MATH 20: PROBABILITY

Lecture 3: Permutations

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PINE

2 East Wheelock Street, Hanover, NH

SMALL PLATES

- Buffalo Cauliflower
- Pine Fries

2 options

LARGE PLATES

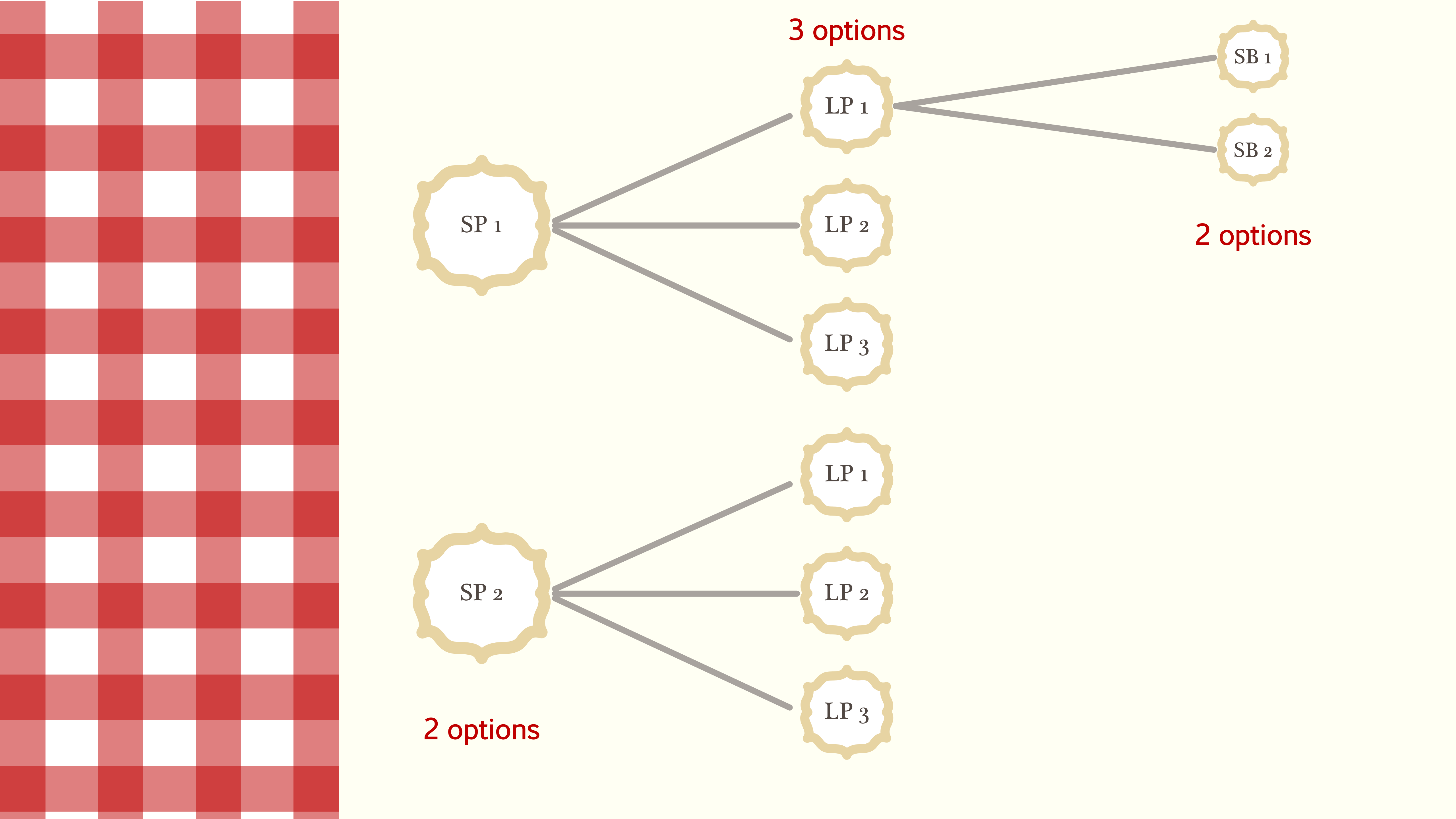
- Crispy Fish Taco
- Maine Lobster Roll
- Hanover Inn Burger

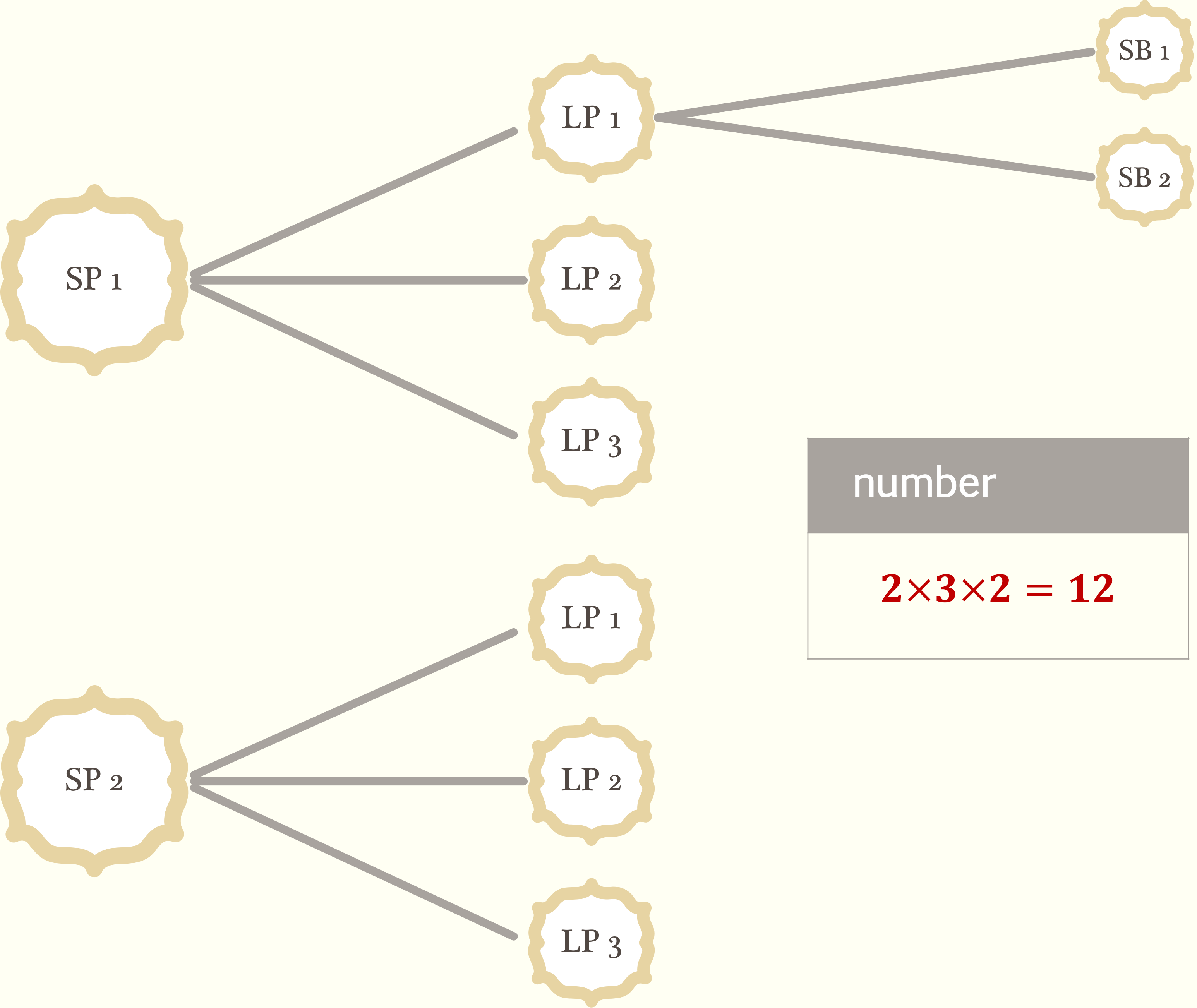
3 options

SWEET BITES

- Citrus Vanilla Cheesecake
- House-Made Ice Creams

2 options


3 options
SB 1
LP 1
SB 2
SP 1
LP 2
2 options
LP 3
LP 1
SP 2
LP 2
LP 3
2 options



number

$2 \times 3 \times 2 = 12$

Le tour du monde en quatre-vingts jours

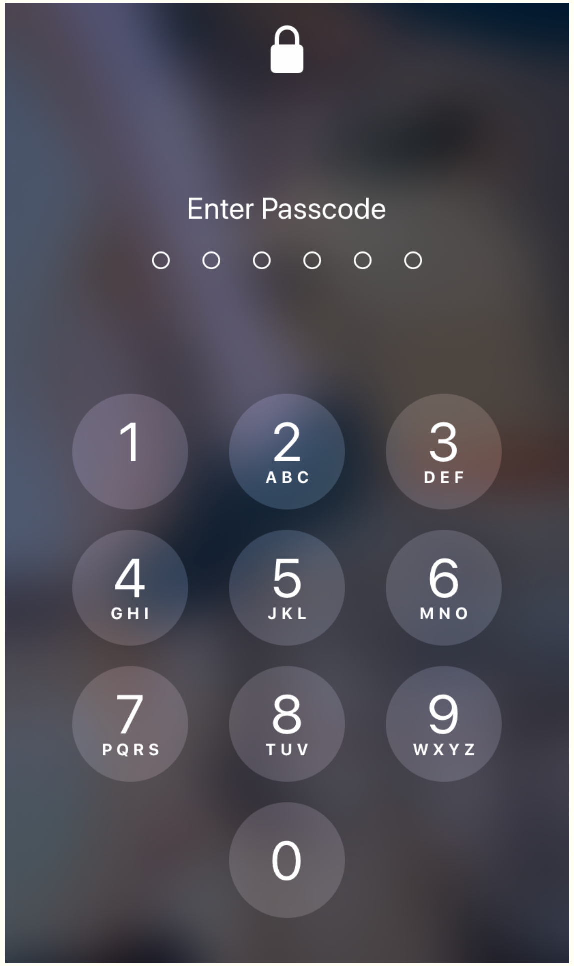
Le tour du monde en quatre-vingts jours

number
$3 \times 2 \times 2 \times 2 \times 1 = 24$

Lockscreen Password

Forgot your password?

Maximum number of attempts required is …

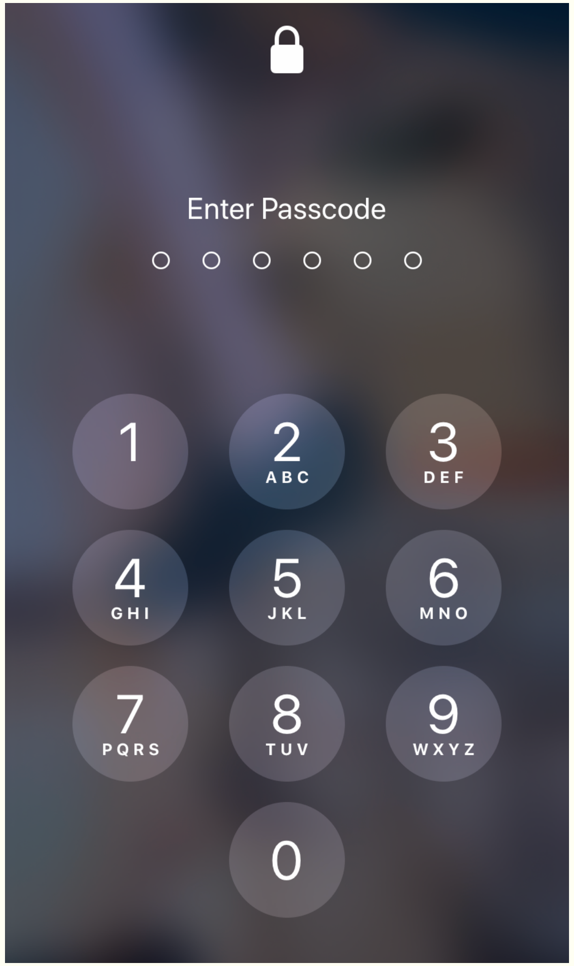


Lockscreen Password

Forgot your password?

Maximum number of attempts required is …

number
$\mathbf{10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6}$



Counting Problems

- Experiment is composed of multiple **independent** stages.
- The numbers of outcomes may be different for each stage.

- Examples: restaurant menu, multi-destination travel, password, name initials…
- Counting technique: **tree diagram**.

What does independent mean here?

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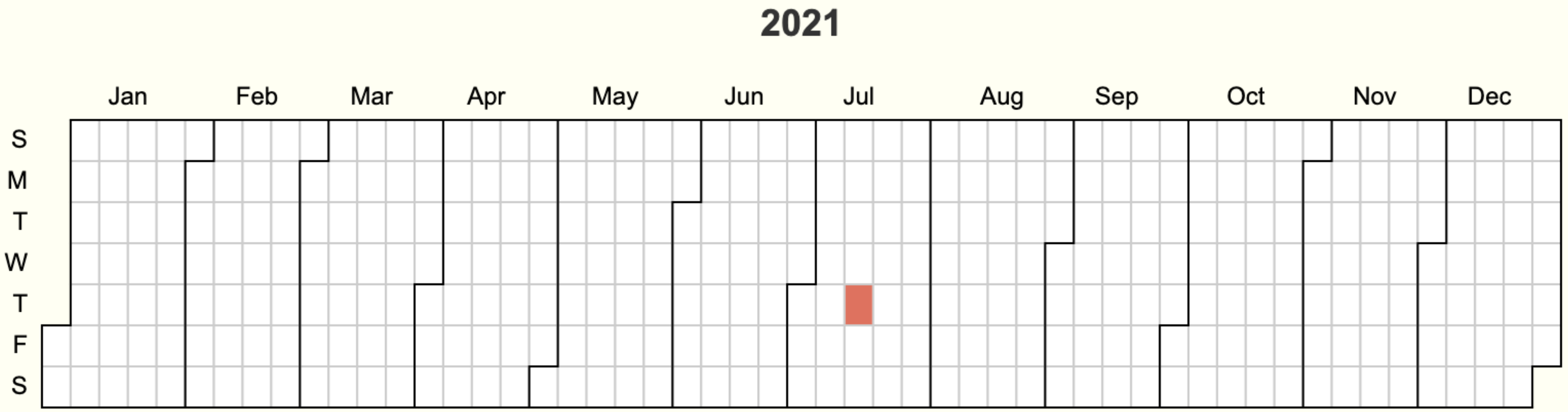
Birthday Problem

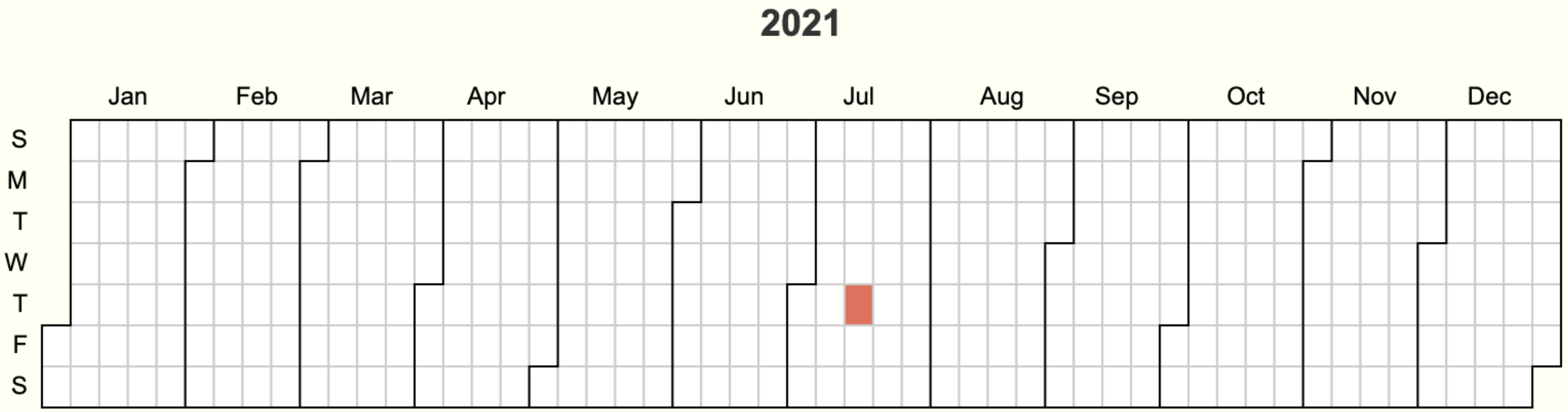
How many students do we need to have in our hours section to make it favorable bet (that is, probability of success greater than $\frac{1}{2}$) that two people in the classroom will have the same birthday?

Birthday Problem



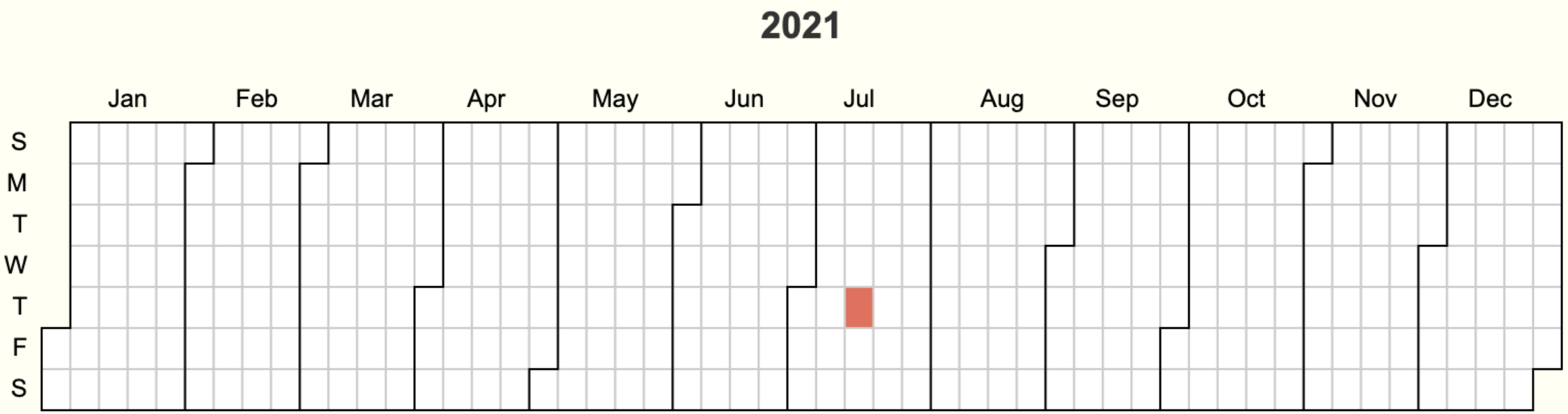
- Ignore leap years in our calculation ($1y = 365\,d$). Assume birthdays are equally likely to fall on any particular day.
- Order the students from 1 to n. There are 💭 possibilities for the birthday of a student. There are 💭 possibilities altogether.

Birthday Problem

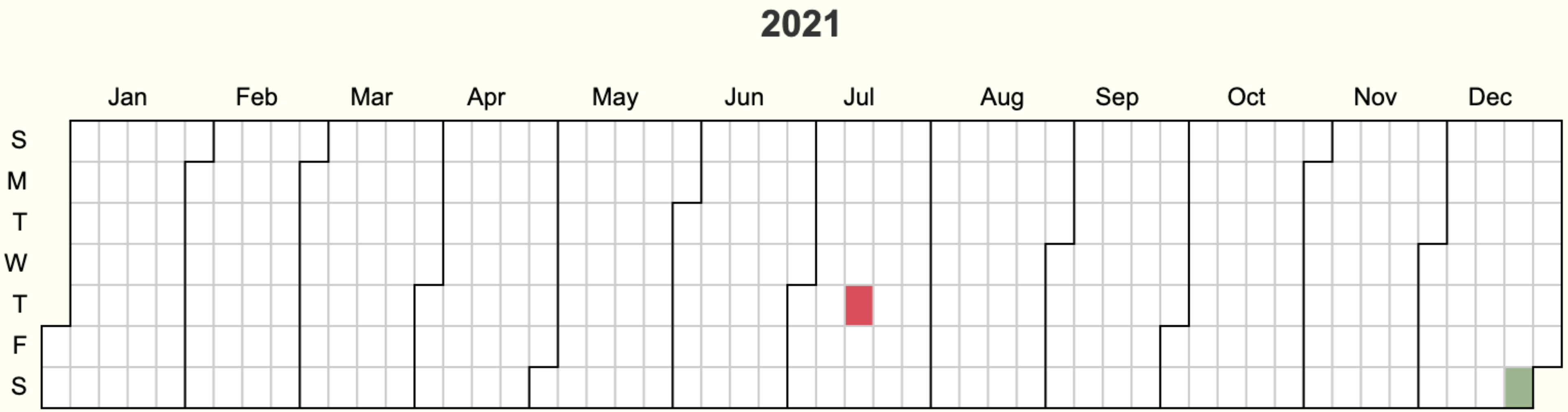


- Ignore leap years in our calculation ($1y = 365\,d$). Assume birthdays are equally likely to fall on any particular day.
- Order the students from 1 to n. There are 365 possibilities for the birthday of a student. There are 365^n possibilities altogether.

Birthdays are different

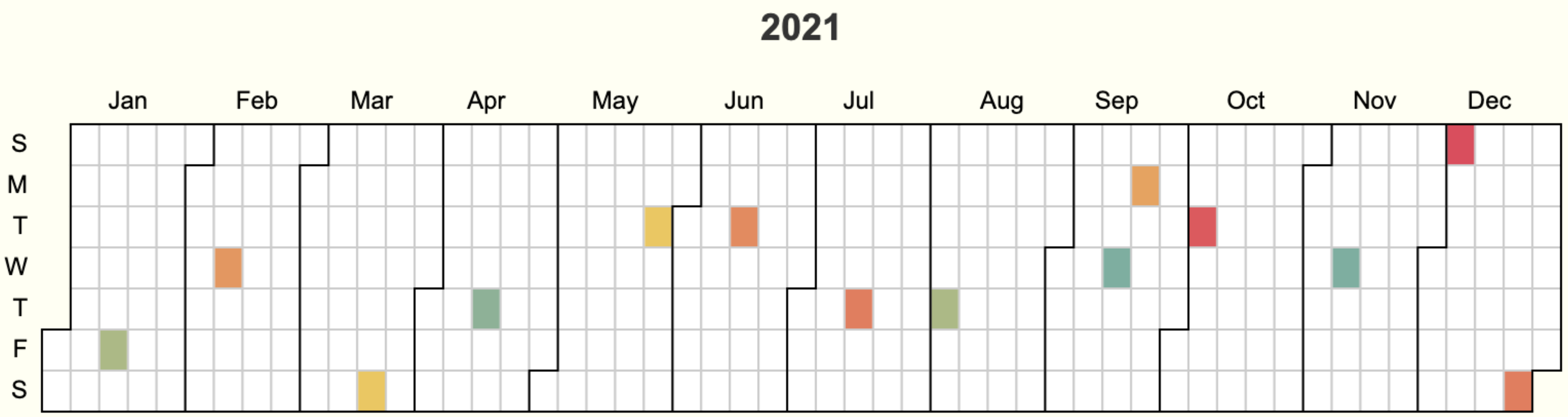


365 possibilities



365×364 possibilities

Birthday Problem



- The probability that all birthdays are different for n students:

$$\frac{(365)_n}{365^n}.$$

We denote the product $k\times(k-1)\times\cdots\times(k-r+1)$ by $(k)_r$ (read 'k down r' or 'k lower r').

Birthday Problem

How many students do we need to have in our hours section to make it favorable bet (that is, probability of success greater than $\frac{1}{2}$) that two people in the classroom will have the same birthday?

probability
$P = 1 - \frac{(365)_n}{365^n} > \frac{1}{2}$
$n = 23, 24, \cdots$

SMALL PLATES

LARGE PLATES

SWEET BITES

Serving Orders

The waiter serve one course at a time. How many possible serving orders are there in total?

Order	Course
1	SP
2	LP
3	SB

Serving Orders

SMALL PLATES

LARGE PLATES

SWEET BITES

Order	1	2	3
Course	SP	LP	SB
Course	SP	SB	LP
Course	LP	SP	SB
Course	LP	SB	SP
Course	SB	SP	LP
Course	SB	LP	SP

$6 = 3 \times 2 \times 1.$

!

Serving Orders

SMALL PLATES = 1

LARGE PLATES = 2

SWEET BITES = 3

Order	1	2	3
Course	1	2	3
Course	1	3	2
Course	2	1	3
Course	2	3	1
Course	3	1	2
Course	3	2	1

$6 = 3 \times 2 \times 1.$

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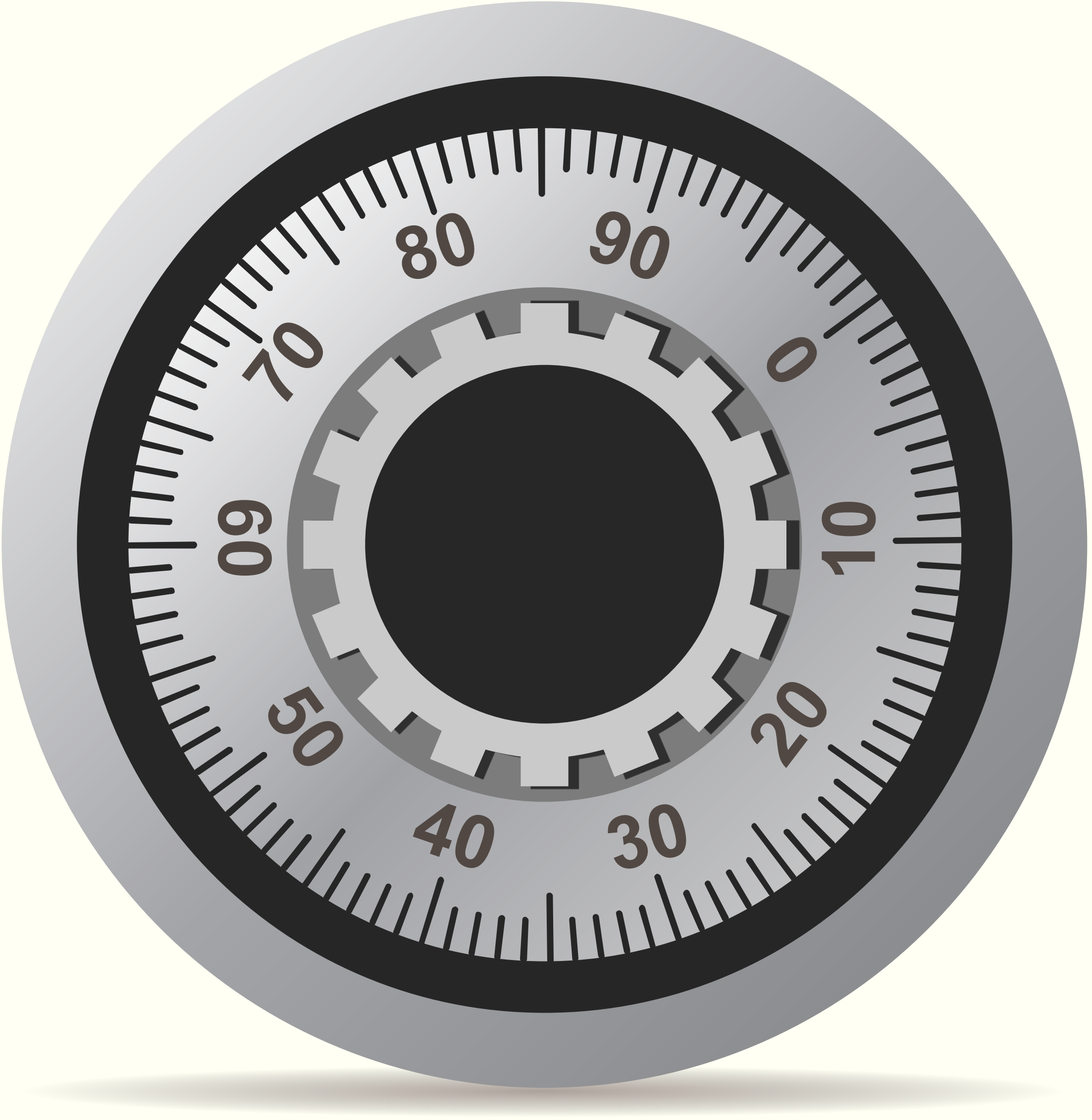
Combination Lock

What is known

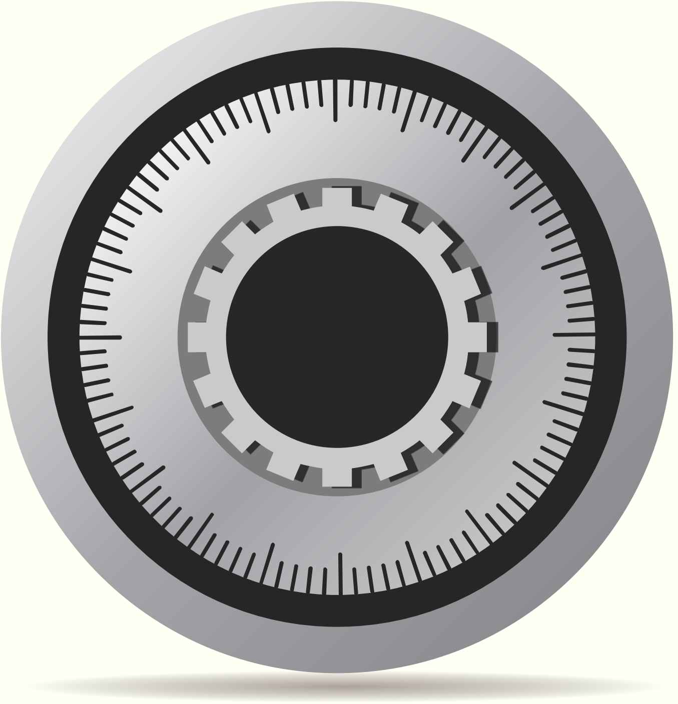
- The password is a combination of four numbers.
- The four numbers are 12, 25, 33, and 77.



What is unknown

The order of these four numbers is unknown.

Password Number Orders



What's the maximum number of tries to open the lock?

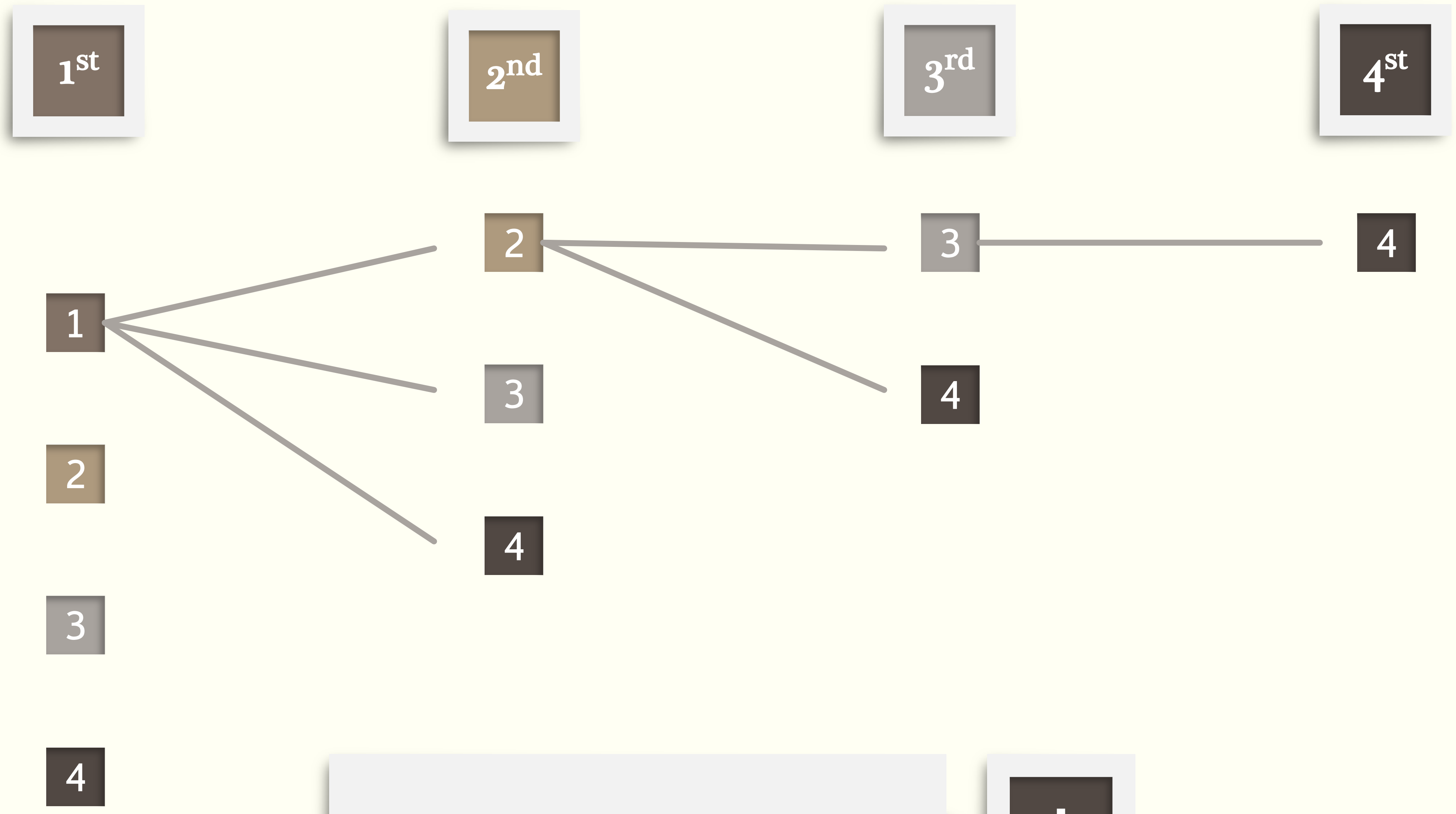
For ease of notation:

- 12 – 1st, 25 – 2nd, 33 – 3rd, 77 – 4st.

Further:

- 12 – 1, 25 – 2, 33 – 3, 77 – 4.

1	2	3	4
1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
...	...	...	...



$$24 = 4\times3\times2\times1.$$

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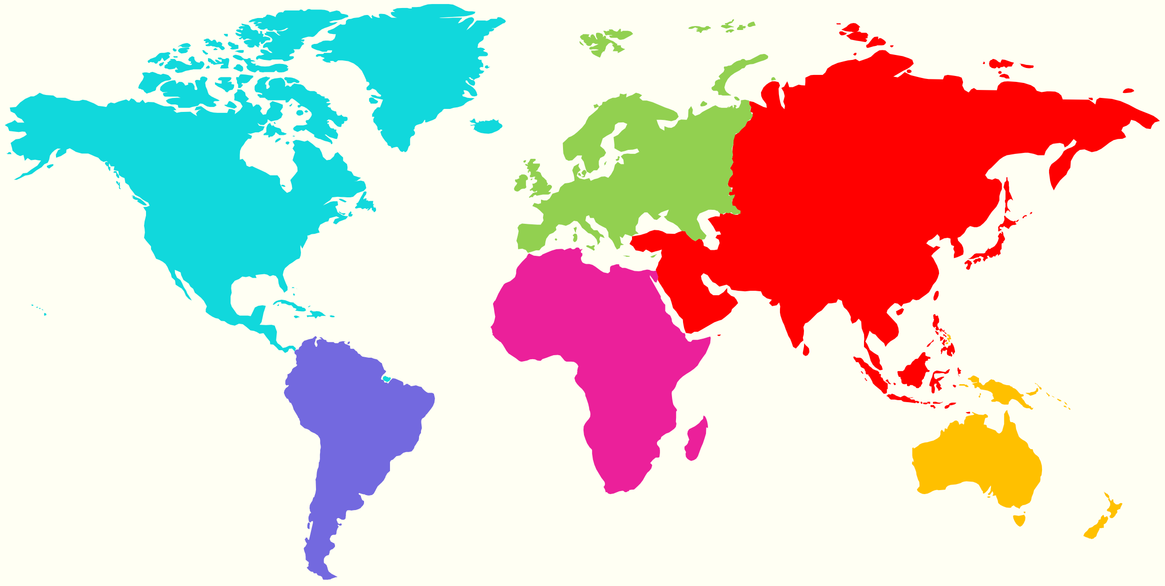
Travel orders



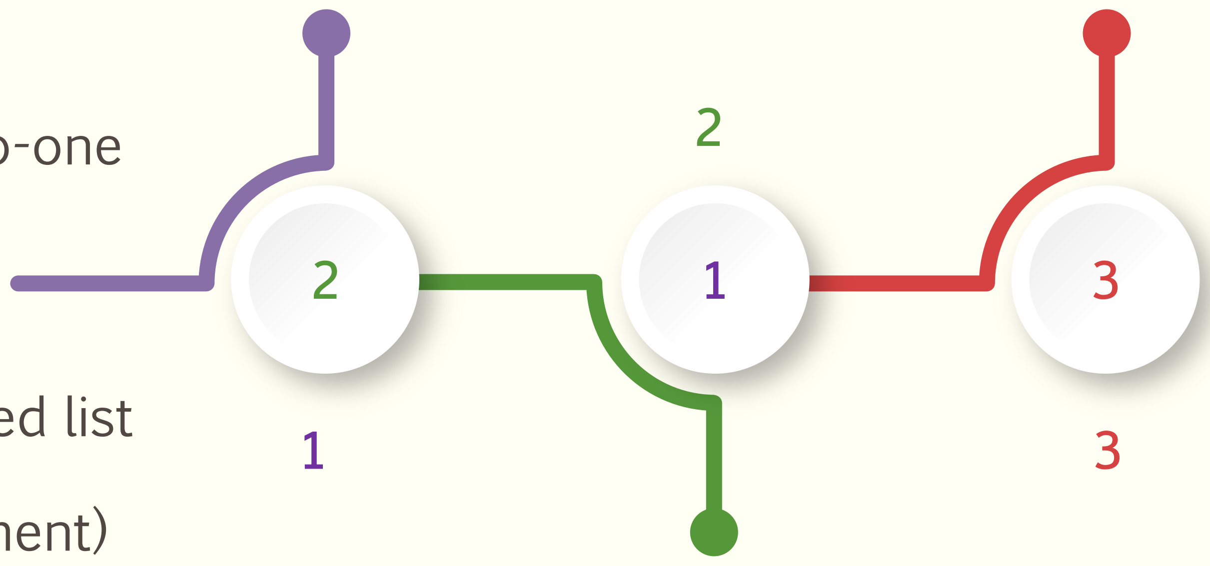
1st stop	United States
2nd stop	Brazil
3rd stop	Egypt
4st stop	China
5st stop	Australia

Mr. Fogg visit one country at a time. How many possible travel orders are there in total?

Permutations

- Let A be any finite set.
- A permutation of A, σ, is a one-to-one mapping of A onto itself.
- For example, if we have an ordered list of elements $A = \{a_1 , a_2, a_3\}$, a possible permutation (re-arrangment) can be prescribed by

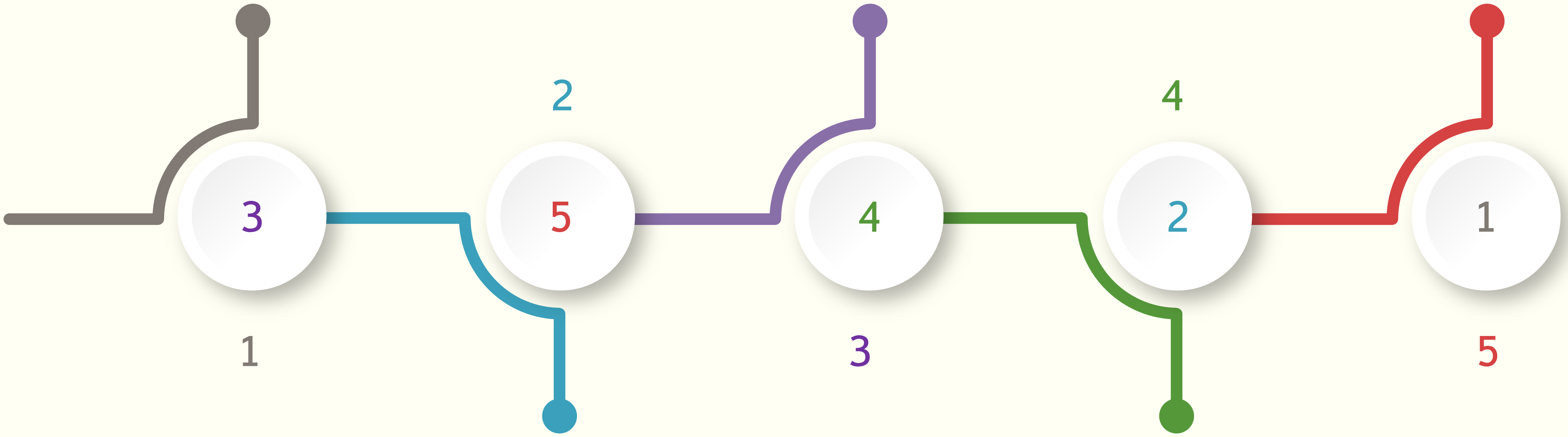
$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$



Permutations

Theorem

The total number of permutations of a set A of n elements is given by $n! = n \times (n-1) \times (n-2) \times \cdots \times 1.$



n factorial $n!$

Stirling's Formula
The sequence $n!$ is asymptotically equal to $n^n e^{-n}\sqrt{2\pi n}$.

!

n	n factorial	Stirling's Formula	ratio
1	1	0.9221	1.0844
2	2	1.919	1.0422
3	6	5.8362	1.0281
4	24	23.5062	1.021
5	120	118.0192	1.0168
6	720	710.0782	1.014
…	…	…	…

Stirling's Formula

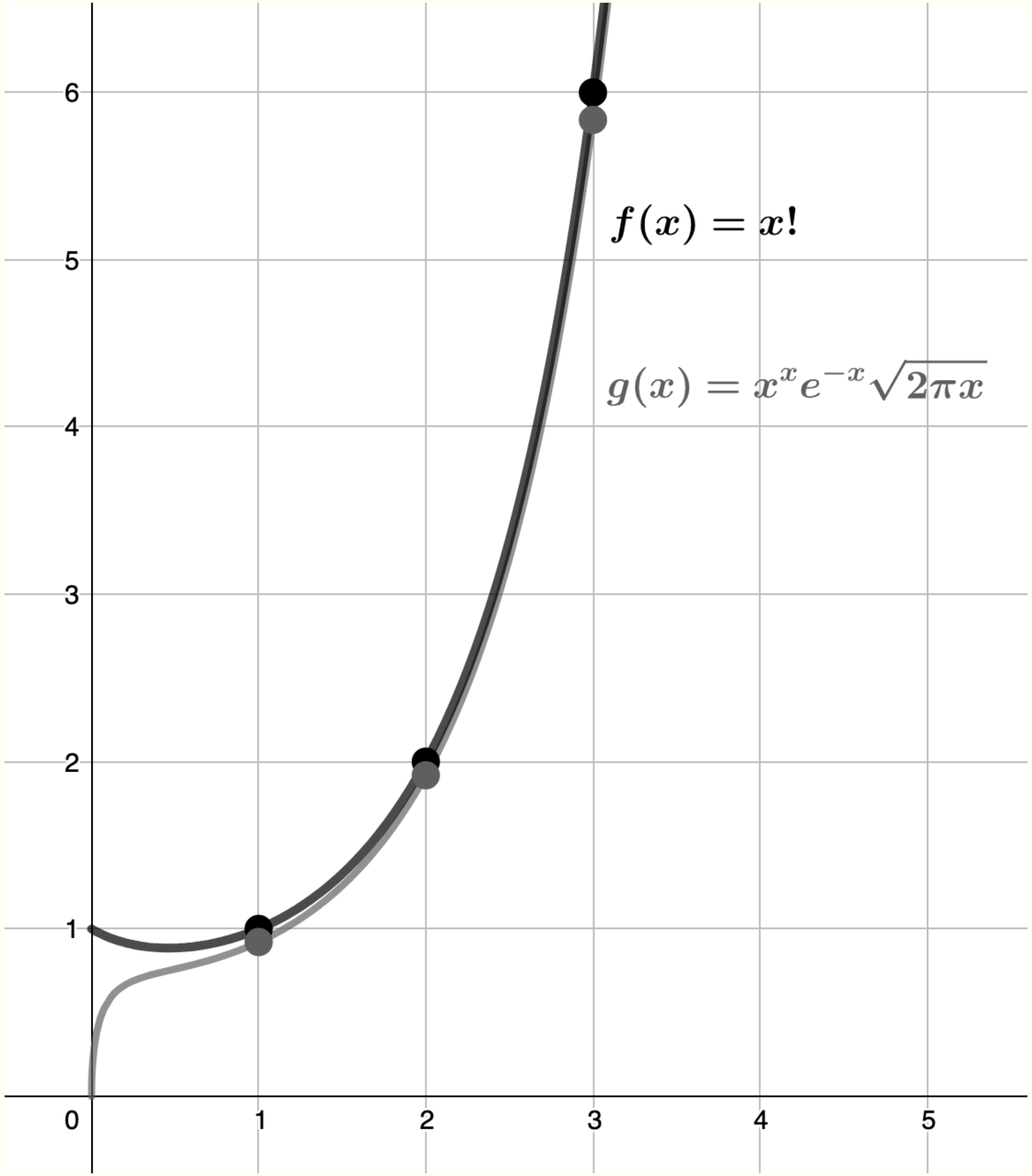
The sequence $n!$ is asymptotically equal to

$n^n e^{-n}\sqrt{2\pi n}.$

Birthday Problem

Apply Stirling's formula to estimate the number of students needed such that $p_n = \frac{(365)_n}{365^n} = \frac{1}{2}.$

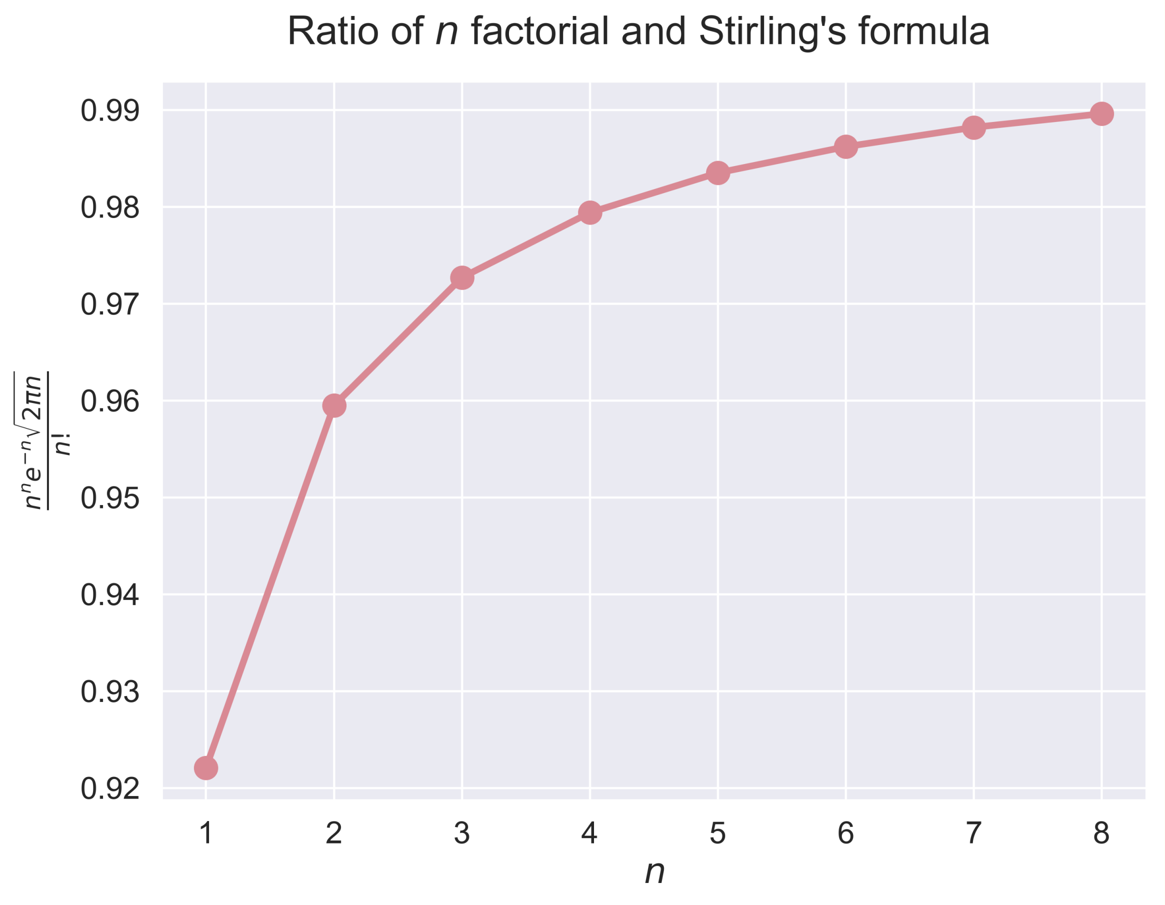
$n = 23.$

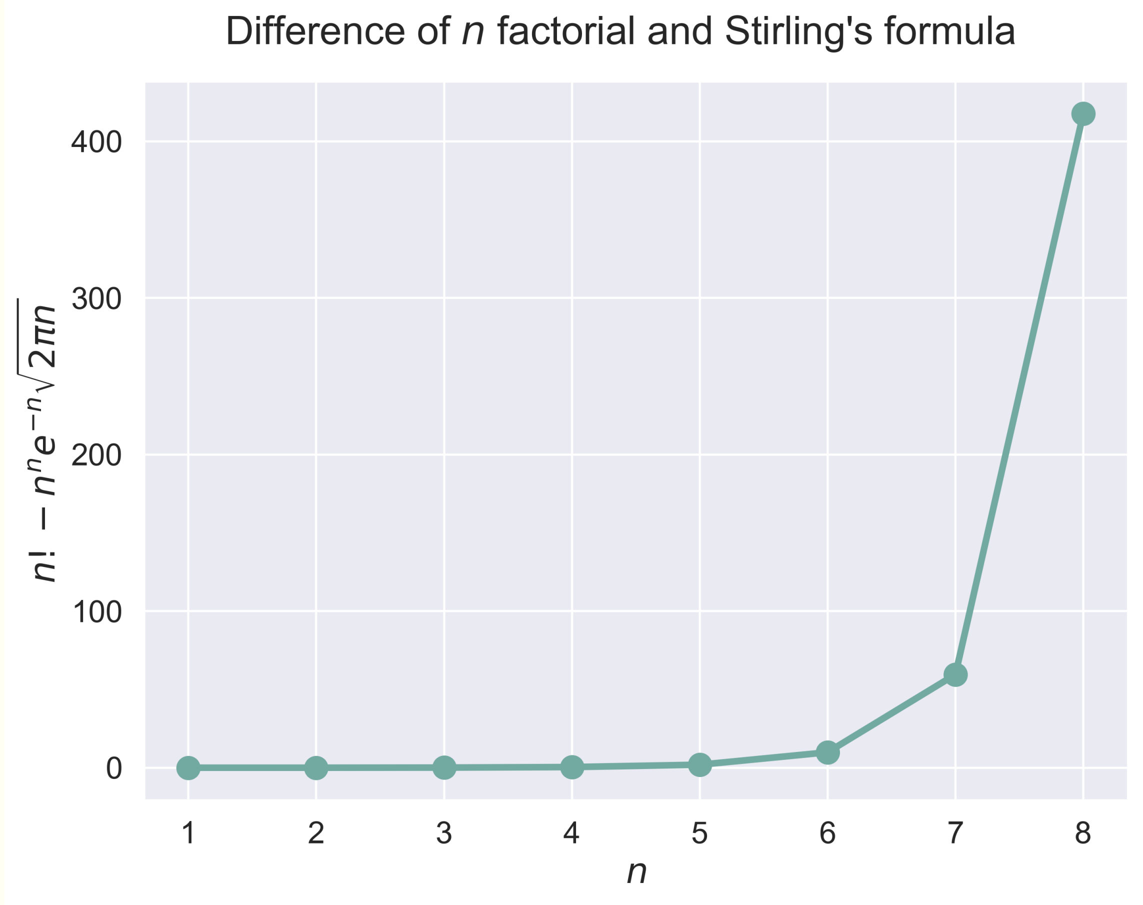


Stirling's formula

n	$n!$	$n^n e^{-n}\sqrt{2\pi n}$	ratio	difference
1	1	0.9221	0.9221	0.0779
2	2	1.919	0.9595	0.081
3	6	5.8362	0.9727	0.1638
4	24	23.5062	0.9794	0.4938
5	120	118.0192	0.9835	1.9808
6	720	710.0782	0.9862	9.9218
7	5040	4980.3958	0.9882	59.6042
8	40320	39902.3955	0.9896	417.6045
…	…	…	…	…

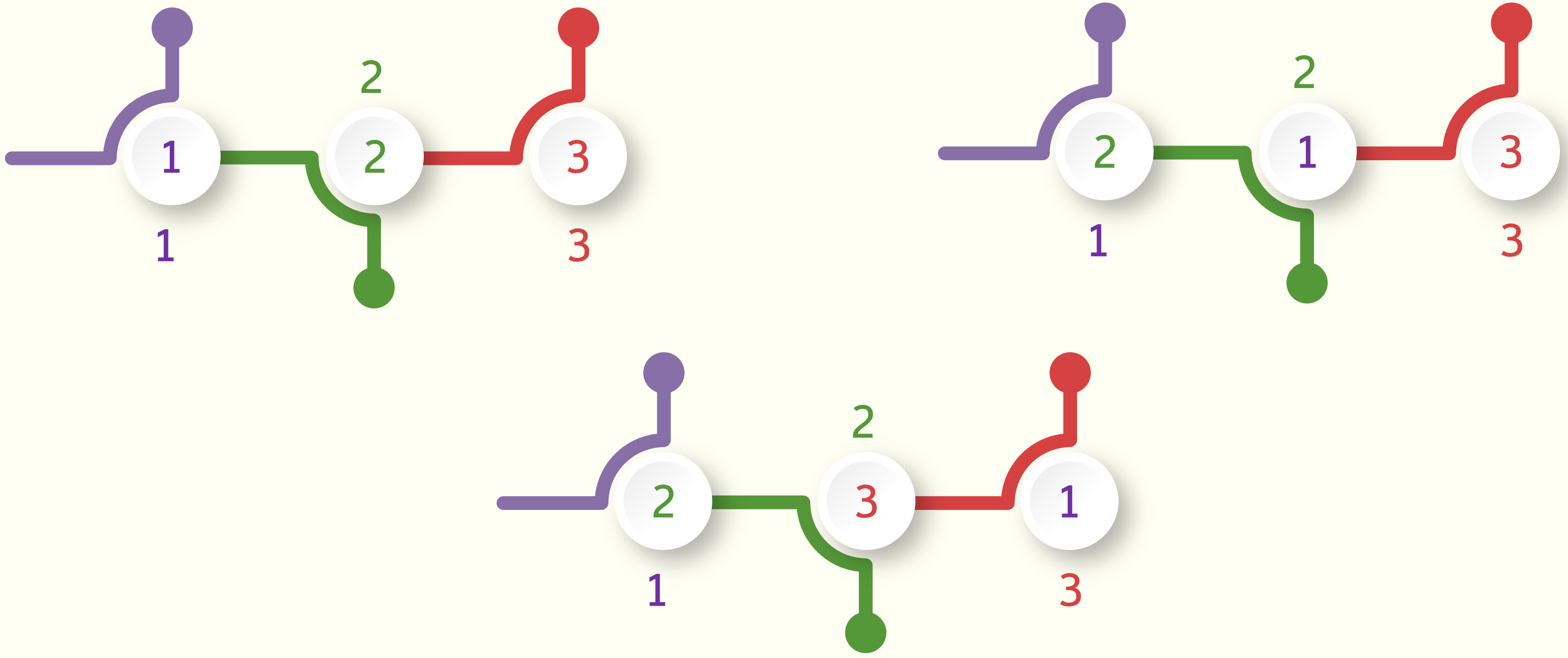
Stirling's formula





Fixed Points

- Since a permutation is a one-to-one mapping of the set onto itself, it is of interest to ask how many points (elements) are mapped onto themselves. Such points are called **fixed points** of the mapping.



Fixed Points

- Let $p_k(n)$ denote the probability that a random permutation of the set $\{1, 2, \cdots, n\}$ has exactly k fixed points.
- What is the probability of no fixed points for a permutation of a set of n elements, $p_0(n)$? This is the famous **hat check** problem.

$$p_0(3) = \cdots.$$

?

1	2	3
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

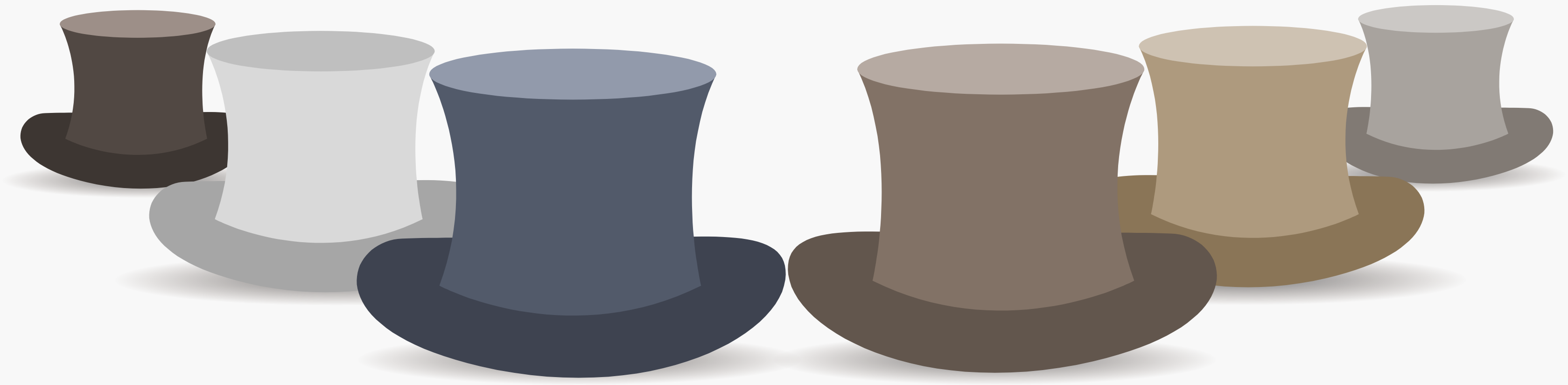
Fixed Points

- Let $p_k(n)$ denote the probability that a random permutation of the set $\{1, 2, \cdots, n\}$ has exactly k fixed points.
- What is the probability of no fixed points for a permutation of a set of n elements, $p_0(n)$? This is the famous **hat check** problem.

$$p_0(3) = \frac{2}{3!} = \frac{1}{3}.$$

?

1	2	3
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1



Hat Check Problem

In a restaurant n hats are checked and they are hopelessly scrambled.

What is the probability that no one gets his own hat back?

Hat Check Problem

- **Number of derangements $D_0(n)$**

  - If there is a derangement, then man $\#1$ will not have his correct hat.

  - We begin by looking at the case where man $\#1$ gets hat $\#2$. Note that this case can be broken down into two subcases:
    a) man $\#2$ gets hat $\#1$, or
    b) man $\#2$ does not get hat $\#1$

Hat Check Problem

- **Number of derangements $D_0(n)$**

- In case (a), for a derangement to occur, we need the remaining $n-2$ men to get the wrong hats. Therefore, the total number of derangements in this subcase is simply $D_0(n-2)$.

- In case (b), for a derangement to occur, man $\#2$ cannot get hat $\#1$ (that's case a), man $\#3$ cannot get hat $\#3$, man $\#i$ cannot get hat $\#i$, etc. In this subcase, the number of derangements is $D_0(n-1)$.

Hat Check Problem

- **Number of derangements $\boldsymbol{D_0(n)}$**
  - We can treat the cases where man $\#1$ receives hat $\#3$, or hat $\#4$, or hat $\#i$, in exactly the same way.
  - Therefore, to account for all possible derangements, there are $n-1$ such possibilities for all the different incorrect hats which man $\#1$ can get.

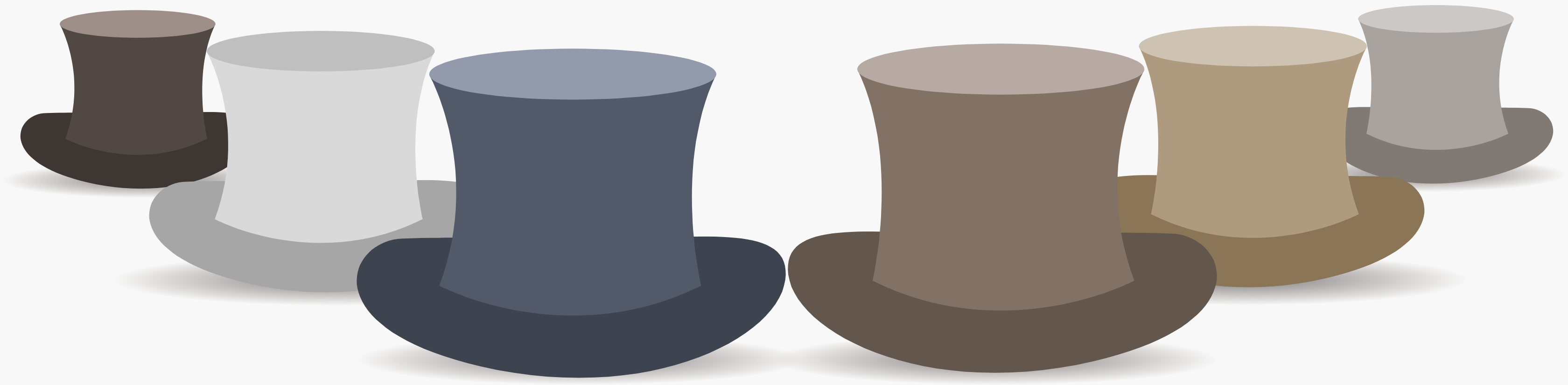
$$D_0(n) = (n-1)[D_0(n-1) + D_0(n-2)].$$

Hat Check Problem

- $D_0(n) = (n-1)[D_0(n-1) + D_0(n-2)]$.
- $p_0(n) = \frac{D_0(n)}{n!}$.
- $p_0(n) = p_0(n-1) - \frac{1}{n}[p_0(n-1) - p_o(n-2)]$.

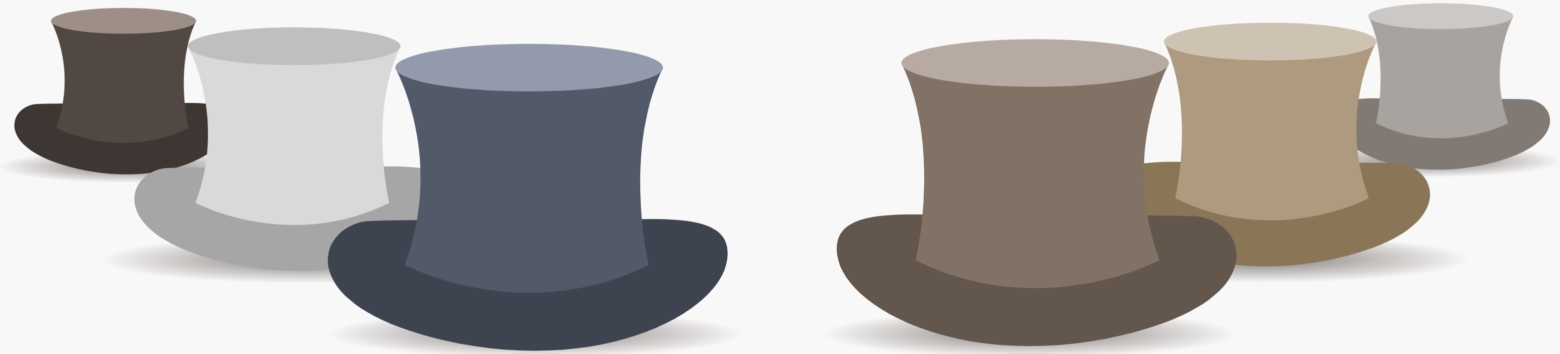
Hat Check Problem

- $p_0(n) = p_0(n-1) - \frac{1}{n}[p_0(n-1) - p_o(n-2)].$
- $p_0(1) = 0,\ p_0(2) = \frac{1}{2}.$
- $p_0(3) = p_0(2) - \frac{1}{3}[p_0(2) - p_o(1)] = \frac{1}{2} - \frac{1}{6}.$
- $p_0(4) = p_0(3) - \frac{1}{4}[p_0(3) - p_o(2)] = \frac{1}{2} - \frac{1}{6} + \frac{1}{24}.$
- $\ldots$



Hat Check Problem

$$p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!}.$$



Hat Check Problem

$$p_0(n) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!}.$$

Taylor Expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots.$$

e^x

Taylor Expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots.$$

$$\boldsymbol{e^x}$$

$$\boldsymbol{p_0(n)}$$

Hat Check Problem

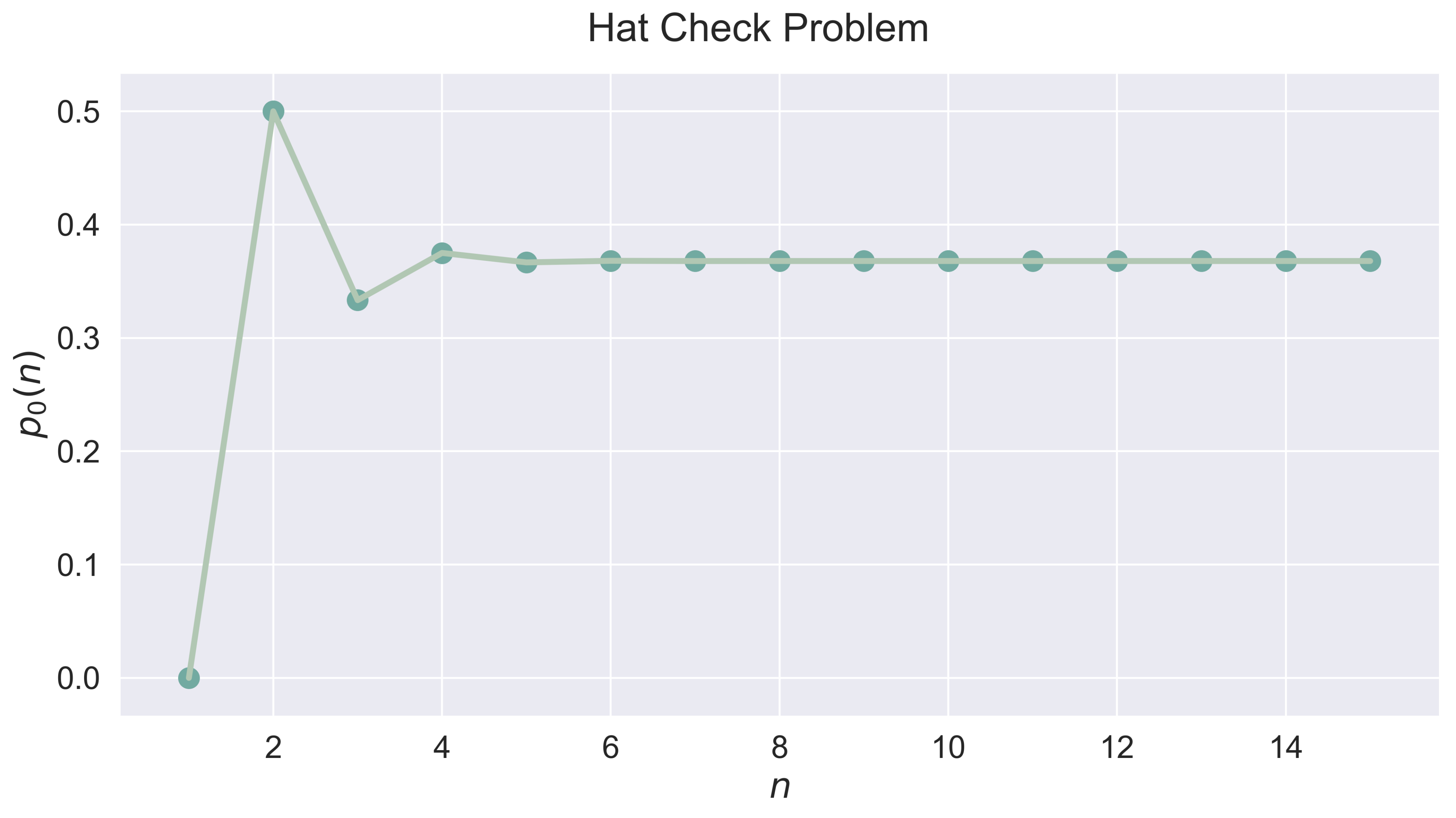
$$x = -1$$

$$e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{(-1)^n}{n!} + \cdots \approx .3679.$$

- **Hat Check Problem (Fixed Point)**

```
hat_check(n = 10)
figure_hat_check(n = 15, fsize = (12, 6), fs = 20)
```

```
# Calculate the probabilities of p_0(n)
def hat_check(n):
    p_1 = 0
    p_2 = 0.5
    p_list = [p_1, p_2]
    while len(p_list) < n:
        # recursive relation
        p_list.append(p_list[-1] - 1/(len(p_list)+1)*(p_list[-1] - p_list[-2]))
    return p_list
```


Hat Check Problem
0.5
0.4
0.3
0.2
0.1
0.0
$p_0(n)$
2
4
6
8
10
12
14
n