MATH 20: PROBABILITY

Lecture 2: Continuous Probability Densities

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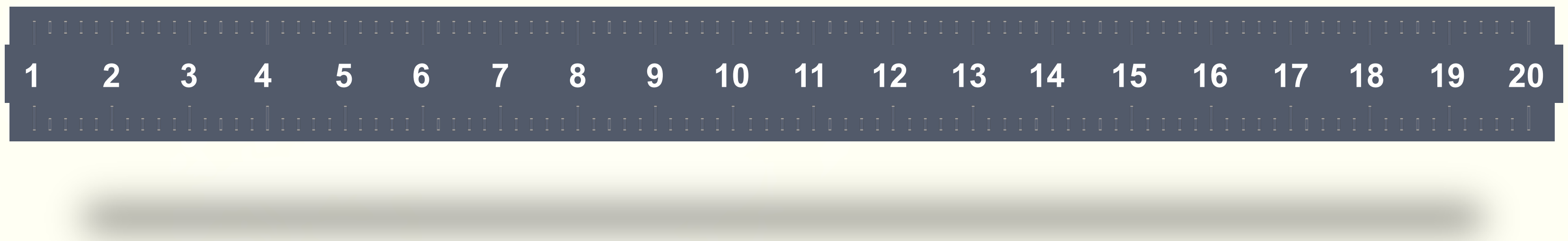
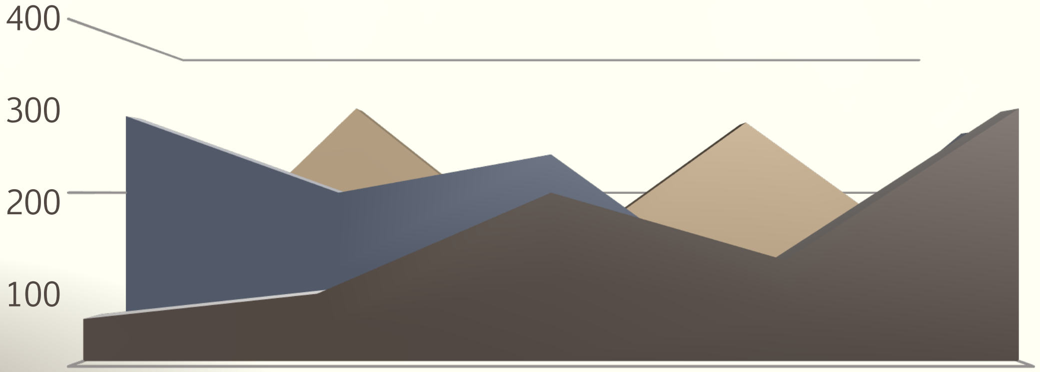
Continuous Sample Space

arrival time	speed
5:45 pm – 6:00 pm	160 mph – 200 mph



length

time



area

velocity (speed)

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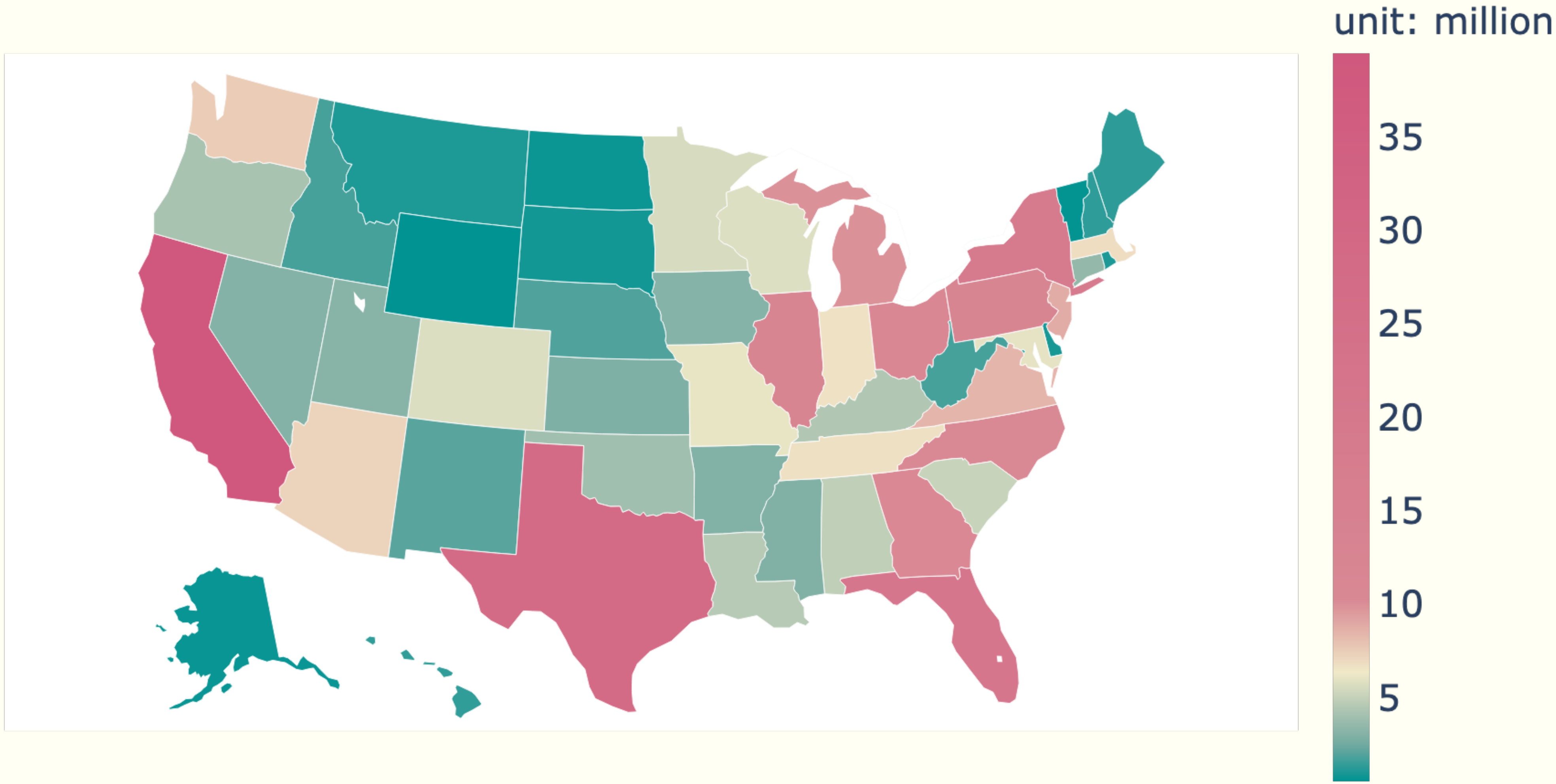
How Much Does a Hershey Kiss Weight?

- A single standard Hershey's Kiss weighs 0.16 ounces.

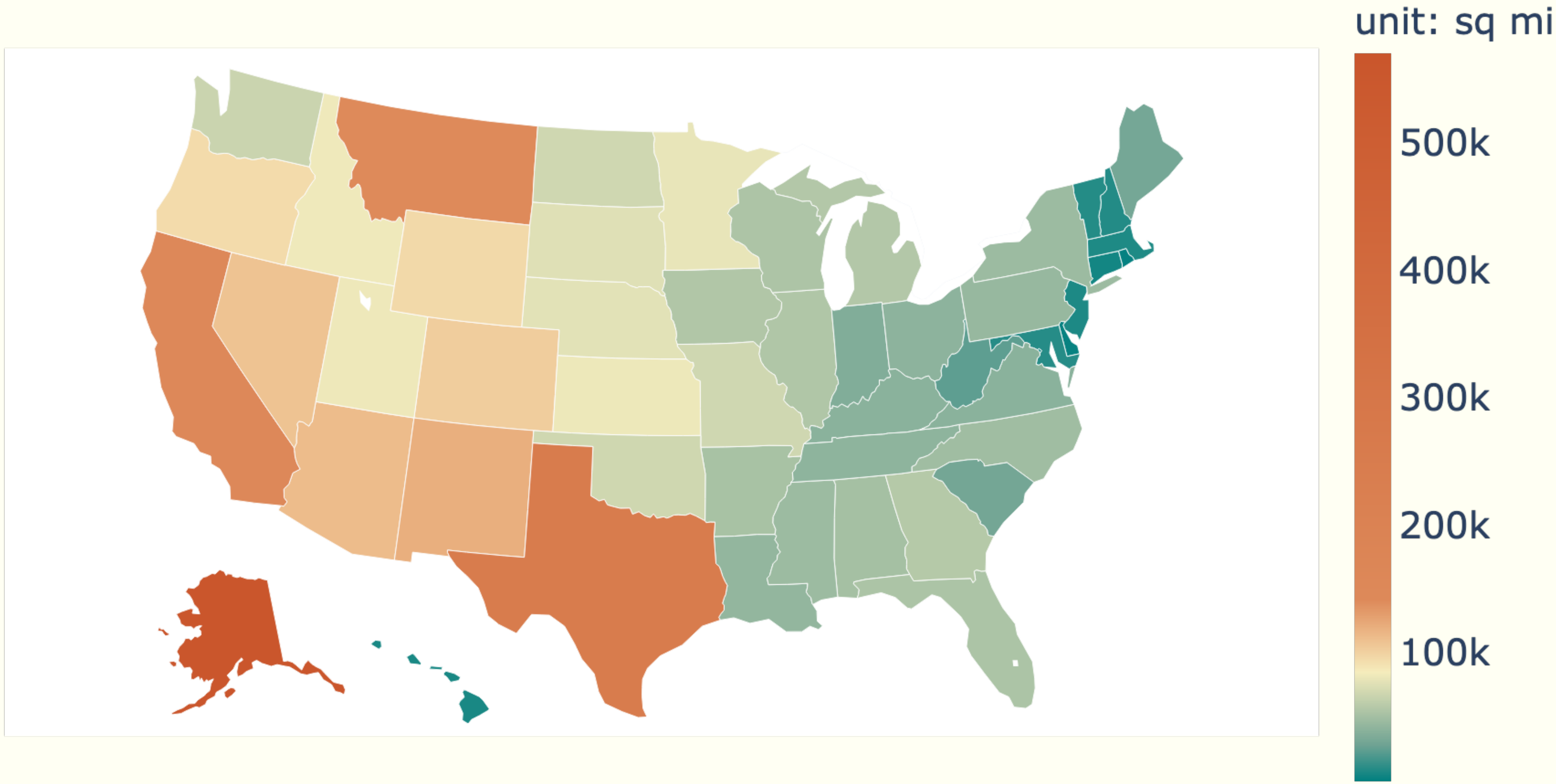
0.1584	0.1577	0.1819	0.1581	0.1438	0.1385	0.1673	0.1611
0.165	0.1452	0.1482	0.1568	0.1603	0.1478	0.1591	0.1519
0.1649	0.1672	0.153	0.1504	0.1587	0.1485	0.1538	0.1498
0.1656	0.1692	0.1477	0.157	0.1574	0.1699	0.1589	0.1487

US: states by census



Randomly select a resident. The probability that he or she lives in New Hampshire is: 0.41%.

US: states by area



Randomly drop a coin from a helicopter. The probability that the coin fall to New Hampshire is: 0.25%.

From Discrete Probabilities to Continuous Probability Densities

- Random variable X
- Ω Sample space
- Elementary outcomes ω
- $m(\omega)$ Probability of an outcome ω occuring
- Events (subsets of Ω) E

A distribution function for X is a real-valued function m whose domain is Ω and which satisfies:

- $m(\omega) \geq 0$, for all $\omega \in \Omega$, and
- $\sum_{\omega \in \Omega} m(\omega) = 1$.

For any subset E of Ω, we define the probability of E to be the number $P(E)$ given by

$$P(E) = \sum_{\omega \in E} m(\omega).$$

01

X

discrete random variable

X

continuous random variable

02

Ω

discrete sample space

Ω

continuous sample space

03

$m(\omega)$

discrete probability distribution

$f(x)$

density function

04

E

events, subsets of Ω

$$P(E) = \sum_{\omega \in E} m(\omega)$$

E

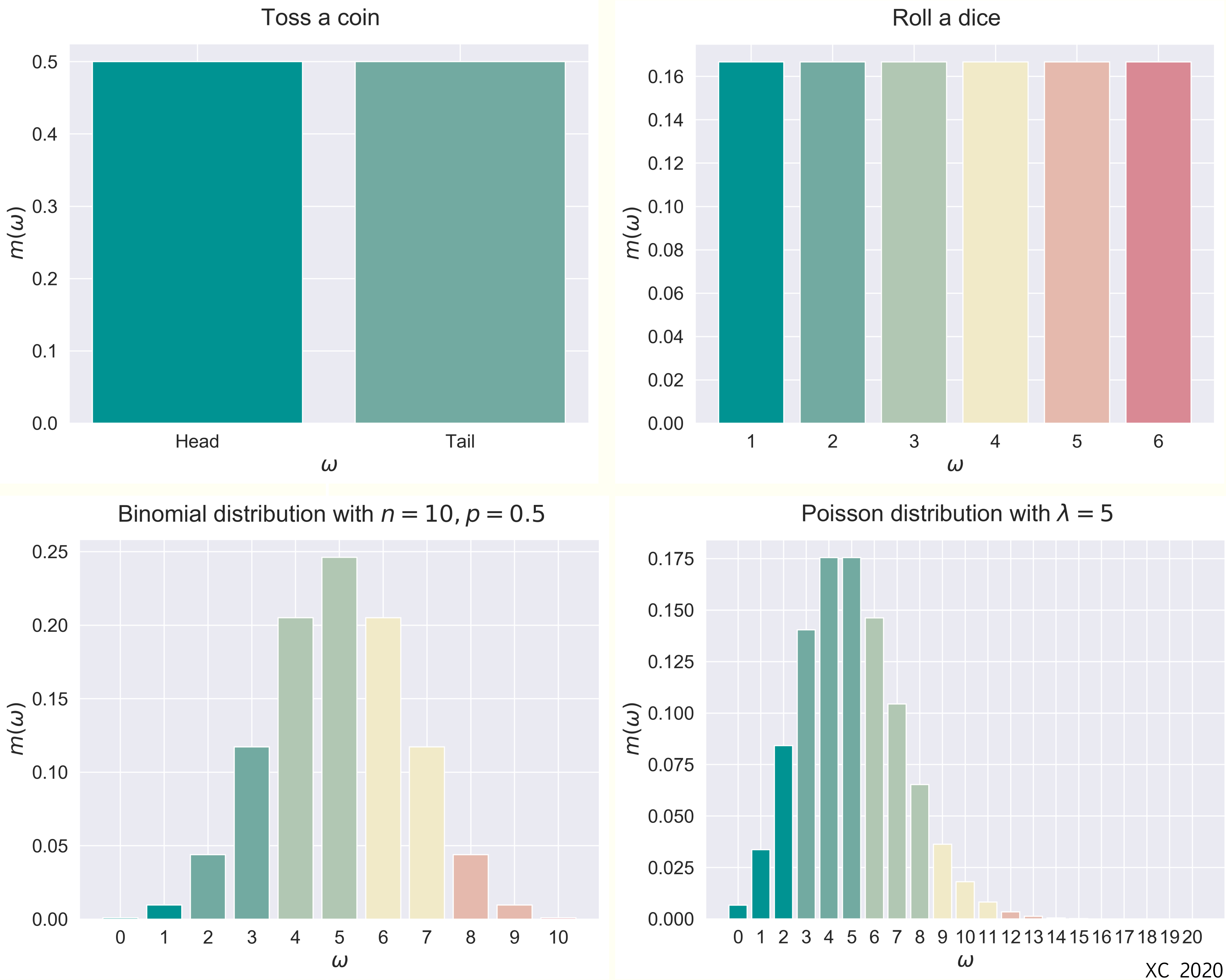
events, subsets of Ω

$$P(x \in E) = \int_E f(x)dx$$

From Discrete Probability Distribution to Density Function

A distribution function for X is a real-valued function m whose domain is Ω and which satisfies:

- $m(\omega) \geq 0$, for all $\omega \in \Omega$, and
- $\sum_{\omega \in \Omega} m(\omega) = 1$.



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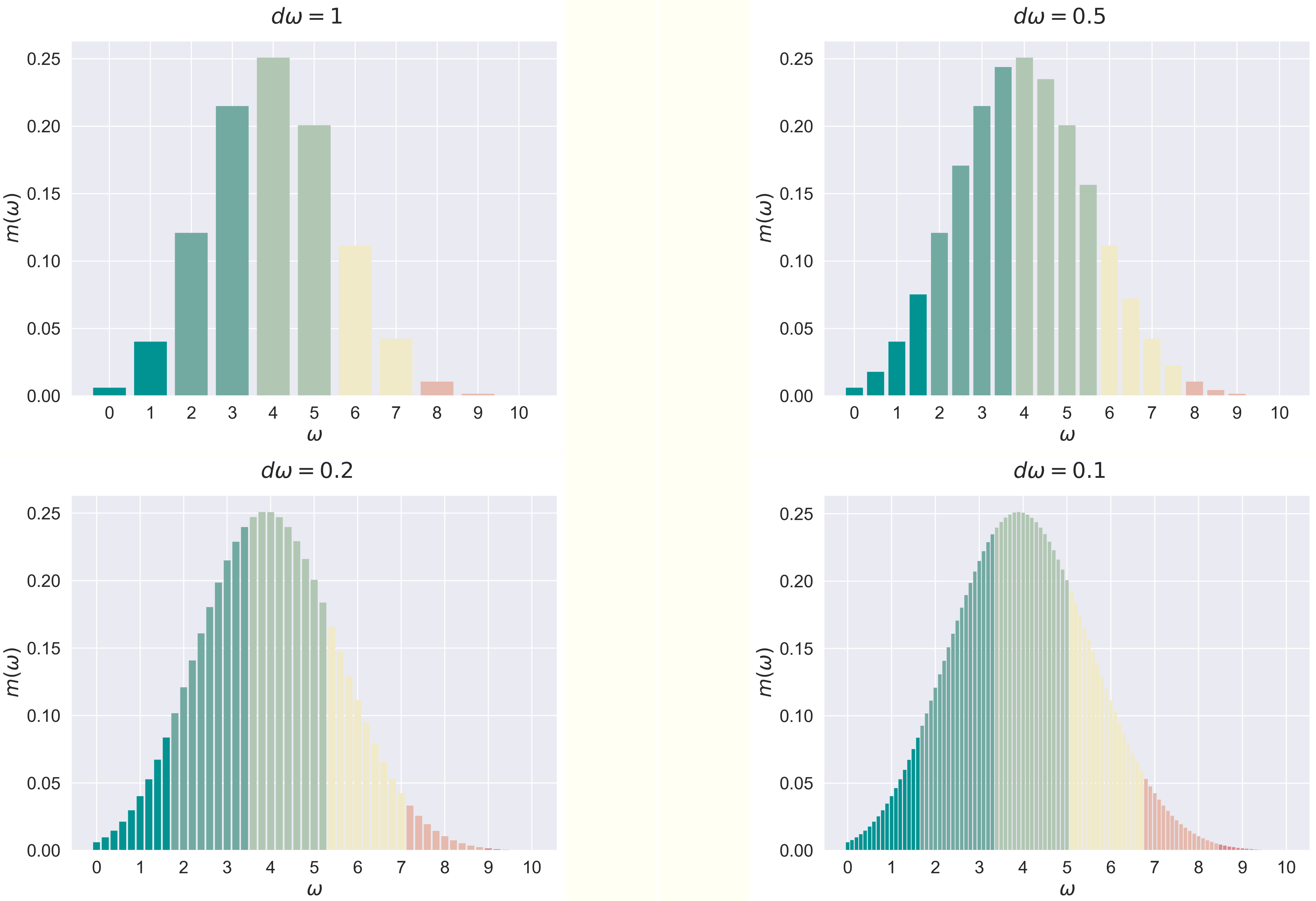
$d\omega = 1$
$d\omega = 0.5$
$d\omega = 0.2$
$d\omega = 0.1$
$m(\omega)$
ω
0.00
0.05
0.10
0.15
0.20
0.25
0 1 2 3 4 5 6 7 8 9 10

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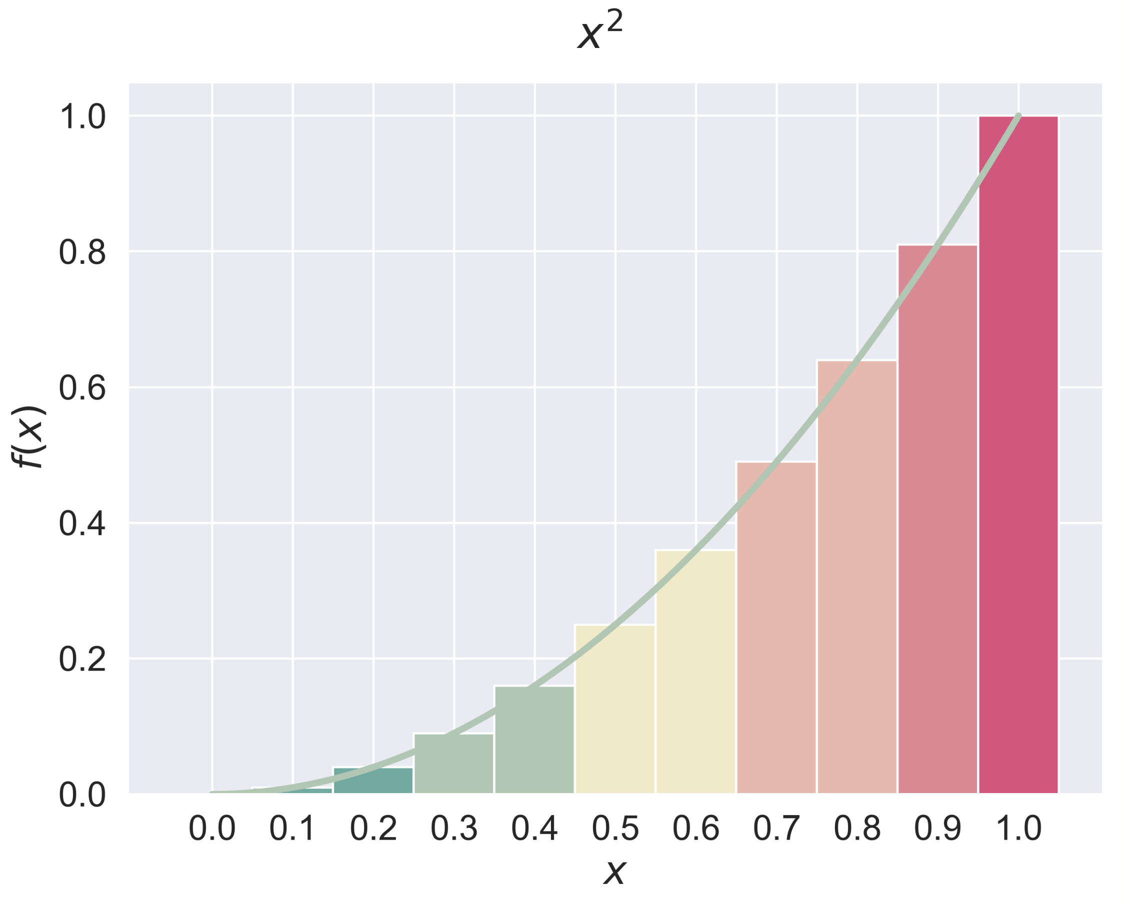
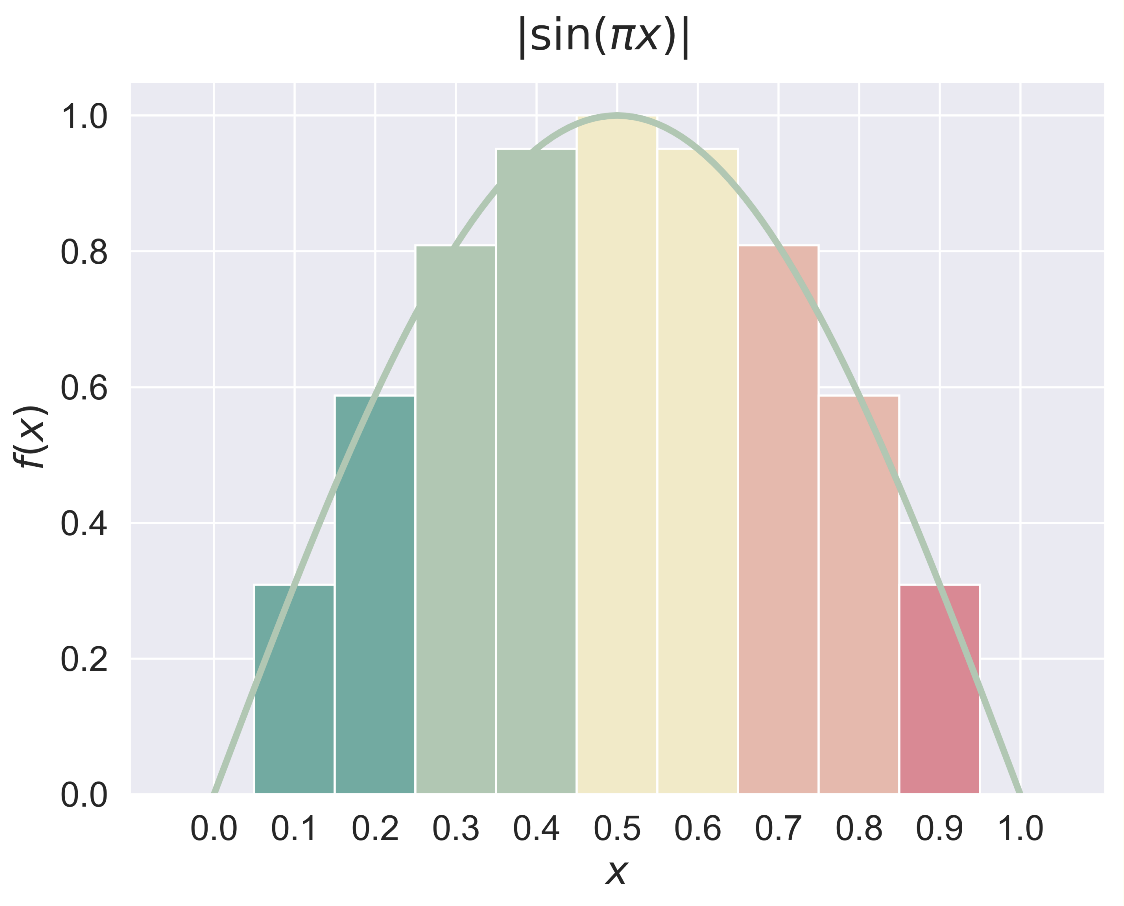
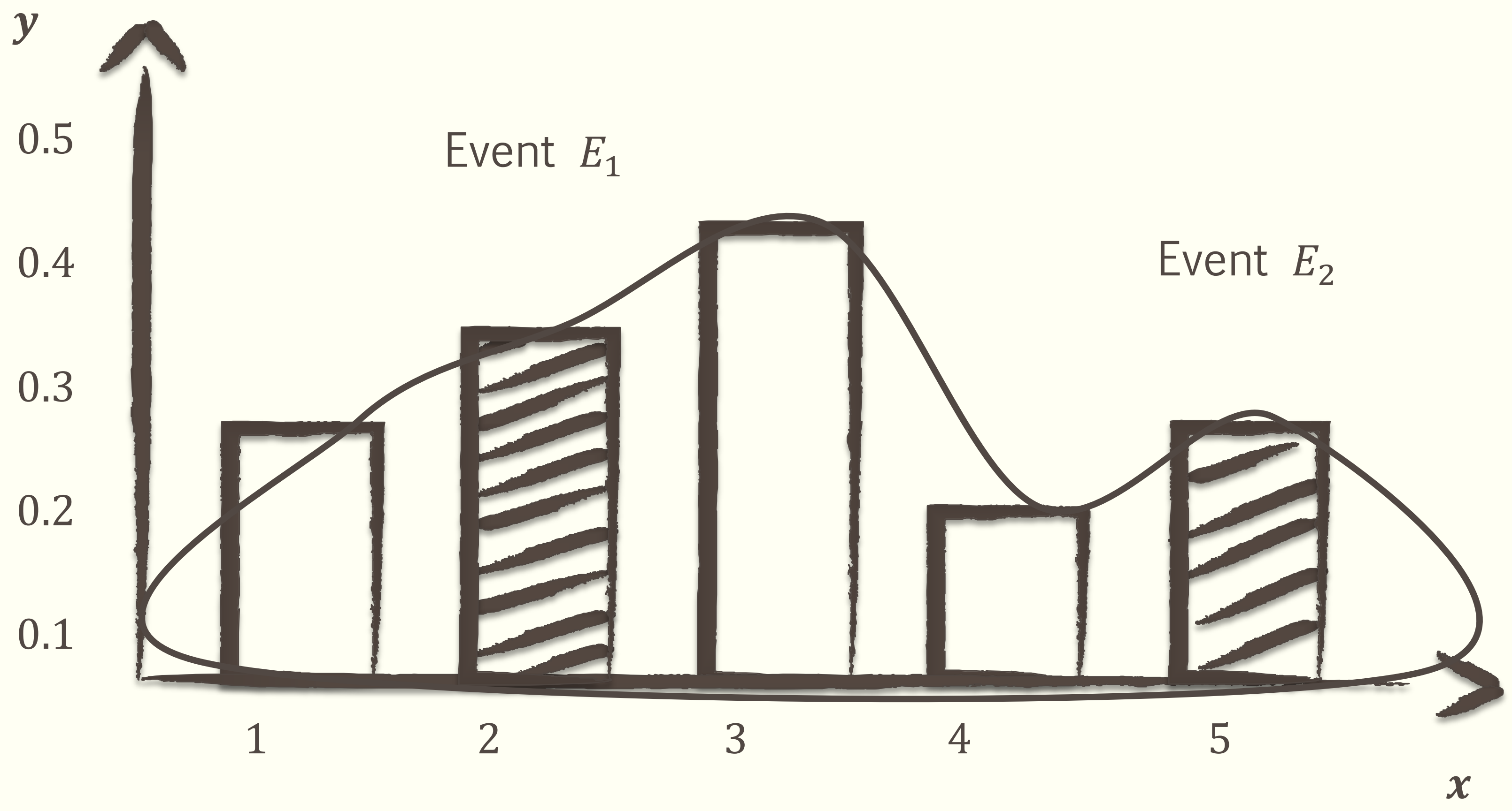
Recall the Riemann Sum in Calculus





A Riemann sum is a certain kind of approximation of an integral by a finite sum. One very common application is approximating the area of functions or lines on a graph, but also the length of curves and other approximations.

Discrete Probability Distribution VS Density Function



discrete random variable VS continuous random variable

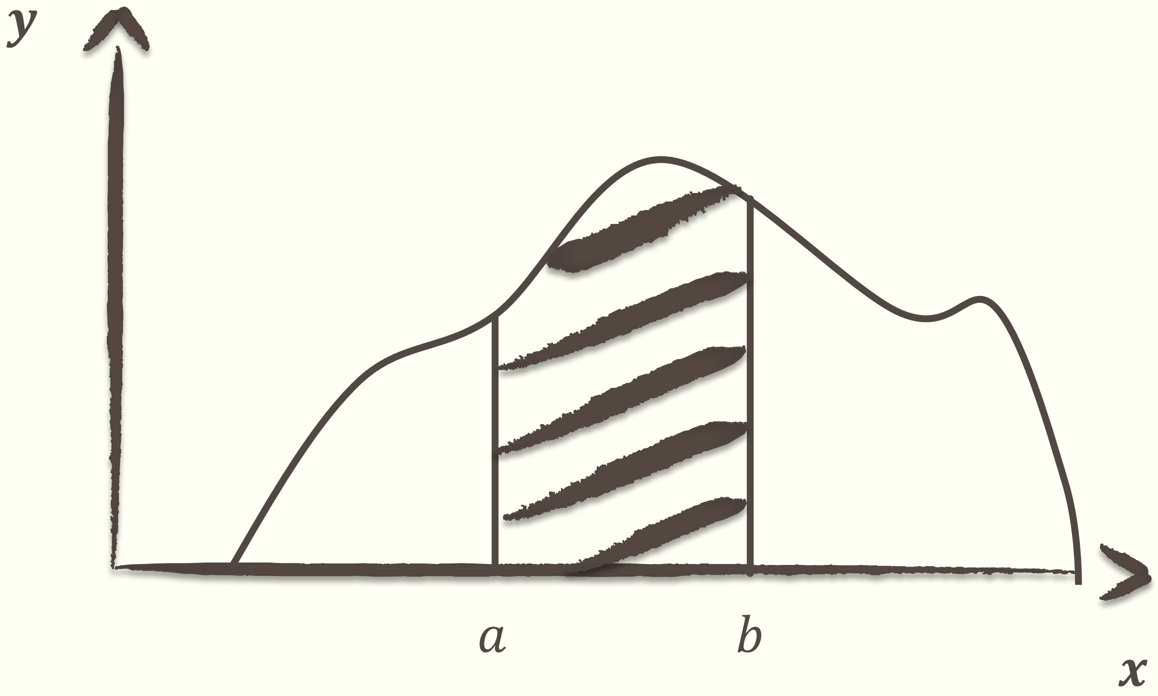
Density Functions of Continuous Random Variable

- Let X be a continuous real-valued random variable. A density function for X is a real-valued function f that satisfies

$$P(a \leq X \leq b) = \int_a^b f(x)dx, \text{ for } a, b \in \mathbb{R}.$$

- If E is a subset of $\mathbb{R}$, then $P(x \in E) = \int_E f(x)dx$.
- In particular, if E is an interval $[a, b]$, the probability that the outcome of the experiment falls in E is given by

$$P([a, b]) = \int_a^b f(x)dx.$$

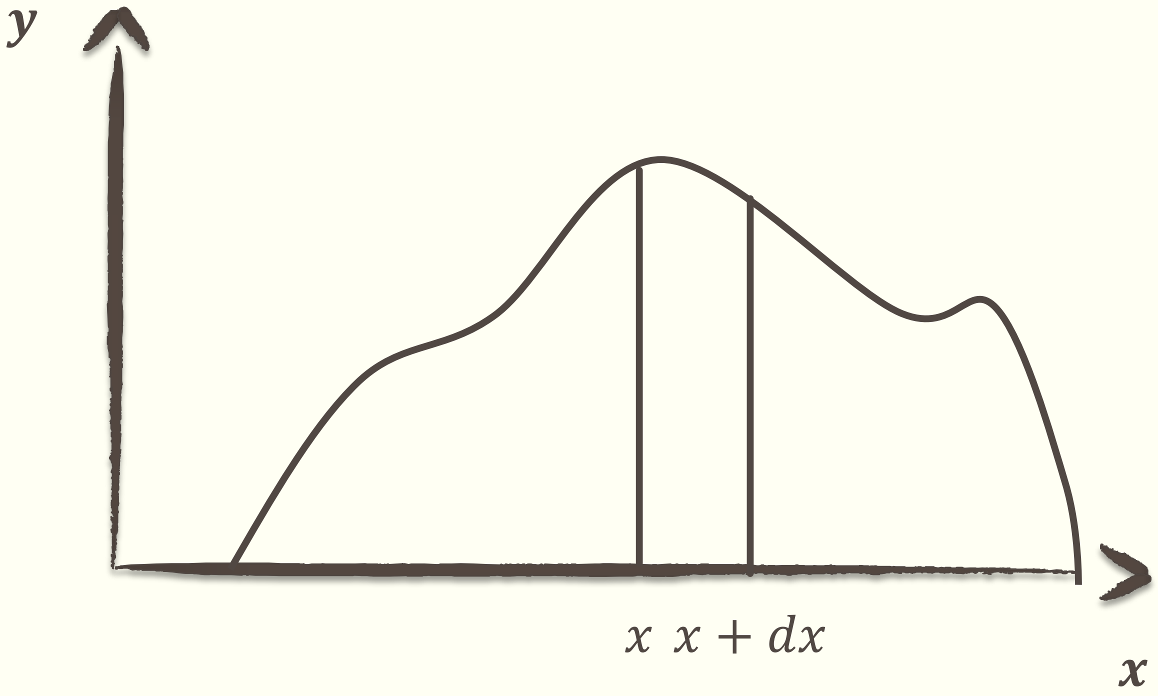


Density Functions of Continuous Random Variable

- The probability of occurrence of an event of the form $[x, x + dx]$, where dx is small, can be estimated by

$$P([x, x + dx]) \approx f(x)dx.$$

- As $dx \to 0$, the above probability approaches 0, so that the probability of a single point x, $P(\{x\})$ is 0.



Cumulative Distribution Function

- Let X be a continuous real-valued random variable. Then the **cumulative distribution function** of X is defined by

$$F_X(x) = P(X \leq x).$$

- If X possesses a density function, then it also has a cumulative distribution function. Their relationship can be expressed in the theorem below.

Let X be a continuous real-valued random variable with density function $f(x)$. Then the function defined by

$$F_X(x) = \int_{-\infty}^{x} f(s)ds,$$

is the cumulative distribution function of X. Furthermore, we have

$$\frac{d}{dx} F_X(x) = f(x).$$

Cumulative Distribution Function

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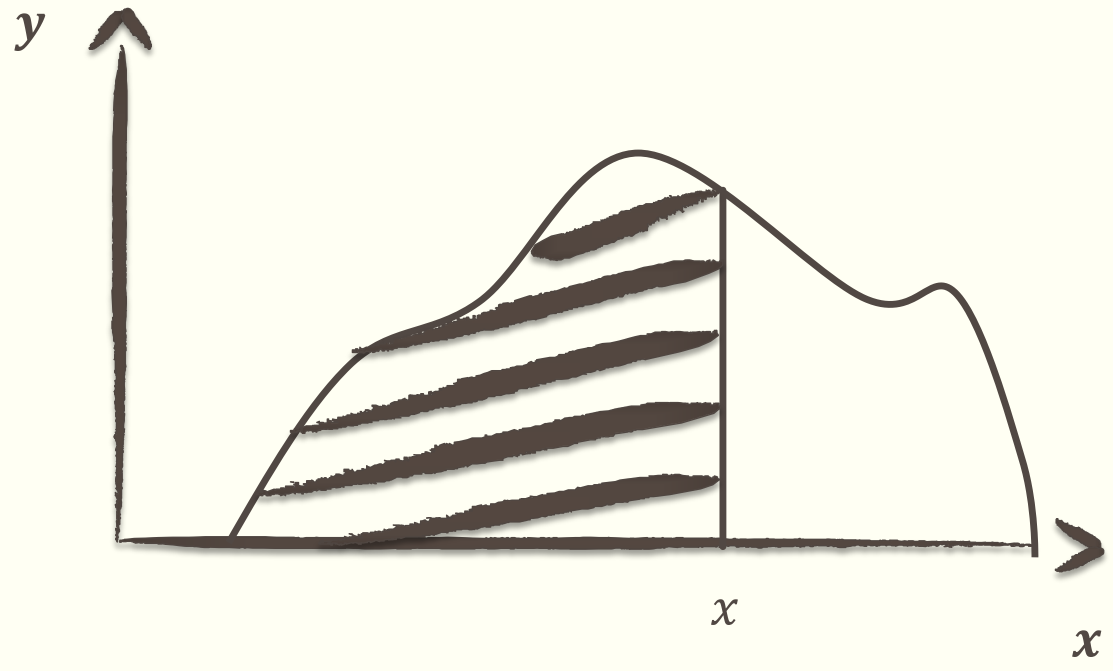


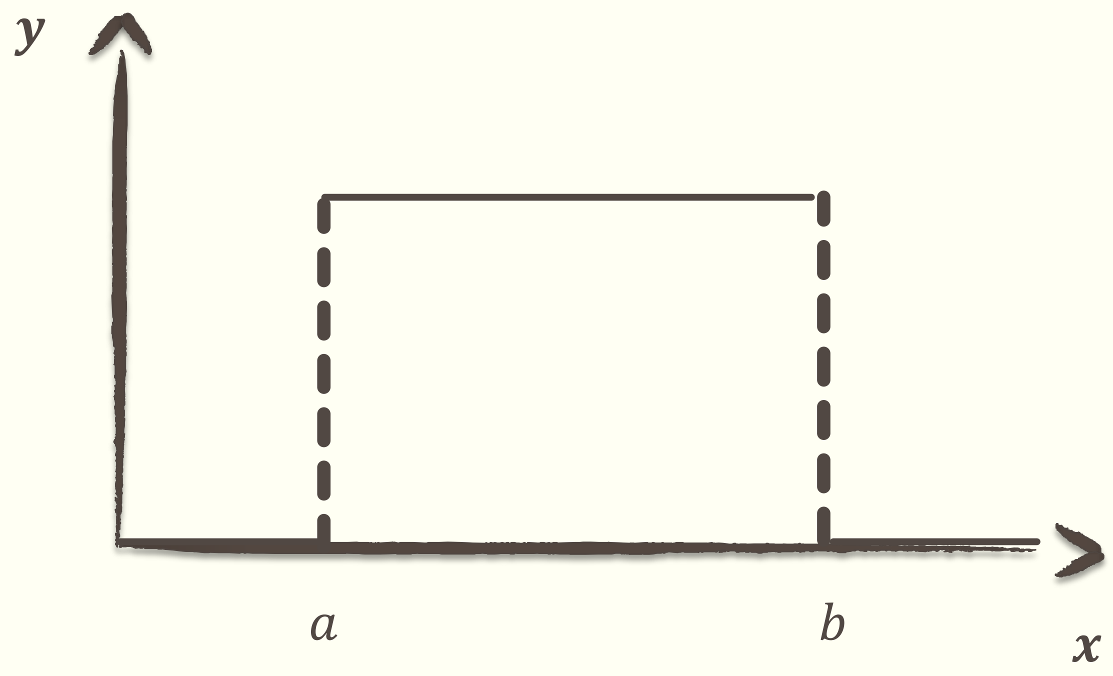
Density function	Cumulative distribution function
	$F_X(x) = x^3, 0 \le x \le 1$
$f(x) = \frac{1}{x}, 1 \le x \le e$	
	$F_X(x) = \frac{1 - \cos \pi x}{2}, 2 \le x \le 3$
$f(x) = e^{-x}, x \ge 0$	

Density function	Cumulative distribution function
$f(x) = 3x^2, 0 \leq x \leq 1$	$F_X(x) = x^3$
$f(x) = \frac{1}{x}, 1 \leq x \leq e$	$F_X(x) = \ln x$
$f(x) = \frac{\pi}{2}\sin \pi x\,, 2 \leq x \leq 3$	$F_X(x) = \frac{1 - \cos \pi x}{2}$
$f(x) = e^{-x}, x \geq 0$	$F_X(x) = 1 - e^{-x}$
$f(x)$ $a \leq x \leq b$	$F_X(x)$ $F_X(a) = 0, F_X(b) = 1$

(Continuous) Uniform Distribution

- The distribution describes an experiment where there is an arbitrary outcome that lies between certain bounds.
- The bounds are defined by the parameters, a and b, which are the minimum and maximum values.
- The probability density function of the continuous uniform distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & x < a \text{ or } x > b \end{cases}$$

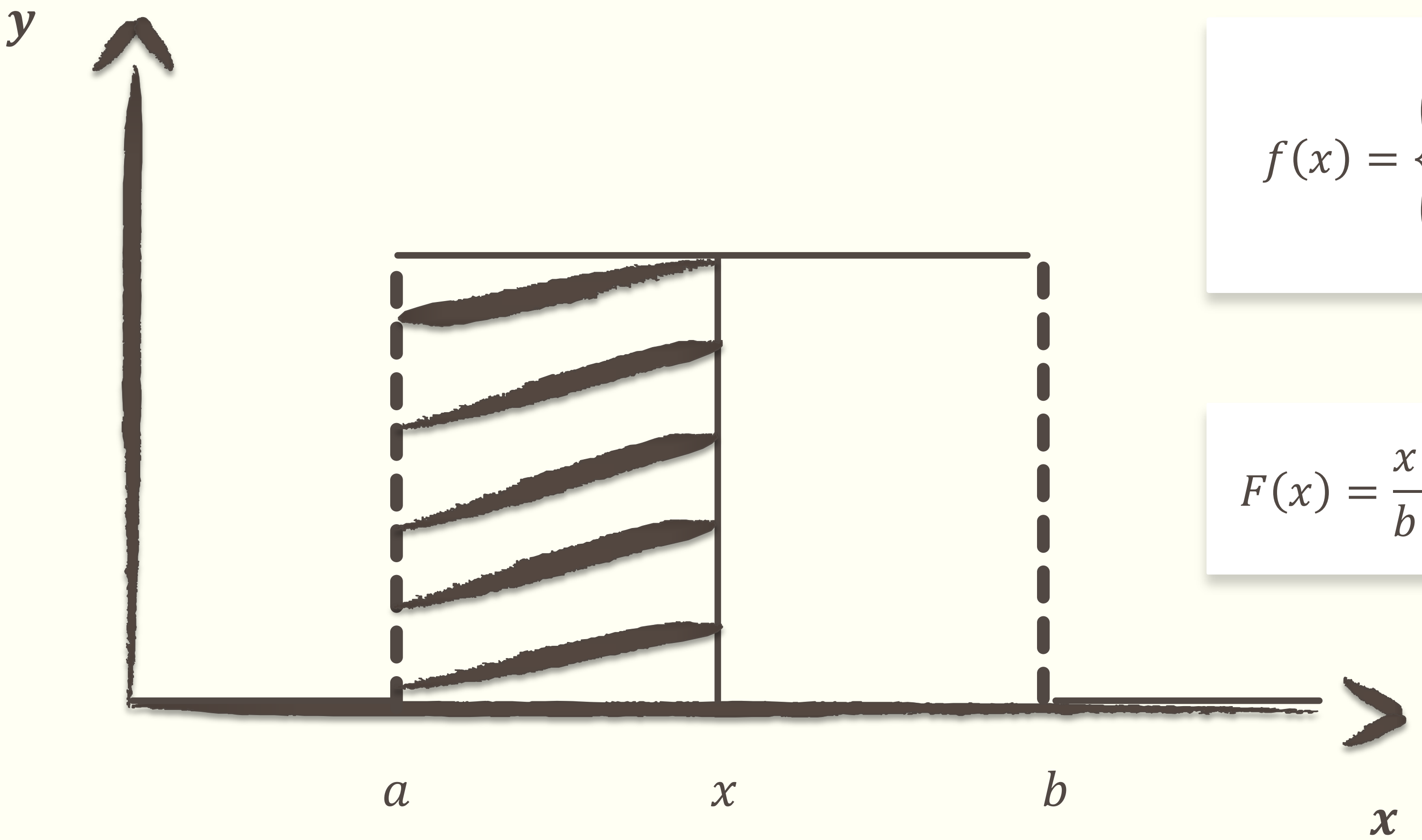


Calculate the Cumulative Distribution Function of the Uniform Distribution

Let X be a continuous real-valued random variable with density function $f(x)$. Then the function defined by

$$F_X(x) = \int_{-\infty}^{x} f(s)ds,$$

is the cumulative distribution function of X.



$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & x < a \text{ or } x > b \end{cases}$$

$$F(x) = \frac{x-a}{b-a}, \qquad a \leq x \leq b$$

Example 1

A real number U is chosen at random from $[0, 1]$ with uniform probability, and then this number is squared.

Let X represent the result, $X = U^2$.

- What is the cumulative distribution function of X?
- What is the density function of X?

00

$F_U(u) = P(U \leq u) = \cdots$

01

$F_X(x) = P(X \leq x) = \cdots$
range or X is ...

02

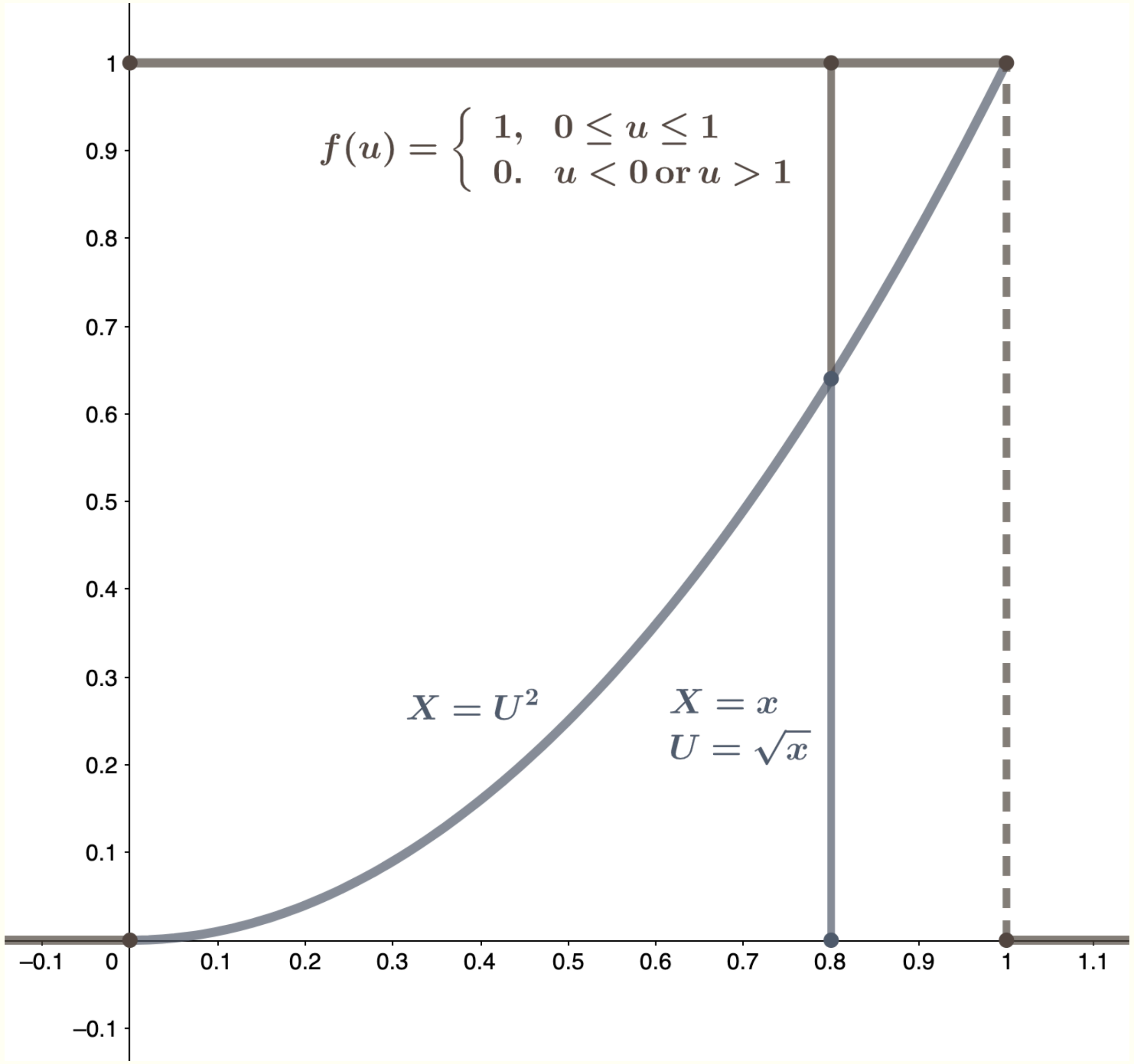
$\frac{d}{dx} F_X(x) = f(x) = \cdots$

01

$F_X(x) = P(X \leq x) = \cdots$
range of X is ...

$\frac{d}{dx} F_X(x) = f(x) = \cdots$

02



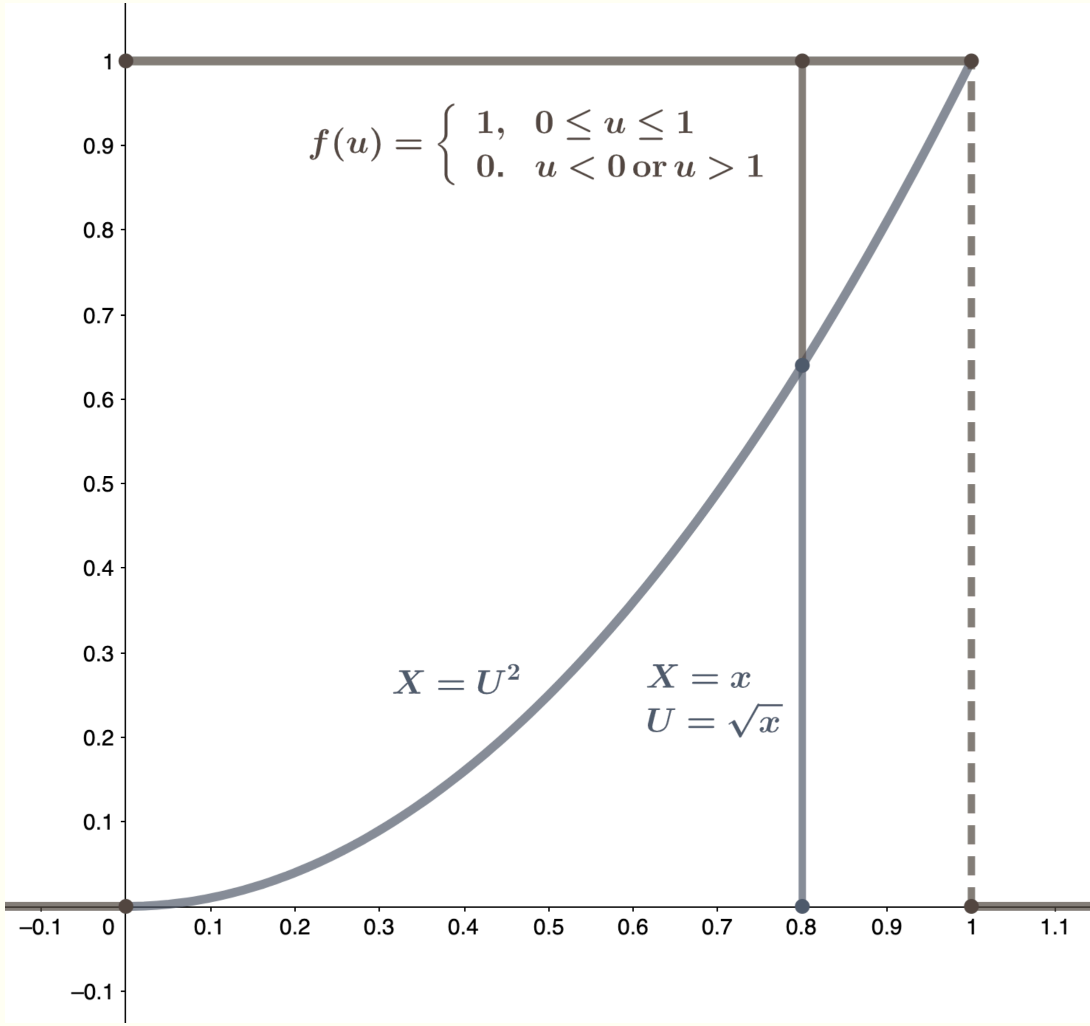
01

$$F_X(x) = P(X \le x) = P(U \le \sqrt{x}) = \sqrt{x}$$

$$0 \le X \le 1$$

$$\frac{d}{dx}F_X(x) = f(x) = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

02



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A real number U is chosen at random from $[0, 1]$ with uniform probability, and then this number is squared.

Let X represent the result,

- $X = \frac{1}{U+1}$.
- $X = \ln(U + 1)$.

- What is the cumulative distribution function of X?
- What is the density function of X?

01

$F_X(x) = P(X \leq x) = \cdots$
range or X is ...

$F_X(x) = P(X \leq x) = P\left(\frac{1}{U+1} \leq x\right) = P(U \geq \frac{1}{x} - 1)$
range or X is $\frac{1}{2} \leq X \leq 1$

$\frac{d}{dx} F_X(x) = f(x) = \cdots$

02

Example 2

Two real numbers X and Y are chosen at random and uniformly from $[0, 1]$. Let $Z = X + Y$.

Please derive expressions for the cumulative distribution and the density function of Z.

01

$F_Z(z) = P(Z \leq z) = \cdots$

range of Z is ...

$\frac{d}{dz}F_Z(z) = f(z) = \cdots$

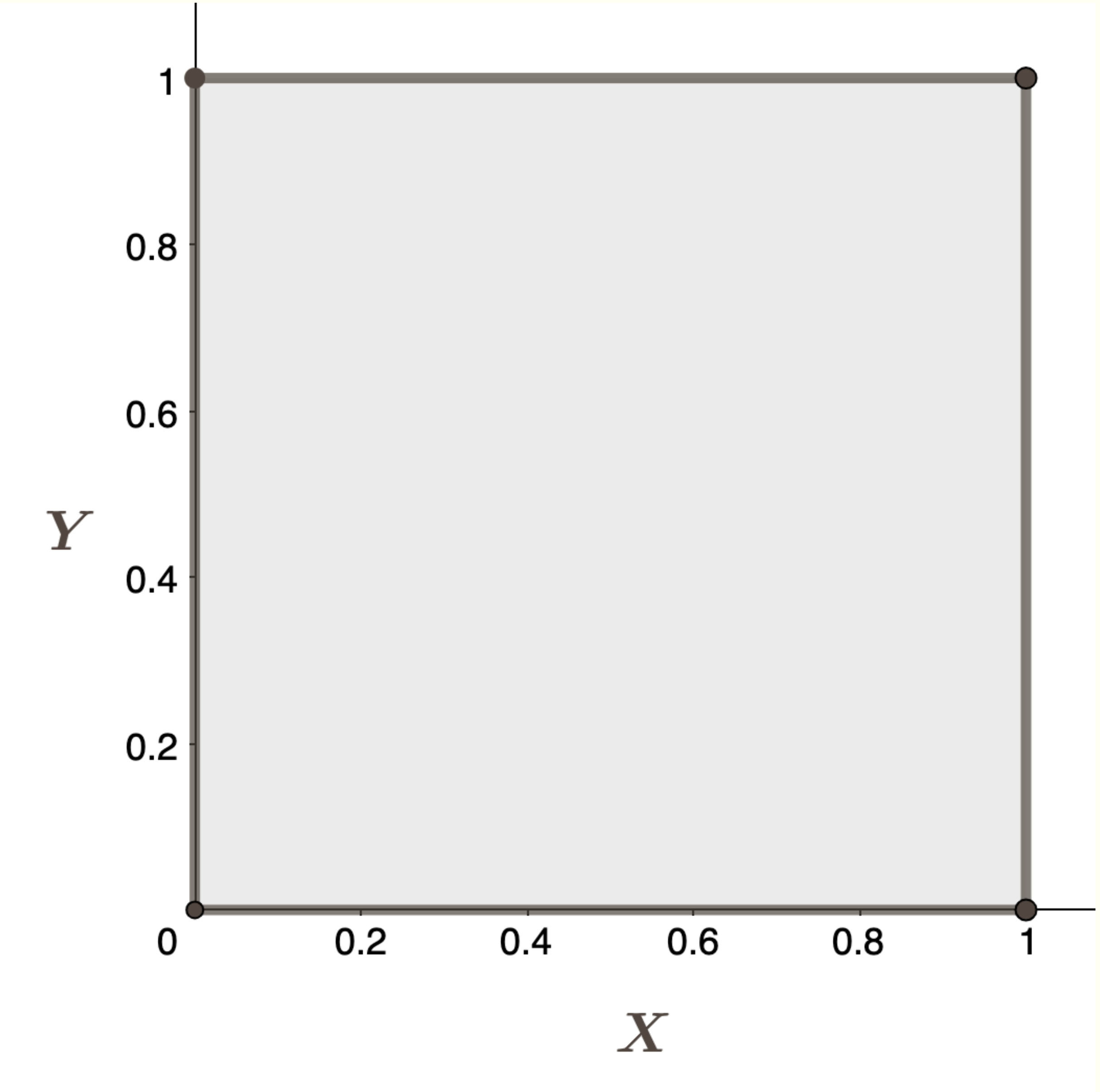
02

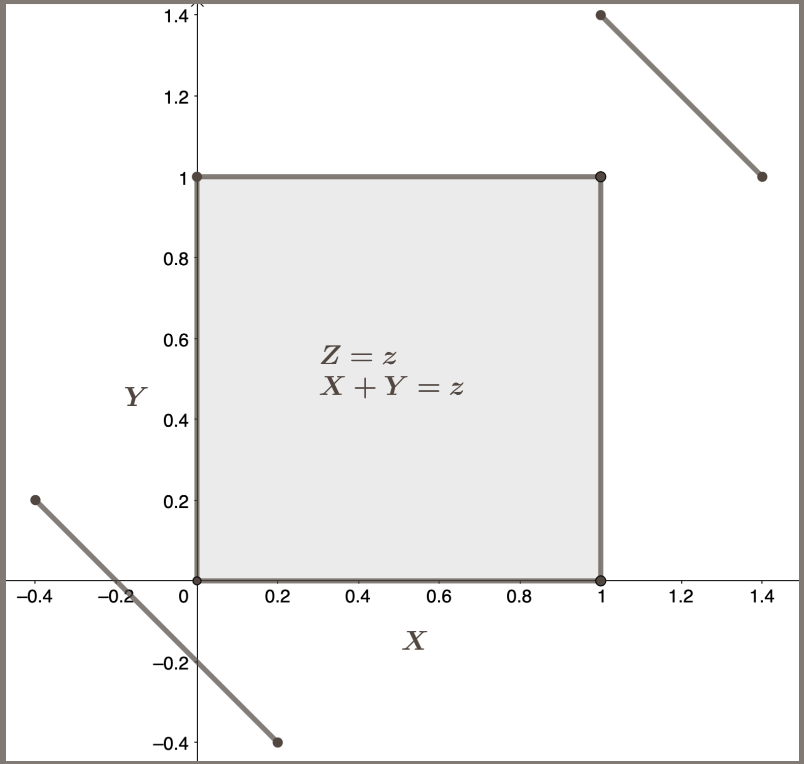
- Two real numbers X and Y are chosen at random and uniformly from $[0,1]$. Let $Z = X + Y$.

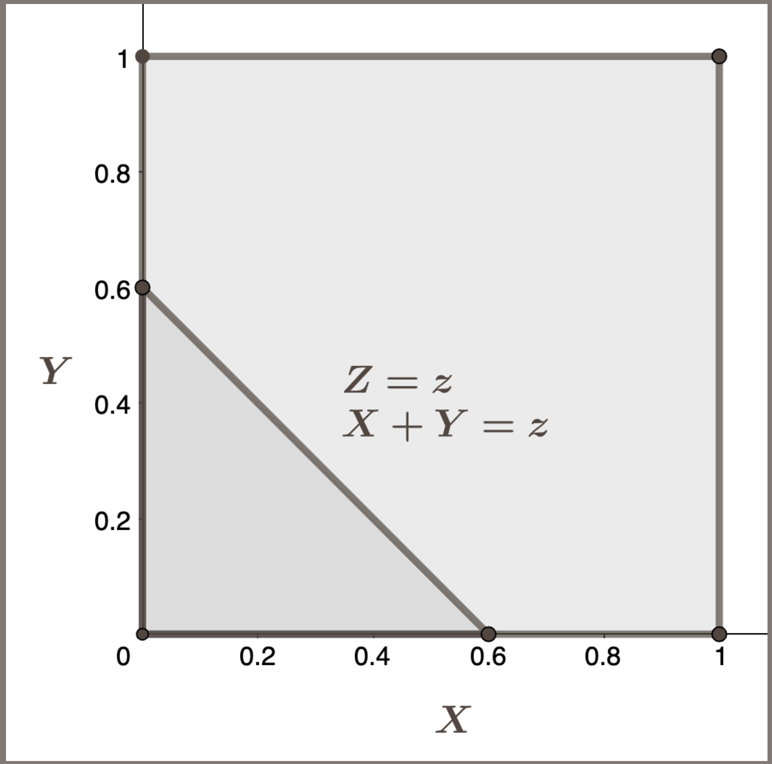
- Please derive expressions for the cumulative distribution and the density function of Z.

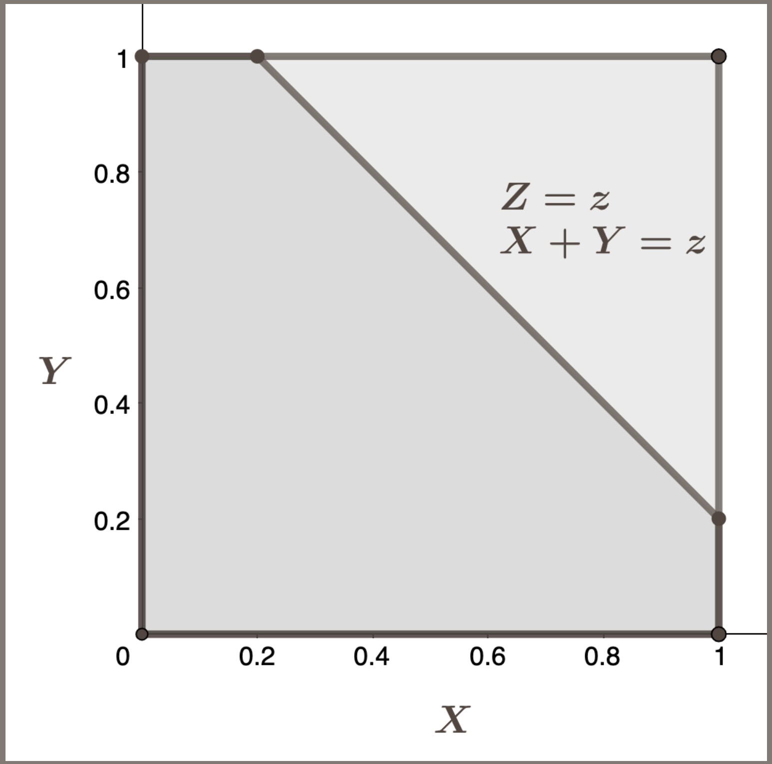
01

$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$



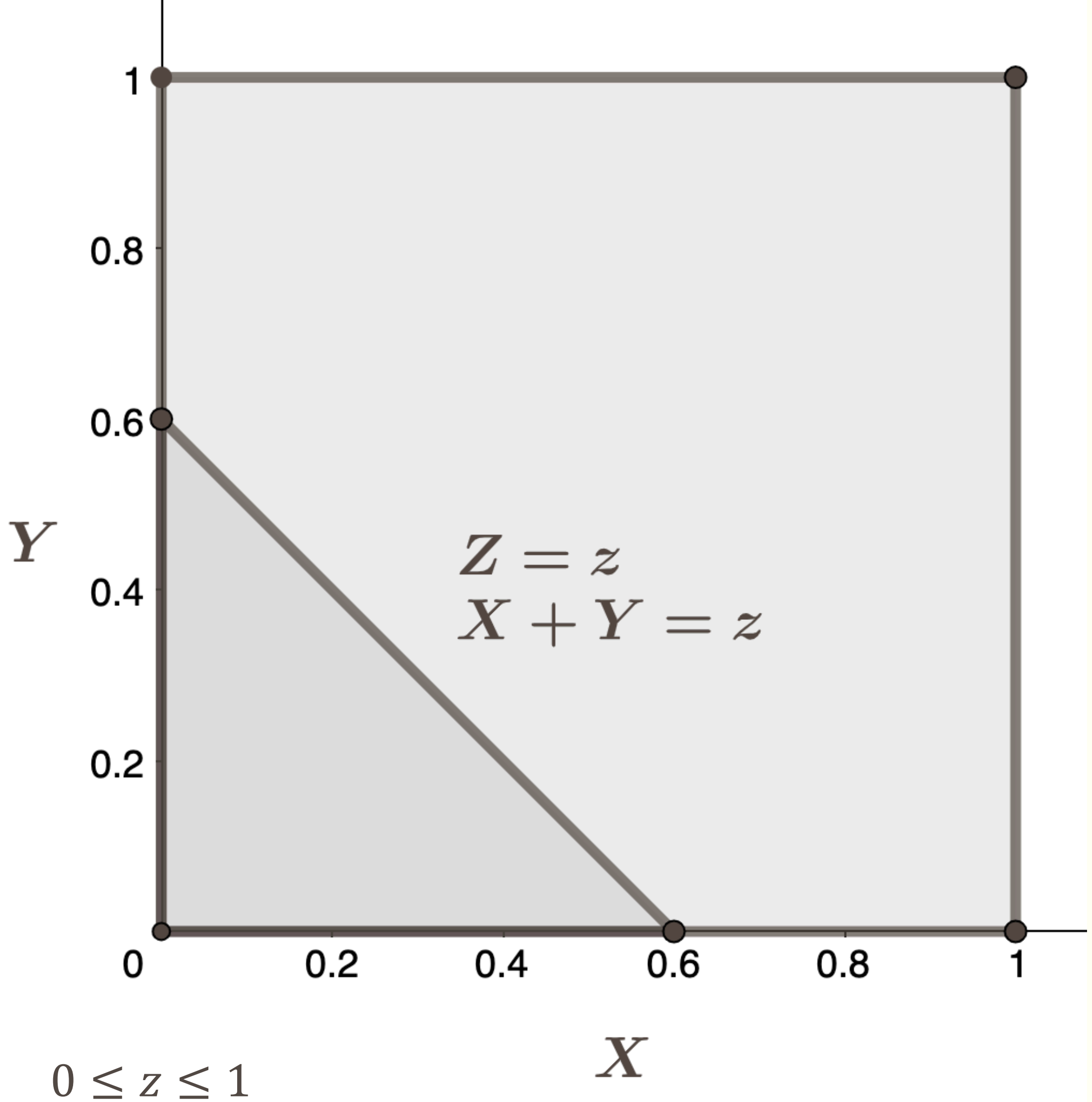




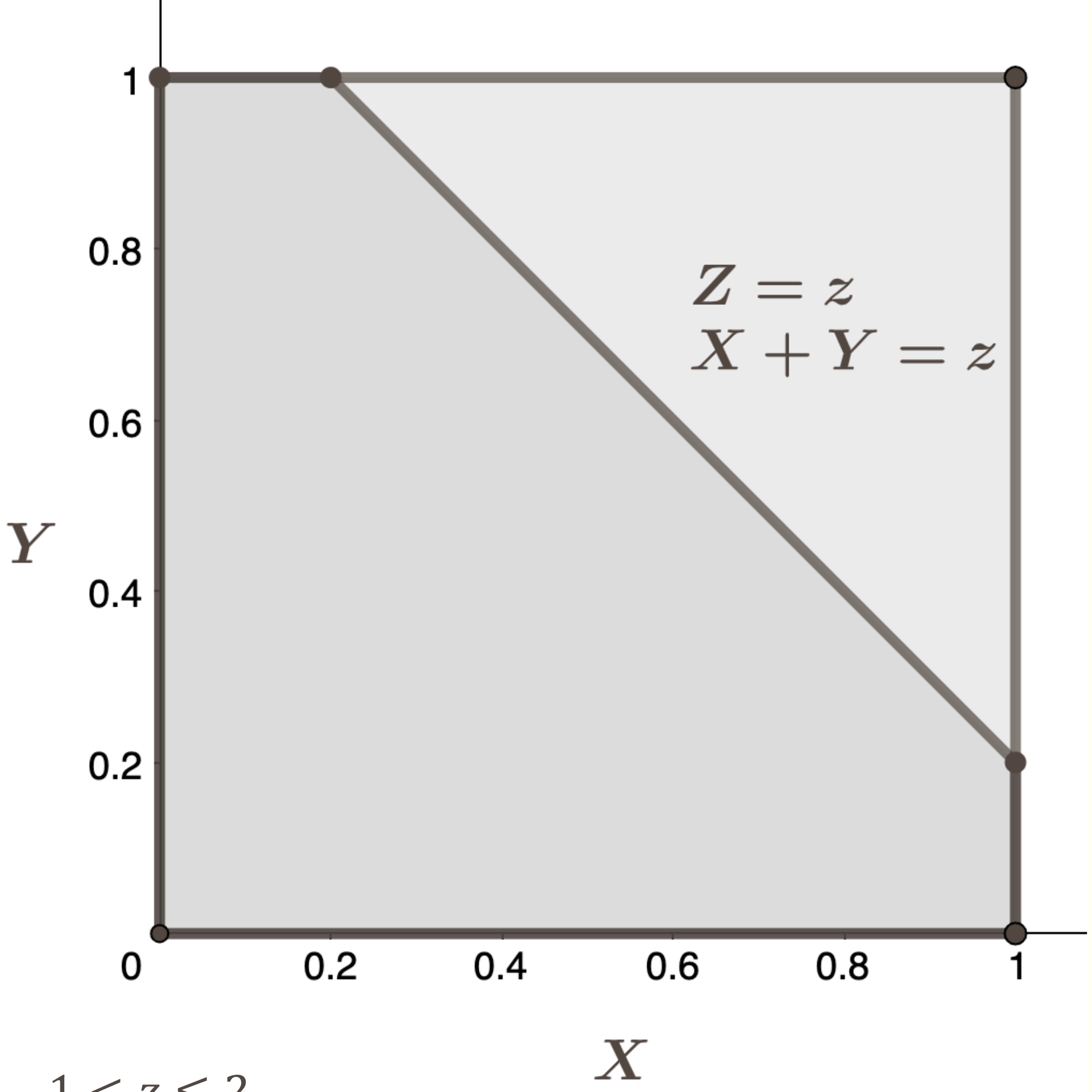


01

range of Z is $0 \leq Z \leq 2$



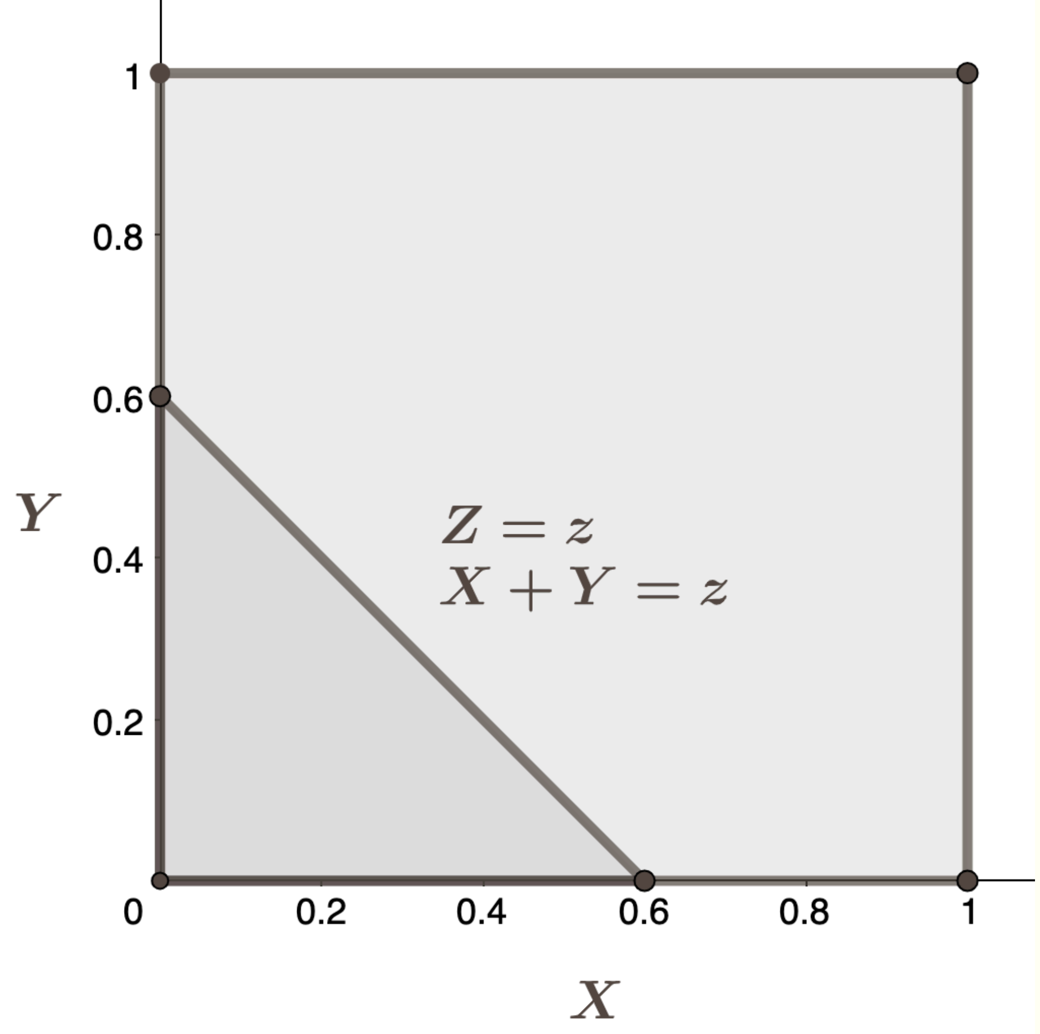
- $0 \le z \le 1$

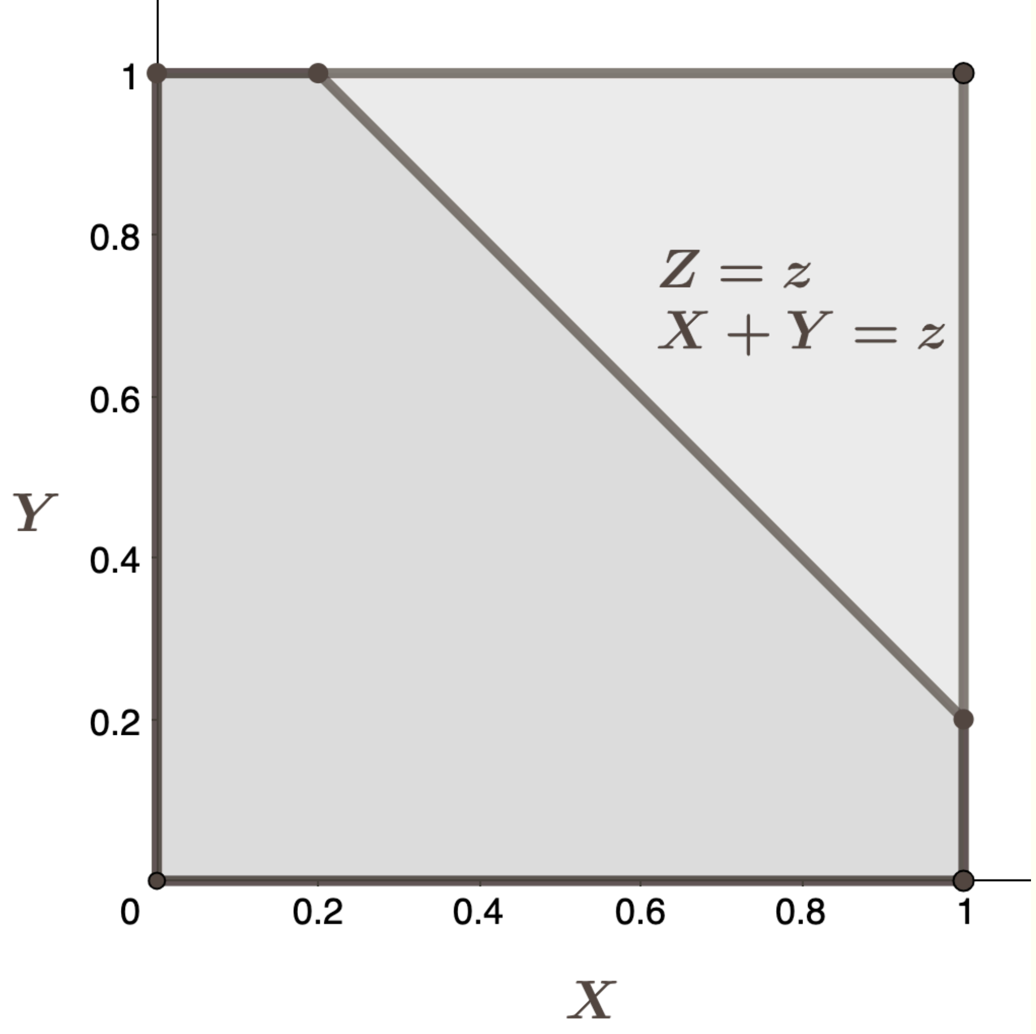


- $1 < z \le 2$

01

$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$





- $0 \le z \le 1$

  $F_Z(z) = P(Z \le z) = P(X + Y \le z) = \frac{1}{2} z^2$

- $1 < z \le 2$

  $F_Z(z) = P(Z \le z) = P(X + Y \le z)$

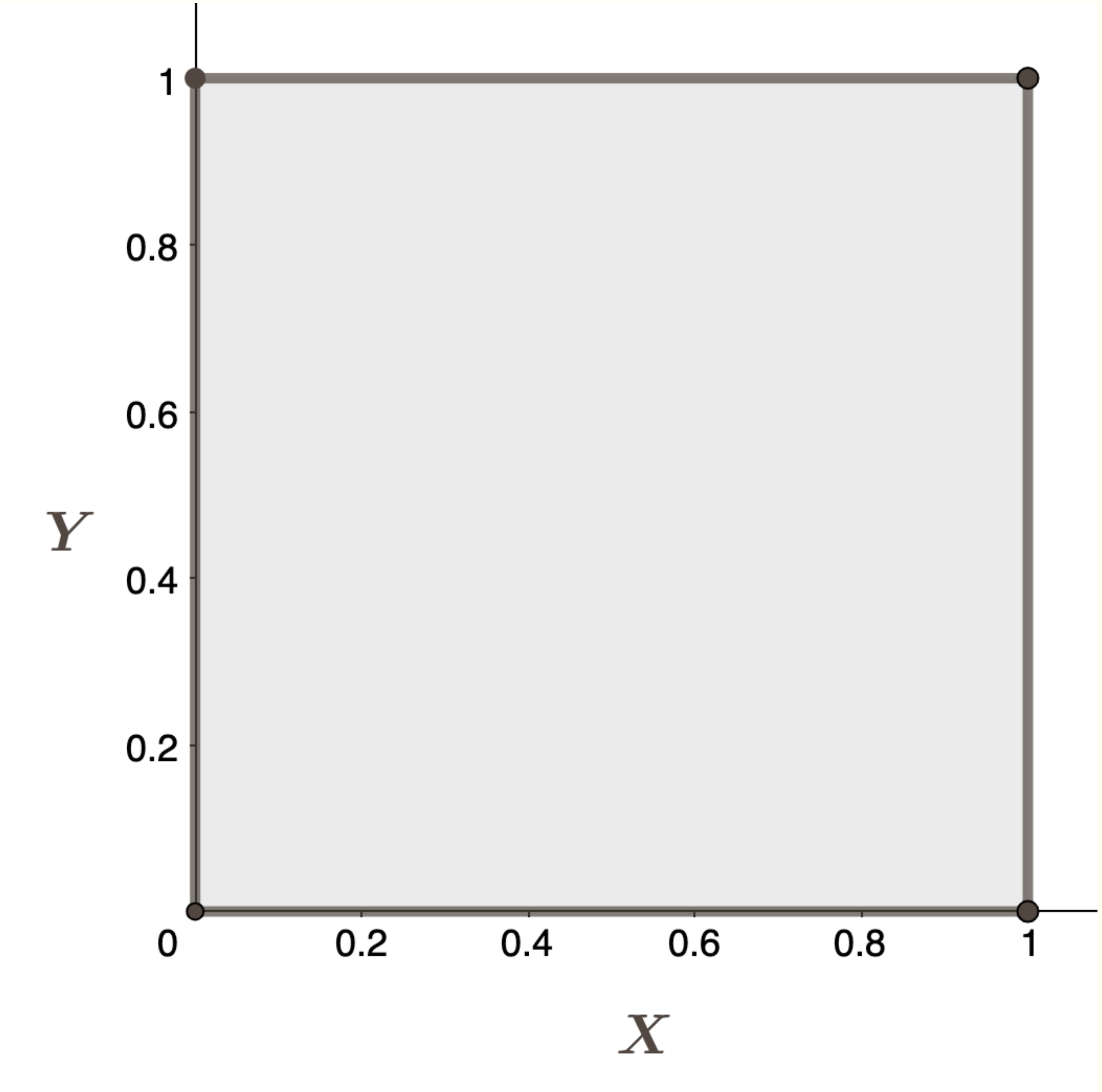
  $= 1 - \frac{1}{2}(2 - z)^2$

- Two real numbers X and Y are chosen at random and uniformly from $[0, 1]$. Let $Z = X + Y$.

- Please derive expressions for the cumulative distribution and the density function of Z.

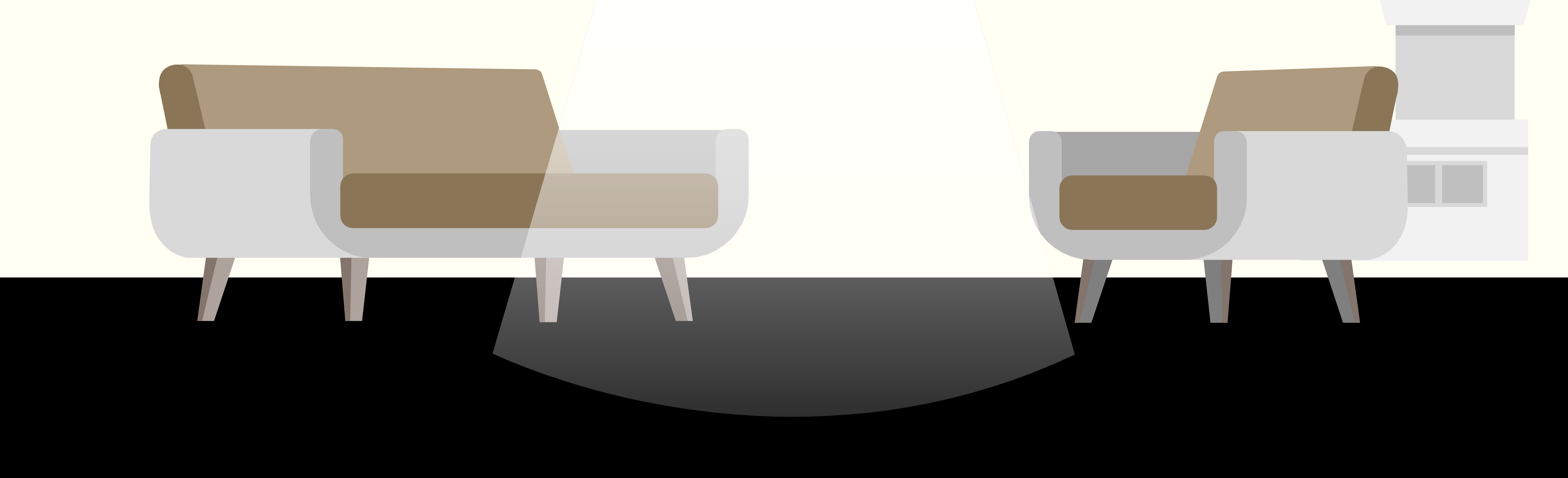
$$F_Z(z) = P(Z \le z) = \begin{cases} \frac{1}{2}z^2, 0 \le z \le 1 \\ 1 - \frac{1}{2}(2-z)^2, 1 < z \le 2 \end{cases}$$

$$\frac{d}{dz}F_Z(z) = f(z) = \begin{cases} z, 0 \le z \le 1 \\ 2 - z, 1 < z \le 2 \end{cases}$$



Example 3

Suppose Mr. and Mrs. Lockhorn agree to meet at the Hanover Inn between 5pm and 6pm on Tuesday.
Suppose each arrives at a time between 5pm and 6pm chosen at random with **uniform probability**.
What is the distribution function for the length of time that the first to arrive has to wait for the other?
What is the density function?



Example 3

01

$Z = \cdots$

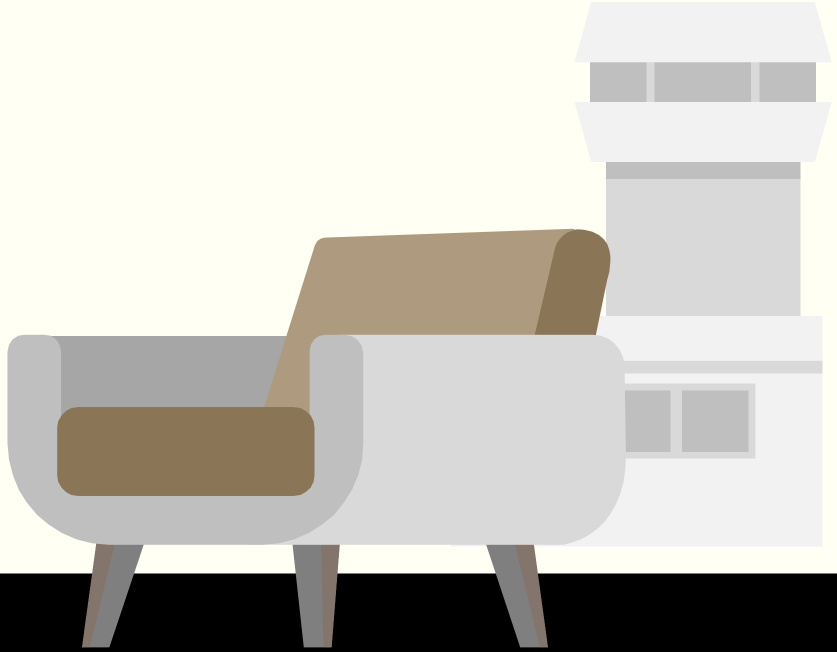
range of Z is …

02

$$F_Z(z) = P(Z \leq z) = \cdots$$

03

$$\frac{d}{dx}F_X(x) = f(x) = \cdots$$



Example 3

$$Z = |X - Y|$$

range of Z is $0 \leq Z \leq 1$

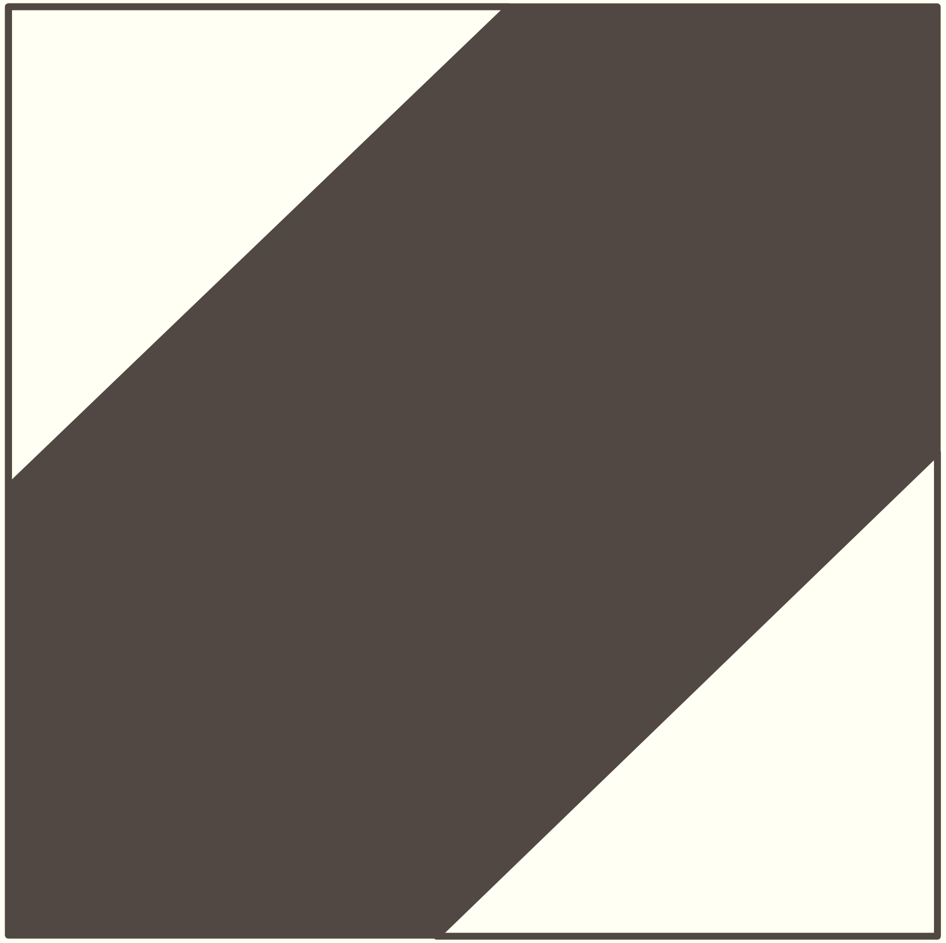
01

02

$$F_Z(z) = P(Z \leq z) = 1 - (1 - z)^2$$

$$\frac{d}{dx} F_X(x) = f(x) = 2(1 - z)$$

03



- Two real numbers X and Y are chosen at random and uniformly from $[0, 1]$.

    - Let $Z = \min(X, Y)$.
    - Let $Z = \max(X, Y)$.

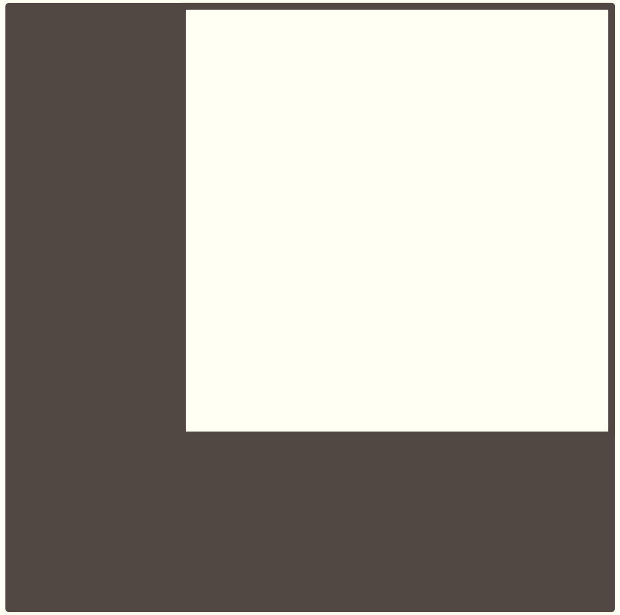
- Please derive expressions for the cumulative distribution and the density function of Z.

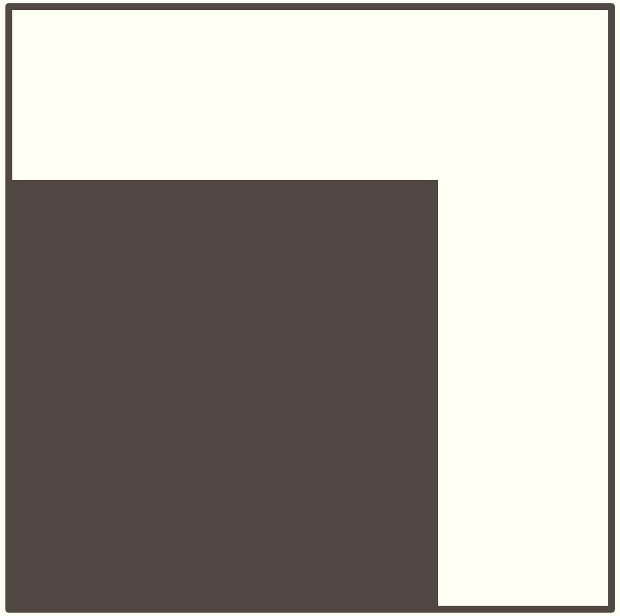
01

$$F_Z(z) = P(Z \leq z)$$

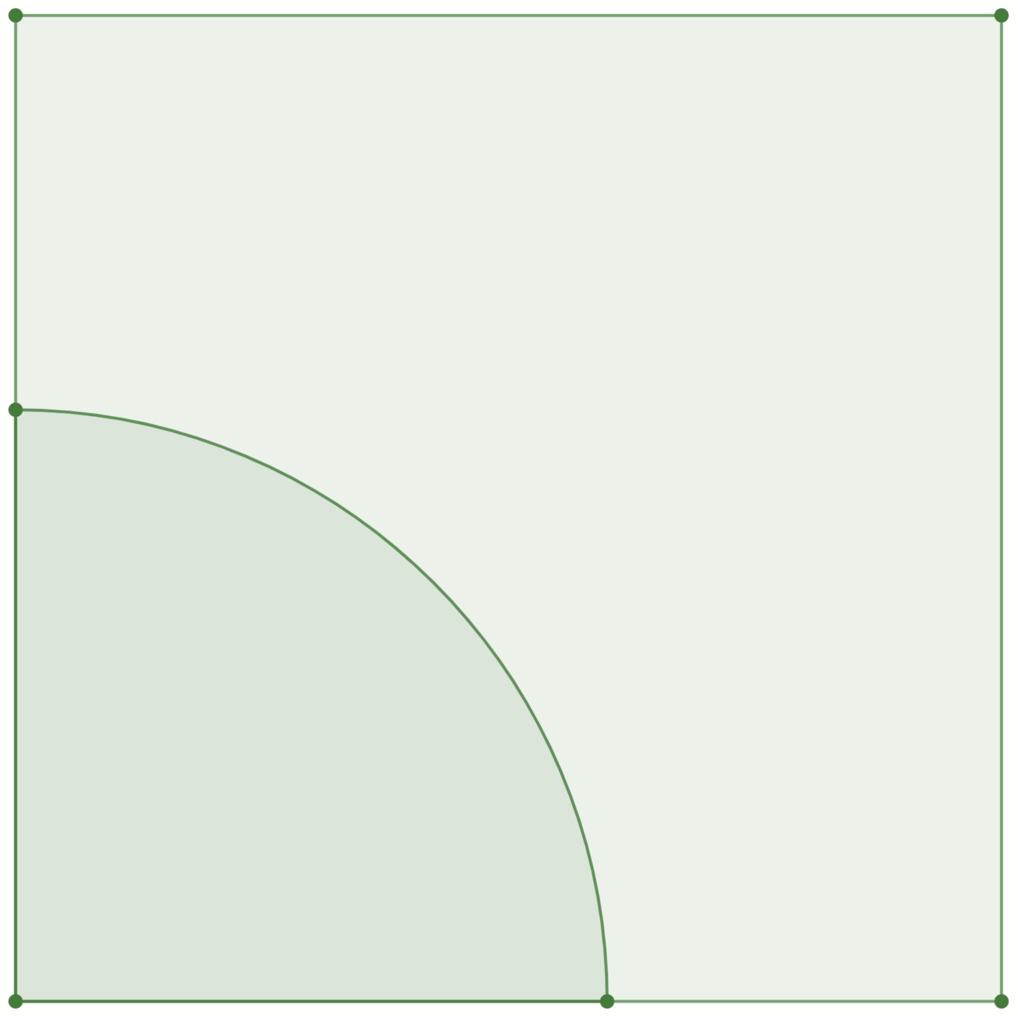
$$\frac{d}{dz} F_Z(z) = f(z) = \cdots$$

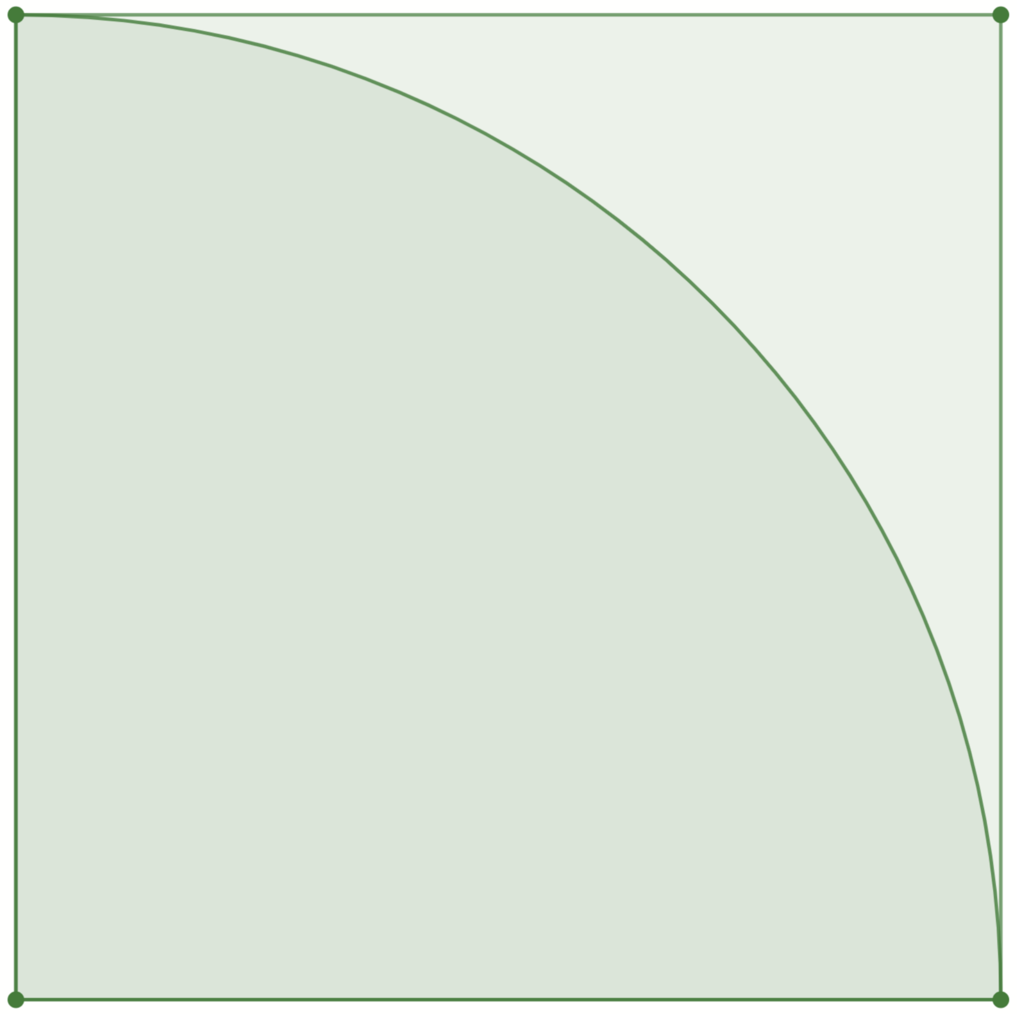
02





$$Z = X^2 + Y^2$$





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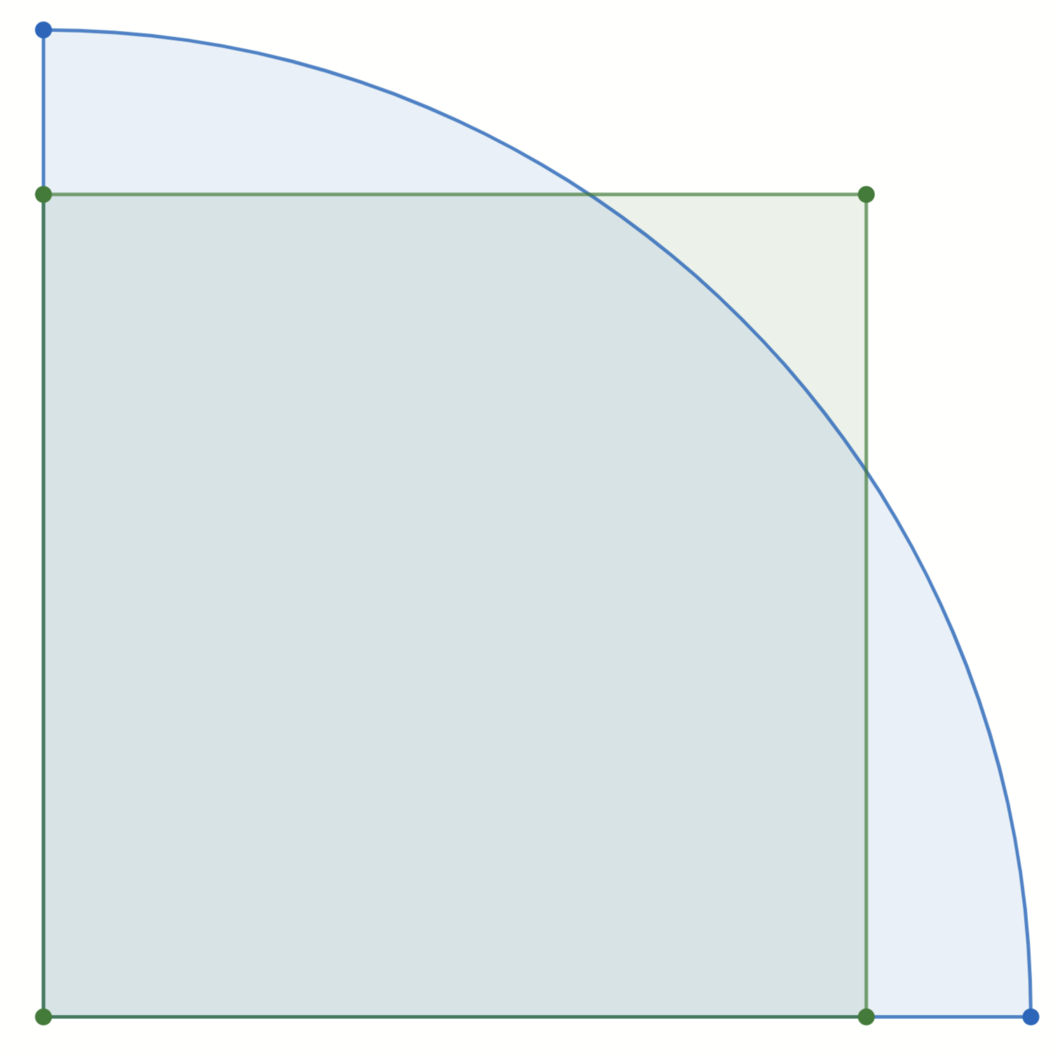
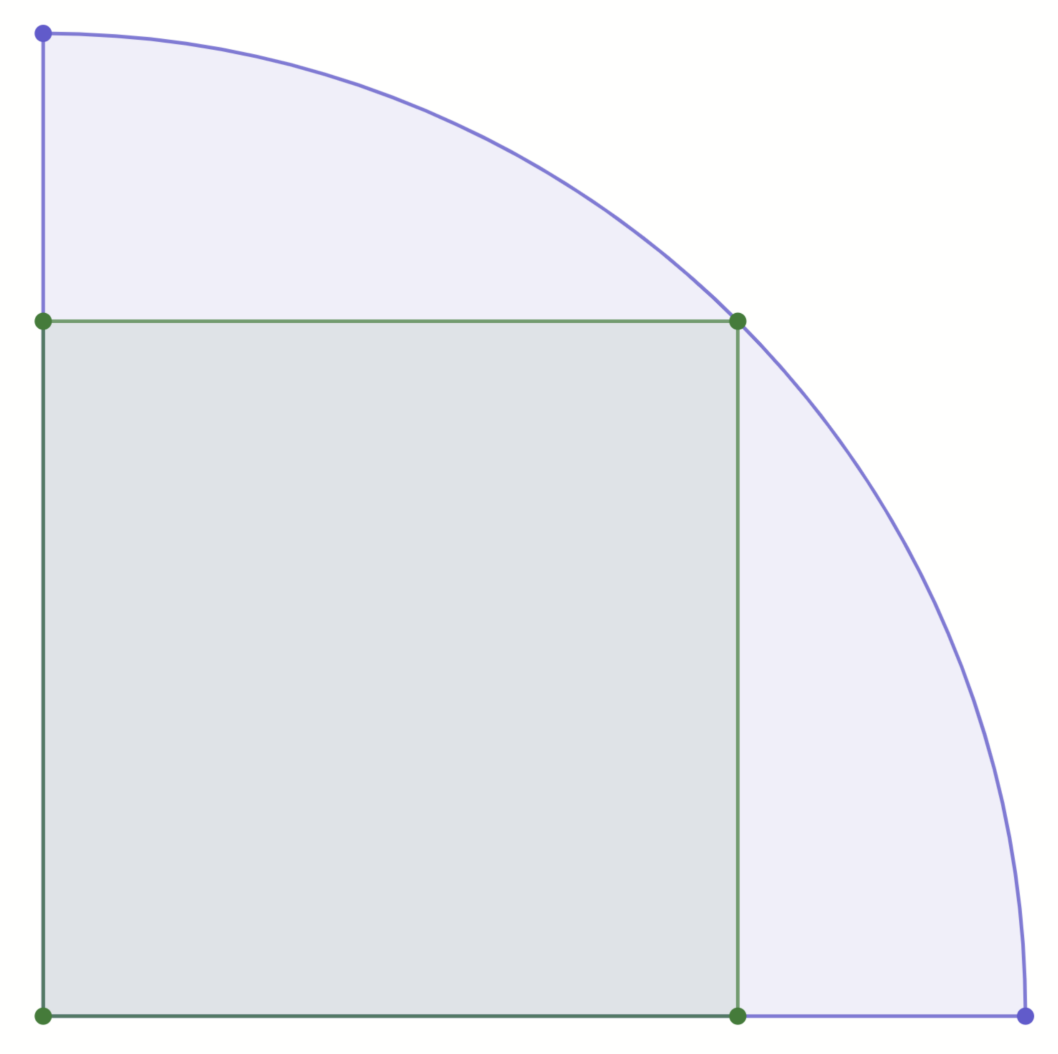
$$Z = X^2 + Y^2$$





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Exponential Distribution

- The exponential distribution is the probability distribution of the **time** between events in a **Poisson point process**. That is, a process in which events occur **continuously and independently at a constant average rate λ**.

- The density function of an exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

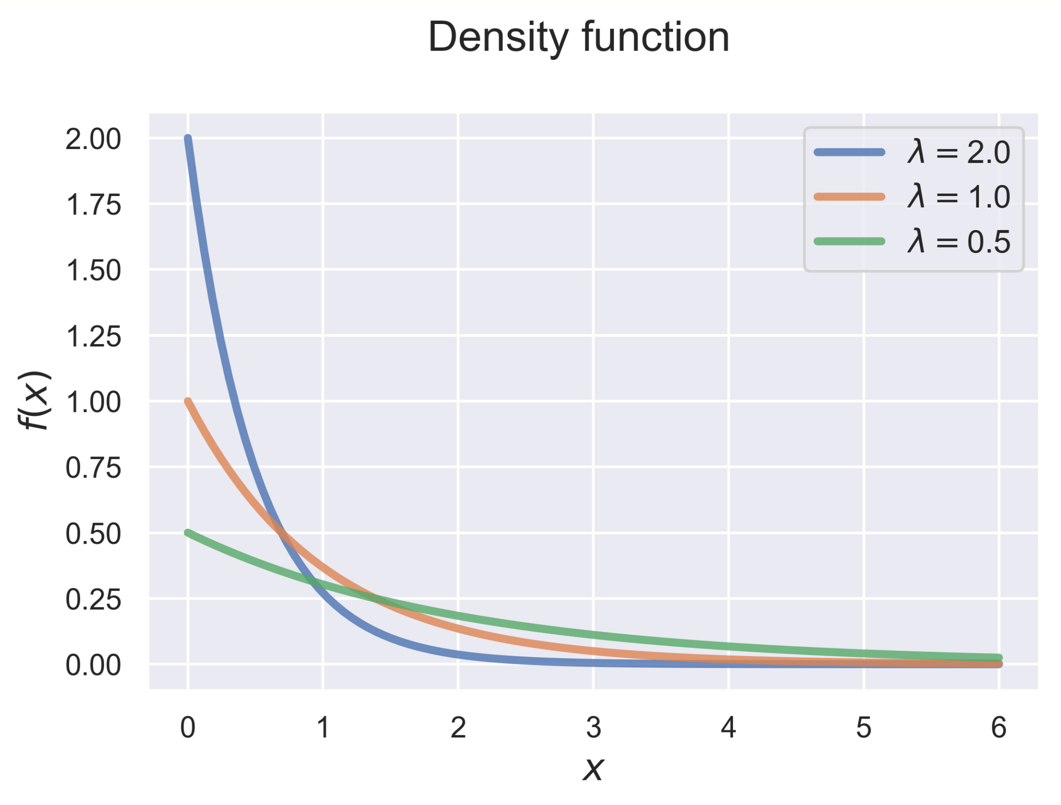
- The cumulative distribution function of an exponential distribution is

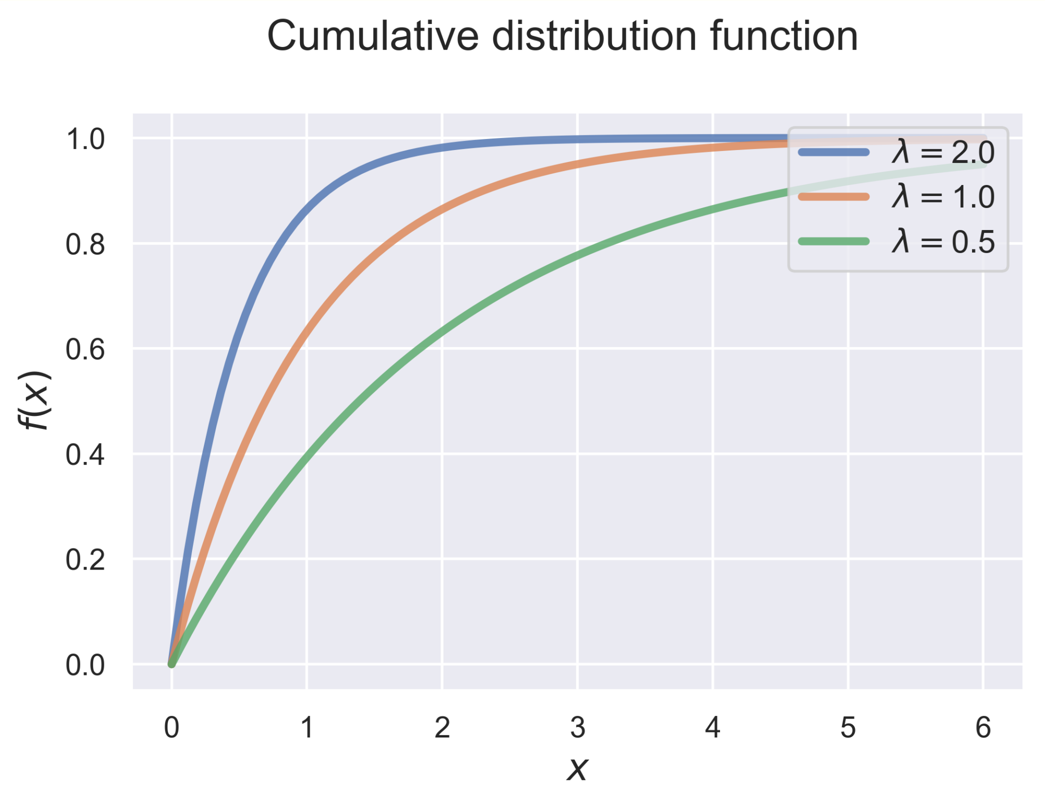
$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Exponential Distribution

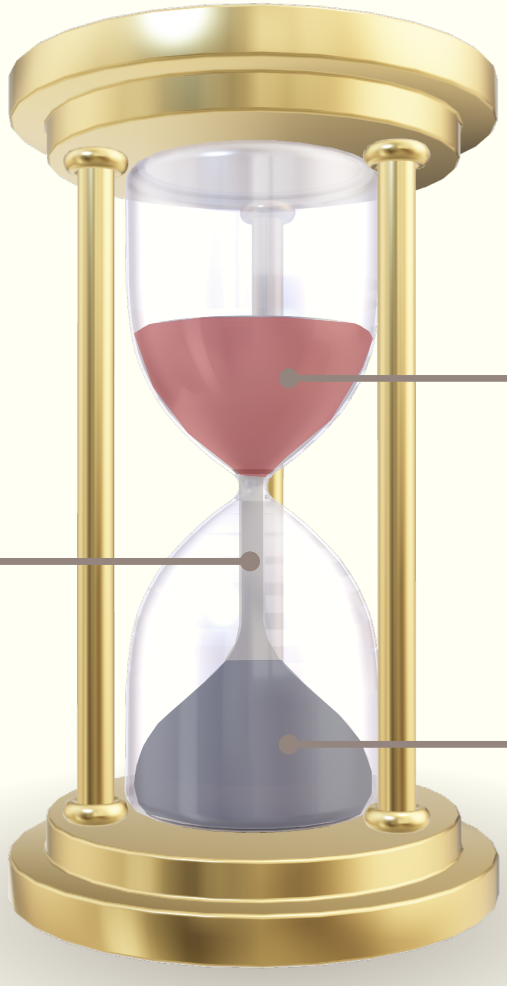
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$





Exponential Distribution



Time

The exponential distribution is often concerned with the amount of time until some specific event occurs.

Density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

The Amount of Time Until Some Specific Event Occurs

the amount of time (beginning now) until an earthquake occurs

the amount of time, in months, a car battery lasts

the number of days ahead travelers purchase their airline tickets

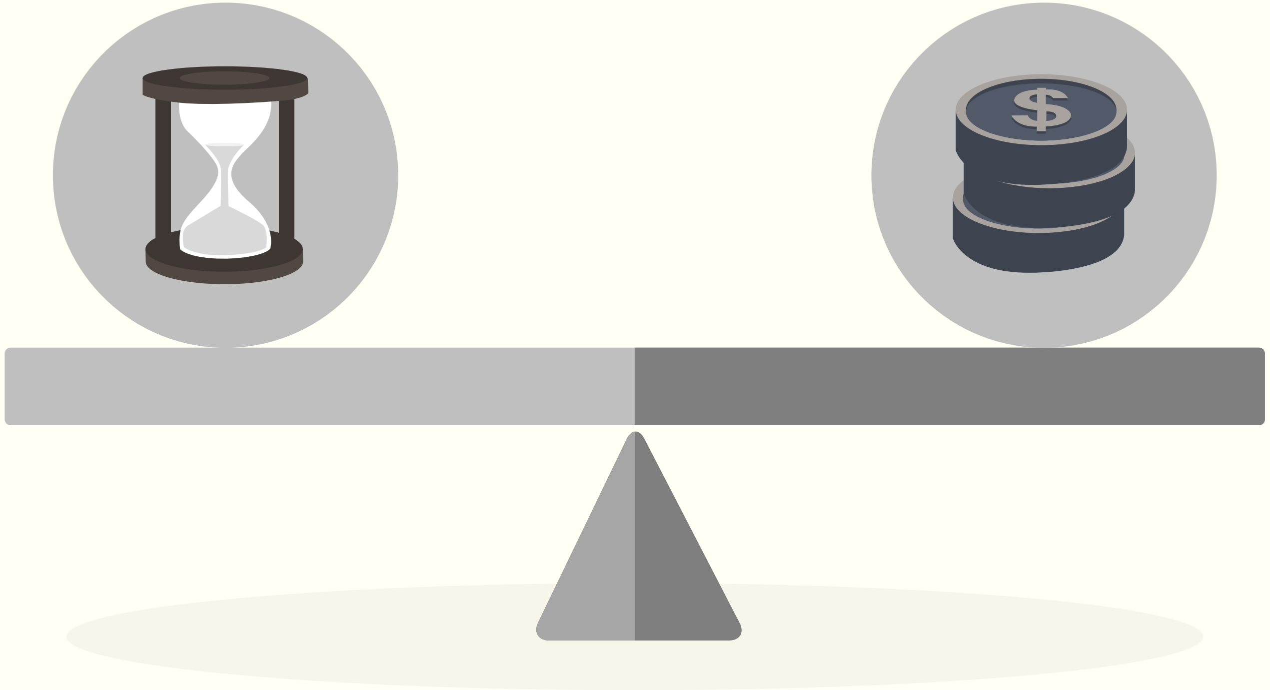
the length, in minutes, of long distance business telephone calls

amount of time (in minutes) a postal clerk spends with his or her customer

Exponential distribution

XC 2020

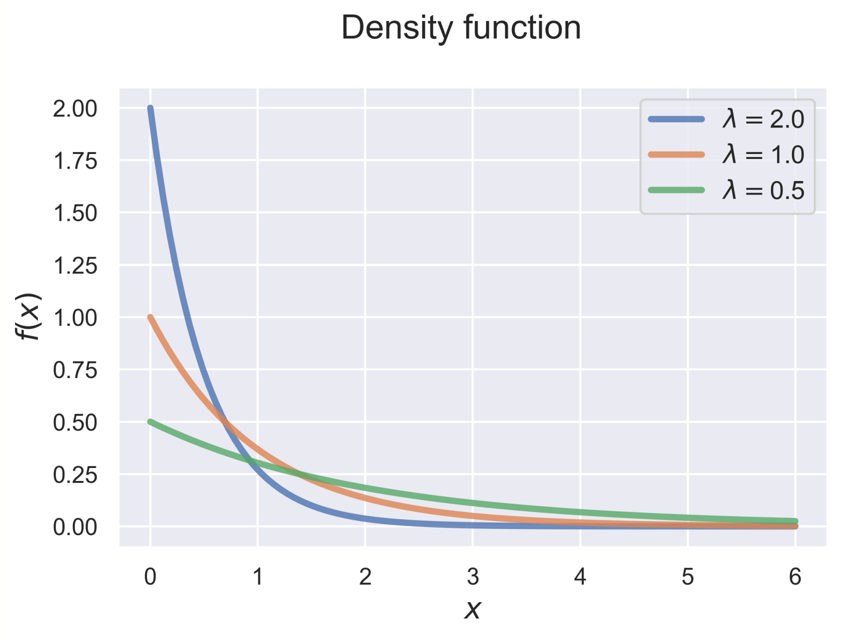
Exponential Distribution



The value of the change that you have in your pocket or purse approximately follows an exponential distribution.

Exponential Distribution

- Values for an exponential random variable occur in the following way.
  - There are fewer large values and more small values.
  - For example, the amount of money customers spend in one trip to the supermarket follows an exponential distribution. There are more people who spend small amounts of money and fewer people who spend large amounts of money.



Example

Let X = amount of time (in minutes) a postal clerk spends with his or her customer. The time is known to have an exponential distribution with the **average** amount of time equal to four minutes.

average: μ

$\mu = 4$

rate: λ

$\lambda = \cdots$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Example

Let X = amount of time (in minutes) a postal clerk spends with his or her customer. The time is known to have an exponential distribution with the **average** amount of time equal to four minutes.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

average: μ	rate: λ
$\mu = 4$	$\lambda = \frac{1}{\mu} = \frac{1}{4}$

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/4}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Example

Find the probability that a clerk spends four to five minutes with a randomly selected customer.

average: μ	rate: λ
$\mu = 4$	$\lambda = \frac{1}{\mu}$

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/4}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Example

Find the probability that a clerk spends four to five minutes with a randomly selected customer.

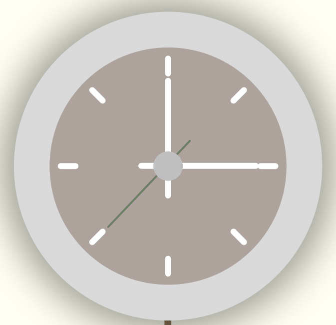
average: μ
$\mu = 4$

rate: λ
$\lambda = \frac{1}{\mu}$

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/4}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$P(4 < X < 5) = \int_4^5 \frac{1}{4} e^{-x/4}\, dx$$
$$= e^{-1} - e^{-\frac{5}{4}} \approx 0.0814$$



1
2
3
4
...

Homework 2

Problem 6

4 pts

Chapter 2.2 Exercise 5

Suppose you are watching a radioactive source that emits particles at a rate described by the exponential density

$$f(t) = \lambda e^{-\lambda t},$$

where $\lambda = 1$, so that the probability $P(0, T)$ that a particle will appear in the next T seconds is $P([0, T]) = \int_0^T \lambda e^{-\lambda t} dt$. Find the probability that a particle (not necessary the first) will appear

(a) within the next second.

(b) within the next 3 seconds.

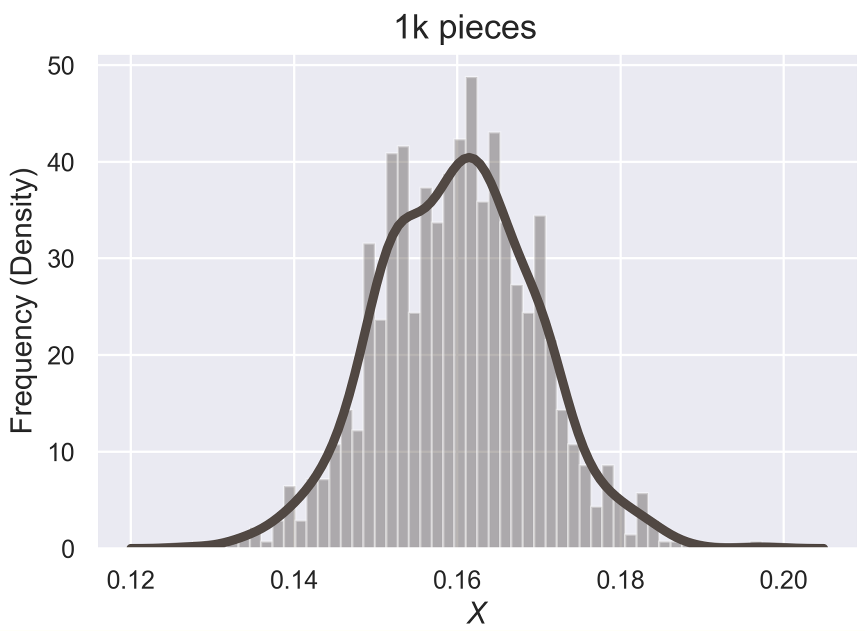
(c) between 3 and 4 seconds from now.

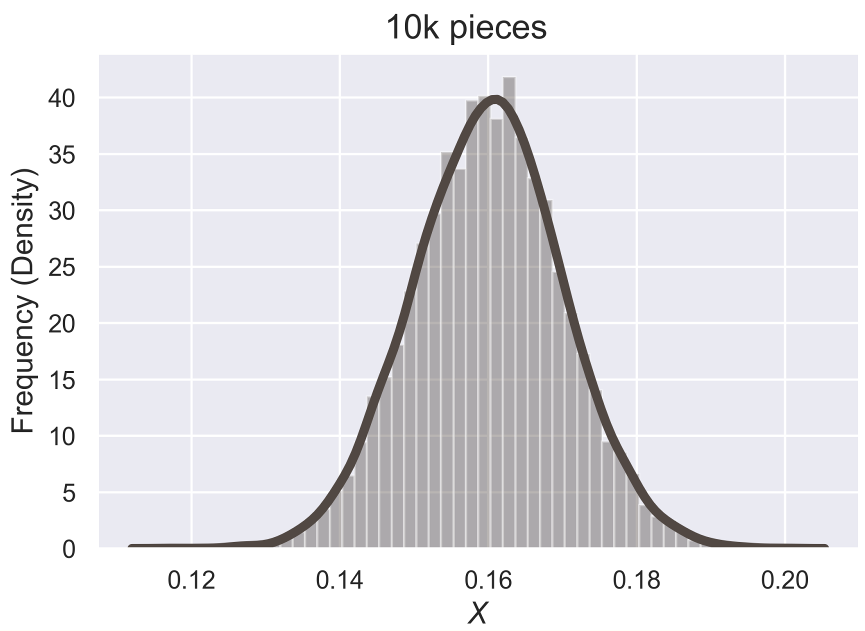
(d) after 4 seconds from now.

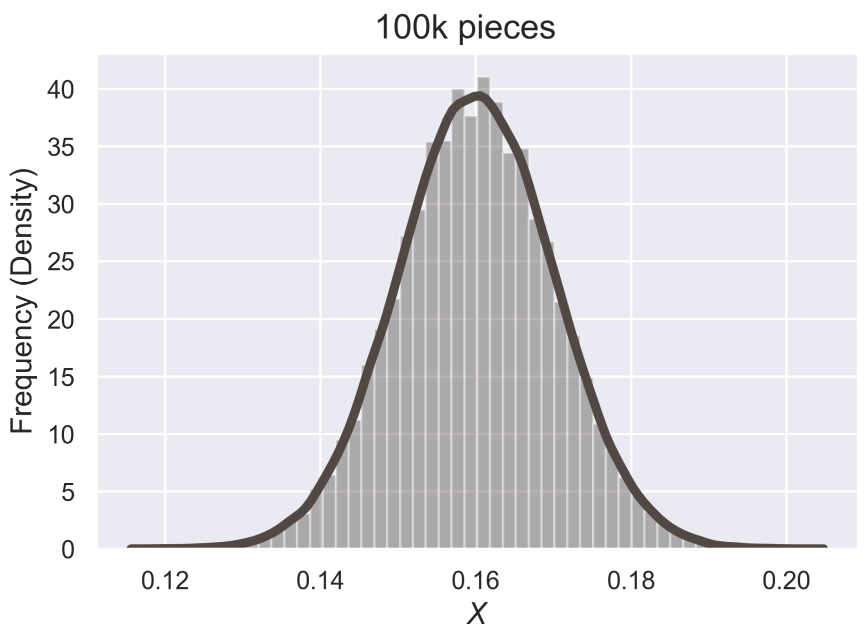
How Much Does a Hershey Kiss Weight?

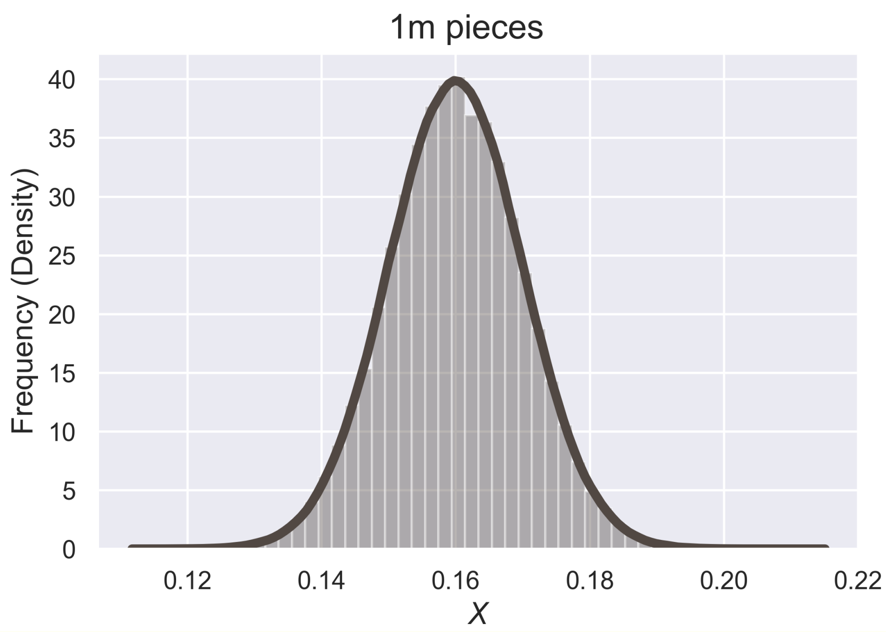
- A single standard Hershey's Kiss weighs 0.16 ounces.
- The data in the table below are the weights of 32 Hershey’s Kiss in a family pack.

0.1584	0.1577	0.1819	0.1581	0.1438	0.1385	0.1673	0.1611
0.165	0.1452	0.1482	0.1568	0.1603	0.1478	0.1591	0.1519
0.1649	0.1672	0.153	0.1504	0.1587	0.1485	0.1538	0.1498
0.1656	0.1692	0.1477	0.157	0.1574	0.1699	0.1589	0.1487


1k pieces
Frequency (Density)
0
10
20
30
40
50
0.12
0.14
0.16
0.18
0.20
X

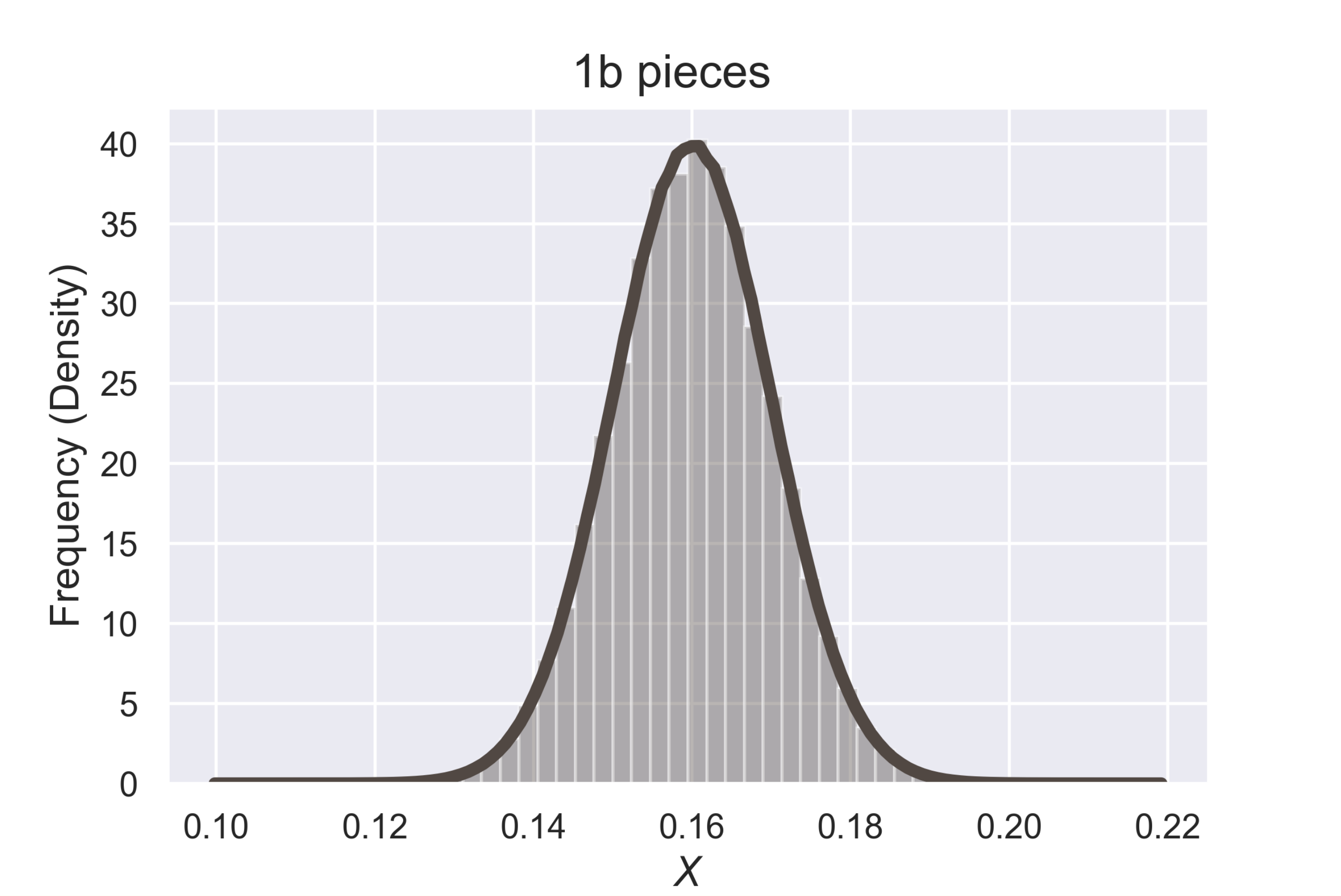

10k pieces
Frequency (Density)
0
5
10
15
20
25
30
35
40
0.12
0.14
0.16
0.18
0.20
X


100k pieces
Frequency (Density)
0
5
10
15
20
25
30
35
40
0.12
0.14
0.16
0.18
0.20
X


1m pieces
Frequency (Density)
0
5
10
15
20
25
30
35
40
0.12
0.14
0.16
0.18
0.20
0.22
X

XC 2020

Normal Density Distribution (Gaussian Distribution)



The normal density function with parameters μ and σ
expectation: μ, standard deviation: σ