MATH 20: PROBABILITY

Generating Functions

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distribution

Random Variable

Password

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distribution

Random Variable

Expected Value & Variance

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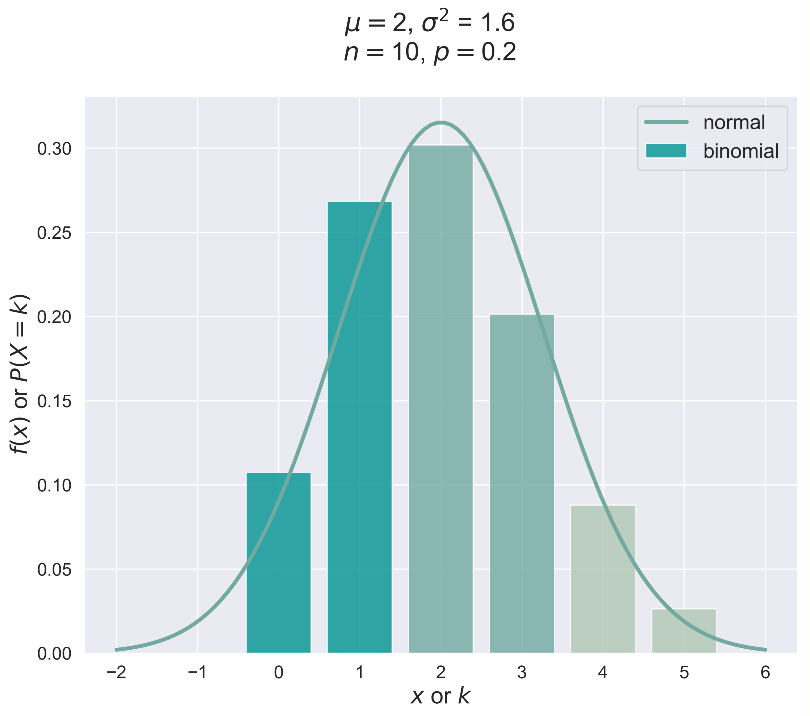
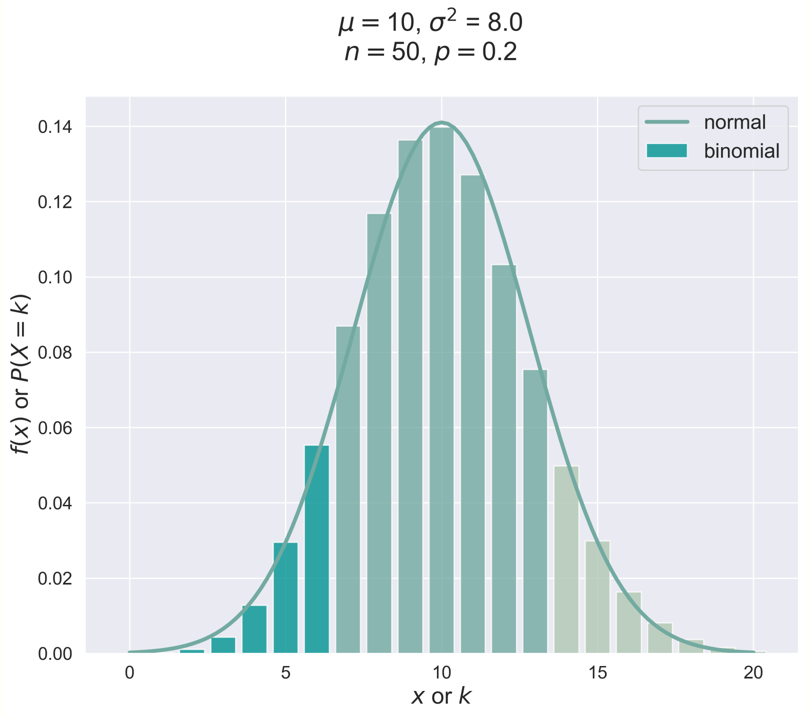
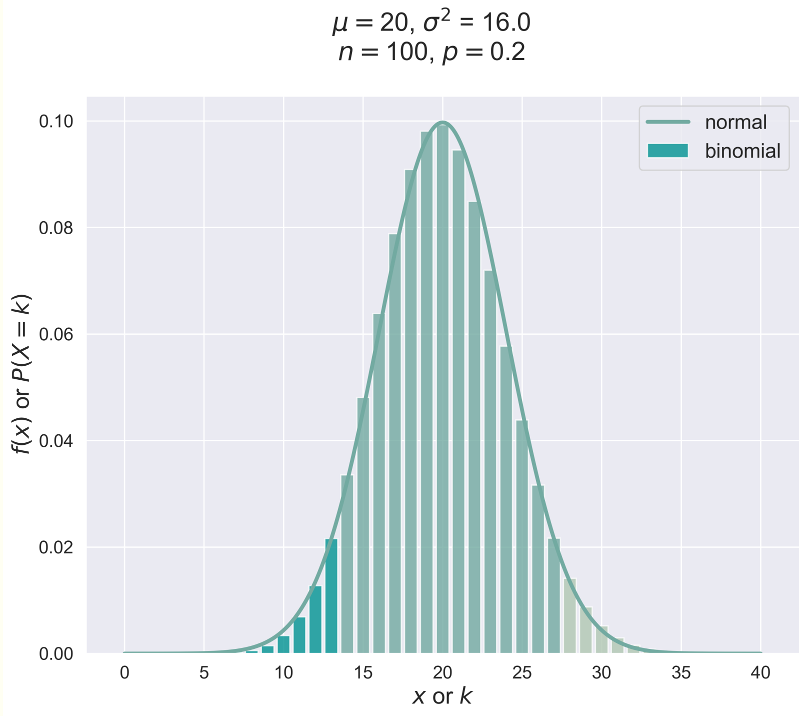
Binomial Distribution and Normal Distribution







Binomial Distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$E(X)$ & $V(X)$

- $np = \mu$
- $np(1-p) = \sigma^2$

distribution

Random Variable

Moments

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GENERATING FUNCTIONS

discrete distribution

Moments

- If X is a random variable with range $\{x_1, x_2, \cdots\}$ of at most countable size, and the distribution function $p = p_X$, we introduce the moments of X, which are numbers defined as follows:

$$\mu_k = k\text{th moment of } X$$

$$= E(X^k)$$

$$= \sum_{j=1}^{+\infty} (x_j)^k p(x_j),$$

provided the sum converges. Here $p(x_j) = P(X = x_j)$.

$\mu_0 = \cdots$

?

kth moment of X
$\mu_k = E(X^k) = \sum_{j=1}^{+\infty} (x_j)^k p(x_j)$
$\mu_0 = E(1) = \sum_{j=1}^{+\infty} p(x_j)$
$\mu_1 = E(X) = \sum_{j=1}^{+\infty} x_j p(x_j)$
$\mu_2 = E(X^2) = \sum_{j=1}^{+\infty} (x_j)^2 p(x_j)$

$$\mu_0 = 1$$

=

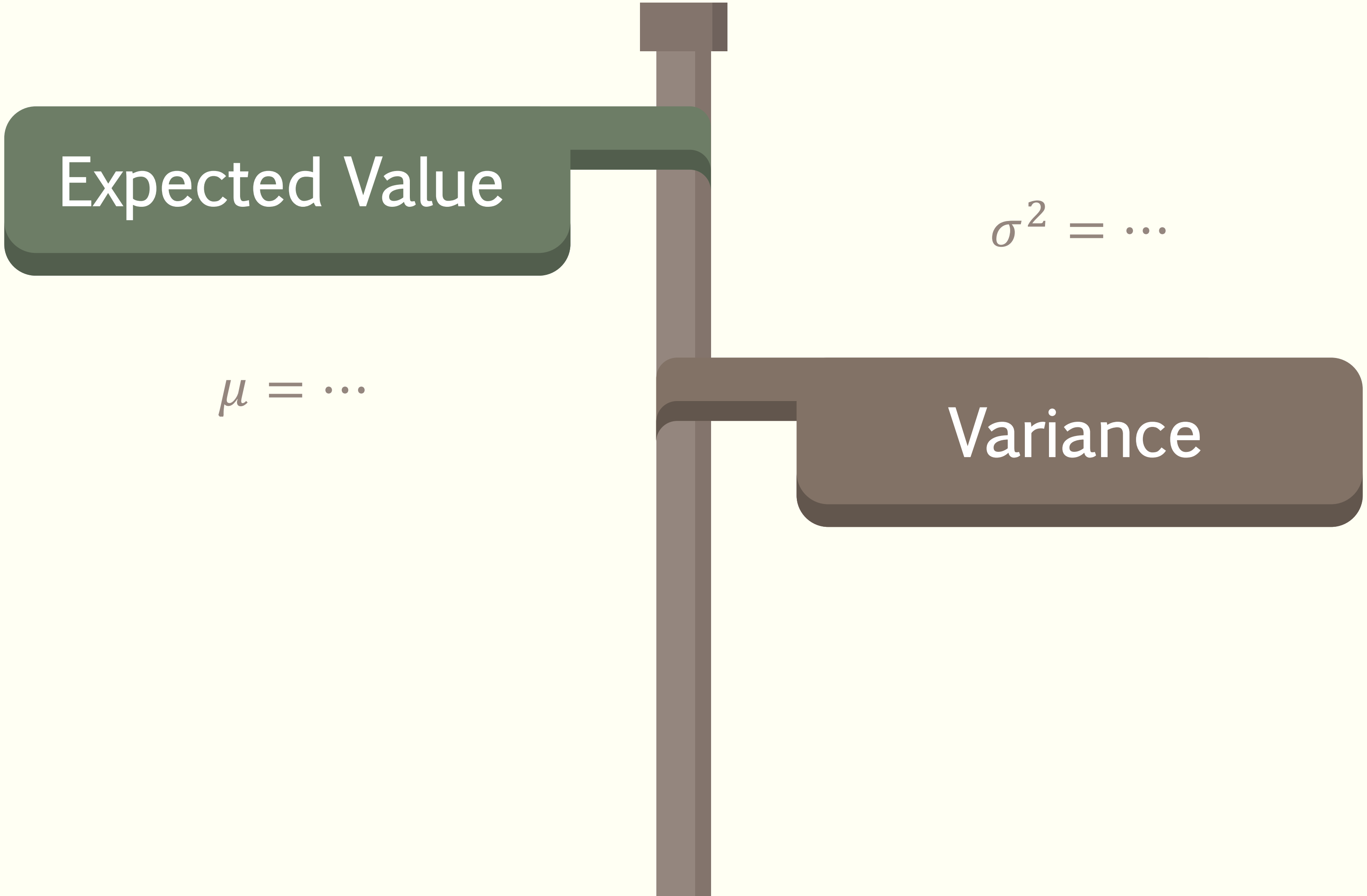
$$\mu_1 = \cdots$$

?

$$\mu_2 = \cdots$$

?

Expected Value & Variance



$\sigma^2 = \cdots$

$\mu = \cdots$

kth moment of X
$\mu_k = E\left(X^k\right) = \sum_{j=1}^{+\infty} \left(x_j\right)^k p(x_j)$
$\mu_1 = E(X) = \sum_{j=1}^{+\infty} x_j p(x_j)$
$\mu_2 = E(X^2) = \sum_{j=1}^{+\infty} \left(x_j\right)^2 p(x_j)$

Expected Value & Variance

Expected Value

$$\mu = \mu_1$$

Variance

$$\sigma^2 = \mu_2 - \mu_1^2$$

kth moment of X
$\mu_k = E\left(X^k\right) = \sum_{j=1}^{+\infty} \left(x_j\right)^k p(x_j)$
$\mu_1 = E(X) = \sum_{j=1}^{+\infty} x_j p(x_j)$
$\mu_2 = E(X^2) = \sum_{j=1}^{+\infty} \left(x_j\right)^2 p(x_j)$

Moment Generating Functions

- We introduce a new variable $t,$ and define a function $g(t)$ as follows:

$$g(t) = E(e^{tX}) = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j).$$

- We call $g(t)$ the moment generating function for X, and think of it as a convenient bookkeeping device for describing the moments of X.

Expected value $E(\phi(X))$

$$\sum_{x \in \Omega} \phi(x) m(x)$$

$$g(t) = E(e^{tX}) = E\left(\sum_{k=0}^{+\infty} \frac{X^k t^k}{k!}\right) = \sum_{k=0}^{+\infty} \frac{E(X^k) t^k}{k!} = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

=

Taylor Expansion

Moment Generating Functions

- If we differentiate $g(t)$ n times and then set $t = 0$, we get μ_n.

$$g(t) = E(e^{tX}) = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

=

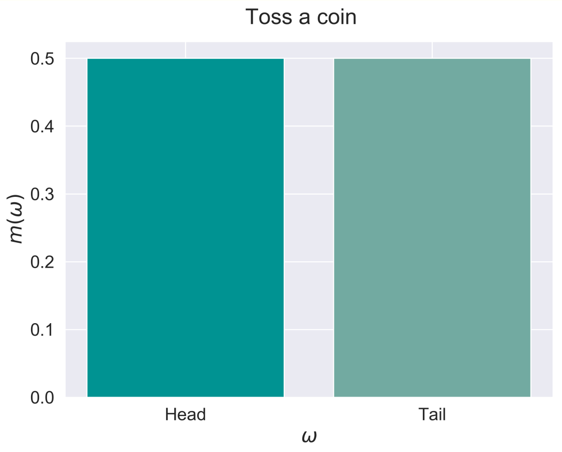
$$\frac{d^n}{dt^n} g(t) = \frac{d^n}{dt^n} \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = \sum_{k=n}^{+\infty} \frac{k!\, \mu_k t^{k-n}}{k!\,(k-n)!} = \sum_{k=n}^{+\infty} \frac{\mu_k t^{k-n}}{(k-n)!}$$

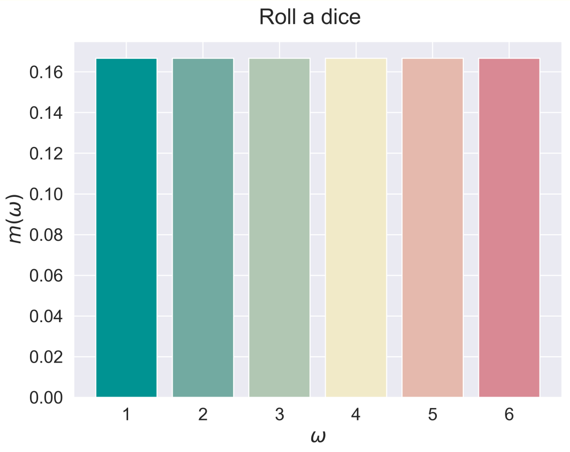
$$\frac{d^n}{dt^n} g(t)|_{t=0} = \sum_{k=n}^{+\infty} \frac{\mu_k t^{k-n}}{(k-n)!}|_{t=0} = \mu_n$$

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

=

Uniform Distribution





$$g(t) = E(e^{tX}) = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

Random variable

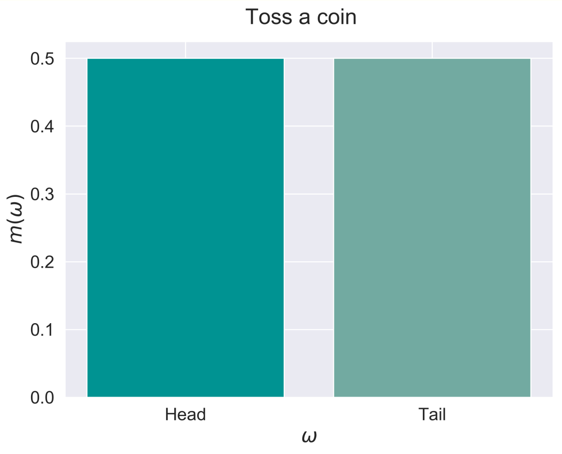
range: $\{1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = \frac{1}{n}$

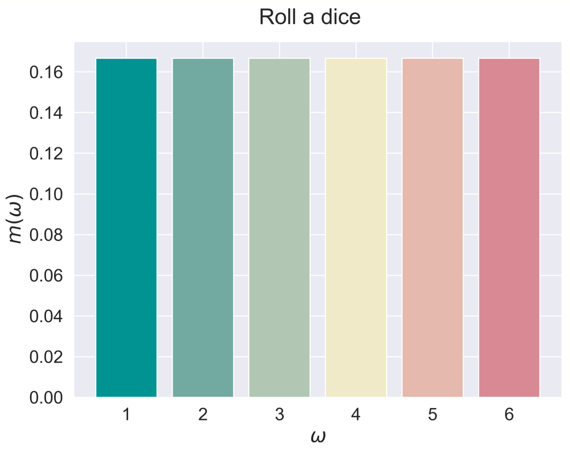
Generating function

$$g(t) = \sum_{j=1}^{n} \frac{1}{n} e^{tj} = \frac{1}{n}(e^t + e^{2t} + e^{3t} + \cdots + e^{nt})$$

$$= \frac{e^t(e^{nt}-1)}{n(e^t-1)}.$$

Uniform Distribution





$$g(t) = E(e^{tX}) = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

=

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

=

Random variable

range: $\{1, 2, 3, \cdots, n\}$

distribution function: $p_X(j) = \frac{1}{n}$

Generating function

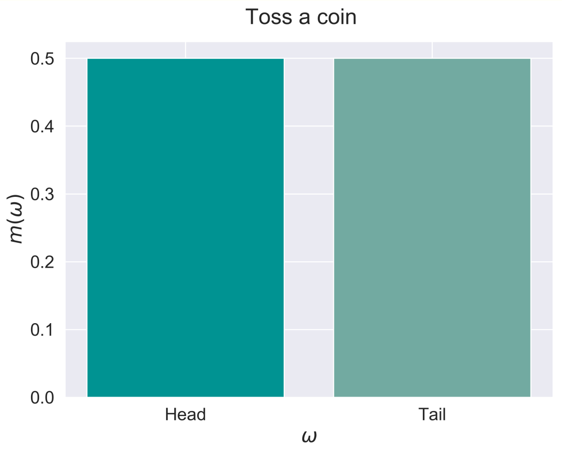
$$g(t) = \sum_{j=1}^{n} \frac{1}{n} e^{tj} = \frac{e^t(e^{nt}-1)}{n(e^t-1)}.$$

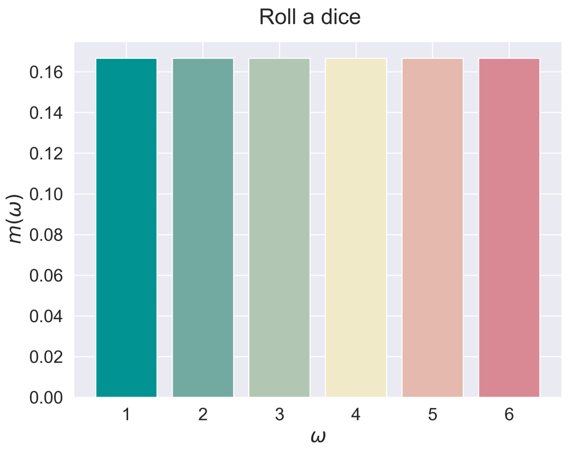
Moments

$$\mu_1 = g'(0) = \frac{1}{n}(1 + 2 + 3 + \cdots + n) = \frac{n+1}{2}.$$

$$\mu_2 = g''(0) = \frac{1}{n}(1 + 4 + 9 + \cdots + n^2) = \frac{(n+1)(2n+1)}{6}.$$

Uniform Distribution





$$\mu = \mu_1$$

$$\sigma^2 = \mu_2 - \mu_1^2$$

Random variable

range: $\{1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = \frac{1}{n}$

Generating function

$$g(t) = \sum_{j=1}^{n} \frac{1}{n} e^{tj} = \frac{e^t(e^{nt}-1)}{n(e^t-1)}.$$

Moments

$$\mu_1 = g'(0) = \frac{1}{n}(1 + 2 + 3 + \cdots + n) = \frac{n+1}{2}.$$

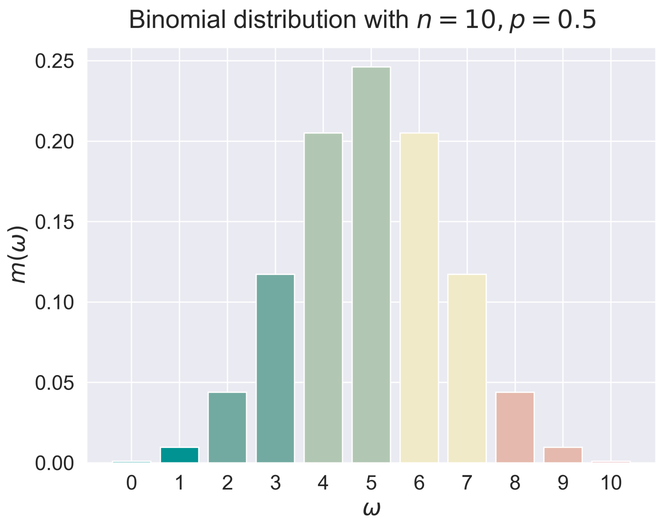
$$\mu_2 = g''(0) = \frac{1}{n}(1 + 4 + 9 + \cdots + n^2) = \frac{(n+1)(2n+1)}{6}.$$

Expected value & variance

$$\mu = \mu_1 = \frac{n+1}{2}.$$

$$\sigma^2 = \mu_2 - \mu_1^2 = \frac{n^2-1}{12}.$$

Binomial Distribution



$$g(t) = E(e^{tX}) = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

Random variable

range: $\{0, 1, 2, 3, \cdots, n\}$

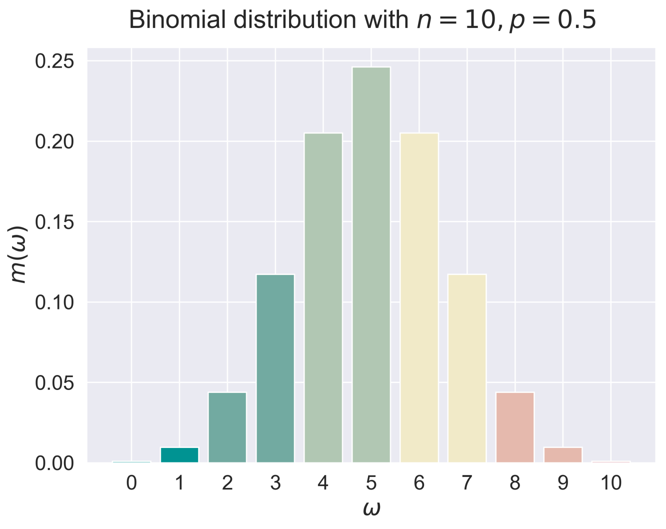
distribution function: $p_X(j) = \binom{n}{j} p^j q^{n-j}$

Generating function

$$g(t) = \sum_{j=1}^{n} e^{tj} \binom{n}{j} p^j q^{n-j} =$$

$$\sum_{j=1}^{n} \binom{n}{j} (pe^t)^j q^{n-j} = (pe^t + q)^n.$$

Binomial Distribution



$$g(t) = E(e^{tX}) = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

Random variable

range: $\{0, 1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = \binom{n}{j} p^j q^{n-j}$

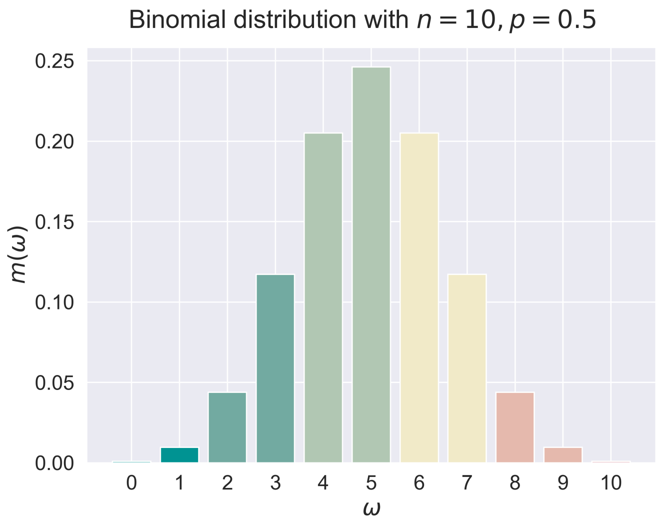
Generating function

$$g(t) = \sum_{j=1}^{n} e^{tj} \binom{n}{j} p^j q^{n-j} = (pe^t + q)^n.$$

Moments

$$\mu_1 = g'(0) = n(pe^t + q)^{n-1} pe^t|_{t=0} = np.$$
$$\mu_2 = g''(0) = n(n-1)p^2 + np.$$

Binomial Distribution



$$\mu = \mu_1$$

$$\sigma^2 = \mu_2 - \mu_1^2$$

Random variable

range: $\{0, 1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = \binom{n}{j} p^j q^{n-j}$

Generating function

$$g(t) = \sum_{j=1}^{n} e^{tj} \binom{n}{j} p^j q^{n-j} = (pe^t + q)^n.$$

Moments

$\mu_1 = g'(0) = np.$
$\mu_2 = g''(0) = n(n-1)p^2 + np.$

Expected value & variance

$\mu = \mu_1 = np.$
$\sigma^2 = \mu_2 - \mu_1^2 = np(1-p).$

Geometric Distribution

$$g(t) = E(e^{tX}) = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

=

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

=

$$\mu = \mu_1$$

$$\sigma^2 = \mu_2 - \mu_1^2$$

Random variable

range: $\{1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = q^{j-1}p$

Generating function

$g(t) = \sum_{j=1}^{n} e^{tj} q^{j-1} p = \frac{pe^t}{1-qe^t}.$

Moments

$\mu_1 = g'(0) = \frac{pe^t}{(1-qe^t)^2}|_{t=0} = \frac{1}{p}.$

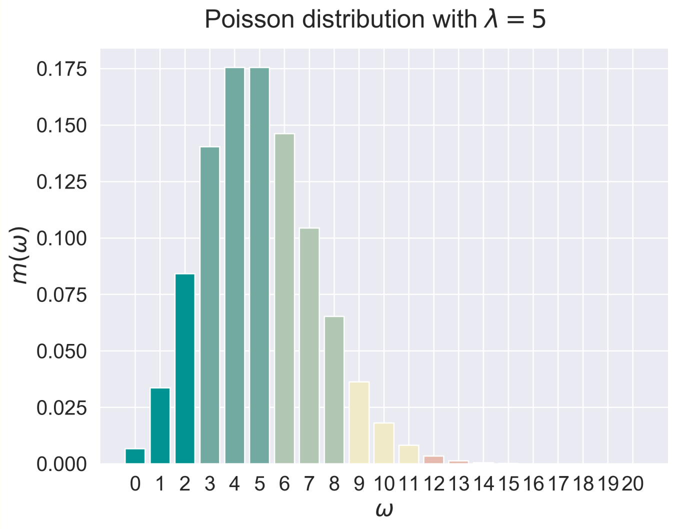
$\mu_2 = g''(0) = \frac{pe^t+pqe^{2t}}{(1-qe^t)^3}|_{t=0} = \frac{1+q}{p^2}.$

Expected value & variance

$\mu = \mu_1 = \frac{1}{p}.$

$\sigma^2 = \mu_2 - \mu_1^2 = \frac{q}{p^2}.$

Poisson Distribution



$$g(t) = E(e^{tX}) = \boxed{\sum_{j=1}^{+\infty} e^{tx_j} p(x_j)} = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

=

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

=

Random variable

range: $\{0, 1, 2, 3, \cdots, n\}$

distribution function: $p_X(j) = e^{-\lambda} \frac{\lambda^j}{j!}$

Generating function

$$g(t) = \sum_{j=1}^{n} e^{tj} e^{-\lambda} \frac{\lambda^j}{j!} = e^{-\lambda} \sum_{j=1}^{n} \frac{(\lambda e^t)^j}{j!} = e^{\lambda(e^t - 1)}.$$

Moments

$$\mu_1 = g'(0) = e^{\lambda(e^t-1)} \lambda e^t|_{t=0} = \lambda.$$

$$\mu_2 = g''(0) = e^{\lambda(e^t-1)} (\lambda^2 e^{2t} + \lambda e^t)|_{t=0} = \lambda^2 + \lambda.$$

Expected value & variance

$$\mu = \mu_1 = \lambda.$$

$$\sigma^2 = \mu_2 - \mu_1^2 = \lambda.$$

Uniform

$$P(X = k) = \frac{1}{n}$$

$$E(X) = \frac{n+1}{2},\ V(X) = \frac{n^2-1}{12}$$

Binomial

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

$$E(X) = np,\ V(X) = npq$$

Geometric

$$P(T = n) = q^{n-1}p$$

$$E(X) = \frac{1}{p},\ V(X) = \frac{1-p}{p^2}$$

Poisson

$$P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

$$E(X) = \lambda,\ V(X) = \lambda$$

Uniform

$$P(X = k) = \frac{1}{n}$$

$$g(t) = \frac{e^t(e^{nt} - 1)}{n(e^t - 1)}$$

Binomial

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

$$g(t) = (pe^t + q)^n$$

Geometric

$$P(T = n) = q^{n-1}p$$

$$g(t) = \frac{pe^t}{1 - qe^t}$$

Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$g(t) = e^{\lambda(e^t - 1)}$$

(b) Assume that X is Poisson distributed with parameter λ. Find $E(X^3)$.

5 pts

$$
\begin{aligned}
E(X^3) &= \sum_{k=0}^{+\infty} k^3 \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{+\infty} k^2 \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=1}^{+\infty} k^2 \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{k=1}^{+\infty} (k^2-1) \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} + \lambda \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{k=2}^{+\infty} (k^2-1) \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} + \lambda \sum_{l=0}^{+\infty} \frac{\lambda^{l}}{l!} e^{-\lambda} \\
&= \lambda \sum_{k=2}^{+\infty} (k+1) \frac{\lambda^{k-1}}{(k-2)!} e^{-\lambda} + \lambda = \lambda^2 \sum_{l=0}^{+\infty} (l+3) \frac{\lambda^{l}}{l!} e^{-\lambda} + \lambda \\
&= \lambda^2 \sum_{l=0}^{+\infty} l \frac{\lambda^{l}}{l!} e^{-\lambda} + 3\lambda^2 \sum_{l=0}^{+\infty} \frac{\lambda^{l}}{l!} e^{-\lambda} + \lambda \\
&= \lambda^3 + 3\lambda^2 + \lambda.
\end{aligned}
$$

Poisson

$$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$$

$$g(t)=e^{\lambda(e^t-1)}.$$

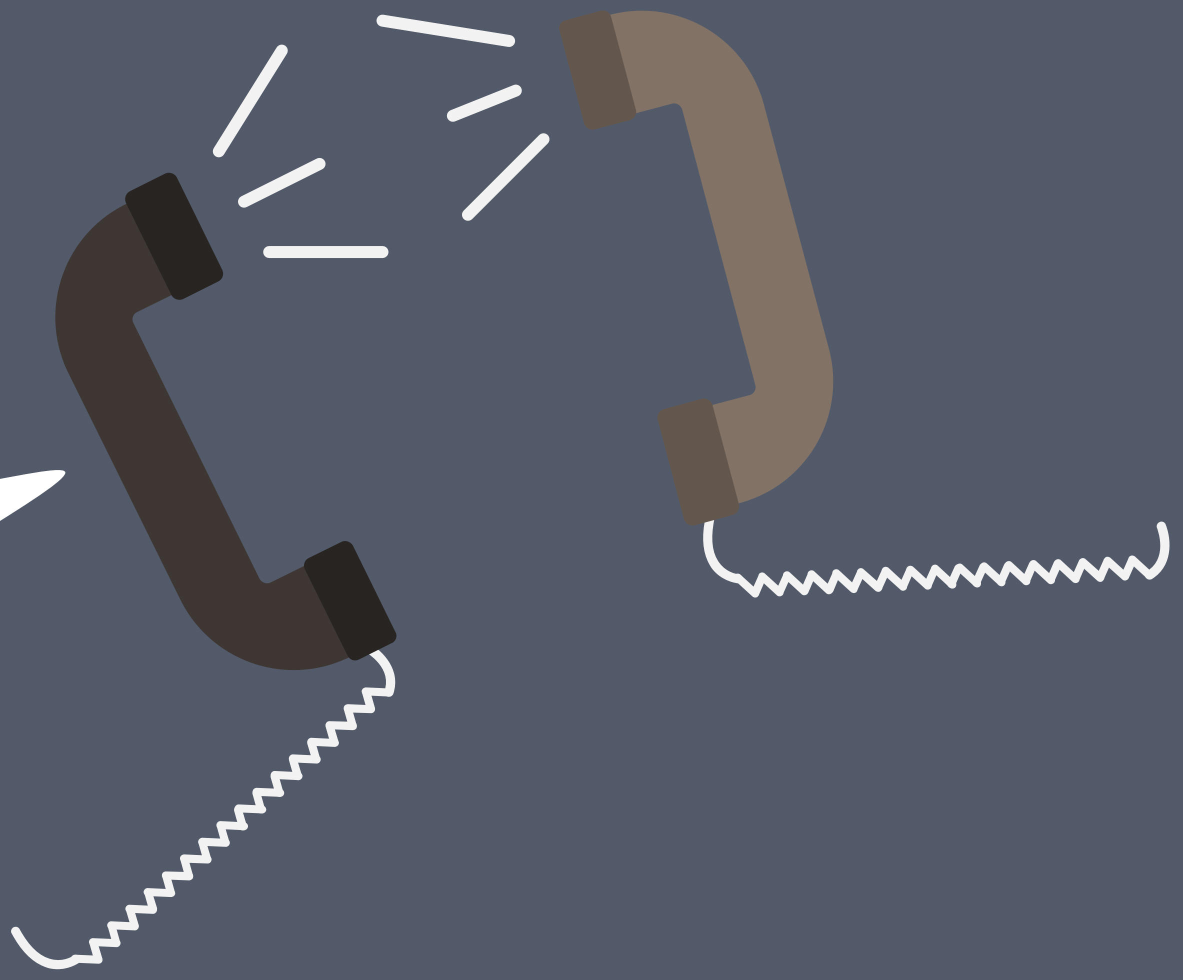
$$g^{(n)}(0)=\frac{d^n}{dt^n}g(t)|_{t=0}=\mu_n$$

=

$$E(X^3)=\mu_3=\frac{d^3}{dt^3}g(t)|_{t=0}=\frac{d^3}{dt^3}e^{\lambda(e^t-1)}|_{t=0}$$
$$=\frac{d}{dt}e^{\lambda(e^t-1)}(\lambda^2e^{2t}+\lambda e^t)|_{t=0}=e^{\lambda(e^t-1)}(\lambda^3e^{3t}+3\lambda^2e^{2t}+\lambda e^t)|_{t=0}$$
$$=\lambda^3+3\lambda^2+\lambda$$

Moment Problem

- Let X be a discrete random variable with finite range $\{x_1, x_2, \cdots, x_n\}$, distribution function p and moment generation function g. Then g is uniquely determined by p, and conversely.



Ordinary Generating Functions

- In the special but important case where the x_j are all **nonnegative integers**, $x_j = j$, we can rewrite the moment generating function in a simpler way:

$$g(t) = E(e^{tX}) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{j=1}^{+\infty} e^{tj} p(j).$$

- We see that $g(t)$ is a polynomial in e^t.
- If we write $z = e^t$, and define the function h by

$$h(z) = \sum_{j=1}^{+\infty} z^j p(j),$$

then $h(z)$ is a polynomial in z containing the same information as $g(t)$.

$$g(t) = h(e^t)$$

=

$$h(z) = g(\ln(z))$$

=

Ordinary Generating Functions

- Moment generating function:

$$g(t) = E(e^{tX}) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{j=1}^{+\infty} e^{tj} p(j).$$

- Ordinary generating function:

$$h(z) = \sum_{j=1}^{+\infty} z^j p(j).$$

$$z = e^t$$

$$g(t) = h(e^t)$$

$$h(z) = g(\ln(z))$$

Ordinary Generating Functions

- Moment generating function:

$$g(t) = E(e^{tX}) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{j=1}^{+\infty} e^{tj} p(j).$$

- Ordinary generating function:

$$h(z) = \sum_{j=1}^{+\infty} z^j p(j).$$

$$h(1) = \sum_{j=1}^{+\infty} p(j) = 1$$

$$h'(1) = \sum_{j=1}^{+\infty} jp(j)$$

$$h''(1) = \sum_{j=1}^{+\infty} j(j-1)p(j) = \sum_{j=1}^{+\infty} j^2 p(j) - \sum_{j=1}^{+\infty} jp(j)$$

$$h(1) = g(0) = 1$$

$$h'(1) = g'(0) = \mu_1$$

$$h''^{(1)} = g''^{(0)} - g'(0) = \mu_2 - \mu_1$$

Ordinary Generating Functions

- Moment generating function:

$$g(t) = E(e^{tX}) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = \sum_{j=1}^{+\infty} e^{tx_j} p(x_j) = \sum_{j=1}^{+\infty} e^{tj} p(j).$$

- Ordinary generating function:

$$h(z) = \sum_{j=1}^{+\infty} z^j p(j).$$

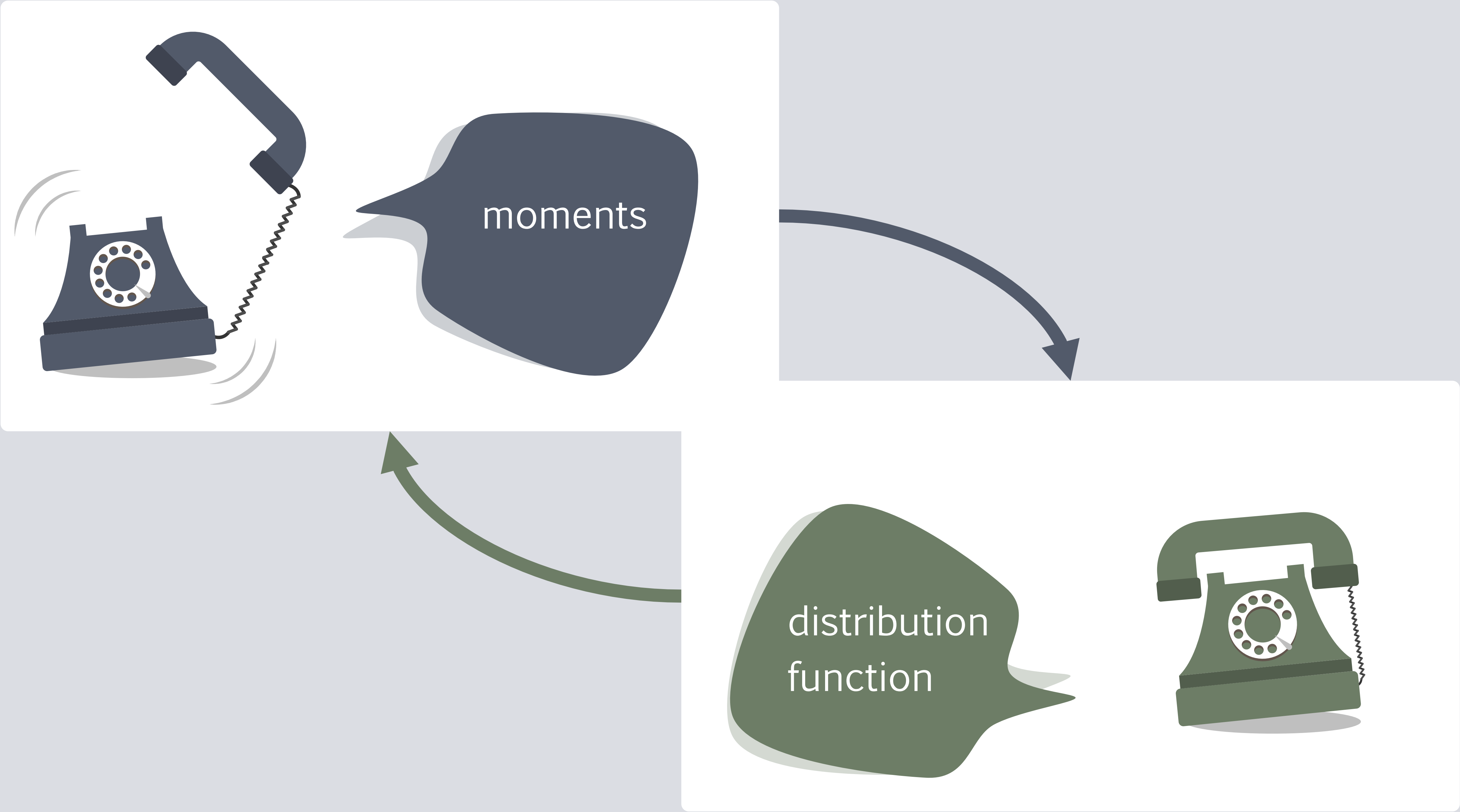
$$g^{(n)}(0) = \frac{d^k}{dt^k} g(t)|_{t=0} = \mu_k$$

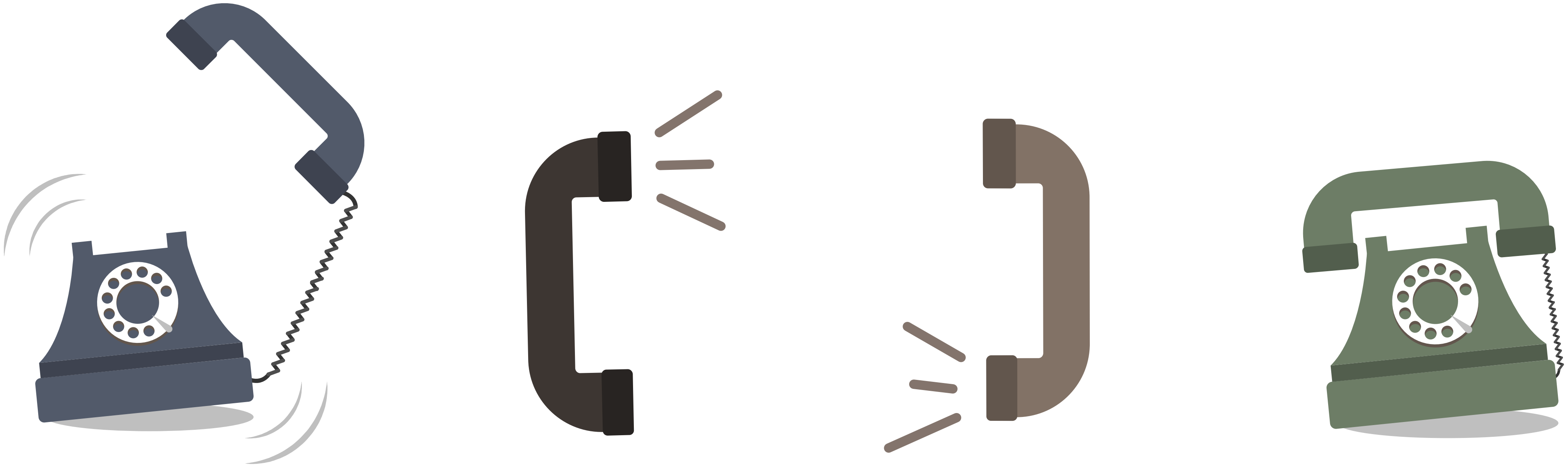
=

Coefficient of z^j in $h(z)$:

$$p(j) = \frac{1}{j!} \frac{d^j}{dz^j} h(z)|_{z=0} = \frac{h^{(j)}(0)}{j!}$$

=


moments
distribution
function



moments → MGF → OGF → distribution function

$$g(t) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

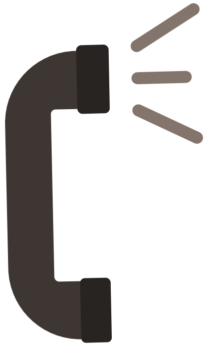
$$h(z) = g(\ln(z))$$

$$p(j) = \frac{h^{(j)}(0)}{j!}$$

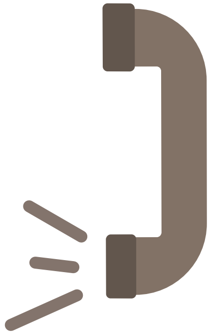
$$\mu_0 = 1,$$
$$\mu_k = \frac{1}{2} + \frac{2^k}{4}, \text{ for } k \geq 1.$$

- $j = \cdots$
- $p(j) = \cdots$

moments



MGF



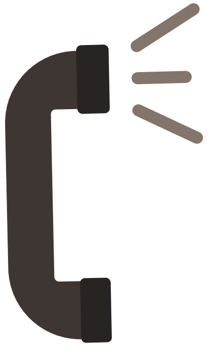
OGF

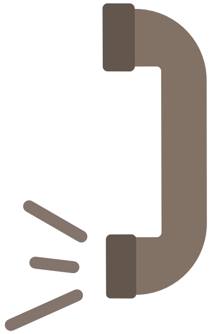
distribution function

$$\mu_0 = 1,$$
$$\mu_k = \frac{1}{2} + \frac{2^k}{4}, \text{ for } k \geq 1.$$

Step 1

$$g(t) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = 1 + \frac{1}{2}\sum_{k=1}^{+\infty} \frac{t^k}{k!} + \frac{1}{4}\sum_{k=1}^{+\infty} \frac{(2t)^k}{k!} = \frac{1}{4} + \frac{1}{2}e^t + \frac{1}{4}e^{2t}$$





moments

MGF

OGF

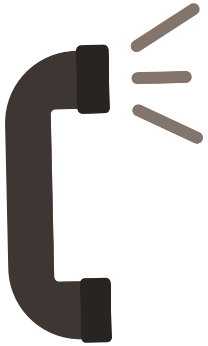
distribution function

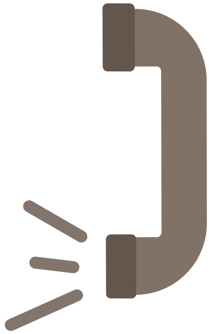
$$\mu_0 = 1,$$
$$\mu_k = \frac{1}{2} + \frac{2^k}{4}, \text{ for } k \geq 1.$$

Step 2

$$h(z) = g(\ln(z))$$
$$= \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2$$

$$g(t) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = 1 + \frac{1}{2}\sum_{k=1}^{+\infty} \frac{t^k}{k!} + \frac{1}{4}\sum_{k=1}^{+\infty} \frac{(2t)^k}{k!} = \frac{1}{4} + \frac{1}{2}e^t + \frac{1}{4}e^{2t}$$





moments

MGF

OGF

distribution function

$$\mu_0 = 1,$$
$$\mu_k = \frac{1}{2} + \frac{2^k}{4}, \text{ for } k \geq 1.$$

$$h(z) = g(\ln(z))$$
$$= \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}z^2$$

Step 3

$$j = 0, 1, 2$$

$$p(j) = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$$

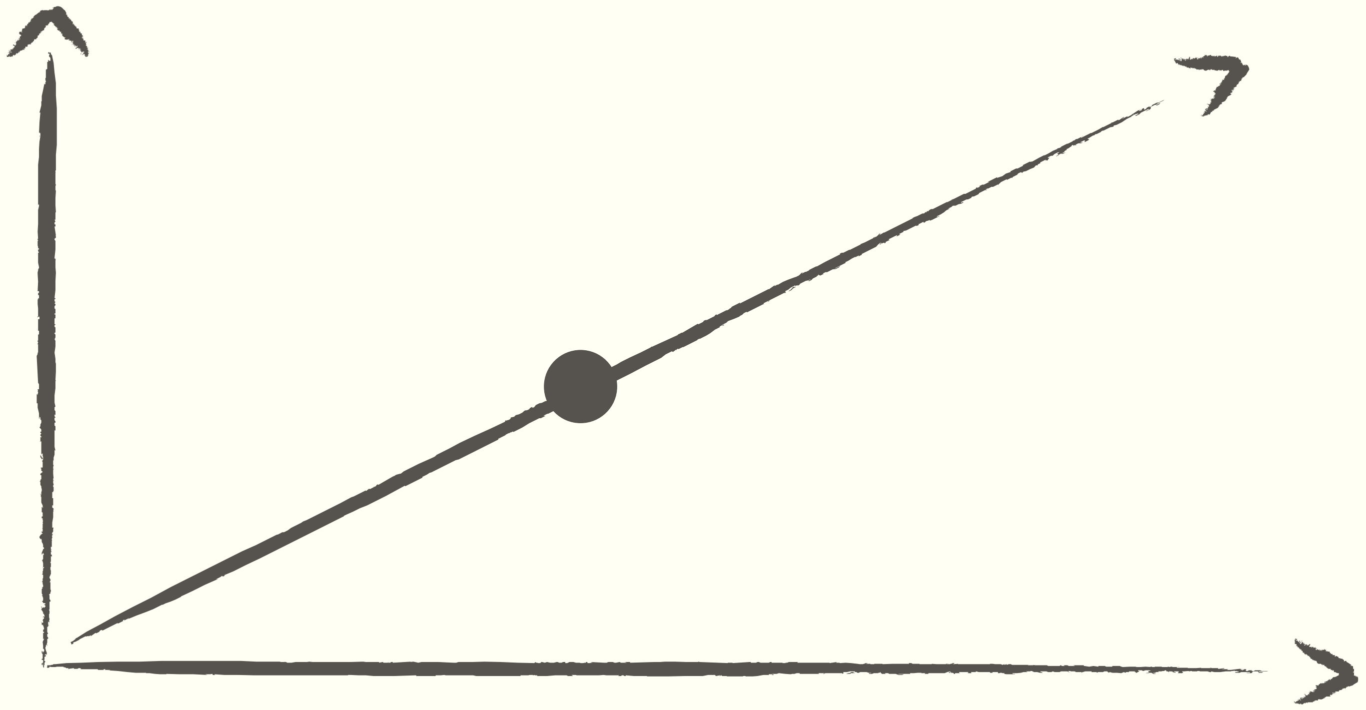
$$g(t) = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!} = 1 + \frac{1}{2}\sum_{k=1}^{+\infty} \frac{t^k}{k!} + \frac{1}{4}\sum_{k=1}^{+\infty} \frac{(2t)^k}{k!} = \frac{1}{4} + \frac{1}{2}e^t + \frac{1}{4}e^{2t}$$

Linearity

$$E(X + Y) = E(X) + E(Y)$$

$$E(cX) = cE(X).$$

$$E(aX + b) = aE(X) + b$$

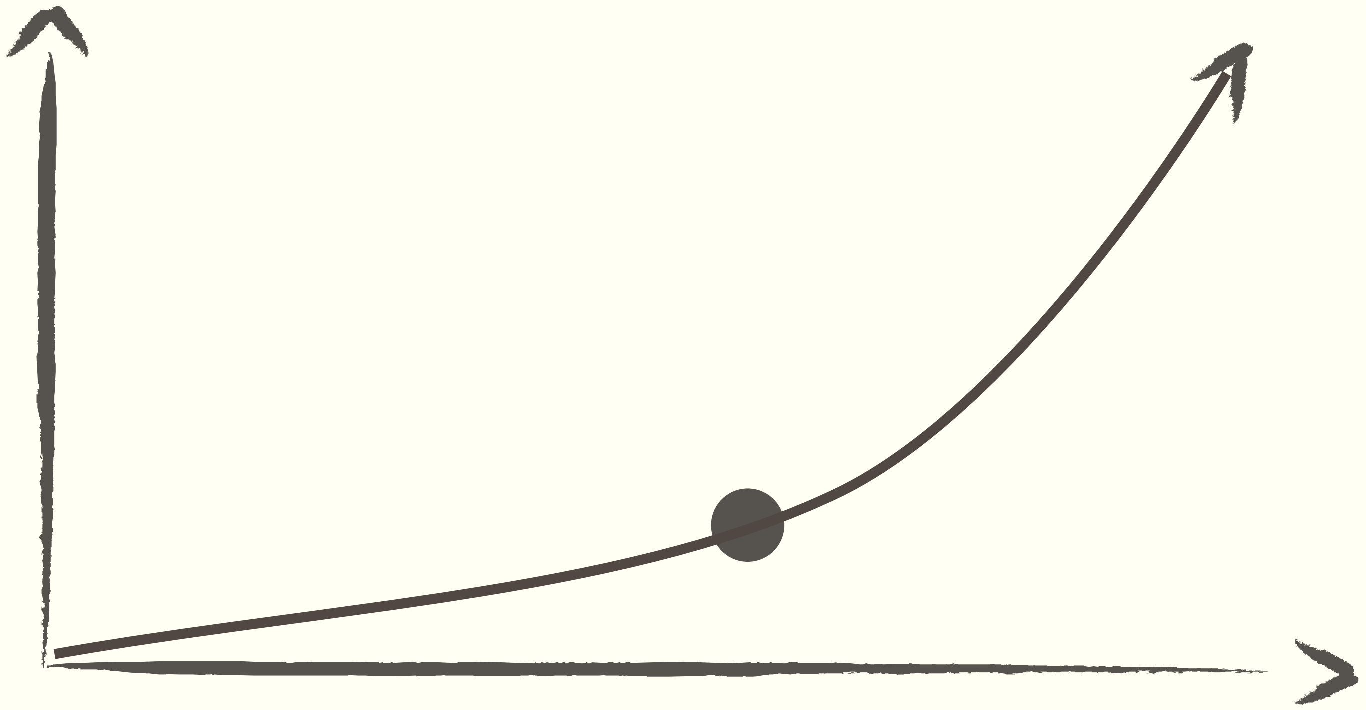


Non-linearity

$$V(cX) = c^2V(X)$$

$$V(X + c) = V(X)$$

$$V(aX + b) = a^2V(X)$$



Properties

- Both the moment generating function g and the ordinary generating function h have many properties useful in the study of random variables, of which we can consider only a few here.

$$g(t) = E(e^{tX})$$

=

$Y = X + a$	$g_Y(t) = E(e^{tY}) = E\left(e^{t(X+a)}\right) = E(e^{ta}e^{tX}) = e^{ta}E(e^{tX}) = e^{ta}g_X(t)$
$Y = bX$	$g_Y(t) = E(e^{tY}) = E\left(e^{tbX}\right) = g_X(bt)$

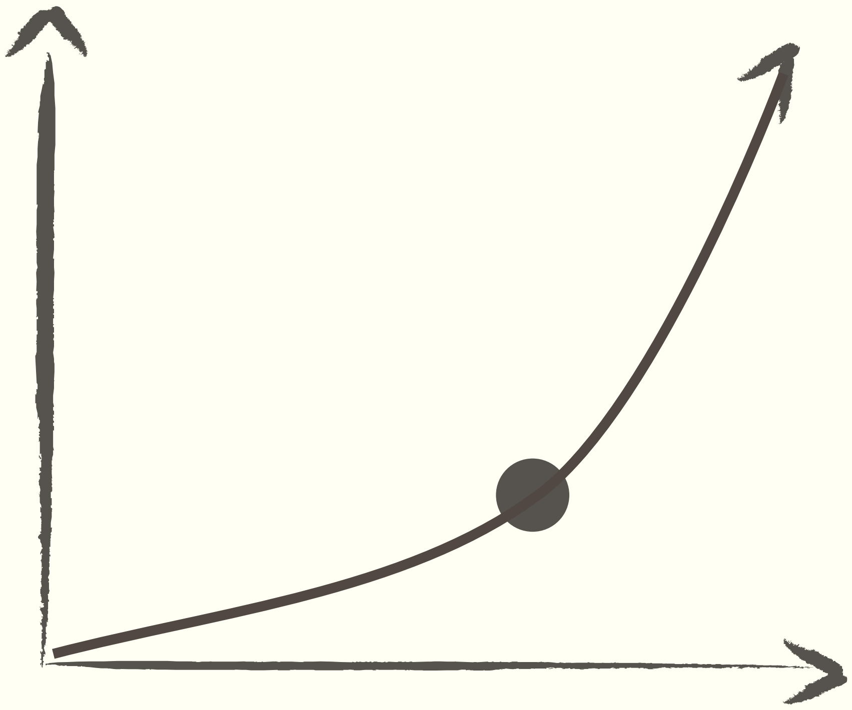
Non-linearity

$$g_{X+a}(t) = e^{ta} g_X(t)$$

$$g_{bX}(t) = g_X(bt)$$

$$Y = \frac{x-\mu}{\sigma}$$

$$g_{\frac{x-\mu}{\sigma}}(t) = e^{-\mu t/\sigma} g_X(\frac{t}{\sigma})$$

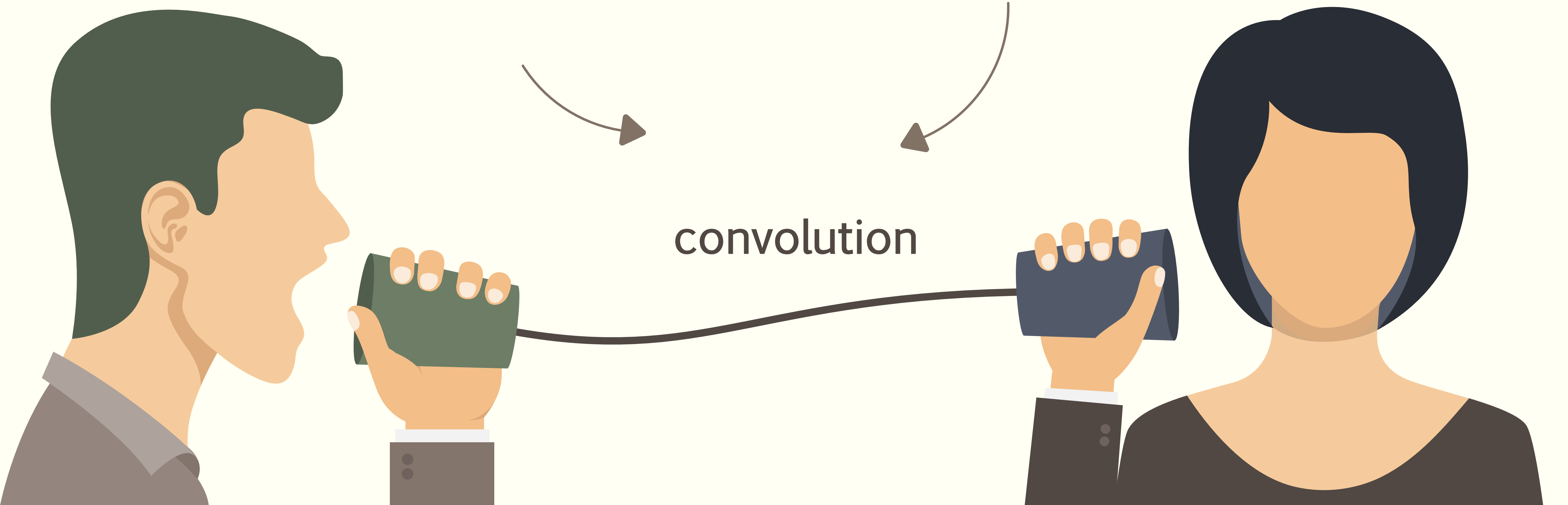


Sum of independent random variables
discrete & continuous

$$m_3(j) = \sum_k m_1(k) m_2(j-k)$$

$$(f * g)(z) = \int_{-\infty}^{+\infty} f(z-y) g(y) dy$$

convolution



Sum of Independent Random Variables

random variable	X	Y	$Z = X + Y$
distribution function (convolution)	p_X	p_Y	$p_Z(j) = \sum_k p_X(k) p_Y(j-k)$
MGF	$g_X(t)$	$g_Y(t)$	$g_Z(t) = \cdots$

$$g(t) = E(e^{tX})$$

=

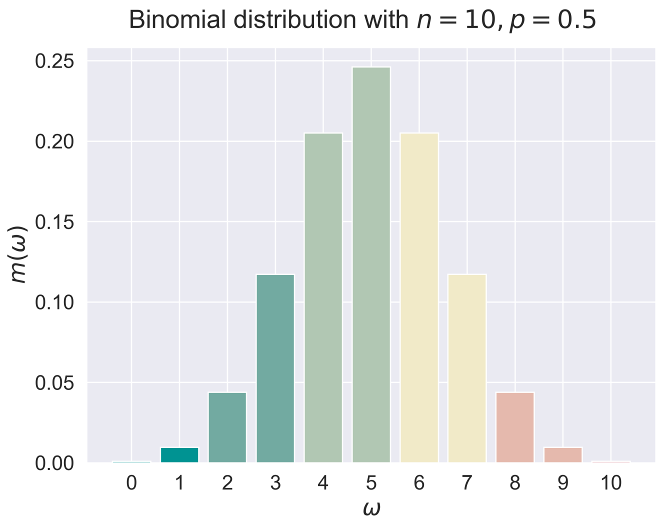
$$g_Z(t) = E\left(e^{t(X+Y)}\right) = E(e^{tX}e^{tY})$$
$$= E(e^{tX})E(e^{tY}) = g_X(t)g_Y(t)$$

random variable	X	Y	$Z = X + Y$
distribution function	p_X	p_Y	$p_Z(j) = \sum_k p_X(k) p_Y(j-k)$
MGF	$g_X(t)$	$g_Y(t)$	$g_Z(t) = g_X(t) g_Y(t)$
OGF	$h_X(z)$	$h_Y(z)$	$h_Z(z) = h_X(z) h_Y(z)$

independent !

$h(z) = g(\ln(z))$ =

Binomial Distribution



$$g_Z(t) = g_X(t)g_Y(t)$$

=

$$h_Z(z) = h_X(z)h_Y(z)$$

=

Random variable

range: $\{0, 1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = p_Y(j) = \binom{n}{j} p^j q^{n-j}$

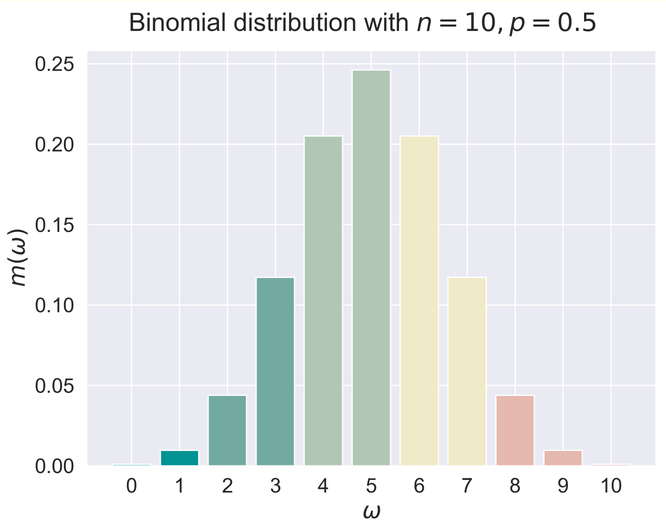
Moment generating function

$$g_X(t) = g_Y(t) = (pe^t + q)^n$$

Ordinary generating function

$$h_X(z) = h_Y(z) = (pz + q)^n$$

Binomial Distribution



$$g_Z(t) = g_X(t)g_Y(t)$$

=

$$h_Z(z) = h_X(z)h_Y(z)$$

=

Random variable

range: $\{0, 1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = p_Y(j) = \binom{n}{j} p^j q^{n-j}$

Moment generating function

$$g_X(t) = g_Y(t) = (pe^t + q)^n$$

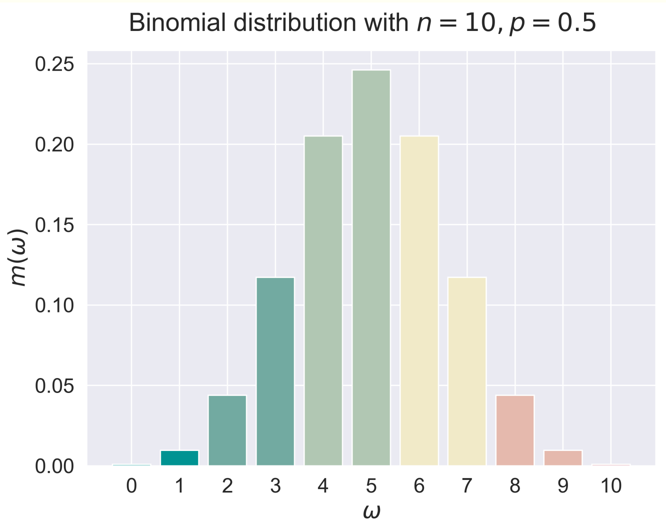
$$g_Z(t) = g_X(t)g_Y(t) = (pe^t + q)^{2n}$$

Ordinary generating function

$$h_X(z) = h_Y(z) = (pz + q)^n$$

$$h_Z(z) = h_X(z)h_Y(z) = (pz + q)^{2n}$$

Binomial Distribution



$$h(z) = \sum_{j=1}^{+\infty} z^j p(j)\,.$$

=

$$h(z) = (pz+q)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} (pz)^j q^{2n-j}$$

Random variable

range: $\{0, 1, 2, 3, \cdots, n\}$
distribution function: $p_X(j) = p_Y(j) = \binom{n}{j} p^j q^{n-j}$

range: $\{0, 1, 2, 3, \cdots, 2n\}$
distribution function: $p_Z(j) = \binom{2n}{j} p^j q^{2n-j}$

Moment generating function

$g_X(t) = g_Y(t) = (pe^t + q)^n$

$g_Z(t) = g_X(t) g_Y(t) = (pe^t + q)^{2n}$

Ordinary generating function

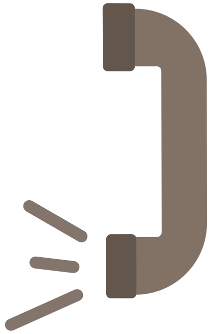
$h_X(z) = h_Y(z) = (pz + q)^n$

$h_Z(z) = h_X(z) h_Y(z) = (pz + q)^{2n}$

$$h(z) = \sum_{j=1}^{+\infty} z^j p(j)\,.$$

$$p(j) = \frac{1}{j!}\frac{d^j}{dz^j} h(z)|_{z=0} = \frac{h^{(j)}(0)}{j!}$$

=



OGF

Ready-made polynomials

Taylor Expansion

distribution function

GENERATING FUNCTIONS

continuous density

Moments

- If X is a continuous random variable defined on the probability space Ω, with density function f_X, then we define the nth moment by the formula

$$\mu_n = E(X^n) = \int_{-\infty}^{+\infty} x^n f_X(x)dx,$$

provided the integral

$$\int_{-\infty}^{+\infty} |x|^n f_X(x)dx,$$

is finite.

- Then just as the discrete case, we see that

$$1 = \mu_0 \qquad \mu = \mu_1 \qquad \sigma^2 = \mu_2 - \mu_1^2$$

Moment Generating Functions

- We introduce a new variable t, and define a function $g(t)$ as follows:

$$g(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx.$$

- We call $g(t)$ the moment generating function for X.

$$g(t) = E(e^{tX}) = E\left(\sum_{k=0}^{+\infty} \frac{X^k t^k}{k!}\right) = \sum_{k=0}^{+\infty} \frac{E(X^k)t^k}{k!} = \sum_{k=0}^{+\infty} \frac{\mu_k t^k}{k!}$$

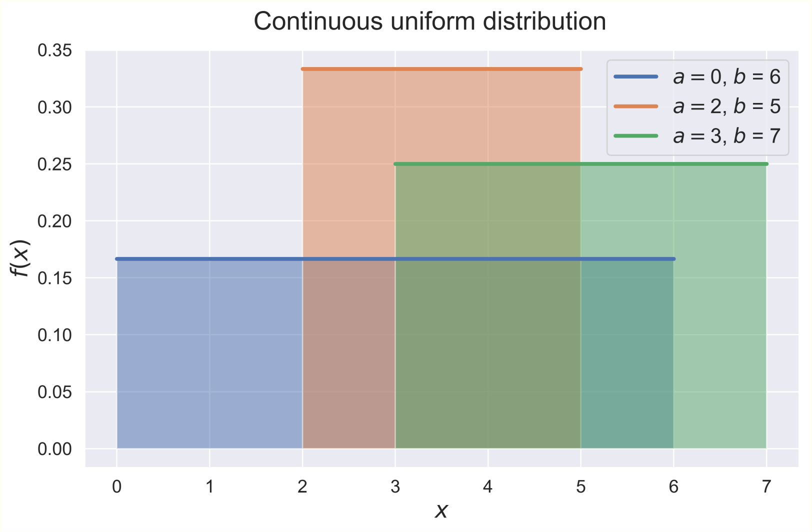
=

$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

=

provided this series converges

Uniform Distribution



$$\mu_n = E(X^n) = \int_{-\infty}^{+\infty} x^n f_X(x)dx$$

$$g(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx$$

Random variable

range: $0 \leq x \leq 1$
density function: $f_X(j) = 1$

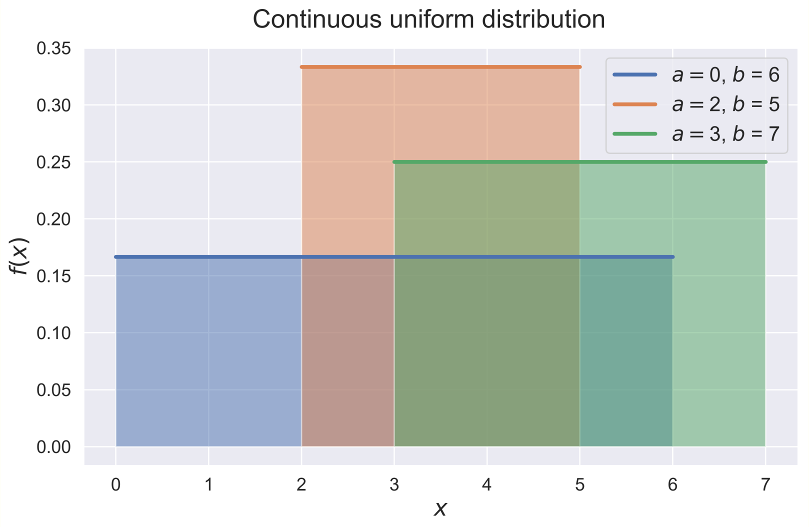
Moment

$$\mu_n = \int_0^1 x^n dx = \frac{1}{n+1}$$

Generating function

$$g(t) = \int_0^1 e^{tx} dx = \frac{e^t - 1}{t}$$

Uniform Distribution



$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

=

Random variable

range: $0 \le x \le 1$
density function: $f_X(x) = 1$

Moment

$$\mu_n = \int_0^1 x^n dx = \frac{1}{n+1}$$

$$\mu_0 = g(0) = 1$$

$$\mu_1 = g'(0) = \frac{1}{2}$$

$$\mu_2 = g''(0) = \frac{1}{3}$$

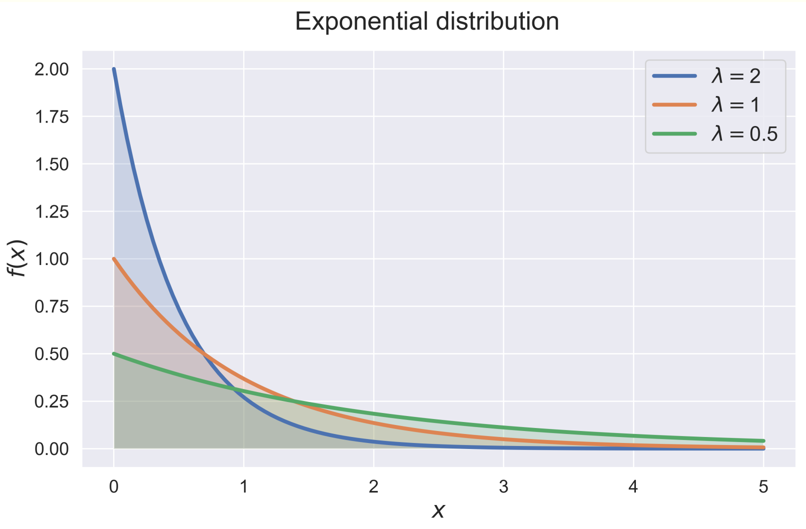
Generating function

$$g(t) = \int_0^1 e^{tx} dx = \frac{e^t - 1}{t}$$

$$\mu = \mu_1 = \frac{1}{2}$$

$$\sigma^2 = \mu_2 - \mu_1^2 = \frac{1}{12}$$

Exponential Distribution



$$\mu_n = E(X^n) = \int_{-\infty}^{+\infty} x^n f_X(x)dx$$

=

$$g(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx$$

=

Random variable

range: $x \geq 0$

density function: $f_X(x) = \lambda e^{-\lambda x}$

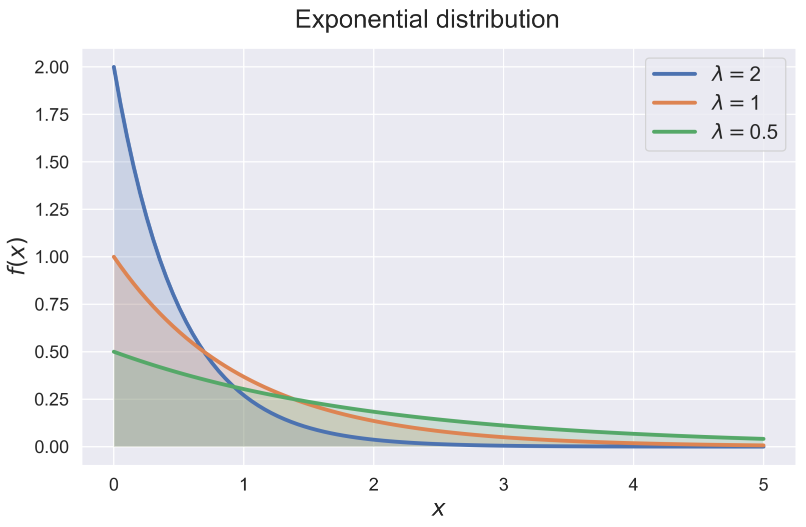
Moment

$$\mu_n = \int_0^{+\infty} x^n \lambda e^{-\lambda x} dx = \frac{n!}{\lambda^n}$$

Generating function

$$g(t) = \int_0^{+\infty} e^{tx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - t}$$

Exponential Distribution



$$g^{(n)}(0) = \frac{d^n}{dt^n} g(t)|_{t=0} = \mu_n$$

Random variable

range: $x \geq 0$
density function: $f_X(x) = \lambda e^{-\lambda x}$

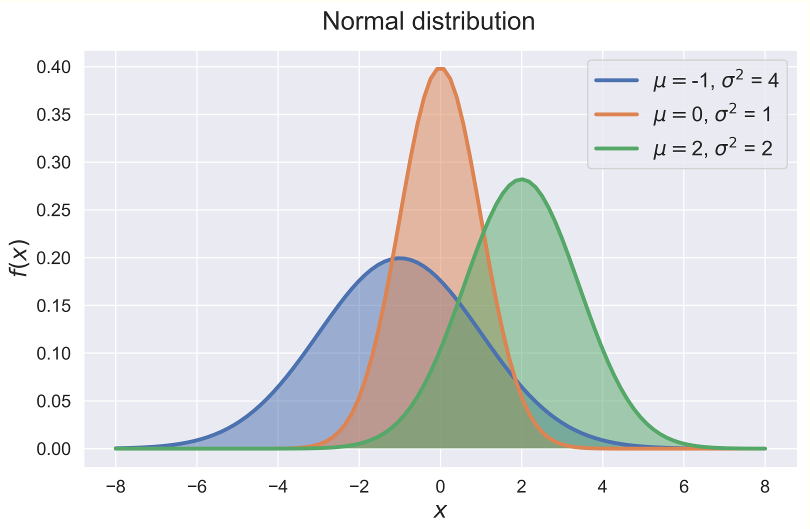
Moment

$$\mu_n = \int_0^{+\infty} x^n \lambda e^{-\lambda x} dx = \frac{\lambda n!}{(\lambda - t)^{n+1}}|_{t=0} = \frac{n!}{\lambda^n}$$

Generating function

$$g(t) = \int_0^{+\infty} e^{tx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - t}$$

Normal Distribution (standard)



$$\mu_n = E(X^n) = \int_{-\infty}^{+\infty} x^n f_X(x) dx$$

=

$$g(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x) dx$$

=

Random variable

range: $-\infty \le x \le +\infty$

density function: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

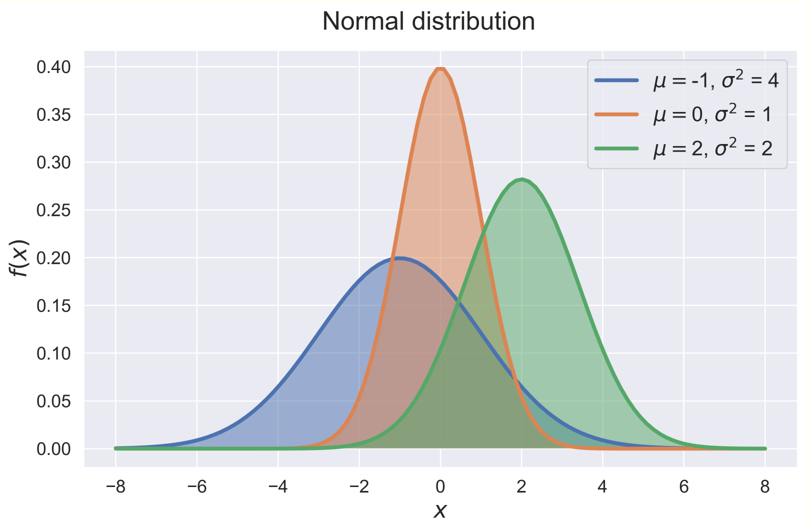
Moment

- $\mu_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-x^2/2} dx = \frac{(2m)!}{2^m m!}$ if $n = 2m$
- $\mu_n = 0$ if $n = 2m + 1$

Generating function

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{tx} e^{-x^2/2} dx = e^{t^2/2}$$

Normal Distribution (general)



$$g(t) = e^{t^2/2}$$

=

$$g_{X+a}(t) = e^{ta} g_X(t)$$
$$g_{bX}(t) = g_X(bt)$$

=

Random variable

range: $-\infty \leq x \leq +\infty$

density function: $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$

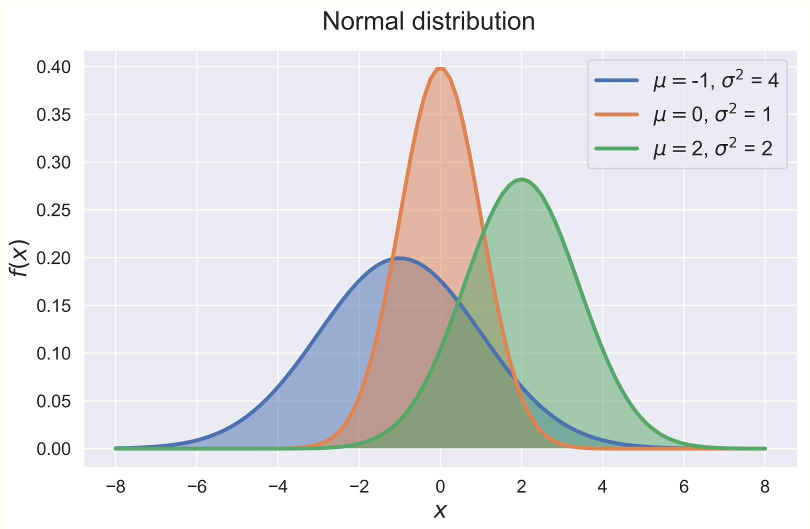
Moment

confluent hypergeometric function

Generating function

$$g(t) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{tx} e^{-(x-\mu)^2/2\sigma^2} dx = e^{\mu t + \sigma^2 t^2/2}$$

Normal Distribution (sum)



$$g_Z(t) = g_X(t)g_Y(t)$$

=

Random variable

range: $-\infty \leq x, y \leq +\infty$
distribution function:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(x-\mu_1)^2/2\sigma_1{}^2}$$

$$f_Y(x) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-(x-\mu_2)^2/2\sigma_2{}^2}$$

Moment generating function

$$g_X(t) = e^{\mu_1 t + \sigma_1{}^2 t^2/2}$$

$$g_Y(t) = e^{\mu_2 t + \sigma_2{}^2 t^2/2}$$

$$g_Z(t) = e^{(\mu_1+\mu_2)t + (\sigma_1{}^2+\sigma_2{}^2)t^2/2}$$

$$E(Z) = \mu_1 + \mu_2$$

$$V(Z) = \sigma_1{}^2 + \sigma_2{}^2$$

Moment Problem

- If X is a bounded random variable, then the moment generating function $g_X(t)$ determines the density function $f_X(x)$ uniquely.