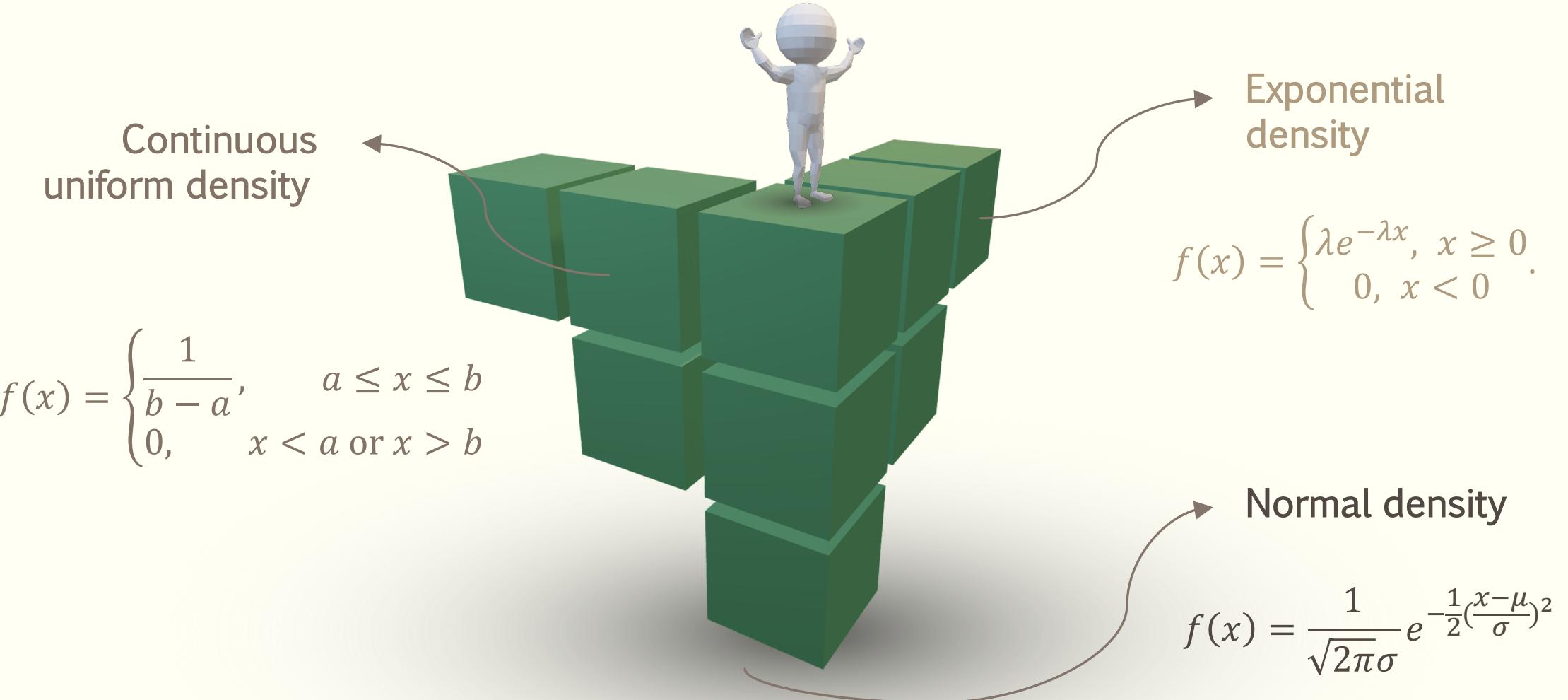


MATH 20: PROBABILITY

Expected Value & Variance of Continuous Random Variables



Important Densities



Continuous uniform

$$f(x) = \frac{1}{b-a} \quad E(X) = \frac{1}{2}(a+b) \quad V(X) = \frac{1}{12}(b-a)^2$$

Exponential

$$f(x) = \lambda e^{-\lambda x} \quad E(X) = \frac{1}{\lambda} \quad V(X) = \frac{1}{\lambda^2}$$

Normal

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \quad E(X) = \mu \quad V(X) = \sigma^2$$

Expected Value

- Let X be a **real-valued** random variable with density function $f(x)$.
- The expected value $\mu = E(X)$ is defined by

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx,$$

provided $\int_{-\infty}^{+\infty} |x|f(x)dx$ is finite.

Discrete expected value $E(X)$

$$\sum_{x \in \Omega} xm(x)$$

Continuous expected value $E(X)$

$$\int_{-\infty}^{+\infty} xf(x)dx$$

Expectation of Functions of Random Variables

- If X is a real-valued random variable and if $\phi: R \rightarrow R$ is a continuous real-valued function with domain $[a, b]$, then

$$E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x)f(x)dx,$$

provided the integral exists.

Discrete expected value $E(\phi(X))$

$$\sum_{x \in \Omega} \phi(x)m(x)$$

Continuous expected value $E(\phi(X))$

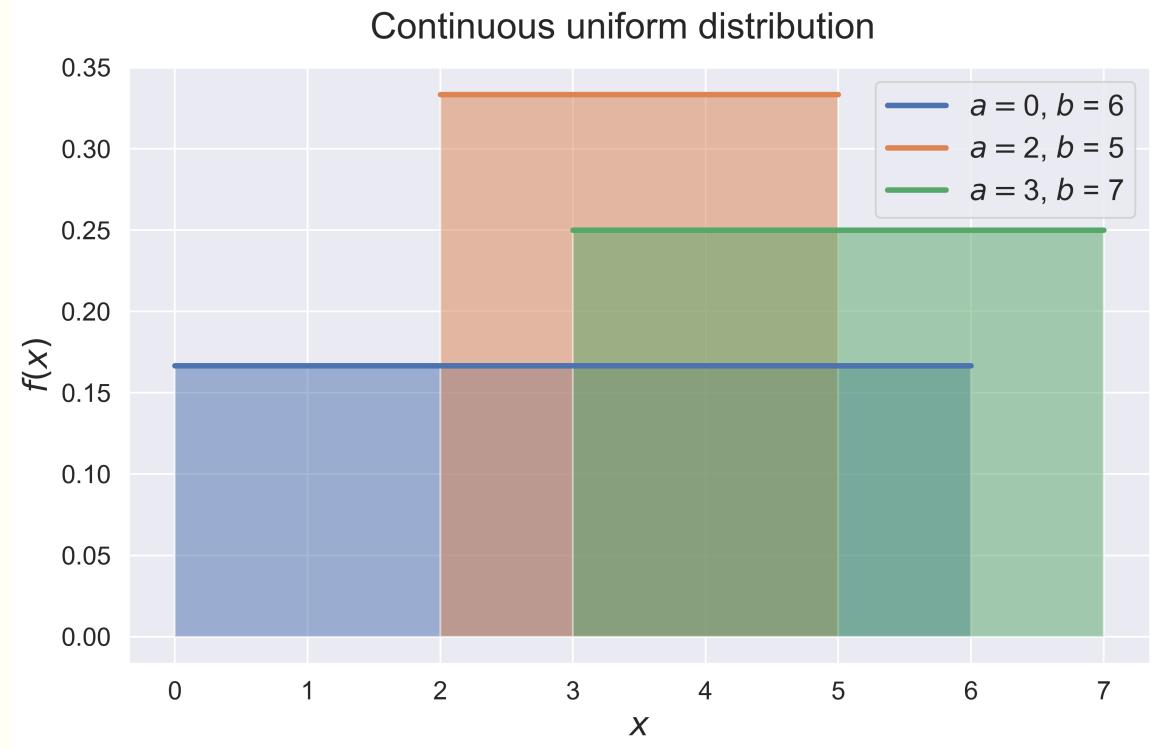
$$\int_{-\infty}^{+\infty} \phi(x)f(x)dx$$

Example 1

Continuous uniform distribution

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

$$E(X) = \dots$$



Continuous uniform distribution

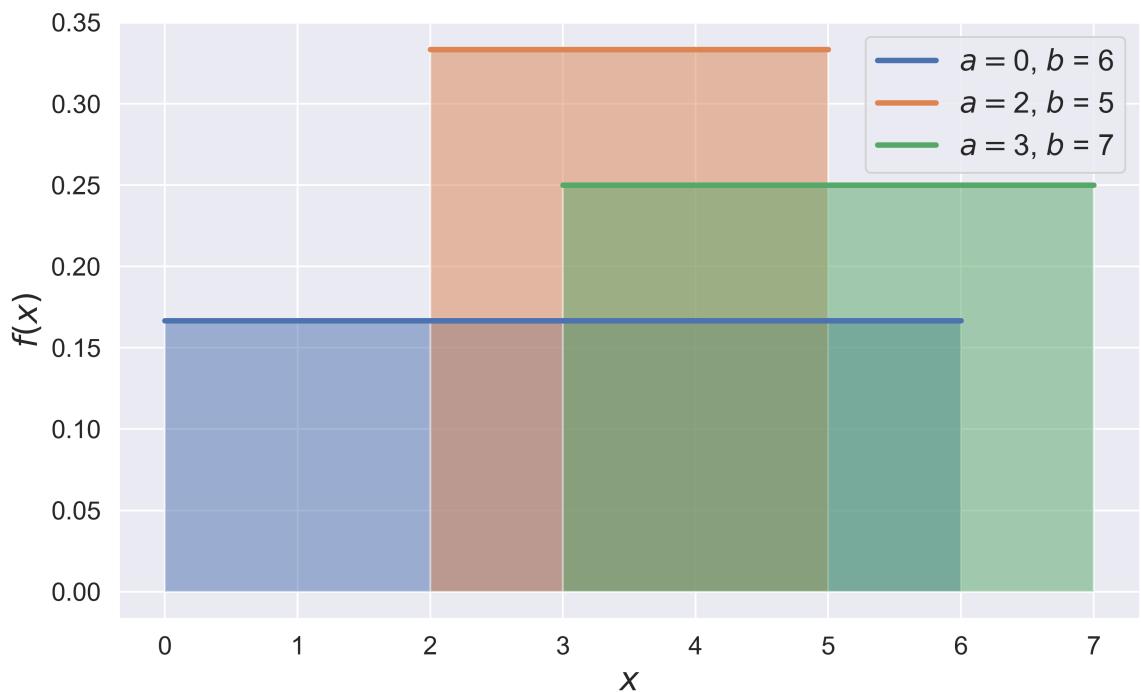
$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \frac{1}{2} x^2 \Big|_a^b = \frac{1}{2} \frac{b^2 - a^2}{b-a}$$

$$= \frac{1}{2}(a+b)$$

Continuous uniform distribution

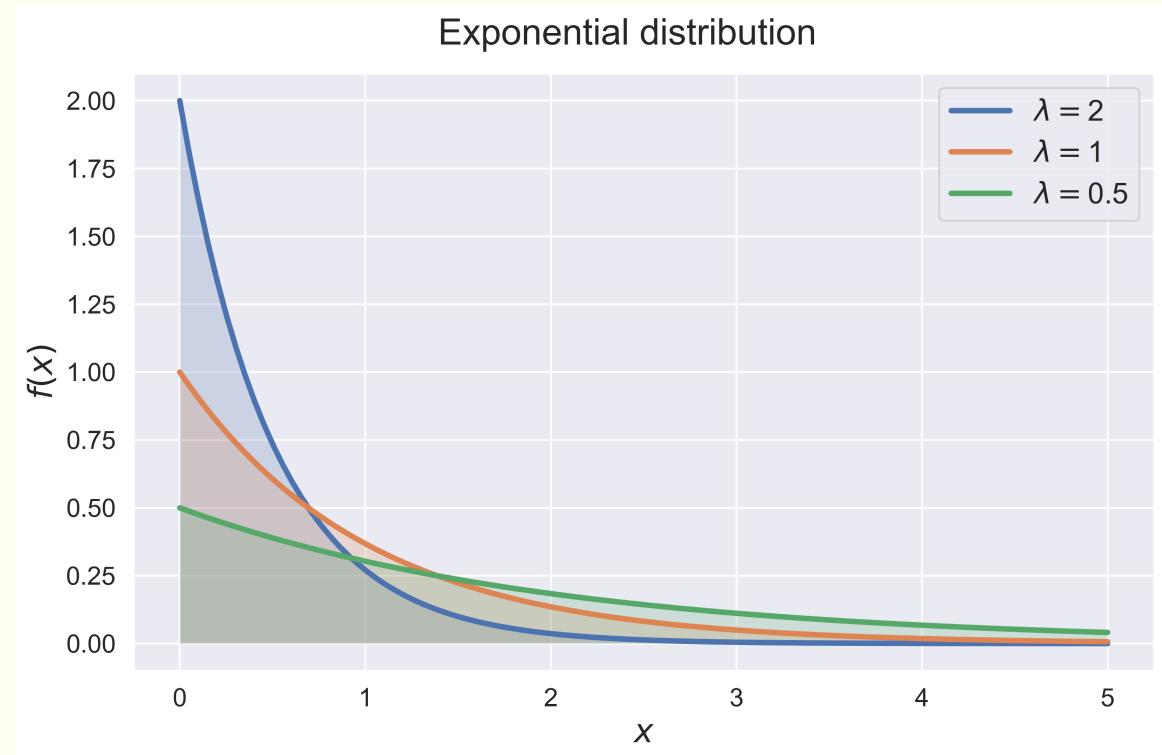


Example 2

Exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

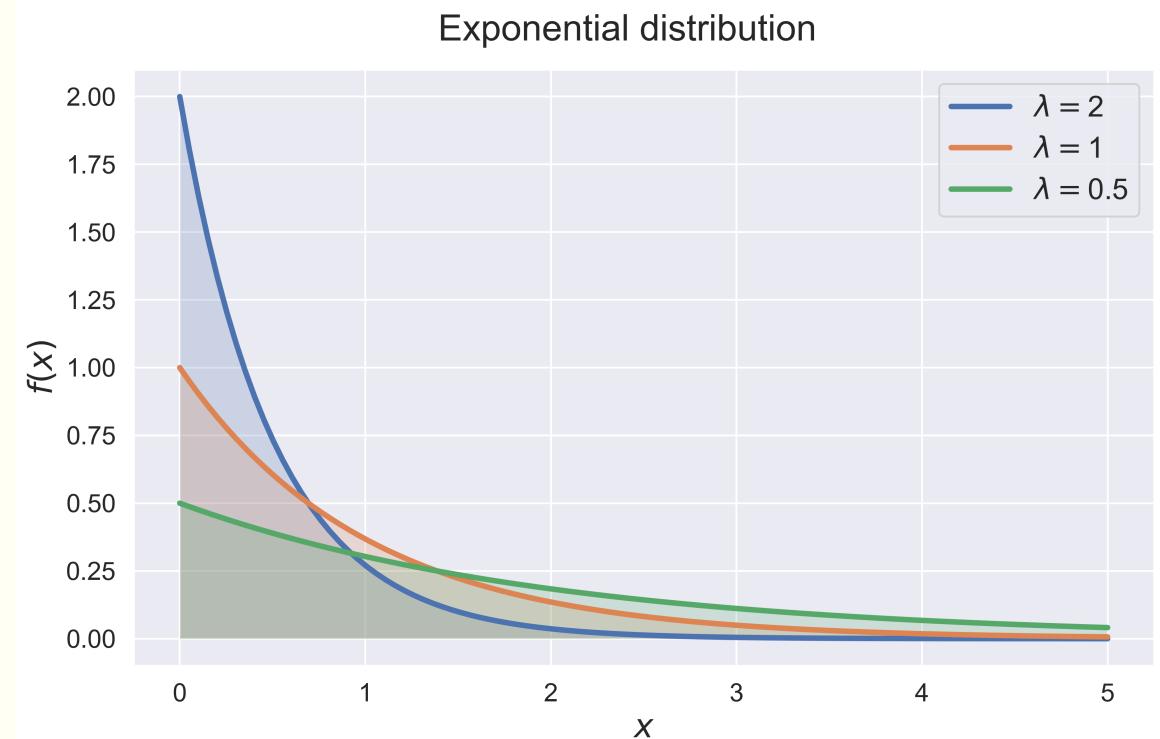
$$E(X) = \dots$$



Exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} xf(x)dx = \int_0^{+\infty} x\lambda e^{-\lambda x}dx \\ &= \int_0^{+\infty} -xde^{-\lambda x} = -xe^{-\lambda x}|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x}dx \\ &= \int_0^{+\infty} e^{-\lambda x}dx = -\frac{1}{\lambda}e^{-\lambda x}|_0^{+\infty} \\ &= \frac{1}{\lambda} \end{aligned}$$



Example 3

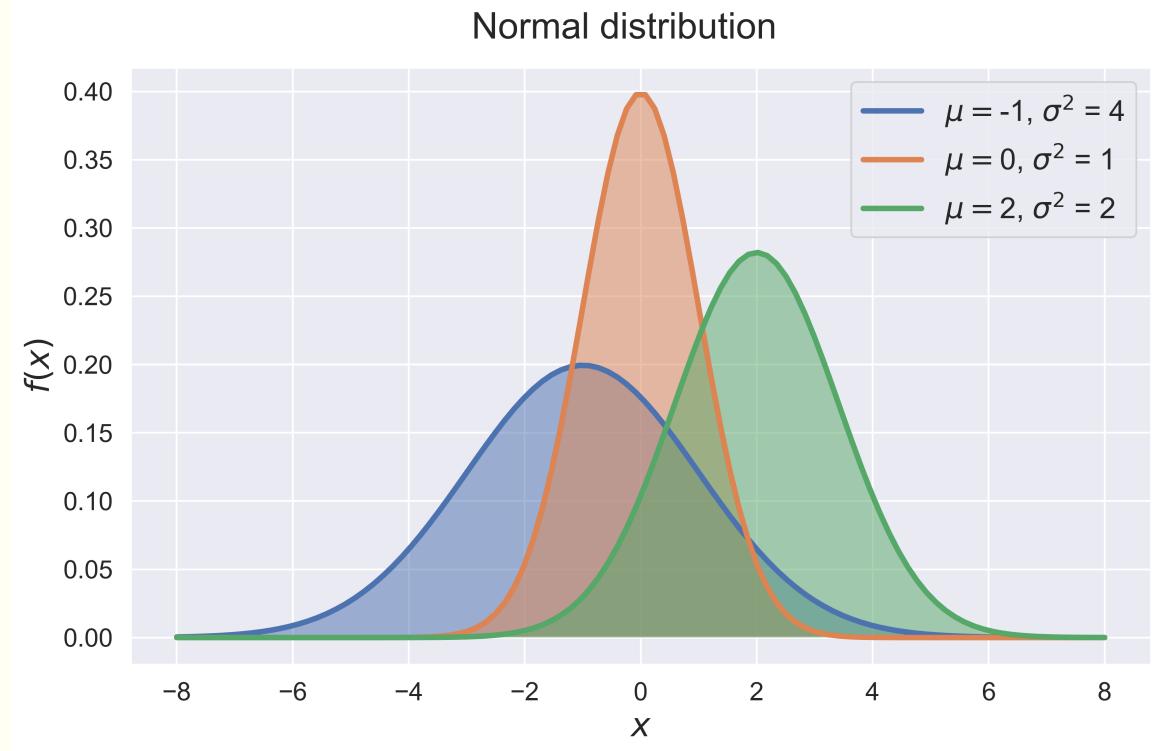
Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

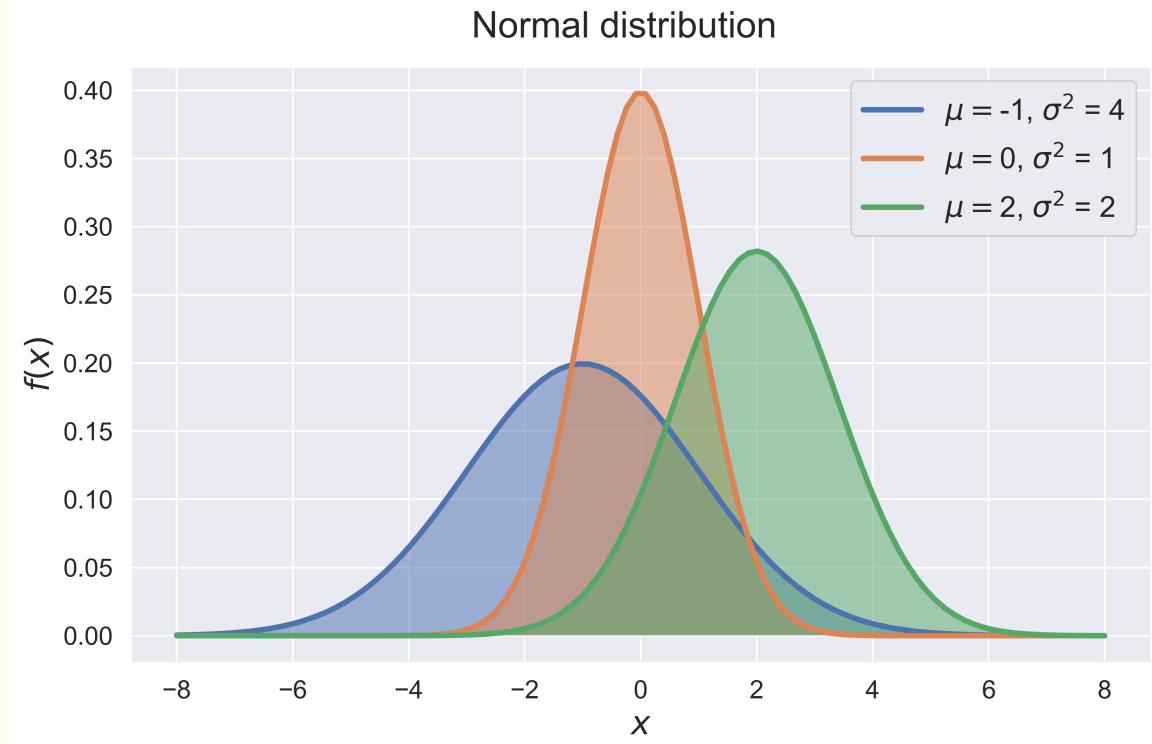
$$E(X) = \dots$$



Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} xe^{-\frac{1}{2}x^2} dx \\ &= 0 \end{aligned}$$



Sum of Random Variables

- If X and Y are real-valued random variables with finite expected values. Then

$$E(X + Y) = E(X) + E(Y),$$

and if c is any constant, then

$$E(cX) = cE(X).$$

- In general, for a linear combination of n real-valued random variables X_i with constants c_i , we have

$$E(c_1X_1 + c_2X_2 + \cdots + c_nX_n) = c_1E(X_1) + c_2E(X_2) + \cdots + c_nE(X_n).$$

Do we need mutual independence of the summands?

?

The Product of Two Random Variables

- Let X and Y be independent real-valued continuous random variables with finite expected values. Then we have

$$E(XY) = E(X)E(Y).$$

- More generally, for n mutually independent random variables X_i , we have

$$E(X_1X_2 \cdots X_n) = E(X_1)E(X_2) \cdots E(X_n).$$

Do we need mutual independence of the factors?

?

Example 3 revisited

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$E(X) = 0$$

Normal distribution

$$Z = \sigma X + \mu$$

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{z-\mu}{\sigma})^2}$$

$$E(Z) = E(\sigma X + \mu) = \mu$$

Conditional Expectation

- Conditional density

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

joint density

- Conditional expected value

$$E(X|Y = y) = \int x f_{X|Y}(x|y) dx.$$

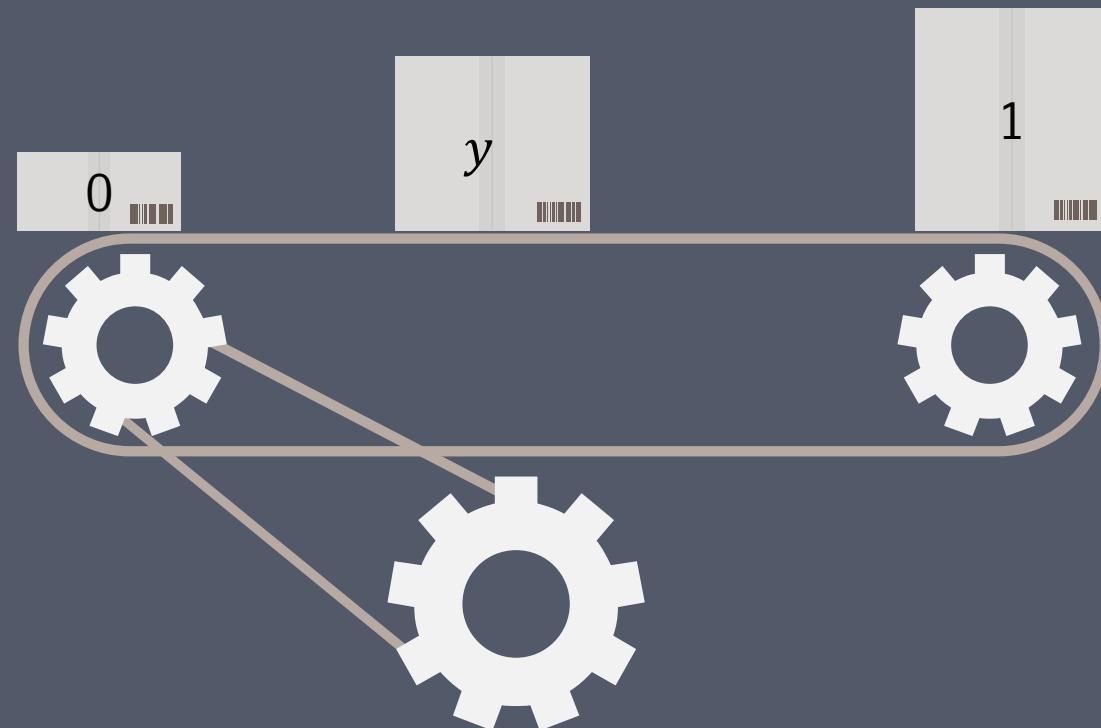
marginal density

Example

- A point Y is chosen at random from $[0, 1]$ uniformly. A second point X is then uniformly and randomly chosen from the interval $[0, Y]$. Find the density and conditional expectation for X .

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$E(X|Y = y) = \int xf_{X|Y}(x|y)dx$$

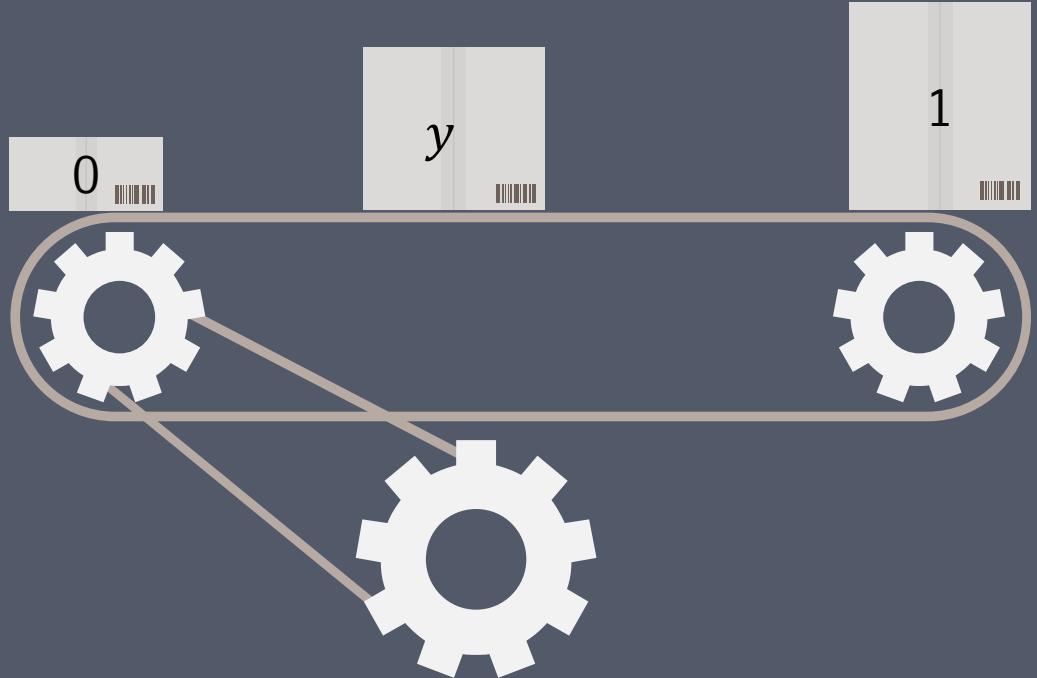


$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$E(X|Y = y) = \int x f_{X|Y}(x|y) dx$$

$$f_{X,Y}(x, y) = \frac{1}{y}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{1}{y}$$



$$f_Y(y) = 1$$

$$E(X|Y = y) = \int_0^y \frac{x}{y} dx = \frac{1}{2}y$$

Conditional Expectation

- Conditional density

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

joint density

- Conditional expected value

$$E(X|Y = y) = \int x f_{X|Y}(x|y) dx.$$

- Expected value

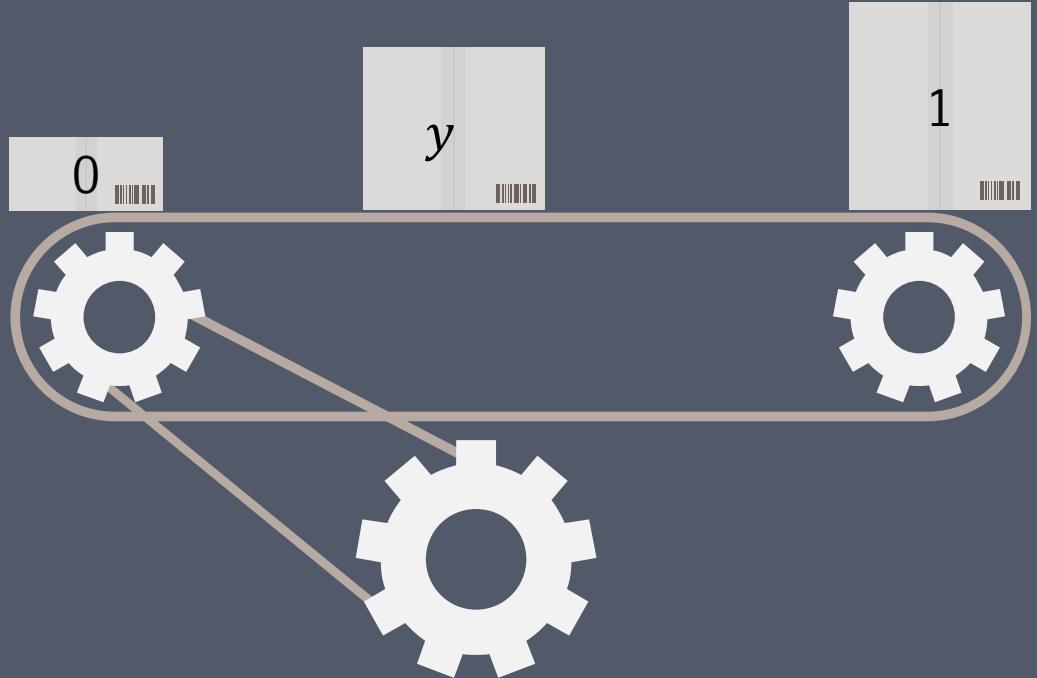
$$E(X) = \int E(X|Y = y) f_Y(y) dy.$$

marginal density

$$E(X) = \int E(X|Y = y)f_Y(y)dy$$

!

$$E(X|Y = y) = \int_0^y \frac{x}{y} dx = \frac{1}{2}y$$



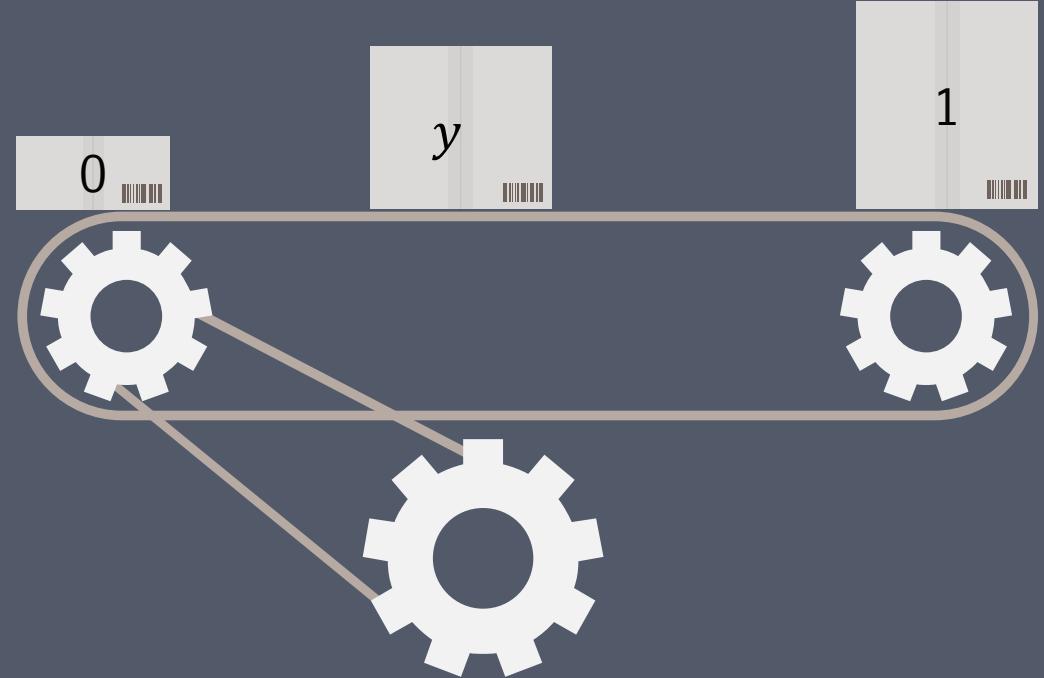
$$E(X) = \int_0^1 \frac{1}{2}y dy = \frac{1}{4}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{y}$$

marginal density: $f_Y(y) = 1$

joint density: $f_{X,Y}(x,y) = \frac{1}{y}$

marginal density: $f_X(x) = \dots$



$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy$$

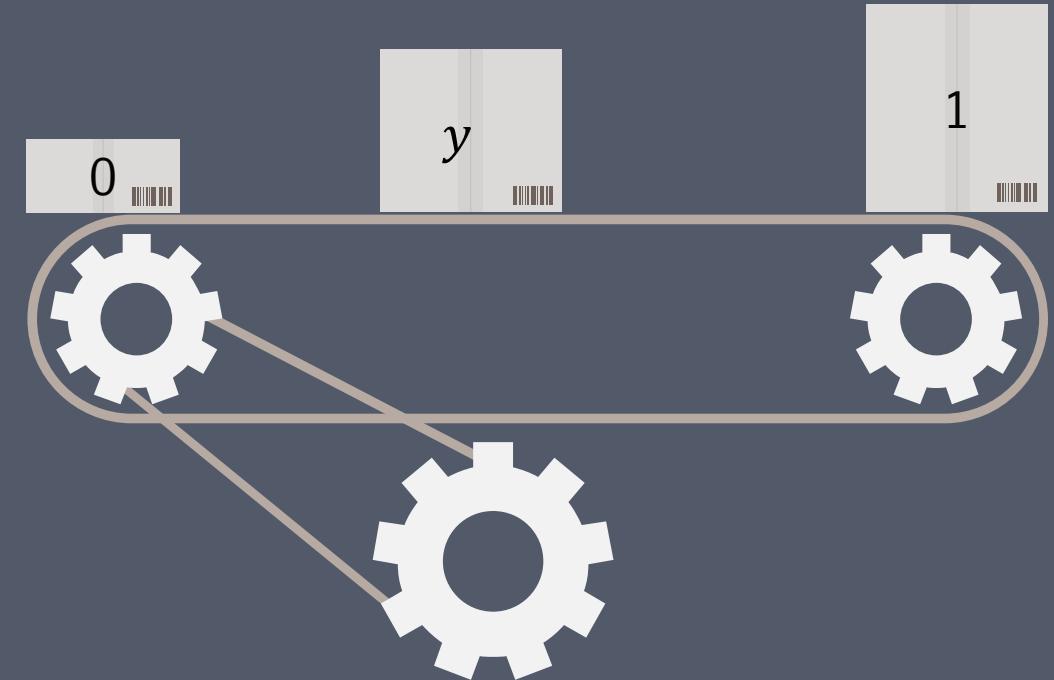
!

marginal density: $f_Y(y) = 1$

joint density: $f_{X,Y}(x,y) = \frac{1}{y}$ $0 \leq x \leq y$

marginal density: $f_X(x)$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy = \int_0^1 f_{X,Y}(x,y) dy \\ &= \int_x^1 \frac{1}{y} dy \\ &= -\ln(x) \end{aligned}$$



$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy$$

!

Continuous uniform

$$f(x) = \frac{1}{b-a} \quad E(X) = \frac{1}{2}(a+b) \quad V(X) = \frac{1}{12}(b-a)^2$$

Exponential

$$f(x) = \lambda e^{-\lambda x} \quad E(X) = \frac{1}{\lambda} \quad V(X) = \frac{1}{\lambda^2}$$

Normal

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \quad E(X) = \mu \quad V(X) = \sigma^2$$

Variance

- Let X be real-valued random variable with density function $f(x)$ and with the expected value $E(X) = \mu$. The variance $\sigma^2 = V(X)$ is defined by

$$\sigma^2 = V(X) = E((X - \mu)^2) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx.$$

Discrete variance $V(X)$

$$\sum_{x \in \Omega} (x - \mu)^2 m(x)$$

Continuous variance $V(X)$

$$\int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx$$

$$V(X) = E(X^2) - \mu^2$$

Properties of Variance

- If X is any random variable and c is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

- If X and Y are independent real-valued random variables, then

$$V(X + Y) = V(X) + V(Y).$$

When do we need independence?

$X + Y$

XY

Do not need

$$E(X + Y) = E(X) + E(Y)$$

...

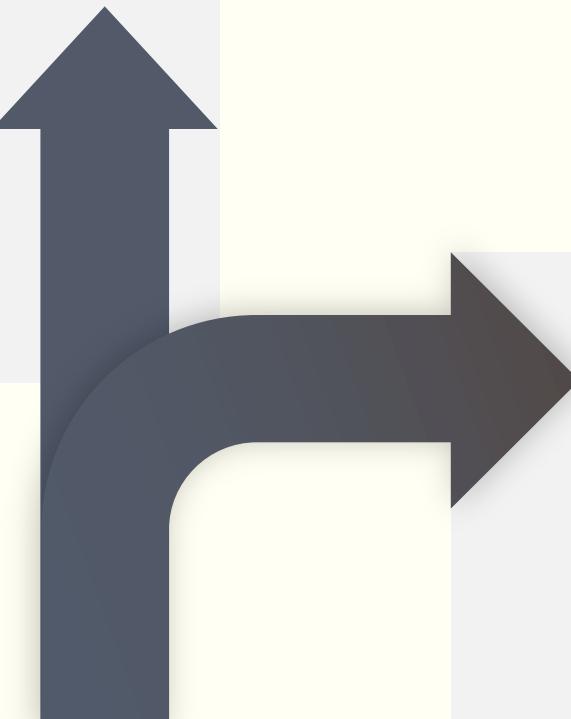
...

Need

$$E(XY) = E(X)E(Y)$$

$$V(X + Y) = V(X) + V(Y)$$

...



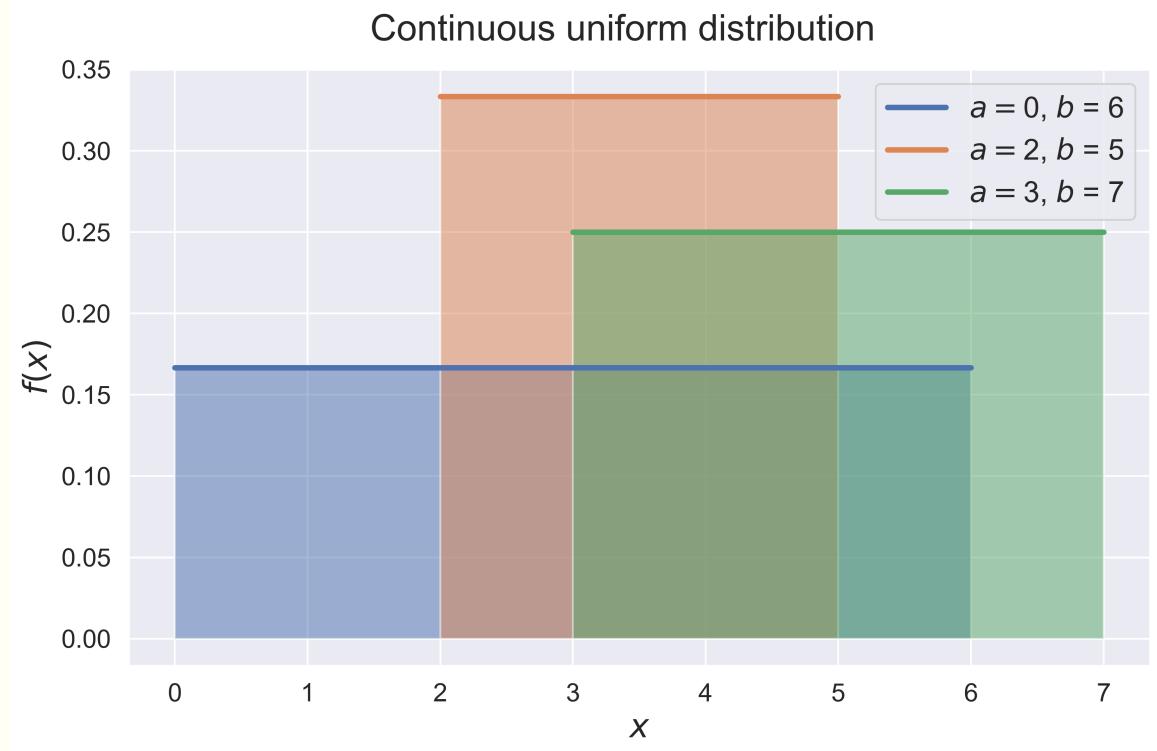
Example 1

Continuous uniform distribution

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

$$E(X) = \frac{1}{2}(a+b)$$

$$V(X) = \dots$$

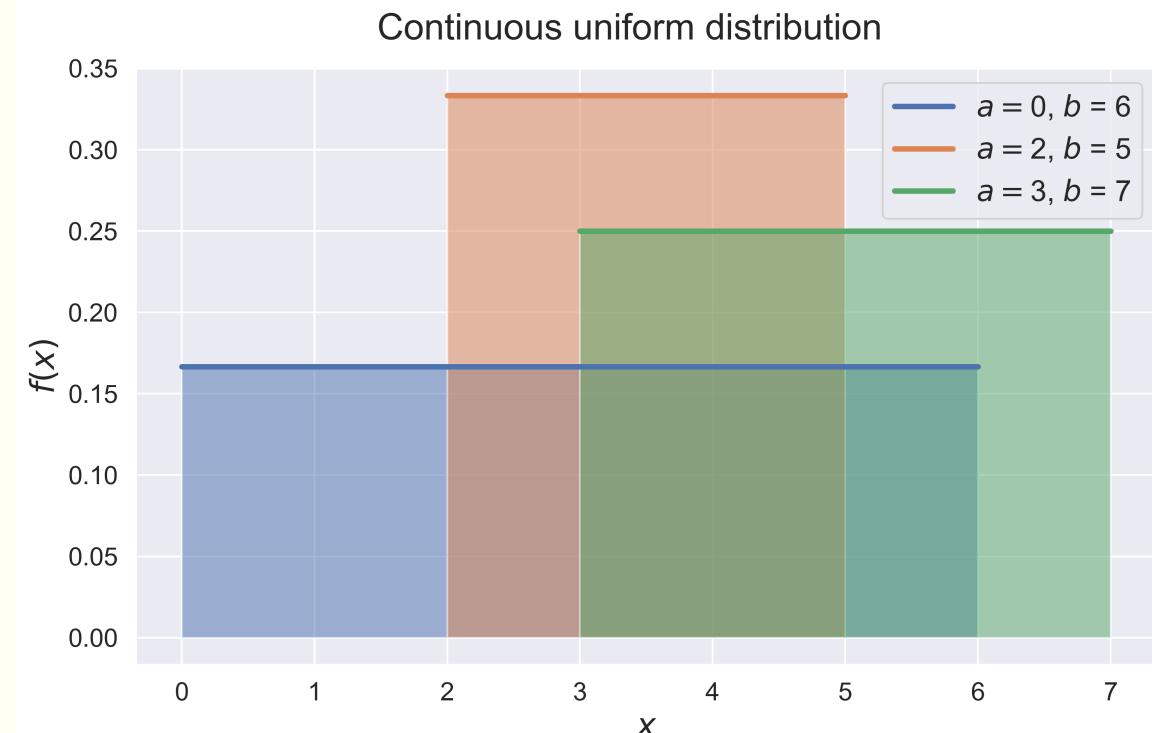


Continuous uniform distribution

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

$$E(X) = \frac{1}{2}(a+b)$$

$$\begin{aligned} V(X) &= E(X^2) - \mu^2 \\ &= \int_a^b x^2 \frac{1}{b-a} dx - \frac{1}{4}(a+b)^2 \\ &= \frac{1}{b-a} \frac{1}{3} x^3 \Big|_a^b - \frac{1}{4}(a+b)^2 \\ &= \frac{1}{3} \frac{b^3 - a^3}{b-a} - \frac{1}{4}(a+b)^2 \\ &= \frac{1}{12} (b-a)^2 \end{aligned}$$

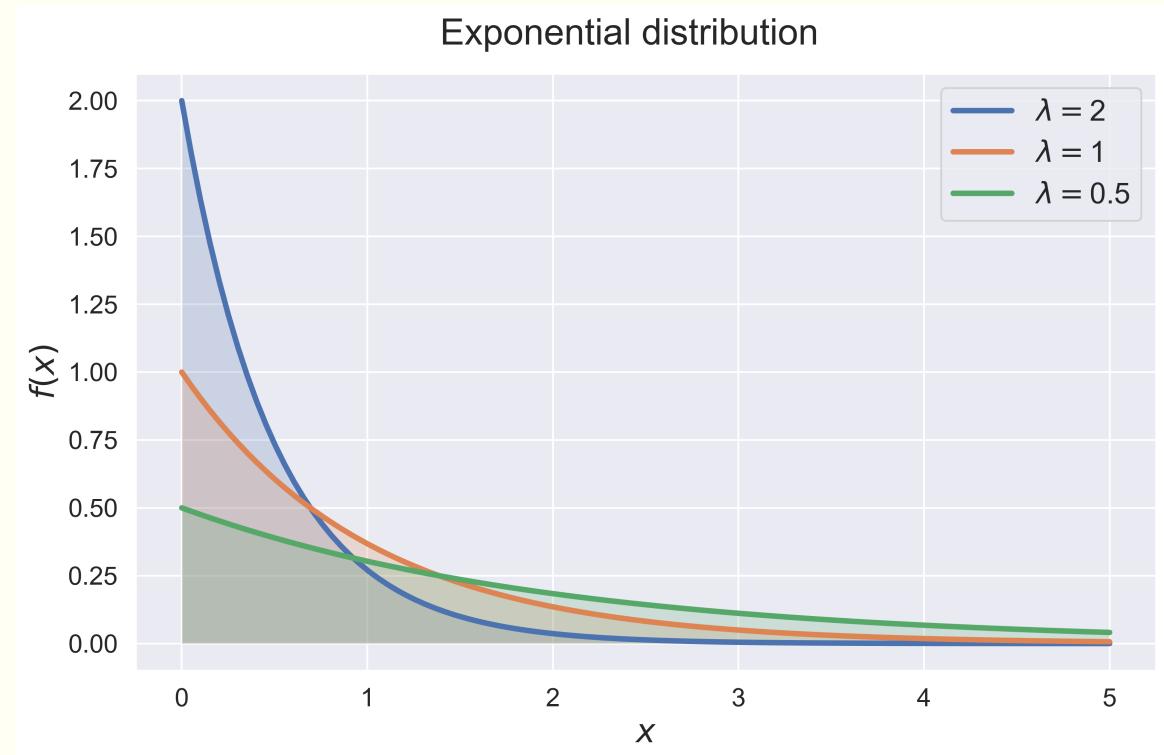


Example 2

Exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$E(X) = \frac{1}{\lambda}$$

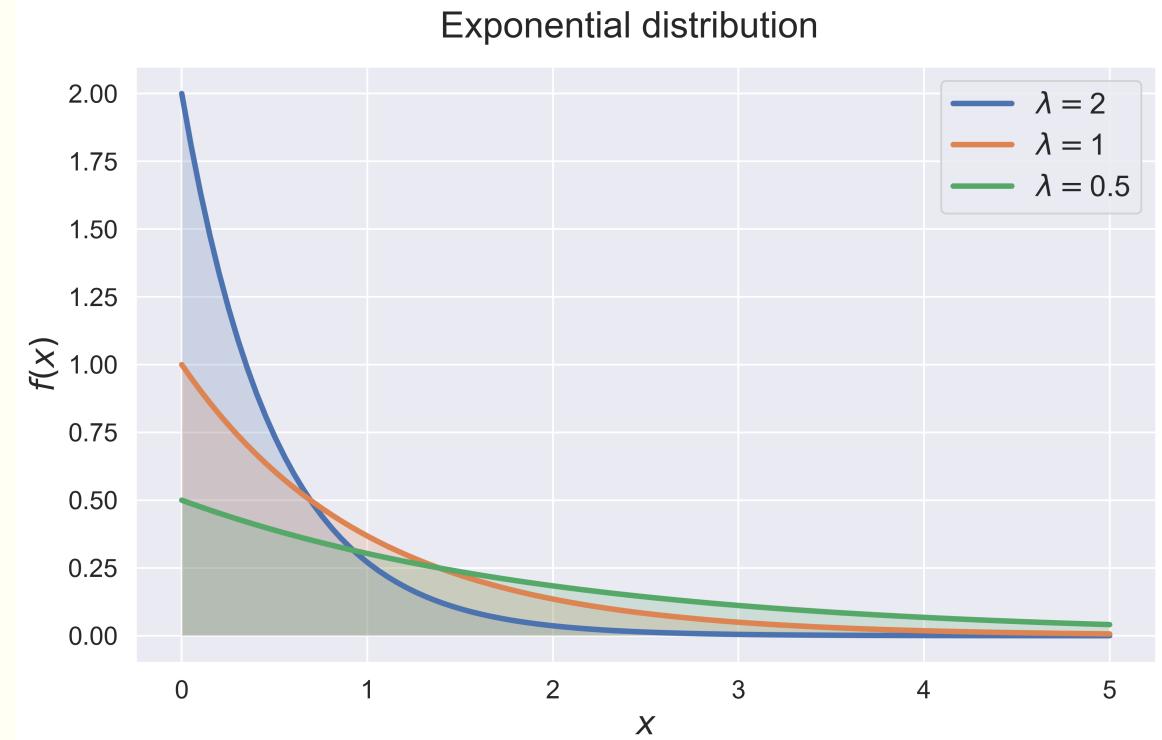


Exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$E(X) = \frac{1}{\lambda}$$

$$\begin{aligned} V(X) &= E(X^2) - \mu^2 = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx - \frac{1}{\lambda^2} \\ &= \int_0^{+\infty} -x^2 de^{-\lambda x} - \frac{1}{\lambda^2} \\ &= -x^2 e^{-\lambda x} \Big|_0^{+\infty} + \int_0^{+\infty} 2x e^{-\lambda x} dx - \frac{1}{\lambda^2} \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \end{aligned}$$



Example 3

Normal distribution

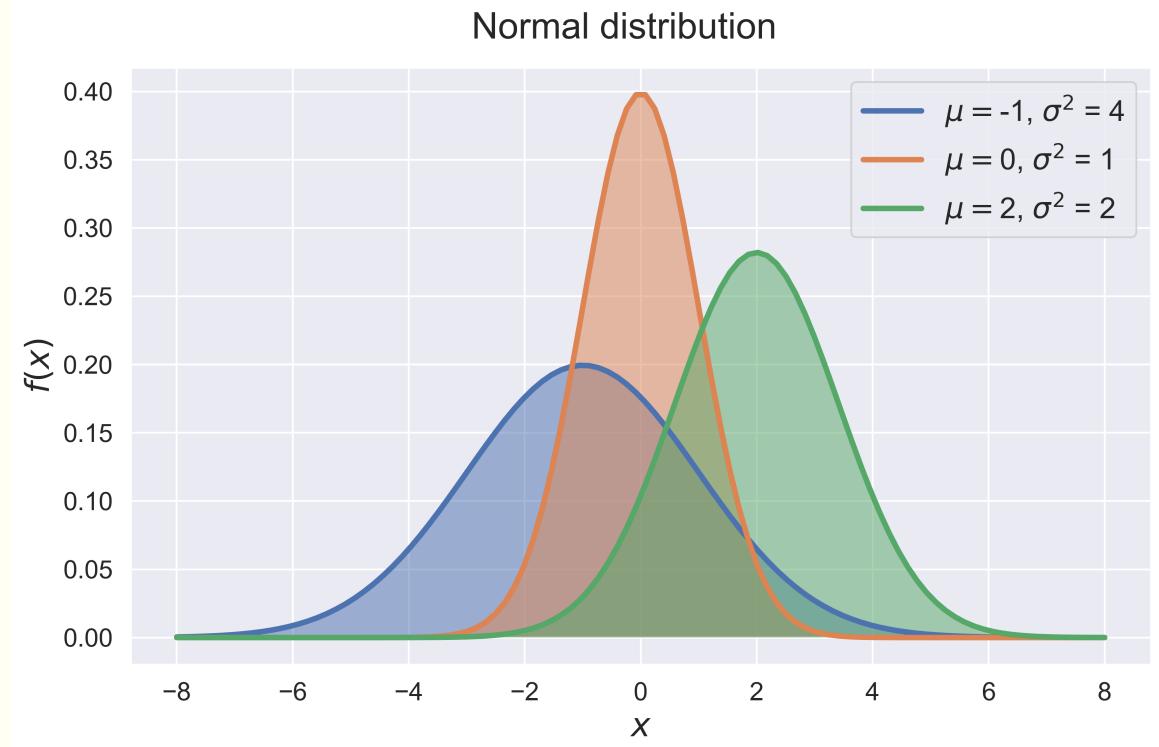
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$E(X) = 0$$

$$V(X) = \dots$$

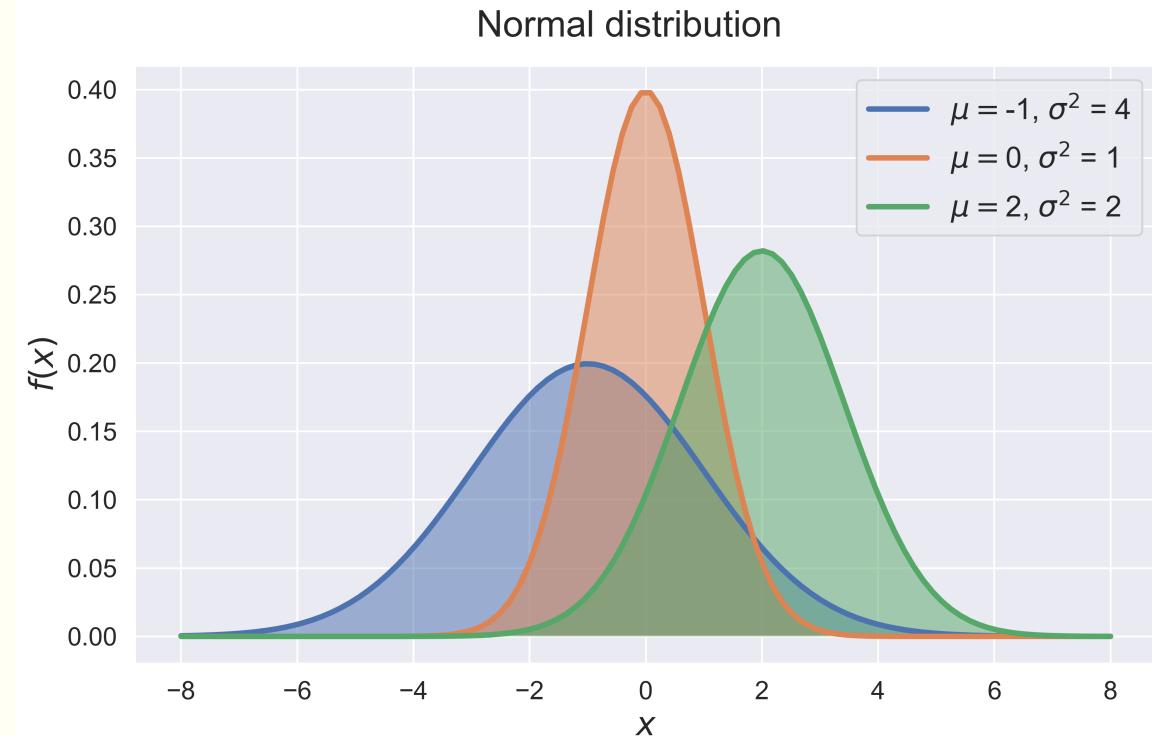


Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$E(X) = 0$$

$$\begin{aligned} V(X) &= E(X^2) - \mu^2 = \int_{-\infty}^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= 2 \int_0^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 2 \int_0^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} d \frac{1}{2}x^2 \\ &= 2 \int_0^{+\infty} -x \frac{1}{\sqrt{2\pi}} de^{-\frac{1}{2}x^2} \\ &= -2x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \Big|_0^{+\infty} + 2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= 2 \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= 1 \end{aligned}$$



Example 3 continued

Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$E(X) = 0$$

$$V(X) = 1$$

Normal distribution

$$Z = \sigma X + \mu$$

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{z-\mu}{\sigma})^2}$$

$$E(Z) = E(\sigma X + \mu) = \mu$$

$$V(X) = \sigma^2$$

Quiz 10: Question 5

Let X be a random variable with range $[-1, 1]$ and let $f_X(x)$ be the density function of X .

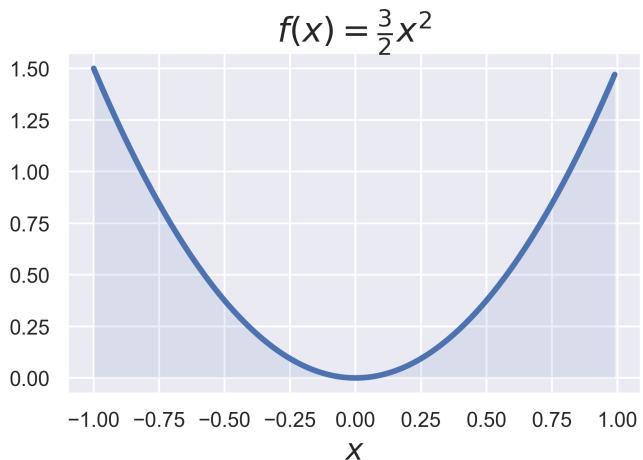
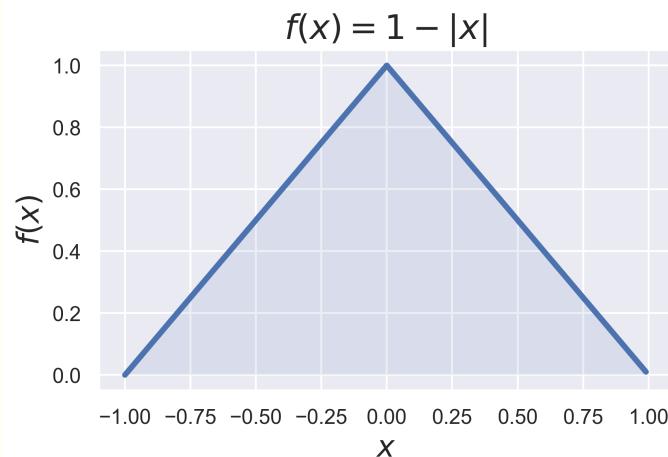
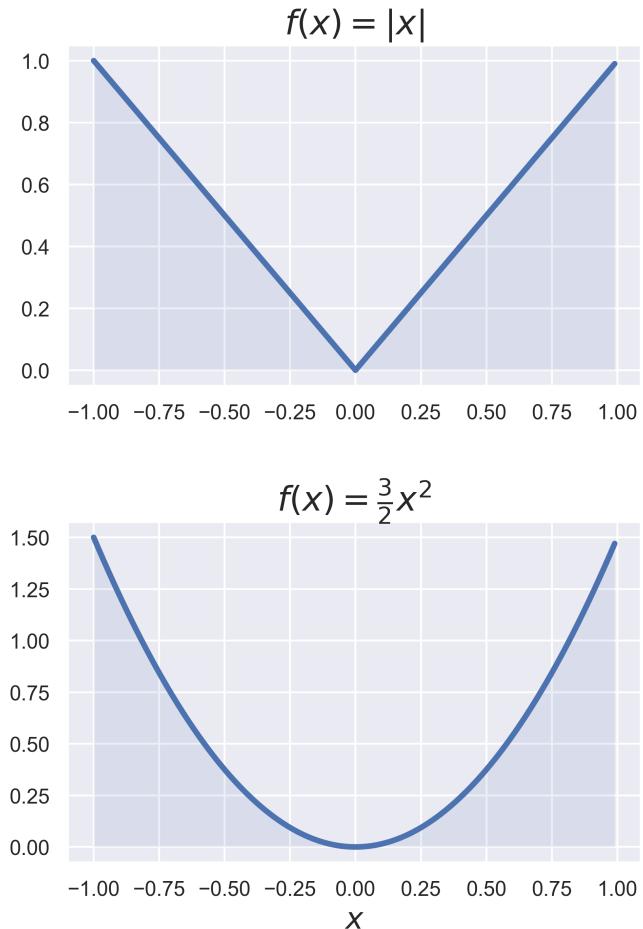
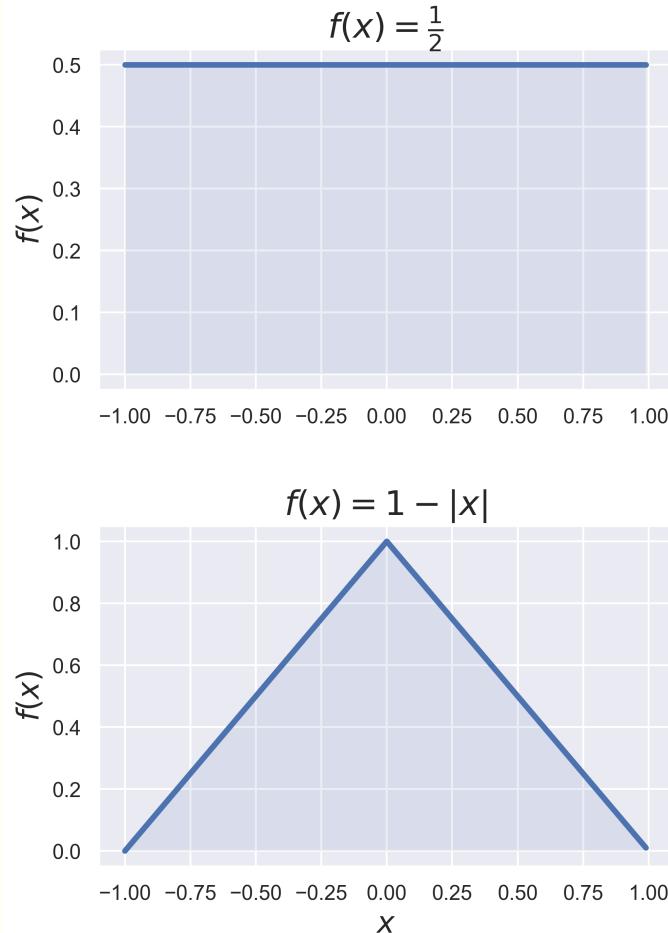
Find $\mu(X)$ and $\sigma^2(X)$ if, for $|X| < 1$,

- $f_X(x) = \frac{1}{2}$
- $f_X(x) = |x|$
- $f_X(x) = 1 - |x|$
- $f_X(x) = \frac{3}{2}x^2$

$$\mu = 0$$

- $\sigma^2 = \frac{1}{3}$
- $\sigma^2 = \frac{1}{2}$
- $\sigma^2 = \frac{1}{6}$
- $\sigma^2 = \frac{3}{5}$

How to compare the variances without calculating them?

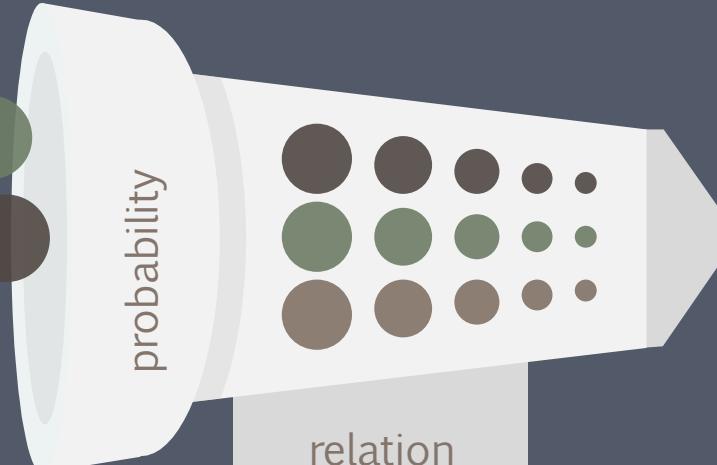


Probability and Statistics

P



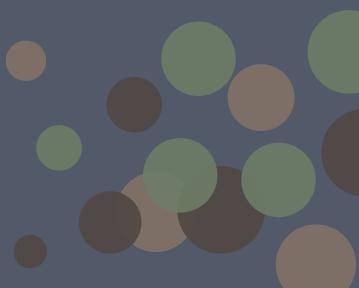
probability



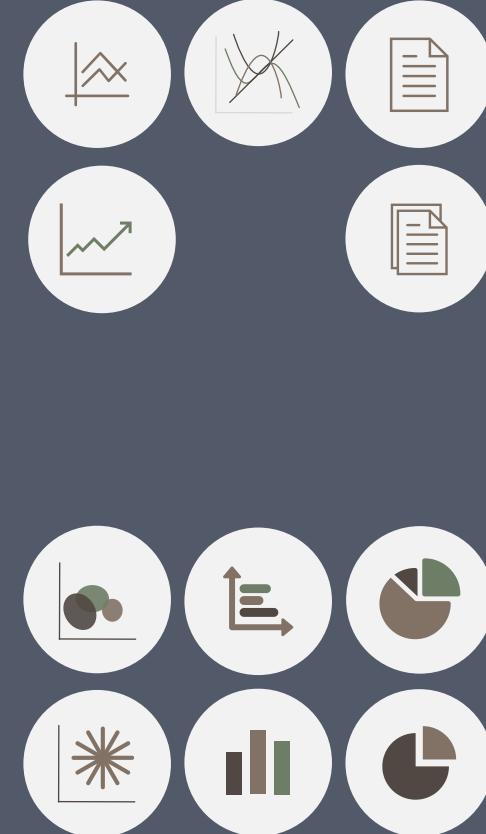
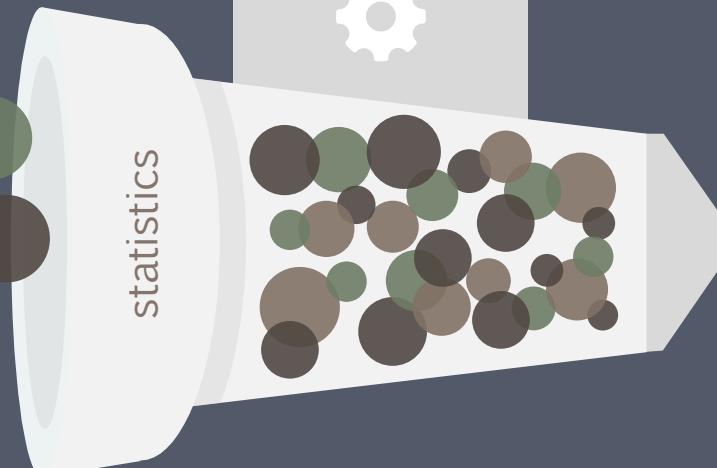
relation



S



statistics



POKEMON

Dataset

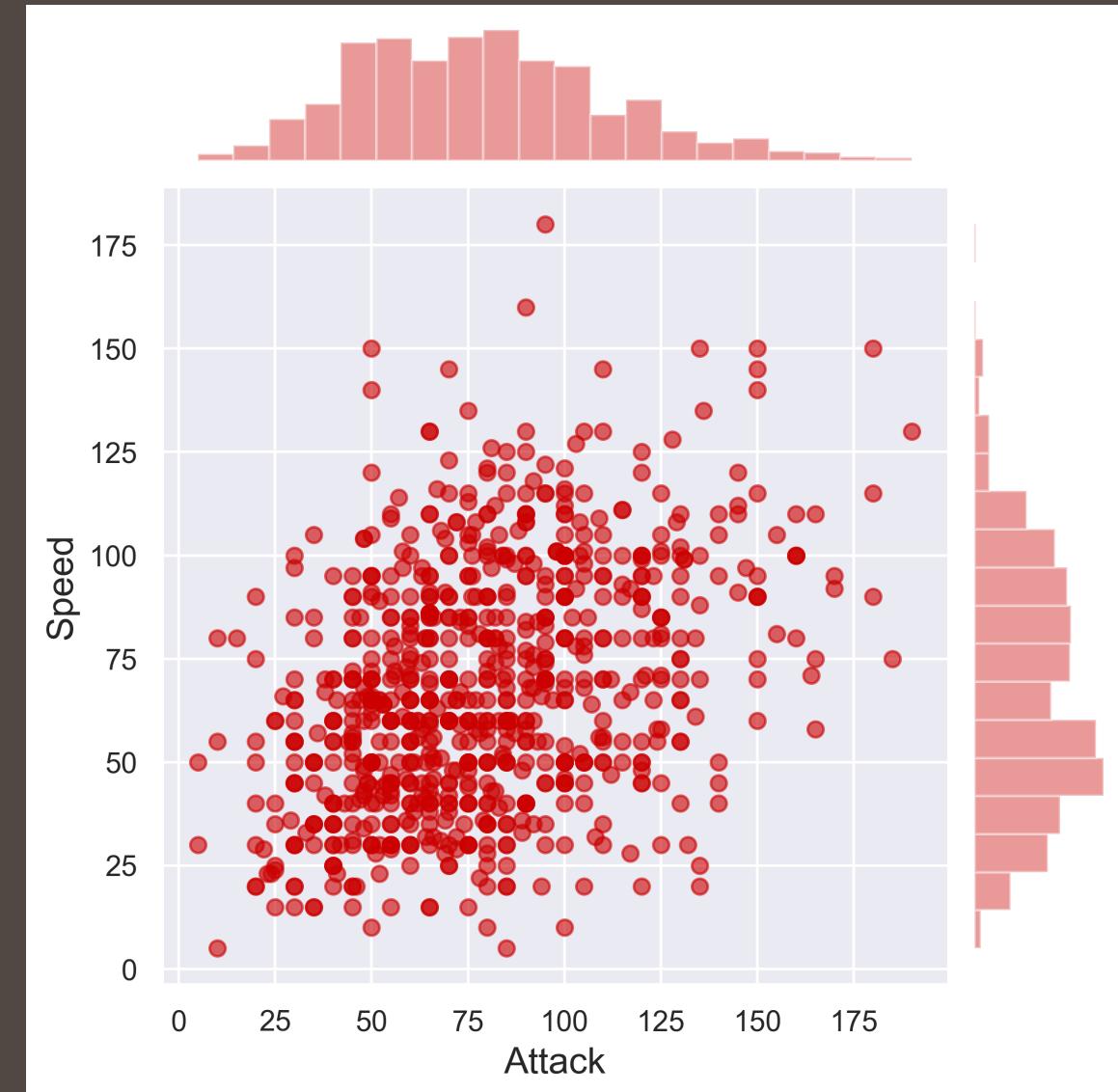
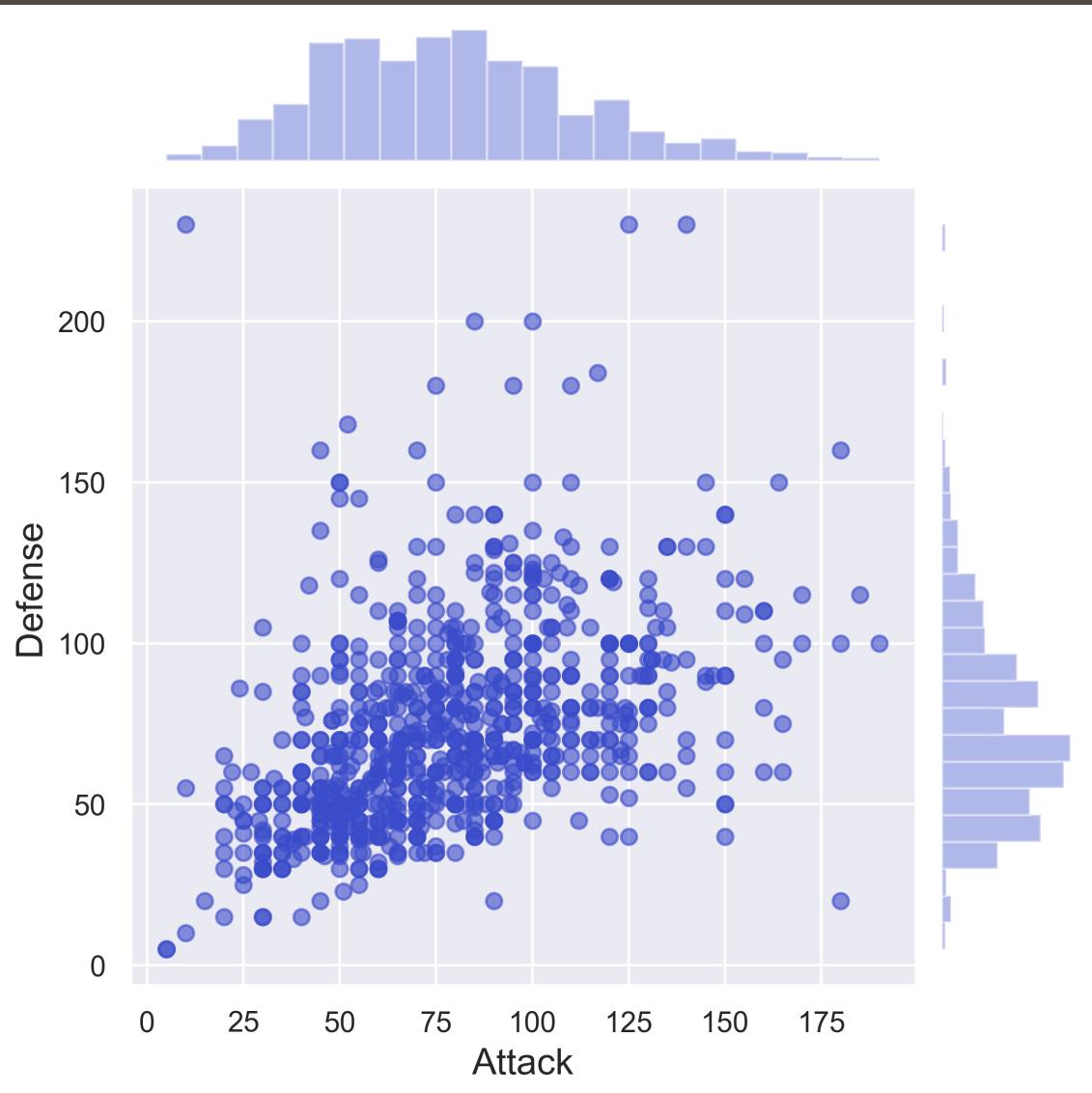


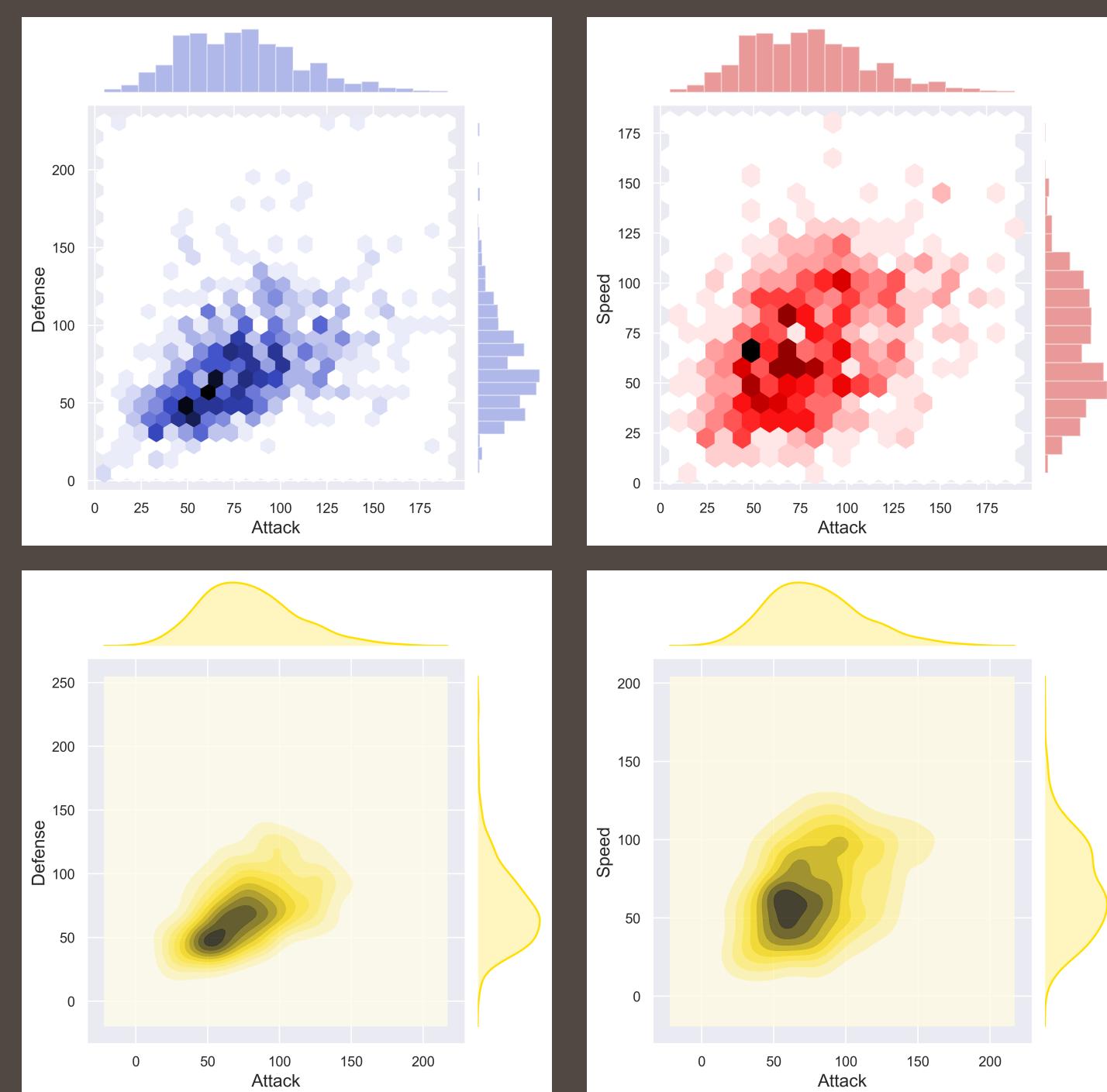
- The data set includes 721 Pokemon, including their number, name, first and second type, and basic stats: HP, Attack, Defense, Special Attack, Special Defense, and Speed.
- This are the raw attributes that are used for calculating how much damage an attack will do in the games.

	Name	Type 1	Type 2	Total	HP	Attack	Defense	Sp. Atk	Sp. Def	Speed	Generation	Legendary
0	Bulbasaur	Grass	Poison	318	45	49	49	65	65	45	1	False
1	Ivysaur	Grass	Poison	405	60	62	63	80	80	60	1	False
2	Venusaur	Grass	Poison	525	80	82	83	100	100	80	1	False
3	VenusaurMega Venusaur	Grass	Poison	625	80	100	123	122	120	80	1	False
4	Charmander	Fire	NaN	309	39	52	43	60	50	65	1	False

Attribute

- Attack: the base modifier for normal attacks (eg. Scratch, Punch)
- Defense: the base damage resistance against normal attacks
- Speed: determines which pokemon attacks first each round





- **Pokemon**

```
figure_pokemon(df = data_pokemon,
               column_A = 'Attack',
               column_B = 'Defense',
               job = 'scatter',
               fs = 16)
```

job = 'hex'

job = 'kde'

Covariance

- Let X and Y be real-valued random variables. The covariance $\text{cov}(X, Y)$ is defined as

$$\text{cov}(X, Y) = E \left((X - \mu(X))(Y - \mu(Y)) \right).$$

- If X and Y are independent, $\text{cov}(X, Y) = 0$. The reverse is not necessarily true.

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y).$$

!

$$V(X + Y) = V(X) + V(Y) + 2\text{cov}(X, Y).$$

!

Proof

Let X and Y be real-valued random variables with expected values $\mu(X) = E(X)$ and $\mu(Y) = E(Y)$.

$$\text{cov}(X, Y) = E((X - \mu(X))(Y - \mu(Y))).$$

$$E(aX + b) = aE(X) + b$$

$$V(X) = E(X^2) - \mu^2$$

$$\begin{aligned}\text{cov}(X, Y) &= E(XY - X\mu(Y) - Y\mu(X) + \mu(X)\mu(Y)) \\ &= E(XY) - \mu(Y)E(X) - \mu(X)E(Y) + \mu(X)\mu(Y) \\ &= E(XY) - E(X)E(Y).\end{aligned}$$

$$\begin{aligned}V(X + Y) &= E((X + Y)^2) - (\mu(X) + \mu(Y))^2 \\ &= E(X^2 + 2XY + Y^2) - (\mu(X))^2 - 2\mu(X)\mu(Y) - (\mu(Y))^2 \\ &= E(X^2) - (\mu(X))^2 + E(Y^2) - (\mu(Y))^2 + 2E(XY) - 2E(X)E(Y) \\ &= V(X) + V(Y) + 2\text{cov}(X, Y).\end{aligned}$$

When do we need independence?

$X + Y$

XY

$\text{COV}(X, Y)$

Do not need

$$E(X + Y) = E(X) + E(Y)$$

$$\text{COV}(X, Y) = 0$$

...

$$X \sim U(-1, 1)$$

$$Y = X^2$$

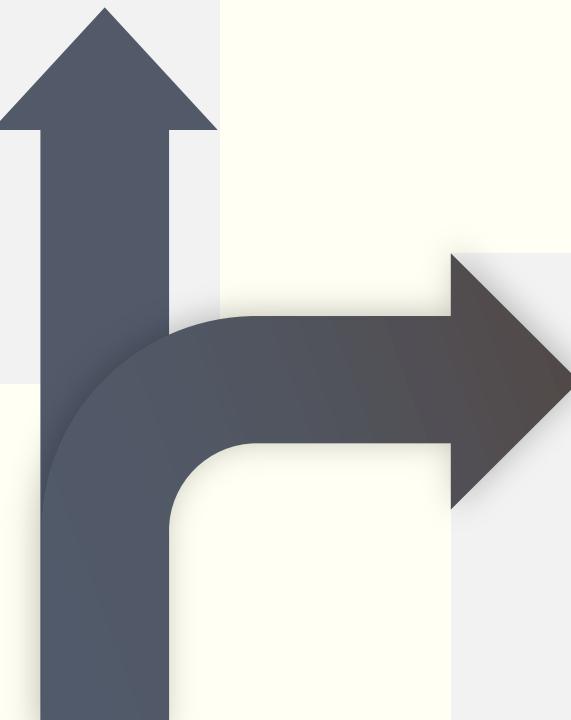
$$\text{COV}(X, Y) = \dots$$

Need

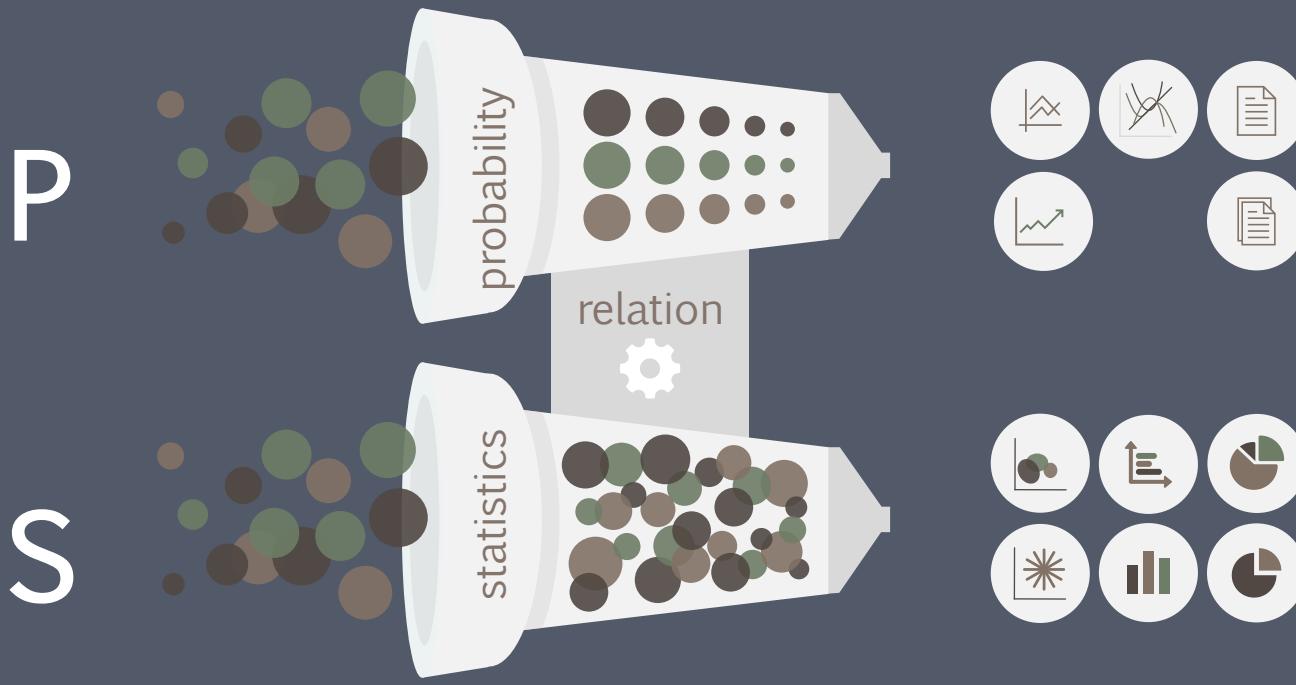
$$E(XY) = E(X)E(Y)$$

$$V(X + Y) = V(X) + V(Y)$$

...



Probability and Statistics



RED WINE

dataset



- The dataset is related to red variants of the Portuguese "Vinho Verde" wine.
- Input variables (based on physicochemical tests):
 - fixed acidity, volatile acidity, citric acid, residual sugar, chlorides, free sulfur dioxide, total sulfur dioxide, density, pH, sulphates and alcohol.
- Output variable (based on sensory data):
 - quality (score between 0 and 10).

	fixed acidity	volatile acidity	citric acid	residual sugar	chlorides	free sulfur dioxide	total sulfur dioxide	density	pH	sulphates	alcohol	quality
0	7.4	0.70	0.00	1.9	0.076	11.0	34.0	0.9978	3.51	0.56	9.4	5
1	7.8	0.88	0.00	2.6	0.098	25.0	67.0	0.9968	3.20	0.68	9.8	5
2	7.8	0.76	0.04	2.3	0.092	15.0	54.0	0.9970	3.26	0.65	9.8	5
3	11.2	0.28	0.56	1.9	0.075	17.0	60.0	0.9980	3.16	0.58	9.8	6
4	7.4	0.70	0.00	1.9	0.076	11.0	34.0	0.9978	3.51	0.56	9.4	5

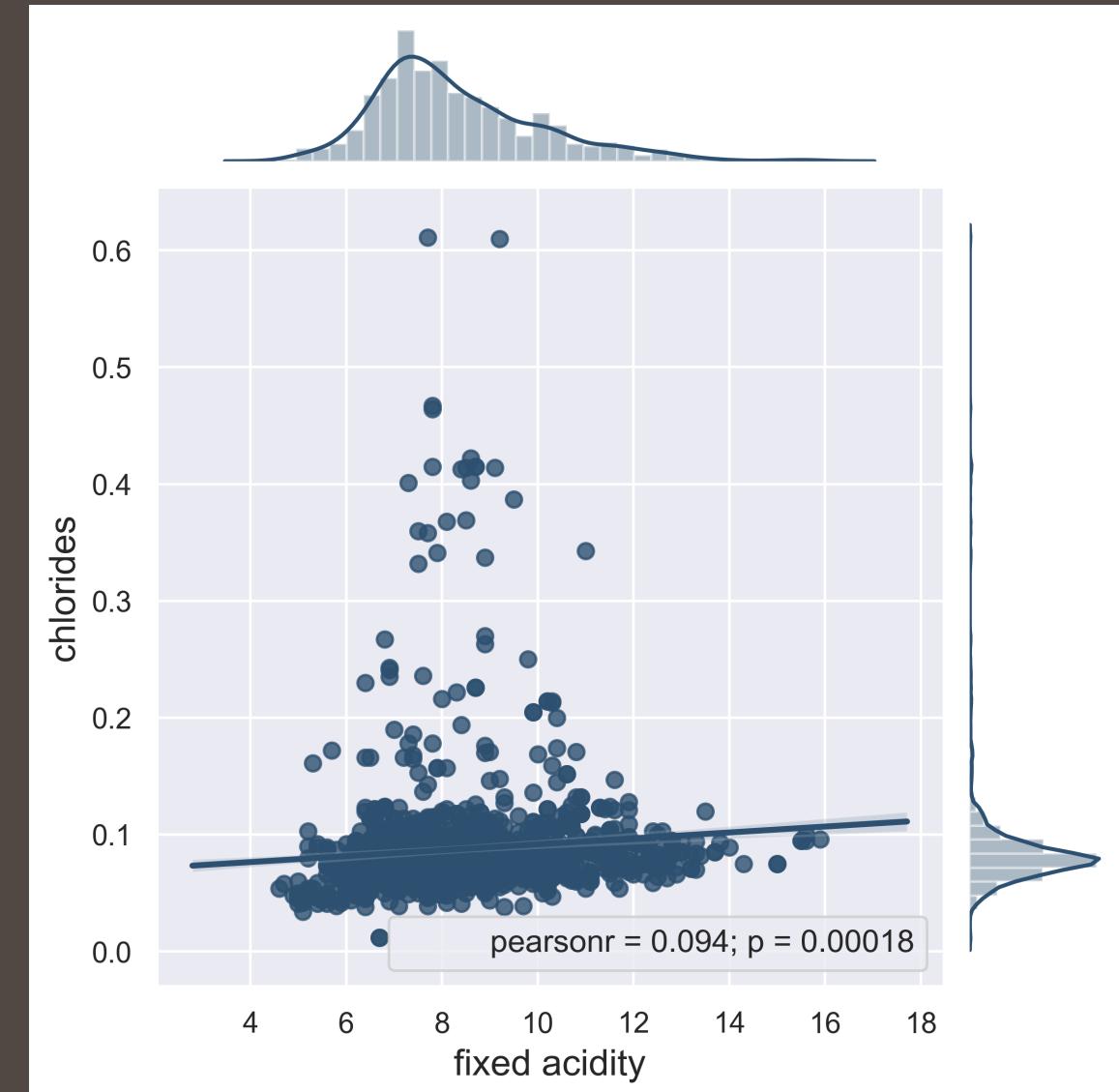
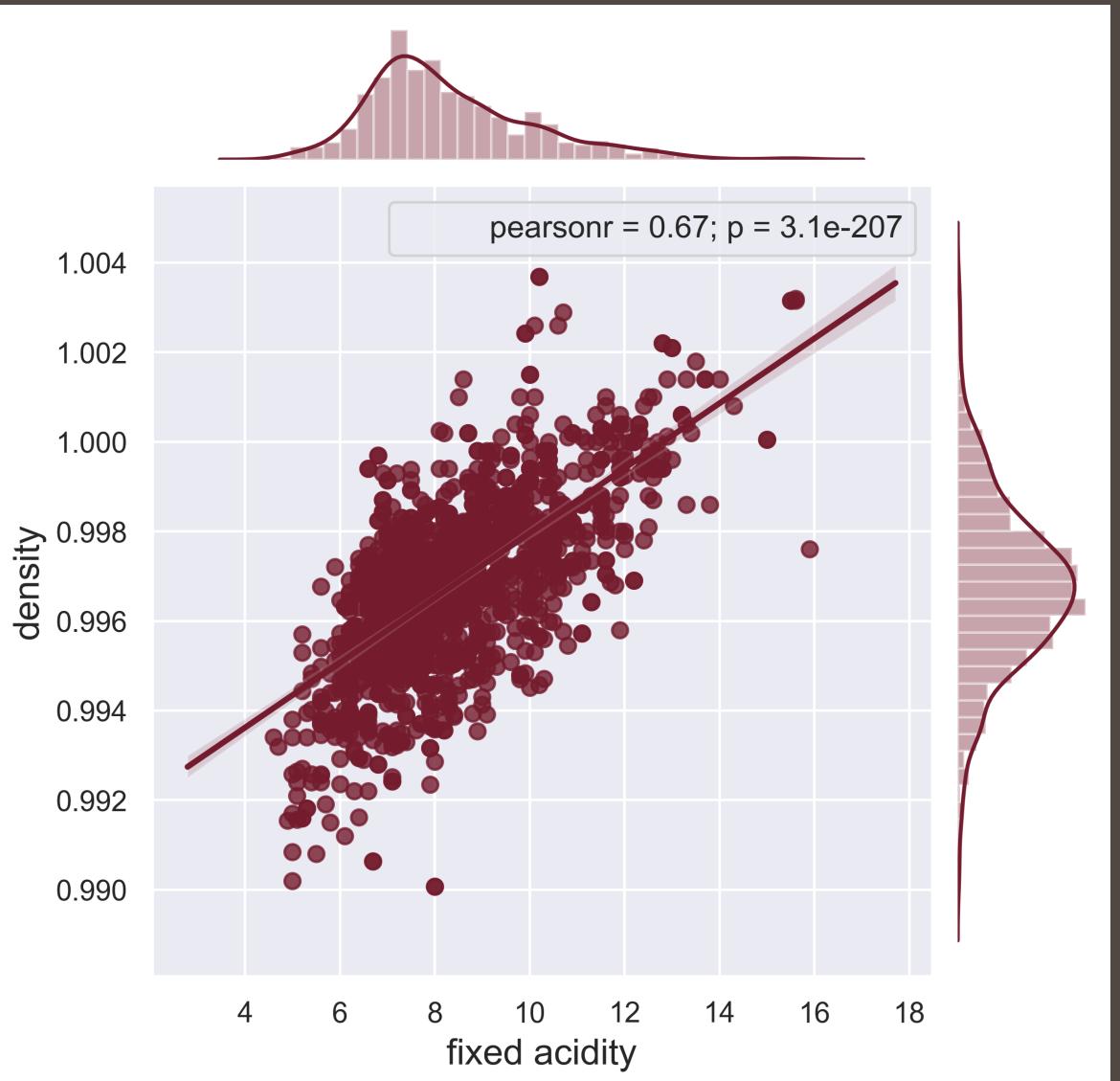
fixed acidity

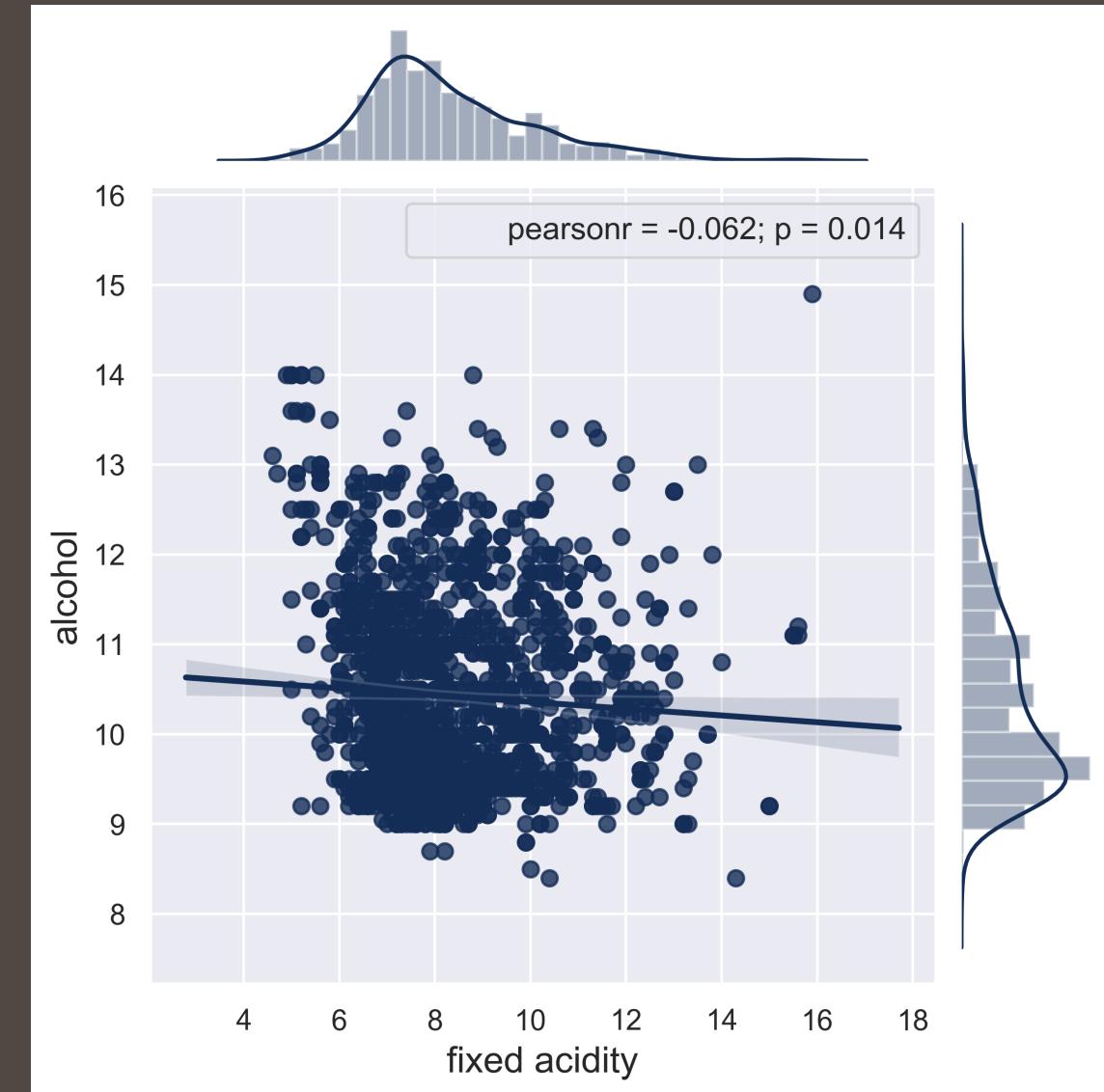
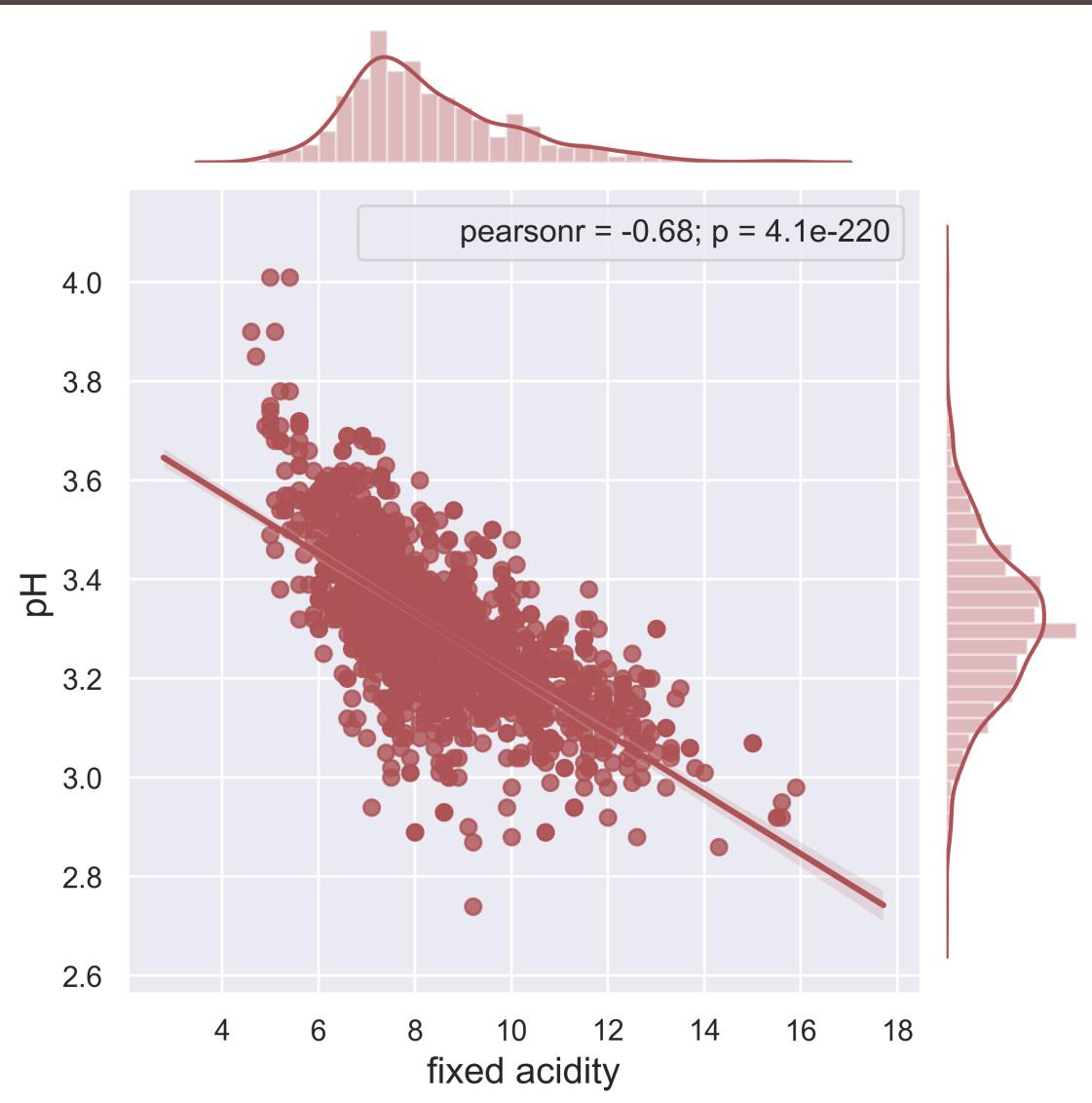
chlorides

density

pH

alcohol





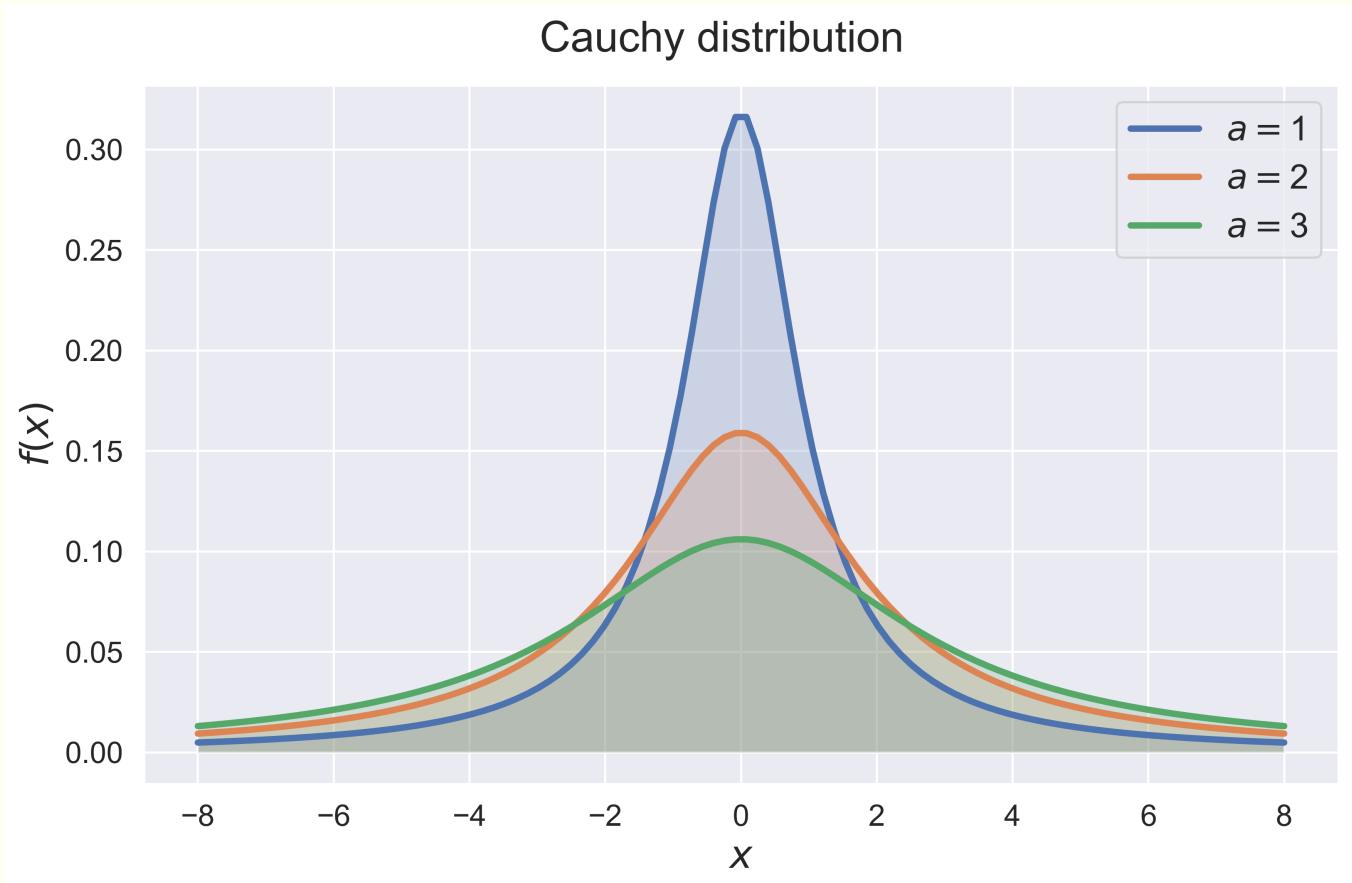
Special example: Cauchy density

Let X be continuous random variable with the Cauchy density function

$$f_X(x) = \frac{a}{\pi} \frac{1}{a^2+x^2}.$$

Does the expectation of X exist?

And its variance?



Special example

Cauchy distribution

$$a = 1$$

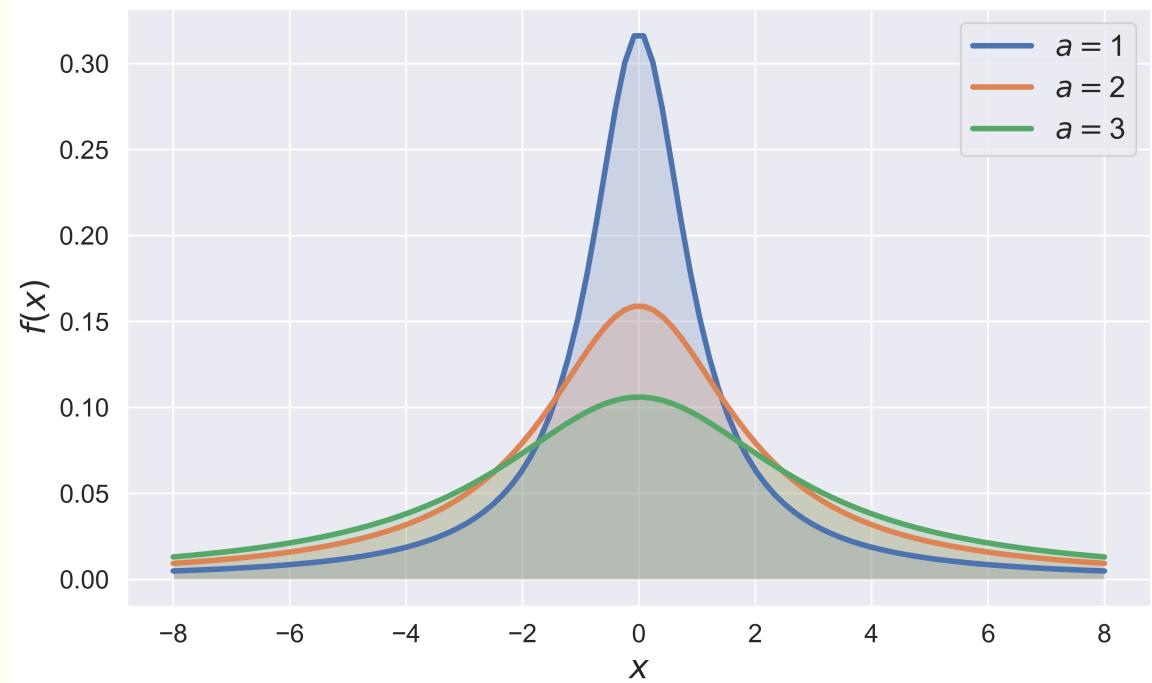
$$f_X(x) = \frac{a}{\pi a^2 + x^2} = \frac{1}{\pi(1 + x^2)}$$

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\pi(1 + x^2)} dx$$

$$\lim_{c \rightarrow +\infty} \int_{-c}^{+c} \frac{x}{\pi(1 + x^2)} dx$$

$$\lim_{c \rightarrow +\infty} \int_c^{2c} \frac{x}{\pi(1 + x^2)} dx$$

Cauchy distribution



Special example

Cauchy distribution

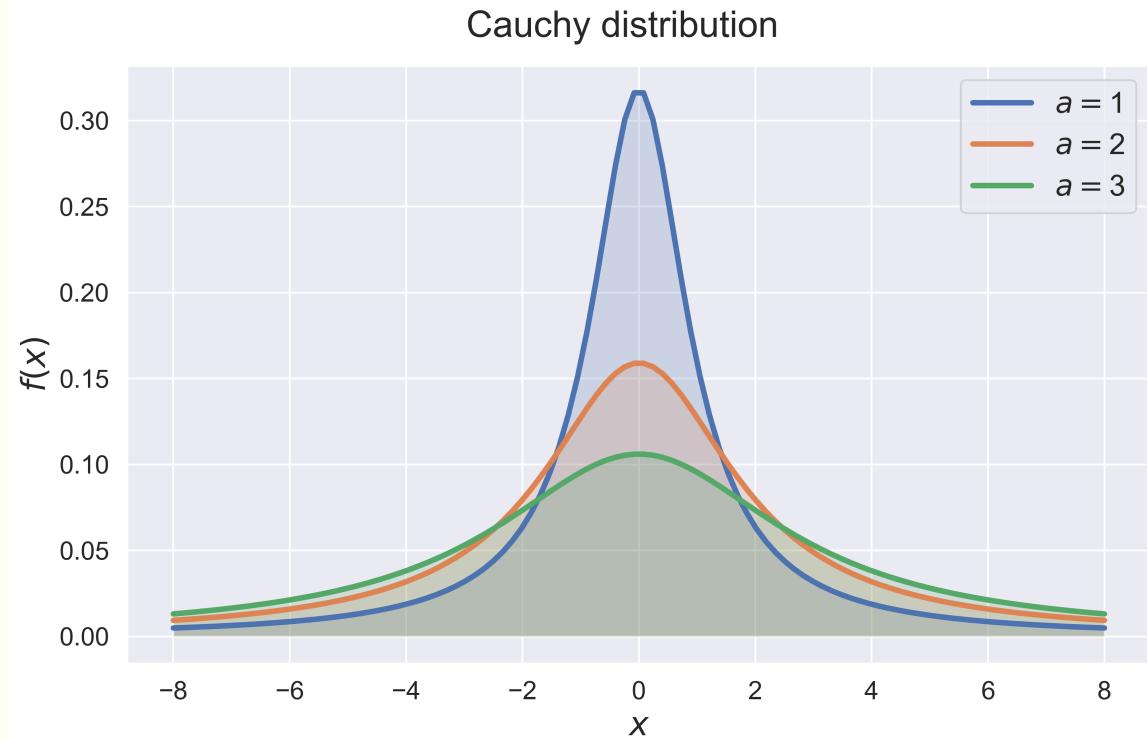
$$a = 1$$

$$f_X(x) = \frac{a}{\pi a^2 + x^2} = \frac{1}{\pi(1 + x^2)}$$

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\pi(1 + x^2)} dx$$

$$\lim_{c \rightarrow +\infty} \int_{-c}^{+c} \frac{x}{\pi(1 + x^2)} dx = 0$$

$$\lim_{c \rightarrow +\infty} \int_c^{2c} \frac{x}{\pi(1 + x^2)} dx = +\infty$$



Special example

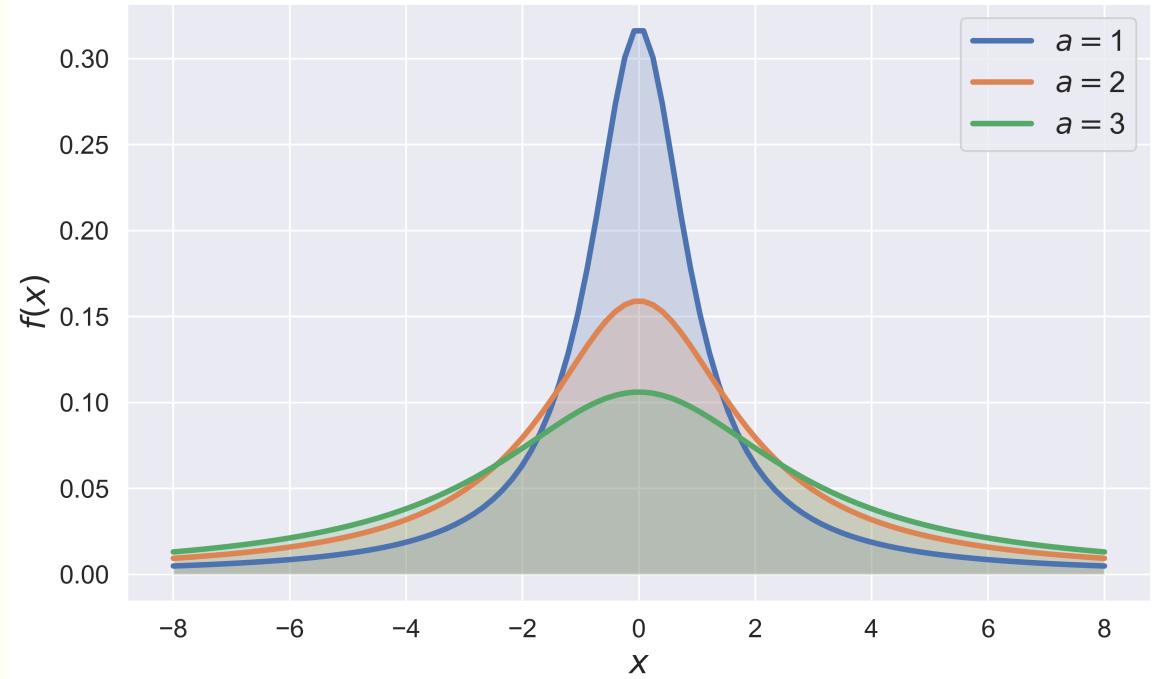
Cauchy distribution

$$a = 1$$

$$f_X(x) = \frac{a}{\pi} \frac{1}{a^2 + x^2} = \frac{1}{\pi(1 + x^2)}$$

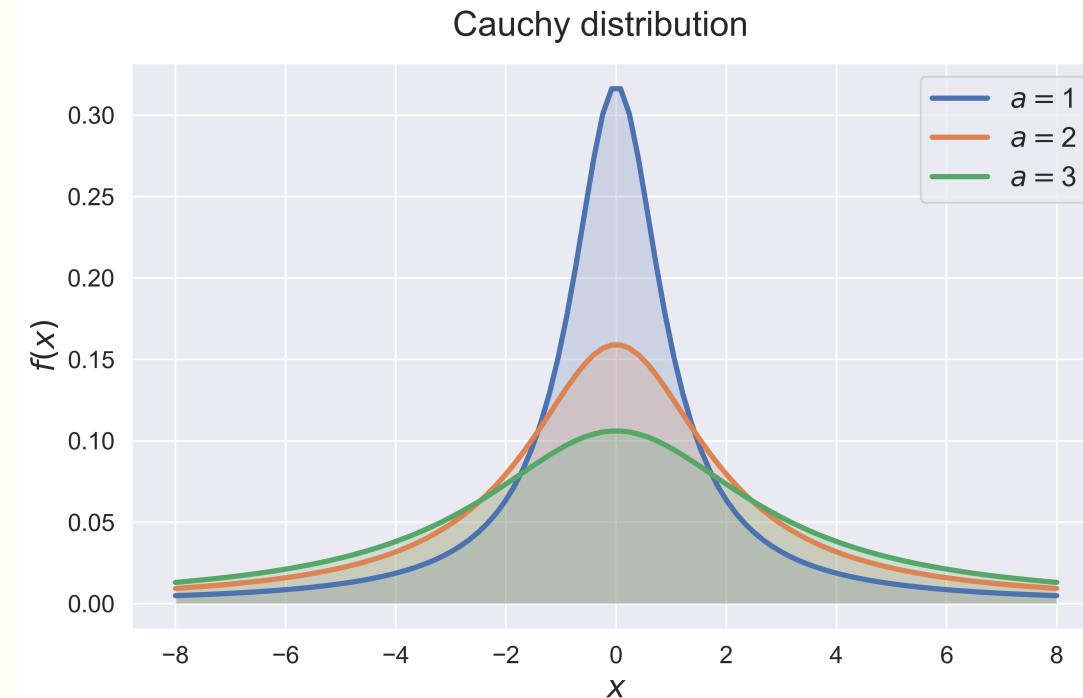
$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} \frac{x^2}{\pi(1 + x^2)} dx \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} 1 - \frac{1}{1 + x^2} dx \\ &= \frac{1}{\pi} \left(\int_{-\infty}^{+\infty} 1 dx - \pi \right) \\ &= +\infty \end{aligned}$$

Cauchy distribution



CAUCHY DENSITY

The Cauchy distribution is often used in statistics as the canonical example of a "pathological" distribution since both its expected value and its variance are undefined.



Final example



- A game is played as follows: a random number X is chosen uniformly from $[0, 1]$.
- Then a sequence Y_1, Y_2, \dots of random numbers is chosen independently and uniformly from $[0, 1]$.
- The game ends the first time that $Y_i > X$. You are paid $(i - 1)$ dollars.
- What is a fair entrance fee for this game?