MATH 20: PROBABILITY

Expected Value of Discrete Random Variables

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Important Distributions

Discrete Uniform Distribution

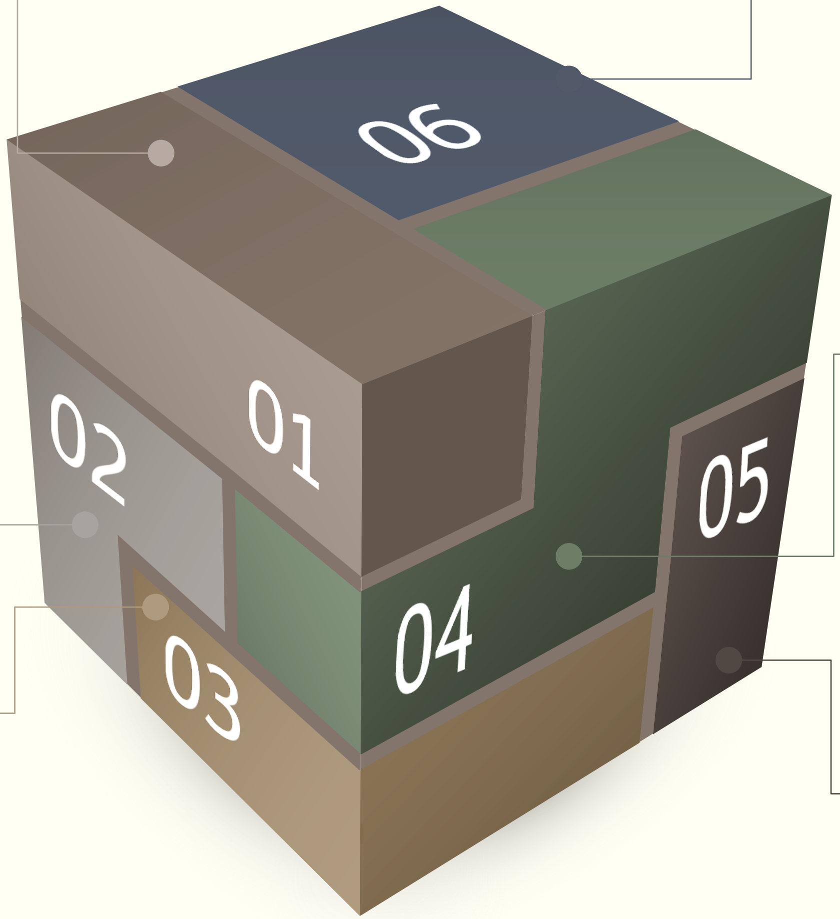
$$m(\omega) = \frac{1}{n}$$

Binomial Distribution

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

Geometric Distribution

$$P(T = n) = q^{n-1} p$$



Hypergeometric Distribution

$$h(N, k, n, x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

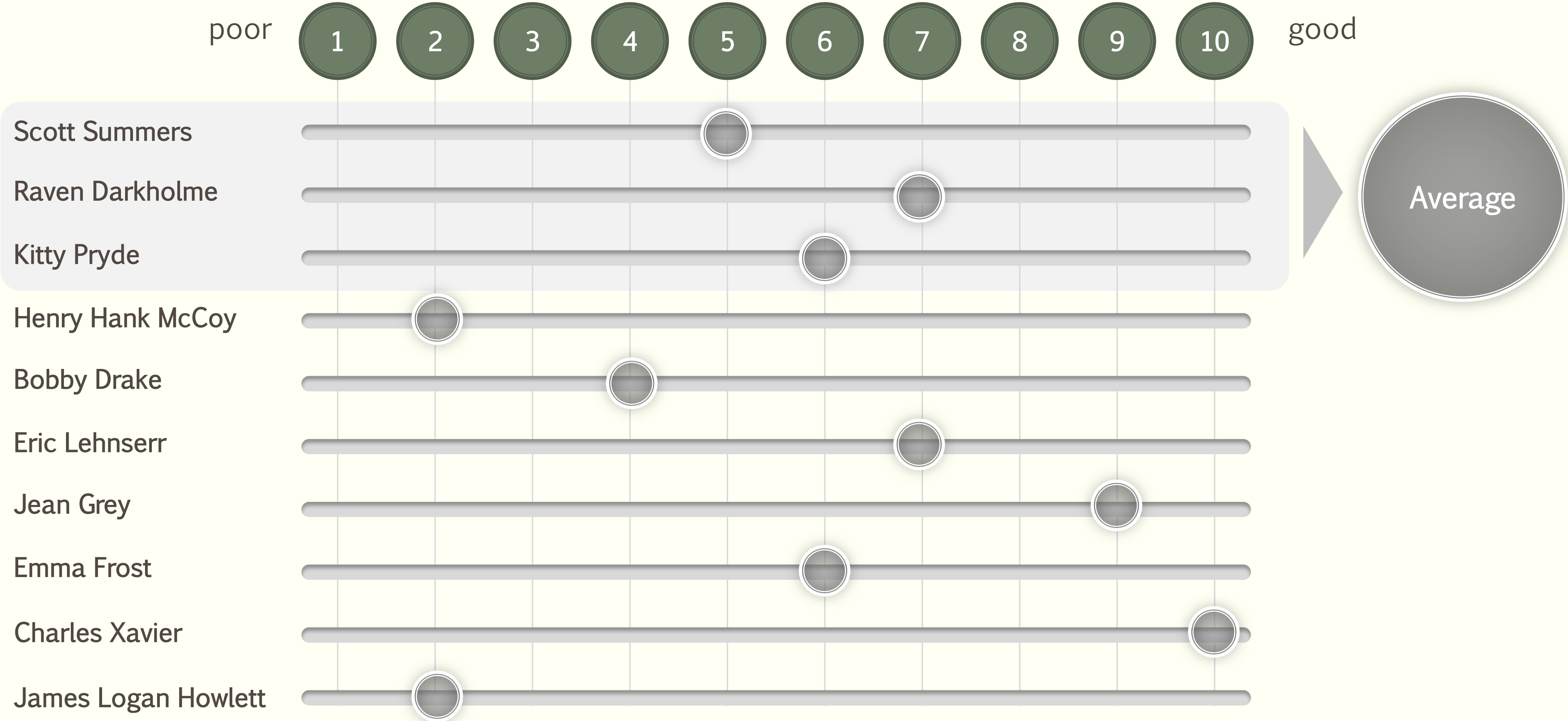
Negative Binomial Distribution

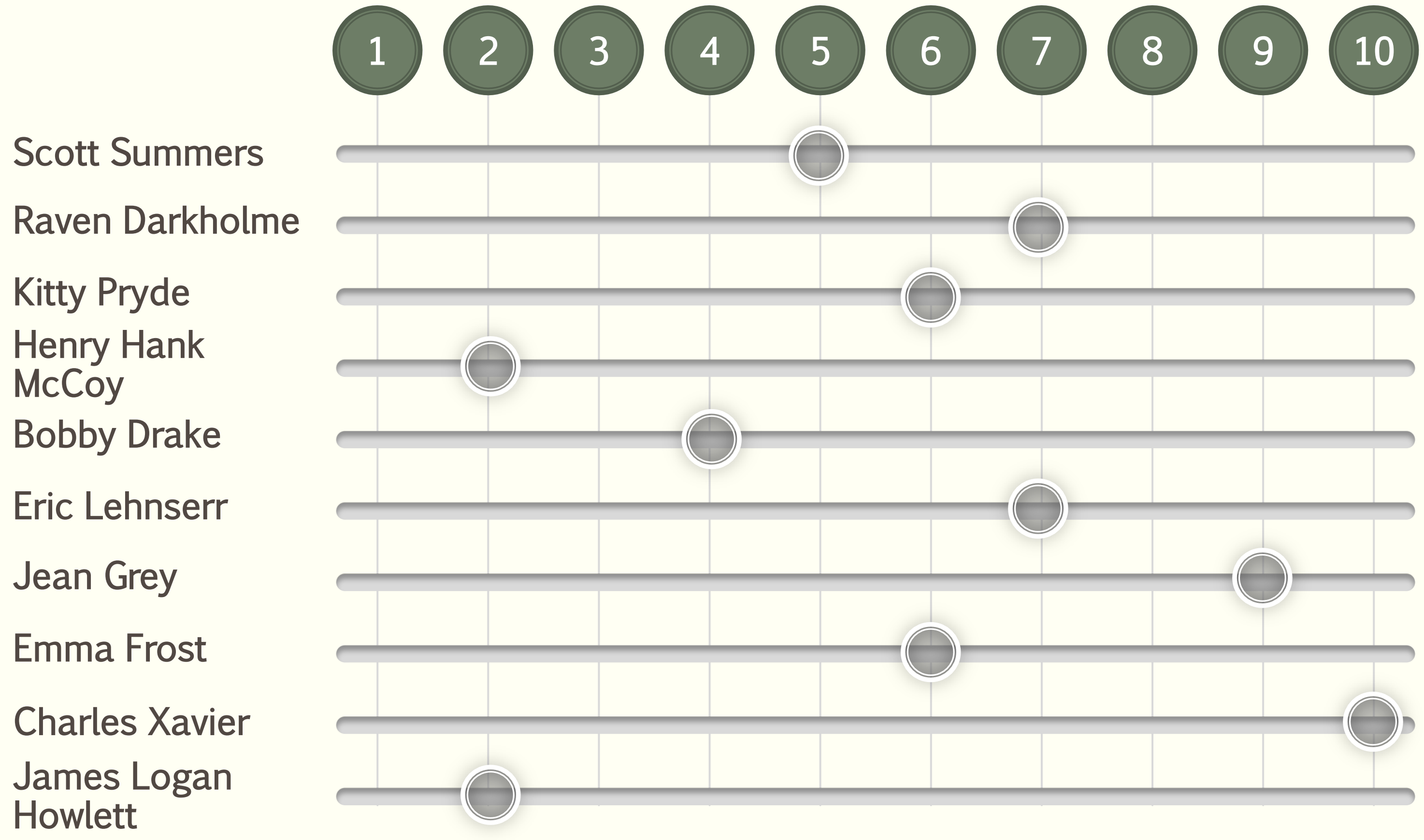
$$u(x, k, p) \\ = \binom{x-1}{k-1} p^k q^{x-k}$$

Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Average Score of a Pre-employment Psychometric Test





Average μ

$$\frac{5+7+6+2+4+7+9+6+10+2}{10} = 5.8$$

$$\frac{2\times 2+4+5+2\times 6+2\times 7+9+10}{10} = 5.8$$

$$\frac{1}{5}\times 2+\frac{1}{10}\times 4+\frac{1}{5}\times 6+\frac{1}{5}\times 7+\frac{1}{10}\times 9+\frac{1}{10}\times 10 = 5.8$$

Average μ

$$\sum_{x\in S} \text{frequency}\times\text{value}$$

Expected Value

- Let X be a **numerically-valued** discrete random variable with sample space Ω and distribution function $m(x)$. The expected value $E(X)$ is defined by

$$E(X) = \sum_{x \in \Omega} x m(x),$$

provided this sum **converges absolutely**.

- The expected value $E(X)$ is often referred to as the mean, and can be denoted by μ for short.
- If the above sum does not converge absolutely, then we say X does not have an expected value.

$$|\sum_{x \in \Omega} x m(x)| < \infty$$

!

Law of Large Numbers

The Law of Large Numbers can help us justify frequency concept of probability and the interpretation of expected value as the average value to be expected in experiments repeated a large number of times.

Expected value $E(X)$

$$\sum_{x\in\Omega} xm(x)$$

Average μ

$$\sum_{x\in S} \text{frequency}\times\text{value}$$

Example 1

Toss a coin
head or tail
$m(x) = \frac{1}{2}$
$E(X) = \cdots$

Example 2

Roll a dice
1, 2, 3, 4, 5, or 6
$m(x) = \frac{1}{6}$
$E(X) = \cdots$

Example 3

Suppose that we flip a coin until a head first appears, and if the number of tosses equals n, then we are paid n dollars. What is the expected value of the payment?

Expected value $E(X)$

$$\sum_{x\in\Omega} x m(x)$$

$$\sum_{n=1}^{+\infty} n q^{n-1} p = \sum_{n=1}^{+\infty} n \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right) = \sum_{n=1}^{+\infty} n \left(\frac{1}{2}\right)^{n} = 2$$

$$(1/2 + 1/4 + 1/8 + \cdots) + (1/4 + 1/8 + 1/16 + \cdots) + (1/8 + 1/16 + \cdots) + \cdots$$
$$= 1 + 1/2 + 1/4 + 1/8 + \cdots = 2$$

Example 3 (continued)

Suppose that we flip a coin until a head first appears, and if the number of tosses equals n, then we are paid 2^n dollars. What is the expected value of the payment?

Expected value $E(X)$

$$\sum_{x\in\Omega} x m(x)$$

$$\sum_{n=1}^{+\infty} 2^n q^{n-1} p = \sum_{n=1}^{+\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{i=1}^{+\infty} 1 = +\infty$$

$$\left|\sum_{x\in\Omega} x m(x)\right| < \infty$$

!

Example 4

Consider the general Bernoulli trial process. As usual, we let $X = 1$ if the outcome is a success and 0 if it is a failure.

Bernoulli trial
$m(x) = \begin{cases} p, & X = 1 \\ 1 - p, & X = 0 \end{cases}$

Expected value $\boldsymbol{E(X)}$
$\sum_{x \in \Omega} x m(x)$
$1 \times p + 0 \times (1 - p) = p$

Binomial

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k} \qquad E(X) = np$$

Geometric

$$P(T = n) = q^{n-1} p \qquad E(X) = \frac{1}{p}$$

Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad E(X) = \lambda$$

Negative binomial

$$u(x,k,p) = \binom{x-1}{k-1} p^k q^{x-k} \qquad E(X) = k\frac{q}{p}$$

Hypergeometric

$$h(N,k,n,x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad E(X) = n\frac{k}{N}$$

Binomial Distribution and Poisson Distribution

Binomial Distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$p = \frac{\lambda t}{n},\ t = 1,\ n \to \infty$$

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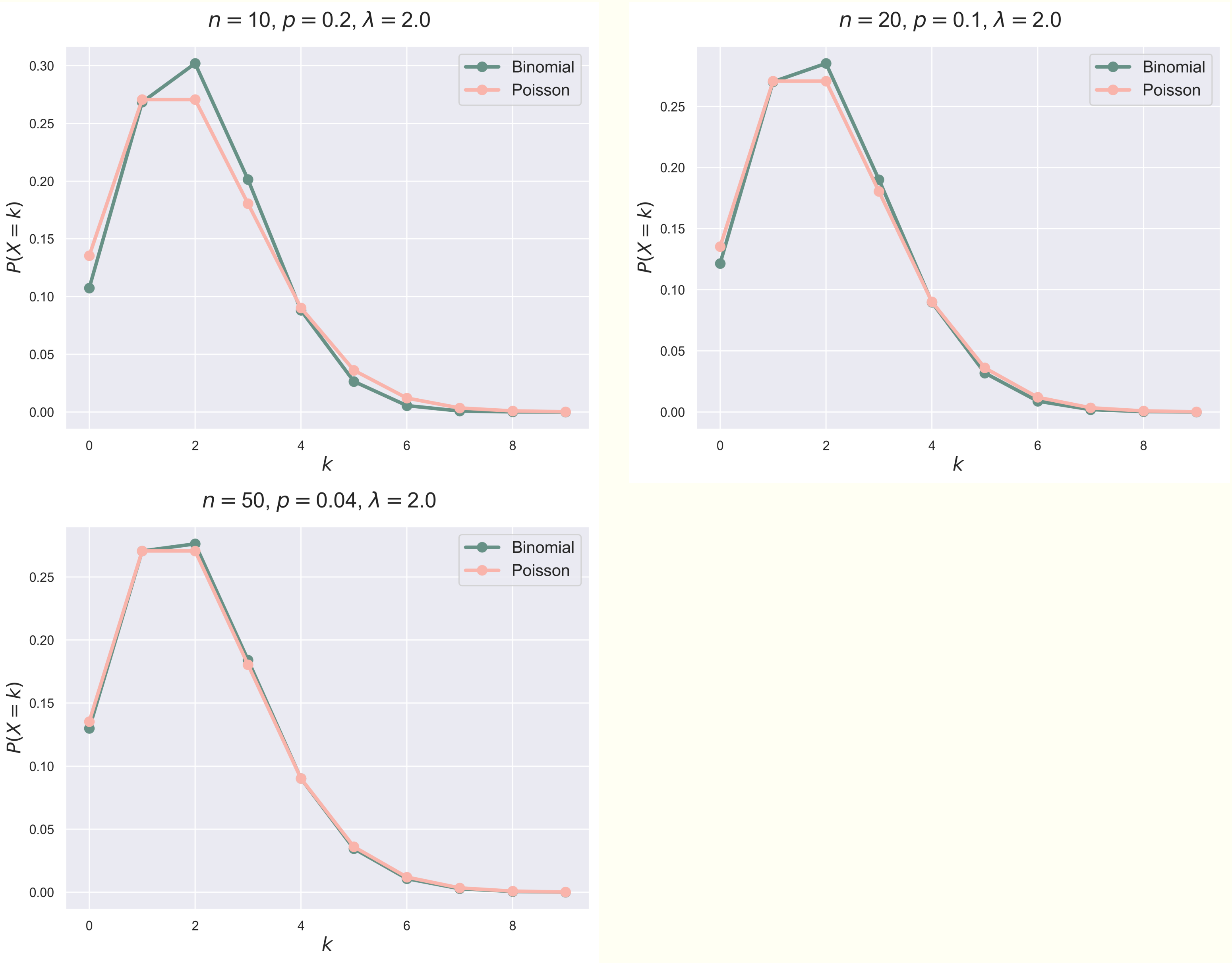


Binomial

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k} \qquad E(X) = np$$

Expected value $E(X)$

$$\sum_{x \in \Omega} x m(x)$$

$$\sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} np \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} q^{n-k} = np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k}$$

$$= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^l q^{n-1-l} = np$$

Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad E(X) = \lambda$$

Expected value $E(X)$

$$\sum_{x \in \Omega} x m(x)$$

$$\sum_{k=0}^{+\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{+\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=1}^{+\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}$$

$$= \lambda \sum_{l=0}^{+\infty} \frac{\lambda^l}{l!} e^{-\lambda} = \lambda$$

Expectation of Functions of Random Variables

- If X is a discrete random variable with sample space Ω and distribution function $m(x)$, and if $\phi\colon \Omega \to R$ is a function, then

$$E(\phi(X)) = \sum_{x\in\Omega} \phi(x)m(x),$$

provided the series converges absolutely.

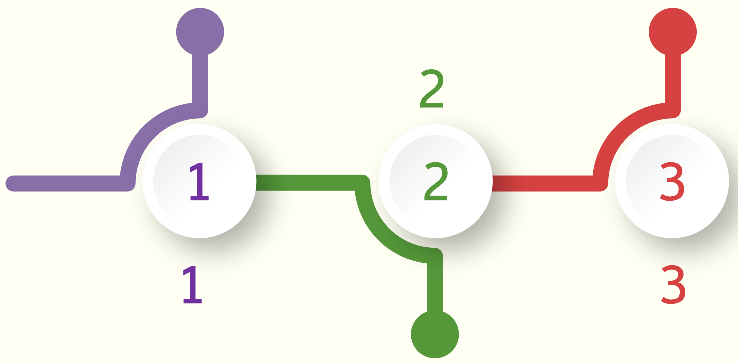
Expected value $\boldsymbol{E(X)}$	Expected value $\boldsymbol{E(\phi(X))}$
$\sum_{x\in\Omega} xm(x)$	$\sum_{x\in\Omega} \phi(x)m(x)$

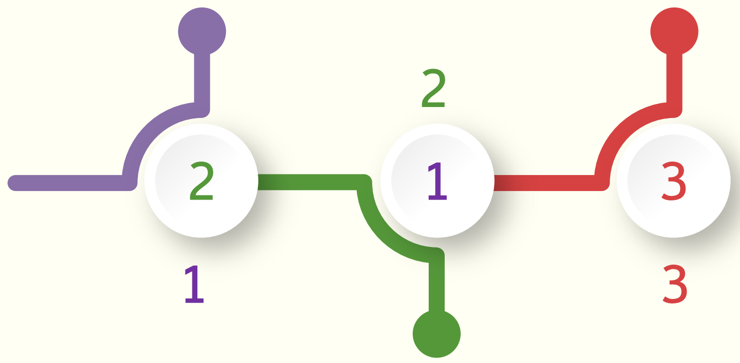
Example 1

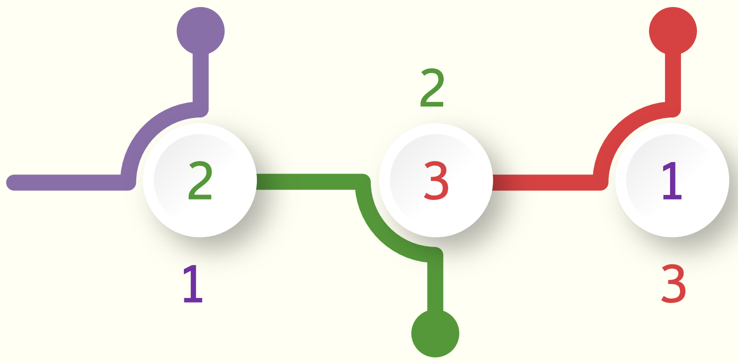
Toss a coin
head or tail
$m(x) = \frac{1}{2}$
$\phi(X) = \begin{cases} 1, & X = \text{Head} \\ 0, & X = \text{Tail} \end{cases}$
$E\big(\phi(X)\big) = \cdots$

Fixed Points

- Since a permutation is a one-to-one mapping of the set onto itself, it is of interest to ask how many points (elements) are mapped onto themselves. Such points are called **fixed points** of the mapping.



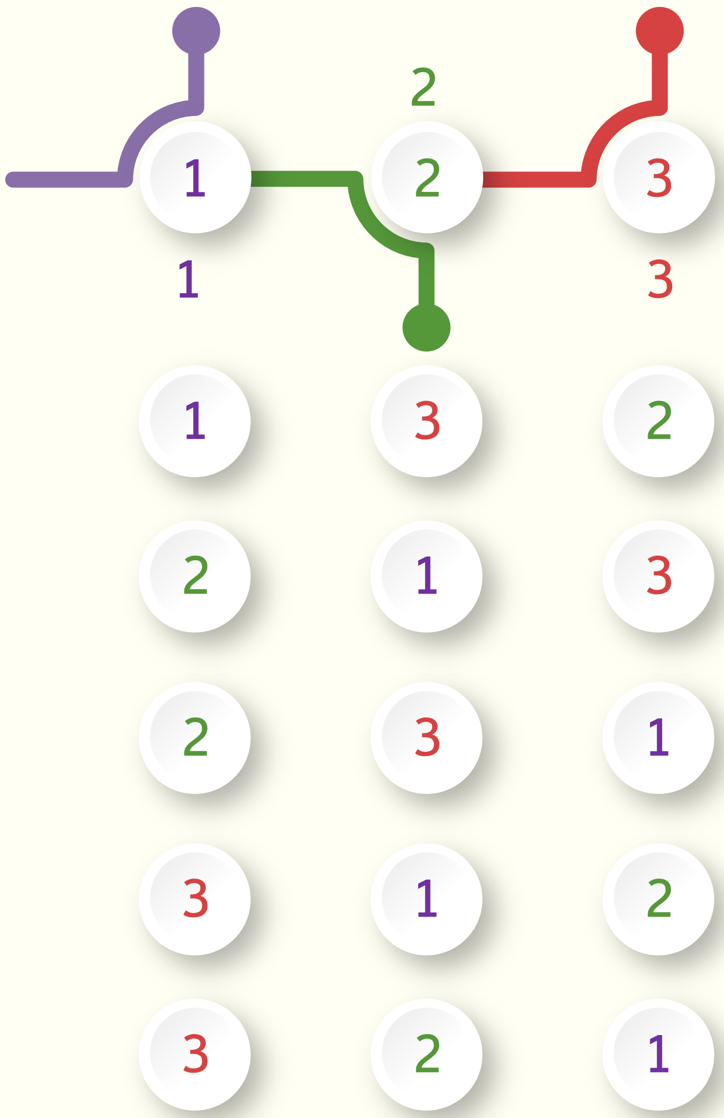




Example 2

Let Y be the number of fixed points in a random permutation of the set $\{a, b, c\}$. To find the expected value of Y, it is helpful to consider the basic random variable associated with this experiment, namely the random variable X which represents the random permutation.

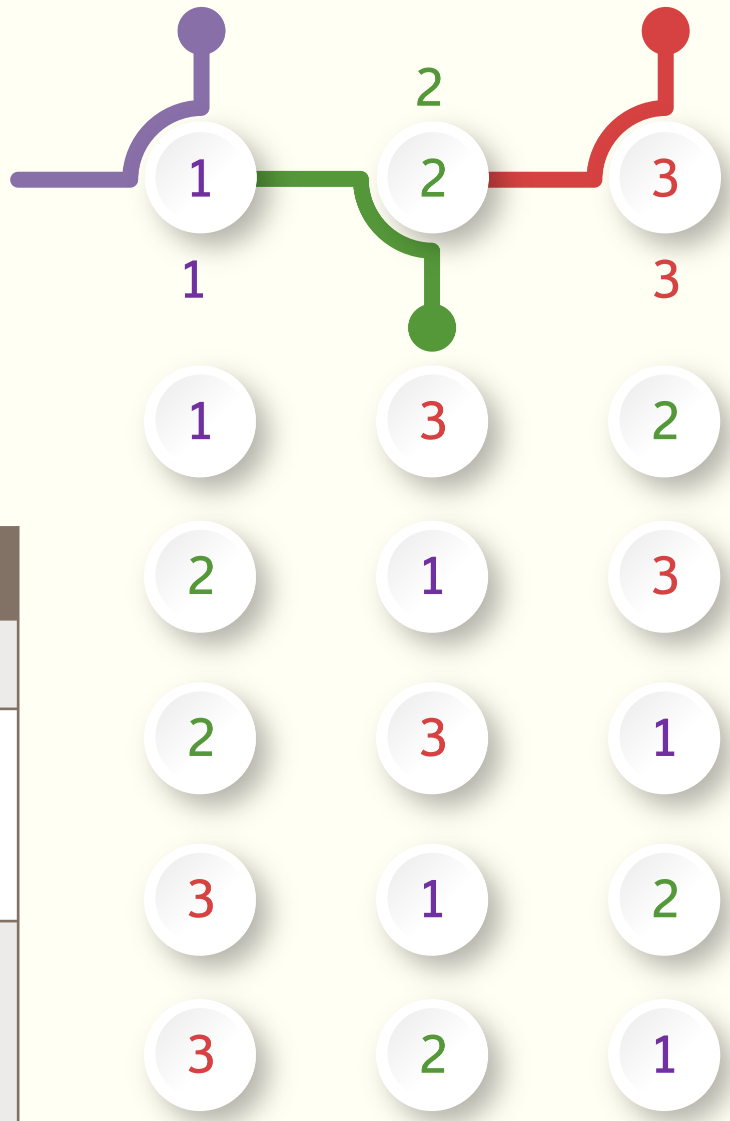
Random permutation
$3! = 6$ possible outcomes
$m(x) = \dfrac{1}{6}$
$E(Y) = \cdots$



Example 2

Let Y be the number of fixed points in a random permutation of the set $\{a, b, c\}$. To find the expected value of Y, it is helpful to consider the basic random variable associated with this experiment, namely the random variable X which represents the random permutation.

Random permutation
$3! = 6$ possible outcomes
$m(x) = \frac{1}{6}$
$E(Y) = 3\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + 0\left(\frac{1}{6}\right) + 0\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right)$ $= 1$



The Sum of Two Random Variables

- Let X and Y be random variables with finite expected values. Then

$$E(X + Y) = E(X) + E(Y),$$

  and if c is any constant, then

$$E(cX) = cE(X).$$

- It can be shown that the expected value of the sum of any finite number of random variables is the sum of the expected values of the individual random variables.

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n).$$

Mutual independence of the summands is not needed.

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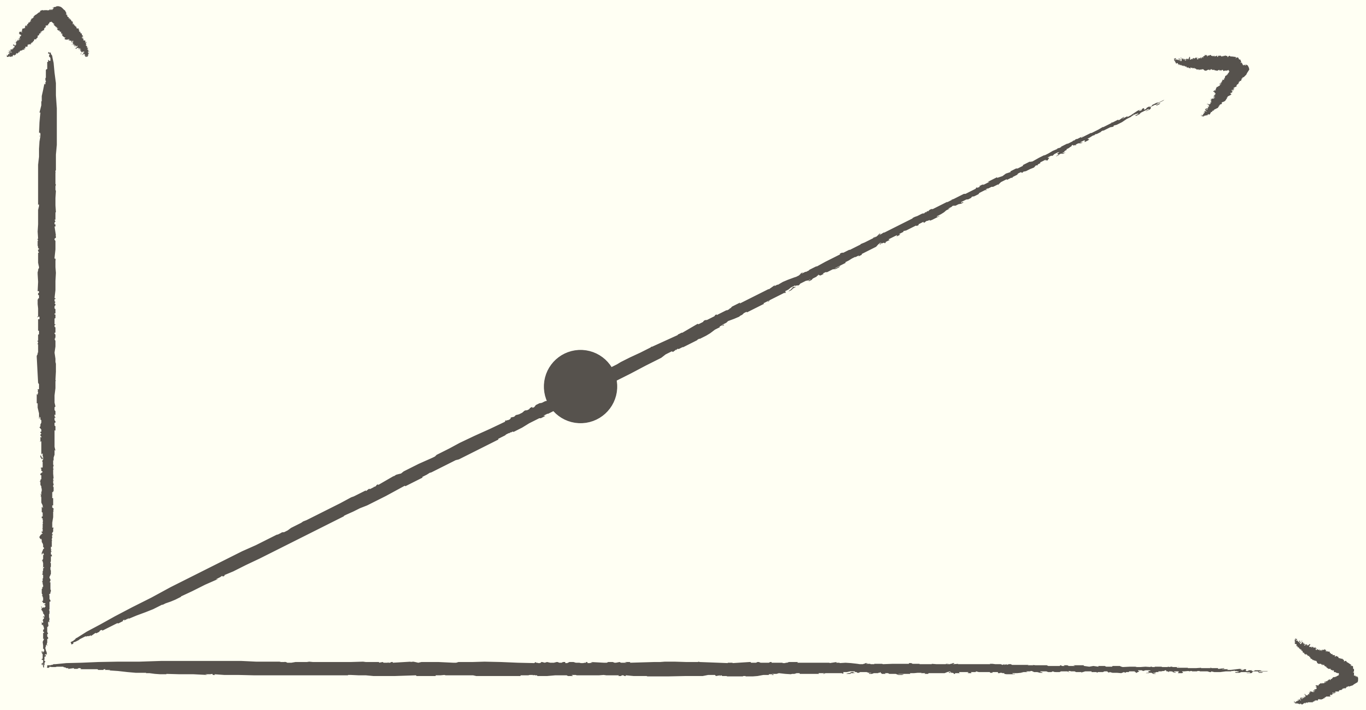
Linearity

$$E(X + Y) = E(X) + E(Y)$$

$$E(cX) = cE(X).$$

$$E(aX + b) = aE(X) + b$$



Proof

Let the sample spaces of X and Y be denoted by Ω_X and Ω_Y, and suppose that

$$\Omega_X = \{x_1, x_2, \cdots\}$$

and

$$\Omega_Y = \{y_1, y_2, \cdots\}.$$

Consider the random variable $X + Y$ to be the result of applying the function $\phi(x, y) = x + y$ to the joint random variable (X, Y). Then according to the Expectation of Functions of Random Variables, we have

$$\sum_k P(X = x_j, Y = y_k) = P(X = x_j)$$

$$\sum_j P(X = x_j, Y = y_k) = P(Y = y_k)$$

$$E(X + Y) = \sum_j \sum_k (x_j + y_k) P(X = x_j, Y = y_k)$$

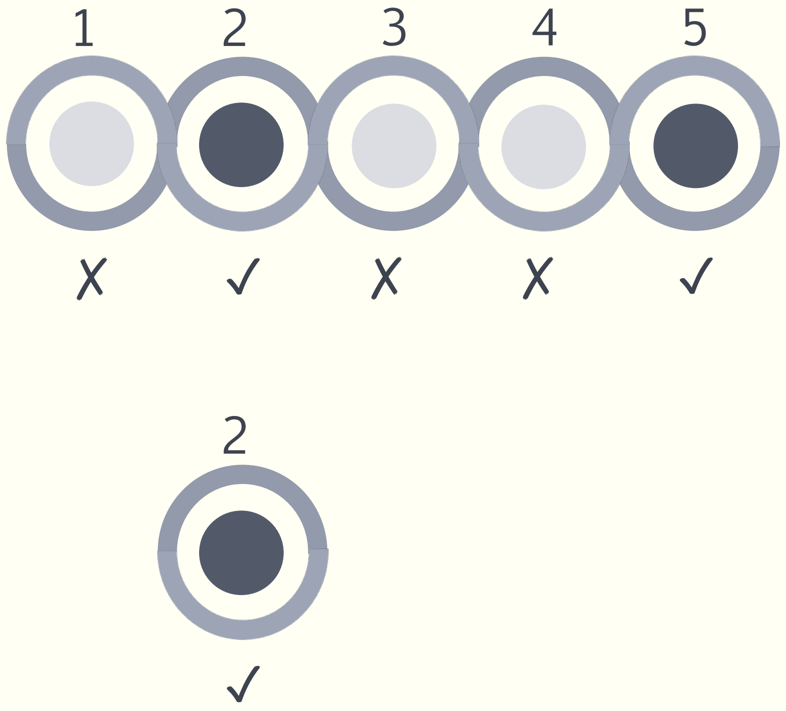
$$= \sum_j \sum_k x_j P(X = x_j, Y = y_k) + \sum_j \sum_k y_k P(X = x_j, Y = y_k)$$

$$= \sum_j x_j P(X = x_j) + \sum_k y_k P(Y = y_k)$$

$$= E(X) + E(Y).$$

Binomial

$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$ $\qquad$ $E(X) = np$



Expected value $E(X)$	
$\sum_{x \in \Omega} x m(x)$	
A single Bernoulli trial	$X_i = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$
	$E(X_i) = 1 \times p + 0 \times (1 - p)$ $= p$
n Bernoulli trials	$E(X)$ $= E(X_1 + X_2 + \cdots + X_n)$ $= np$

Number of fixed points

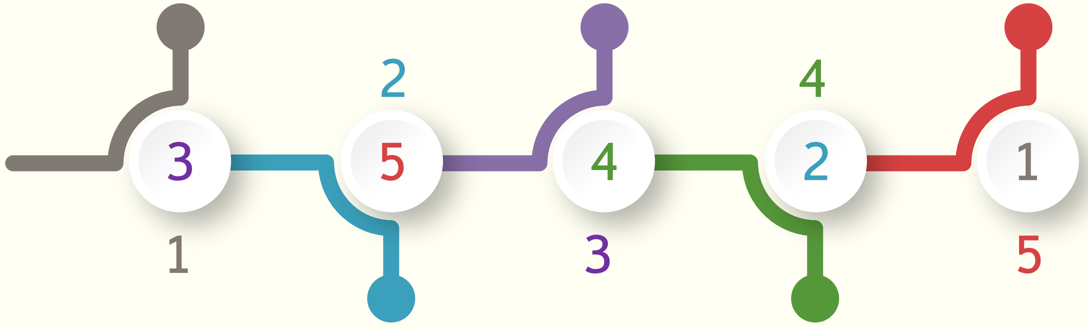
We now give a very quick way to calculate the average number of fixed points in a random permutation of the set $\{1, 2, 3, \ldots, n\}$.

Let Z denote the random permutation.

For each i, $1 \le i \le n$, let X_i equal 1 if Z fixes i, and 0 otherwise.

$$X_i = \begin{cases} 1, & Z \text{ fixes } i \\ 0, & \text{otherwise} \end{cases}$$

let F denote the number of fixed points in Z.



$$F = \cdots$$

?

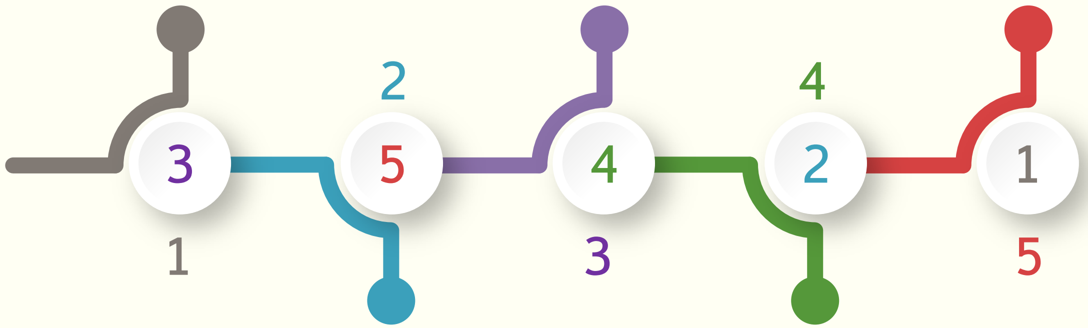
Number of fixed points

Let Z denote the random permutation.

For each i, $1 \le i \le n$, let X_i equal 1 if Z fixes i, and 0 otherwise.

$$X_i = \begin{cases} 1, & Z \text{ fixes } i \\ 0, & \text{otherwise} \end{cases}$$

let F denote the number of fixed points in Z.



$$F = X_1 + X_2 + \cdots + X_n$$

!

$$E(F) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

!

Number of fixed points

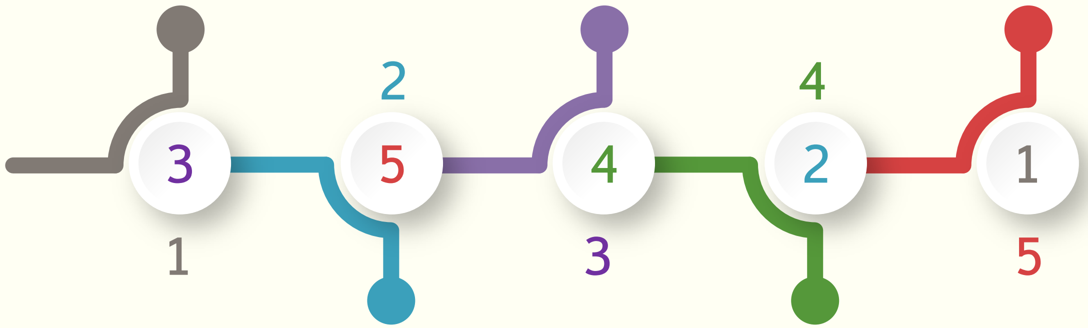
Let Z denote the random permutation.

For each i, $1 \leq i \leq n$, let X_i equal 1 if Z fixes i, and 0 otherwise.

$$X_i = \begin{cases} 1, & Z \text{ fixes } i \\ 0, & \text{otherwise} \end{cases}$$

$$E(X_i) = \frac{1}{n}$$

let F denote the number of fixed points in Z.



$$F = X_1 + X_2 + \cdots + X_n$$

!

$$E(F) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

!

$$E(F) = n \times \frac{1}{n} = 1$$

!

The Product of Two Random Variables

- Let X and Y be random variables with finite expected values. If X and Y are independent random variables, then

$$E(XY) = E(X)E(Y).$$

Mutual independence of the summands is needed.

!

Proof

Let the sample spaces of X and Y be denoted by Ω_X and Ω_Y, and suppose that

$$\Omega_X = \{x_1, x_2, \cdots\}$$

and

$$\Omega_Y = \{y_1, y_2, \cdots\}.$$

Consider the random variable XY to be the result of applying the function $\phi(x, y) = xy$ to the joint random variable (X, Y). Then according to the Expectation of Functions of Random Variables, we have

$$P\left(X = x_j, Y = y_k\right) = (X = x_j)P(Y = y_k)$$

$$\begin{aligned} E(XY) &= \sum_j \sum_k \left(x_j y_k\right) P(X = x_j, Y = y_k) \\ &= \sum_j \sum_k \left(x_j y_k\right) P(X = x_j) P(Y = y_k) \\ &= \Big(\sum_j x_j P(X = x_j)\Big)\Big(\sum_k y_k P(Y = y_k)\Big) \\ &= E(X)E(Y). \end{aligned}$$

August 2020

Sun	Mon	Tue	Wed	Thu	Fri	Sat
26	27	28	29	30	31	01
02	03	04	05	06	07	08
09	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31	01	02	03	04	05

Midterm 2

Open book

Scope: Chapter 4, 5, 6

- conditional probability
- distributions and densities
- expected value and variance

Materials: Slides, homework, quizzes, textbook

Date & Time: August 10, 3 hours, 24 hours

Office hours: August 10, August 11

Homework due: 11:00 pm August 14

Conditional Expectation

- If F is any event and X is a random variable with sample space $\Omega = \{x_1, x_2, \cdots\}$, then the conditional expectation given F is defined by

$$E(X|F) = \sum_j x_j P(X = x_j|F).$$

- Let X be a random variable with sample space Ω. If $F_1, F_2, \cdots, F_r$ are events such that $F_i \cap F_j = \emptyset$ for $i \neq j$ and $\Omega = \cup_j F_j$, then

$$E(X) = \sum_j E\left(X|F_j\right) P(F_j).$$

Proof

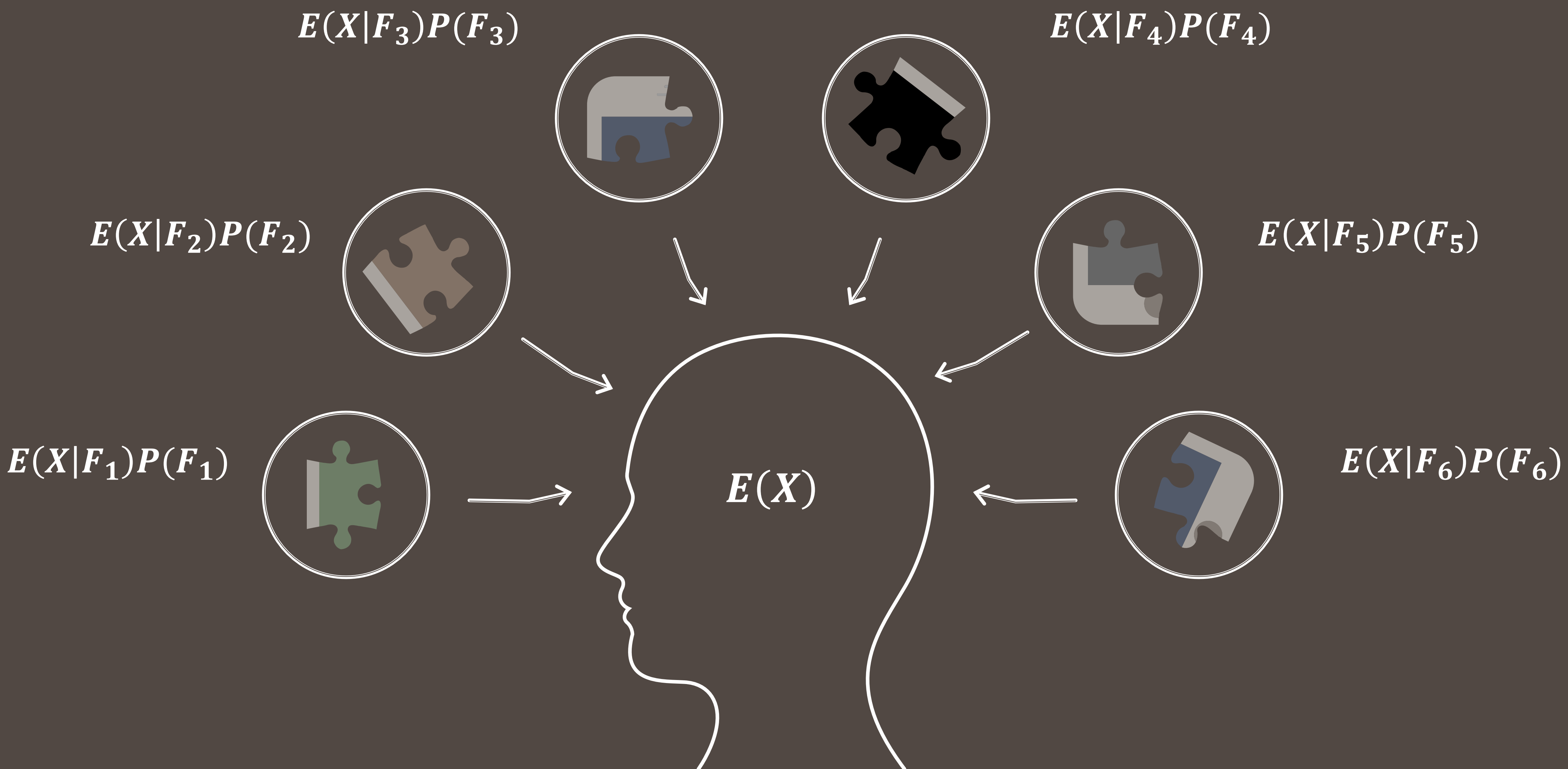
We have

$$E(X|F) = \sum_j x_j P(X = x_j|F)$$

$$\sum_j P(X = x_k \cap F_j) = P(X = x_k)$$

$$\begin{aligned}\sum_j E(X|F_j)P(F_j) &= \sum_j \sum_k x_k P(X = x_k|F_j)P(F_j) \\ &= \sum_j \sum_k x_k P(X = x_k \cap F_j) \\ &= \sum_k x_k \sum_j P(X = x_k \cap F_j) \\ &= \sum_k x_k P(X = x_k) \\ &= E(X)\end{aligned}$$

Why We Need Conditional Expectation?



EXAMPLE

Farming Sim

Seasons

There are four seasons in the game that alternate every 17 minutes: spring, summer, fall, and winter. The current season restricts what crops are available for planting and allow you to harvest certain kinds of trees.

Weather

There is a 25% chance that a season will have special weather.

Rain will slowly water all of your planted crops.

Droughts will cause watered plants to dry out three times as fast.

Snow will cancel out the growth speed bonus that water provides.

The chances of the specific weathers in the different seasons are:

	Rain	Droughts	Snow
Spring	100%		
Summer	20%	80%	
Fall	100%		
Winter	20%		80%

Seasons

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The chances of the specific weathers in the different seasons are:

Special weather			
	Rain	Droughts	Snow
Spring	100%		
Summer	20%	80%	
Fall	100%		
Winter	20%		80%

All weather				
	Normal	Rain	Droughts	Snow
Spring	75%	25%		
Summer	75%	5%	20%	
Fall	75%	25%		
Winter	75%	5%		20%

Seasons

There are four seasons in the game that alternate every 17 minutes: spring, summer, fall, and winter. The current season restricts what crops are available for planting and allow you to harvest certain kinds of trees.

Weather

There is a 25% chance that a season will have special weather.

Rain will slowly water all of your planted crops.

Droughts will cause watered plants to dry out three times as fast.

Snow will cancel out the growth speed bonus that water provides.

The chances of the specific weathers in the different seasons are:

All weather (a season)				
	Normal	Rain	Droughts	Snow
Spring	3/4	1/4		
Summer	3/4	1/20	1/5	
Fall	3/4	1/4		
Winter	3/4	1/20		1/5

All weather (a year)				
	Normal	Rain	Droughts	Snow
Spring	3/16	1/16		
Summer	3/16	1/80	1/20	
Fall	3/16	1/16		
Winter	3/16	1/80		1/5

All weather (a year)				
	Normal	Rain	Droughts	Snow
Spring	3/16	1/16		
Summer	3/16	1/80	1/20	
Fall	3/16	1/16		
Winter	3/16	1/80		1/20

	Normal	Rain	Droughts	Snow
income	1000	500	200	100

	Normal	Rain	Droughts	Snow	Total
Spring	$3/16$	$1/16$	0	0	$1/4$
Summer	$3/16$	$1/80$	$1/20$	0	$1/4$
Fall	$3/16$	$1/16$	0	0	$1/4$
Winter	$3/16$	$1/80$	0	$1/5$	$1/4$
Total	$3/4$	$3/20$	$1/20$	$1/20$	$1/4$

	Normal	Rain	Droughts	Snow
income	1000	500	200	100

$$E(X|F) = \sum_j x_j P(X = x_j|F).$$

!

$$E(X) = \sum_j E(X|F_j)P(F_j).$$

!

- X: income

- F_1, F_2, F_3, F_4: seasons
spring, summer, fall, winter

$$E(X|F_i) = \cdots$$

$$E(X) = \cdots$$

	Normal	Rain	Droughts	Snow	Total
Spring	3/16	1/16	0	0	1/4
Summer	3/16	1/80	1/20	0	1/4
Fall	3/16	1/16	0	0	1/4
Winter	3/16	1/80	0	1/5	1/4
Total	3/4	3/20	1/20	1/20	1/4

- X: income

- F_1, F_2, F_3, F_4: seasons spring, summer, fall, winter

	Normal	Rain	Droughts	Snow
income	1000	500	200	100

$$E(X|F) = \sum_j x_j P(X = x_j|F).$$

!

$$E(X) = \sum_j E(X|F_j)P(F_j).$$

!

$$E(X|\text{spring}) = 1000\times\frac{3}{4} + 500\times\frac{1}{4} = \frac{3500}{4}$$

$$E(X|\text{summer}) = 1000\times\frac{3}{4} + 500\times\frac{1}{20} + 200\times\frac{1}{5} = \frac{16300}{20}$$

$$E(X|\text{fall}) = 1000\times\frac{3}{4} + 500\times\frac{1}{4} = \frac{3500}{4}$$

$$E(X|\text{winter}) = 1000\times\frac{3}{4} + 500\times\frac{1}{20} + 100\times\frac{1}{5} = \frac{15900}{20}$$

$$E(X) = E(X|\text{spring})P(\text{spring}) + E(X|\text{summer})P(\text{summer}) + E(X|\text{fall})P(\text{fall}) + E(X|\text{winter})P(\text{winter})$$
$$= \left(\frac{3500}{4} + \frac{16300}{20} + \frac{3500}{4} + \frac{15900}{20}\right)\times\frac{1}{4} = 840$$

	Normal	Rain	Droughts	Snow	Total
Spring	3/16	1/16	0	0	1/4
Summer	3/16	1/80	1/20	0	1/4
Fall	3/16	1/16	0	0	1/4
Winter	3/16	1/80	0	1/5	1/4
Total	3/4	3/20	1/20	1/20	1/4

	Normal	Rain	Droughts	Snow
income	1000	500	200	100

$$E(X|F) = \sum_j x_j P(X = x_j|F).$$

!

$$E(X) = \sum_j E(X|F_j)P(F_j).$$

!

- X: income

- F_1, F_2, F_3, F_4: weather
normal, rain, droughts, snow

$$E(X|F_i) = \cdots$$

$$E(X) = \cdots$$

	Normal	Rain	Droughts	Snow	Total
Spring	3/16	1/16	0	0	1/4
Summer	3/16	1/80	1/20	0	1/4
Fall	3/16	1/16	0	0	1/4
Winter	3/16	1/80	0	1/5	1/4
Total	3/4	3/20	1/20	1/20	1/4

- X: income

- F_1, F_2, F_3, F_4: weather
normal, rain, droughts, snow

	Normal	Rain	Droughts	Snow
income	1000	500	200	100

$$E(X|F) = \sum_j x_j P(X = x_j|F).$$

!

$$E(X) = \sum_j E(X|F_j)P(F_j).$$

!

$$E(X|\text{normal}) = 1000,\ E(X|\text{rain}) = 500,$$

$$E(X|\text{droughts}) = 200,\ E(X|\text{snow}) = 100$$

$$P(\text{normal}) = \frac{3}{16}\times 4 = \frac{3}{4},\ P(\text{rain}) = \frac{3}{20}$$

$$P(\text{droughts}) = \frac{1}{20},\ P(\text{snow}) = \frac{1}{20}$$

$$E(X) = E(X|\text{normal})P(\text{normal}) + E(X|\text{rain})P(\text{rain}) + E(X|\text{droughts})P(\text{droughts}) + E(X|\text{snow})P(\text{snow})$$
$$= 1000\times\frac{3}{4} + 500\times\frac{3}{20} + 200\times\frac{1}{20} + 100\times\frac{1}{20} = 840$$

MARTINGALES

Fairness to a Player

Martingales

- Let $S_1, S_2, \cdots, S_n$ be the sequential outcomes of a repeated experiment (such as fair games), we have

$$\boldsymbol{E(S_n | S_{n-1} = a, \cdots, S_2 = t, S_1 = r) = S_{n-1}}.$$

Mathematical Finance

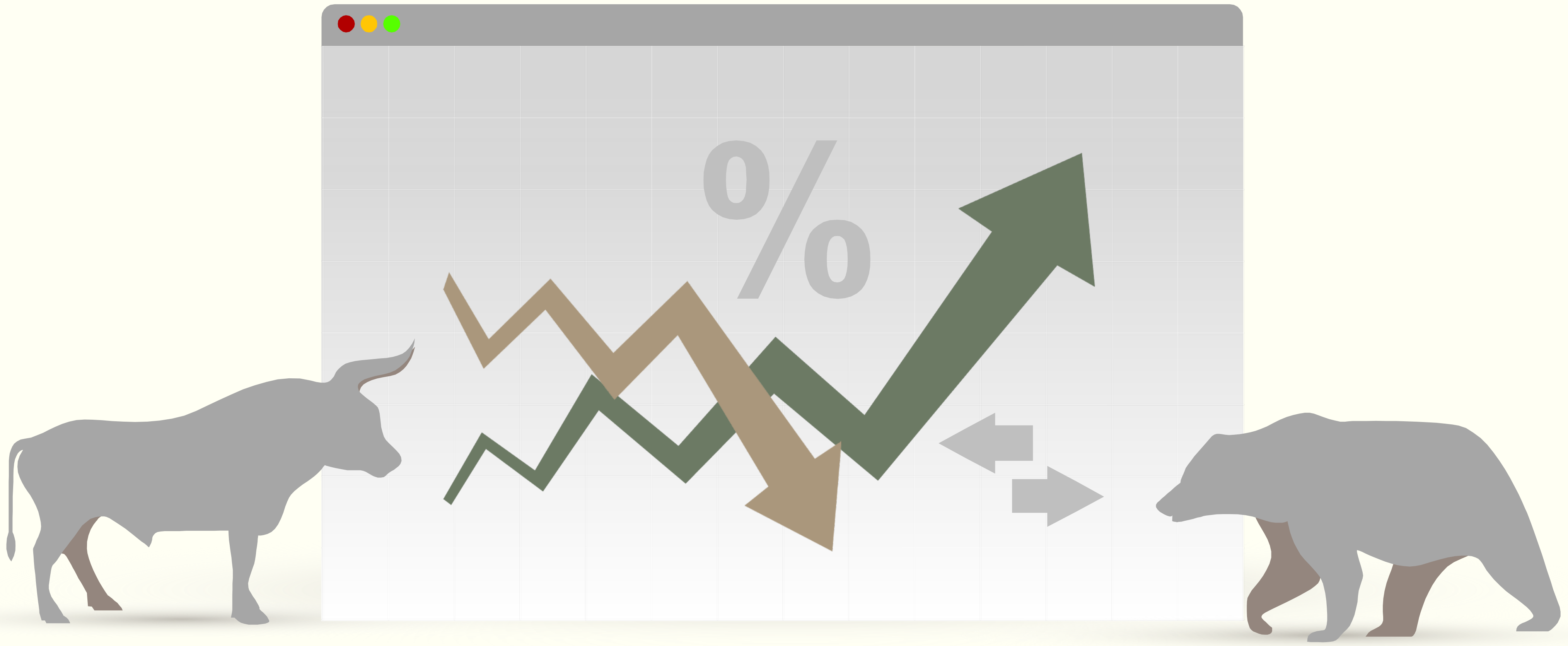
Samuelson suggested in 1965 that the stock prices follow a martingale: **Proof That Properly Anticipated Prices Fluctuate Randomly**.

“Given all I know today, expected price tomorrow is the price today.”

Martingales in Finance

$$E(S_n | S_{n-1} = a, \cdots, S_2 = t, S_1 = r) = S_{n-1}.$$

!



Many asset prices are believed to behave approximately like martingales, at least in the short term.

Martingales in Finance

$$E(S_n | S_{n-1} = a, \cdots, S_2 = t, S_1 = r) = S_{n-1}.$$

!

Efficient market hypothesis

- New information is instantly absorbed into the stock value, so expected value of the stock tomorrow should be the value today. If it were higher, statistical arbitrageurs would bid up today's price until this was not the case.
- But there are some caveats: interest, risk premium, etc.

Martingales in Finance

$$E(S_n|S_{n-1} = a, \cdots, S_2 = t, S_1 = r) = S_{n-1}.$$

!

Risk neutral probability

- According to the **fundamental theorem of asset pricing,** the discounted price $\frac{X(n)}{A(n)}$, where A is a risk-free asset, is a martingale with respected to **risk neutral probability**.