MATH 20: PROBABILITY

Fundamental Theorems of Probability Theory

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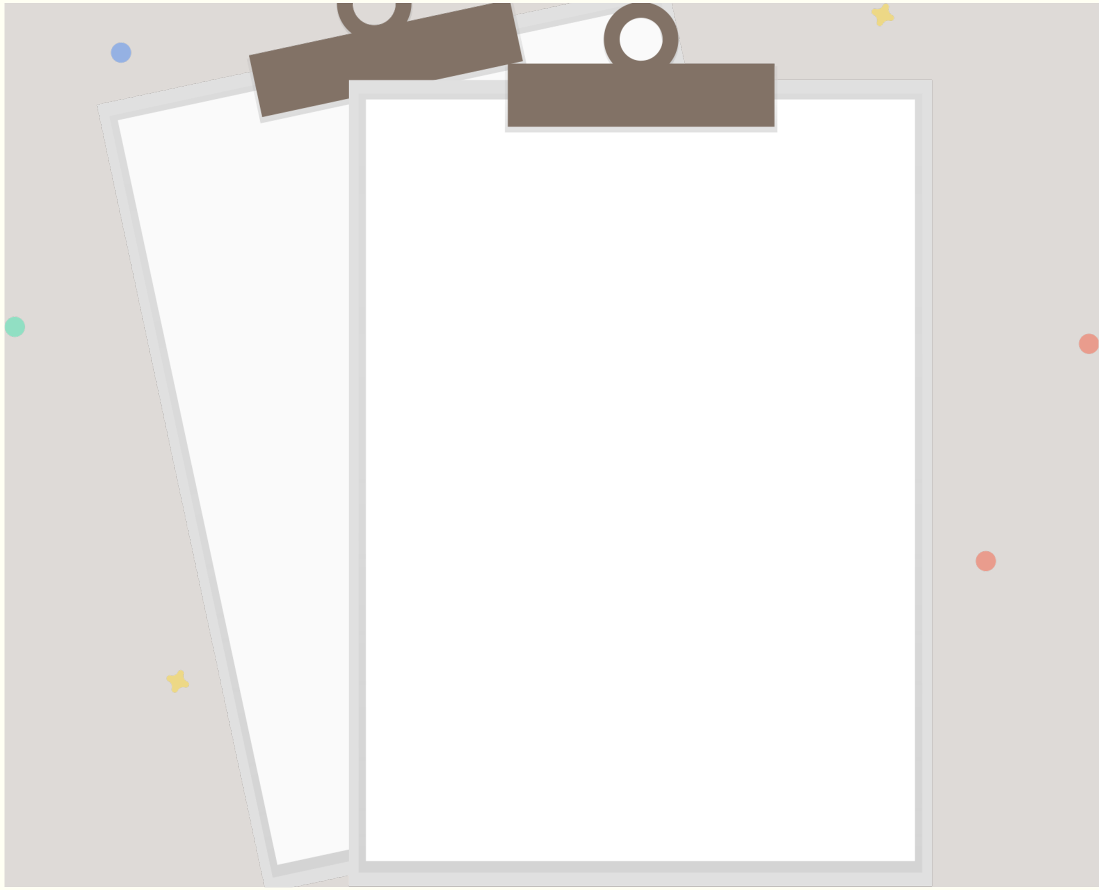


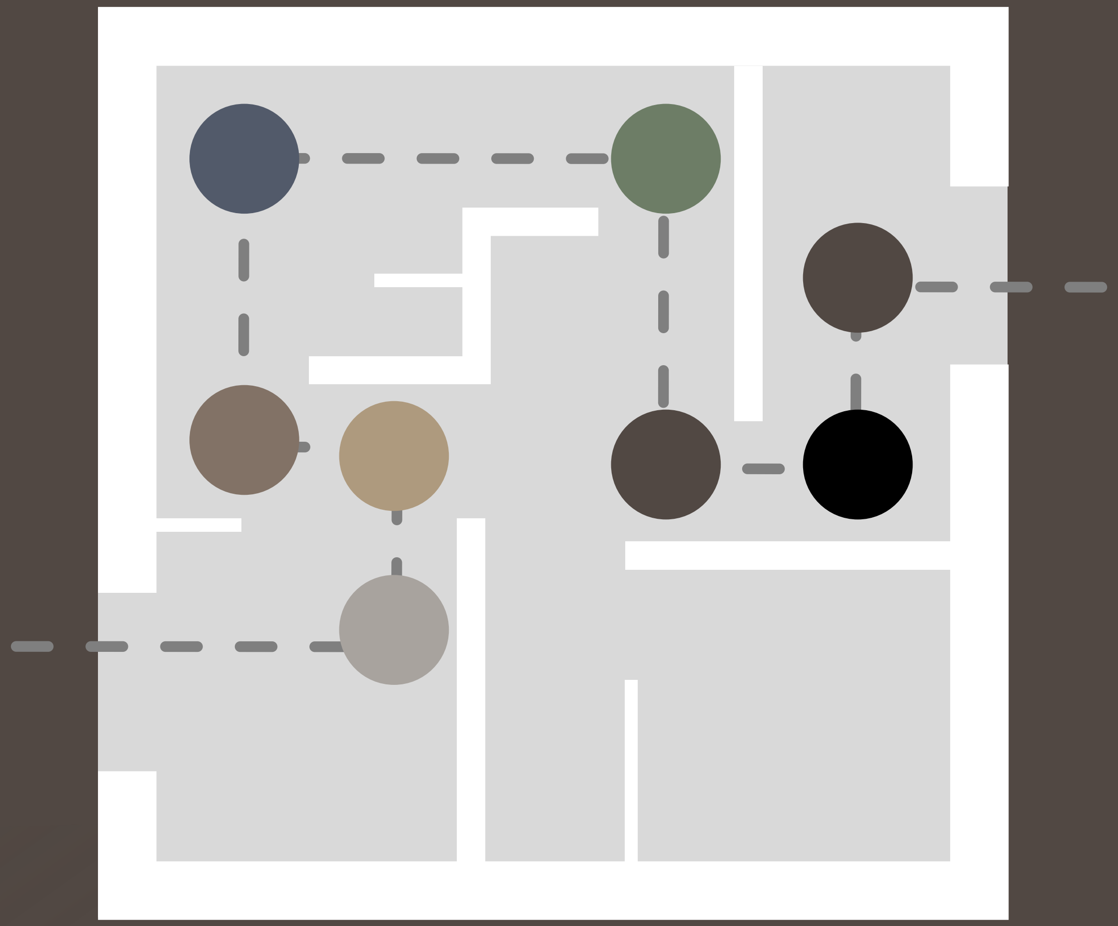
Fundamental Theorems of Probability Theory

Law of Large Numbers

Central Limit Theorem

Law of Large Numbers

- frequency as interpretation of probability
  - convergence of the **sample mean**



LAW OF LARGE NUMBERS FOR DISCRETE RANDOM VARIABLES

discrete random variables

Let X be a nonnegative discrete random variable with expected value $E(X)$, and let $\varepsilon > 0$ be any positive number. Then

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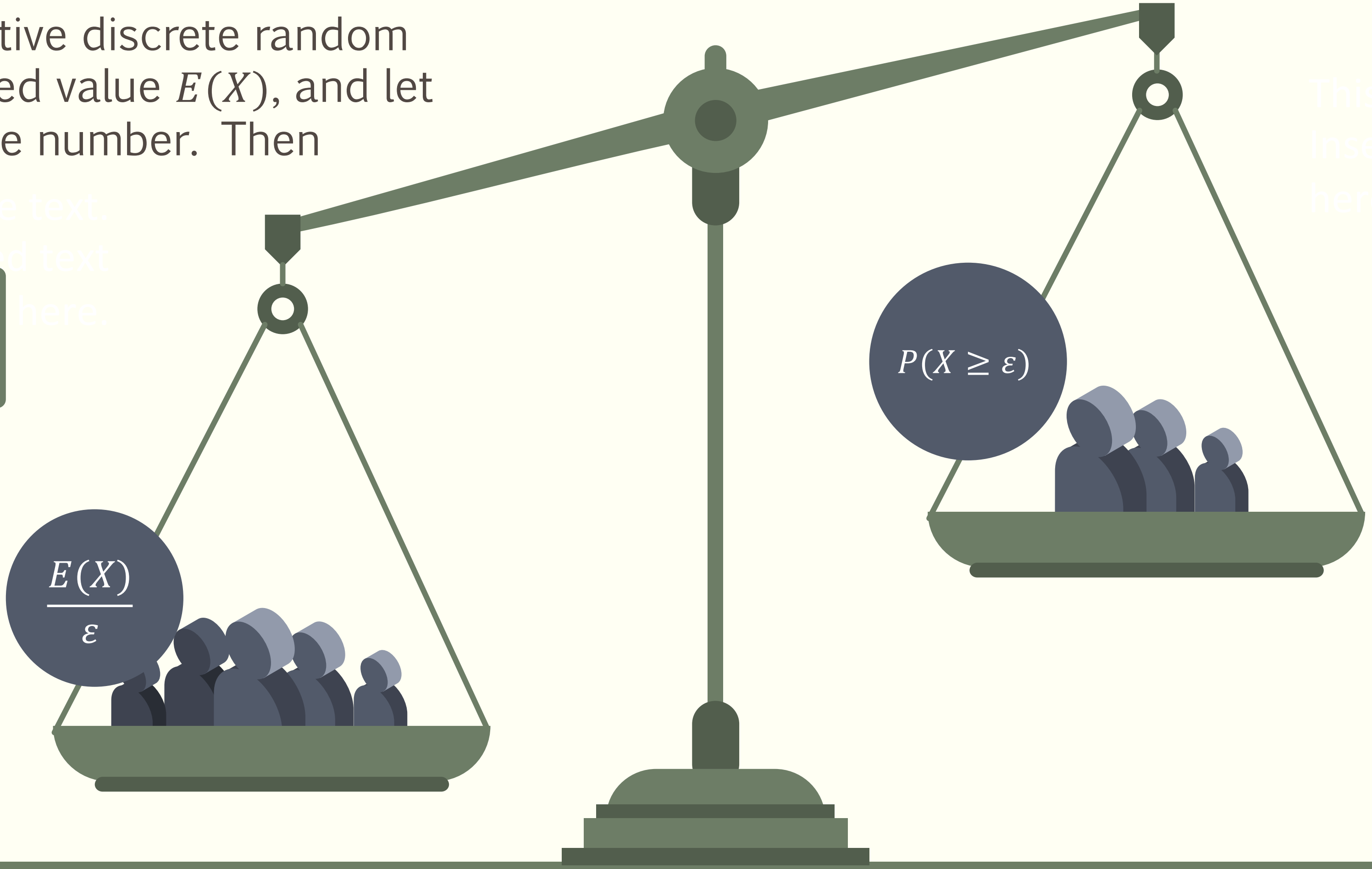
X: nonnegative



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Markov inequality

$$P(X \geq \varepsilon) \leq \frac{E(X)}{\varepsilon}$$

Proof

Let X be a **positive** discrete random variable with expected value $E(X)$, and let $\varepsilon > 0$ be any positive number.

$$P(X \geq \varepsilon) \leq \frac{E(X)}{\varepsilon}$$

$$\sum_{x} m(x) = 1$$

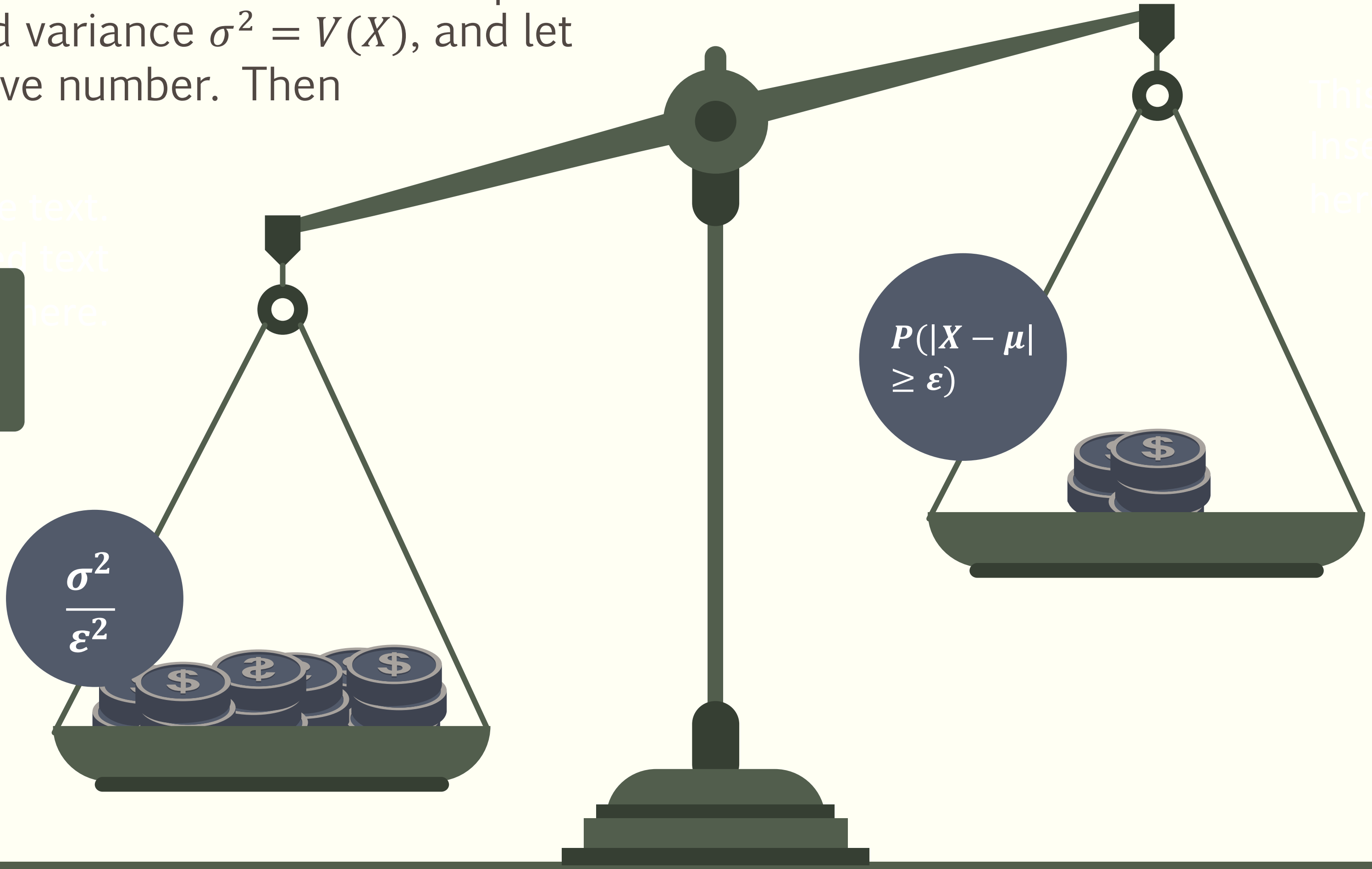
$$P(X \geq \varepsilon) = \sum_{x \geq \varepsilon} m(x)$$

$$\sum_{x} x m(x) = E(X)$$

$$E(X) = \sum_{x} x m(x) \geq \sum_{x \geq \varepsilon} x m(x) \geq \varepsilon \sum_{x \geq \varepsilon} m(x) = \varepsilon P(X \geq \varepsilon)$$

Let X be a discrete random variable with expected value $\mu = E(X)$ and variance $\sigma^2 = V(X)$, and let $\varepsilon > 0$ be any positive number. Then

X: not necessarily nonnegative



Chebyshev inequality

$$P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

Proof

Let X be a discrete random variable with expected value $\mu = E(X)$ and variance $\sigma^2 = V(X)$, and let $\varepsilon > 0$ be any positive number.

$$\boldsymbol{P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}}$$

$$\boldsymbol{\sum_x m(x) = 1}$$

$$P(|\boldsymbol{X} - \boldsymbol{\mu}| \geq \boldsymbol{\varepsilon}) = \sum_{|x-\mu|\geq\varepsilon} m(x)$$

$$\boldsymbol{\sum_x (x-\mu)^2 m(x) = V(X)}$$

$$\sigma^2 = V(X) = \sum_x (x-\mu)^2 m(x) \geq \sum_{|x-\mu|\geq\varepsilon} (x-\mu)^2 m(x)$$

$$\geq \varepsilon^2 \sum_{|x-\mu|\geq\varepsilon} m(x) = \varepsilon^2 P(|X-\mu| \geq \varepsilon)$$

Markov inequality

$$P(X \geq \varepsilon) \leq \frac{E(X)}{\varepsilon}$$

X: nonnegative

$E(X)$: known

Chebyshev inequality

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

X: not necessarily nonnegative

$E(X)$: known
$V(X)$: known

Example 1

- Let X be any random variable which takes on values $0, 1, 2, \cdots, n$ and has $E(X) = V(X) = 1$. Show that for any positive integer k,

$$P(X \geq k + 1) \leq \frac{1}{k^2}.$$

Markov Inequality
$P(X \geq \varepsilon) \leq \dfrac{E(X)}{\varepsilon}$

Chebyshev Inequality
$P(\lvert X - \mu \rvert \geq \varepsilon) \leq \dfrac{\sigma^2}{\varepsilon^2}$

Example 1

- Let X be any random variable which takes on values $0, 1, 2, \cdots, n$ and has $E(X) = V(X) = 1$. Show that for any positive integer k,

$$P(X \geq k+1) \leq \frac{1}{k^2}.$$

Chebyshev Inequality
$P(\lvert X-\mu\rvert \geq \varepsilon) \leq \dfrac{\sigma^2}{\varepsilon^2}$
$\mu = \sigma^2 = 1, \varepsilon = k$
$P(\lvert X-1\rvert \geq k) = P(X \geq k+1) \leq \dfrac{1}{k^2}$

Example 2

- Choose X with distribution

$$m(x) = \begin{pmatrix} -\varepsilon & \varepsilon \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$$P(|X| \geq \varepsilon) = 1$$

Chebyshev Inequality
$P(\lvert X - \mu \rvert \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$
$\mu = 0, \sigma^2 = \varepsilon^2$
$P(\lvert X \rvert \geq \varepsilon) \leq \frac{\varepsilon^2}{\varepsilon^2} = 1$

Law of Large Numbers

- Let $X_1, X_2, \cdots, X_n$ be an independent trials process, with same finite expected value $\mu = E(X_j)$ and finite variance $\sigma^2 = V(X_j)$.
- Let $S_n = X_1 + X_2 + \cdots + X_n$. Then for any $\varepsilon > 0$,

$$P(|\frac{S_n}{n} - \mu| \geq \varepsilon) \to 0, \text{ as } n \to +\infty.$$

Large Numbers

- Equivalently,

$$\lim_{n \to \infty} P(\left|\frac{S_n}{n} - \mu\right| < \varepsilon) = 1.$$

As $\frac{S_n}{n}$ is an average of the individual outcomes, the LLN is often referred to as the **law of averages**.

!

Law of Large Numbers

- Let $X_1, X_2, \cdots, X_n$ be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$.
- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sum, and $A_n = \frac{S_n}{n}$ be the average. Then

$$E(A_n) = \mu$$

=

$$V(A_n) = \frac{\sigma^2}{n},\ D(A_n) = \frac{\sigma}{\sqrt{n}}$$

=

$$n \to +\infty$$

$$V(A_n) \to 0$$

$$D(A_n) \to 0$$

Proof

Let $X_1, X_2, \cdots, X_n$ be an independent trials process, with same finite expected value $\mu = E(X_j)$ and finite variance $\sigma^2 = V(X_j)$.

$$P(|\frac{S_n}{n} - \mu| \geq \varepsilon) \to 0,\ \text{as } n \to +\infty.$$

$$\boldsymbol{E\left(\frac{S_n}{n}\right) = \mu}$$

$$\boldsymbol{V\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}}$$

$$\boldsymbol{P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}}$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}$$

$$n \to +\infty$$

$$\frac{\sigma^2}{n\varepsilon^2} \to 0$$

Law of Large Numbers

convergence of the sample mean

(Strong)
Law of Large
Numbers

$$P\left(\lim_{n\to\infty}\frac{S_n}{n}=\mu\right)=1$$

$$\lim_{n\to\infty}P(\left|\frac{S_n}{n}-\mu\right|<\varepsilon)=1$$

(Weak)
Law of Large
Numbers

Example 3

- Consider the general Bernoulli trial process.
- As usual, we let $X = 1$ if the outcome is a success and 0 if it is a failure.

Bernoulli trial

$$m(x) = \begin{cases} p, & X = 1 \\ 1 - p, & X = 0 \end{cases}$$

Expected value $\boldsymbol{E(X)}$

$$\sum_{x \in \Omega} x m(x) = 1 \times p + 0 \times (1 - p) = p$$

Variance $V(X)$

$$E(X^2) - \mu^2 = p - p^2 = p(1 - p)$$

Example 3

- Now consider n Bernoulli trials.
- Then $S_n = \sum_{i=1}^{n} X_i$ is the number of successes in n trials and $\mu = E\left(\frac{S_n}{n}\right) = E(X_i) = p$.

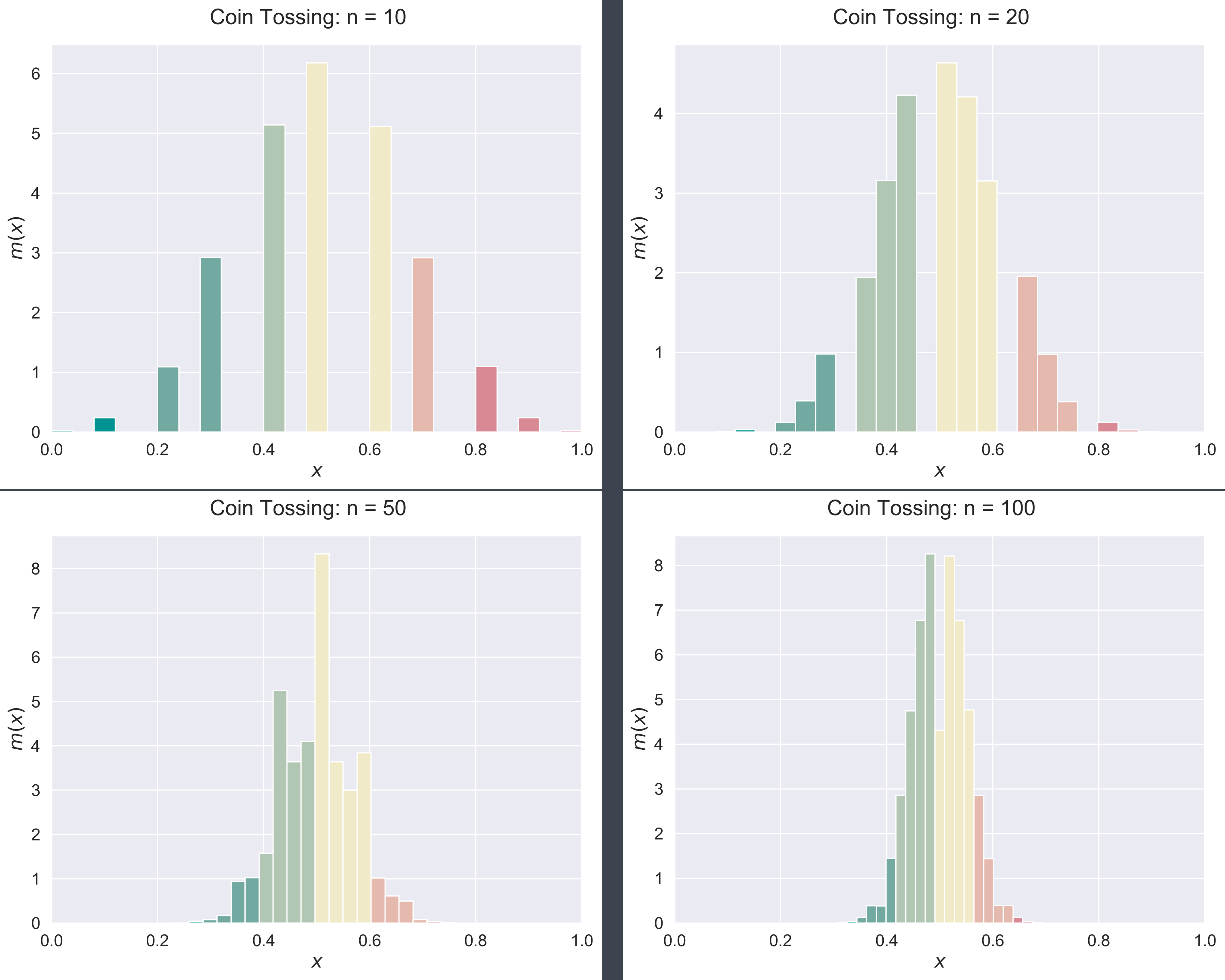
Law of Large Numbers
$\lim_{n\to\infty} P\left(\left\|\frac{S_n}{n} - \mu\right\| < \varepsilon\right) = 1$
$\mu = p$
$\lim_{n\to\infty} P\left(\left\|\frac{S_n}{n} - p\right\| < \varepsilon\right) = 1$

Coin Tossing

$$n \to +\infty$$

$$\mu = \frac{1}{2}$$

$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$

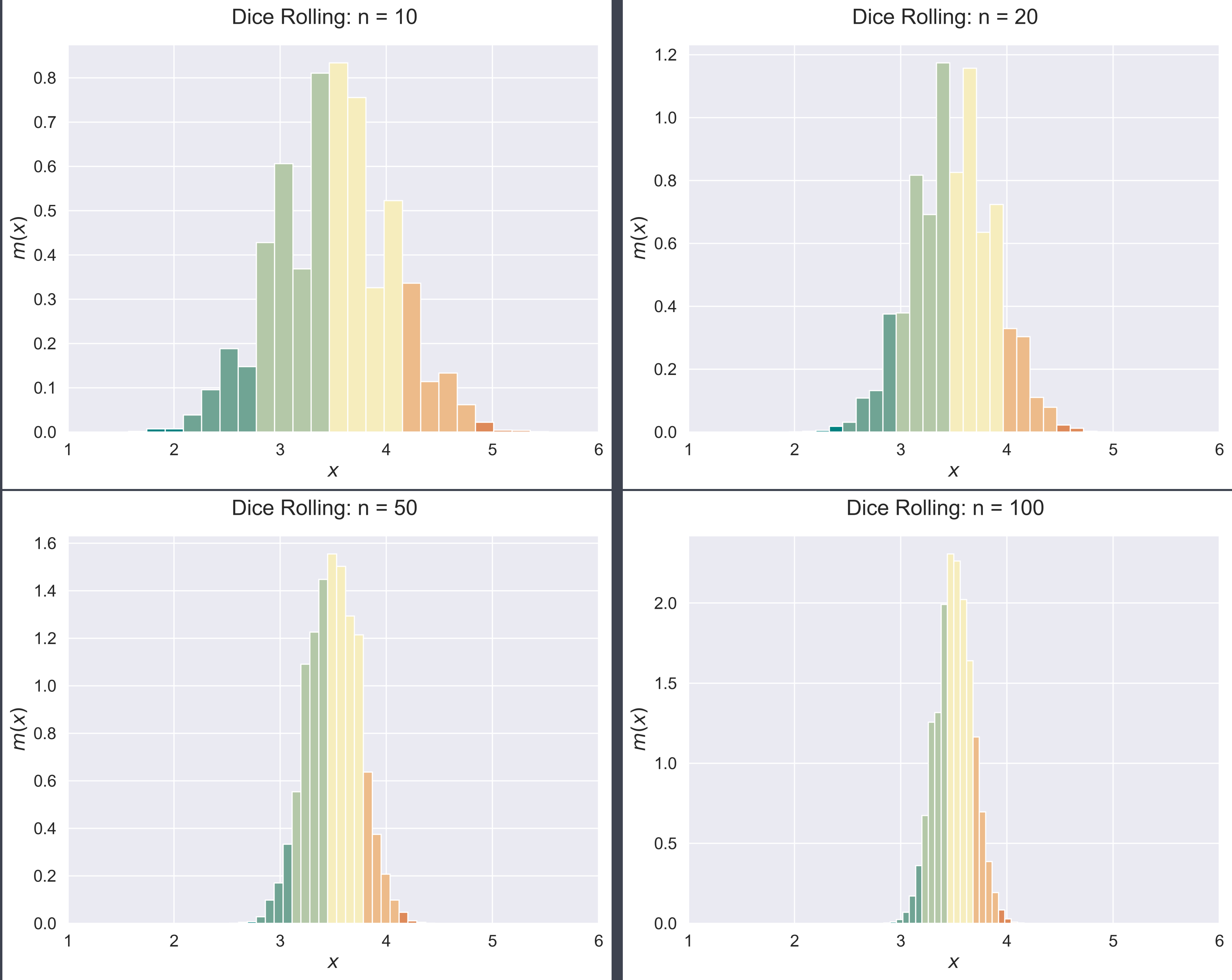


Dice Rolling

$$n \to +\infty$$

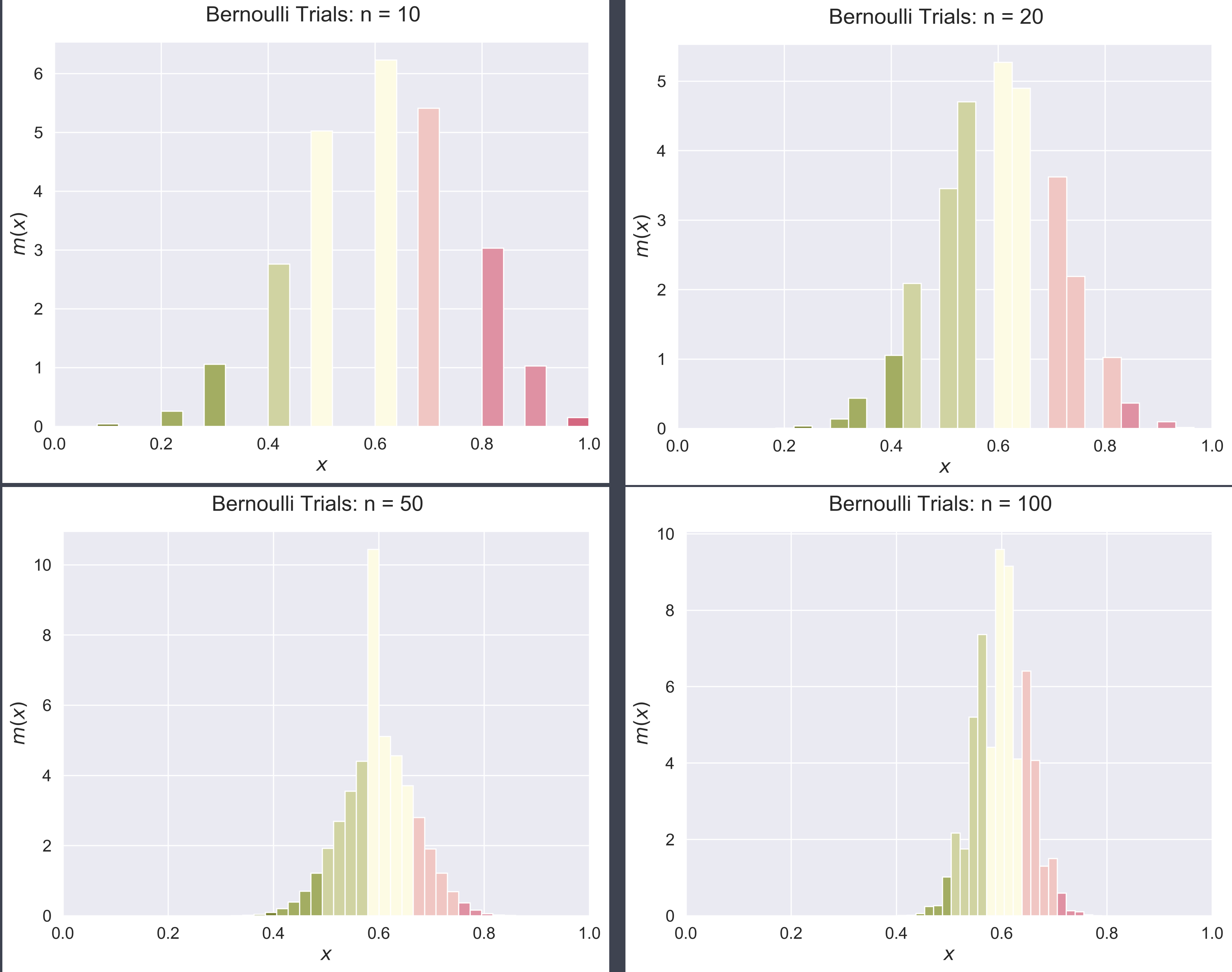
$$\mu = \frac{7}{2}$$

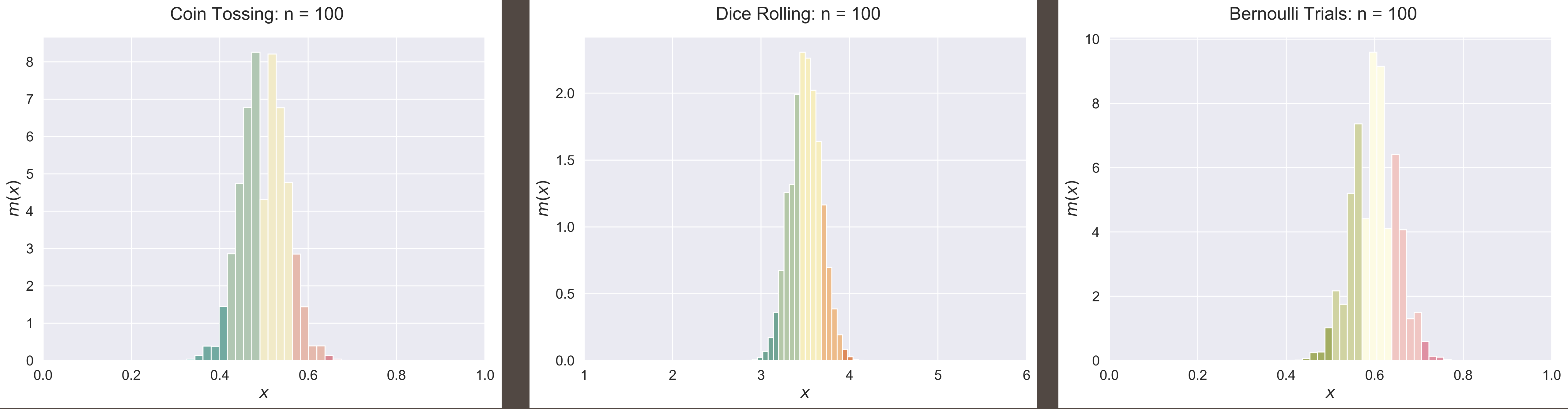
$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$



Bernoulli Trials

We can start with a random experiment about which little can be predicted and, by taking averages, obtain an experiment in which the outcome can be predicted with a high degree of certainty.





Law of Large Numbers

- frequency as interpretation of probability
  - convergence of the **sample mean**

$$S_n = X_1 + X_2 + \cdots + X_n,\ A_n = \frac{S_n}{n}$$

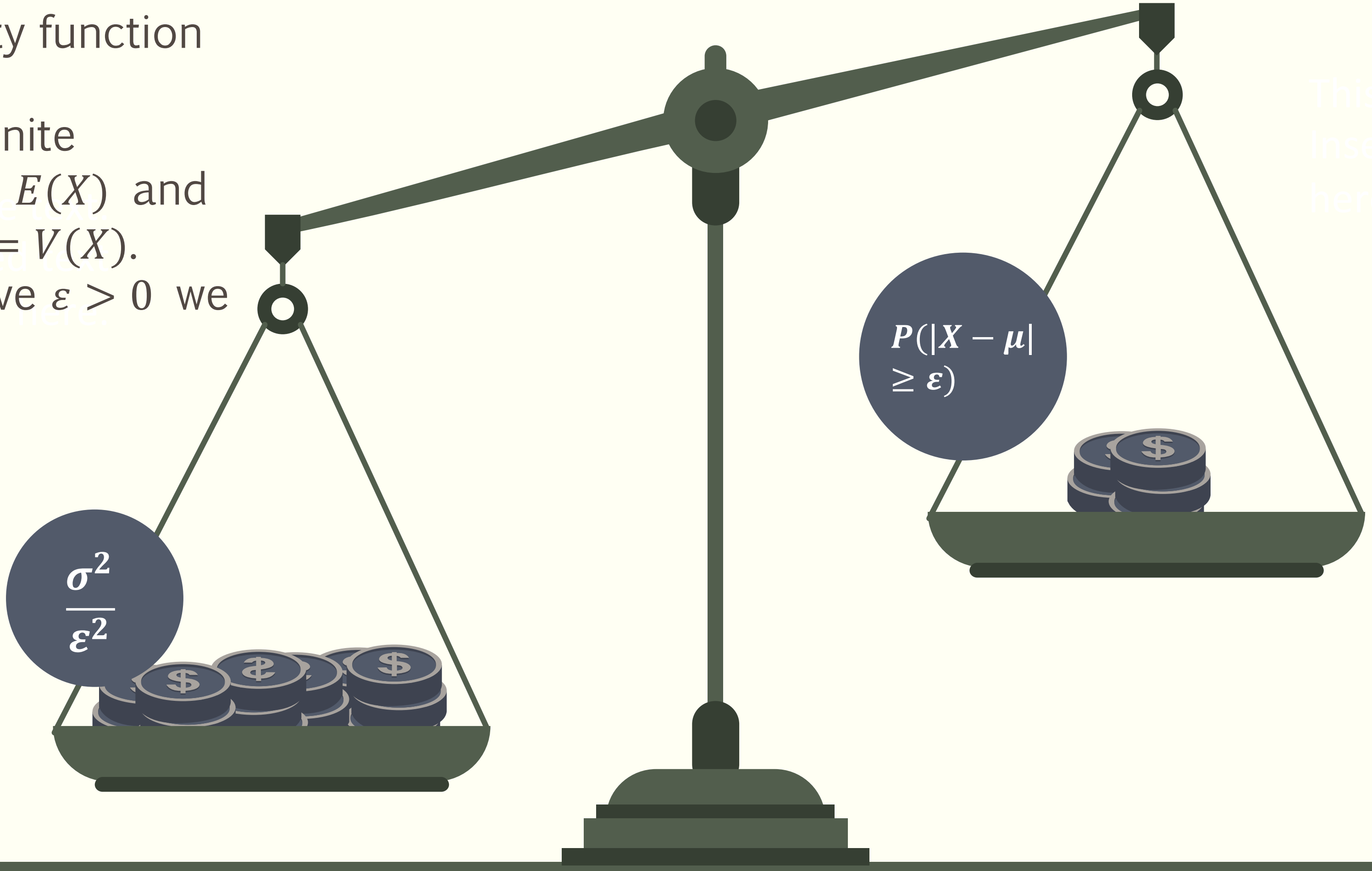
$$E(A_n) = \mu,\ V(A_n) = \frac{\sigma^2}{n}$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}$$

LAW OF LARGE NUMBERS FOR CONTINUOUS RANDOM VARIABLES

continuous random variables

Let X be a continuous random variable with density function $f(x)$.
Suppose X has a finite expected value $\mu = E(X)$ and fintite variance $\sigma^2 = V(X)$.
Then for any positive $\varepsilon > 0$ we have



finite expected value

finite variance

Chebyshev inequality

$$P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

Law of Large Numbers

- Let $X_1, X_2, \cdots, X_n$ be an independent trials process with a continuous density function f, **finite** expected value μ and **finite** variance σ^2.
- Let $S_n = X_1 + X_2 + \cdots + X_n$. Then for any $\varepsilon > 0$,

$$P(|\frac{S_n}{n} - \mu| \geq \varepsilon) \to 0, \text{ as } n \to +\infty.$$

- Equivalently,

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) = 1.$$

As $\frac{S_n}{n}$ is an average of the individual outcomes, the LLN is often referred to as the **law of averages**.

!

Example 4

- Suppose we choose at random n numbers from the interval $[0, 1]$ with uniform distribution.
- Let X_i describes the ith choice.

Uniform distribution
$f(x) = \dfrac{1}{b-a}, \qquad a \le x \le b$
$f(x) = 1, \qquad 0 \le x \le 1$

Expected value $E(X)$

$$E(X) = \frac{1}{2}(a+b) = \frac{1}{2}(0+1) = \frac{1}{2}$$

Variance $V(X)$

$$V(X) = E(X^2) - \mu^2$$
$$= \frac{1}{12}(b-a)^2 = \frac{1}{12}(1-0)^2 = \frac{1}{12}$$

Example 4

- Suppose we choose at random n numbers from the interval $[0, 1]$ with uniform distribution.
- Let X_i describes the ith choice and $S_n = \sum_{i=1}^{n} X_i$.

Expected value

$$\boldsymbol{E}(\frac{S_n}{n}) = \frac{1}{2}$$

Variance

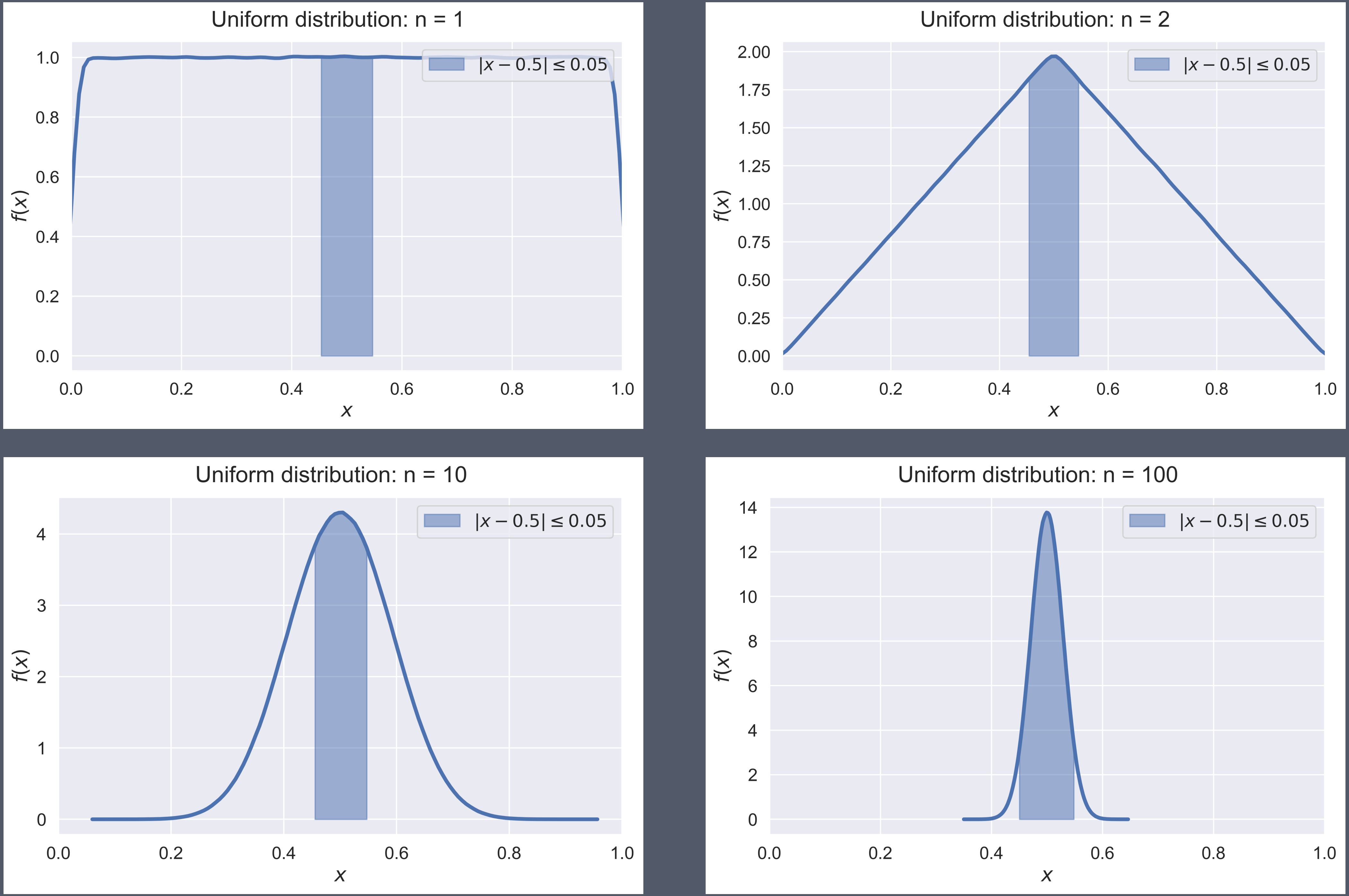
$$V\left(\frac{S_n}{n}\right) = \frac{1}{12n}$$

Chebyshev Inequality

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

$$\mu = \frac{1}{2}, \sigma^2 = \frac{1}{12}$$

$$P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \varepsilon\right) \leq \frac{1}{12n\varepsilon^2}$$


Uniform distribution: n = 1
Uniform distribution: n = 2
Uniform distribution: n = 10
Uniform distribution: n = 100
$|x - 0.5| \leq 0.05$
$f(x)$
x
0.0
0.2
0.4
0.6
0.8
1.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
0
1
2
3
4
6
8
10
12
14

Example 5

- Suppose we choose n real numbers at random, using a normal distribution with mean 0 and variance 1. Then
  - $\mu = E(X_i) = 0$
  - $\sigma^2 = V(X_i) = 1$

Expected value

$$\boldsymbol{E}\left(\frac{S_n}{n}\right) = 0$$

Variance

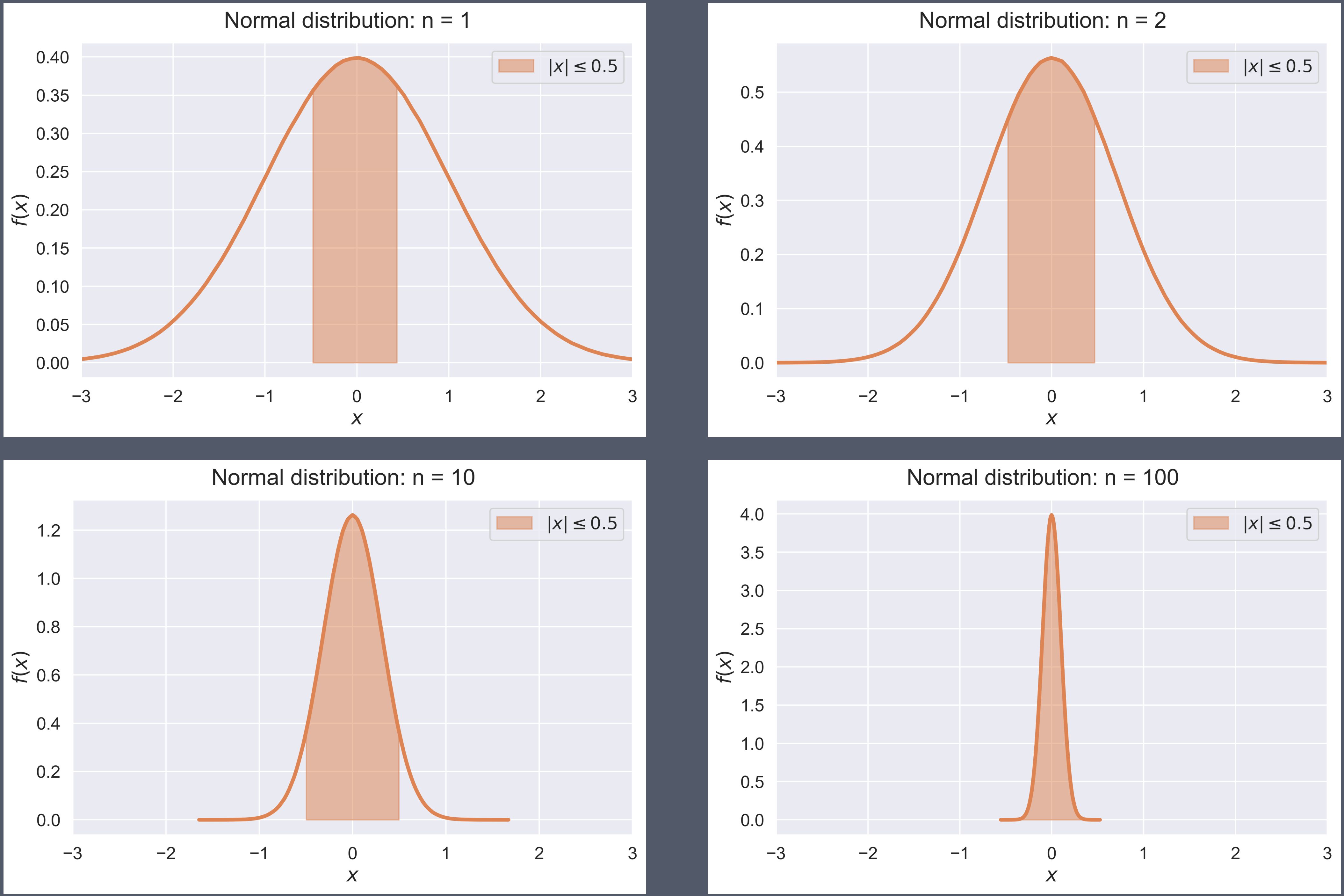
$$V\left(\frac{S_n}{n}\right) = \frac{1}{n}$$

Chebyshev Inequality

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

$$\mu = 0, \sigma^2 = \frac{1}{n}$$

$$P\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) \leq \frac{1}{n\varepsilon^2}$$



Normal distribution: n = 1
Normal distribution: n = 2
Normal distribution: n = 10
Normal distribution: n = 100
|x| ≤ 0.5
f(x)
x

Example 6

- Suppose we choose n numbers from $(-\infty, +\infty)$ with a Cauchy density with parameter $a = 1$.
- We know that for the Cauchy density the expected value and variance are undefined.

Cauchy distribution
$f(x) = \dfrac{1}{\pi a(1 + \frac{x}{a})^2}$
$f(x) = \dfrac{1}{\pi(1 + x)^2}$

Expected value $E(X)$
undefined

Variance $V(X)$
undefined

Example 6

- In this case, the density function for $A_n = \frac{S_n}{n}$ is given by

Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x)^2}$$

The density function for A_n is the same for all n.

!

The Law of Large Numbers does not hold.

!

Why?

Example 6

- In this case, the density function for $A_n = \frac{S_n}{n}$ is given by

Cauchy distribution

$$f(x) = \frac{1}{\pi(1+x)^2}$$

The density function for A_n is the same for all n.

!

The Law of Large Numbers does not hold.

!

Why?

finite expected value and finite variance!

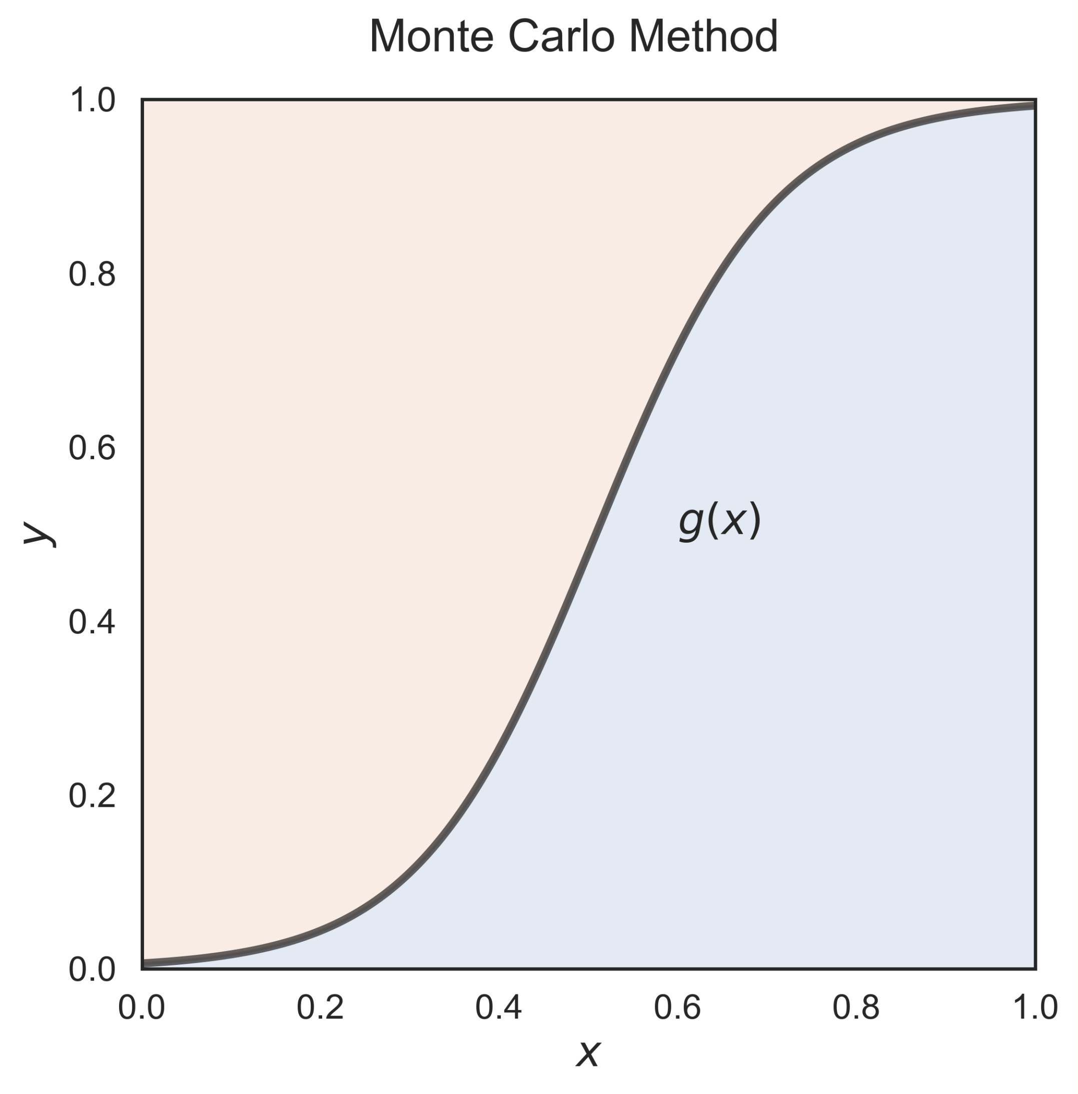
MONTE CARLO METHOD

estimate the area of the region

Monte Carlo method

- Let $g(x)$ be continuous function defined for $x \in [0, 1]$ with values in $[0, 1]$.

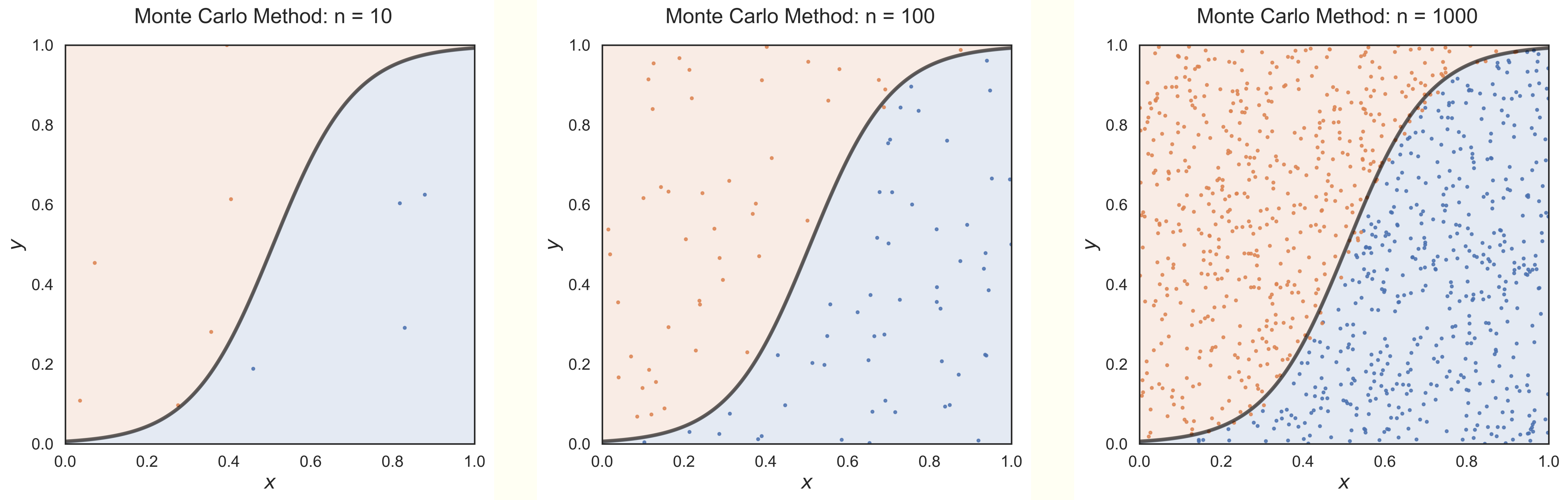
- How to estimate the area of the region under the graph of $g(x)$?



Monte Carlo method: naive

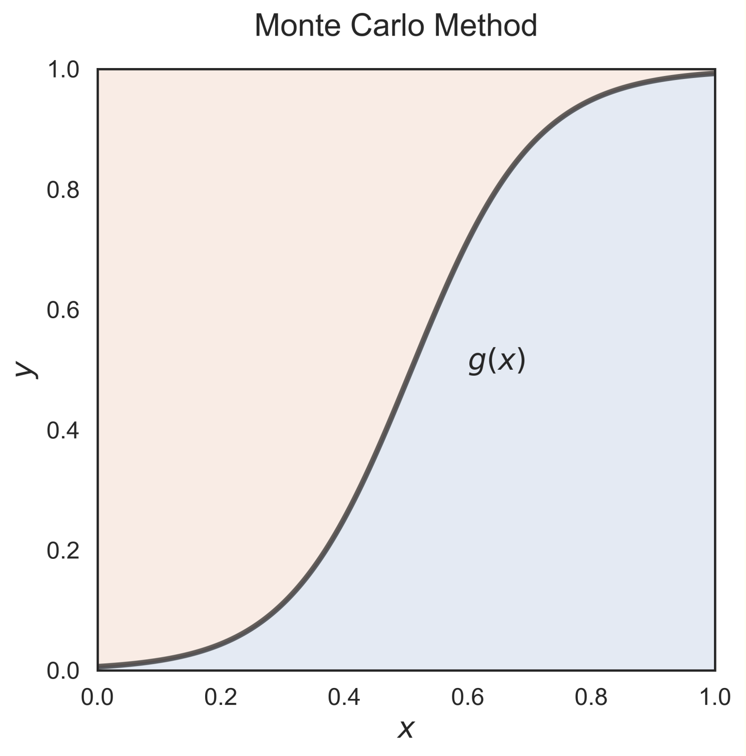
- Let $g(x)$ be continuous function defined for $x \in [0,1]$ with values in $[0,1]$.
- How to estimate the area of the region under the graph of $g(x)$?

Choose a large number of random values for x and y with uniform distribution and seeing what fraction of the points $P(x,y)$ fall inside the region under the graph.



Monte Carlo method: advanced

- Let $g(x)$ be continuous function defined for $x \in [0, 1]$ with values in $[0, 1]$.
- How to estimate the area of the region under the graph of $g(x)$?



Choose a large number of independent values X_n at random from $[0, 1]$ with uniform density.

Set $Y_n = g(X_n)$.

Area: A

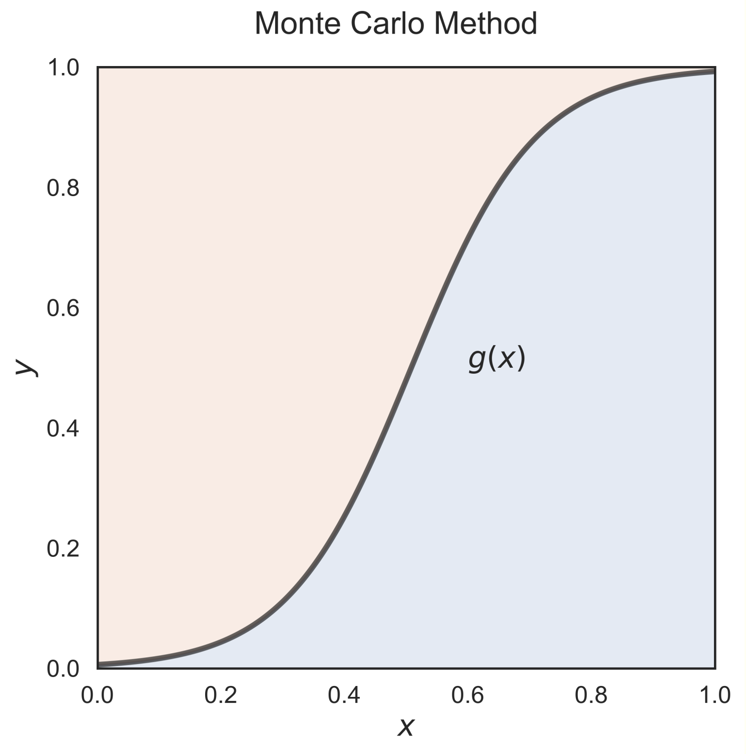
$$\int_0^1 g(x)dx$$

Expected value: μ

$$E(Y_n) = E\big(g(X_n)\big) = \int_0^1 g(x)f(x)dx = \int_0^1 g(x)dx$$

Monte Carlo method: advanced

- Let $g(x)$ be continuous function defined for $x \in [0,1]$ with values in $[0,1]$.
- How to estimate the area of the region under the graph of $g(x)$?



Choose a large number of independent values X_n at random from $[0,1]$ with uniform density.

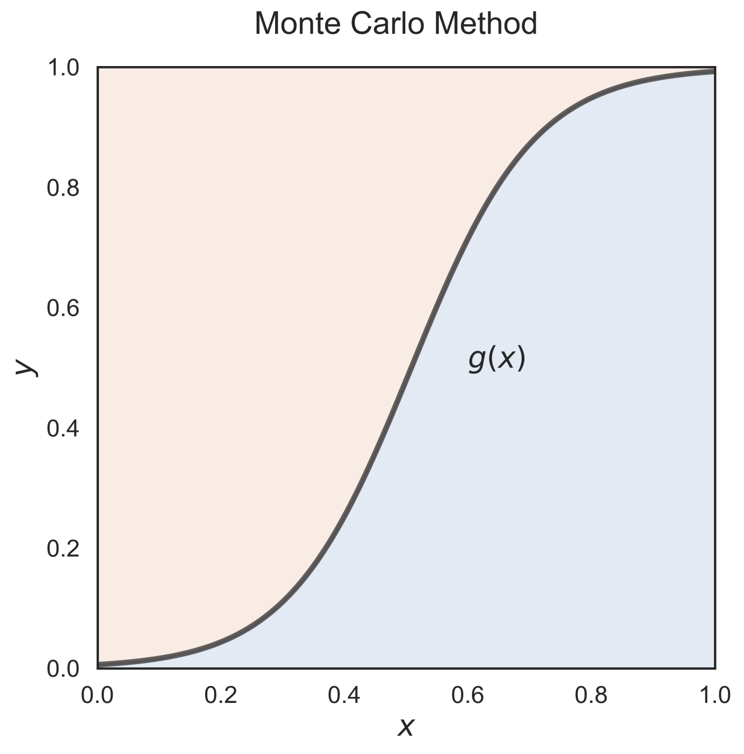
Set $Y_n = g(X_n)$.

Area: A	Expected value: μ
$\int_0^1 g(x)dx$	$E(Y_n) = \\ = \int_0^1 g(x)dx$

$$A = \mu \approx \frac{1}{n}\sum_{i=1}^{n} Y_n = \frac{1}{n}\sum_{i=1}^{n} g(X_n)$$

Monte Carlo method: advanced

- Let $g(x)$ be continuous function defined for $x \in [0,1]$ with values in $[0,1]$.
- How to estimate the area of the region under the graph of $g(x)$?

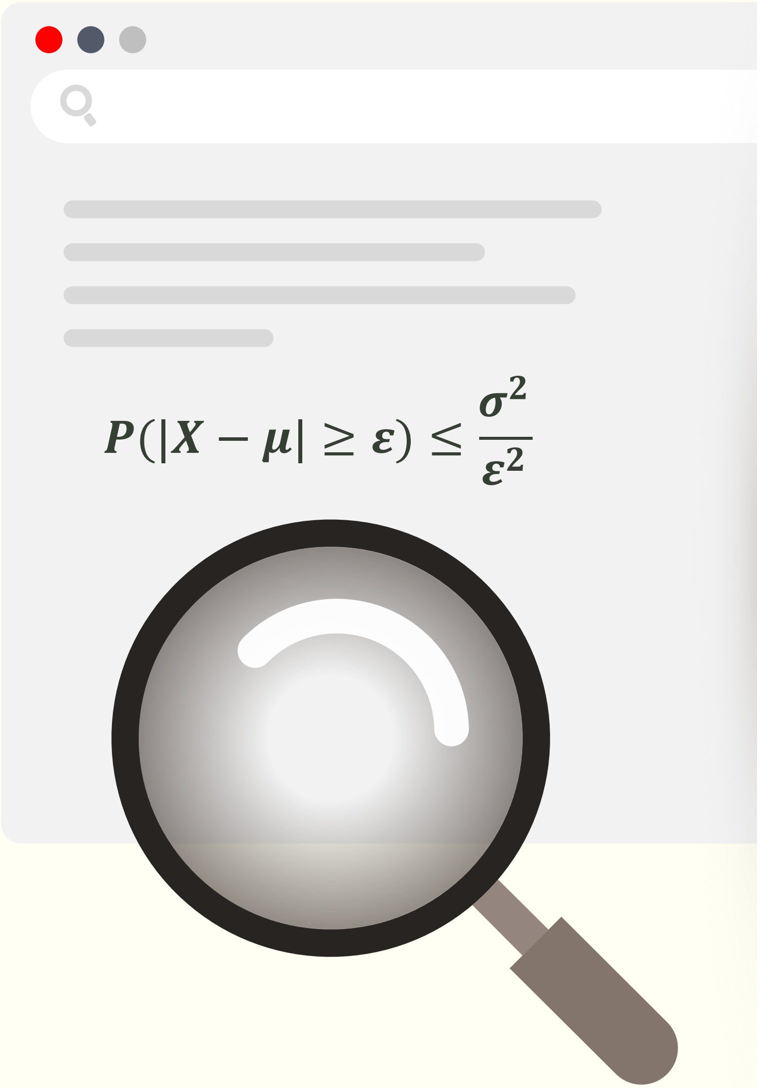


Area: A	Expected value: μ
$\int_0^1 g(x)dx$	$E(Y_n) = = \int_0^1 g(x)dx$

$$A = \mu \approx \frac{1}{n}\sum_{i=1}^{n} Y_n = \frac{1}{n}\sum_{i=1}^{n} g(X_n) = A_n$$

$$\sigma^2 = E\left((Y_n - \mu)^2\right) = \int_0^1 (g(x) - \mu)^2 dx \leq 1$$

$$P(|A_n - A| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \leq \frac{1}{n\varepsilon^2}$$



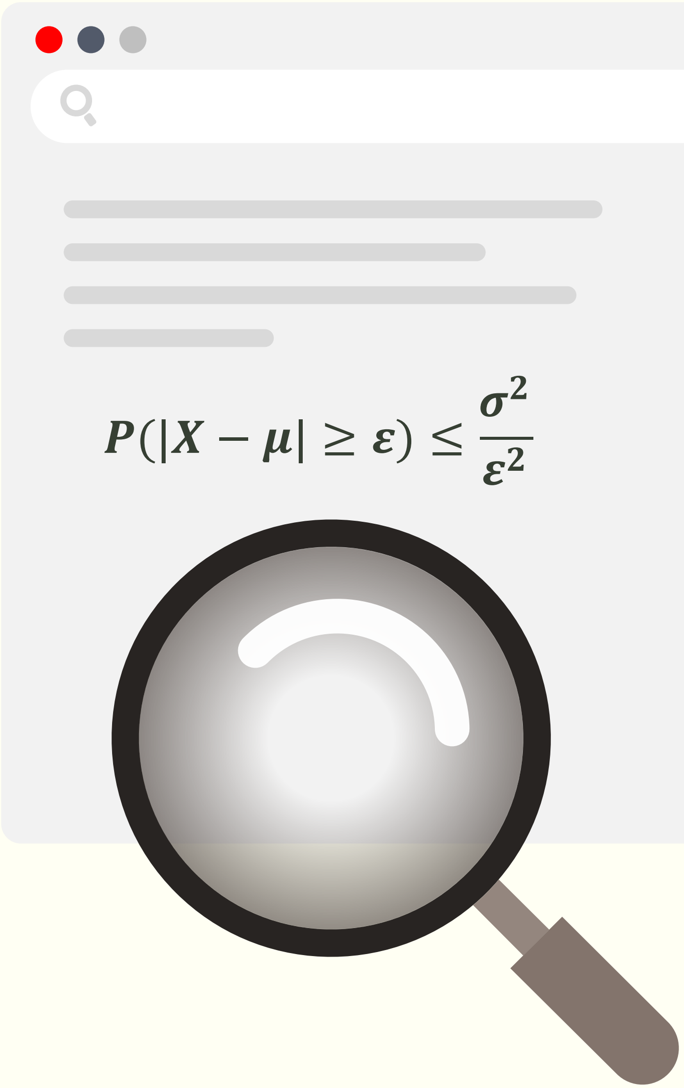
Quiz 12 Question 3

A student's score on a particular probability final is a random variable with values of $[0, 100]$, mean 70, and variance 25.

Using Chebyshev's Inequality, find a lower bound for the probability that the student's score will fall between 65 and 75.

$$P(65 \leq X \leq 75) = P(|X - 70| \leq 5) = 1 - P(|X - 70| \geq 5)$$
$$\geq 1 - \frac{25}{5^2} = 1 - 1 = 0$$

not interesting



Quiz 12 Question 4

A student's score on a particular probability final is a random variable with values of $[0, 100]$, mean 70, and variance 25.

If 10 students take the final, find a lower bound for the probability that the class average will fall between 65 and 75.

$$P(65 \leq A_n \leq 75) = P(|A_n - 70| \leq 5) = 1 - P(|A_n - 70| \geq 5)$$
$$\geq 1 - \frac{25/10}{5^2} = 1 - \frac{1}{10} = \frac{9}{10}$$