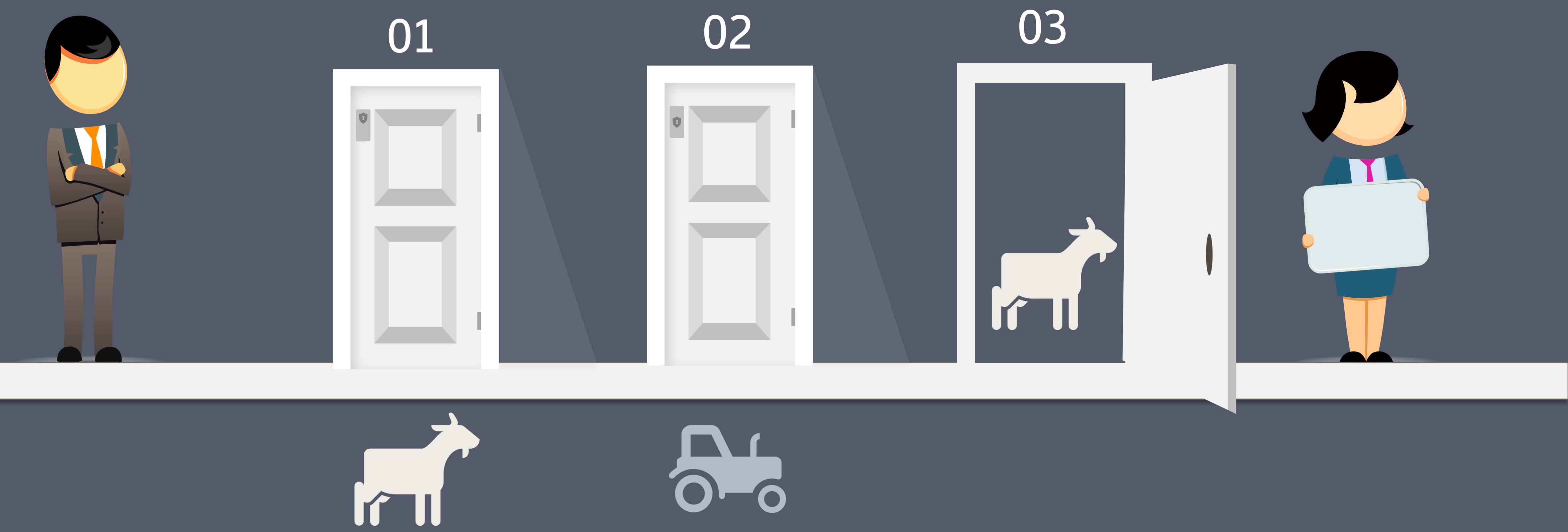
Monty Hall Problem

- You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.



Monty Hall Problem

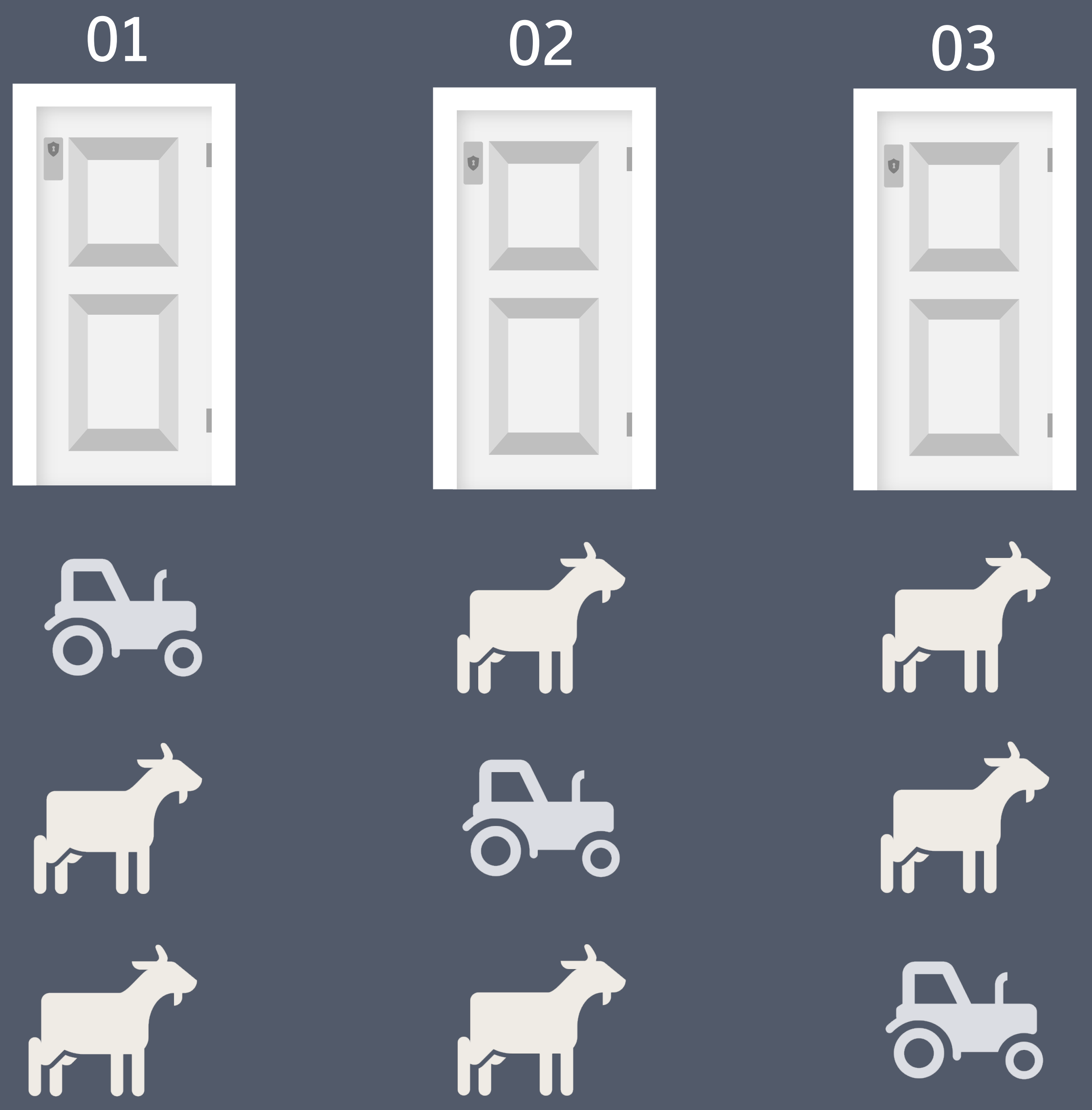
- He then says to you, "Do you want to pick door No. 2?”
- Is it to your advantage to switch your choice?

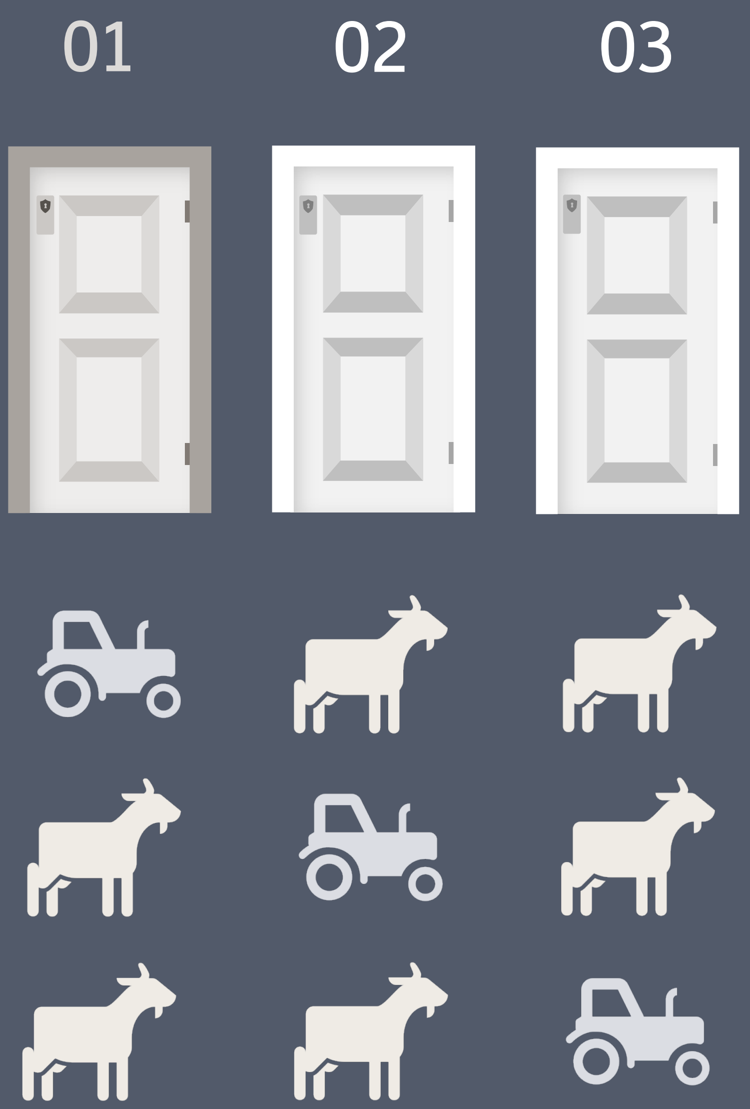


Monty Hall Problem

How many possible arrangements of one car and two goats behind three doors?

Find where the car is:
$\binom{3}{1} = 3.$





What is your probability of winning the car after you pick door No.1?

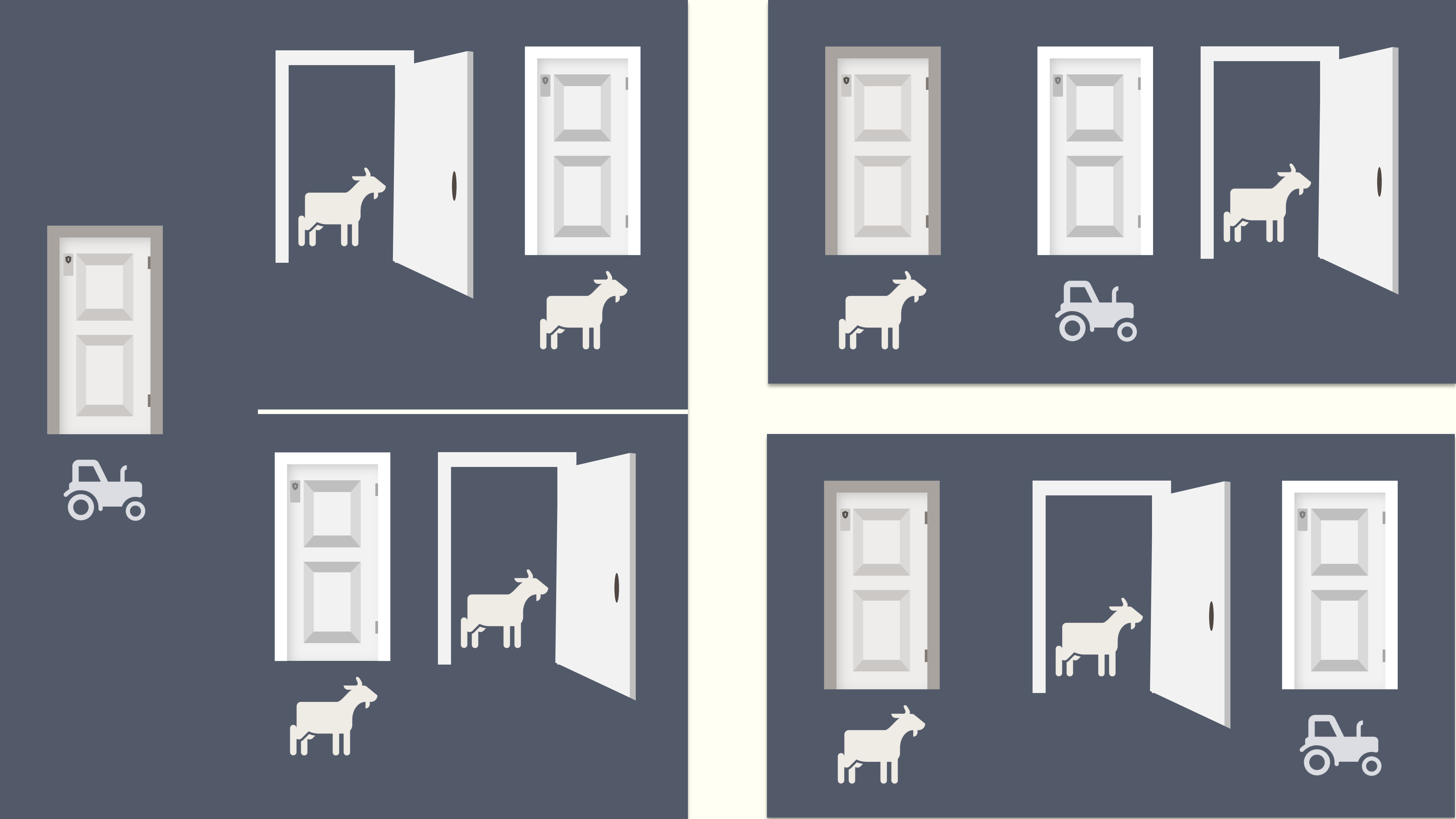
?

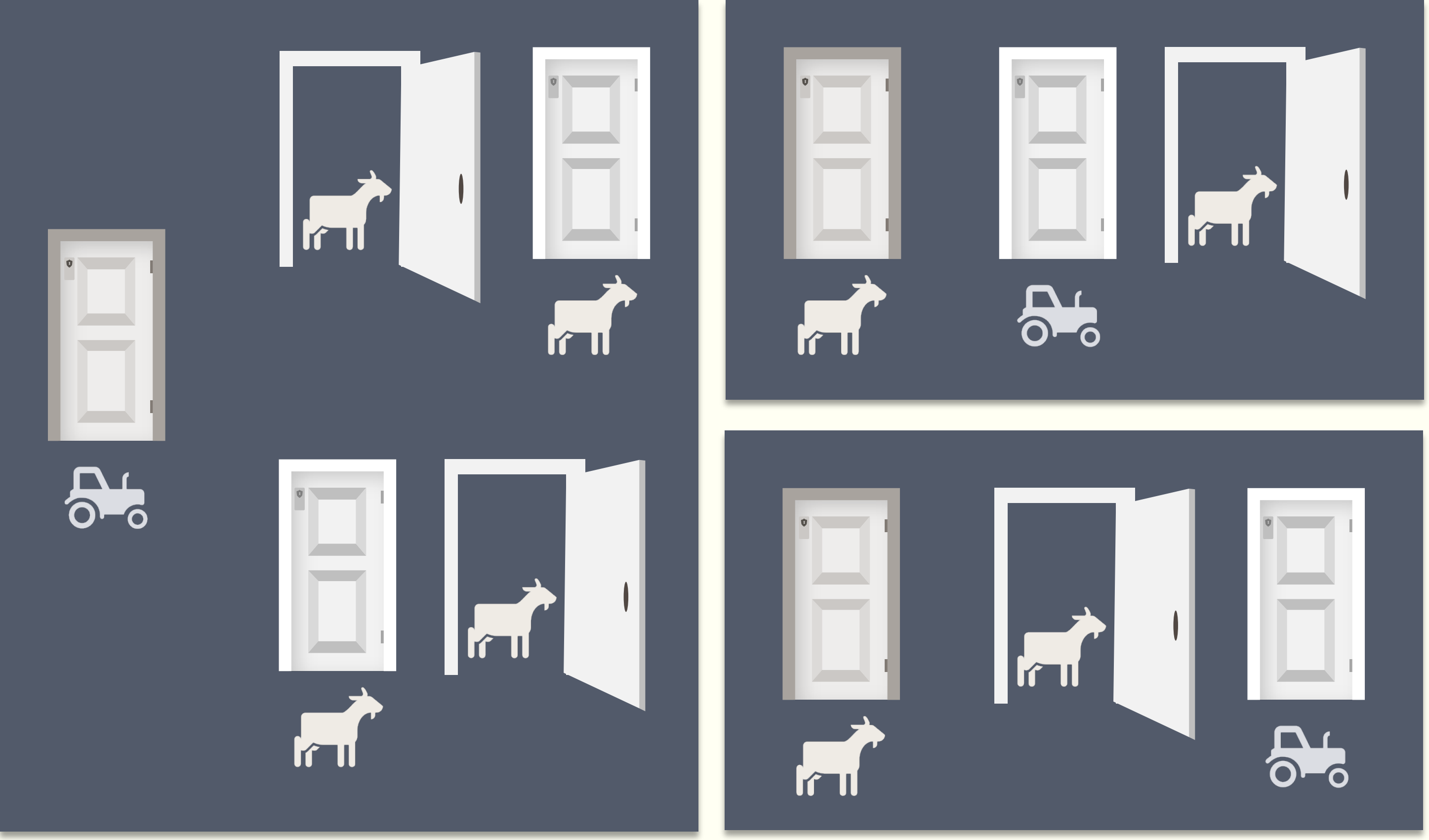
What is your probability of winning if you stick to your initial choice?

?

What is your probability of winning if you switch?

?





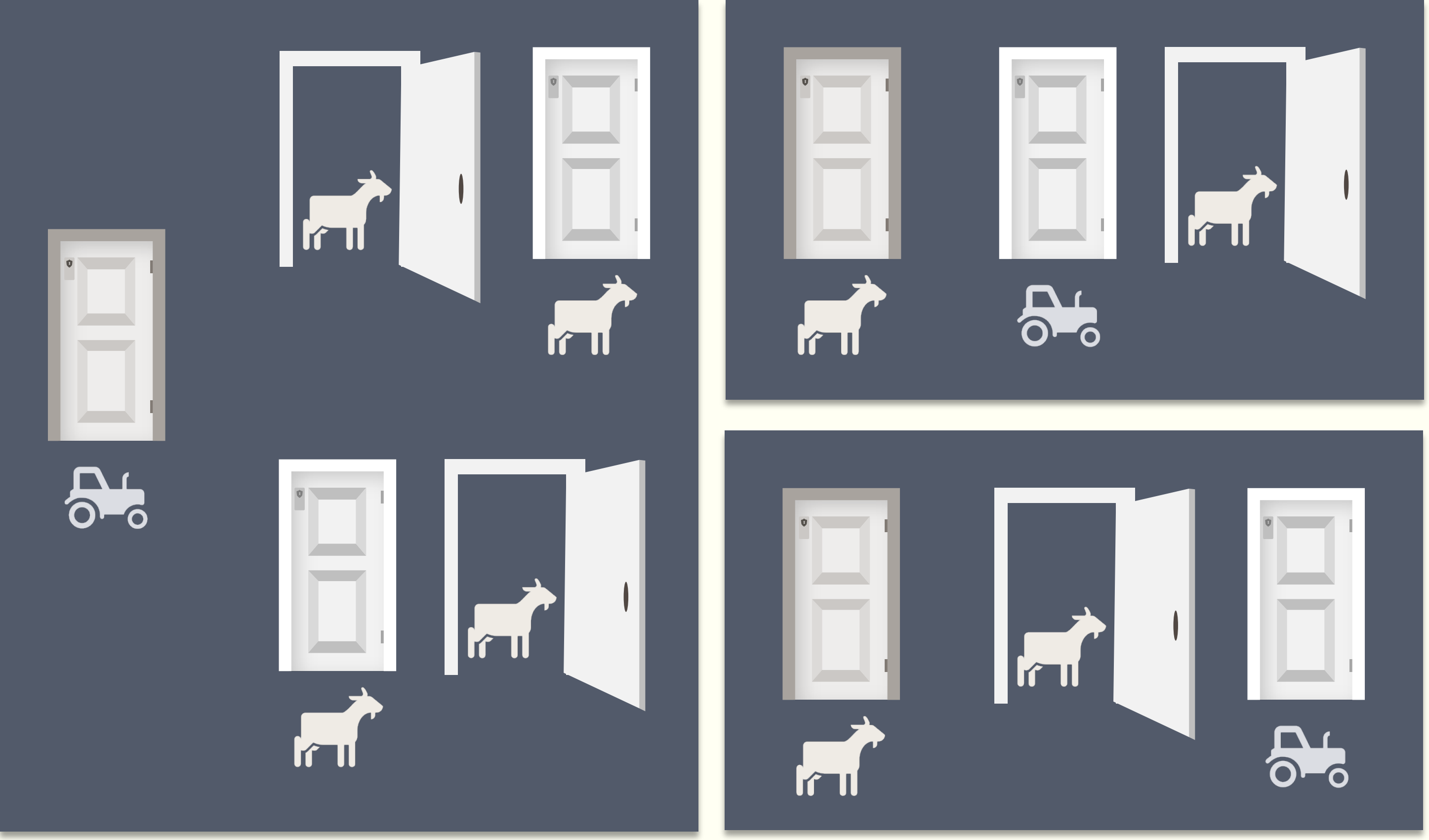
What is your probability of winning if you stick to your initial choice?

?

What is your probability of winning if you switch?

?

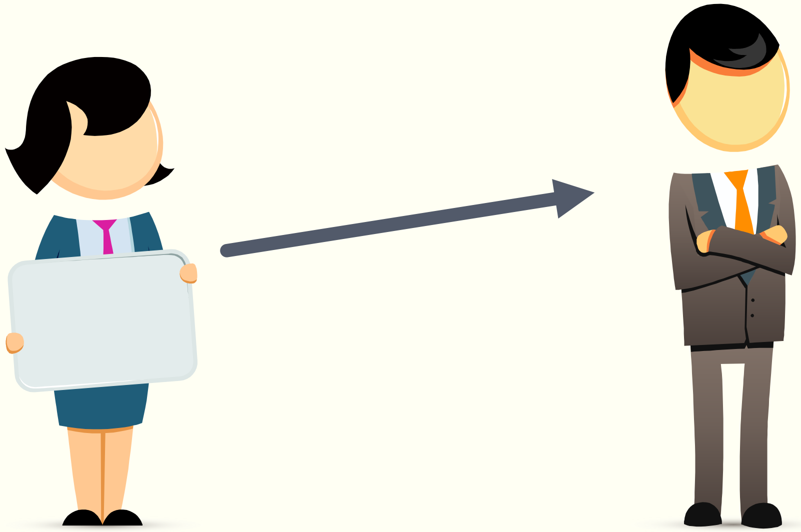
Behind door 1	Behind door 2	Behind door 3	Staying at door 1	Switching
car	goat	goat	win car	win goat
goat	car	goat	win goat	win car
goat	goat	car	win goat	win car



Behind doors	Staying	Switching
car goat goat	win car	win goat
goat car goat	win goat	win car
goat goat car	win goat	win car
probability of winning	$\frac{1}{3}$	$\frac{2}{3}$

there is more **information** about doors 2 and 3 than was available at the beginning of the game!

!



Colorful Balls in a Jar

Experiment

Blindly pick up a ball.

Outcome X

- $P(X = \text{blue ball}) = \frac{4}{12} = \frac{1}{3}$
- $P(X = \text{green ball}) = \frac{2}{12} = \frac{1}{6}$
- $P(X = \text{brown ball}) = \frac{6}{12} = \frac{1}{2}$

Colorful Balls in a Jar

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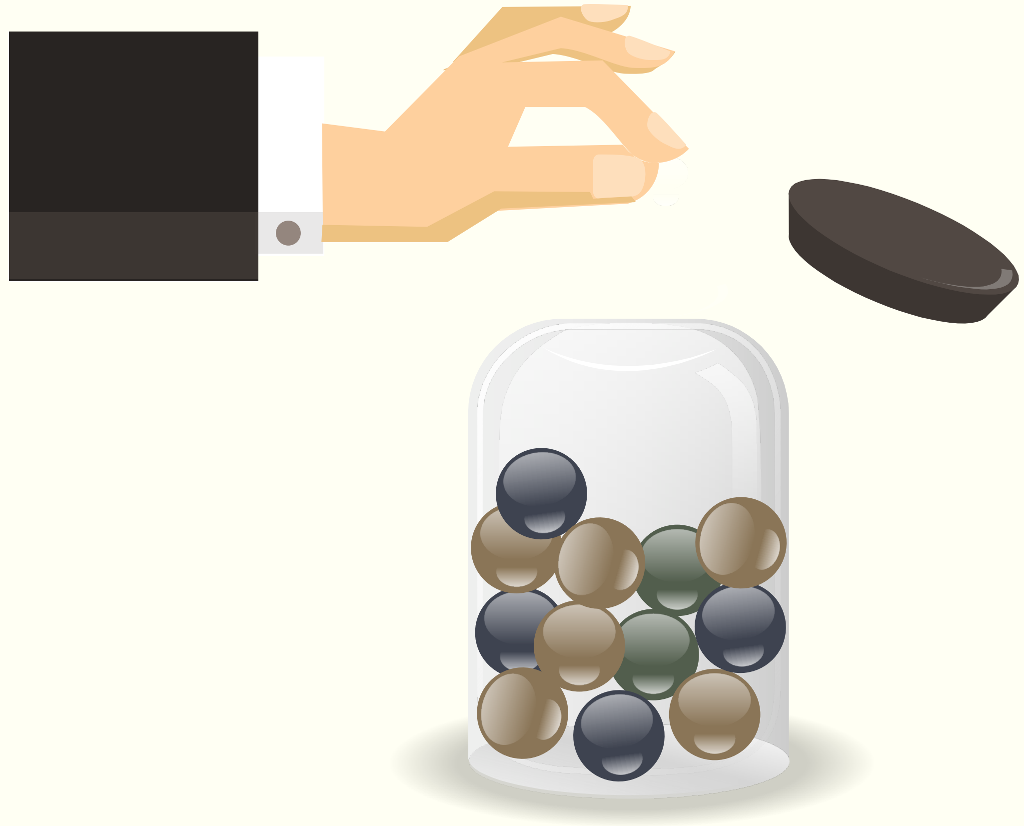
Event E: the ball is blue or green

$$P(E) = \frac{1}{2}$$

Event F: the ball is brown or green

$$P(E) = \frac{2}{3}$$



Event E: the ball is blue or green

$$P(E) = \frac{1}{2}$$

Event F: the ball is brown or green

$$P(E) = \frac{2}{3}$$

The ball is picked up and we are told that Event E has occurred.

Outcome X

- $P(X = \text{blue ball}) = \frac{4}{12} = \frac{1}{3}$
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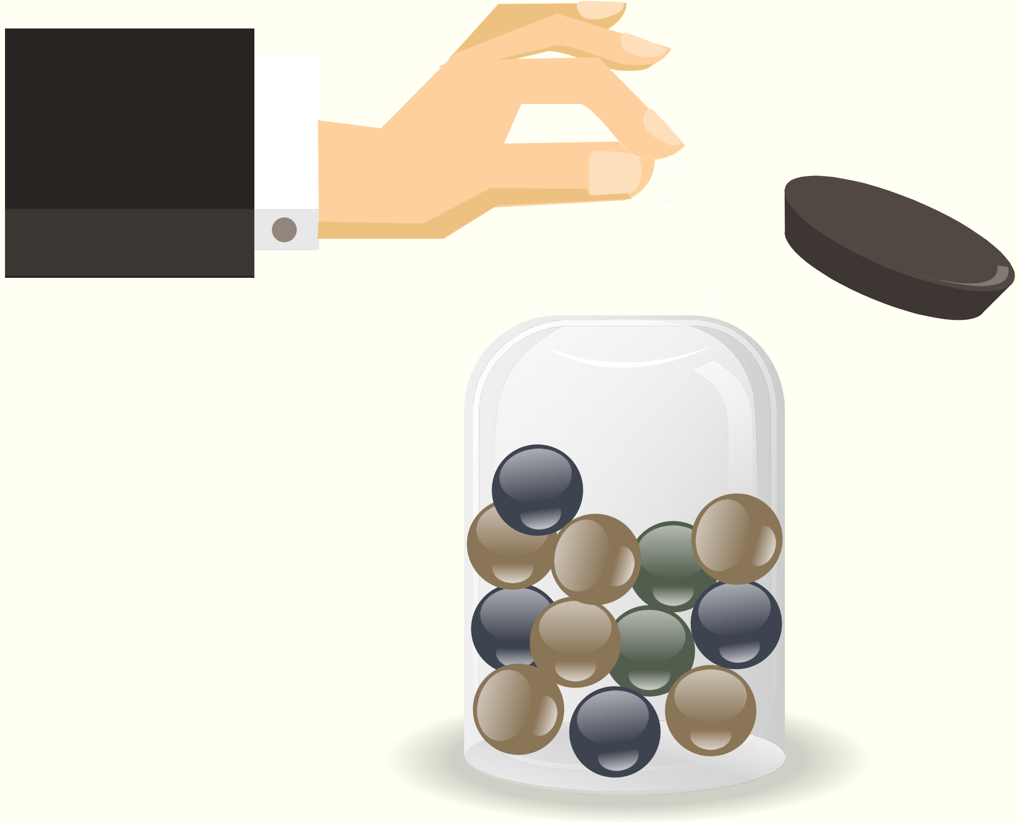
- $P(X = \text{brown ball}) = \frac{6}{12} = \frac{1}{2}$

Event F given E: the ball is brown or green known that the ball is blue or green

$$P(E|F) = \cdots$$



Outcome X

- $P(X = \text{blue ball}) = \frac{4}{12} = \frac{1}{3}$
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Event E: the ball is blue or green

$$P(E) = \frac{1}{2}$$

Event F: the ball is brown or green

$$P(E) = \frac{2}{3}$$

The ball is picked up and we are told that Event E has occurred.

!

Event F given E: the ball is brown or green known that the ball is blue or green

$$P(E|F) = \frac{2}{6} = \frac{1}{3}$$

Discrete Conditional Probability

- Suppose we assign a distribution function to a sample space and then learn that an event E has occurred.
- We call the new probability for an event F the **conditional probability** of F **given** E and denote it by $P(F|E)$.

there is more **information** about the event than was available at the beginning of the experiment.

!

How to compute $P(F|E)$?

?

Discrete Conditional Probability

- Let $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be the original sample space with distribution function $m(\omega_j)$ assigned.
- Given that the event E has occurred ($P(E) > 0$), the conditional probability

$$m(\omega_k|E) = cm(\omega_k)$$

for all ω_k in E.

$$\sum_E m(\omega_k|E) = c\sum_E m(\omega_k) = cP(E) = 1 \Longrightarrow c = \frac{1}{P(E)}.$$

- Thus, we define the conditional distribution given E

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}.$$

Discrete Conditional Probability

- Let $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be the original sample space with distribution function $m(\omega_j)$ assigned.
- Given that the event E has occurred $(P(E) > 0)$, the conditional probability

$$m(\omega_k|E) = \cdots$$

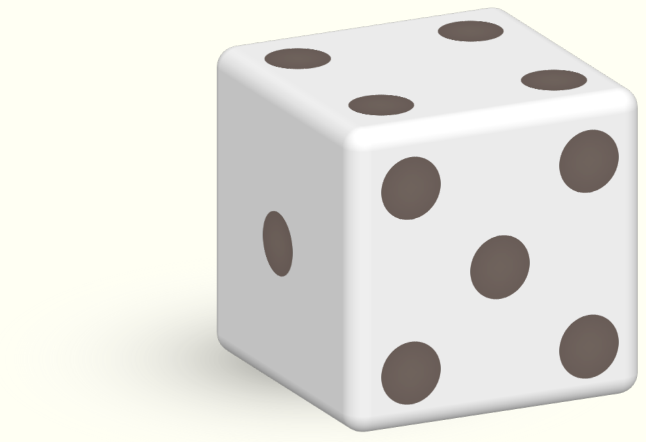
ω_k in E

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

ω_k not in E

$$m(\omega_k|E) = \cdots$$

A dice



Experiment

Rolling a die once.

Sample Space $\boldsymbol{\Omega}$

$\Omega = \{1, 2, 3, 4, 5, 6\}$

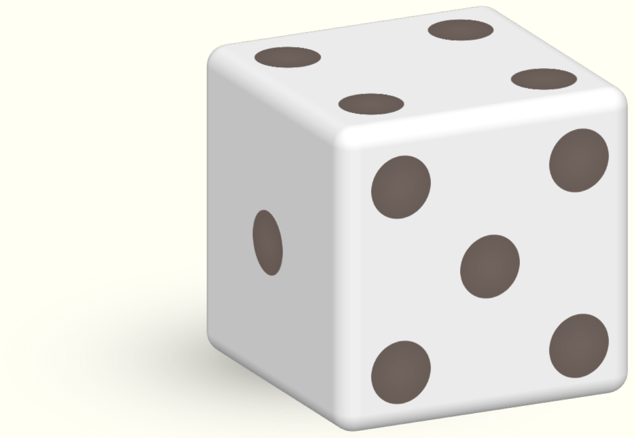
Distribution Function $\boldsymbol{m(\omega_j)}$

$$m(1) = m(2) = m(3) = m(4) = m(5) = m(6) = \frac{1}{6}$$

Event $\boldsymbol{E}$: $\boldsymbol{X > 4}$

$$P(E) = \frac{2}{6} = \frac{1}{3}$$

A dice



Event E: $X > 4$

$$P(E) = \frac{2}{6} = \frac{1}{3}$$

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

?

Sample Space Ω

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

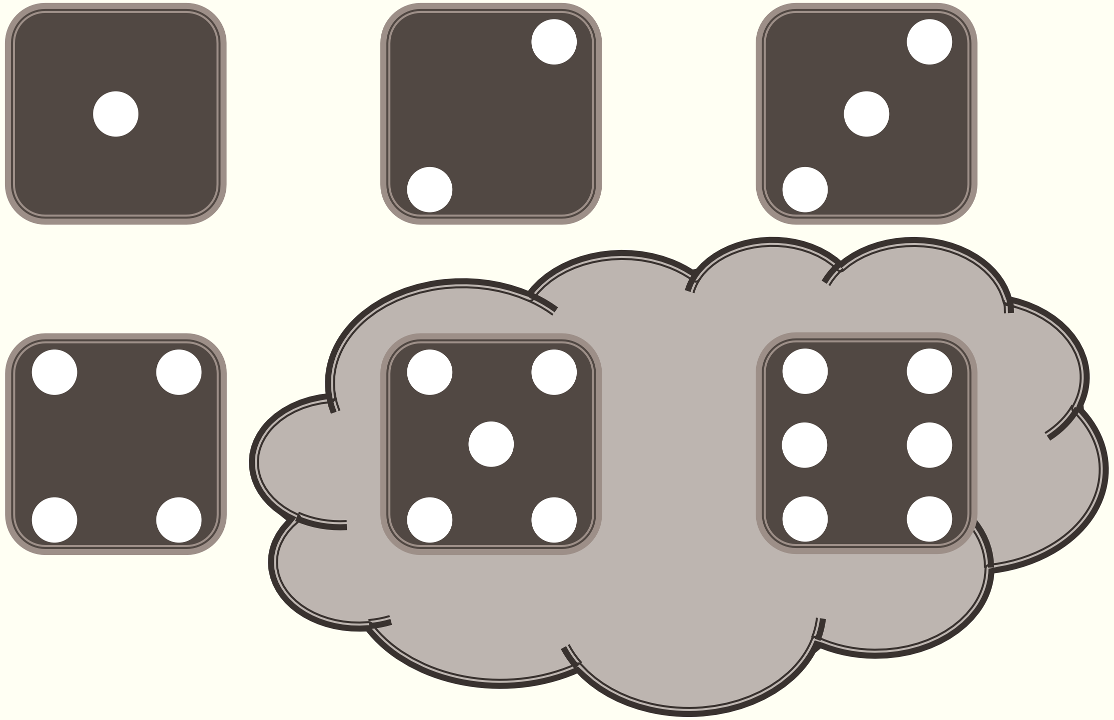
Distribution Function $m(\omega_j)$

$$m(1) = m(2) = m(3) = m(4) = m(5) = m(6) = \frac{1}{6}$$

A dice

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

?



Sample Sub-Space

- $\{1, 2, 3, 4\}$
- $\{5, 6\}$

Distribution Function $m(\omega_j)$

$$m(\omega_j) = \frac{1}{6}$$

Event E: $X > 4$

$$P(E) = \frac{2}{6} = \frac{1}{3}$$

Conditional Probability

- $m(1|E) = m(2|E) = m(3|E) = m(4|E) = 0$
- $m(5|E) = m(6|E) = \frac{1/6}{1/3} = \frac{1}{2}$

Conditional Probability Formula

ω_k in E

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

ω_k not in E

$$m(\omega_k|E) = 0$$

The condition probability of a general event F given that E occurs,

$$P(F|E) = \sum_{F\cap E} m(\omega_k|E) = \sum_{F\cap E} \frac{m(\omega_k)}{P(E)} = \frac{P(F\cap E)}{P(E)}.$$

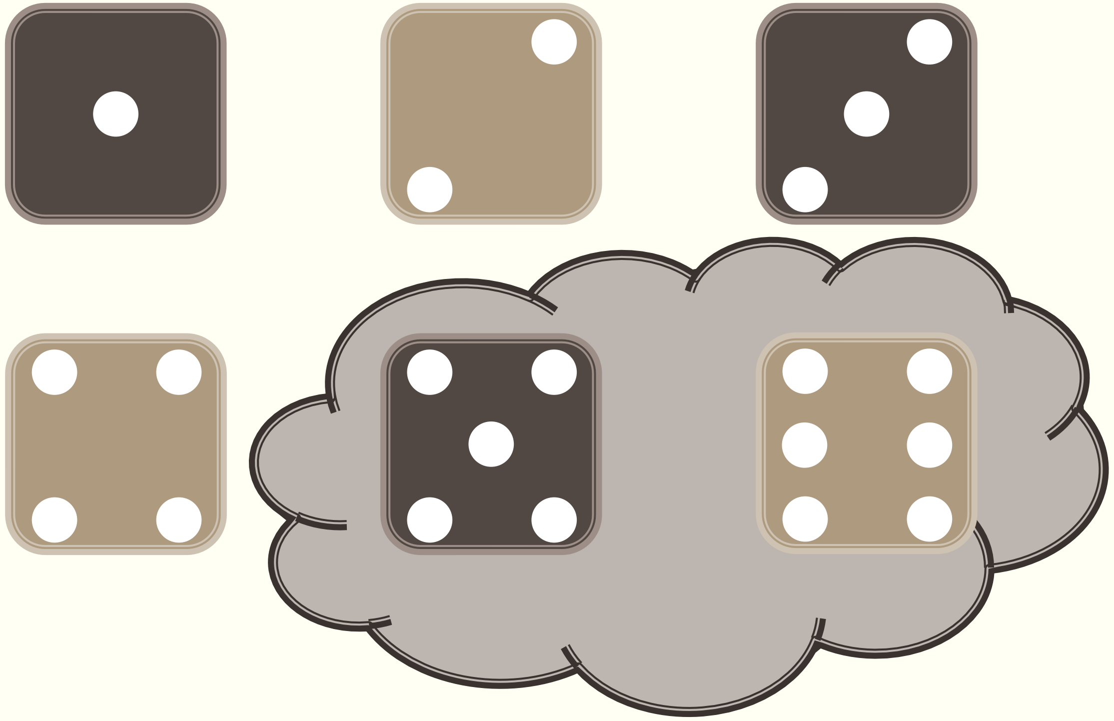
=

$$P(F|E) = \frac{P(F\cap E)}{P(E)}.$$

=

A dice

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$



Event E: $X > 4$

$$P(E) = \frac{2}{6} = \frac{1}{3}$$

Event F: X is even

$$P(F) = \frac{3}{6} = \frac{1}{2}$$

Event $E \cap F$: $X = 6$

$$P(E \cap F) = \frac{1}{6}$$

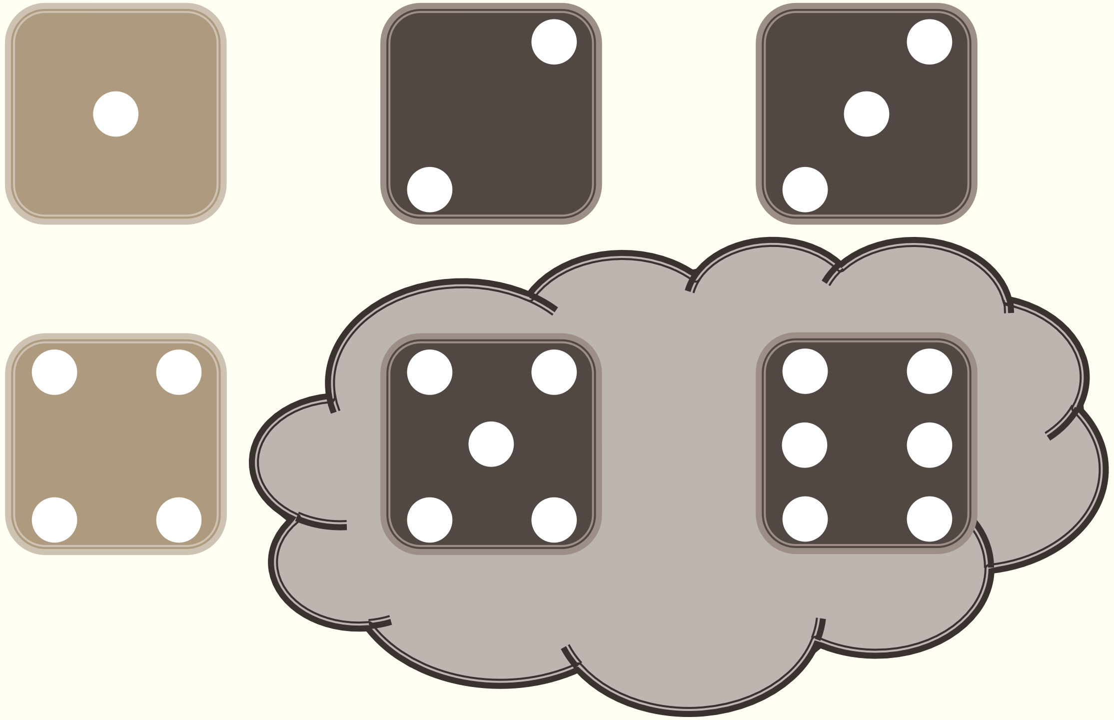
Conditional Probability

$$P(F|E) = \frac{1/6}{1/3} = \frac{1}{2}$$

A dice

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

=



Event $\boldsymbol{E}$: $\boldsymbol{X} > \mathbf{4}$

$$P(E) = \frac{2}{6} = \frac{1}{3}$$

Event $\boldsymbol{F}$: $\boldsymbol{X}$ is a square number

$$P(F) = \frac{2}{6} = \frac{1}{3}$$

Event $\boldsymbol{E} \cap \boldsymbol{F}$: $\emptyset$

$$P(E \cap F) = 0$$

Conditional Probability

$$P(F|E) = \frac{0}{2} = \frac{0}{1/3} = 0$$

Boy or Girl

What is the probability that a family of two children has

- two boys given that it has at least one boy?
- two boys given that the first child is a boy?
- one girl given that it has at least one boy?
- one girl given that the first child is a boy?

Boy or Girl

- two boys given that it has at least one boy?
- two boys given that the first child is a boy?
- one girl given that it has at least one boy?
- one girl given that the first child is a boy?

$$P(F|E) = \frac{P(F\cap E)}{P(E)}.$$

=

F	E
two boys	at least one boy
one girl	the first child is a boy

Boy or Girl

Experiment

A family of two children.

Sample Space $\boldsymbol{\Omega}$

$$\Omega = \{BB, BG, GB, GG\}$$

Distribution Function $\boldsymbol{m(\omega_j)}$

$$m(BB) = m(BG) = m(GB) = m(GG) = \frac{1}{4}$$

Event: at least one boy

$P = \cdots$

Event: the first child is a boy

$P = \cdots$

Event: two boys

$P = \cdots$

Event: one girl

$P = \cdots$

Boy or Girl

What is the probability that a family of two children has
- two boys given that it has at least one boy?

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

Event $\boldsymbol{E}$: at least one boy

$$P(E) = 1 - \frac{1}{4} = \frac{3}{4}$$

Event $\boldsymbol{F}$: two boys

$$P(F) = \frac{1}{4}$$

Event $\boldsymbol{E} \cap \boldsymbol{F}$: two boys

$$P(E \cap F) = \frac{1}{4}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Boy or Girl

What is the probability that a family of two children has

- two boys given that the first child is a boy?

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

=

Event $\boldsymbol{E}$: the first is a boy

$$P(E) = \frac{1}{2}$$

Event $\boldsymbol{F}$: two boys

$$P(F) = \frac{1}{4}$$

Event $\boldsymbol{E \cap F}$: two boys

$$P(E \cap F) = \frac{1}{4}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Boy or Girl

What is the probability that a family of two children has

- one girl given that it has at least one boy?

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

=

Event $\boldsymbol{E}$: at least one boy

$$P(E) = 1 - \frac{1}{4} = \frac{3}{4}$$

Event $\boldsymbol{F}$: one girl

$$P(F) = \frac{1}{2}$$

Event $\boldsymbol{E} \cap \boldsymbol{F}$: one boy and one girl

$$P(E \cap F) = \frac{1}{2}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/2}{3/4} = \frac{2}{3}.$$

Boy or Girl

What is the probability that a family of two children has

- one girl given that the first child is a boy?

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

Event $\boldsymbol{E}$: the first is a boy

$$P(E) = \frac{1}{2}$$

Event $\boldsymbol{F}$: one girl

$$P(F) = \frac{1}{2}$$

Event $\boldsymbol{E \cap F}$: the first is a boy and the second is a girl

$$P(E \cap F) = \frac{1}{4}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

What is the probability that a family of two children has

- two boys given that it has at least one boy?

$$P(F) = \frac{1}{4}, P(F|E) = \frac{1}{3}.$$

- two boys given that the first child is a boy?

$$P(F) = \frac{1}{4}, P(F|E) = \frac{1}{2}.$$

- one girl given that it has at least one boy?

$$P(F) = \frac{1}{2}, P(F|E) = \frac{2}{3}.$$

- one girl given that the first child is a boy?

$$P(F) = \frac{1}{2}, P(F|E) = \frac{1}{2}.$$

Independent Events

Tow events E and F are independent if both E and F have positive probability and if
$P(E|F) = P(E)$, and $P(F|E) = P(F)$.

=

What is the probability that a family of two children has

- one girl given that the first child is a boy?

$$P(F) = \frac{1}{2}, P(F|E) = \frac{1}{2}.$$

Boy or Girl

A Philosophical Essay on Probabilities

I have seen men, ardently desirous of having a son, who could learn only with anxiety of the births of boys in the month when they expected to become fathers.
Imagining that the ratio of these births to those of girls ought to be the same at the end of each month, they judged that **the boys already born would render more probable the births next of girls**.

Pierre-Simon Laplace

Black or Red

Monte Carlo Casino

in a game of roulette at the Monte Carlo Casino on August 18, 1913, the ball fell in black 26 times in a row.
This was an extremely uncommon occurrence: the probability of a sequence of either red or black occurring 26 times in a row is around 1 in 66.6 million, assuming the mechanism is unbiased.
Gamblers lost millions of francs betting against black, reasoning incorrectly that the streak was causing an imbalance in the randomness of the wheel, and that it had to be followed by a long streak of red.

Gambler's Fallacy

The **gambler's fallacy**, also known as the **Monte Carlo fallacy** or the fallacy of the maturity of chances, is the erroneous belief that **if a particular event occurs more frequently than normal during the past it is less likely to happen in the future (or vice versa),** when it has otherwise been established that the probability of such events does not depend on what has happened in the past.
Such events, having the quality of historical independence, are referred to as **statistically independent**. The fallacy is commonly associated with gambling, where it may be believed, for example, that the next dice roll is more than usually likely to be six because there have recently been less than the usual number of sixes.

Tow events E and F are independent if both E and F have positive probability and if
$P(E|F) = P(E)$, and $P(F|E) = P(F)$.

=

Independent Events

Tow events E and F are independent if both E and F have positive probability and if $P(E|F) = P(E)$, and $P(F|E) = P(F)$.

=

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

Theorem

If $P(E) > 0$ and $P(F) > 0$, the E and F are independent if and only if

$$P(E \cap F) = P(E)P(F).$$

T

Theorem

If $P(E) > 0$ and $P(F) > 0$, the E and F are independent if and only if
$$P(E \cap F) = P(E)P(F).$$

Extension to a finite set of n events

A set of events $\{A_1, A_2, \cdots, A_n\}$ is said to be mutually independent if for any subset $\{A_i, A_j, \cdots, A_m\}$ of these events we have
$$P\left(A_i \cap A_j \cap \cdots A_m\right) = P(A_i)P(A_j) \cdots P(A_m).$$
Or equivalently, if for any sequence $\bar{A}_1, \bar{A}_2, \cdots, \bar{A}_n$, with $\bar{A}_i = A_i$ or $\bar{A}_i = \tilde{A}_i$,
$$P(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \bar{A}_n) = P(\bar{A}_1)P(\bar{A}_2) \cdots P(\bar{A}_n).$$

Weather Forecast

Event-1: rain

$$P(\text{rain}) = 0.6$$

Event-2: windy & cloudy

$$P(\text{windy \& cloudy}) = 0.48$$

Event-3: other people carrying umbrellas

$$P(\text{umbrella}) = 0.56$$

Event-4: the dentist is open today

$$P(\text{dentist}) = \frac{5}{7}$$

Weather Forecast

Event-1: rain

$$P(\text{rain}) = 0.6$$

Event-2: windy & cloudy

$$P(\text{windy \& cloudy}) = 0.48$$
$$P(\text{rain} \mid \text{windy \& cloudy}) = 0.85$$

Event-3: other people carrying umbrellas

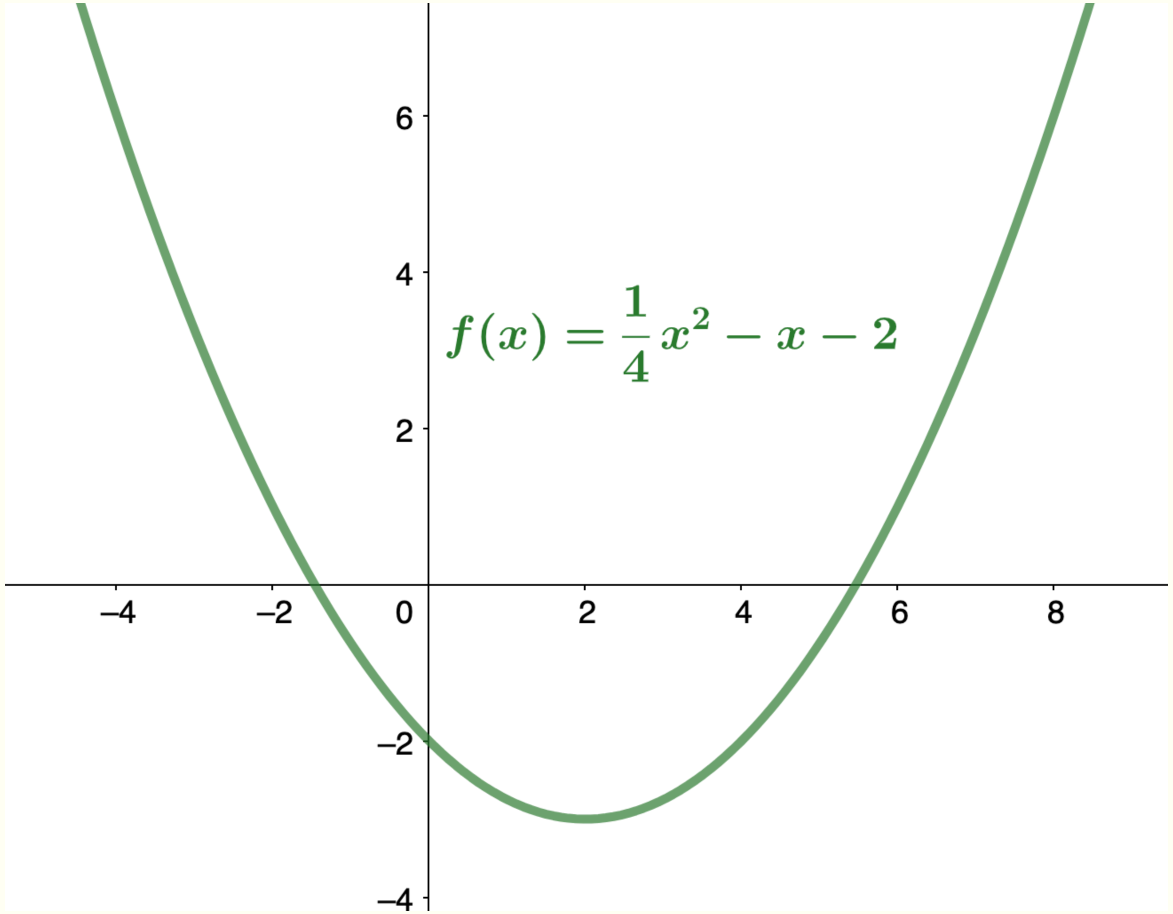
$$P(\text{umbrellas}) = 0.56$$
$$P(\text{rain} \mid \text{umbrellas}) = 0.9$$

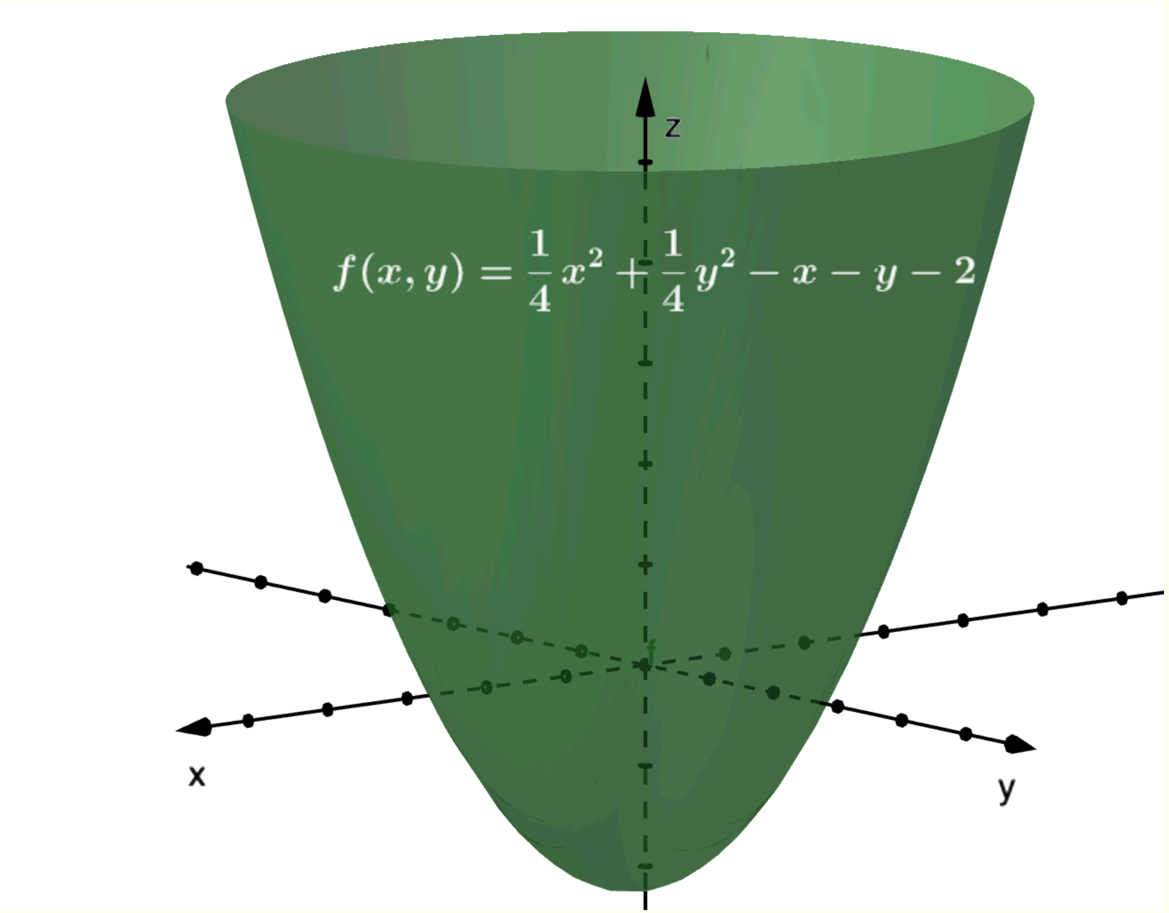
Event-4: the dentist is open today

$$P(\text{dentist}) = \frac{5}{7}$$
$$P(\text{rain} \mid \text{dentist}) = 0.6$$

JOINT DISTRIBUTION FUNCTIONS

joint random variable


$f(x) = \frac{1}{4}x^2 - x - 2$
6
4
2
0
–2
–4
–4
–2
2
4
6
8


$f(x, y) = \frac{1}{4}x^2 + \frac{1}{4}y^2 - x - y - 2$
z
x
y

Joint Distribution Functions

- Let $X_1, X_2, \cdots, X_n$ be random variables associated with an experiment. Suppose that the sample space (the set of possible outcomes) of X_i is the set R_i.
- The joint random variable $X = (X_1, X_2, \cdots, X_n)$ is defined to be the random variable whose outcomes consist of ordered n-tuples of outcomes, with the ith coordinate lying in the set R_i. The sample space Ω of X is the Cartesian product of the R_i's:

$$\Omega = R_1 \times R_2 \times \cdots \times R_n.$$

Joint Distribution Functions

- The joint distribution function of X gives the probability of each of the outcomes. The distributions of the individual random variables are called **marginal distributions**.
- The random variables $X_1, X_2, \cdots, X_n$ are **mutually independent** if

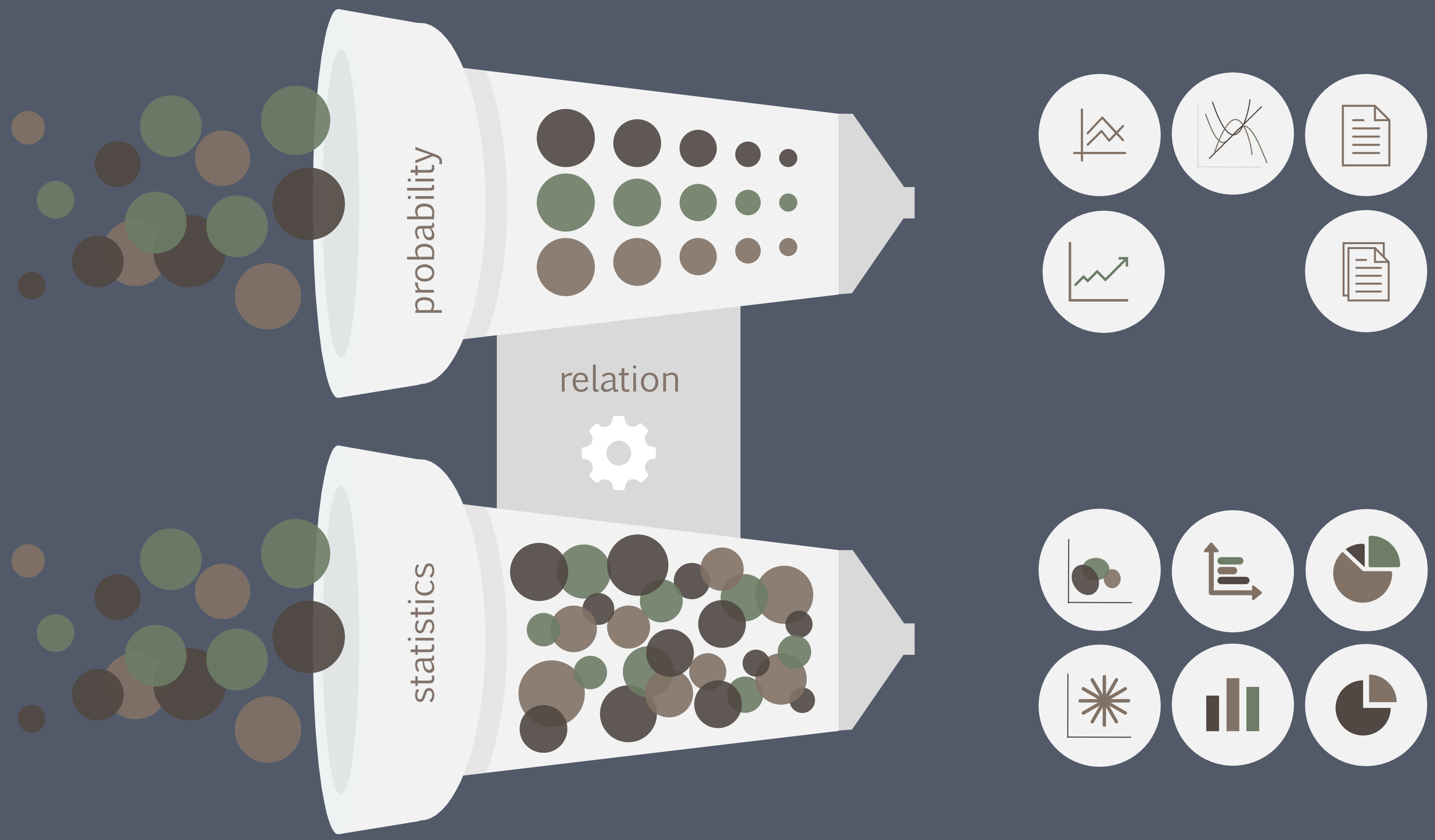
$$P(X_1 = r_1, X_2 = r_2, \cdots, X_n = r_n) = P(X_1 = r_1)P(X_2 = r_2)\cdots P(X_n = r_n),$$

for any choice of $r_1, r_2, \cdots, r_n$.

Probability and Statistics

P

S



In a group of 60 people, the numbers who do or do not smoke and do or do not have cancer are reported as shown below.

Cancer

	Not Smoke	Smoke	Total
Not Cancer	40	10	50
Cancer	7	3	10
Total	47	13	60

Smoke

Let Ω be the sample space consisting of these 60 people. A person is chosen at random from the group. Let $C(\omega) = 1$ if this person has cancer and 0 if not, and $S(\omega) = 1$ if this person smokes and 0 if not.

the **marginal distributions** of C and S

Cancer

Cancer

- Random Variable: C
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(C = 0) = \frac{50}{60} = \frac{5}{6}$$

$$P(C = 1) = \frac{10}{60} = \frac{1}{6}$$

Smoke

- Random Variable: S
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(S = 0) = \frac{47}{60}$$

$$P(S = 1) = \frac{13}{60}$$

Smoke

Cancer and Smoke

- Random Variable: $X = (C, S)$
- Sample Space: $\Omega = \{0,1\}\times\{0,1\}$
- Joint Distribution Function:

the **joint distribution** of $\{C, S\}$

Cancer

		S	
		0	1
C	0	40/60	10/60
	1	7/60	3/60

Smoke

Cancer

- Random Variable: C
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(C=0) = \frac{50}{60} = \frac{5}{6}$$
$$P(C=1) = \frac{10}{60} = \frac{1}{6}$$

Smoke

- Random Variable: S
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(S=0) = \frac{47}{60}$$
$$P(S=1) = \frac{13}{60}$$

the **marginal distributions** of C and S

Cancer

		S	
		0	1
C	0	40/60	10/60
	1	7/60	3/60

the **joint distribution** of $\{C, S\}$

Smoke

Cancer

- Random Variable: C
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(C=0)=\frac{50}{60}=\frac{5}{6}$$
$$P(C=1)=\frac{10}{60}=\frac{1}{6}$$

Smoke

- Random Variable: S
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(S=0)=\frac{47}{60}$$
$$P(S=1)=\frac{13}{60}$$

the marginal distributions of C and S

		S = 0	S = 1
C	0	40/60	10/60
	1	7/60	3/60

the joint distribution of $\{C, S\}$

Tow events E and F are independent if both E and F have positive probability and if $P(E|F) = P(E)$, and $P(F|E) = P(F)$.

=

Are C and S independent?

?

$$P(C=1|S=1)=\cdots$$

$$P(S=1|C=1)=\cdots$$

Cancer

- Random Variable: C
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(C = 0) = \frac{50}{60} = \frac{5}{6}$$

$$P(C = 1) = \frac{10}{60} = \frac{1}{6}$$

Smoke

- Random Variable: S
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(S = 0) = \frac{47}{60}$$

$$P(S = 1) = \frac{13}{60}$$

The marginal distributions of C and S

		S	
		0	1
C	0	40/60	10/60
	1	7/60	3/60

the joint distribution of $\{C, S\}$

Tow events E and F are independent if both E and F have positive probability and if $P(E|F) = P(E)$, and $P(F|E) = P(F)$.

=

Are C and S independent?

?

$$P(C = 1|S = 1) = \frac{3}{13}$$

$$P(S = 1|C = 1) = \frac{3}{10}$$

Cancer

- Random Variable: C
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(C=0) = \frac{50}{60} = \frac{5}{6}$$
$$P(C=1) = \frac{10}{60} = \frac{1}{6}$$

Smoke

- Random Variable: S
- Sample Space: $\{0, 1\}$
- Distribution Function:

$$P(S=0) = \frac{47}{60}$$
$$P(S=1) = \frac{13}{60}$$

The marginal distributions of C and S

		S = 0	S = 1
C	0	40/60	10/60
	1	7/60	3/60

the joint distribution of $\{C, S\}$

If $P(E) > 0$ and $P(F) > 0$, the E and F are independent if and only if $P(E \cap F) = P(E)P(F)$.

=

Are C and S independent?

=

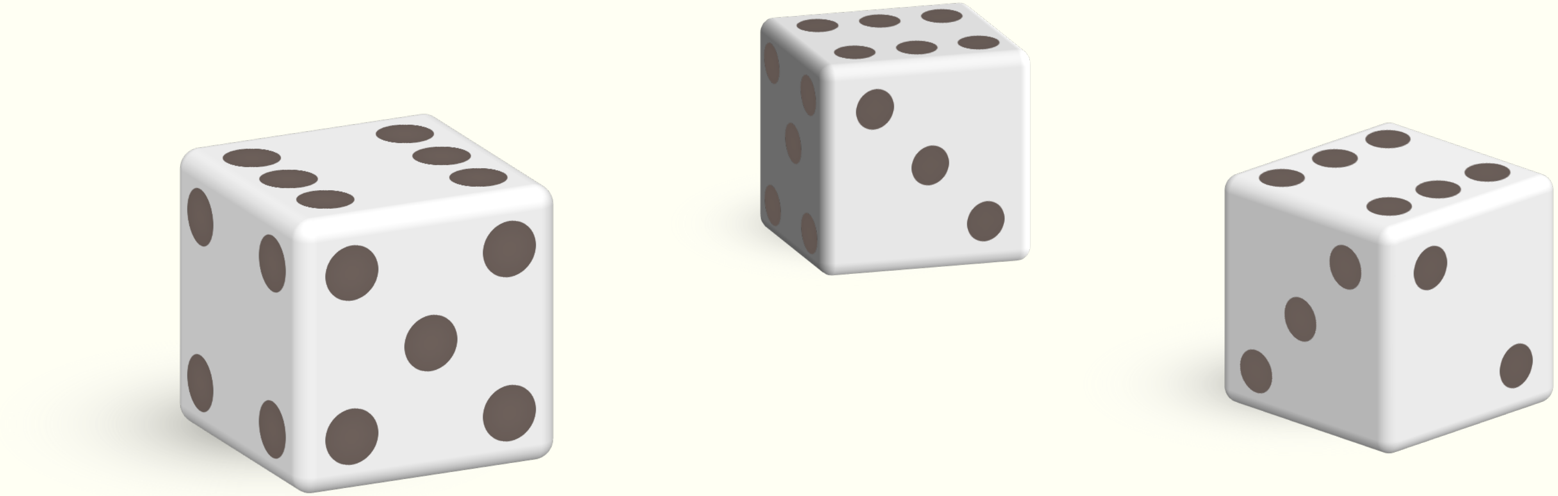
$$P(C=1, S=1) = \frac{3}{60}$$

$$P(C=1)P(S=1) = \frac{1}{6}\frac{13}{60} = \frac{13}{360}$$

INDEPENDENT TRIALS PROCESS

mutually independent & same distribution

Independent Trials Process



A sequence of random variables $X_1, X_2, \cdots, X_n$ that are **mutually independent** and that have **the same distribution** is called a sequence of independent trials or independent trials process.

Independent Trials Process

- A sequence of random variables $X_1, X_2, \cdots, X_n$ that are **mutually independent** and that have **the same distribution** is called a sequence of independent trials or independent trials process.
- Suppose we repeat n times for a simple experiment that has a sample space $R = \{r_1, r_2, \cdots, r_s\}$, and a distribution function

$$m_X = \begin{pmatrix} r_1 & r_2 & \cdots & r_s \\ p_1 & p_2 & \cdots & p_s \end{pmatrix}.$$

$$R = \{1, 2, 3, 4, 5, 6\}$$

$$m_X = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

Independent Trials Process

- We can describe the entire experiment using the sample space $\Omega = R \times R \times \cdots \times R$, consisting of all possible sequences $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ where ω_j is chosen from R.

- The joint distribution is assigned to be the product distribution

$$m(\omega) = m(\omega_1)m(\omega_2)\cdots m(\omega_n),$$

with $m(\omega_j) = p_k$ when $\omega_j = r_k$.

- Let X_j denote the jth coordinate of the outcome ω, the random variable $X_1, X_2, \cdots, X_n$ form an independent trials process.