MATH 20: PROBABILITY

Variance of Discrete Random Variables

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Important Distributions

Discrete Uniform Distribution

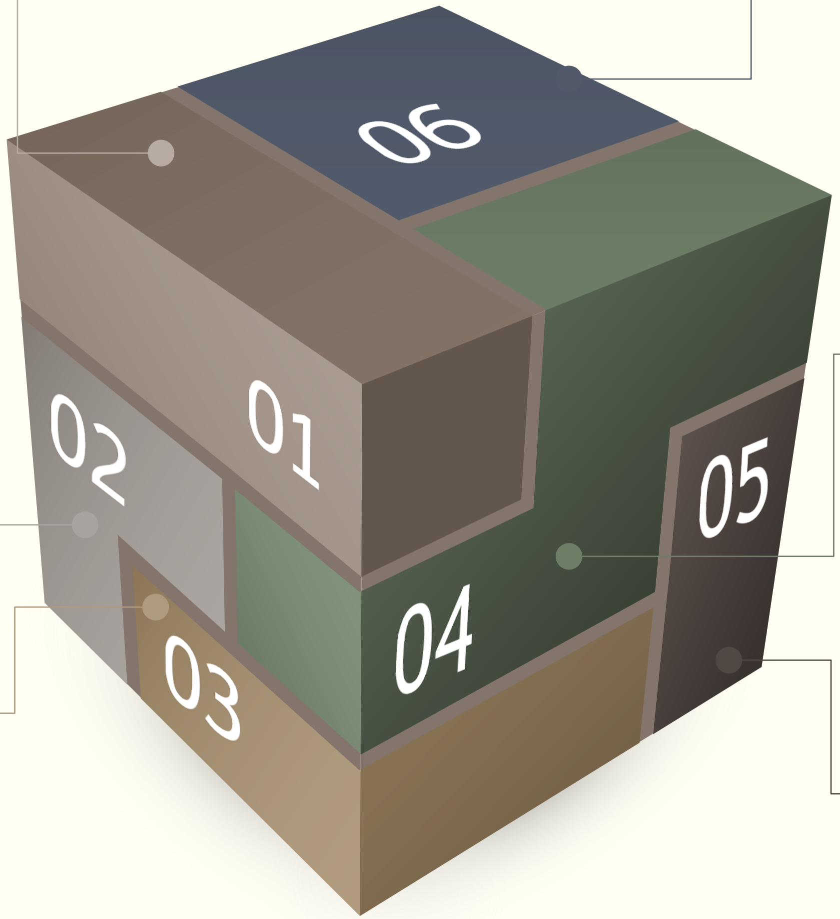
$$m(\omega) = \frac{1}{n}$$

Binomial Distribution

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

Geometric Distribution

$$P(T = n) = q^{n-1} p$$



Hypergeometric Distribution

$$h(N, k, n, x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

Negative Binomial Distribution

$$u(x, k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Binomial

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k} \qquad E(X) = np$$

Geometric

$$P(T = n) = q^{n-1}p \qquad E(X) = \frac{1}{p}$$

Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad E(X) = \lambda$$

Negative binomial

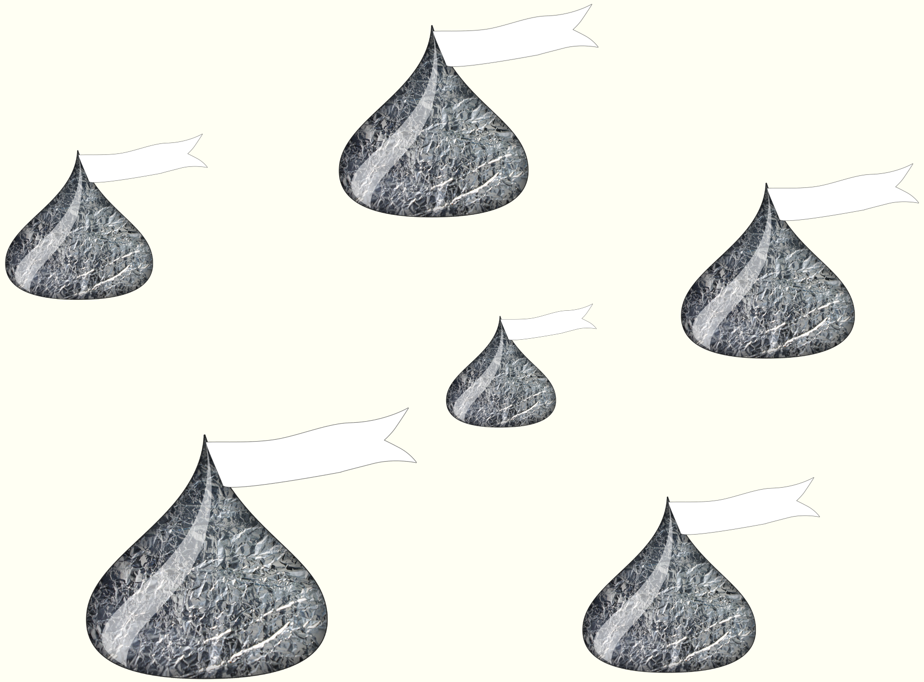
$$u(x,k,p) = \binom{x-1}{k-1} p^k q^{x-k} \qquad E(X) = k\frac{q}{p}$$

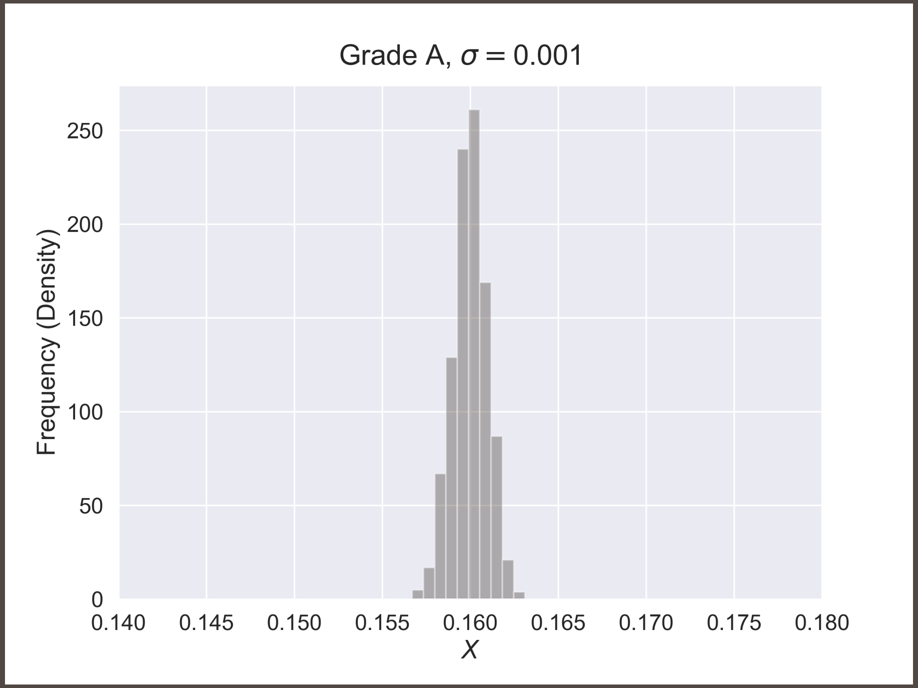
Hypergeometric

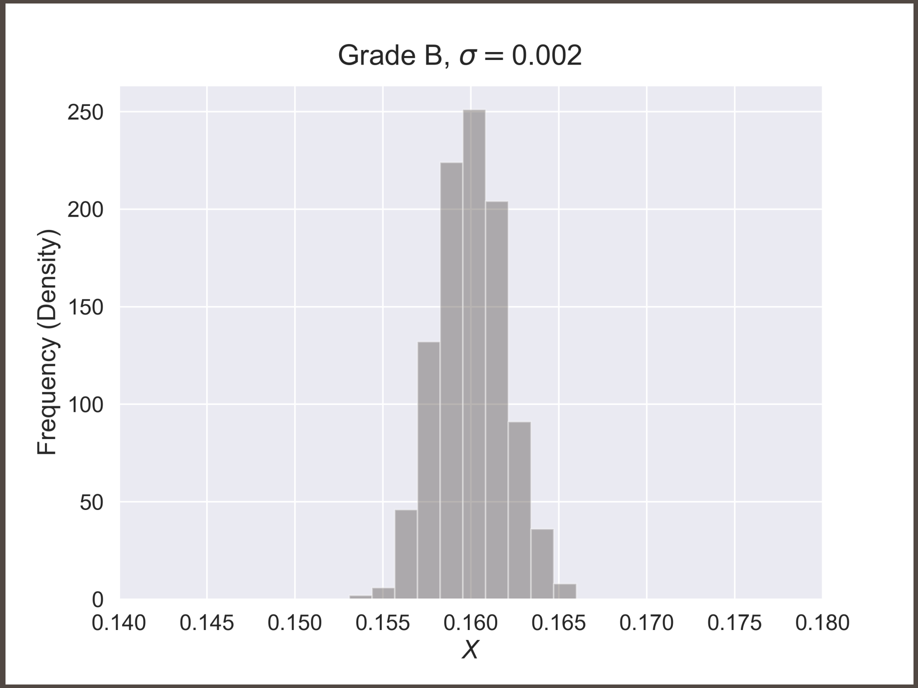
$$h(N,k,n,x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad E(X) = n\frac{k}{N}$$

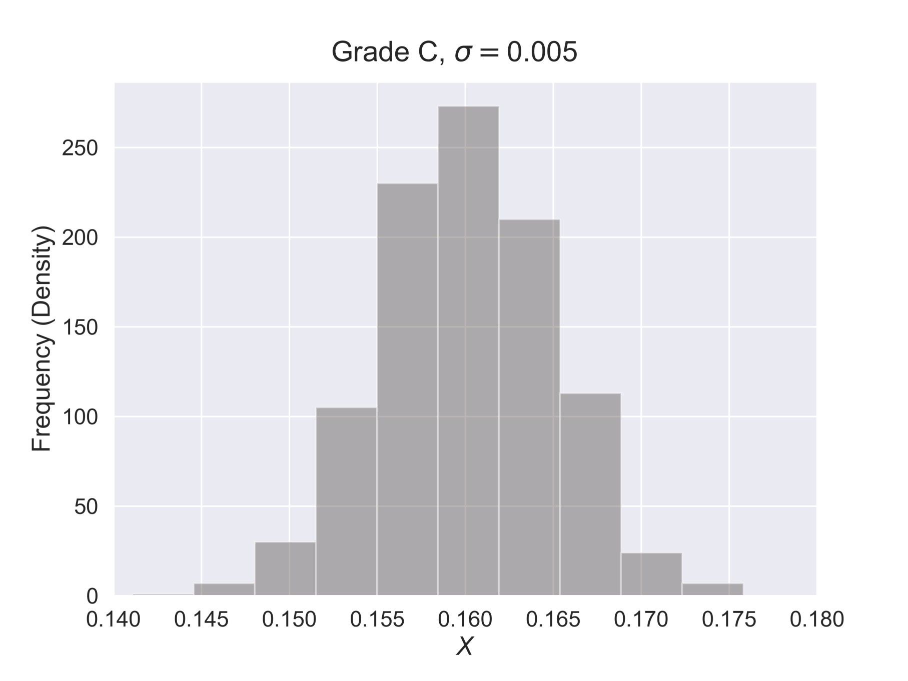
How Much Does a Hershey Kiss Weight?

- A single standard Hershey's Kiss weighs 0.16 ounces.









Same μ, different σ.

How to measure the quality of products?

?

Home Made Versus Factory Made

Variance of Discrete Random Variables

- Let X be a numerically-valued random variable with expected value $\mu = E(X)$. Then the variance of X, denoted by $V(X)$, is

$$V(X) = E((X-\mu)^2).$$

Expected value $E(X)$

$$\sum_{x\in\Omega} x m(x)$$

Expected value $E(\phi(X))$

$$\sum_{x\in\Omega} \phi(x) m(x)$$

Variance $V(X)$

$$\sum_{x\in\Omega} (x-\mu)^2 m(x)$$

Variance of Discrete Random Variables

- Let X be a numerically-valued random variable with expected value $\mu = E(X)$. Then the variance of X, denoted by $V(X)$, is

$$V(X) = E((X-\mu)^2).$$

- Standard deviation of X, denoted by $D(X)$, is $D(X) = \sqrt{V(X)}$.
- We often write σ for $D(X)$ and σ^2 for $V(X)$.

Variance $V(X)$

$$\sum_{x \in \Omega} (x-\mu)^2 m(x)$$

Variance of Discrete Random Variables

- Let X be a numerically-valued random variable with expected value $\mu = E(X)$. Then the variance of X, denoted by $V(X)$, is

$$V(X) = E((X - \mu)^2).$$

- If X is any random variable with $\mu = E(X)$, then

$$V(X) = E(X^2) - \mu^2.$$

Variance $V(X)$
$\sum_{x \in \Omega} (x - \mu)^2 m(x)$
$\sum_{x \in \Omega} x^2 m(x) - \sum_{x \in \Omega} x m(x)$

Proof

Let X be a numerically-valued random variable with expected value $\mu = E(X)$.

$$V(X) = \sum_{x\in\Omega} (x-\mu)^2 m(x)$$

$$\sum_{x\in\Omega} x^2 m(x) = E(X^2)$$

$$\sum_{x\in\Omega} x m(x) = \mu$$

$$\sum_{x\in\Omega} m(x) = 1$$

$$\begin{aligned} V(X) &= \sum_{x\in\Omega} (x-\mu)^2 m(x) \\ &= \sum_{x\in\Omega} (x^2 - 2\mu x + \mu^2) m(x) \\ &= \sum_{x\in\Omega} x^2 m(x) - 2\mu \sum_{x\in\Omega} x m(x) + \mu^2 \sum_{x\in\Omega} \mu^2 m(x) \end{aligned}$$

$$\begin{aligned} V(X) &= \sum_{x\in\Omega} x^2 m(x) - 2\mu \sum_{x\in\Omega} x m(x) + \mu^2 \sum_{x\in\Omega} \mu^2 m(x) \\ &= E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

Example 1

Toss a coin
head or tail
1 or 0
$m(x) = \frac{1}{2}$
$\mu = E(X) = \frac{1}{2}$
$\sigma^2 = V(X) = \cdots$

Example 1

Toss a coin
head or tail
1 or 0
$m(x) = \frac{1}{2}$
$\mu = E(X) = \frac{1}{2}$
$\sigma^2 = V(X) = \frac{1}{4}$

Example 2

Roll a dice
1, 2, 3, 4, 5, or 6
$m(x) = \frac{1}{6}$
$\mu = E(X) = \frac{7}{2}$
$\sigma^2 = V(X) = \cdots$

Example 2

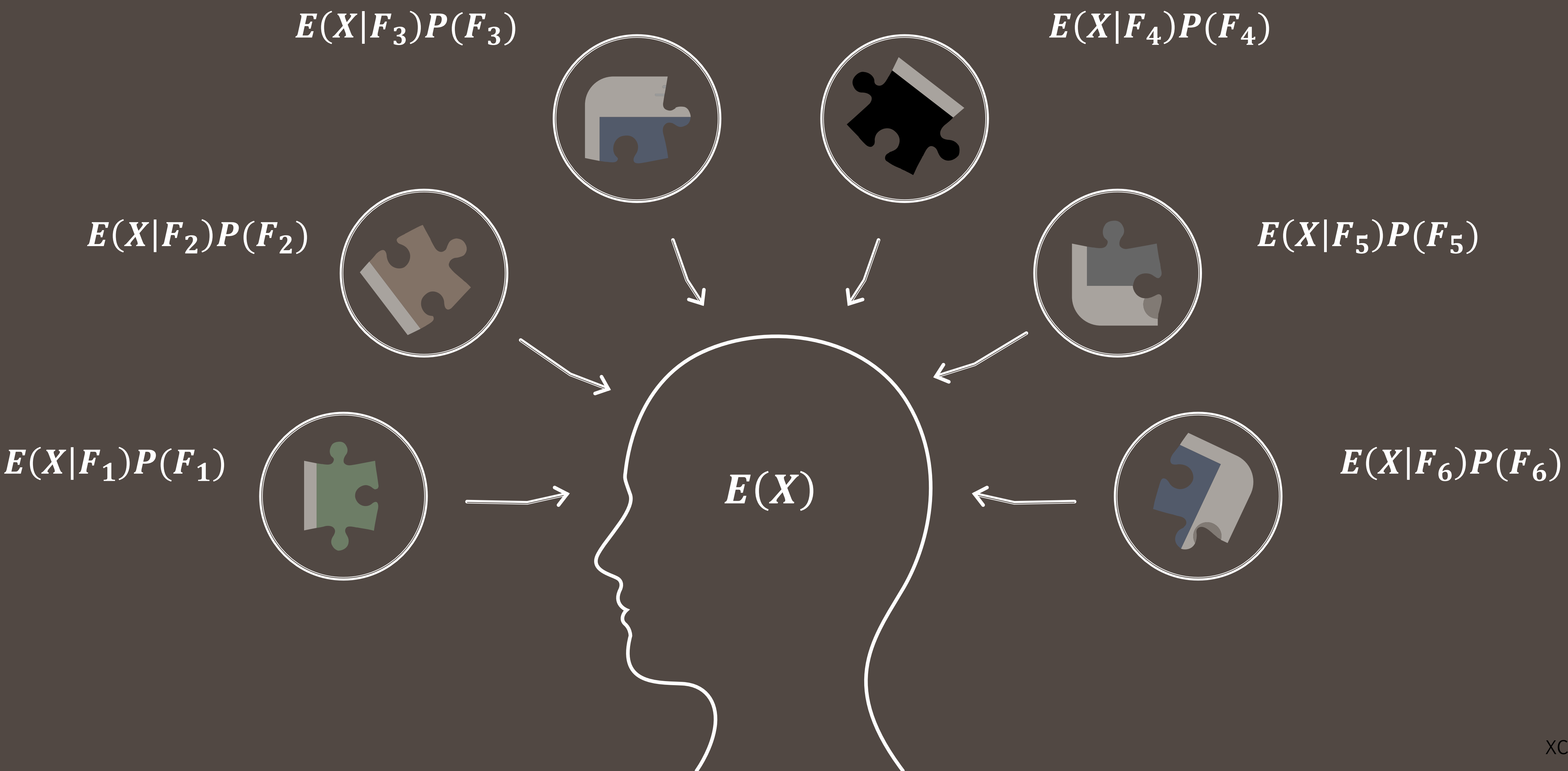
Roll a dice
1, 2, 3, 4, 5, or 6
$m(x) = \frac{1}{6}$
$\mu = E(X) = \frac{7}{2}$
$\sigma^2 = V(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$

Quiz 9

What is the expected number of dice tosses needed to get two consecutive six's?

Number of tosses
2, 3, 4, …
$E(X) = E(X\|1)P(1) + E(X\|2) + E(X\|3)P(3) + E(X\|4)P(4) + E(X\|5)P(5) + E(X\|6)P(6)$
$E(X) = \frac{5}{6}E(X\|1) + \frac{1}{6}E(X\|6) = \frac{5}{6}\big(1 + E(X)\big) + \frac{1}{6}(\frac{5}{6}E(X\|61) + \frac{1}{6}E(X\|66))$
$E(X) = \frac{5}{6}\big(1 + E(X)\big) + \frac{1}{6}[\frac{5}{6}\big(2 + E(X)\big) + \frac{2}{6}]$

Is the distribution function a must for calculating expectation?



Example 3

Consider the general Bernoulli trial process. As usual, we let $X = 1$ if the outcome is a success and 0 if it is a failure.

Expected value $E(X)$

$$\sum_{x\in\Omega} xm(x) = 1\times p + 0\times(1-p) = p$$

Bernoulli trial

$$m(x) = \begin{cases} p, & X = 1 \\ 1-p, & X = 0 \end{cases}$$

Variance $V(X)$

$$E(X^2) - \mu^2 = \cdots$$

Example 3

Consider the general Bernoulli trial process. As usual, we let $X = 1$ if the outcome is a success and 0 if it is a failure.

Expected value $E(X)$

$$\sum_{x \in \Omega} x m(x) = 1 \times p + 0 \times (1 - p) = p$$

Bernoulli trial

$$m(x) = \begin{cases} p, & X = 1 \\ 1 - p, & X = 0 \end{cases}$$

Variance $V(X)$

$$E(X^2) - \mu^2 = p - p^2$$

Binomial

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

$$E(X) = np,\ V(X) = npq$$

Geometric

$$P(T=n) = q^{n-1}p$$

$$E(X) = \frac{1}{p},\ V(X) = \frac{1-p}{p^2}$$

Poisson

$$P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

$$E(X) = \lambda,\ V(X) = \lambda$$

Negative binomial

$$u(x,k,p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$E(X) = k\frac{q}{p},\ V(X) = k\frac{q}{p^2}$$

Hypergeometric

$$h(N,k,n,x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$E(X) = n\frac{k}{N},\ V(X) = \frac{nk(N-k)(N-n)}{N^2(N-1)}$$

Binomial Distribution and Poisson Distribution

Binomial Distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

$$E(X) = np,\ V(X) = npq$$

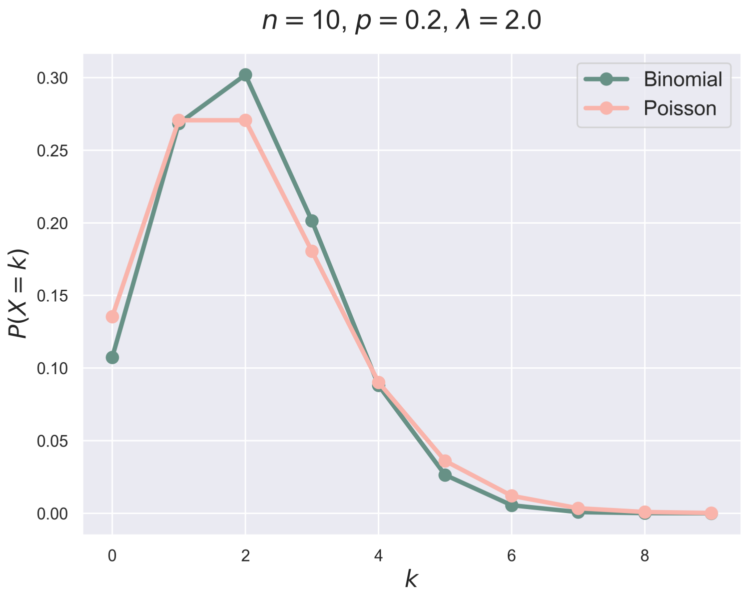
Poisson Distribution

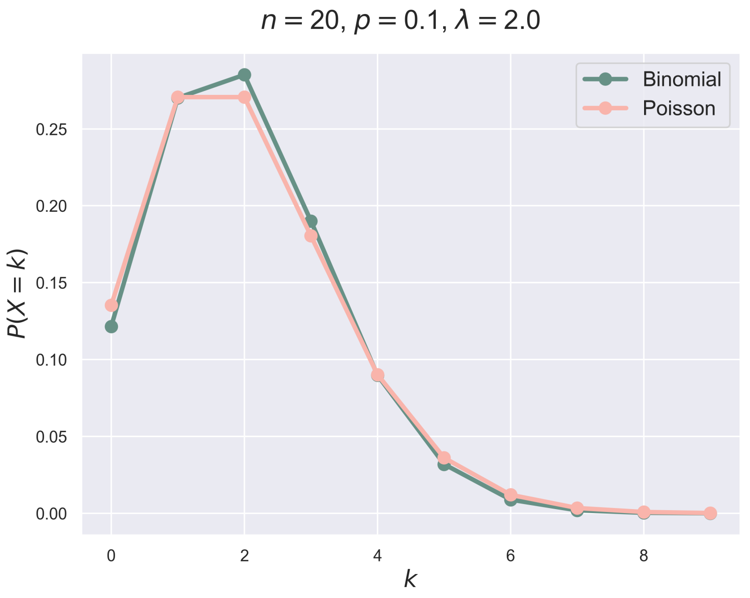
$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

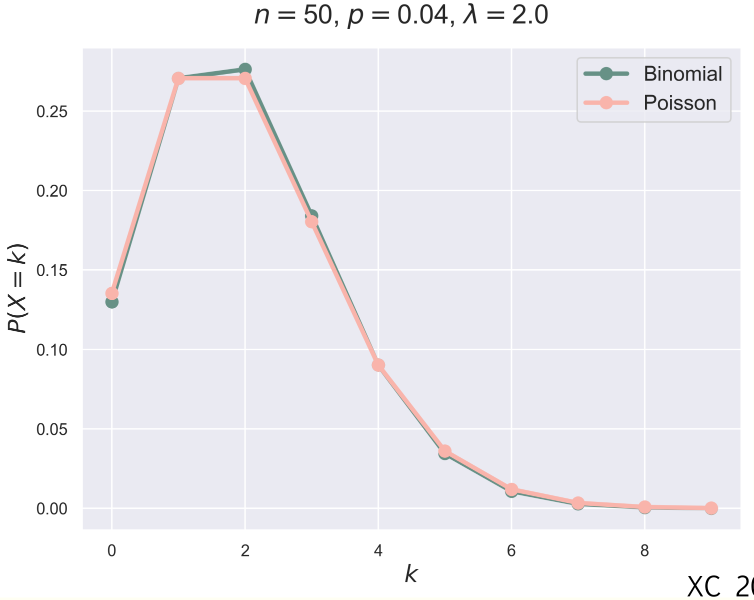
$$E(X) = \lambda,\ V(X) = \lambda$$

$$p = \frac{\lambda t}{n},\ t = 1,\ n \to \infty,\ p \to 0$$

=







Binomial

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k} \qquad E(X) = np$$

Variance $V(X)$

$$E(X^2) - \mu^2 = \sum_{x\in\Omega} x^2 m(x) - \mu^2$$

$$\sum_{x\in\Omega} x^2 m(x) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k^2 \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k^2 \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} npk \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} q^{n-k} = np \sum_{k=1}^{n} (k-1+1) \binom{n-1}{k-1} p^{k-1} q^{n-k}$$

$$= np \sum_{l=0}^{n-1} l \binom{n-1}{l} p^l q^{n-1-l} + np \sum_{l=0}^{n-1} \binom{n-1}{l} p^l q^{n-1-l}$$

Binomial

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k} \qquad E(X) = np$$

Variance $V(X)$

$$E(X^2) - \mu^2 = \sum_{x\in\Omega} x^2 m(x) - \mu^2$$

- $\sum_{x\in\Omega} x^2 m(x) = n(n-1)p^2 \sum_{l=1}^{n-1} \binom{n-2}{l-1} p^{l-1} q^{n-1-l} + np = n(n-1)p^2 + np$
- $\mu^2 = n^2 p^2$
- $\sum_{x\in\Omega} x^2 m(x) - \mu^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$

Poisson

$$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda} \qquad E(X)=\lambda$$

Variance $V(X)$

$$E(X^2)-\mu^2=\sum_{x\in\Omega}x^2m(x)-\mu^2$$

$$\sum_{x\in\Omega}x^2m(x)=\sum_{k=0}^{+\infty}k^2\frac{\lambda^k}{k!}e^{-\lambda}=\sum_{k=1}^{+\infty}k\frac{\lambda^k}{k!}e^{-\lambda}$$

$$=\sum_{k=1}^{+\infty}\lambda k\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}=\sum_{k=1}^{+\infty}\lambda(k-1+1)\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}$$

$$=\lambda\sum_{l=0}^{+\infty}l\frac{\lambda^l}{l!}e^{-\lambda}+\lambda\sum_{l=0}^{+\infty}\frac{\lambda^l}{l!}e^{-\lambda}$$

Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$E(X) = \lambda$$

Variance $V(X)$

$$E(X^2) - \mu^2 = \sum_{x \in \Omega} x^2 m(x) - \mu^2$$

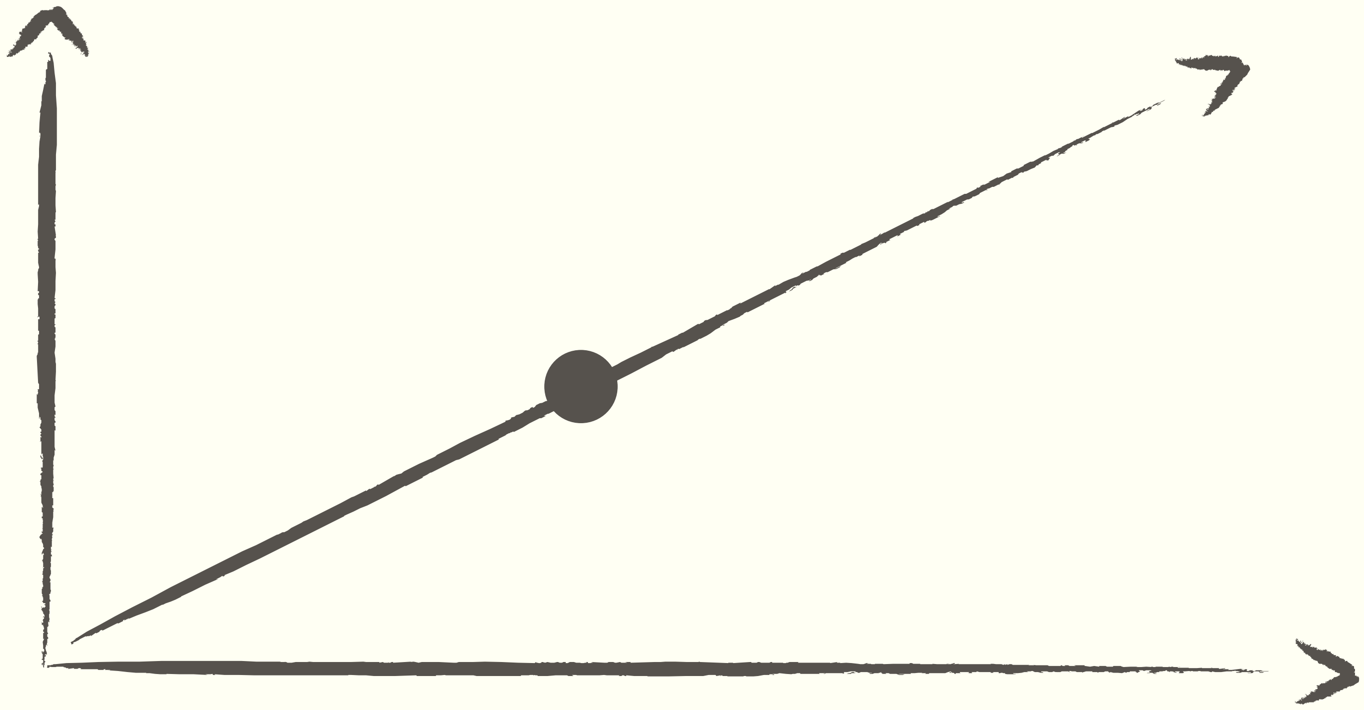
- $\sum_{x\in\Omega} x^2 m(x) = \lambda \sum_{l=0}^{+\infty} l \frac{\lambda^l}{l!} e^{-\lambda} + \lambda \sum_{l=0}^{+\infty} \frac{\lambda^l}{l!} e^{-\lambda} = \lambda^2 \sum_{l=1}^{+\infty} \frac{\lambda^{l-1}}{(l-1)!} e^{-\lambda} + \lambda = \lambda^2 + \lambda$
- $\mu^2 = \lambda^2$
- $\sum_{x\in\Omega} x^2 m(x) - \mu^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Linearity

$$E(X + Y) = E(X) + E(Y)$$

$$E(cX) = cE(X).$$

$$E(aX + b) = aE(X) + b$$



Properties of Variance

- If X is any random variable and c is any constant, then

$$V(cX) = c^2 V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

Variance
$V(X) = E(X^2) - \mu^2$ $= E(X^2) - [E(X)]^2$
$V(cX) = E\left((cX)^2\right) - [E(cX)]^2$

Properties of Variance

- If X is any random variable and c is any constant, then

$$V(cX) = c^2 V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

Variance
$V(X) = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$
$V(cX) = E\big((cX)^2\big) - [E(cX)]^2 = E(c^2X^2) - [cE(X)]^2$ $= c^2E(X^2) - c^2[E(X)]^2 = c^2(E(X^2) - [E(X)]^2)$ $= c^2V(X)$

Properties of Variance

- If X is any random variable and c is any constant, then

$$V(cX) = c^2 V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

Variance
$V(X) = E\big((X - \mu)^2\big) = E\big((X - E(X))^2\big)$
$V(X + c) = E\big((X + c - E(X + c))^2\big)$

Properties of Variance

- If X is any random variable and c is any constant, then

$$V(cX) = c^2 V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

Variance

$$V(X) = E\big((X-\mu)^2\big) = E\big((X - E(X))^2\big)$$

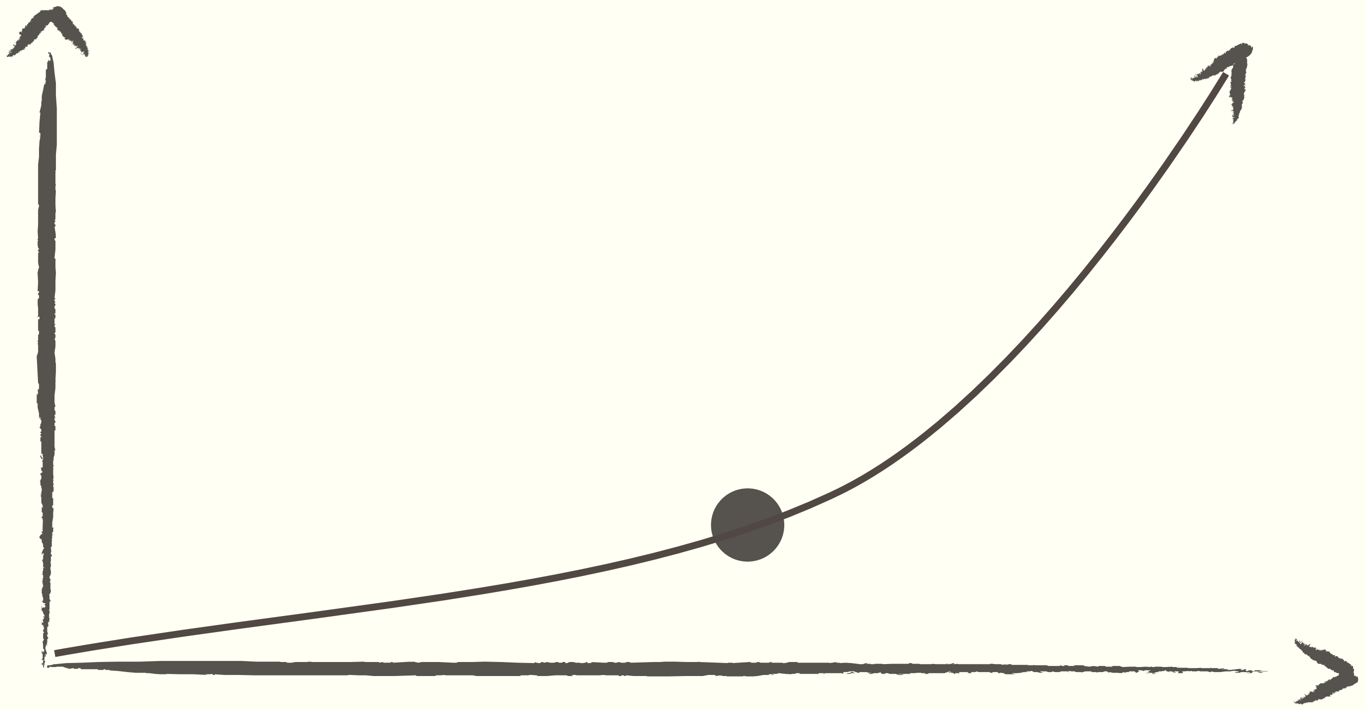
$$\begin{aligned} V(X+c) &= E\big((X + c - E(X+c))^2\big) = E\big((X + c - E(X) - c)^2\big) \\ &= E\left(\left(X - E(X)\right)^2\right) \\ &= V(X). \end{aligned}$$

Non-linearity

$$V(cX) = c^2 V(X)$$

$$V(X + c) = V(X)$$

$$V(aX + b) = a^2 V(X)$$



When do we need independence?

$X + Y$	XY

Do not need

$E(X + Y) = E(X) + E(Y)$

...

...

$\mathbf{V}(\mathbf{X} + \mathbf{Y}) = \mathbf{V}(\mathbf{X}) + \mathbf{V}(\mathbf{Y})$

Need

$E(XY) = E(X)E(Y)$

...

When do we need independence?

$X + Y$

XY

Do not need

$E(X+Y) = E(X) + E(Y)$

...

...

Need

$E(XY) = E(X)E(Y)$

$V(X+Y) = V(X) + V(Y)$

...

Proof

Let X and Y be random variables with finite expected values.
And X and Y are independent random variables.

$$V(X) = E(X^2) - \mu^2$$

$$E(X) = a$$

$$E(Y) = b$$

$$E(X+Y) = a+b$$

$$E(XY) = E(X)E(Y)$$

$$\begin{aligned} V(X+Y) &= E\big((X+Y)^2\big) - (a+b)^2 \\ &= E(X^2+2XY+Y^2) - (a+b)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - a^2 - 2ab - b^2 \end{aligned}$$

$$\begin{aligned} V(X+Y) &= E(X^2) + 2E(X)E(Y) + E(Y^2) - a^2 - 2ab - b^2 \\ &= E(X^2) + 2ab + E(Y^2) - a^2 - 2ab - b^2 \\ &= E(X^2) - a^2 + E(Y^2) - b^2 \\ &= V(X) + V(Y) \end{aligned}$$

Properties of Variance

- If X is any random variable and c is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

- Let X and Y be two **independent** random variables. Then

$$V(X + Y) = V(X) + V(Y).$$

- It can be shown that the variance of the sum of any number of mutually independent random variables is the sum of the individual variances.

Properties of Variance

- Let $X_1, X_2, \cdots, X_n$ be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sum, and $A_n = \frac{S_n}{n}$ be the average. Then

$$V(cX) = c^2 V(X)$$

independent

$$V(X + Y) = V(X) + V(Y).$$

$$E(S_n) = \cdots$$

?

$$V(S_n) = \cdots$$

?

Properties of Variance

- Let $X_1, X_2, \cdots, X_n$ be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sum, and $A_n = \frac{S_n}{n}$ be the average. Then

$V(cX) = c^2 V(X)$

independent

$V(X + Y) = V(X) + V(Y).$

$E(S_n) = n\mu$ =

$V(S_n) = n\sigma^2$ =

$E(A_n) = \cdots$?

$V(A_n) = \cdots, D(A_n) = \cdots$?

Properties of Variance

- Let $X_1, X_2, \cdots, X_n$ be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sum, and $A_n = \frac{S_n}{n}$ be the average. Then

$E(S_n) = n\mu$ =

$E(A_n) = \mu$ =

$V(S_n) = n\sigma^2$ =

$V(A_n) = \frac{\sigma^2}{n}, D(A_n) = \frac{\sigma}{\sqrt{n}}$ =

Example

Bernoulli trial

$$m(x) = \begin{cases} p, & X = 1 \\ 1 - p, & X = 0 \end{cases}$$

$$\boldsymbol{E}(\boldsymbol{X}) = p$$

$$V(X) = p - p^2$$

Binomial

n independent Bernoulli trials

$$E(S_n) = n\mu$$ =

$$V(S_n) = n\sigma^2$$ =

Law of Large Numbers

- Let $X_1, X_2, \cdots, X_n$ be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$.
- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sum, and $A_n = \frac{S_n}{n}$ be the average. Then

$E(A_n) = \mu$ =

$V(A_n) = \frac{\sigma^2}{n}, D(A_n) = \frac{\sigma}{\sqrt{n}}$ =

$n \to +\infty$

$V(A_n) \to \cdots$

$D(A_n) \to \cdots$