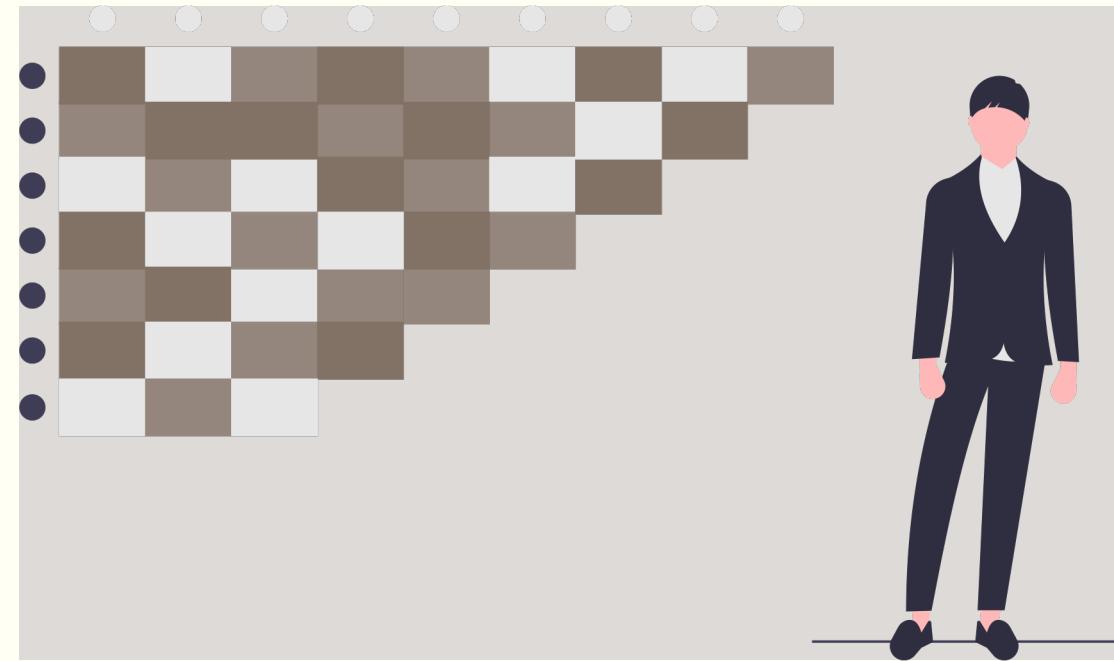




MATH 20: PROBABILITY

Sums of Independent Random Variables

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Syllabus

Sums of Random
Variables

Law of Large Numbers

Central Limit Theorem

Generating Functions

Markov Chains

Quiz



Quiz



Homework (due Fri 28)



Quiz



Final



Mon 17

Tue 18

Wed 19

Thu 20

Fri 21

Sat 22

Sun 23

Mon 24

Tue 25

Wed 26

...

Sun 30

Example from previous lectures

Two real numbers X and Y are chosen at random and uniformly from $[0, 1]$. Let $Z = X + Y$.

Please derive expressions for the cumulative distribution and the density function of Z .

01

$F_Z(z) = P(Z \leq z) = \dots$
range of Z is ...

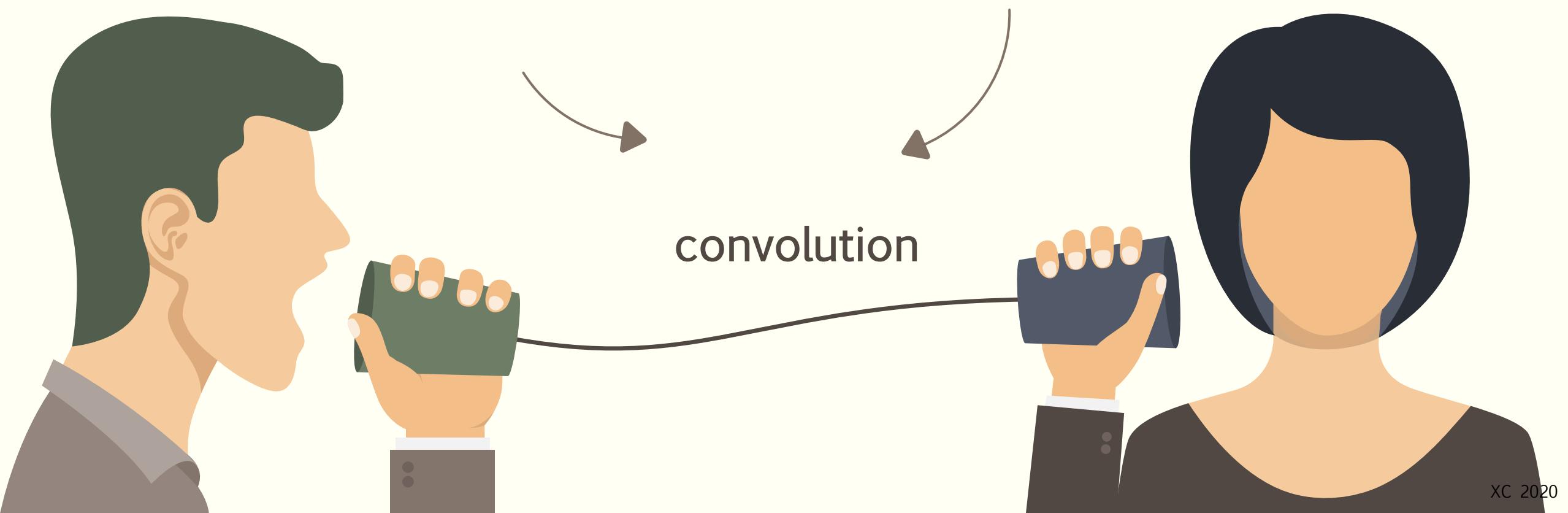
$$\frac{d}{dz} F_Z(z) = f(z) = \dots$$

02

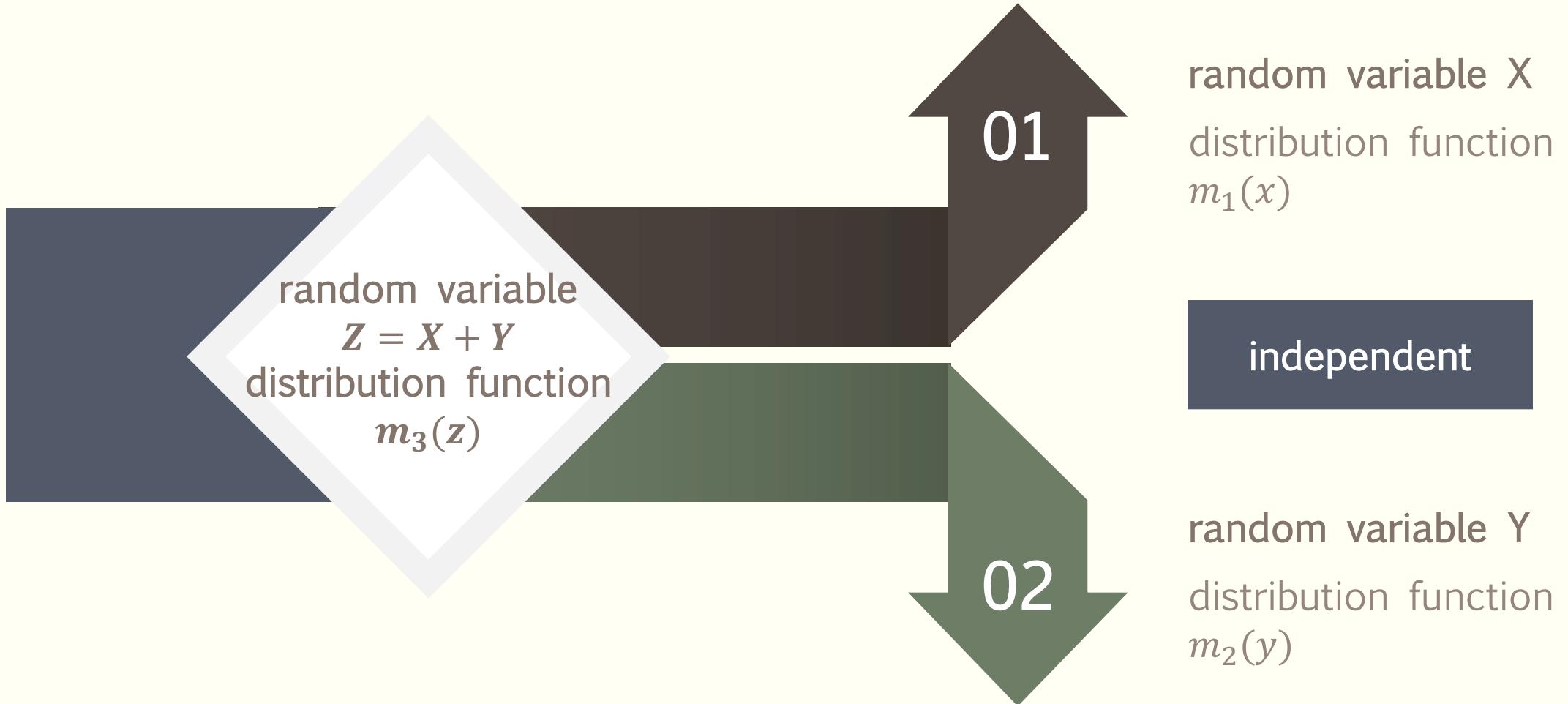
Sum of independent random variables discrete & continuous

$$m_3(j) = \sum_k m_1(k)m_2(j - k)$$

$$(f * g)(z) = \int_{-\infty}^{+\infty} f(z - y)g(y)dy$$

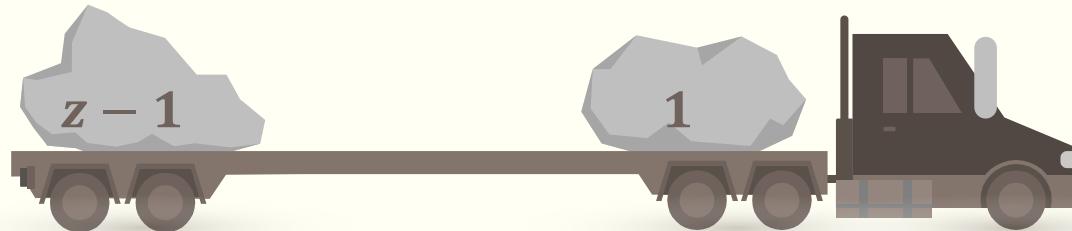


Sum of discrete random variables

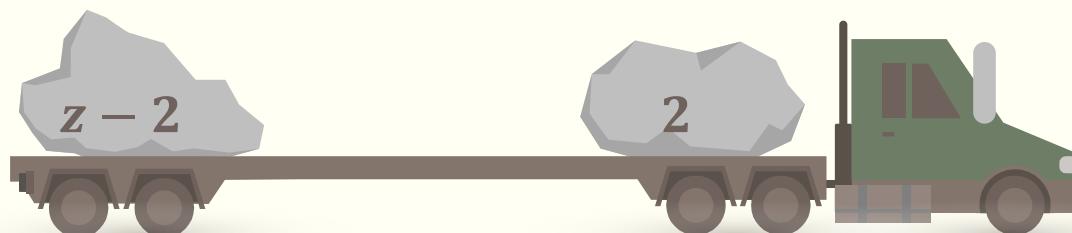


$$Z = X + Y$$

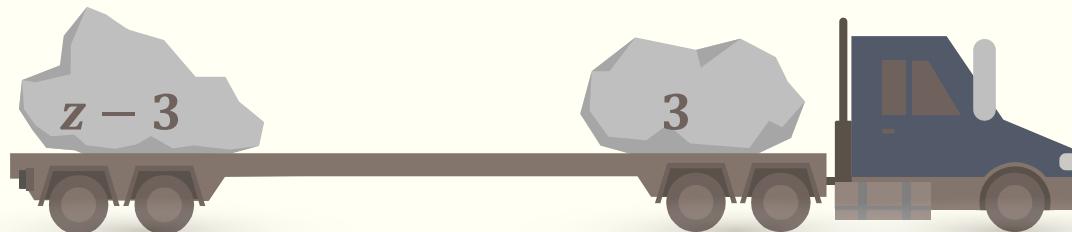
the probability that Z takes on the value z



$$z = 1 + (z - 1)$$



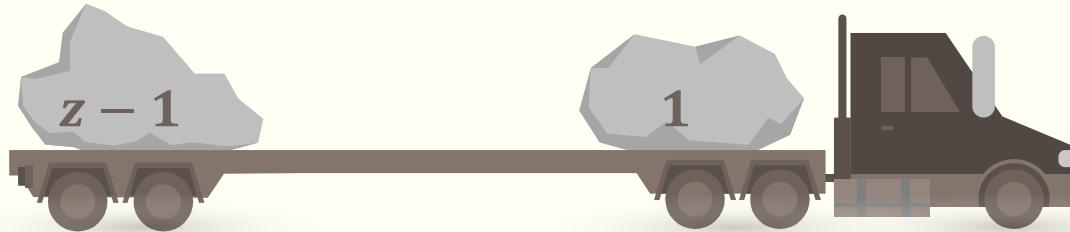
$$z = 2 + (z - 2)$$



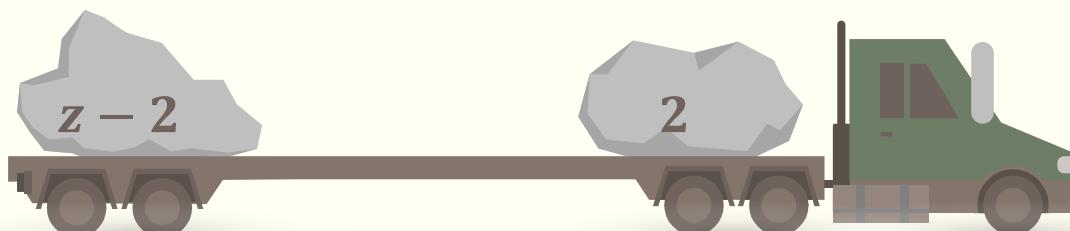
$$z = 3 + (z - 3)$$

$$Z = X + Y$$

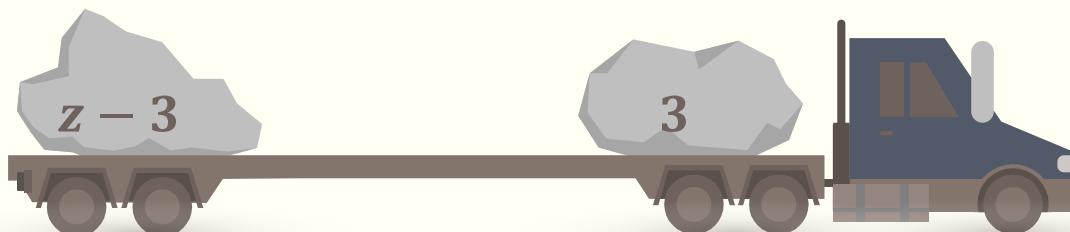
the probability that Z takes on the value z



$$z = 1 + (z - 1)$$



$$z = 2 + (z - 2)$$



$$z = 3 + (z - 3)$$

$$\begin{aligned} P(Z = z) \\ = \sum_{k=-\infty}^{+\infty} P(X = k)P(Y = z - k) \end{aligned}$$

Sum of discrete random variables

- Let X and Y be two **independent** integer-valued random variables, with distribution functions $m_1(x)$ and $m_2(x)$ respectively.
- Then the **convolution** of $m_1(x)$ and $m_2(x)$ is the distribution function $m_3 = m_1 * m_2$ given by

$$m_3(j) = \sum_k m_1(k)m_2(j - k),$$

for $j = \dots, -2, -1, 0, 1, 2, \dots$.

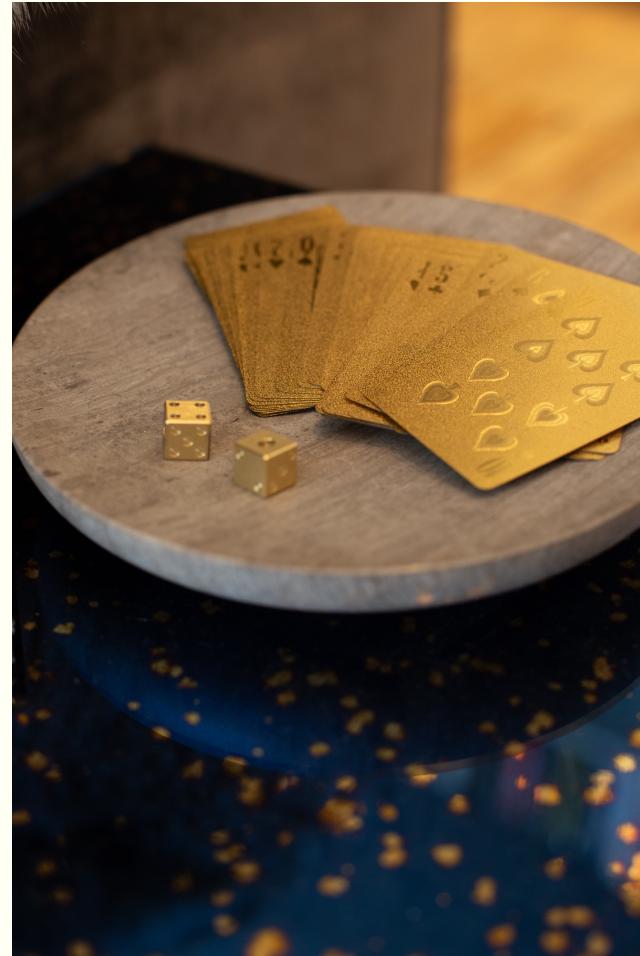
- The function $m_3(x)$ is the distribution function of the random variable $Z = X + Y$.

Example

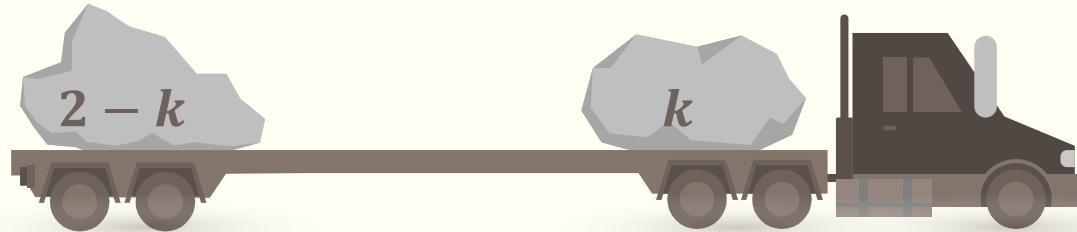
- A die is rolled twice. Let X_1 and X_2 be the outcomes, and let $S_2 = X_1 + X_2$ be the sum of these outcomes.
- Then X_1 and X_2 have the common distribution function:

$$m = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{array} \right).$$

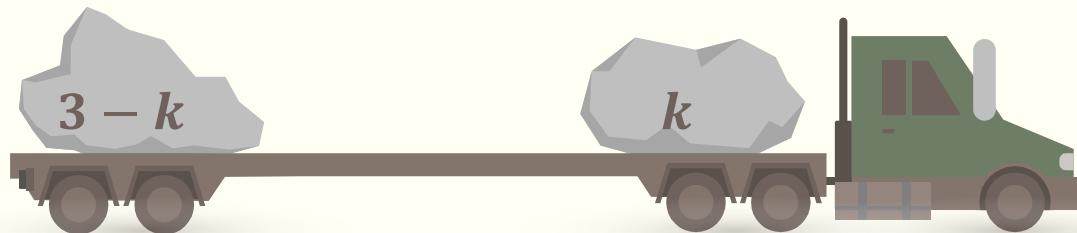
- The distribution function of S_2 is then the convolution of this distribution with itself.



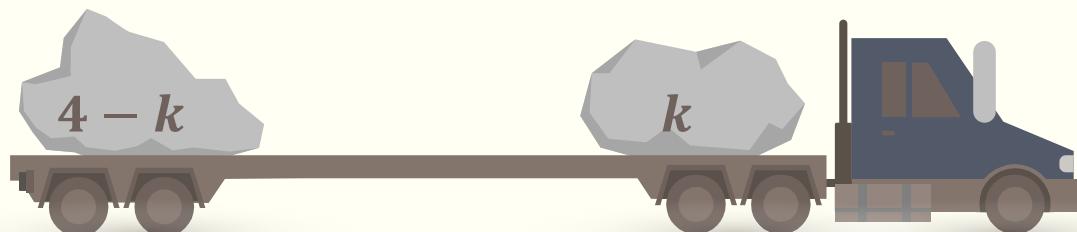
$$S_2 = X_1 + X_2$$



$$P(S_2 = 2) = m(1)m(1)$$



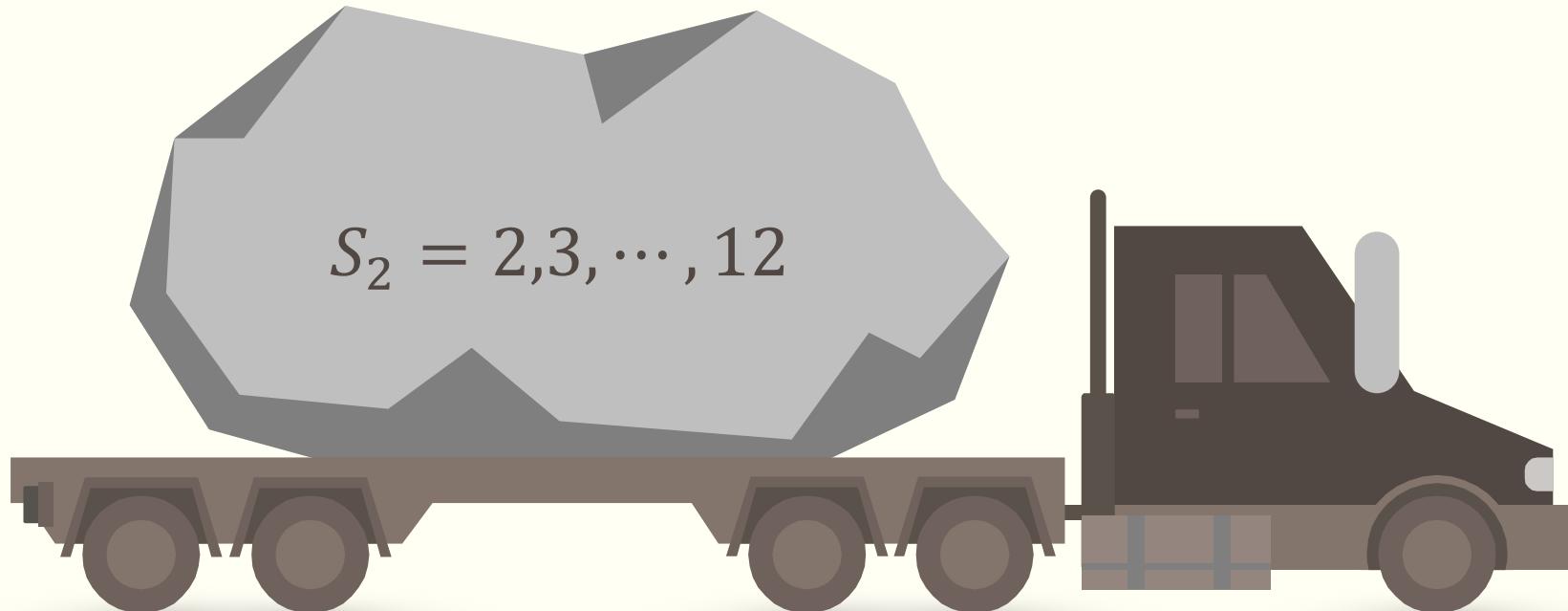
$$P(S_2 = 3) = m(1)m(2) + m(2)m(1)$$



$$P(S_2 = 4) = \dots$$

Example

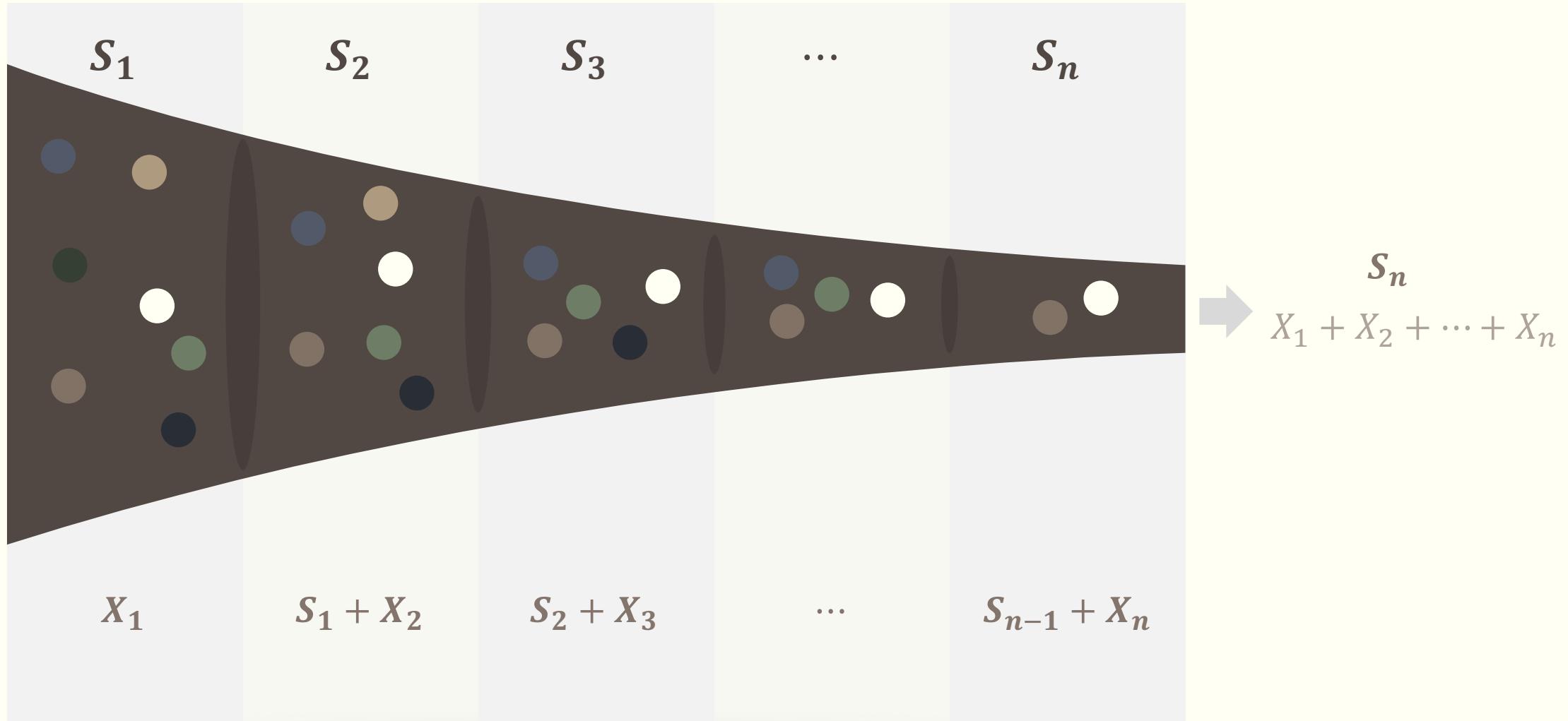
$$S_2 = X_1 + X_2$$



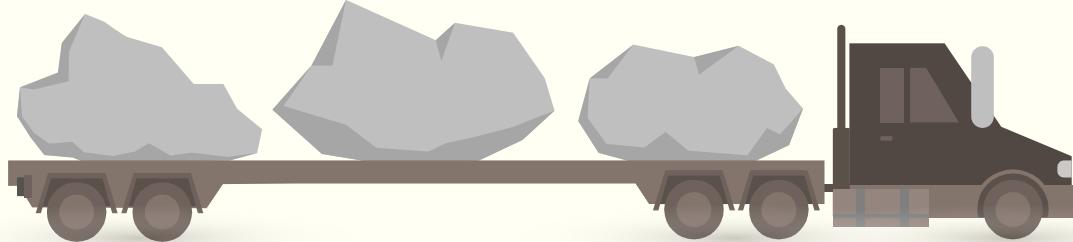
S_2	$P(S_2)$
2 and 12	$\frac{1}{36}$
3 and 11	$\frac{2}{36}$
4 and 10	$\frac{3}{36}$
5 and 9	$\frac{4}{36}$
6 and 8	$\frac{5}{36}$
7	$\frac{6}{36}$

Sum of n discrete random variables

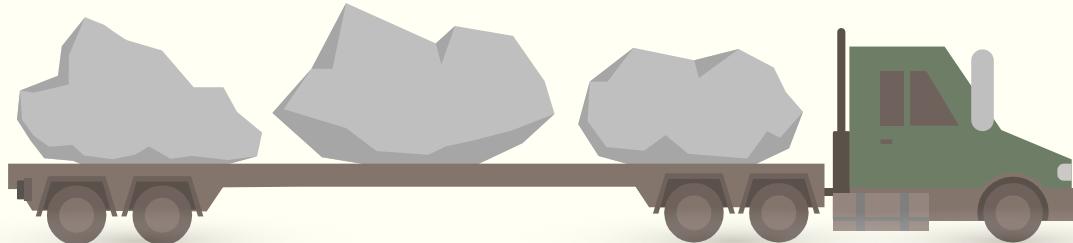
independent random variables



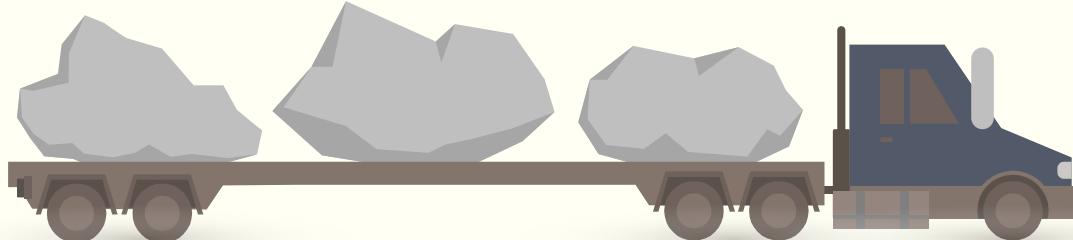
$$S_3 = X_1 + X_2 + X_3 = S_2 + X_3$$



$$P(S_3 = 3) = P(S_2 = 2)P(X_3 = 1)$$



$$P(S_3 = 4) = P(S_2 = 2)P(X_3 = 2) + P(S_2 = 3)P(X_3 = 1)$$

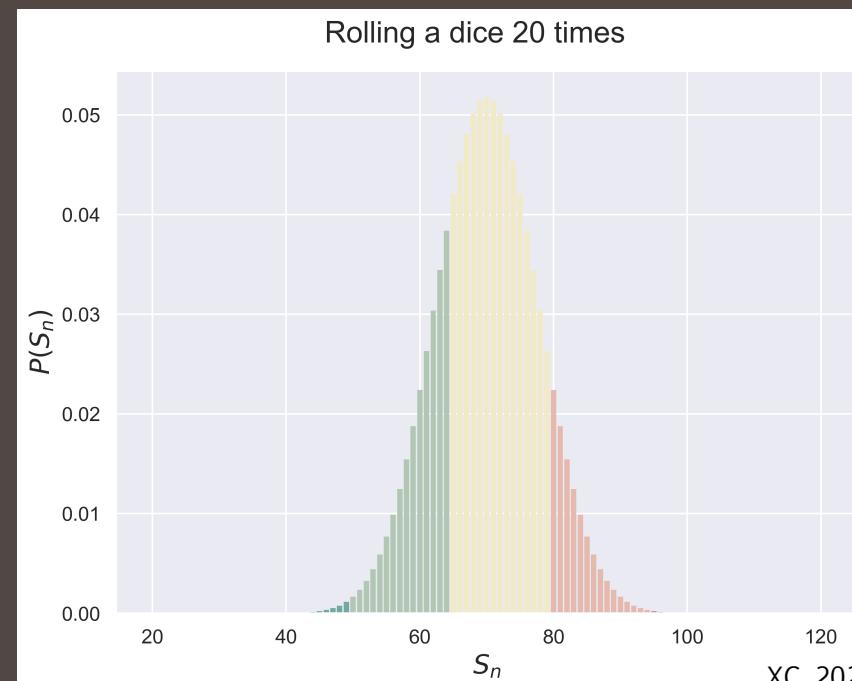
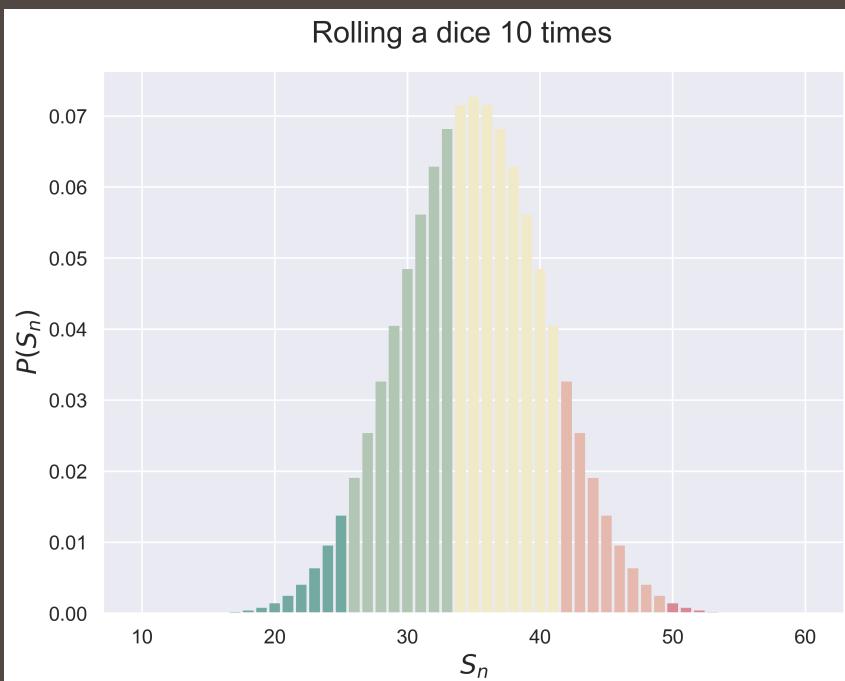
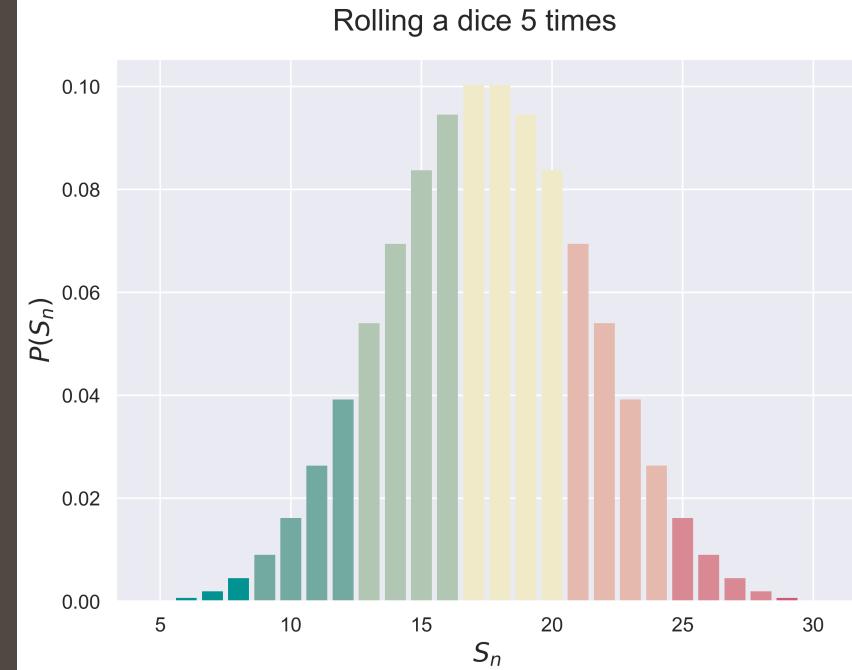
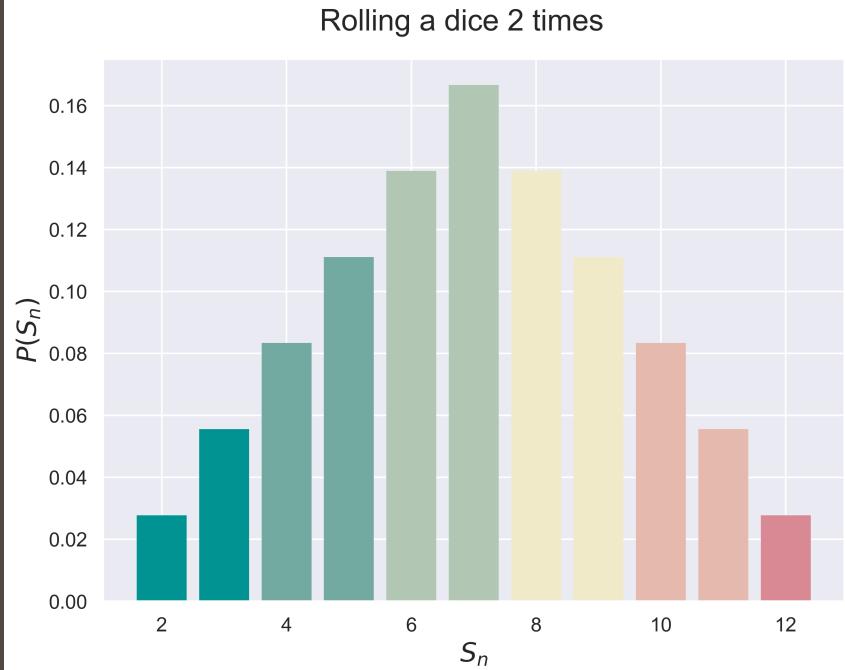


$$P(S_3 = 5) = \dots$$

Bell-shaped curve

$$n \rightarrow \infty$$
$$P(S_n) \rightarrow \dots$$

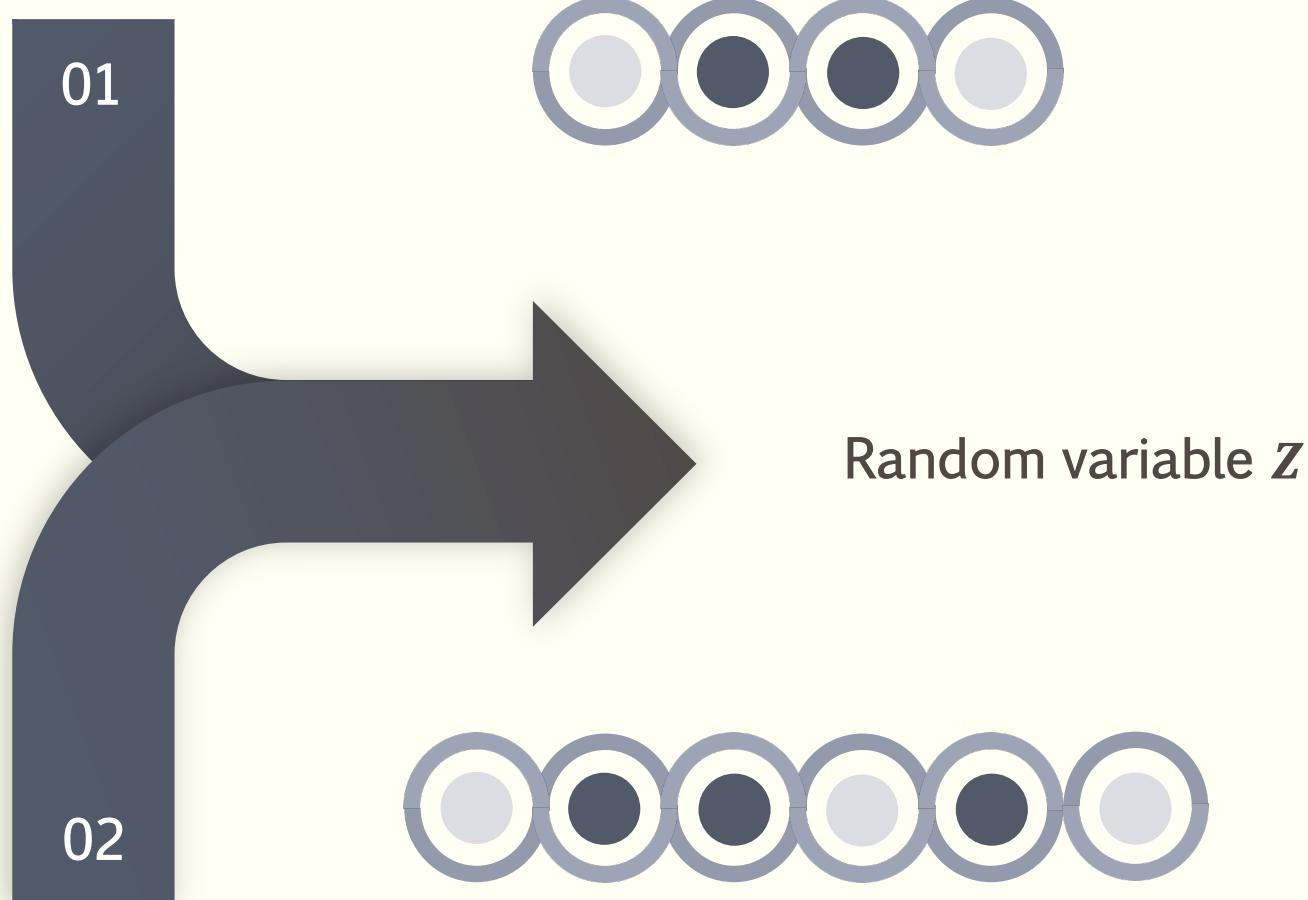
Central Limit Theorem



The convolution of two binomial distributions

Random variable X
binomial distribution
parameters:
 m and p

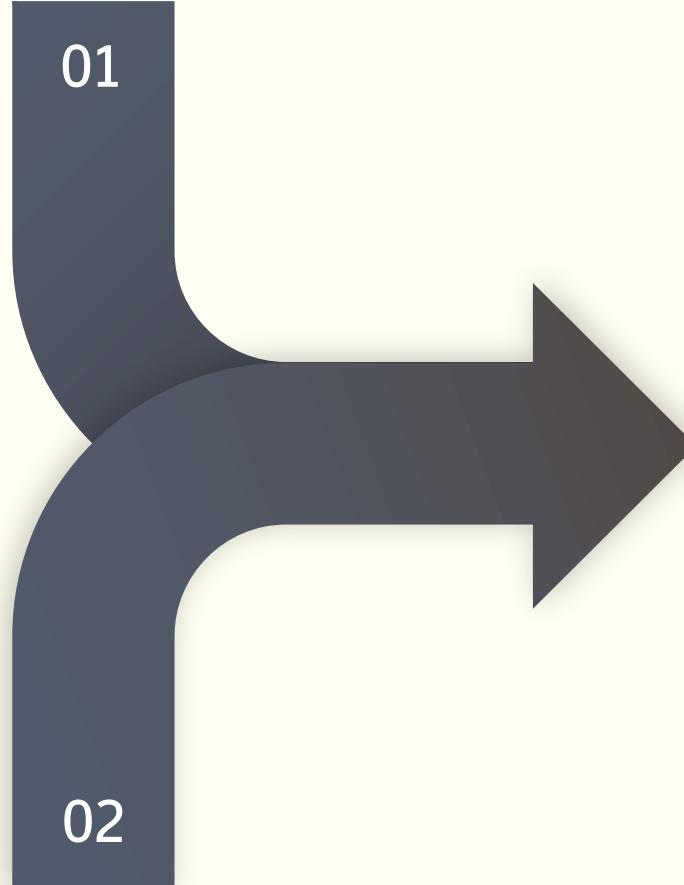
Random variable Y
binomial distribution
parameters:
 n and p



The convolution of two binomial distributions

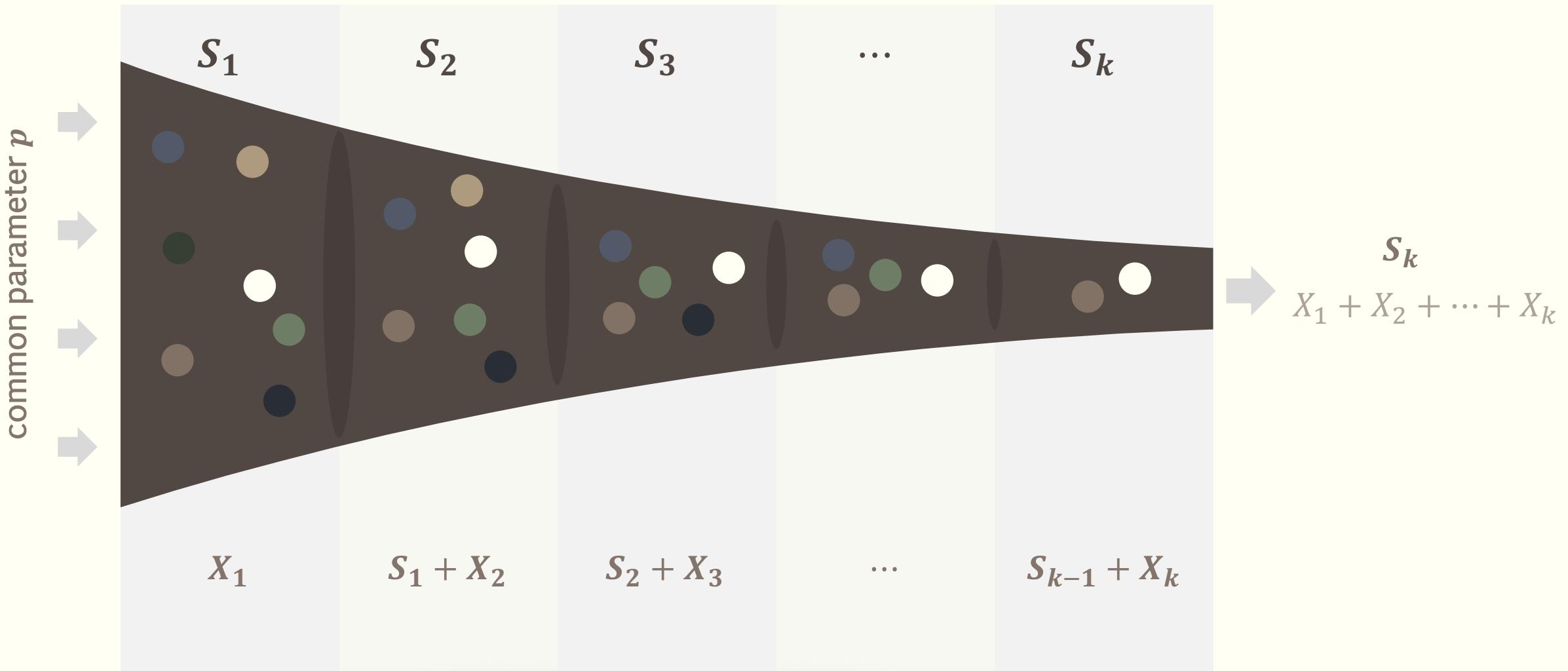
Random variable X
binomial distribution
parameters:
 m and p

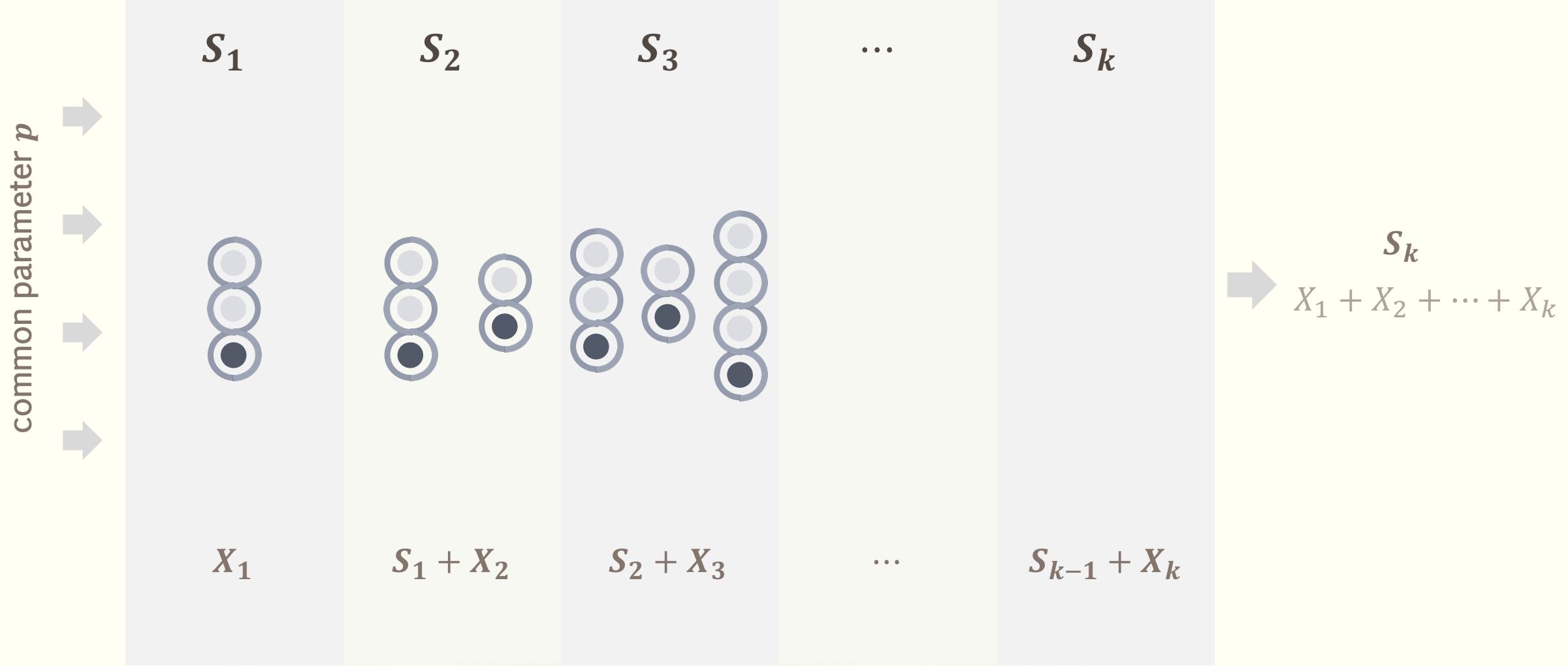
Random variable Y
binomial distribution
parameters:
 n and p



Random variable Z
binomial distribution
parameters:
 $m + n$ and p

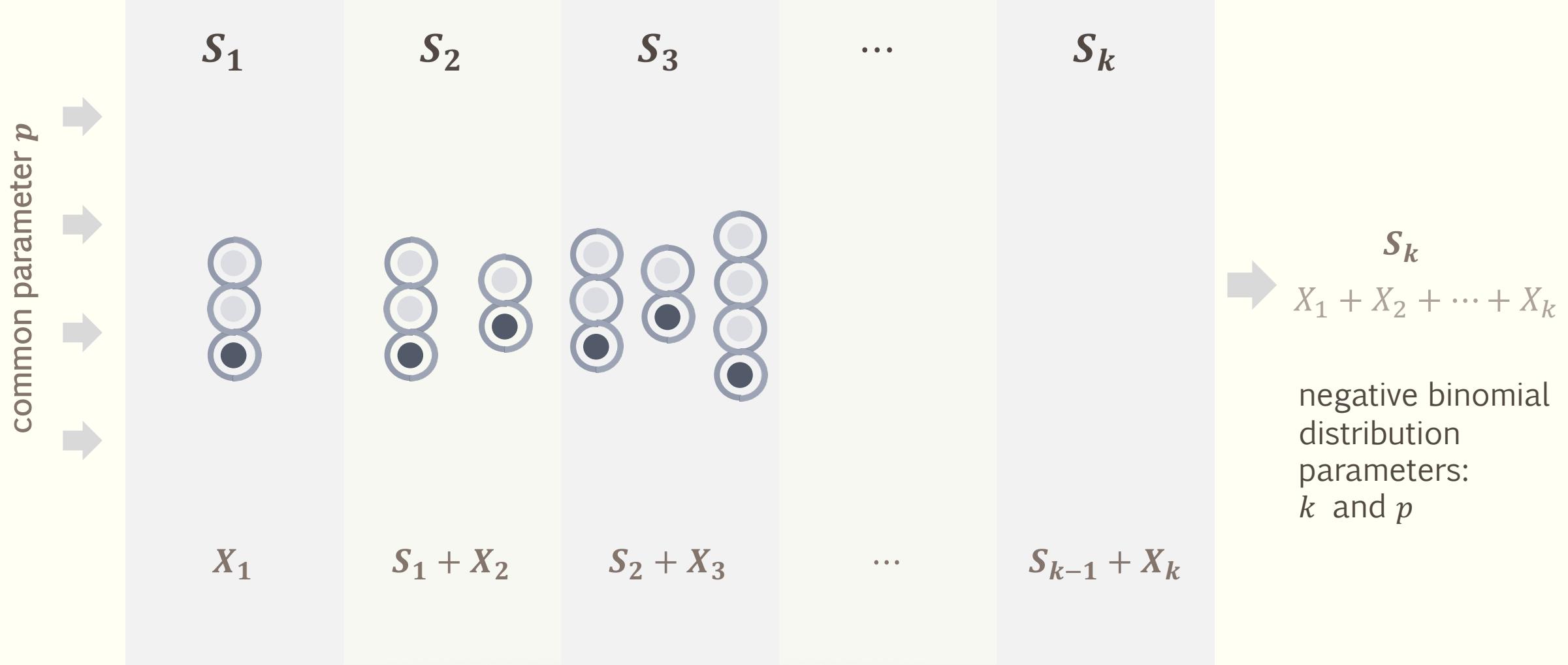
The convolution of k geometric distributions





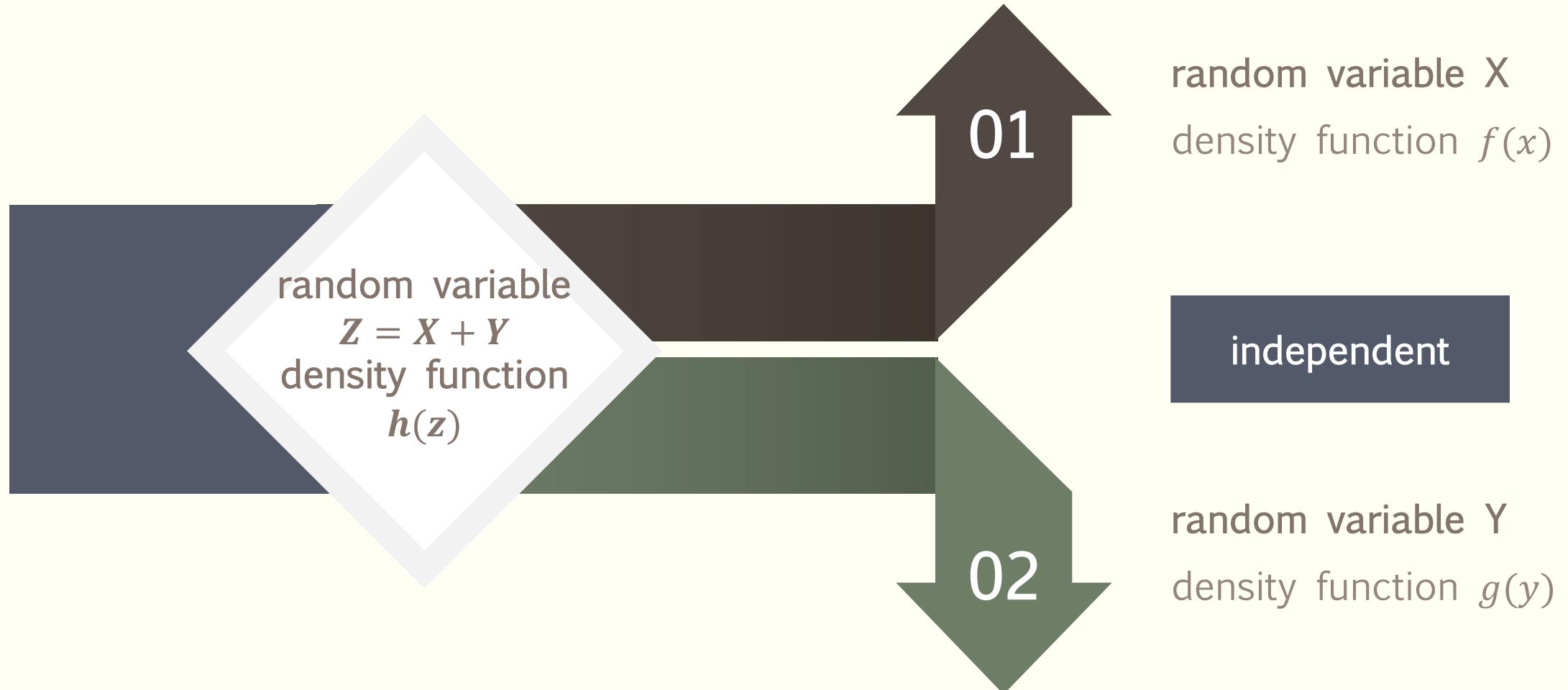
X_i : the number of trials up to and including **the first success**

S_K : ...



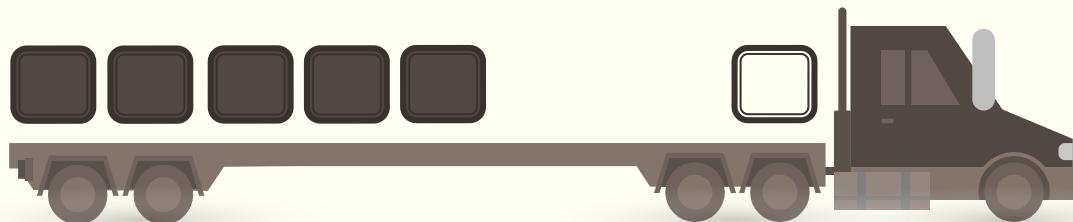
X_i : the number of trials up to and including **the first success**
 S_K : the number of trials up to and include the k th successes

Sum of continuous random variables



$$Z = X + Y$$

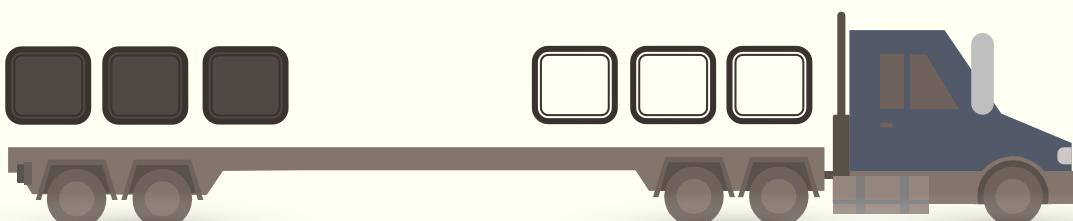
the probability that Z takes on the value z



$$z = 1 + (z - 1)$$



$$z = 2 + (z - 2)$$



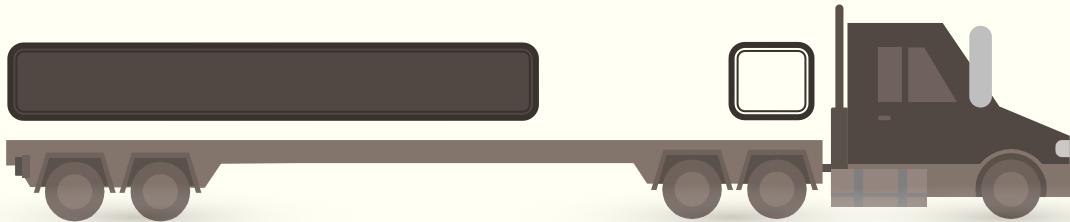
$$z = 3 + (z - 3)$$

discrete

A vertical double-headed arrow pointing both up and down, positioned to the right of the trucks. The word "discrete" is written vertically next to it, oriented from bottom to top.

$$Z = X + Y$$

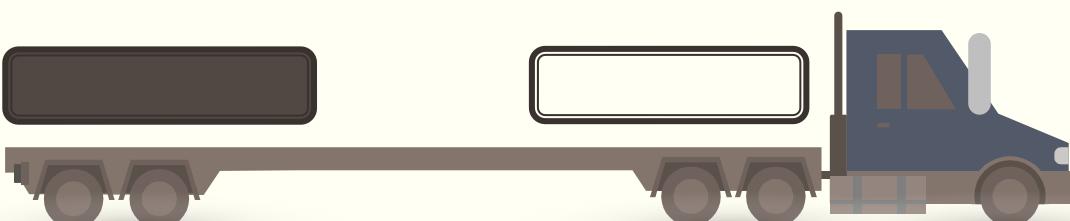
the probability that Z takes on the value z



$$z = a + (z - a)$$



$$z = b + (z - b)$$



$$z = c + (z - c)$$

continuous

A vertical double-headed arrow pointing upwards and downwards, indicating a continuous range or interval.

Convolution

- Let X and Y be two continuous random variables with density functions $f(x)$ and $g(y)$, respectively.
- Assume that both $f(x)$ and $g(y)$ are defined for all real numbers.
- Then the **convolution** $f * g$ of f and g is the function given by

$$(f * g)(z) = \int_{-\infty}^{+\infty} f(z - y)g(y)dy.$$



Sum of continuous random variables

- Let X and Y be two **independent** random variables with density functions $f_X(x)$ and $f_Y(y)$ defined for all x .
- Then the sum $Z = X + Y$ is a random variable with density function $f_Z(z)$, where f_Z is the convolution of f_X and f_Y .

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{+\infty} f_X(z - y)f_Y(y)dy.$$



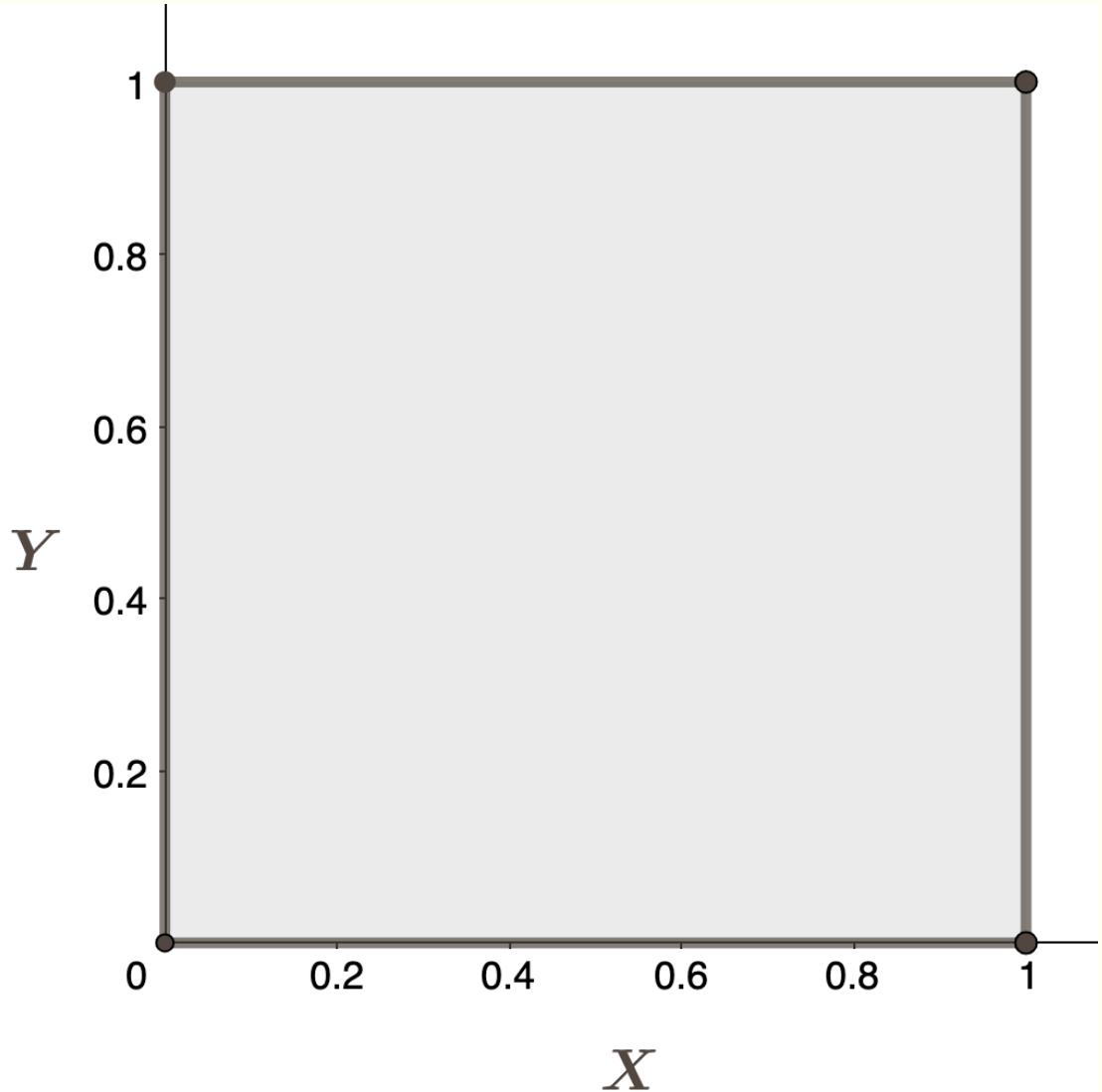
Example 1: uniform

- Suppose we choose independently two numbers at random from the interval $[0, 1]$ with uniform probability density.
- What is the density of their sum?

Uniform distribution

$$f_X(x) = f_Y(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z - y) f_Y(y) dy$$



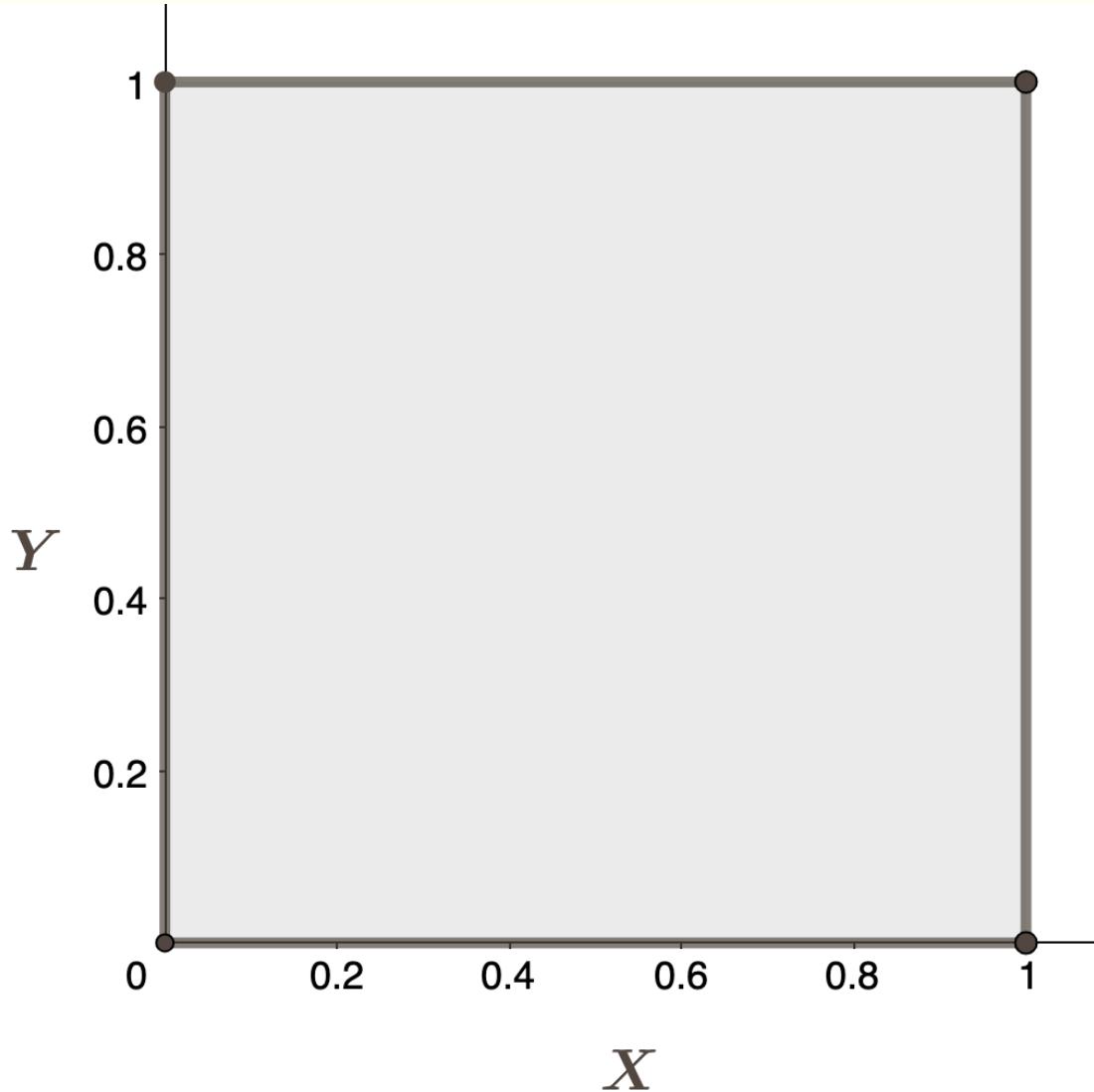
Example 1: uniform

Uniform distribution

$$f_X(x) = f_Y(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z - y) f_Y(y) dy$$

$$f_Z(z) = \int_0^1 f_X(z - y) dy$$



Example 1: uniform

Uniform distribution

$$f_X(x) = f_Y(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy$$

$$f_Z(z) = \int_0^1 f_X(z-y)dy$$

$$0 \leq z - y \leq 1, z - 1 \leq y \leq z$$

$$0 \leq z \leq 1$$

$$f_Z(z) = \int_0^z dy = z$$

$$1 \leq z \leq 2$$

$$f_Z(z) = \int_{z-1}^1 dy = 2 - z$$

Example 1: uniform

Uniform distribution

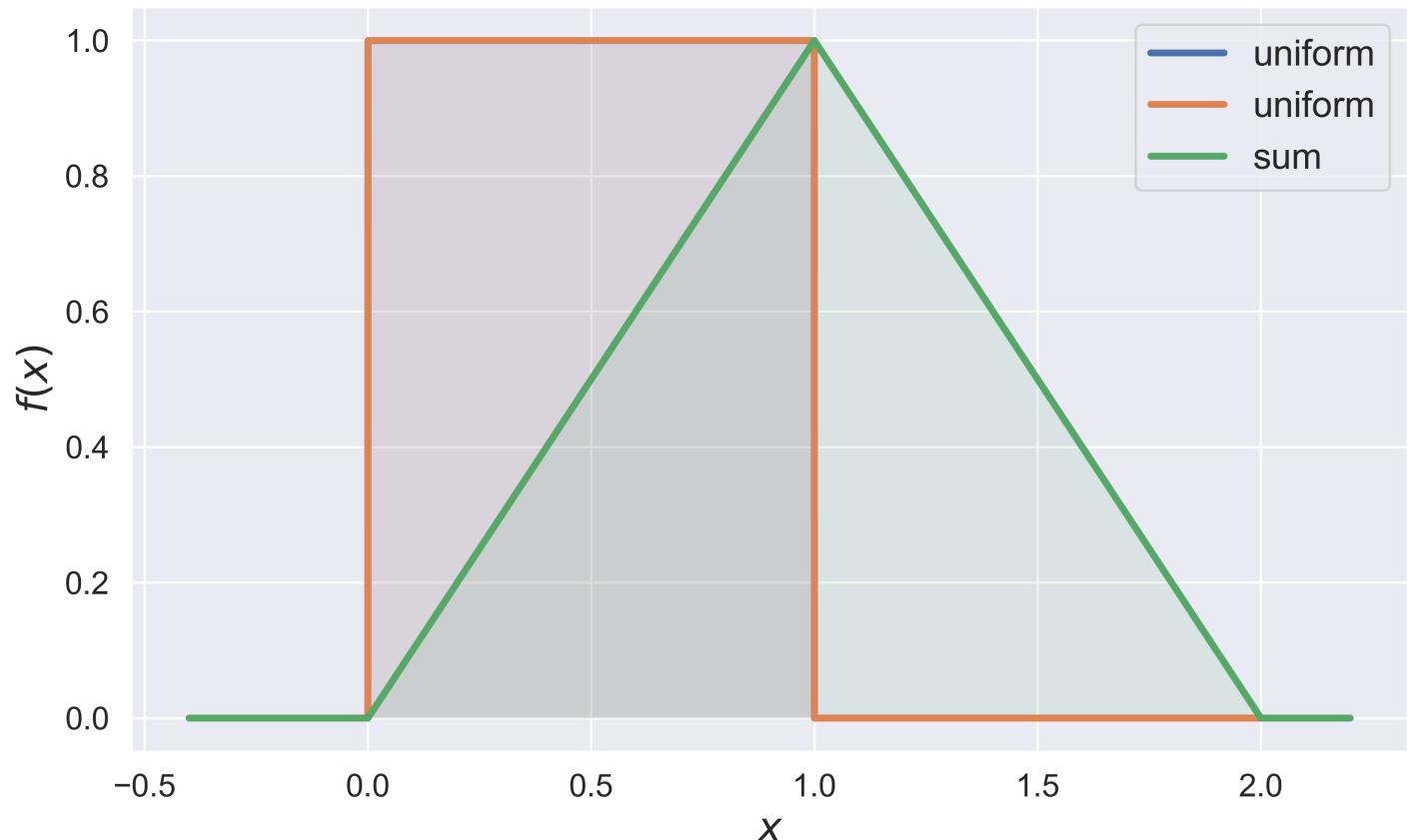
$$f_X(x) = f_Y(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z - y) f_Y(y) dy$$

$$f_Z(z) = \int_0^1 f_X(z - y) dy$$

$$f_Z(z) = \begin{cases} z, & 0 \leq z \leq 1 \\ 2 - z, & 1 \leq z \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Sum of continuous random variables



Example 2: exponential

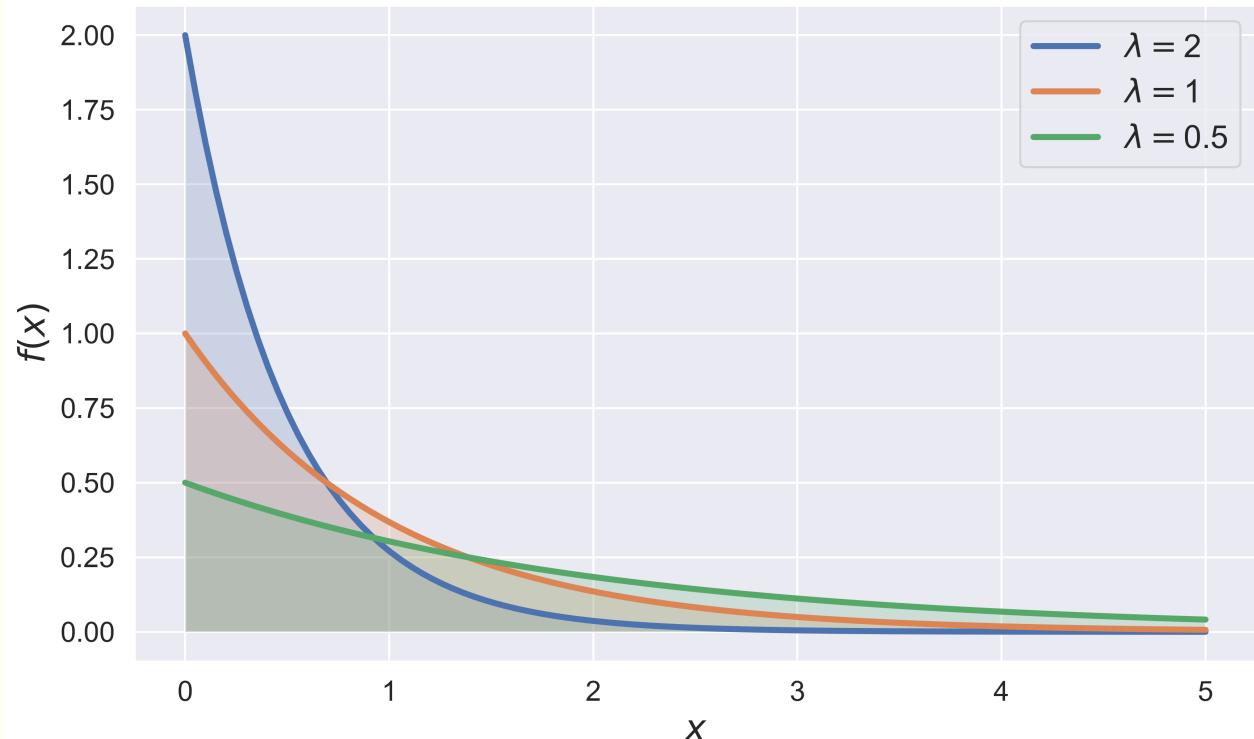
- Suppose we choose two numbers at random from the interval $[0, \infty)$ with an exponential density with parameter λ .
- What is the density of their sum?

Exponential distribution

$$f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0. & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z - y) f_Y(y) dy$$

Exponential distribution



Example 2: exponential

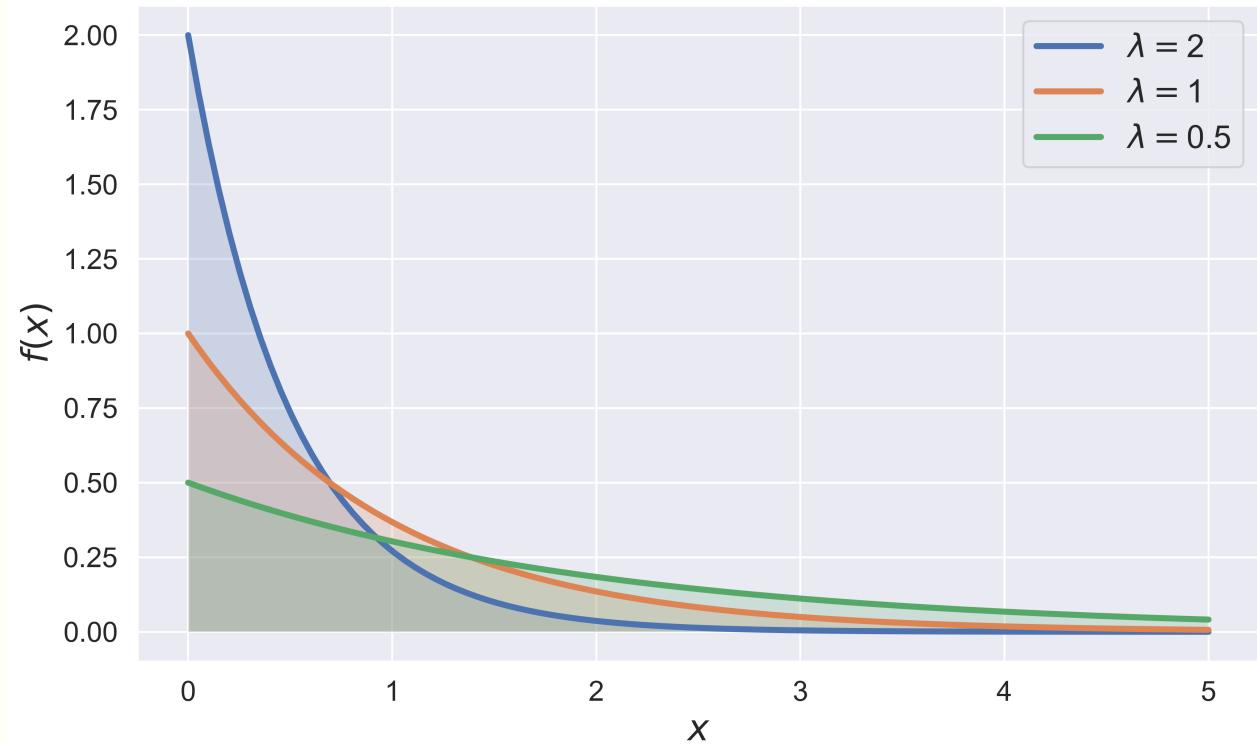
Exponential distribution

$$f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy$$

$$f_Z(z) = \int_0^z f_X(z-y)f_Y(y)dy$$

Exponential distribution



Example 2: exponential

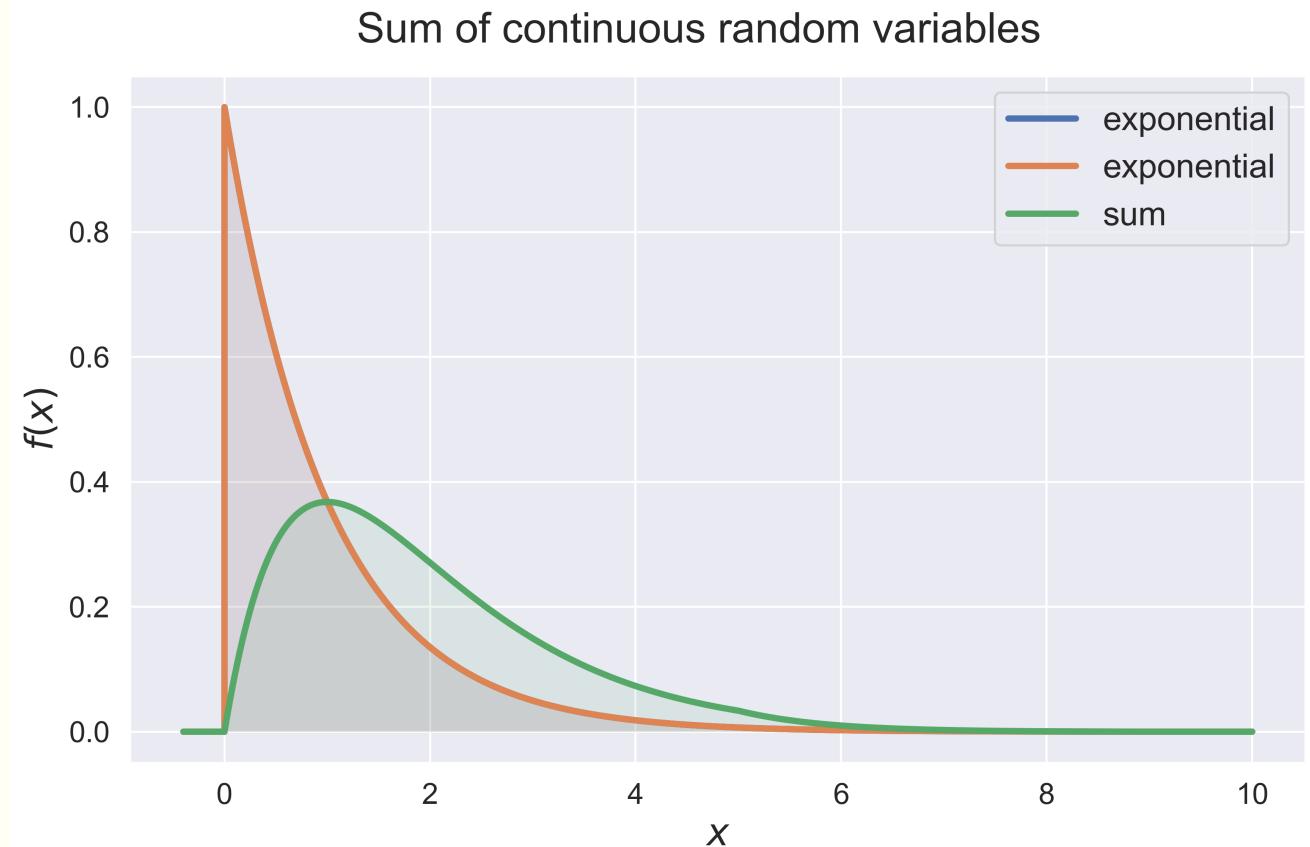
Exponential distribution

$$f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy$$

$$\begin{aligned} f_Z(z) &= \int_0^z \lambda e^{-\lambda(z-y)} \lambda e^{-\lambda y} dy \\ &= \int_0^z \lambda^2 e^{-\lambda z} dy \\ &= \lambda^2 z e^{-\lambda z} \end{aligned}$$

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & z \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

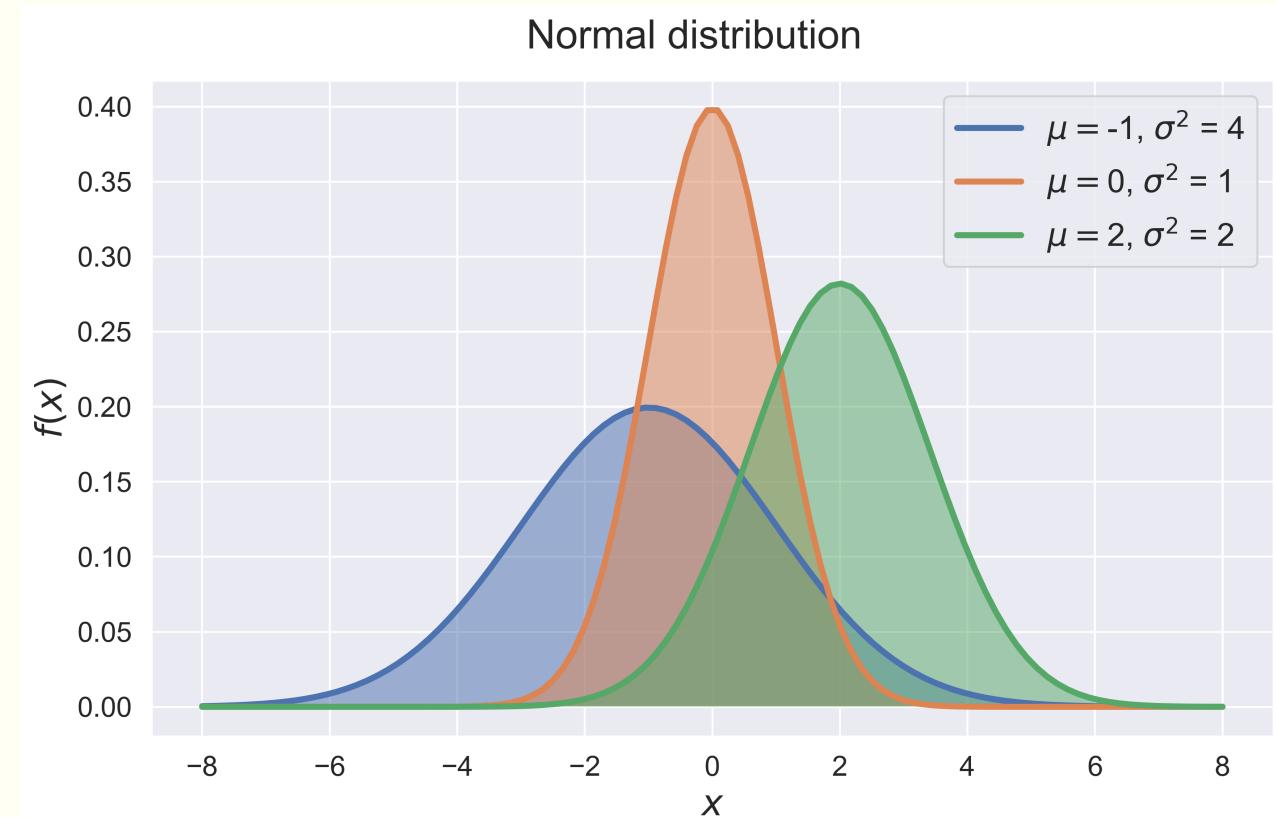


Example 3: normal

- Suppose X and Y are two independent random variables, each with the standard normal density.
- What is the density of their sum?

Normal distribution

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy$$



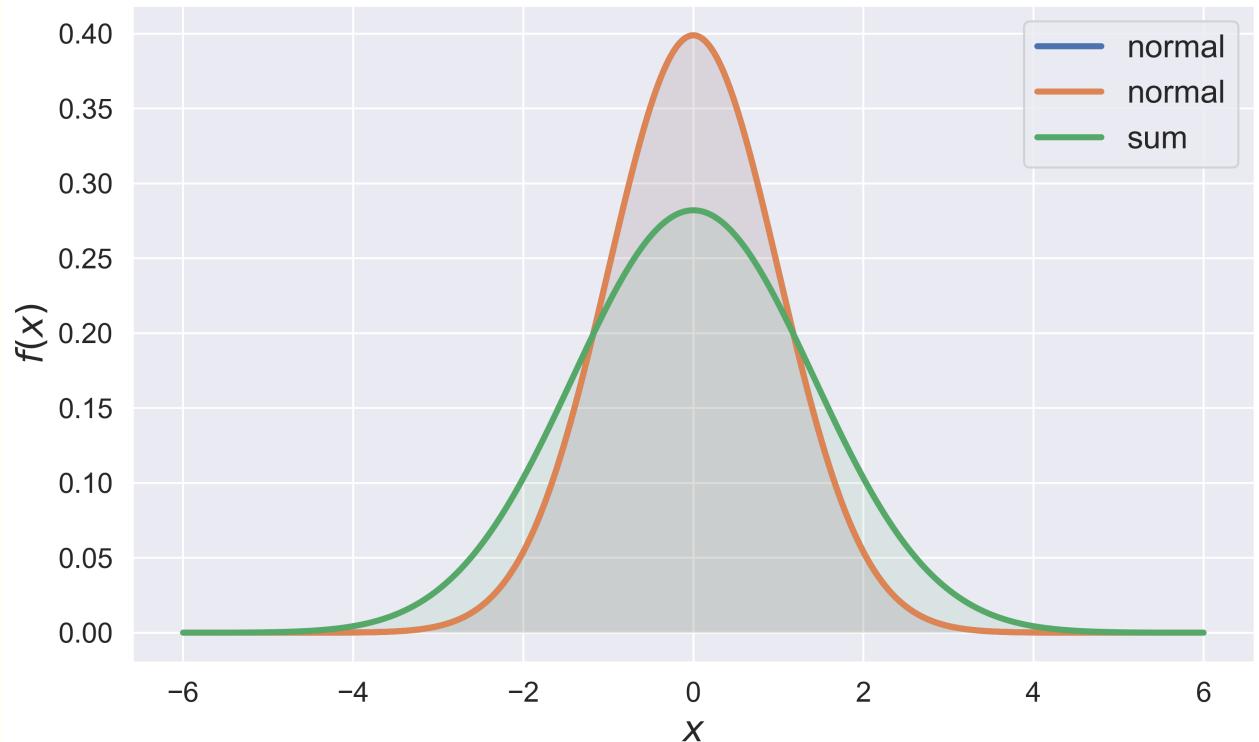
Example 3: normal

Normal distribution

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

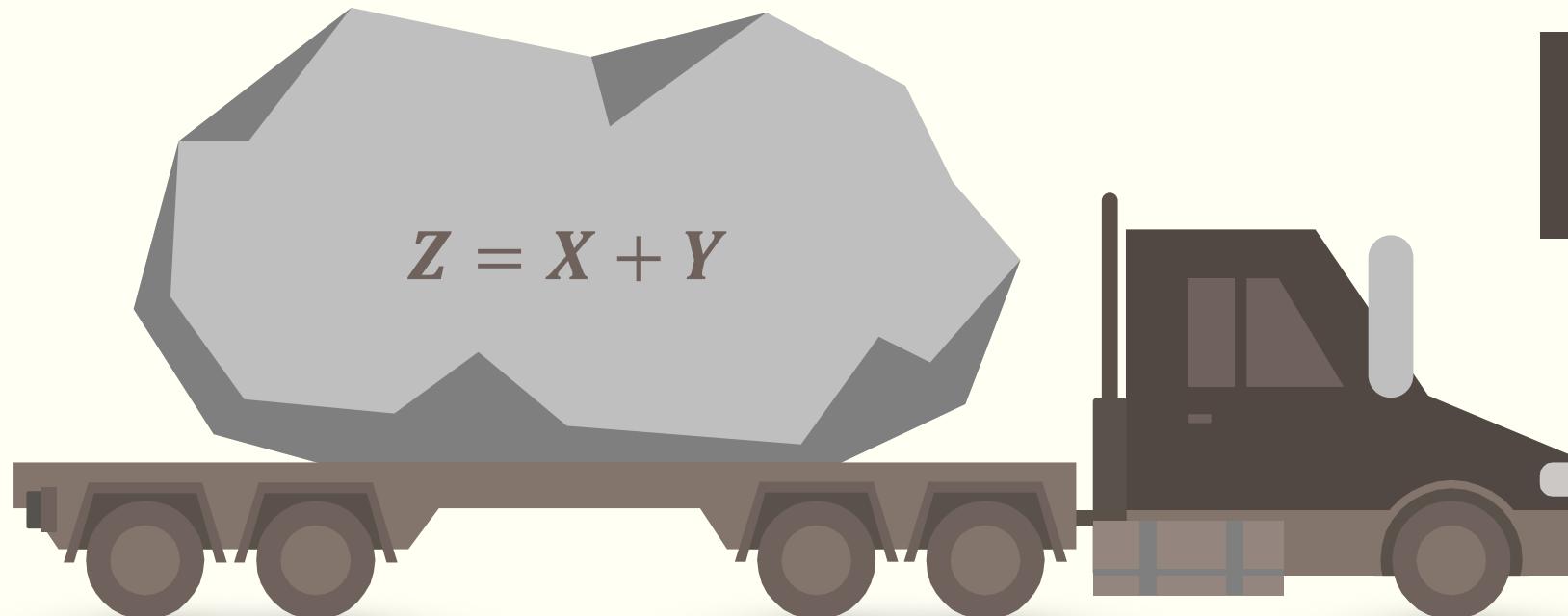
$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(z-y)^2/2} e^{-y^2/2} dy \\ &= \frac{1}{\sqrt{4\pi}} e^{-z^2/4} \end{aligned}$$

Sum of continuous random variables



Sum of Two Independent Normal Random Variables

convolution

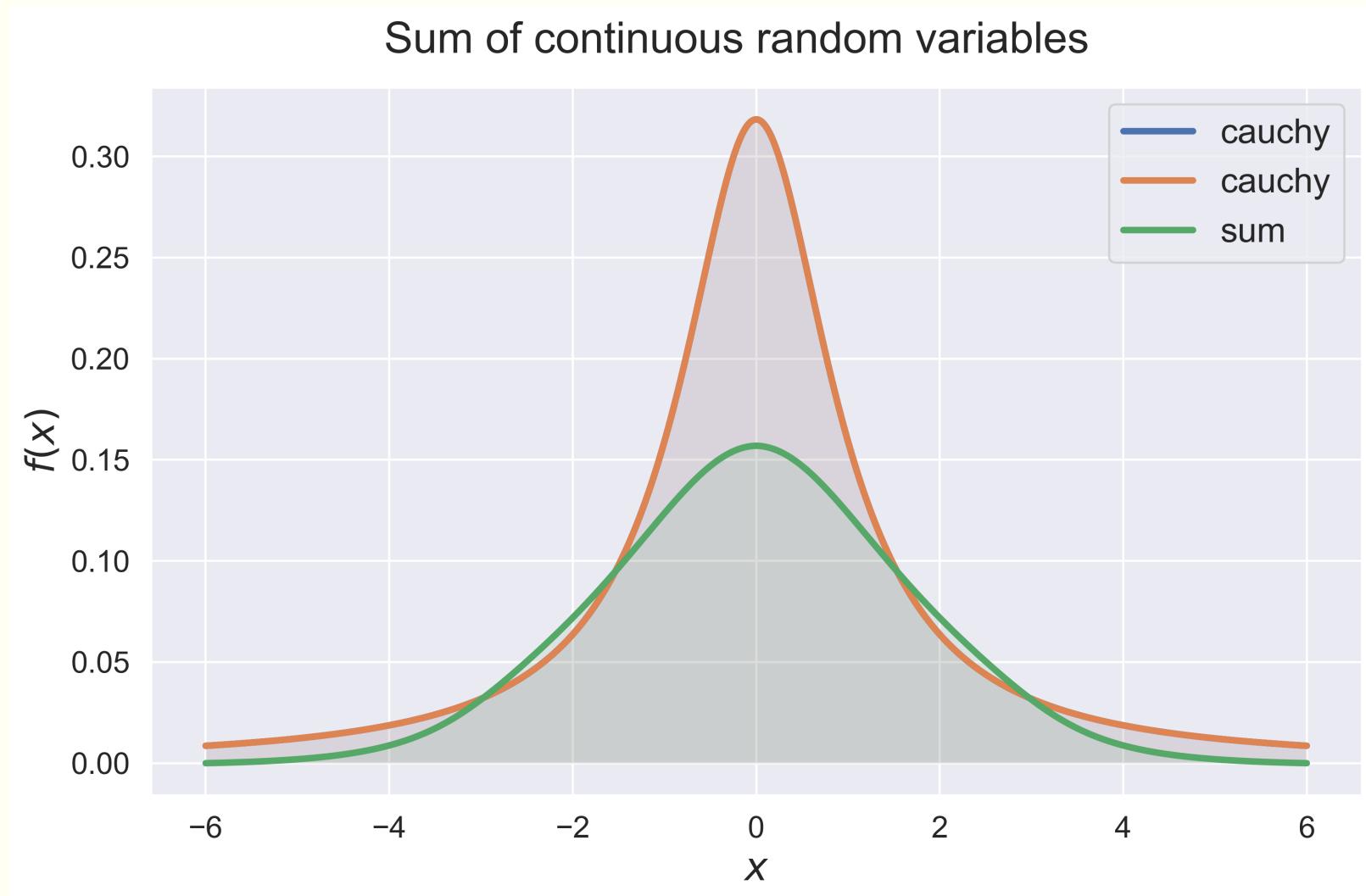


- Means: μ_1 and μ_2
- Variances: σ_1^2 and σ_2^2

Normal + Normal = Normal

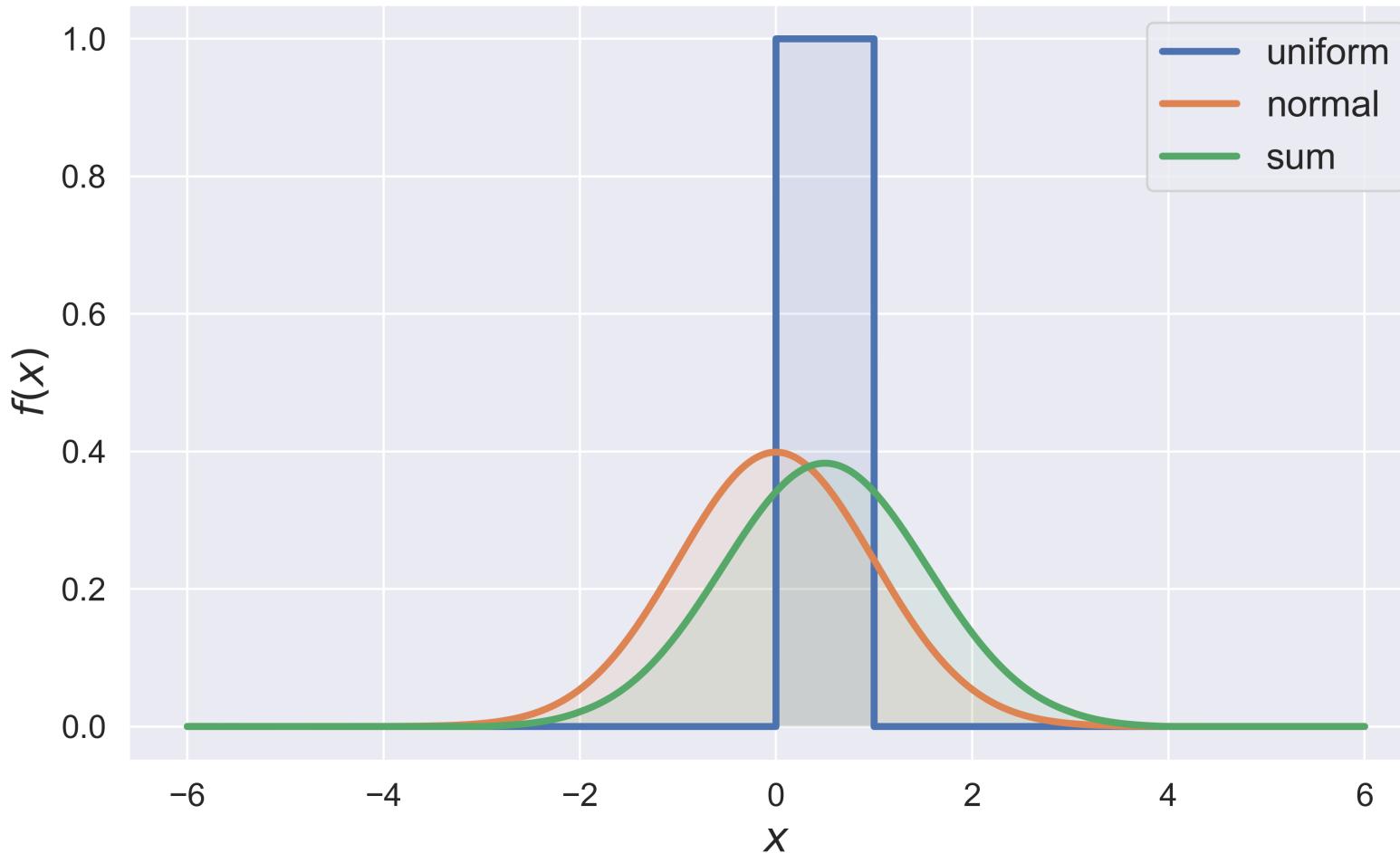
- Means: $\mu_1 + \mu_2$
- Variances: $\sigma_1^2 + \sigma_2^2$

Sum of Two Independent Random Variables (quiz)



Sum of Two Independent Random Variables

Sum of continuous random variables



Slides 0817

- Sum of Independent Random Variables.

```
convolution(dist_1 = 'uniform',
            dist_2 = 'normal',
            interval = [-3, 3])
figure_convolution(dist_1 = 'uniform',
                    dist_2 = 'uniform',
                    interval = [-0.2, 1.1],
                    fsize = (10, 6), fs = 20)
```