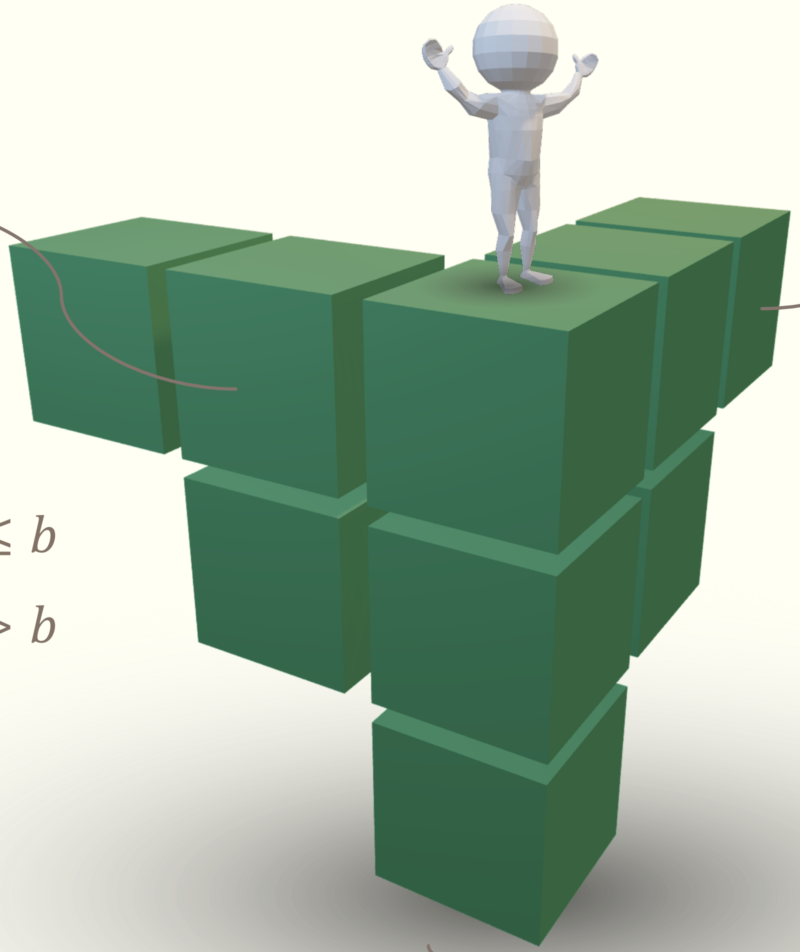
IMPORTANT DENSITIES

- Continuous uniform density
- Exponential density
- Normal density

Important Densities

Continuous uniform density

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & x < a \text{ or } x > b \end{cases}$$



Exponential density

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}.$$

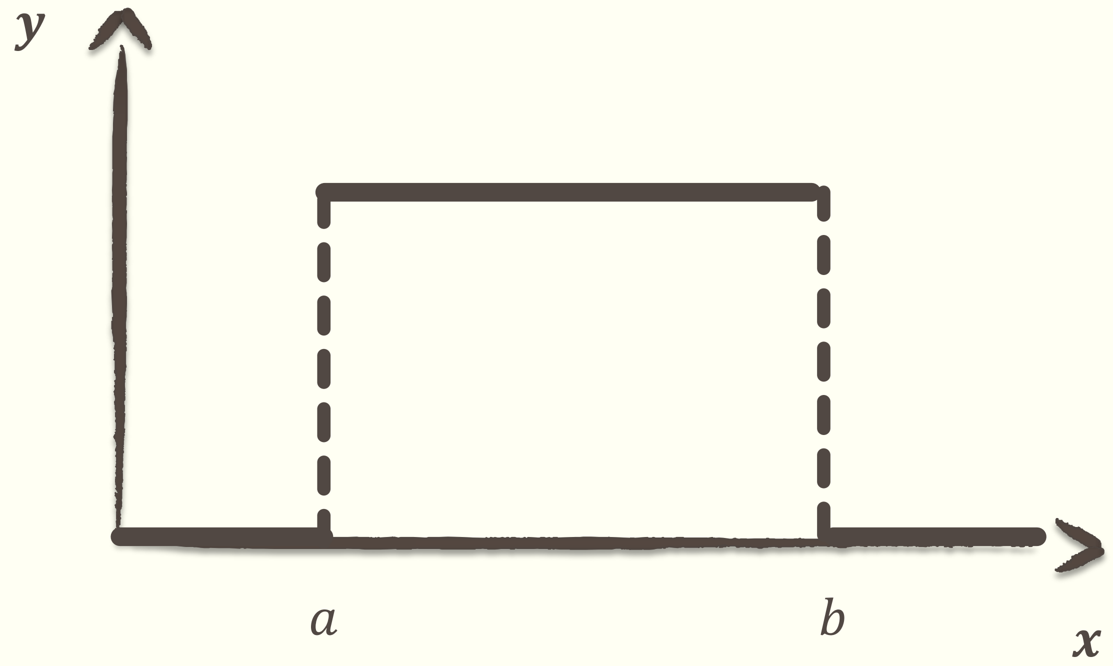
Normal density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Continuous Uniform Distribution

- The distribution describes an experiment where there is an arbitrary outcome that lies between certain bounds.
- The bounds are defined by the parameters, a and b, which are the minimum and maximum values.
- The probability density function of the continuous uniform distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & x < a \text{ or } x > b \end{cases}$$

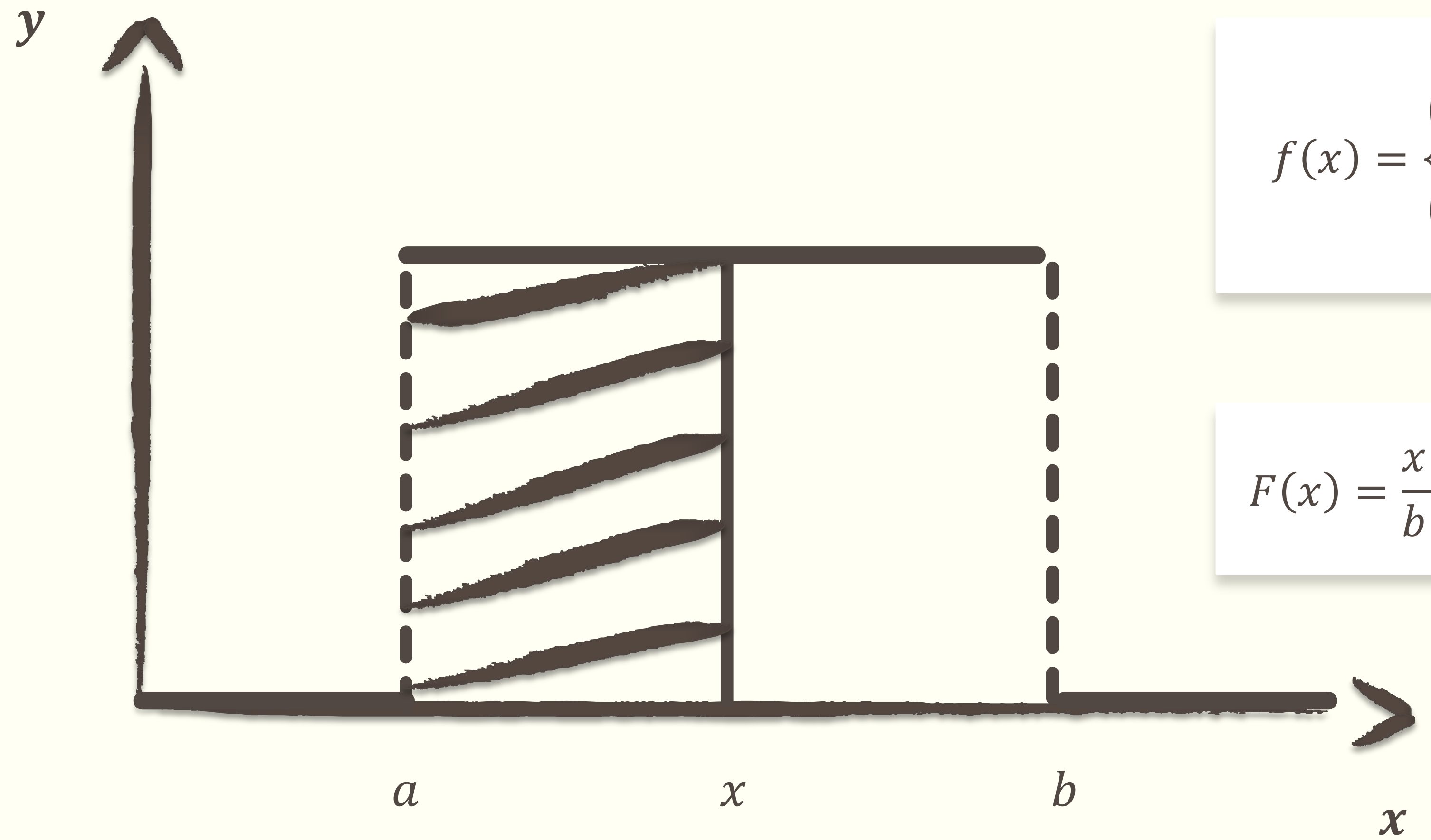


Calculate the Cumulative Distribution Function of the Uniform Distribution

Let X be a continuous real-valued random variable with density function $f(x)$. Then the function defined by

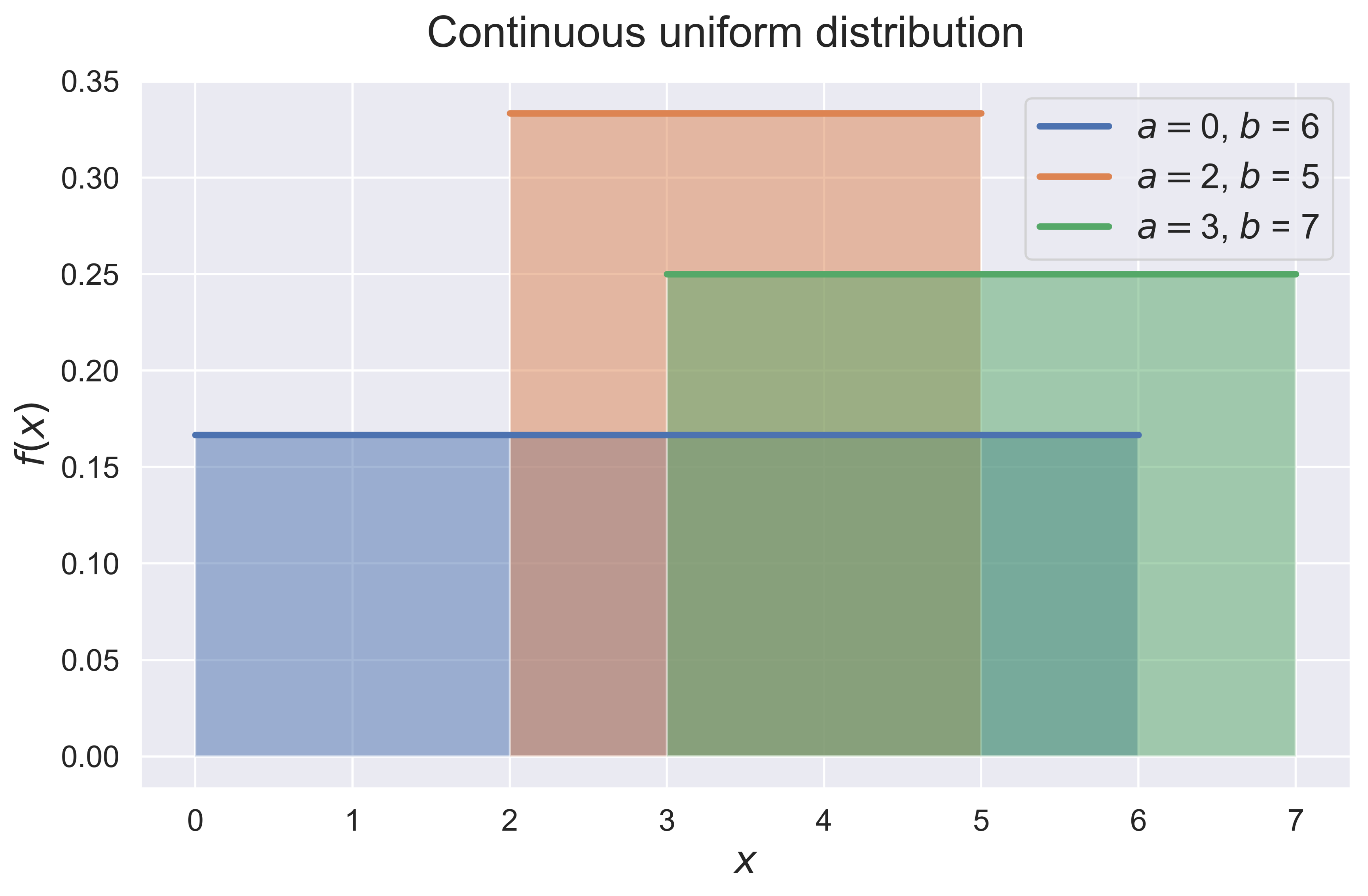
$$F_X(x) = \int_{-\infty}^{x} f(s)ds,$$

is the cumulative distribution function of X.



$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0, & x < a \text{ or } x > b \end{cases}$$

$$F(x) = \frac{x-a}{b-a}, \qquad a \leq x \leq b$$



Continuous uniform distribution
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
f(x)
0
1
2
3
4
5
6
7
x
a = 0, b = 6
a = 2, b = 5
a = 3, b = 7

Example 1

A real number U is chosen at random from $[0,1]$ with uniform probability, and then this number is squared.

Let X represent the result, $X = U^3$.

- What is the cumulative distribution function of X?
- What is the density function of X?

$F_U(u) = P(U \leq u) = \cdots$

00

01

$F_X(x) = P(X \leq x) = \cdots$

range or X is ...

$\frac{d}{dx} F_X(x) = f(x) = \cdots$

02

01

$$F_X(x) = P(X \leq x) = P(U \leq \sqrt[3]{x}) = \sqrt[3]{x}$$

$$0 \leq X \leq 1$$

$$\frac{d}{dx} F_X(x) = f(x) = \frac{d}{dx} \sqrt[3]{x} = \frac{1}{3\sqrt[3]{x^2}}$$

02

Example 2

Two real numbers X and Y are chosen at random and uniformly from $[0,1]$. Let $Z = X + Y$.

Please derive expressions for the cumulative distribution and the density function of Z.

01

$F_Z(z) = P(Z \le z) = \cdots$

range of Z is …

$$\frac{d}{dz}F_Z(z) = f(z) = \cdots$$

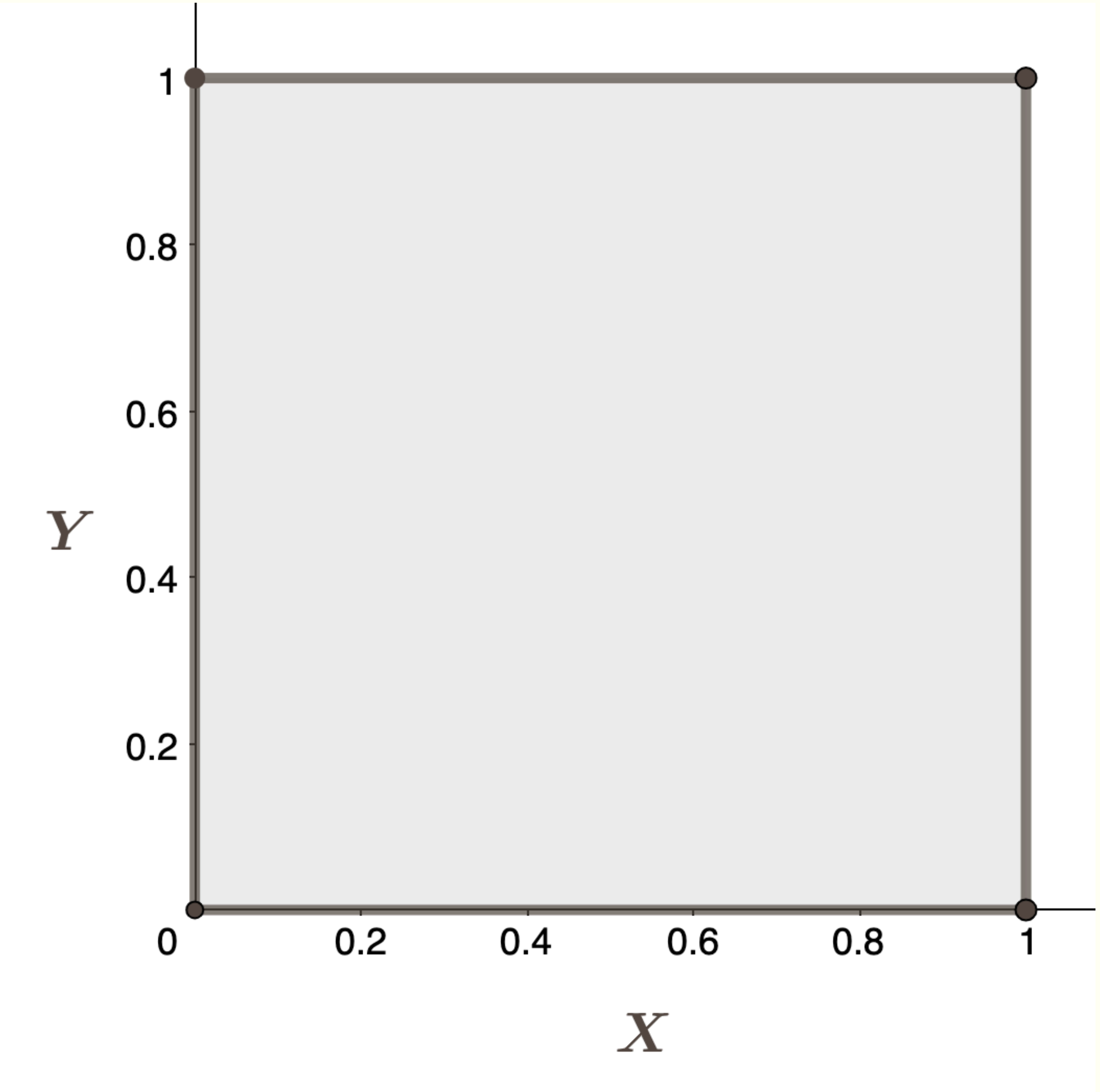
02

- Two real numbers X and Y are chosen at random and uniformly from $[0, 1]$. Let $Z = X + Y$.

- Please derive expressions for the cumulative distribution and the density function of Z.

01

$$F_Z(z) = P(Z \leq z)$$
$$= P(X + Y \leq z)$$



Choose a number U from the unit interval $[0, 1]$ with uniform distribution.
Consider the following random variables:

- $X = U + 2$
- $Y = 2U$
- $Z = |U - \frac{1}{2}|$

Which one of them follows a uniform distribution as well?

01

$F_Z(z) = P(Z \leq z) = \cdots$
range or Z is ...

$\frac{d}{dz} F_Z(z) = f(z) = \cdots$

02

Choose a number U from the unit interval [0, 1] with uniform distribution.
Consider the following random variables:

- $X = U + 2$
- $Y = 2U$
- $Z = |U - \frac{1}{2}|$

Which one of them follows a uniform distribution as well?

01

$$F_Z(z) = P(Z \le z) = P\left(\left|U - \frac{1}{2}\right| \le z\right) = 2z$$

range or Z is $[0, \frac{1}{2}]$

$$\frac{d}{dz}F_Z(z) = f(z) = \frac{d}{dz}(2z) = 2$$

02

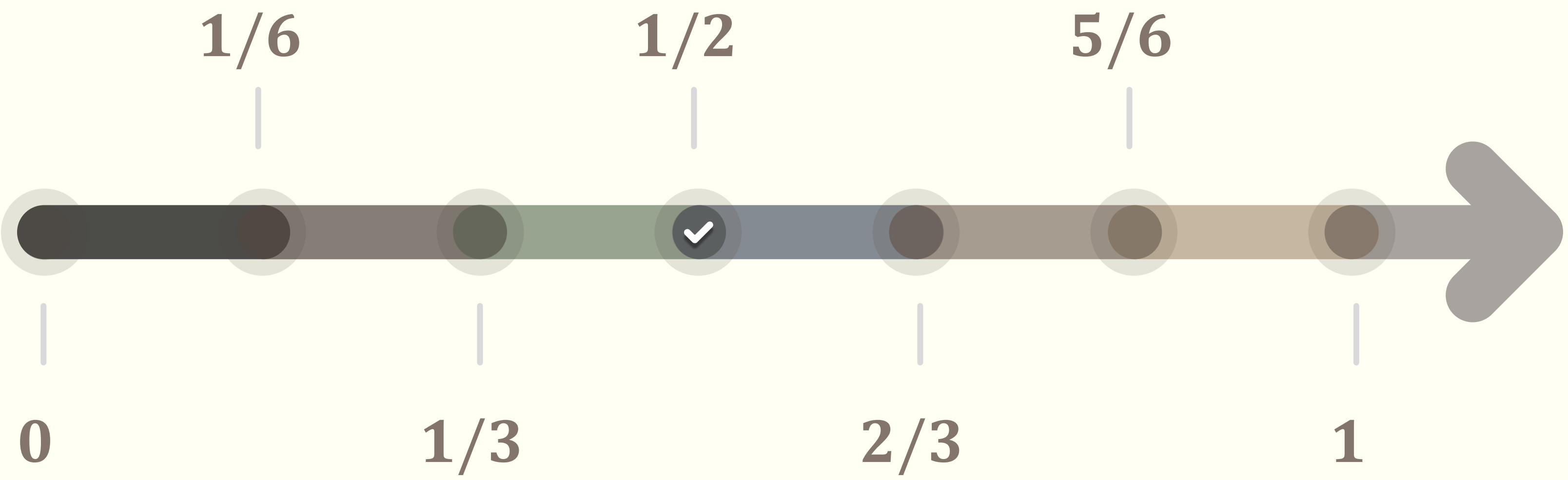
Uniform distribution

Consider the following random variable:

- $Z = |U - \frac{1}{2}|$

$P(Z = 0) = P(U = \frac{1}{2})$ =

$P\left(Z = \frac{1}{6}\right) = P\left(U = \frac{1}{3}\right) + P(U = \frac{2}{3})$ =



$P(Z = 0) = \frac{1}{2} P(Z = \frac{1}{6})$

It cannot be uniform?

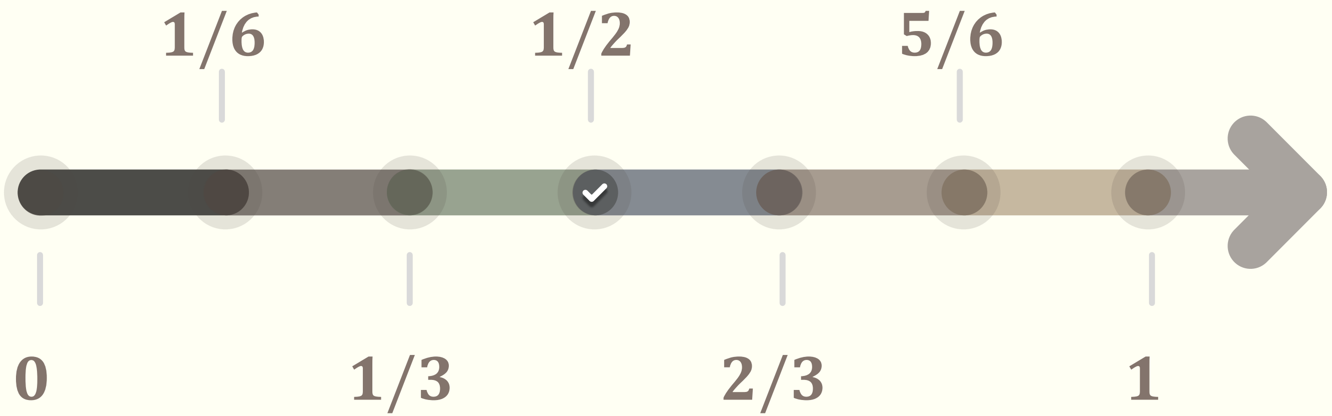
Uniform distribution

Consider the following random variable:

- $Z = |U - \frac{1}{2}|$

$$P(Z = 0) = P\left(U = \frac{1}{2}\right) = 0$$

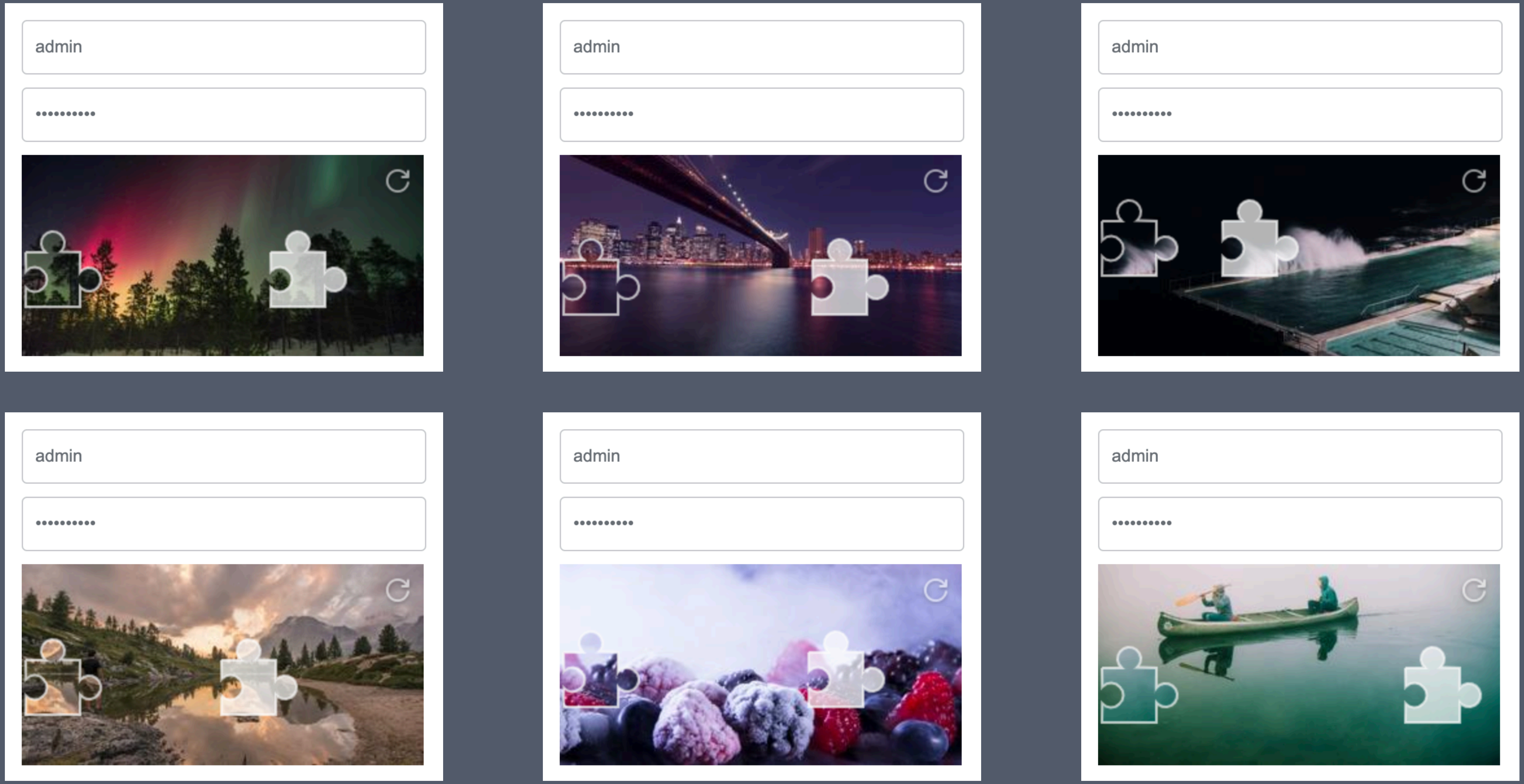
$$P\left(Z = \frac{1}{6}\right) = P\left(U = \frac{1}{3}\right) + P\left(U = \frac{2}{3}\right) = 0$$



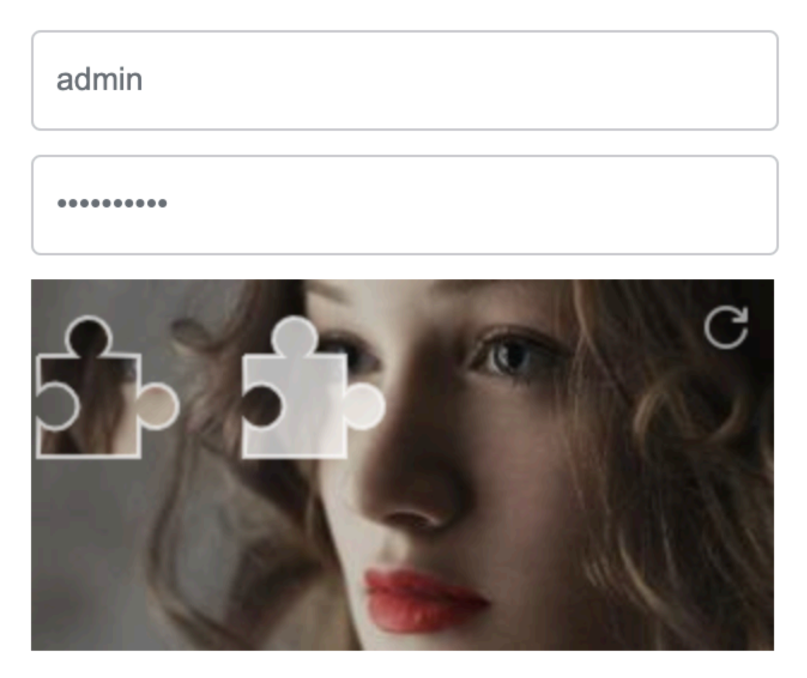
$$P(Z = 0) = P\left(Z = \frac{1}{6}\right) = 0$$

For continuous random variable, the probability at a single point cannot be used to decide the distribution!

Sliding Verification Code



Sliding Verification Code




```
canvasInit() {
      // # generate random number
      const random = (min, max) => {
        return Math.floor(Math.random() * (max - min + 1) + min);
      };
      //x: 254, y: 109
      let mx = random(127, 244),
        bx = random(10, 128),
        y = random(10, 99);

      this.slider = { mx, bx };

      this.draw(mx, bx, y);
    },
```

Javascript: Math.random()

The Math.random() function returns a floating-point, pseudo-random number in the range 0 to less than 1 (inclusive of 0, but not 1) with approximately **uniform distribution** over that range, which you can then scale to your desired range.

Exponential Distribution

- The exponential distribution is the probability distribution of the **time** between events in a **Poisson point process**. That is, a process in which events occur continuously and independently **at a constant average rate λ**.
- The density function of an exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

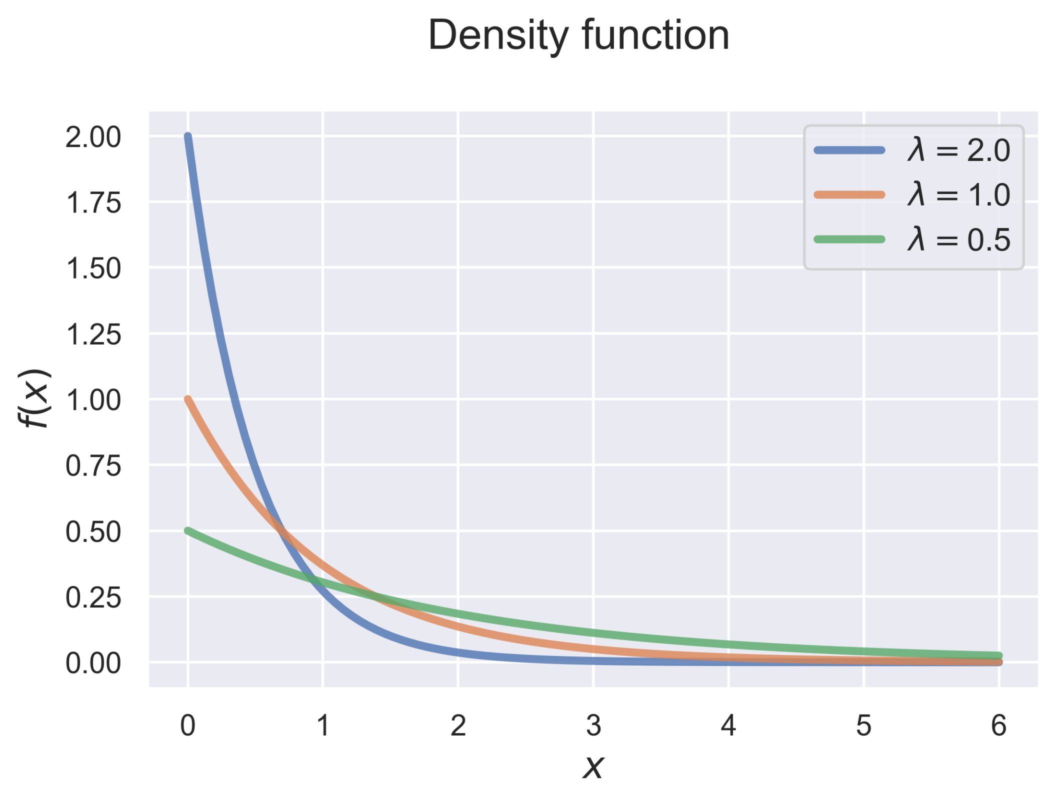
- The cumulative distribution function of an exponential distribution is

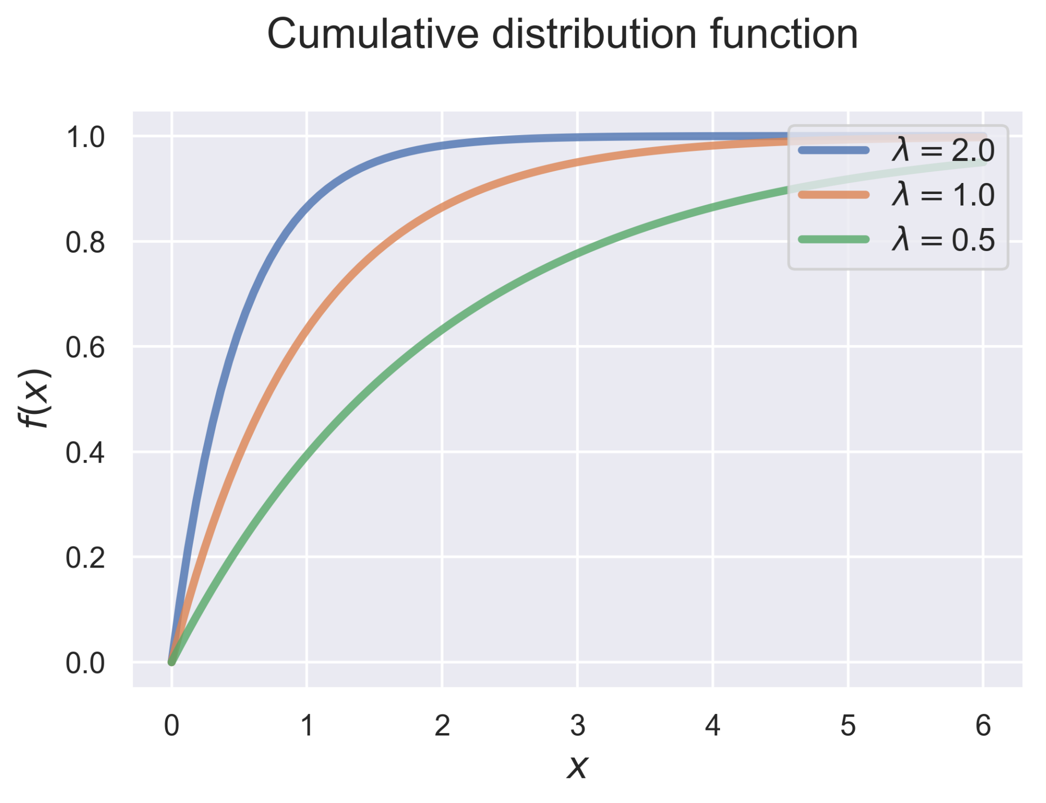
$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$





Exponential Distribution

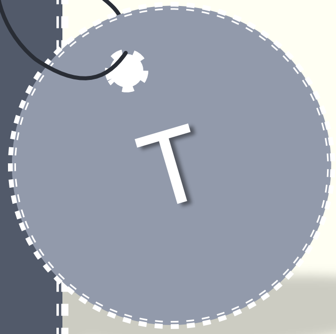
$$P(T > t) = e^{-\lambda t}$$

$$P(T > r + s | T > r) = \frac{P(T > r + s \cap T > r)}{P(T > r)} = \frac{P(T > r + s)}{P(T > r)}$$

$$= \frac{e^{\lambda-(r+s)}}{e^{-\lambda r}} = e^{-\lambda s} = P(T > s)$$

Memoryless Property

$$P(T > r + s | T > r) = P(T > s)$$



Poisson Process

Binomial distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Poisson process: the occurrence of each event is **exponentially distributed**. By time t, how many such events have happened? The number of events are **Poisson-distributed**.

Poisson Process

- A Poisson Process is a model for a series of discrete event where the **average time** between events is known, but the exact timing of events is random.

- The arrival of an event is independent of the event before (waiting time between events is **memoryless**).

- A Poisson process meets the following criteria (in reality many phenomena modeled as Poisson processes do not meet these exactly):

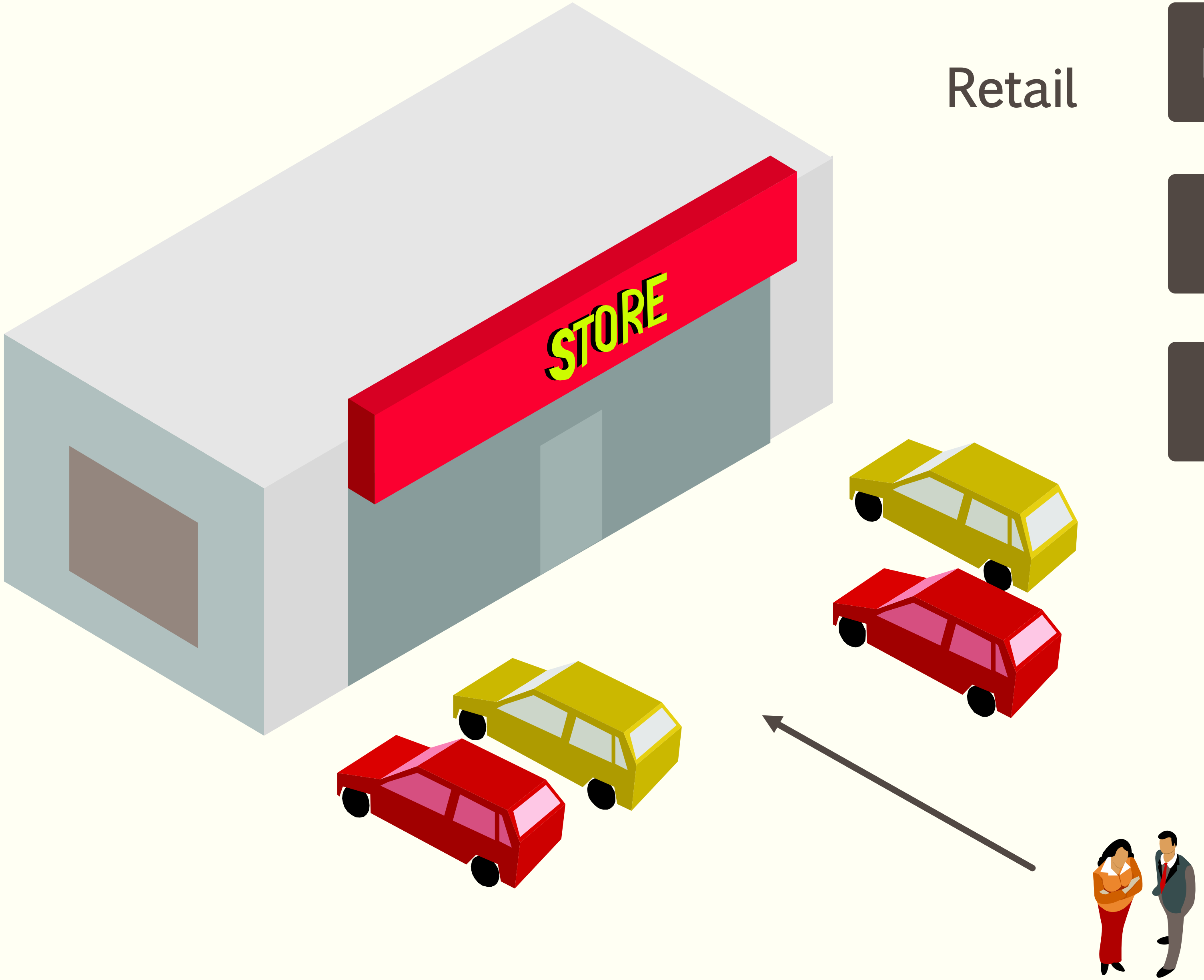
    - Events are **independent** of each other. The occurrence of one event does not affect the probability another event will occur.
    - The average rate (events per time period) is constant.
    - Two events cannot occur at the same time.

Retail

Events are independent of each other.

The average rate is constant.

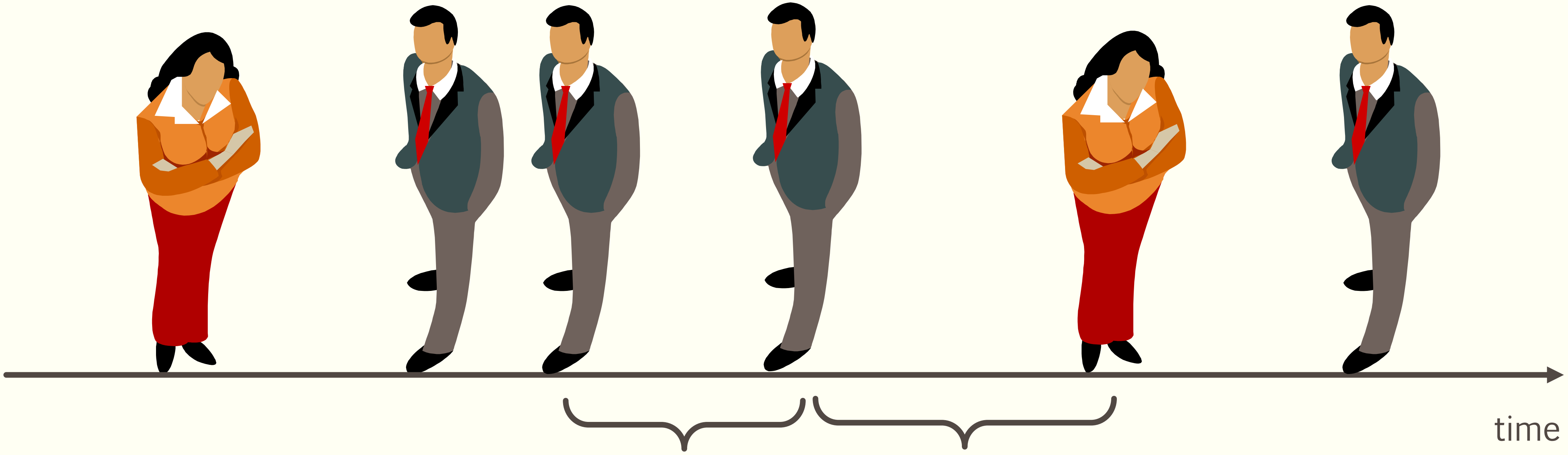
Two events cannot occur at the same time.



Customers

Poisson Process

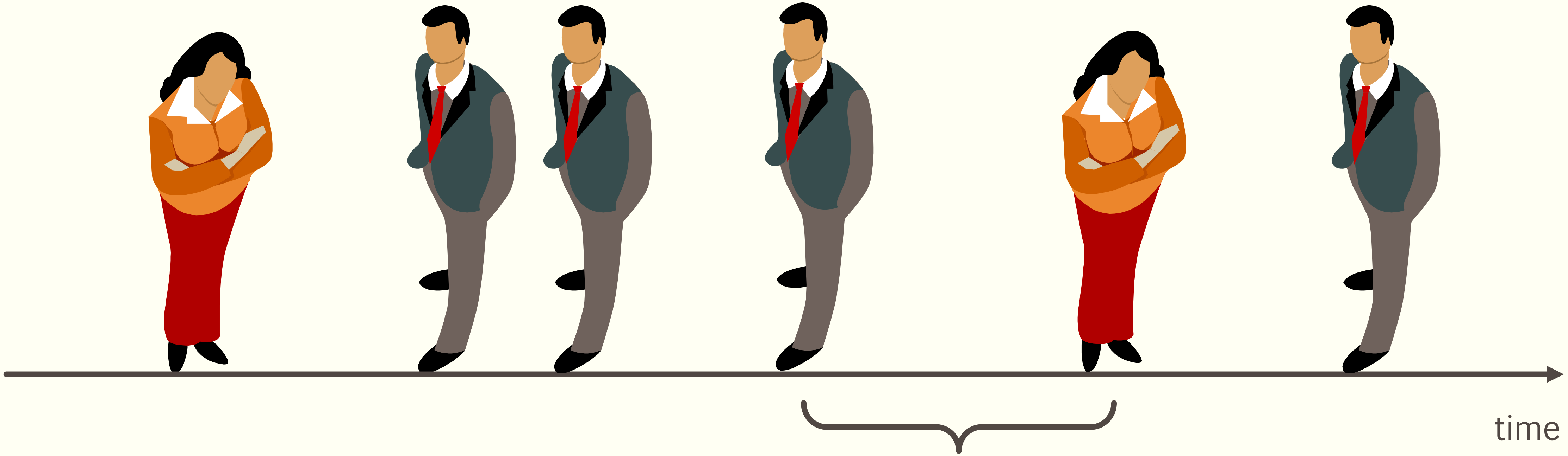
- A Poisson Process is a model for a series of discrete event where the **average time** between events is known, but the exact timing of events is random.
- The arrival of an event is independent of the event before (waiting time between events is **memoryless**).



Waiting time

- An intriguing part of a Poisson process involves figuring out how long we have to wait until the next event (this is sometimes called the interarrival time).
- The waiting time follows an exponential distribution:

$$f(t) = \lambda e^{-\lambda t}$$
$$P(T > t) = e^{-\lambda t}$$



Number of events

$$np = \lambda t,\ t = 1,\ n \to \infty,\ p \to 0$$

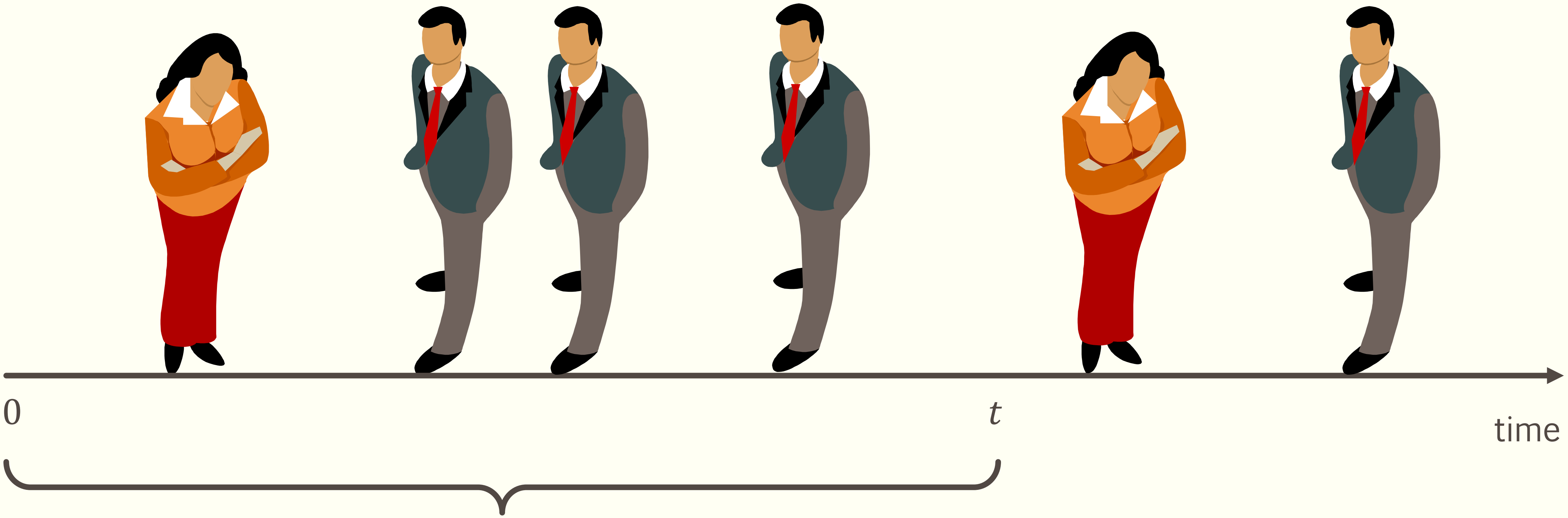
=

- By time t, how many such events have happened? The number of events are Poisson-distributed as

$$P(N = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

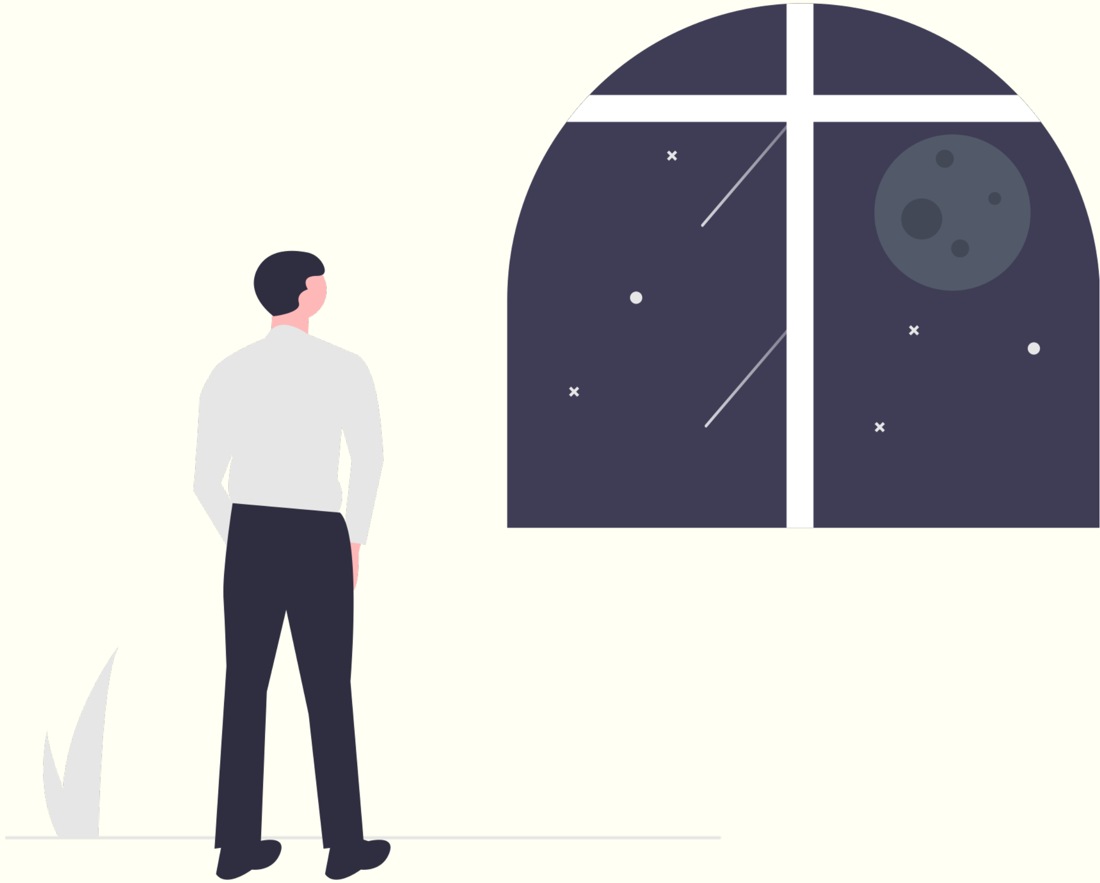
Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$



EXAMPLE

The probability of seeing a shooting star in 1 hour is 91%. What is the probability of seeing a shooting star in 30 minutes?



Poisson distribution

- By time t, how many such events have happened? The number of events are Poisson-distributed as

$$P(N = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$t \; = \; 1$ hour
$P(X \geq 1) = 1 - e^{-\lambda t} = 91\%$

$t \; = \; 30$ minutes
$P(X \geq 1) = 1 - e^{-\lambda t} = \cdots$

$t \;=\;$ 1 hour	
$P(X \geq 1) = 1 - e^{-\lambda t} = 91\%$	$e^{-\lambda t} = 9\% = 0.09$

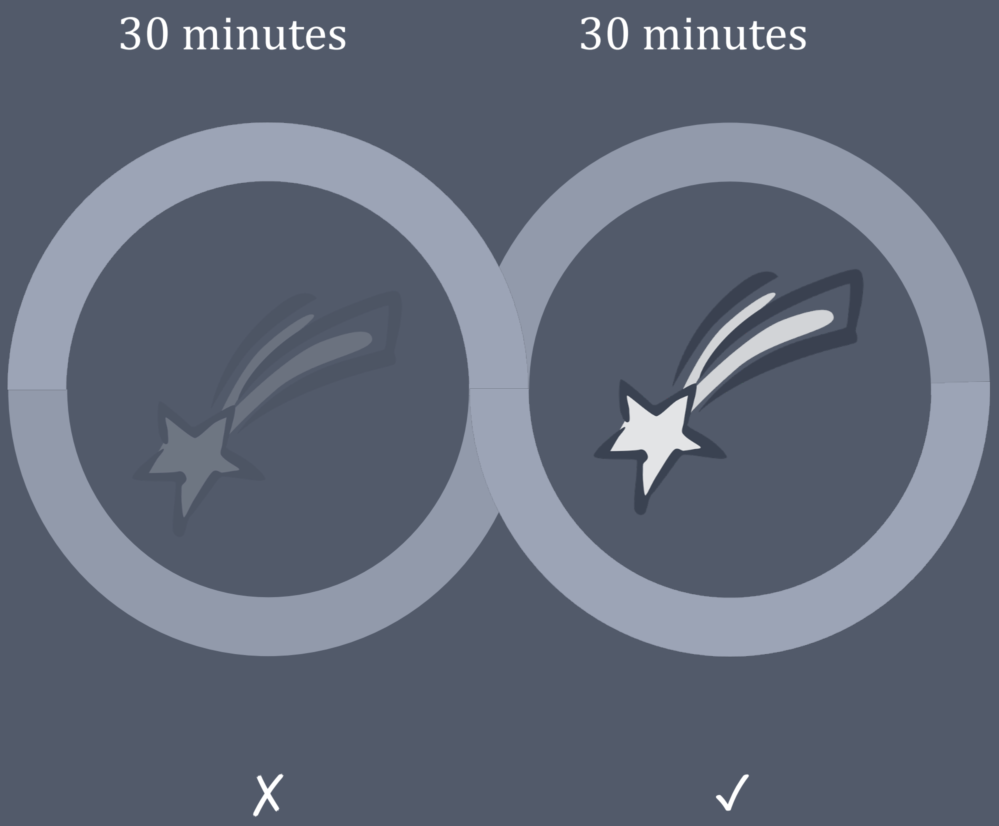
$t \;=\;$ 30 minutes	
$P(X \geq 1) = 1 - e^{-\lambda t} = 70\%$	$e^{-\lambda t} = 0.3$

Binomial distribution

- success: a shooting star
- failure: no shooting star
- probability of a success: p

$t = 1$ hour
$P(X = 0) = (1 - p)^2 = 0.09$

$t = 30$ minutes
$P(X = 0) = 1 - p = 0.3$

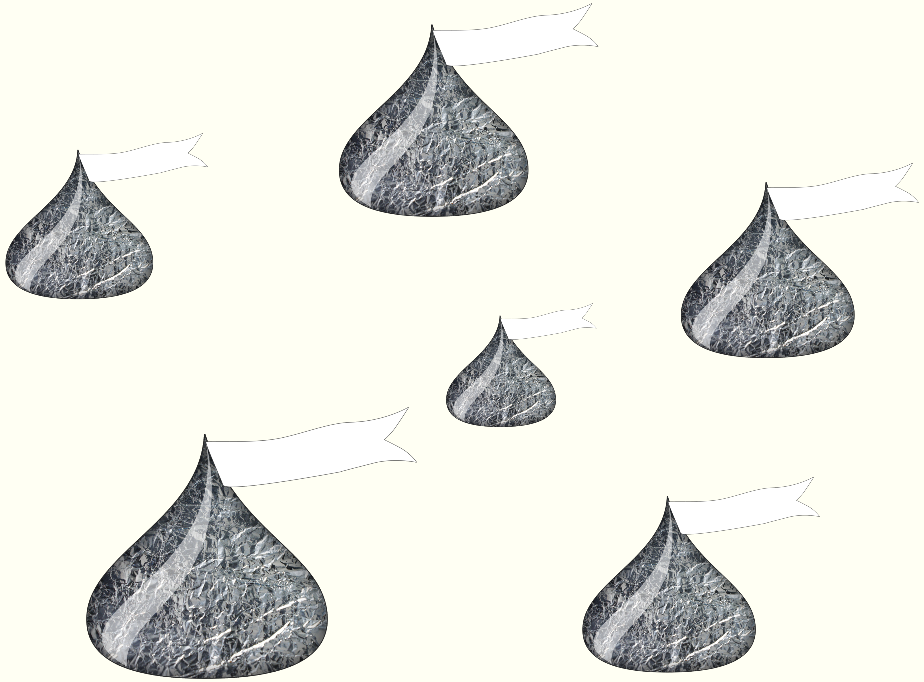


NORMAL DISTRIBUTION

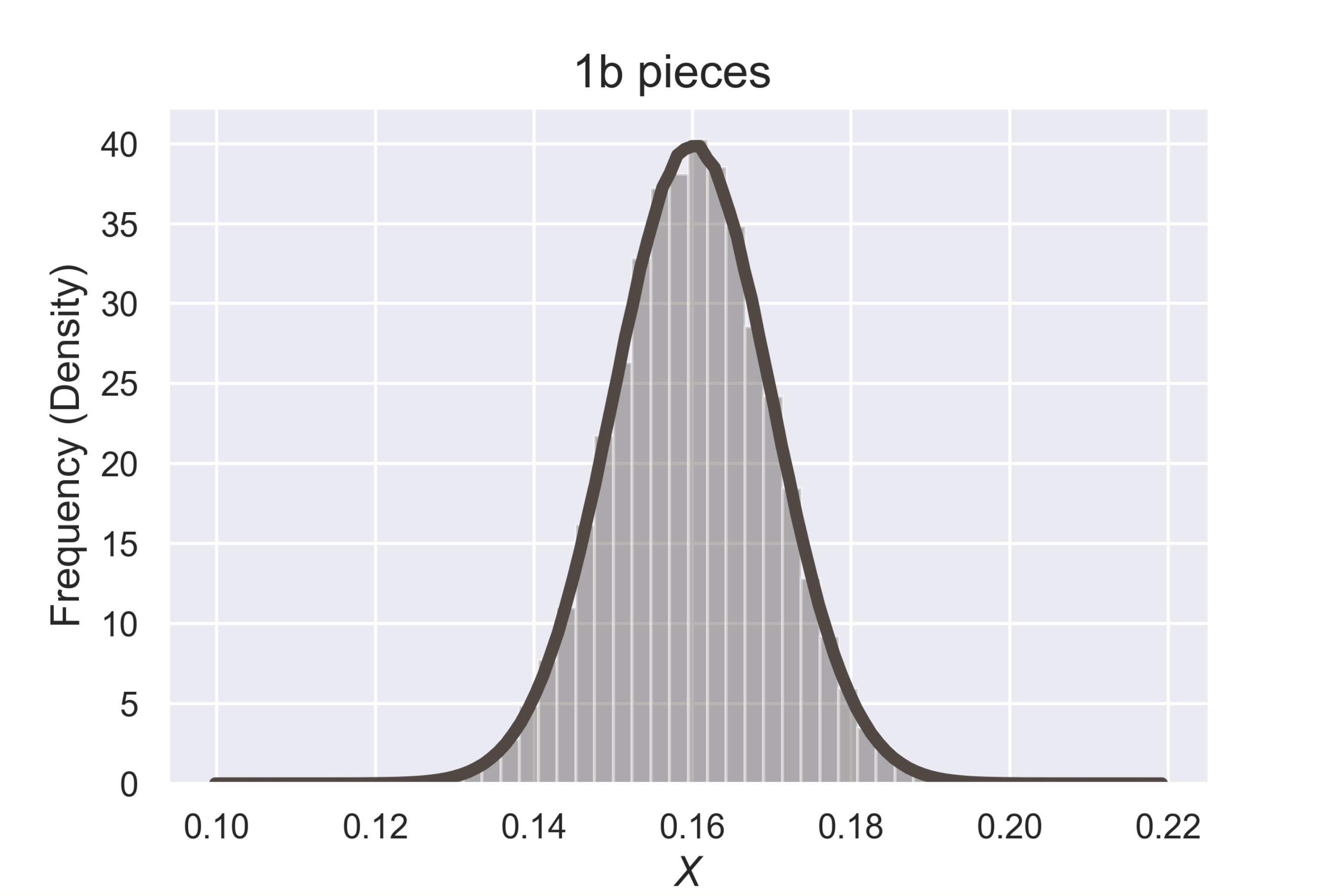
- expectation: μ
- standard deviation: σ

How Much Does a Hershey Kiss Weight?

- A single standard Hershey's Kiss weighs 0.16 ounces.



Normal Density Distribution (Gaussian Distribution)



- Three-sigma limits is a statistical calculation that refers to data within three standard deviations from a mean.
- In business applications, three-sigma refers to processes that operate efficiently and produce items of the highest quality.
- Three-sigma limits are used to set the upper and lower control limits in statistical quality control charts.
- Control charts are used to establish limits for a manufacturing or business process that is in a state of statistical control.

The normal density function with parameters μ and σ
expectation: μ standard deviation: σ

Normal Density Distribution

- The normal density function with parameters μ and σ:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

- Its cumulative distribution

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/2\sigma^2} du$$

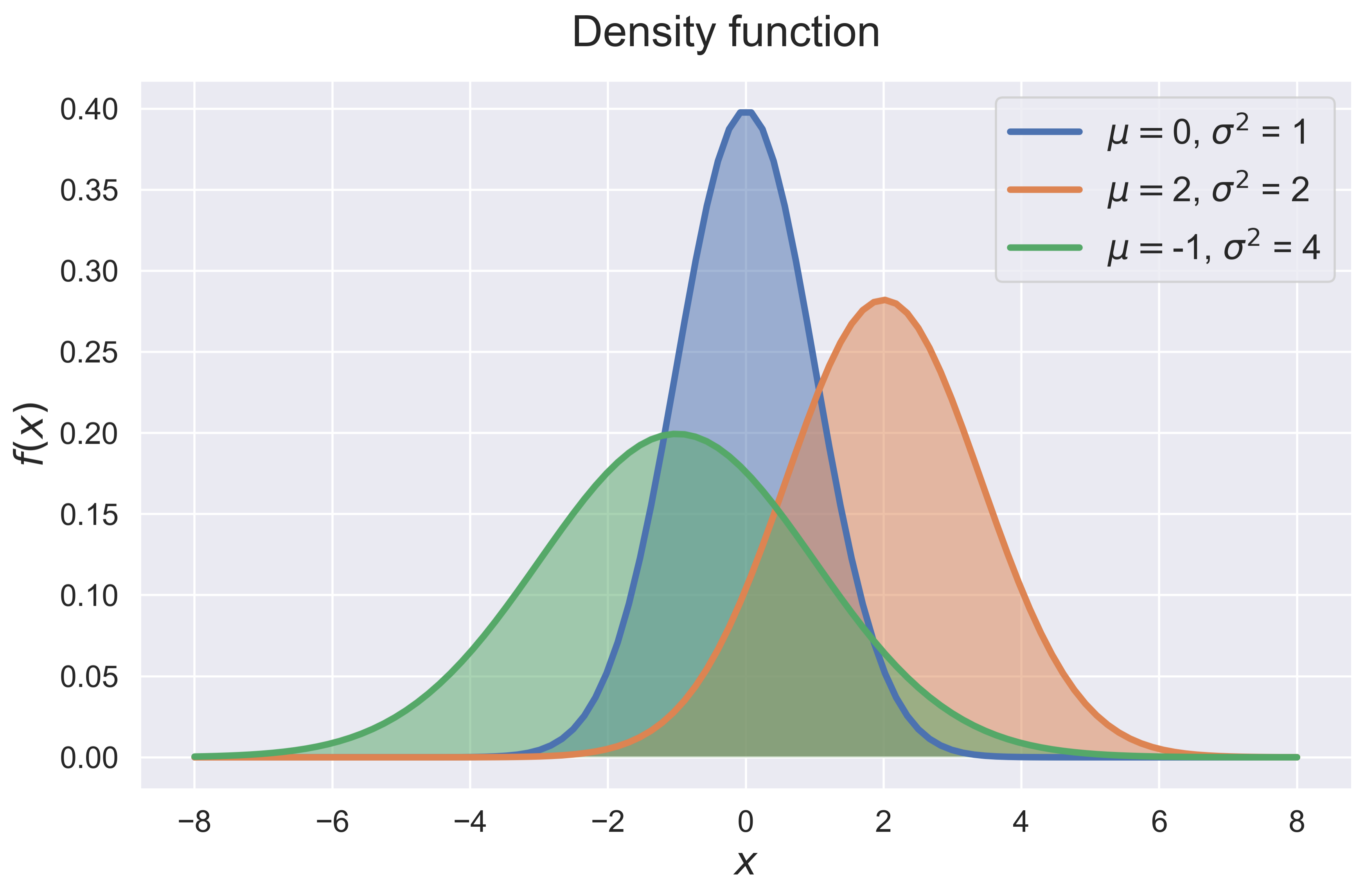
- The standard normal distribution Z is the normal density function with $\mu = 0$ and $\sigma = 1$. A general normal distribution X can be written as

$$X = \sigma Z + \mu$$

Bell Curve

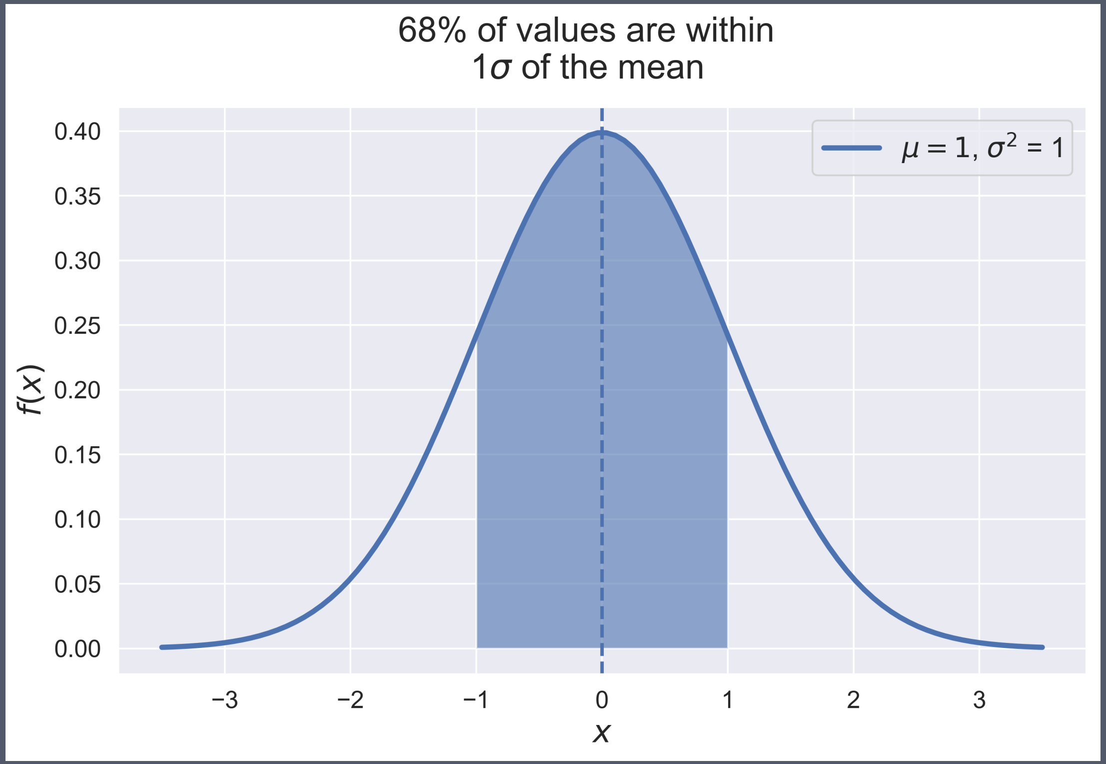
It is often called a "Bell Curve" because it looks like a bell.

Standard normal distribution Z
$\mu = 0$ and $\sigma = 1$
General normal distribution X
$X = \sigma Z + \mu$



Standard Deviation

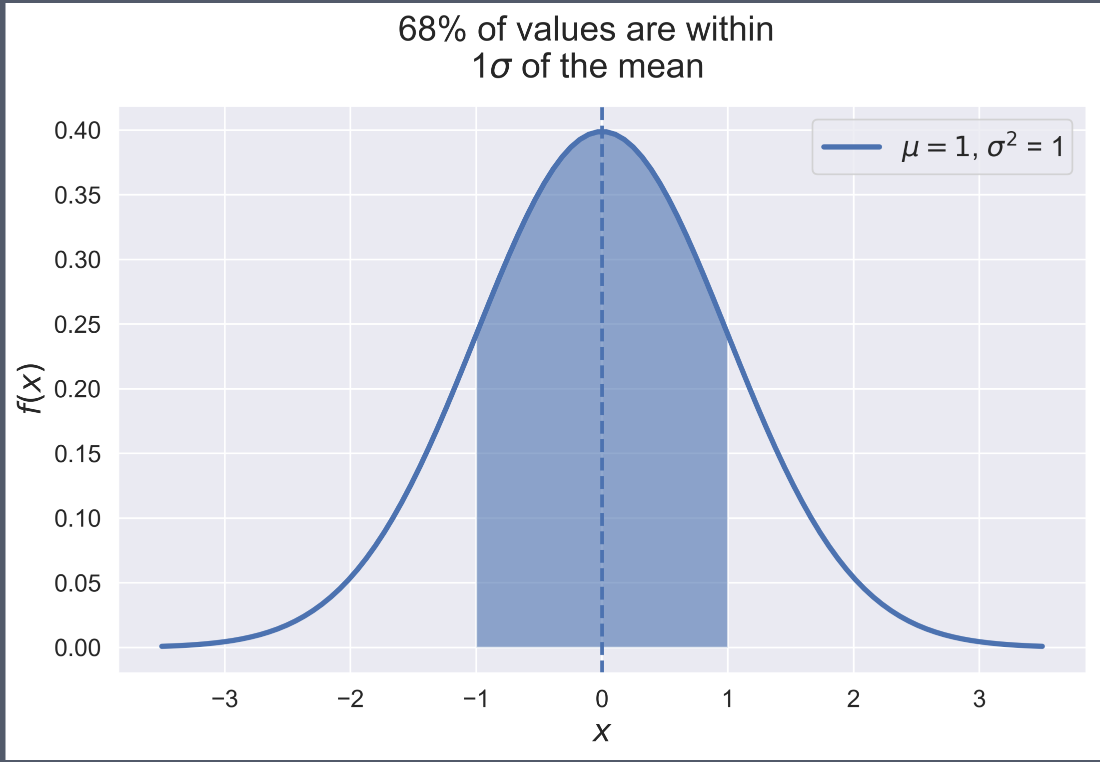
The standard deviation is a measure of how spread out numbers are.

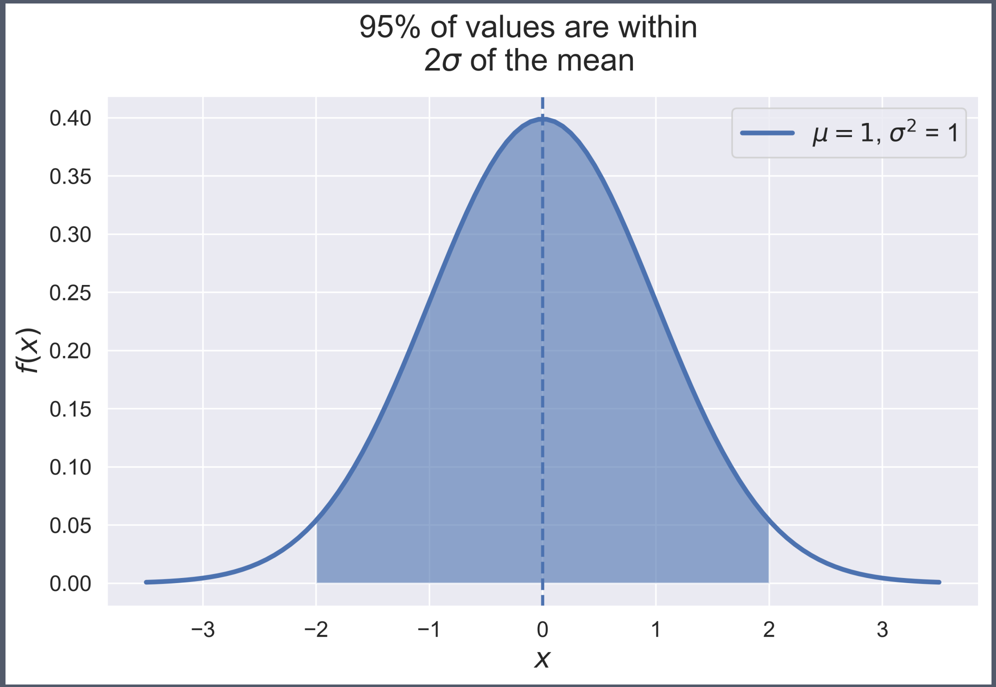


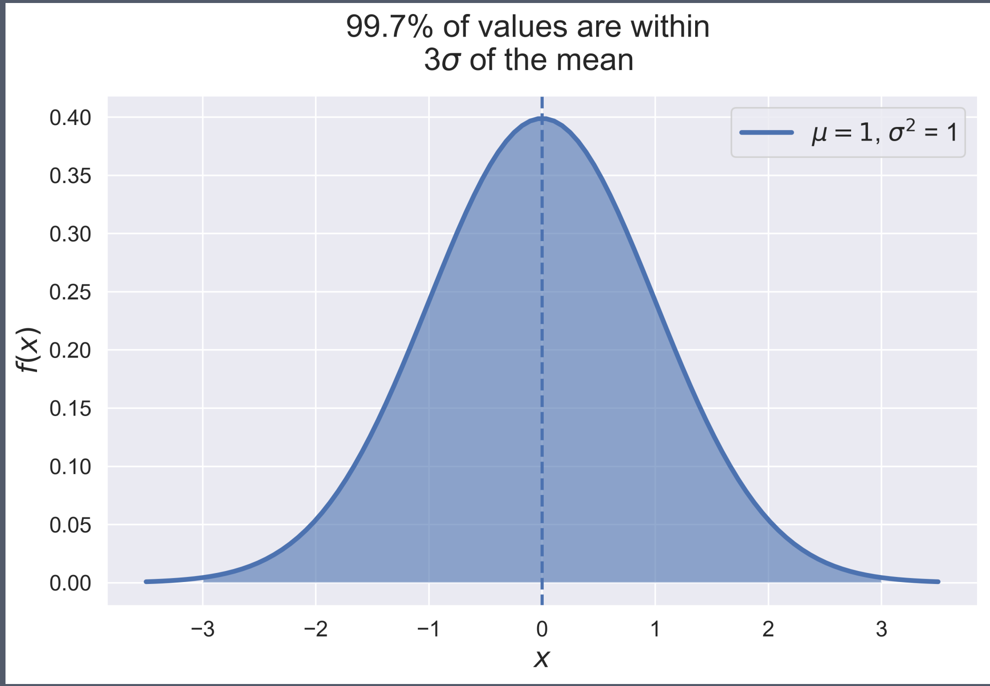
Standard normal distribution Z
$\mu = 0$ and $\sigma = 1$
$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$
within 1 σ of the mean
$P(\lvert Z-\mu\rvert \le \sigma) = P(\lvert Z\rvert \le 1) = \int_{\mu-\sigma}^{\mu+\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/2\sigma^2} du = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$

Standard Deviation

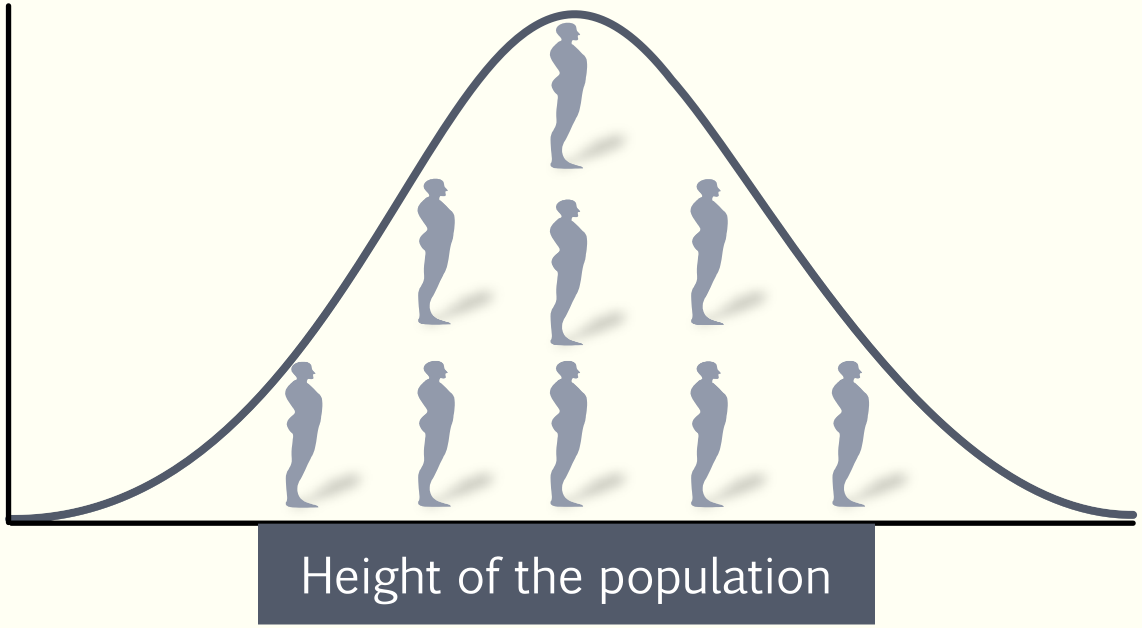
The standard deviation is a measure of how spread out numbers are.

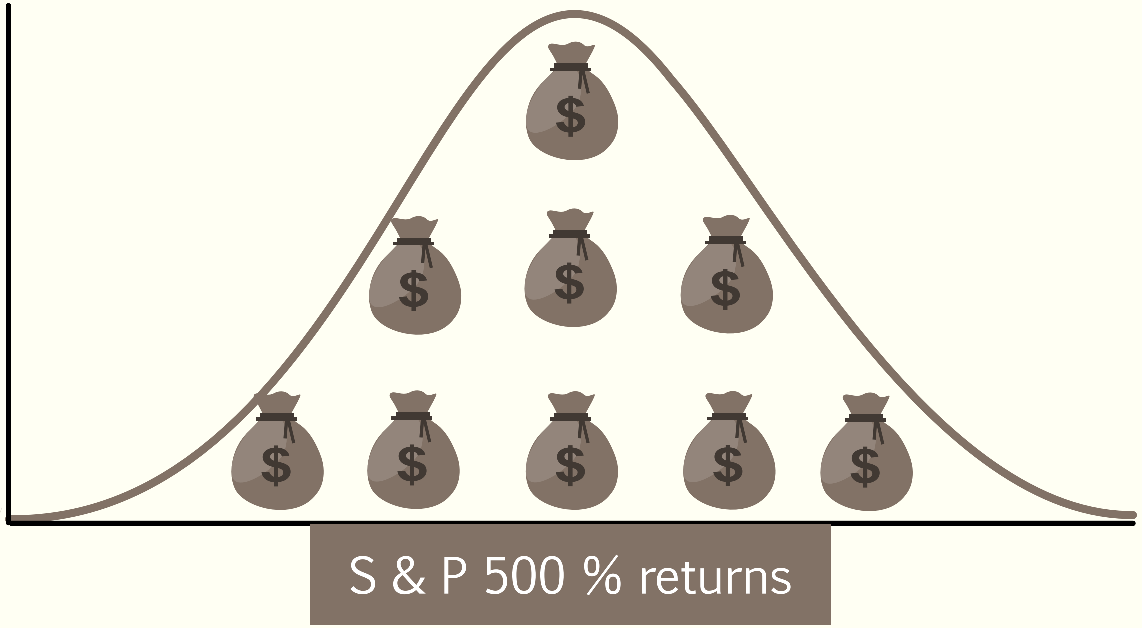




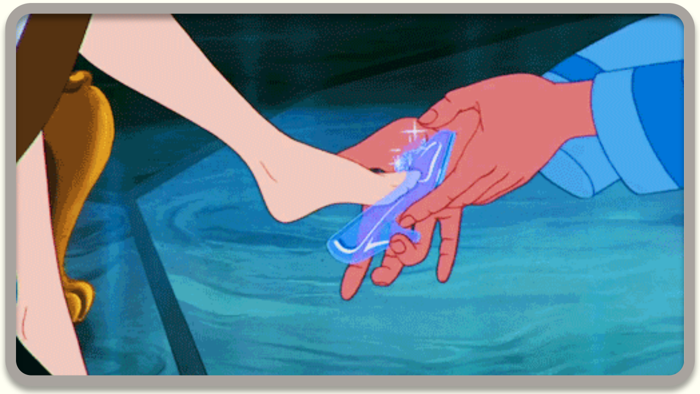


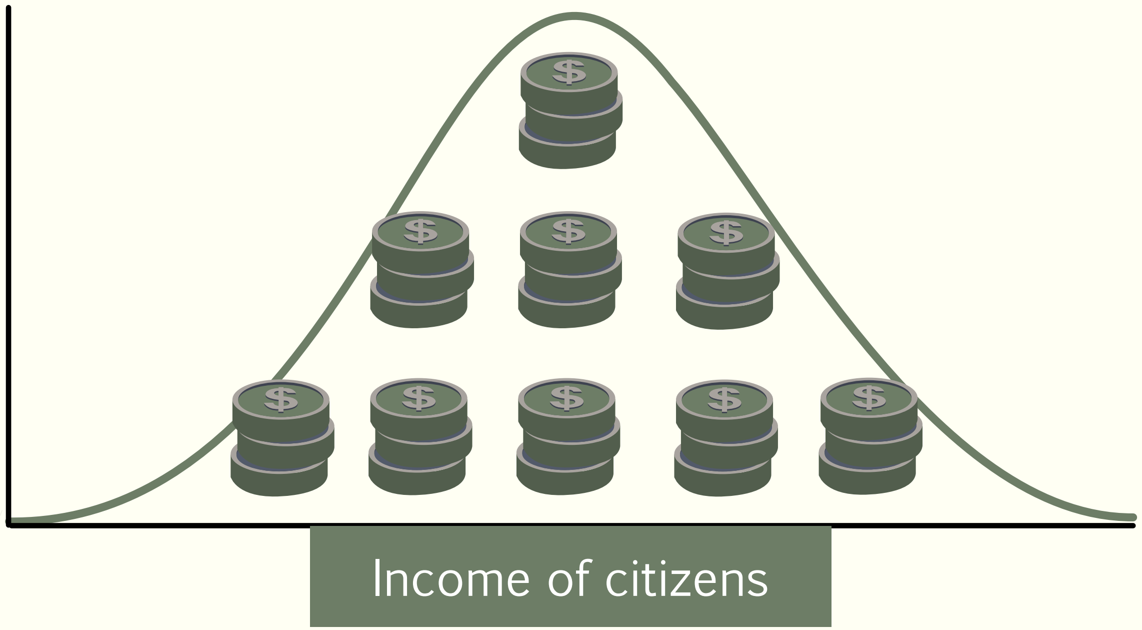
Examples of Normal Distribution

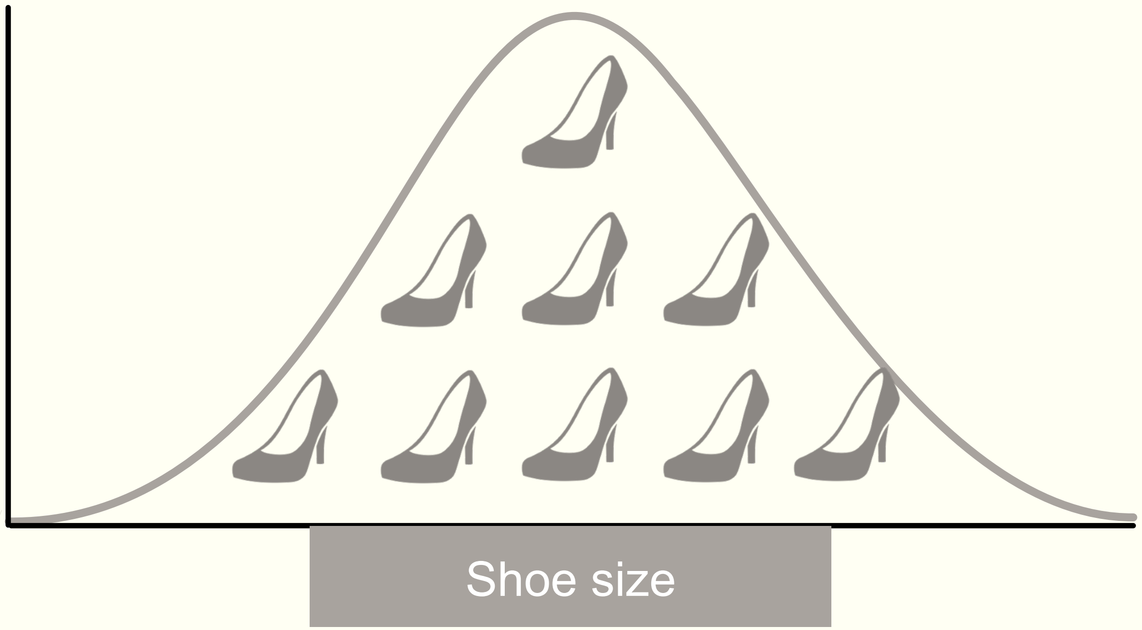




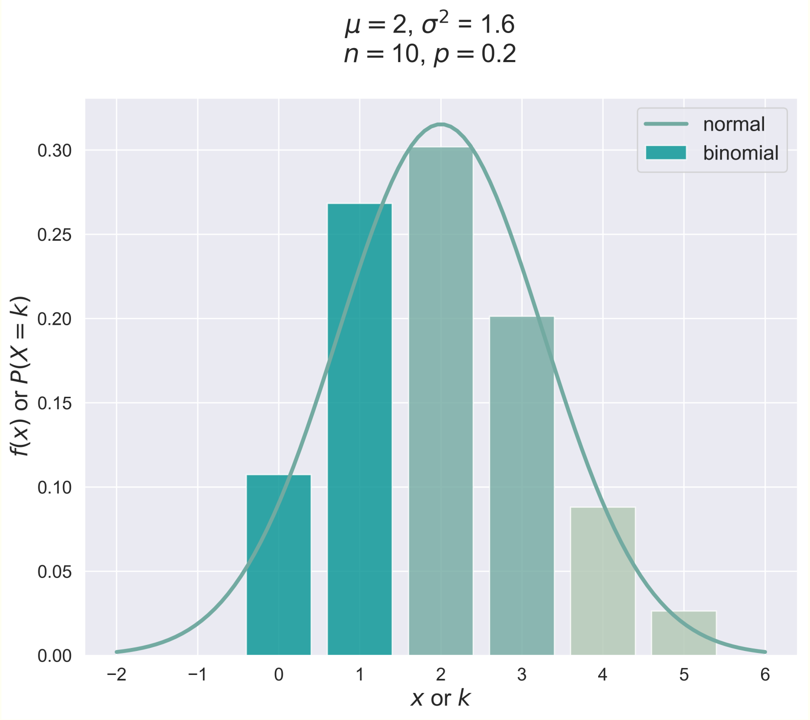
Examples of Normal Distribution

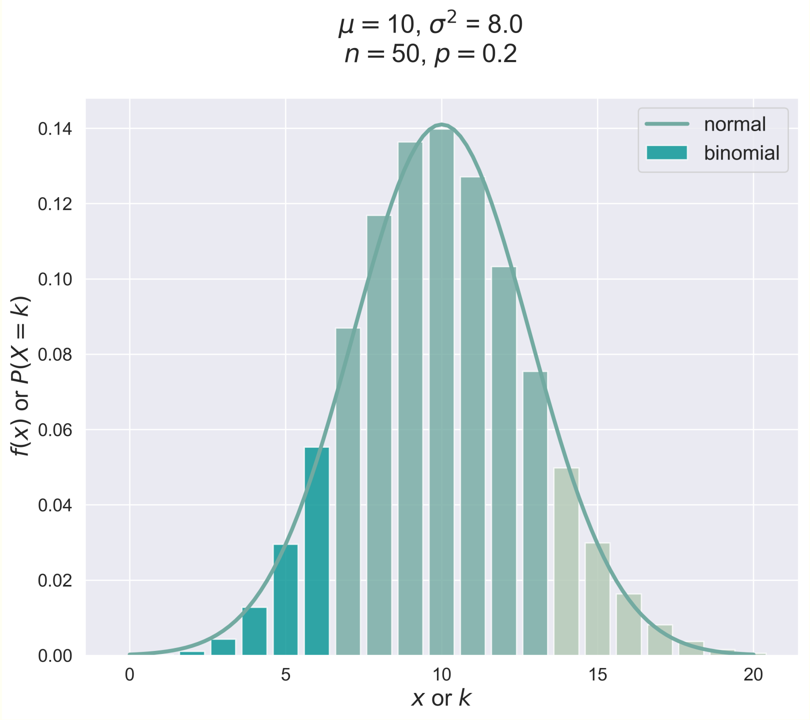


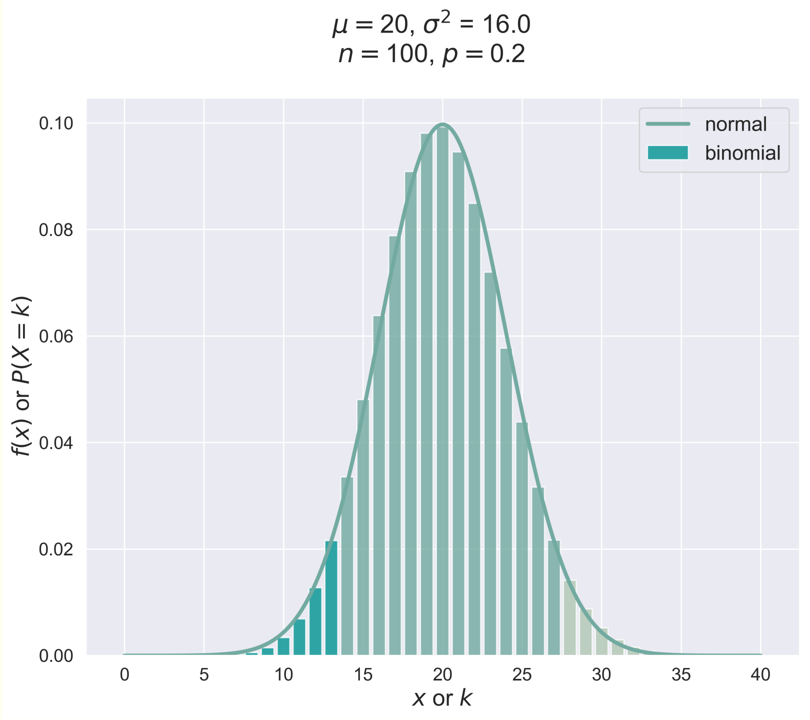




Binomial Distribution and Normal Distribution







Binomial Distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

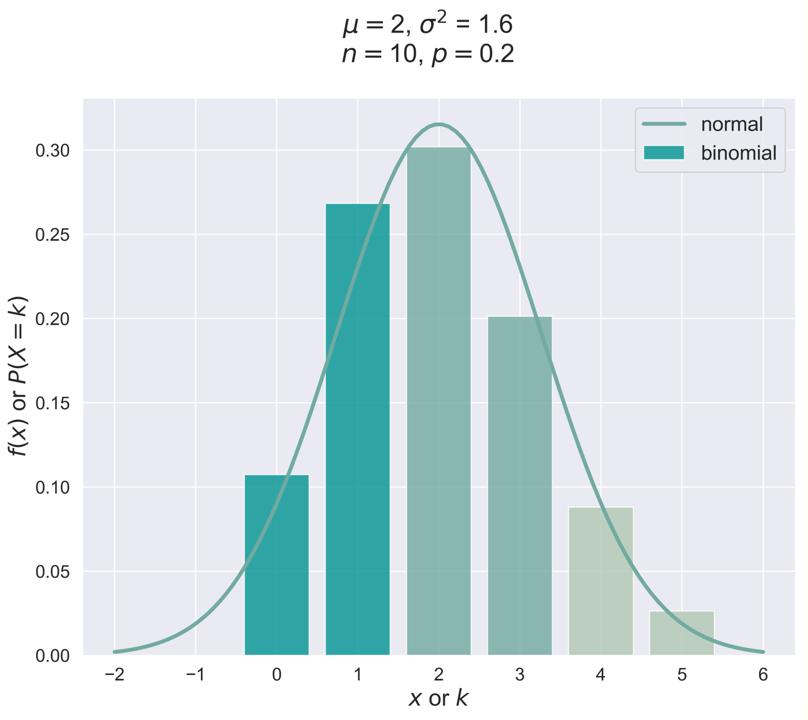
Normal Distribution

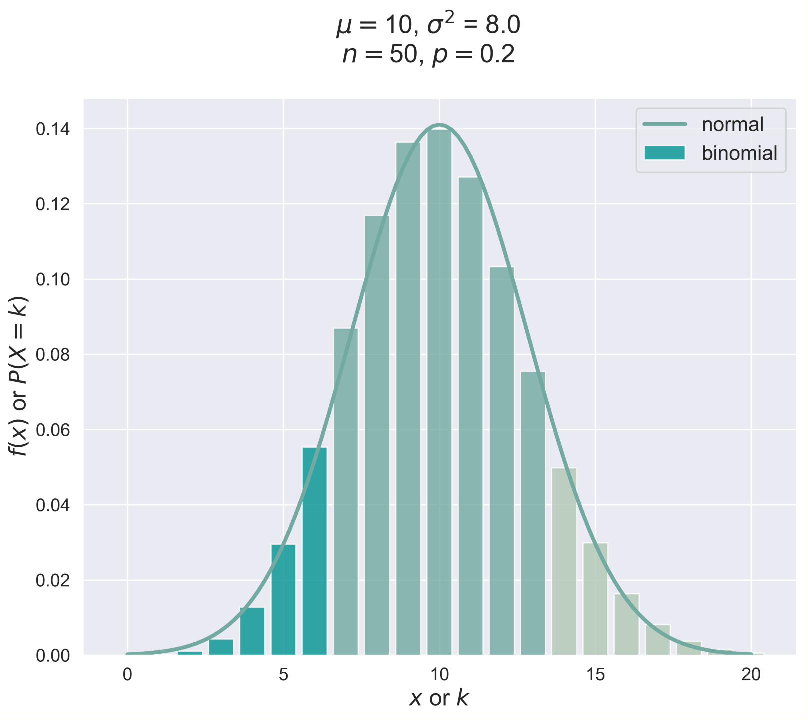
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

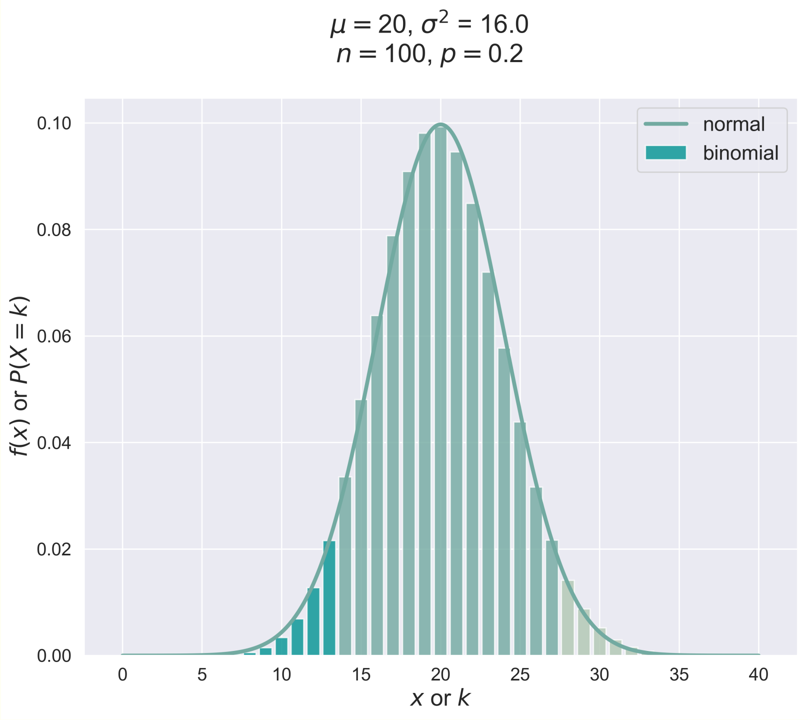
$n \to \infty$

- $np = \cdots$
- $np(1-p) = \cdots$

Binomial Distribution and Normal Distribution







Binomial Distribution

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$n \to \infty$

- $np = \mu$
- $np(1-p) = \sigma^2$