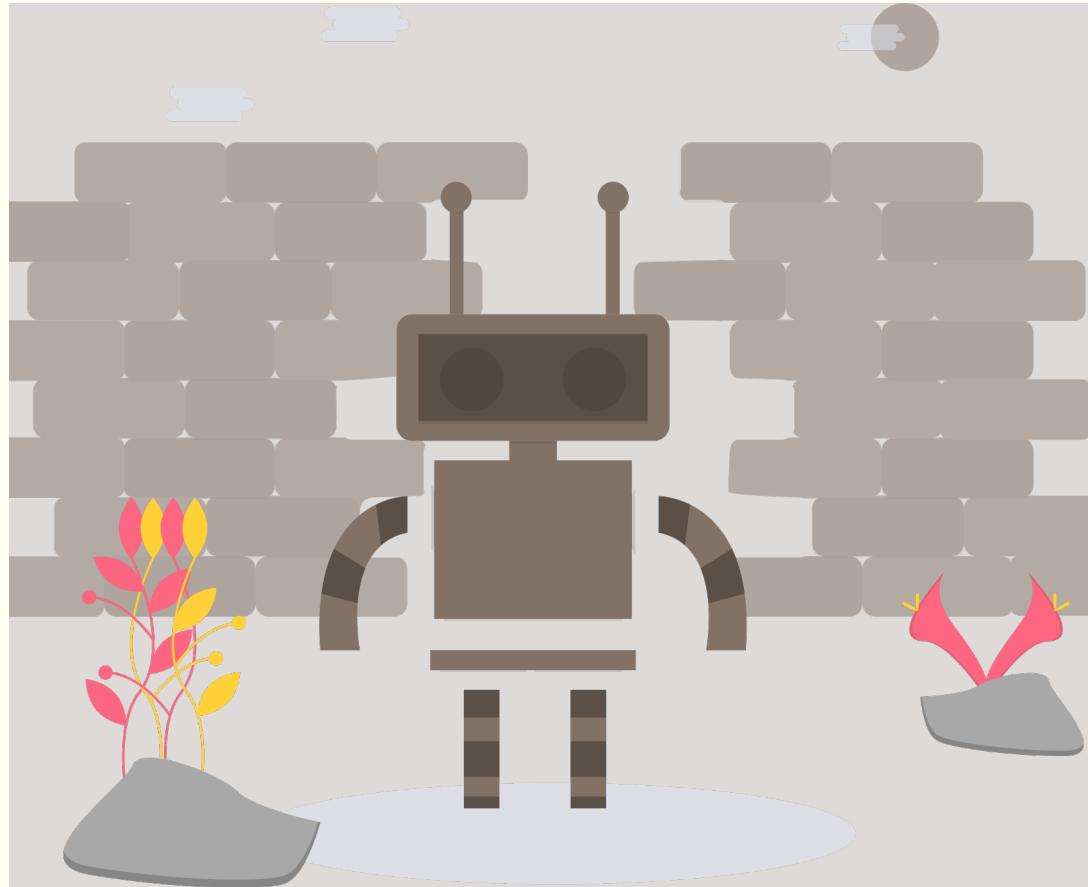




MATH 20: PROBABILITY

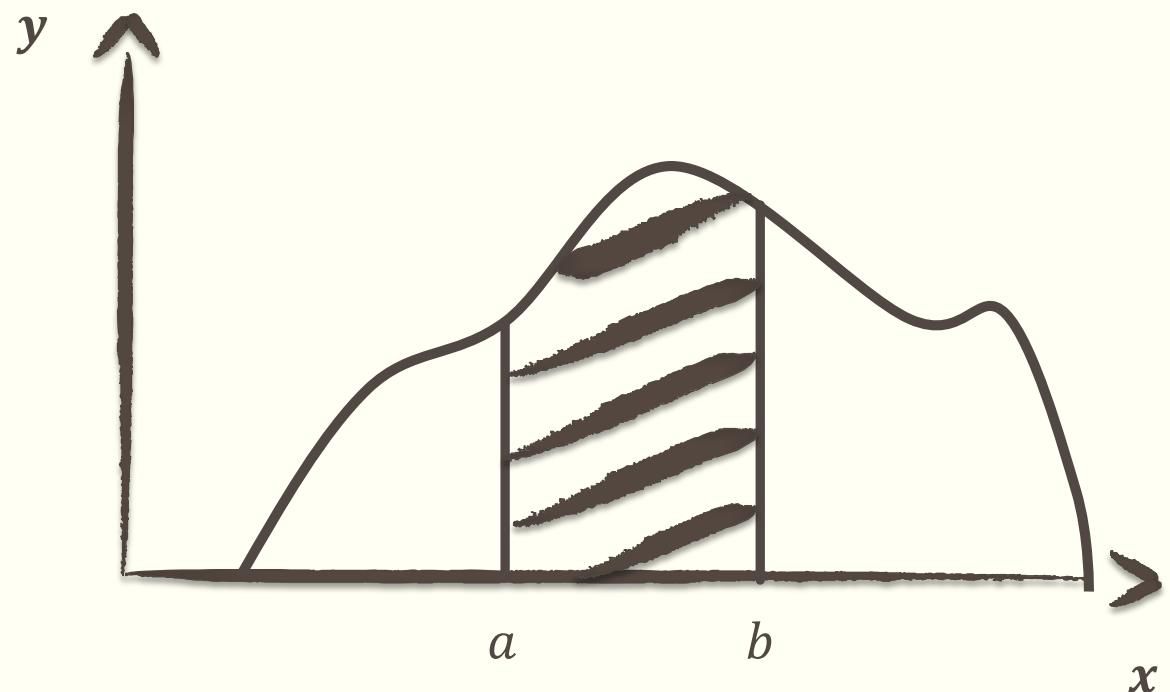
Continuous Conditional Probability



Density Functions of Continuous Random Variable

- Assume X is continuous random variable with density function $f(x)$, and if E is an event with positive probability, we define conditional density function (which is normalized to have integral 1 over E):

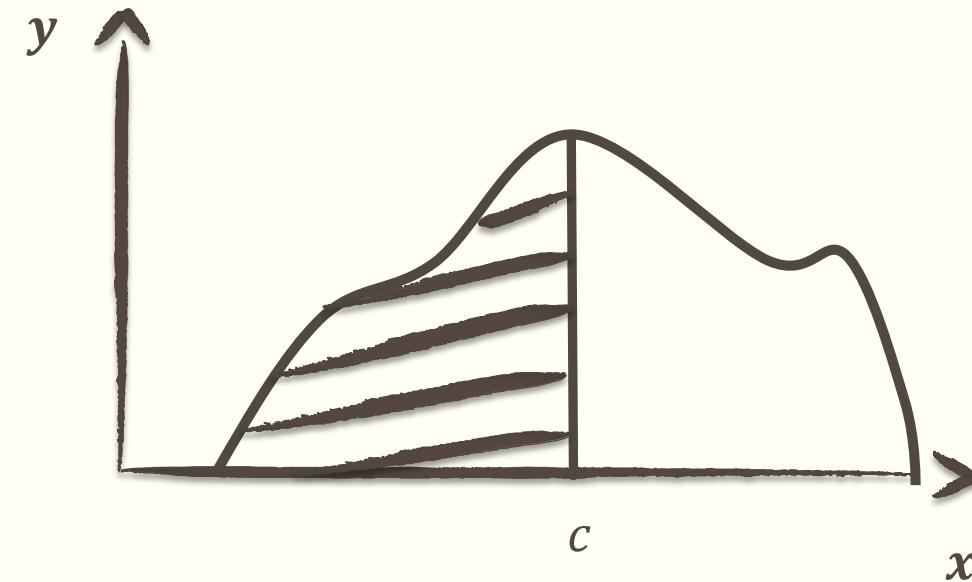
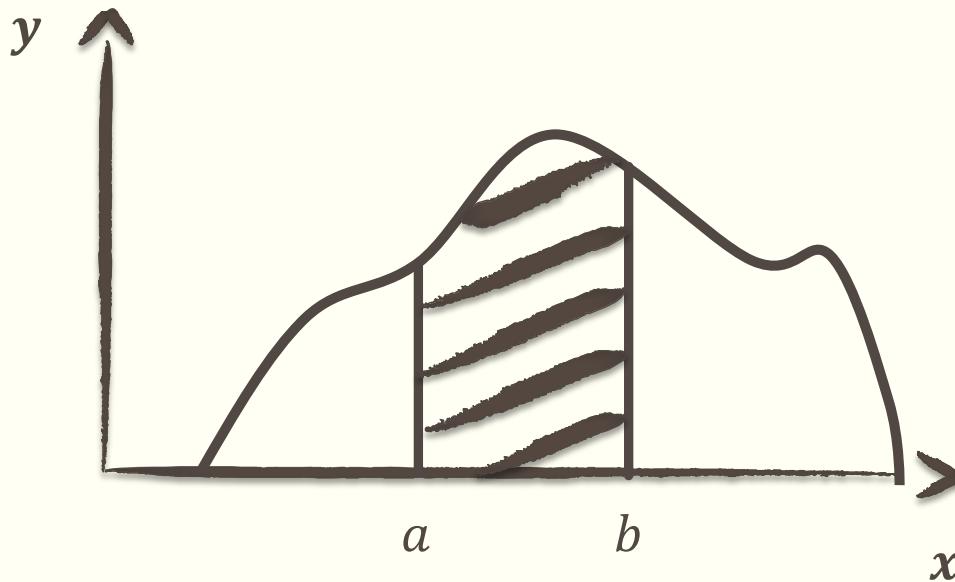
$$f(x|E) = \begin{cases} f(x)/P(E), & x \in E \\ 0, & x \notin E \end{cases}$$



Density Functions of Continuous Random Variable

- For any event F , the conditional probability of F given E is given by

$$P(F|E) = \int_F f(x|E)dx = \int_{E \cap F} \frac{f(x)}{P(E)} dx = \frac{P(E \cap F)}{P(E)}.$$



Density Functions of Continuous Random Variable

- For any event F , the conditional probability of F given E is given by

$$P(F|E) = \int_F f(x|E)dx = \int_{E \cap F} \frac{f(x)}{P(E)} dx = \frac{P(E \cap F)}{P(E)}.$$

- Event E and F are independent

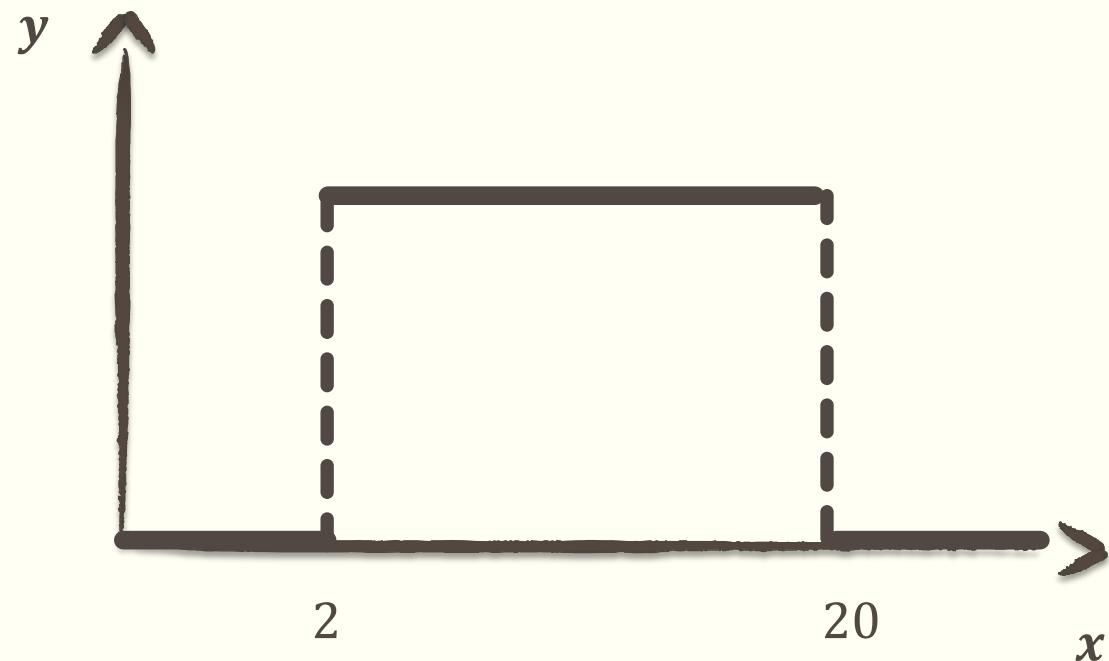
\Leftrightarrow

$$P(E \cap F) = P(E)P(F)$$

\Leftrightarrow

$$P(F|E) = P(F) \text{ and } P(E|F) = P(E).$$

Uniform Distribution



$$f(x) = \begin{cases} \frac{1}{18}, & 2 \leq x \leq 20 \\ 0, & x < 2 \text{ or } x > 20 \end{cases}$$

Event $E: X > 10$

$$P(X > 10) = \frac{5}{9}$$

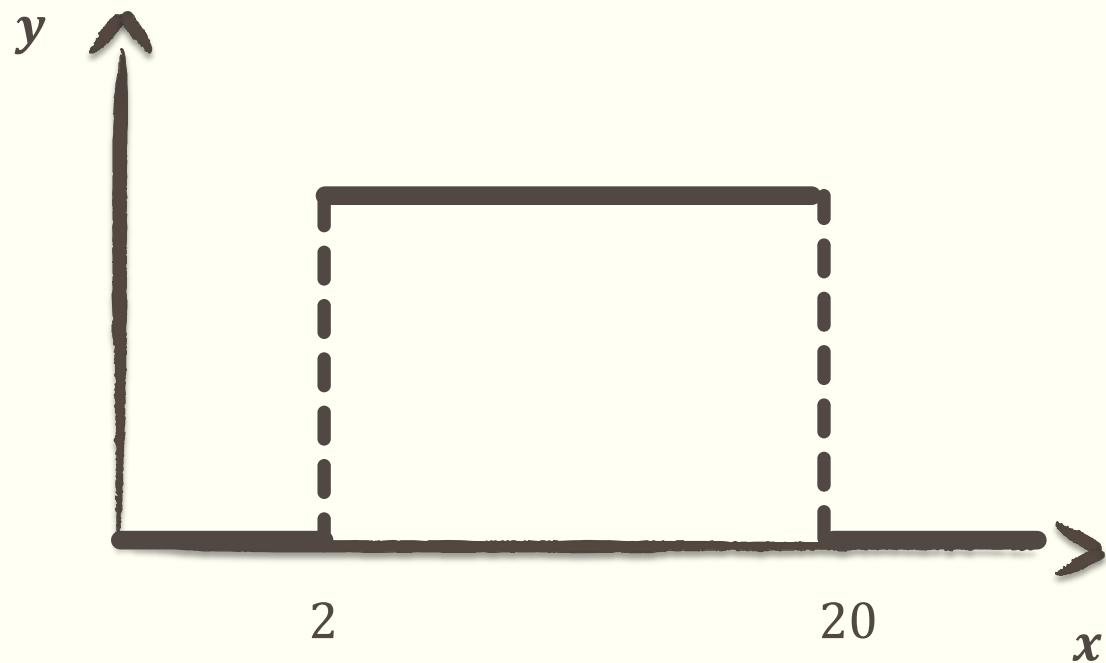
Event $F: X > 12$

$$P(X > 12) = \frac{4}{9}$$

Event $F|E: X > 12|X > 10$

$$P(X > 12|X > 10) = \dots$$

Uniform Distribution



$$f(x) = \begin{cases} \frac{1}{18}, & 2 \leq x \leq 20 \\ 0, & x < 2 \text{ or } x > 20 \end{cases}$$

Event $E: X > 10$

$$P(X > 10) = \frac{5}{9}$$

Event $F: X > 12$

$$P(X > 12) = \frac{4}{9}$$

Event $F|E: X > 12|X > 10$

$$\begin{aligned} P(X > 12|X > 10) &= \frac{P(X > 12 \cap X > 10)}{P(X > 10)} \\ &= \frac{P(X > 12)}{P(X > 10)} = \frac{4}{5} \end{aligned}$$

Exponential Distribution

- The exponential distribution is the probability distribution of the **time** between events in a Poisson point process. That is, a process in which events occur continuously and independently at a constant average rate λ .
- The density function of an exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

- The cumulative distribution function of an exponential distribution is

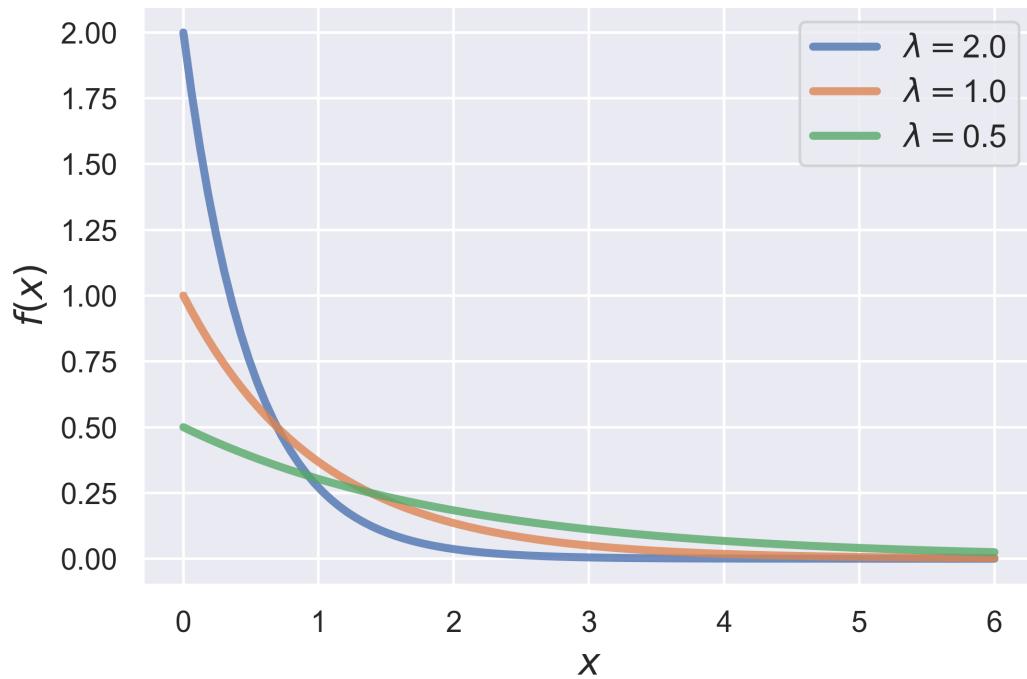
$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Exponential Distribution

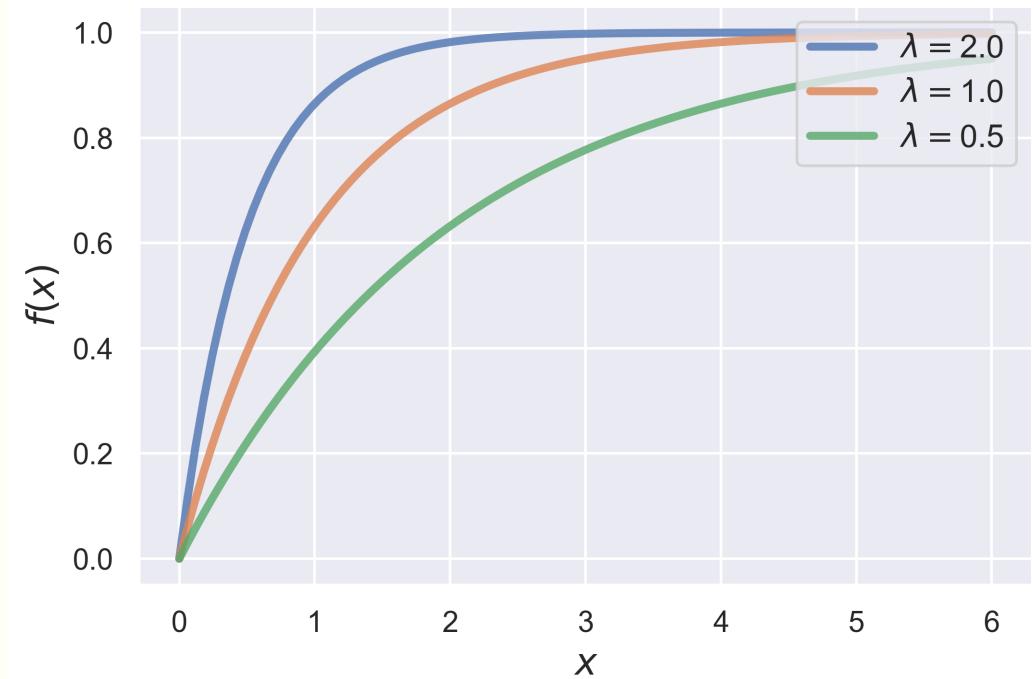
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

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Density function

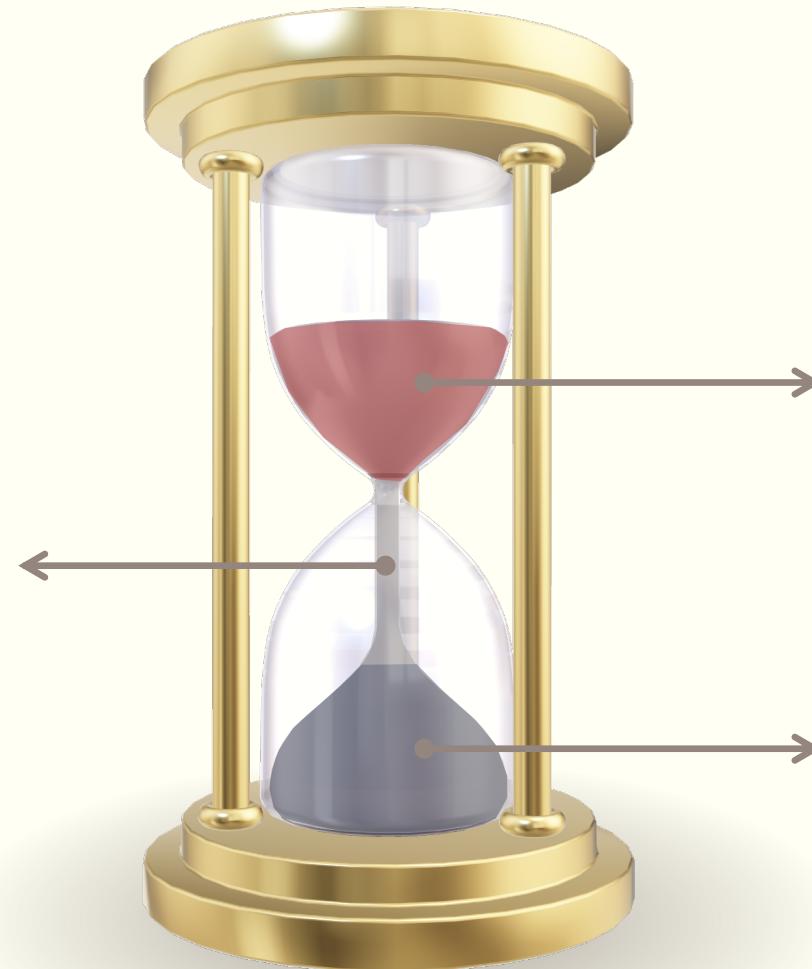


Cumulative distribution function



Exponential Distribution

Density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$


Time

The exponential distribution is often concerned with the amount of time until some specific event occurs.

Cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The Life Span of a Lightbulb

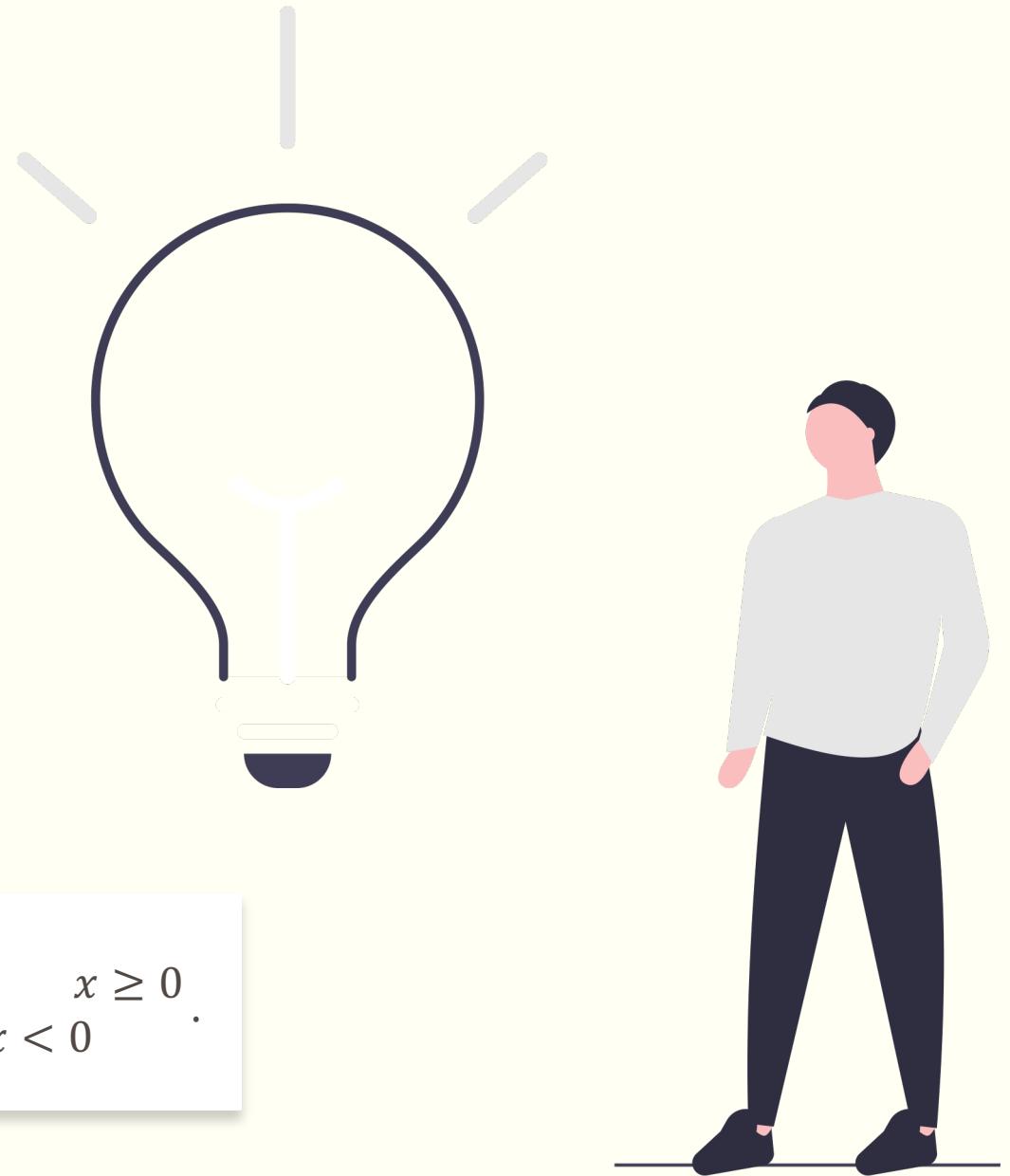
- Assume the life span of a lightbulb is a random variable t with an exponential density function. The average lifetime is 30 months.

average: μ
$\mu = 30$

rate: λ
$\lambda = \frac{1}{\mu}$

$$f(x) = \begin{cases} \frac{1}{30} e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$



The Life Span of a Lightbulb



- If the lightbulb is already lit for 15 months, how long it will continue to last until burning out?

$$f(x) = \begin{cases} \frac{1}{30} e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

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$$F(x) = \begin{cases} 1 - e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Event $E: X > 15$

$$\begin{aligned} P(X > 15) &= 1 - P(X \leq 15) = e^{-15/30} \\ &= e^{-1/2} \end{aligned}$$

Event $F: X > 15 + s$

$$\begin{aligned} P(X > 15 + s) &= 1 - P(X \leq 15 + s) \\ &= e^{-(15+s)/30} = e^{-1/2} e^{-s/30} \end{aligned}$$

The Life Span of a Lightbulb

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Event $F|E: X > 15 + s | X > 15$

$$P(X > 15 + s | X > 15) = \frac{P(X > 15 + s \cap X > 15)}{P(X > 15)} = \frac{P(X > 15 + s)}{P(X > 15)} = \frac{e^{-1/2}e^{-s/30}}{e^{-1/2}} = e^{-s/30}$$

The Life Span of a Lightbulb

$$f(x) = \begin{cases} \frac{1}{30}e^{-x/30}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

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Event $H: X > s$

$$P(X > s) = 1 - P(X \leq s) = e^{-s/30}$$

The Life Span of a Lightbulb

Event $F|E: X > 15 + s | X > 15$

$$P(X > 15 + s | X > 15) = \frac{P(X > 15 + s \cap X > 15)}{P(X > 15)} = \frac{P(X > 15 + s)}{P(X > 15)} = \frac{e^{-1/2} e^{-s/30}}{e^{-1/2}} = e^{-s/30}$$

Event $H: X > s$

$$P(X > s) = 1 - P(X \leq s) = e^{-s/30}$$

$$P(X > 15 + s | X > 15) = P(X > s)$$

=

The Life Span of a Lightbulb

- Assume the life span of a lightbulb is a random variable t with an exponential density function.
- If the lightbulb is already lit for time r , what is the probability that it will not burn out for further time s ?

average: μ

rate: λ

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$P(X > r + s | X > r) = \dots$$



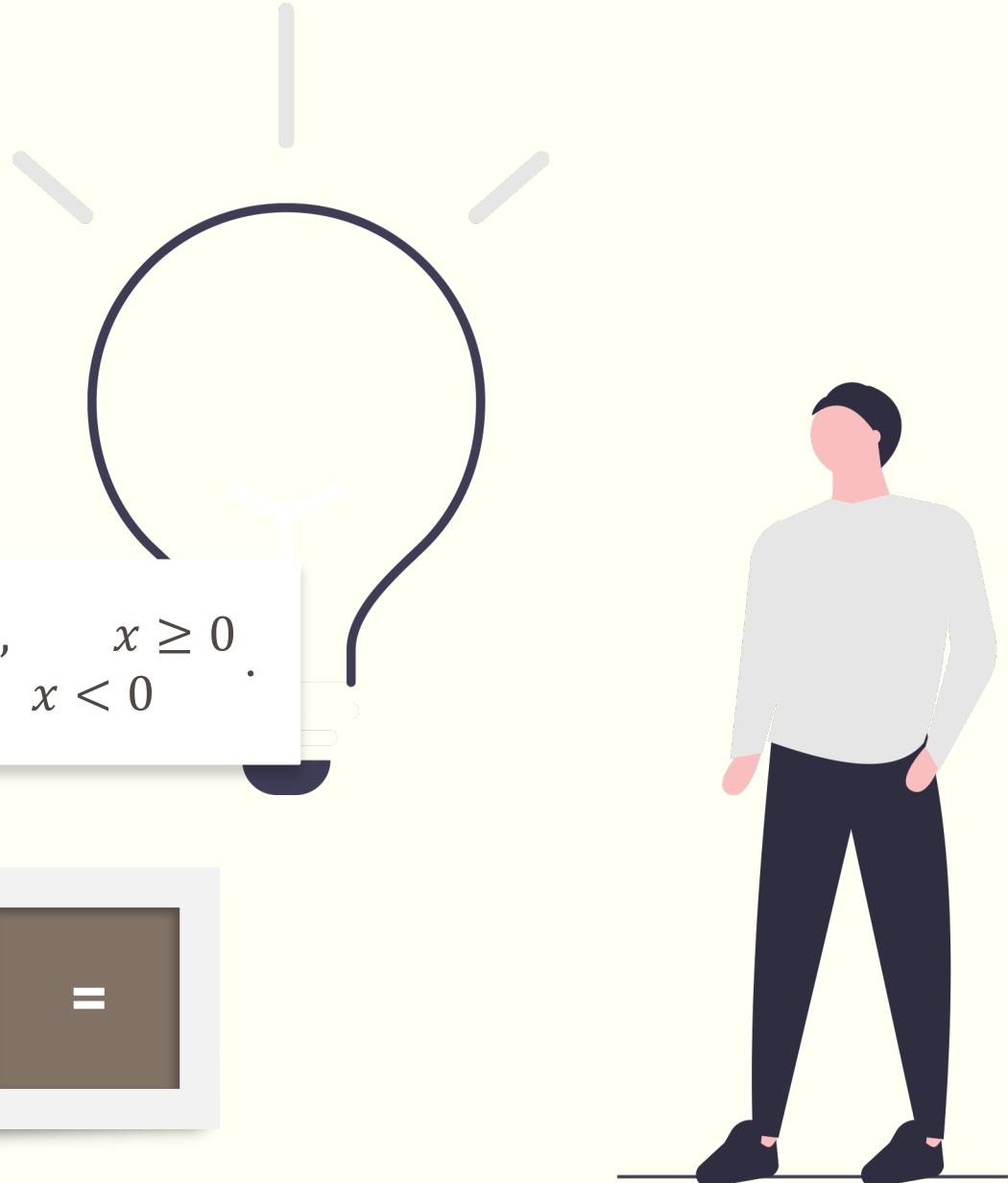
The Life Span of a Lightbulb

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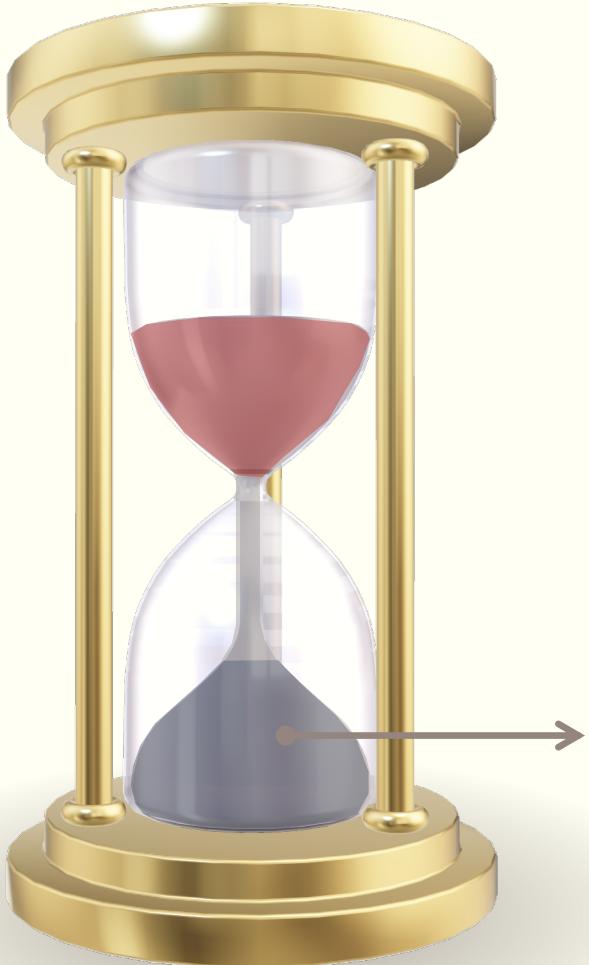
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$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

$$P(X > r + s | X > r) = e^{-\lambda s} = P(X > s)$$



Exponential Distribution

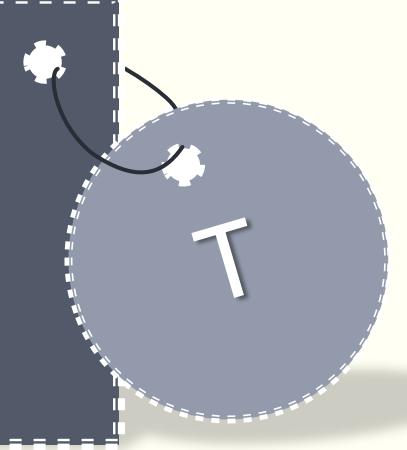


Time

- The amount of time we have to wait for an occurrence does not depend on how long we have already waited.
- The memoryless property says that knowledge of what has occurred in the past has no effect on future probabilities.

Memoryless Property

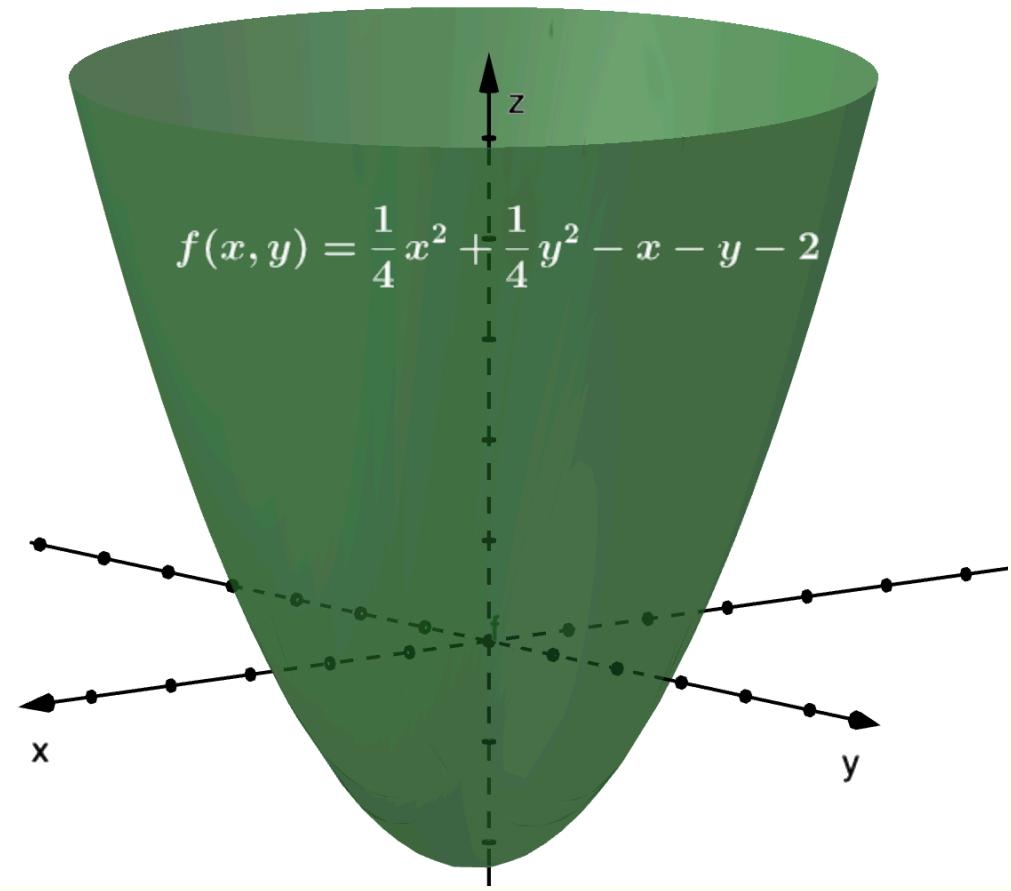
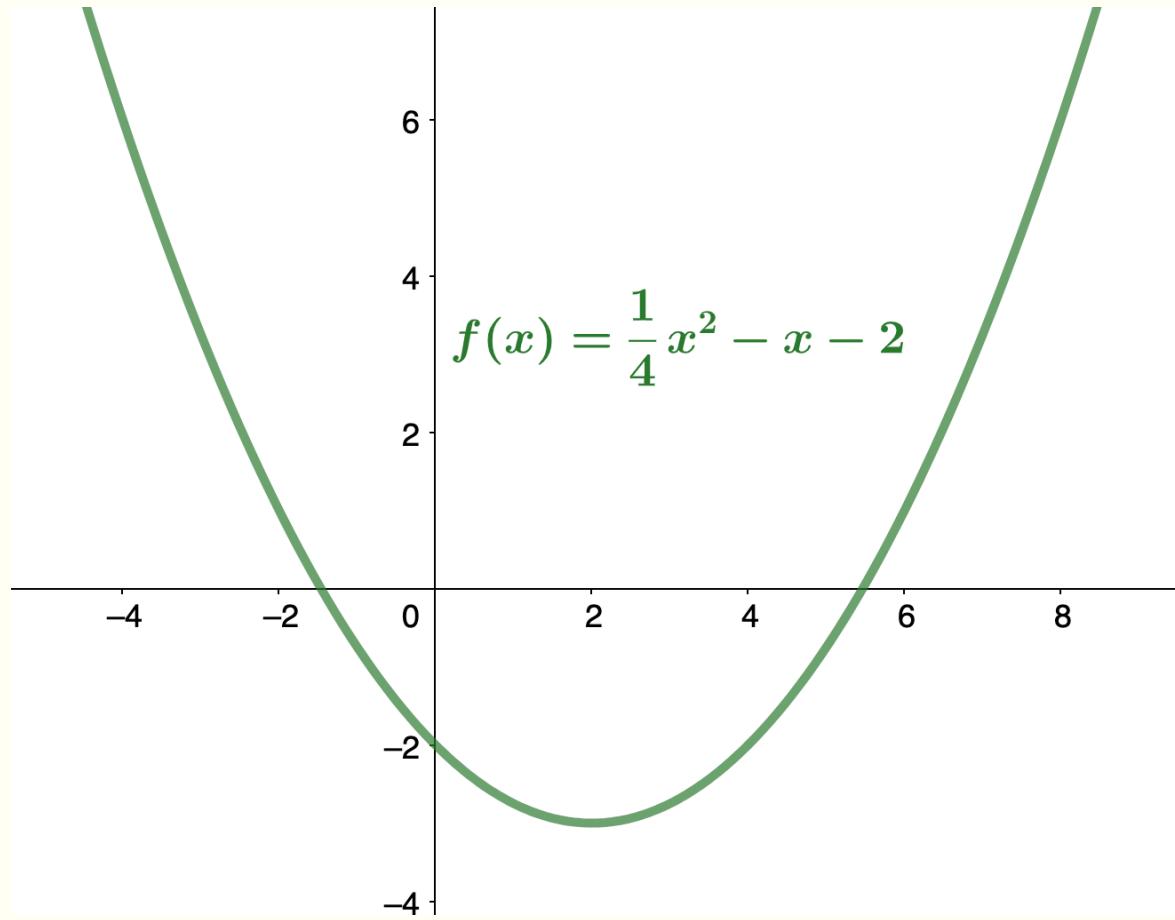
$$P(X > r + s | X > r) = P(X > s)$$





JOINT DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS

For continuous random variables



Joint Density and Cumulative Distribution Functions

- Let X_1, X_2, \dots, X_n be continuous random variables associated with an experiment. And let $X = (X_1, X_2, \dots, X_n)$.
- The joint cumulative distribution function of X is defined by

$$F(x_1, x_2, \dots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n).$$

- The joint density function of X , $f(x_1, x_2, \dots, x_n)$, satisfies the following equations:

$$F(x_1, x_2, \dots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(t_1, t_2, \dots, t_n) dt_n dt_{n-1} \cdots dt_1.$$

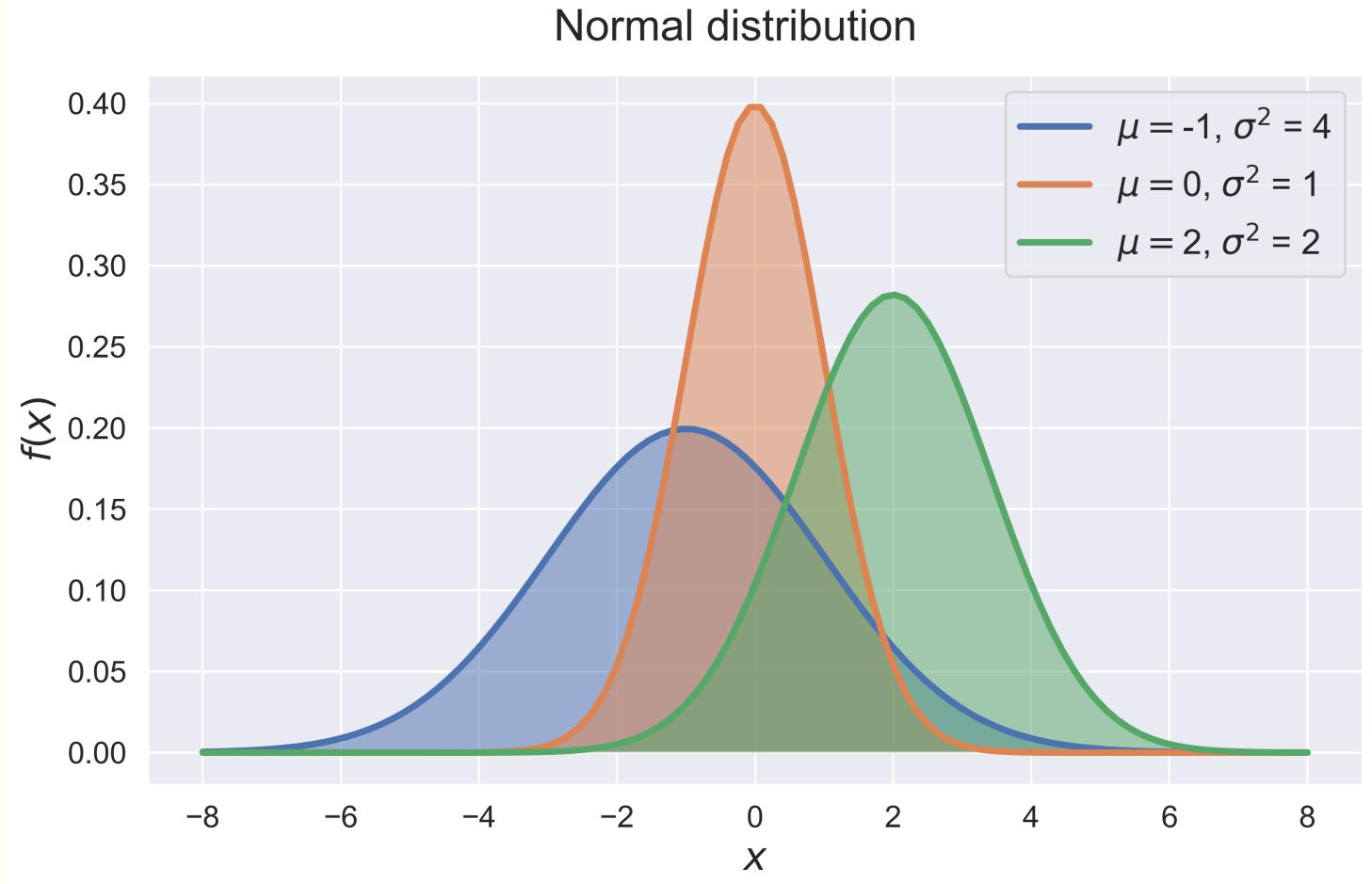
- Therefore we have

$$f(x_1, x_2, \dots, x_n) = \frac{\partial^n F(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}.$$

Normal Distribution

Single variate

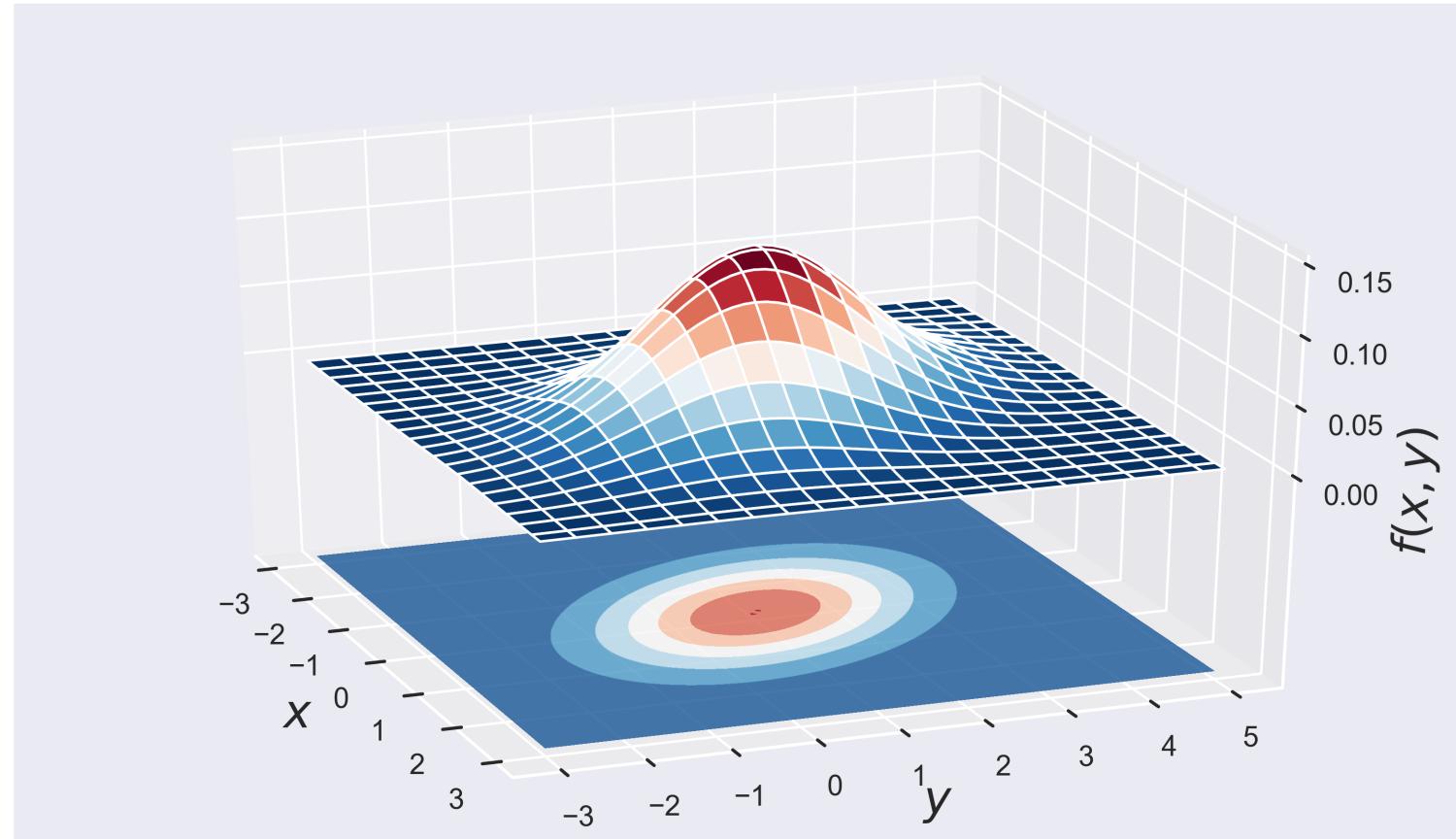
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

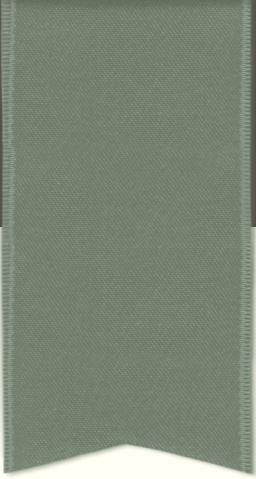


Bivariate

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{(y - \mu_y)^2}{\sigma_y^2} - \frac{2\rho(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y} \right]\right)$$

Bivariate density function





INDEPENDENT RANDOM VARIABLES

For continuous random variables

Independent Random Variables

- Let X_1, X_2, \dots, X_n be continuous random variables with cumulative distribution functions $F_1(x), F_2(x), \dots, F_n(x)$. And let $X = (X_1, X_2, \dots, X_n)$.
- These random variables are **mutually independent** if the joint cumulative distribution of X is the product of individual cumulative distribution distributions $F_i(x_i)$,

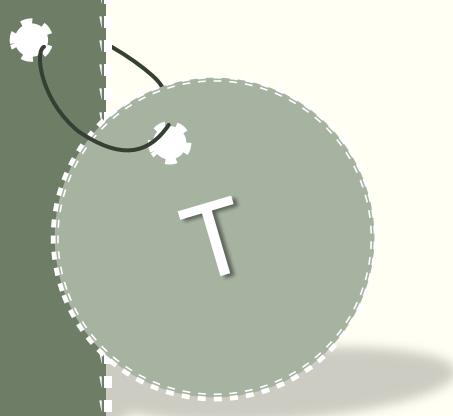
$$F(x_1, x_2, \dots, x_n) = F_1(x_1)F_2(x_2) \cdots F_n(x_n),$$

for any choice of x_1, x_2, \dots, x_n .

Theorem

Let X_1, X_2, \dots, X_n be continuous random variables with density functions $f_1(x), f_2(x), \dots, f_n(x)$.

Then these random variables are **mutually independent** if and only if $f(x_1, x_2, \dots, x_n) = f_1(x_1)f_2(x_2) \cdots f_n(x_n)$, for any choice of x_1, x_2, \dots, x_n .

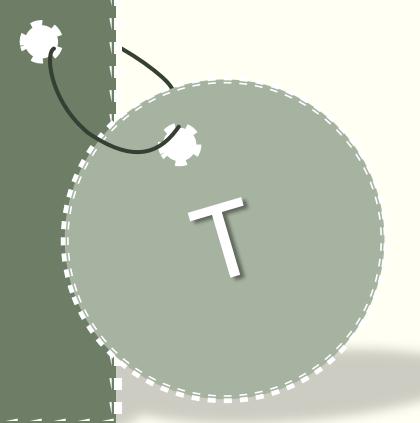


Independent Random Variables

Theorem

Let X_1, X_2, \dots, X_n be continuous random variables with density functions $f_1(x), f_2(x), \dots, f_n(x)$.

Then these random variables are **mutually independent** if and only if $f(x_1, x_2, \dots, x_n) = f_1(x_1)f_2(x_2) \cdots f_n(x_n)$, for any choice of x_1, x_2, \dots, x_n .

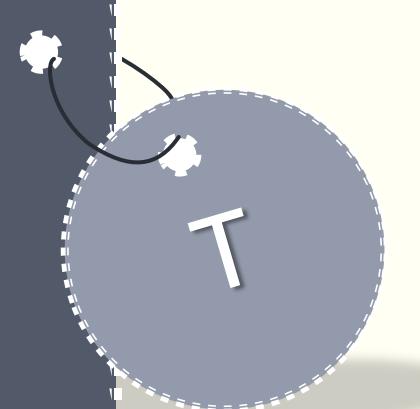


Theorem

Let X_1, X_2, \dots, X_n be mutually independent continuous random variables.

Let $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$ be continuous functions.

Then $\phi_1(X_1), \phi_2(X_2), \dots, \phi_n(X_n)$ are mutually independent.



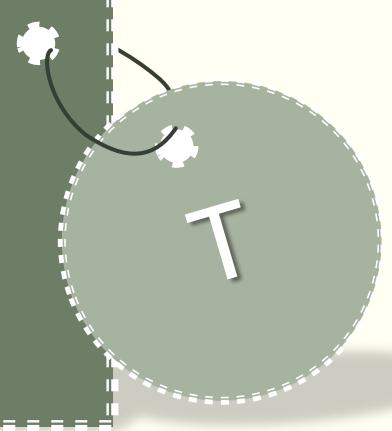
Normal Distribution

Single variate

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Theorem

... these random variables are **mutually independent** if and only if
 $f(x_1, x_2, \dots, x_n) = f_1(x_1)f_2(x_2) \cdots f_n(x_n)$,
for any choice of x_1, x_2, \dots, x_n .



Bivariate

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{(y - \mu_y)^2}{\sigma_y^2} - \frac{2\rho(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y} \right]\right)$$

How?

?

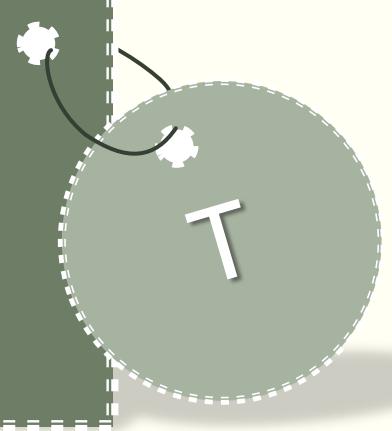
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Theorem

... these random variables are **mutually independent** if and only if
 $f(x_1, x_2, \dots, x_n) = f_1(x_1)f_2(x_2) \cdots f_n(x_n)$,
for any choice of x_1, x_2, \dots, x_n .



Bivariate

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right]\right)$$

How?

?

$\rho = \dots$

=

Normal Distribution

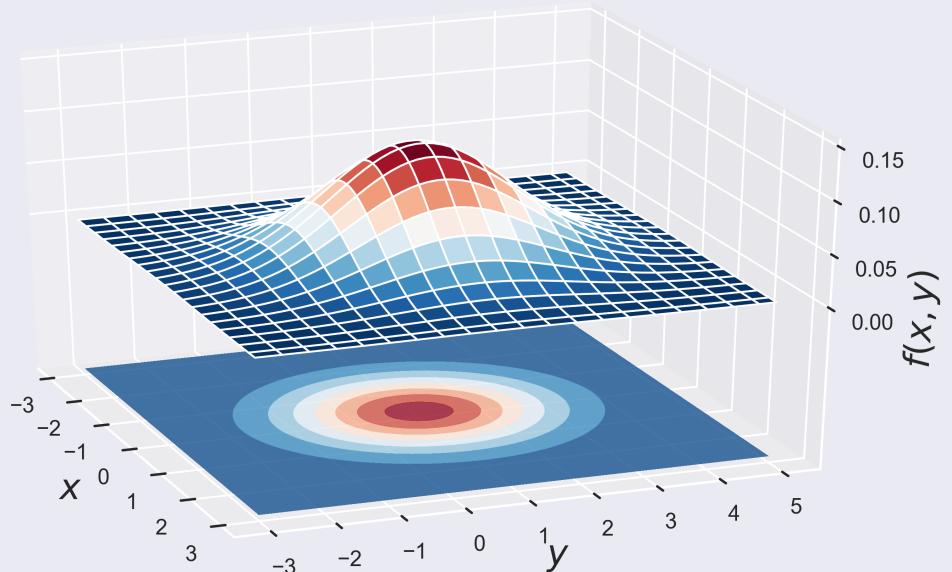
independent
 $\rho = 0$

=

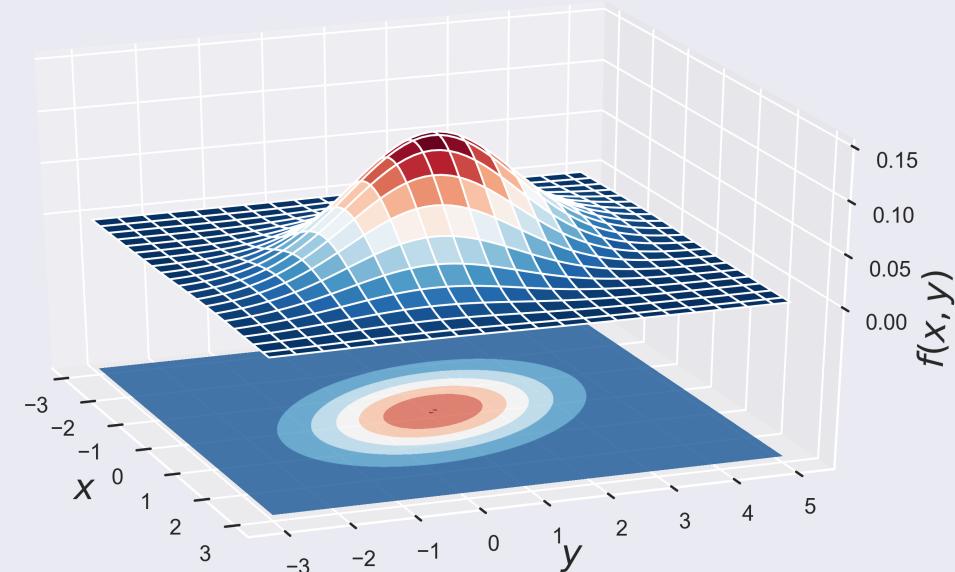
dependent
 $\rho \neq 0$

≠

Bivariate density function



Bivariate density function



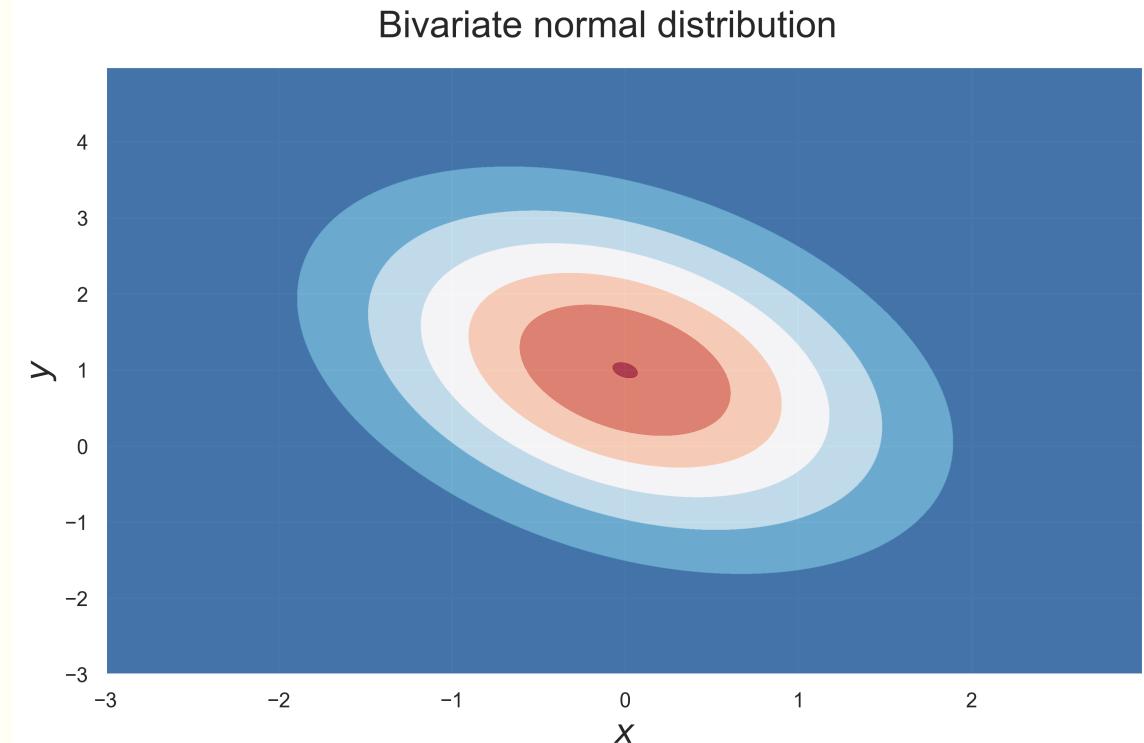
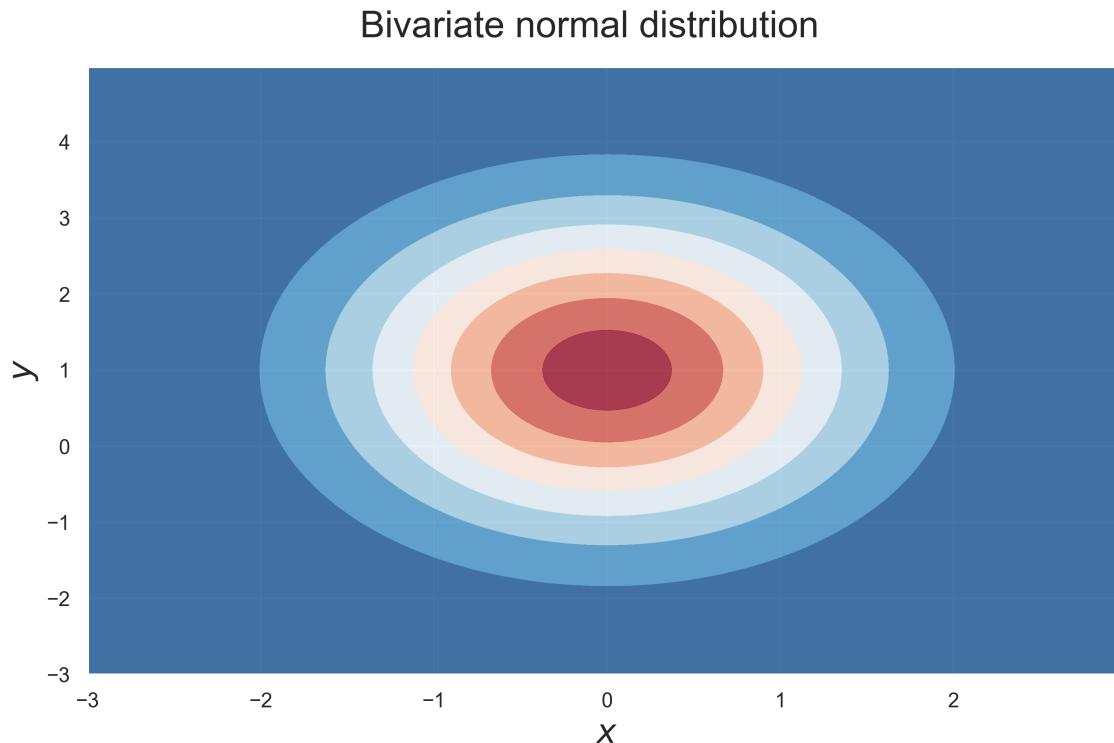
Normal Distribution

independent
 $\rho = 0$

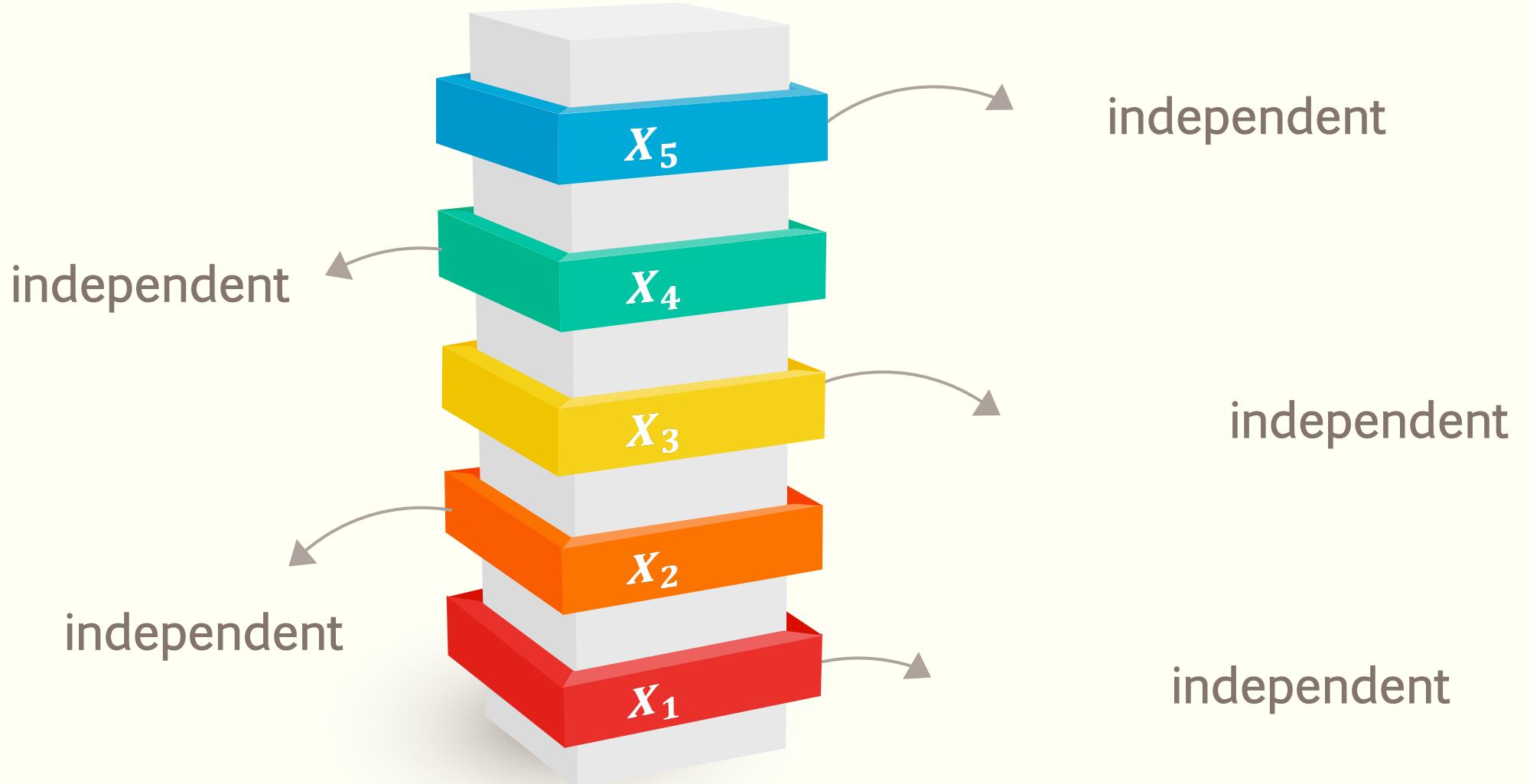
=

dependent
 $\rho \neq 0$

≠



Independent Random Variables



Example

- Suppose you choose two numbers x and y , independently at random from the interval $[0, 1]$.
- Given that their sum lies in the interval $[0, 1]$, find the probability that

$$xy < \frac{1}{2}.$$

independently

?

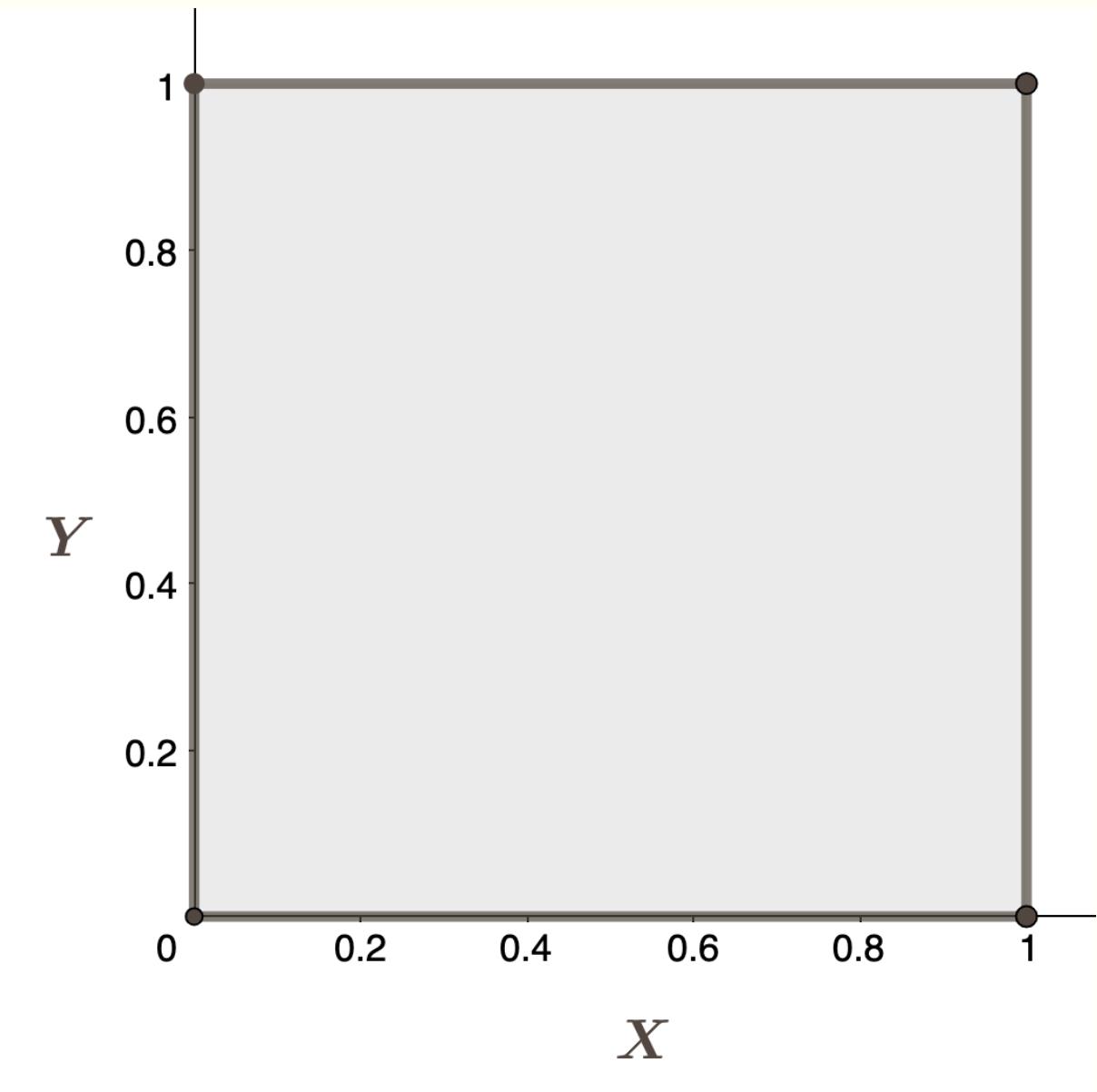
given that

?

Example

independently

!



Example

independently

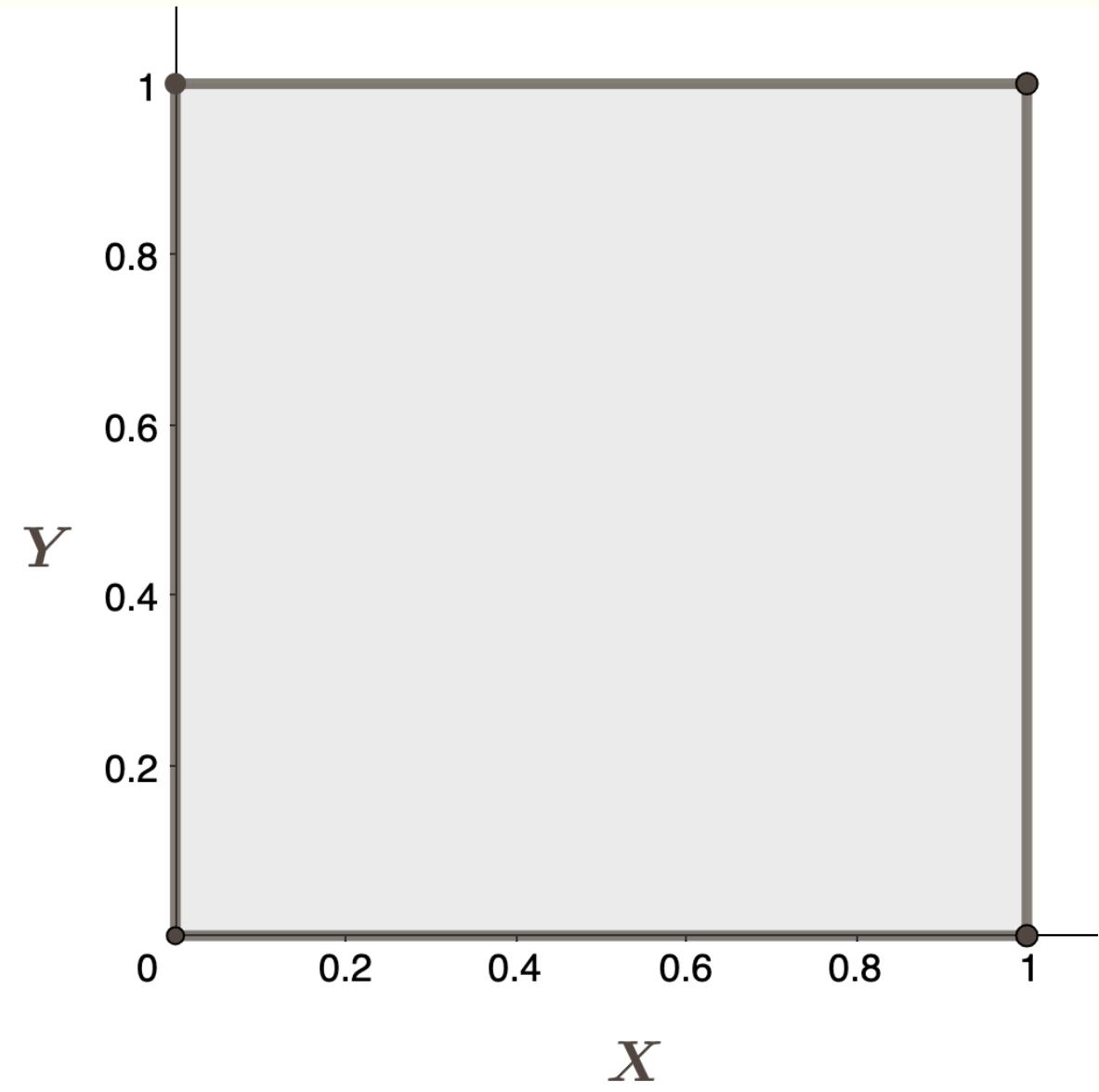
!

given that

!

Event $E: X + Y \in [0, 1]$

$$P(0 \leq X + Y \leq 1) = \dots$$



Example

independently

!

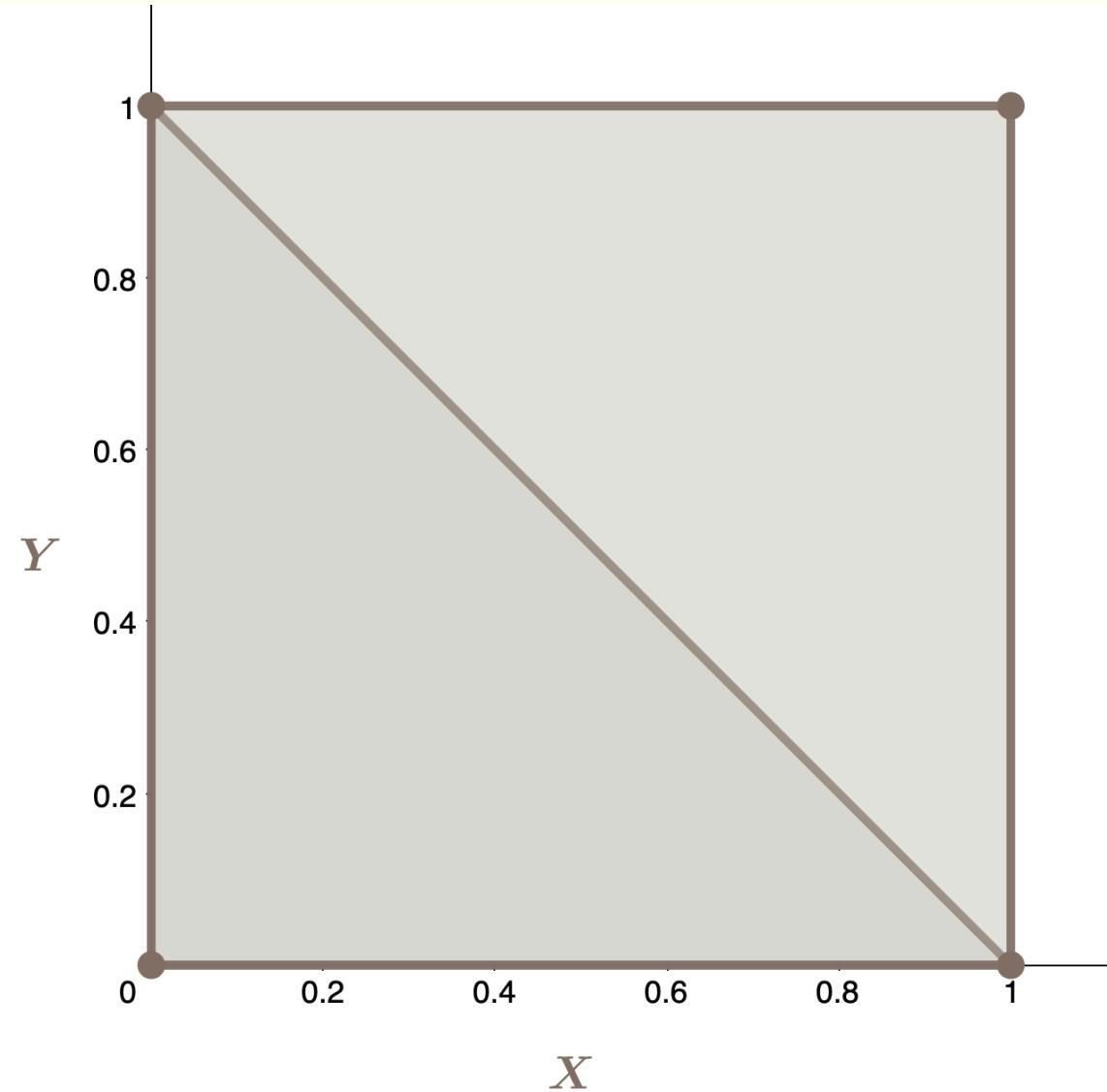
given that

!

Event $E: X + Y \in [0, 1]$

lines

$$P(0 \leq X + Y \leq 1) = \frac{1}{2}$$



Example

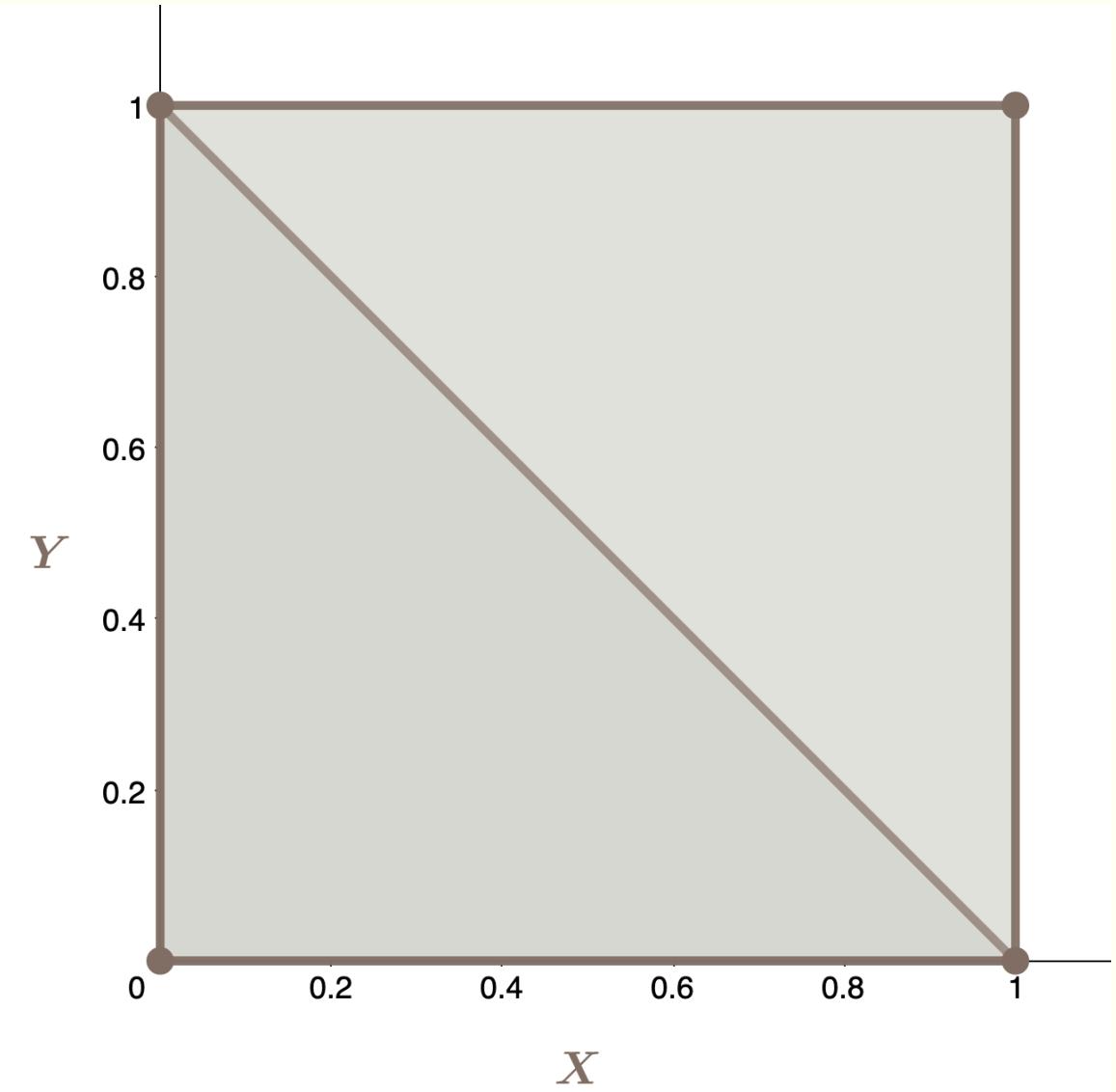
Event $E: X + Y \in [0, 1]$

lines

$$P(0 \leq X + Y \leq 1) = \frac{1}{2}$$

Event $F: XY < \frac{1}{2}$

$$P\left(XY < \frac{1}{2}\right)$$



Example

Event $E: X + Y \in [0, 1]$

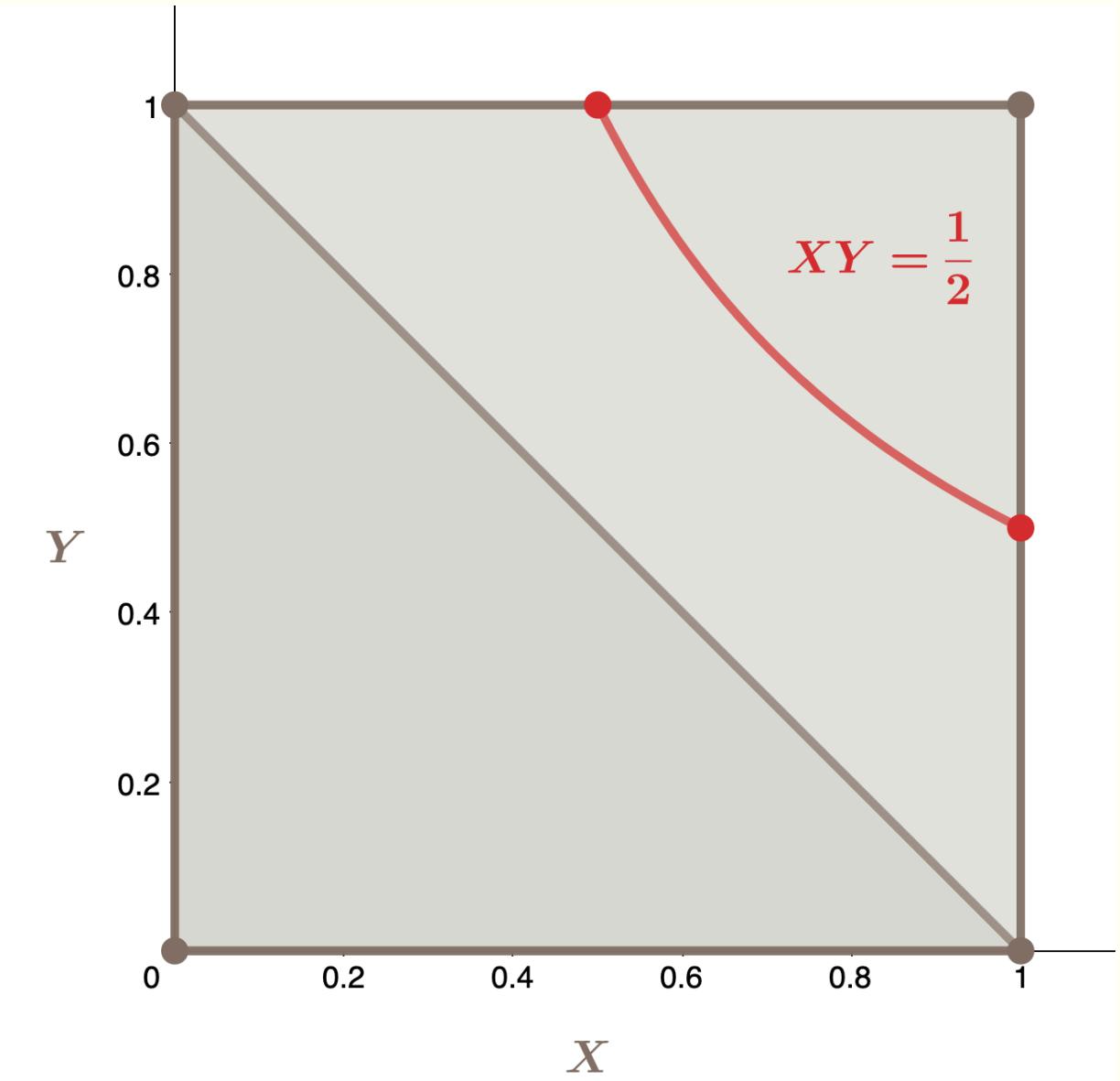
lines

$$P(0 \leq X + Y \leq 1) = \frac{1}{2}$$

Event $F: XY < \frac{1}{2}$

hyperbolas

$$P\left(XY < \frac{1}{2}\right)$$



Example

Event $E: X + Y \in [0, 1]$

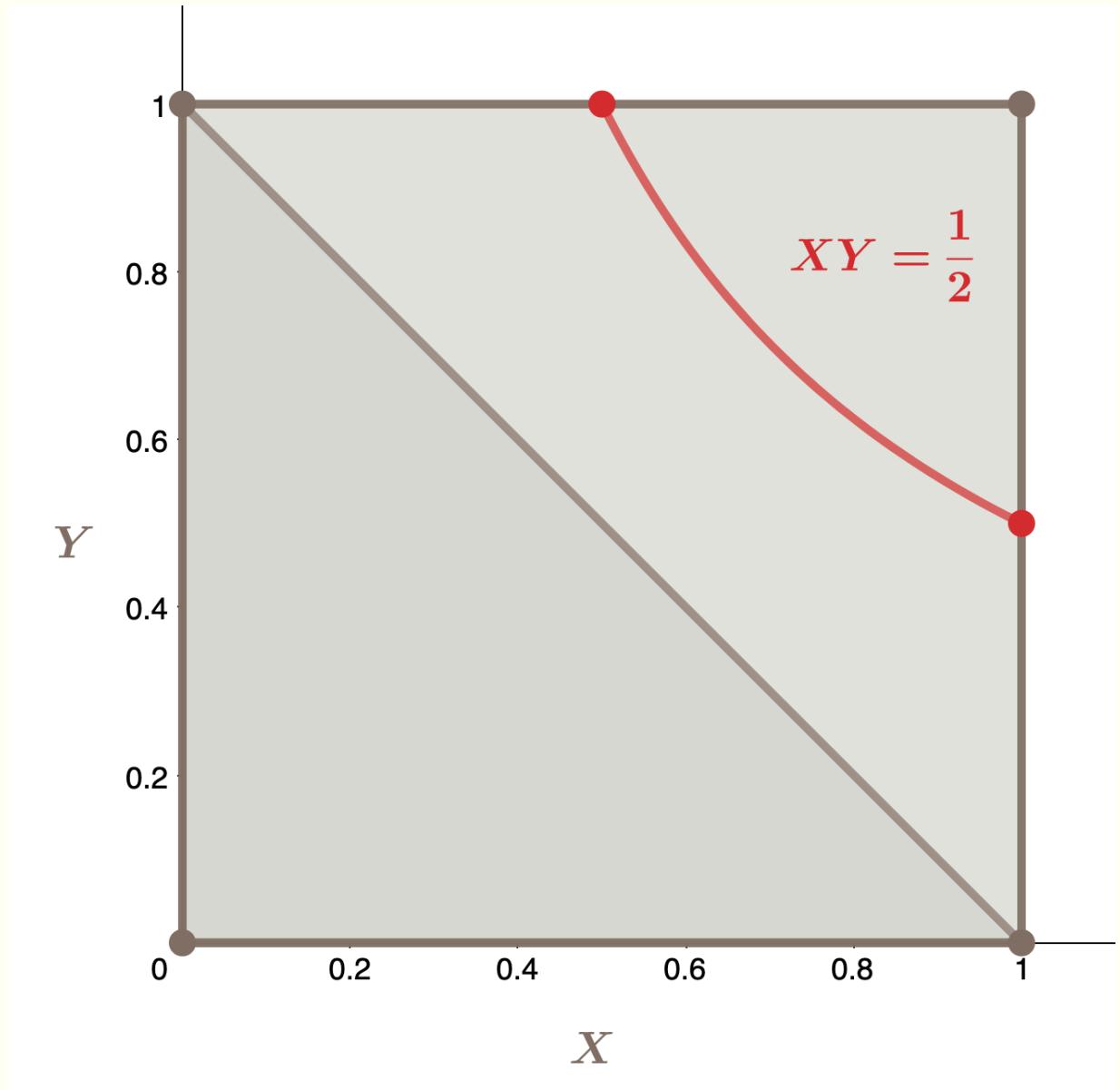
$$P(0 \leq X + Y \leq 1) = \frac{1}{2}$$

Event $F: XY < \frac{1}{2}$

$$P\left(XY < \frac{1}{2}\right)$$

Event $F|E: XY < \frac{1}{2} | X + Y \in [0, 1]$

$$P\left(XY < \frac{1}{2} \middle| 0 \leq X + Y \leq 1\right) = \dots$$



Example

Event $E: X + Y \in [0, 1]$

$$P(0 \leq X + Y \leq 1) = \frac{1}{2}$$

Event $F: XY < \frac{1}{2}$

$$P\left(XY < \frac{1}{2}\right)$$

Event $F|E: XY < \frac{1}{2} | X + Y \in [0, 1]$

$$P\left(XY < \frac{1}{2} \mid 0 \leq X + Y \leq 1\right) = 1$$

