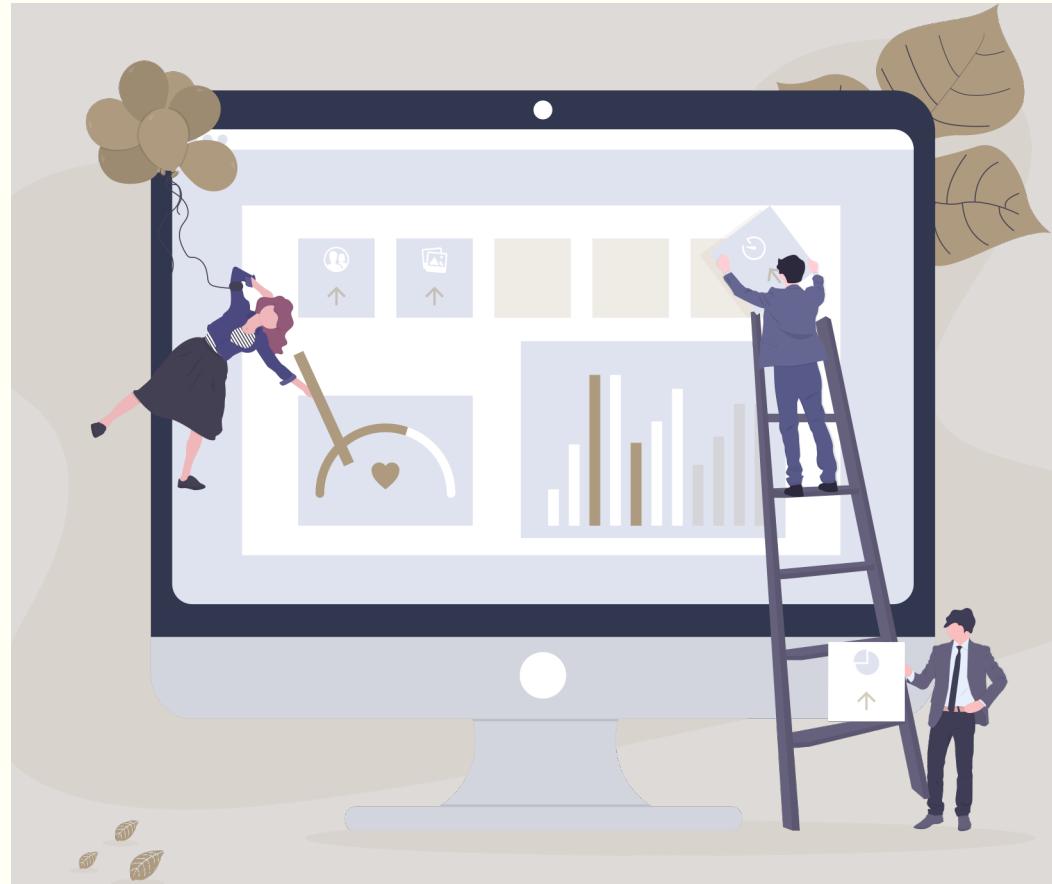




# MATH 20: PROBABILITY

Variance of Discrete Random Variables

Xingru Chen  
[xingru.chen.gr@dartmouth.edu](mailto:xingru.chen.gr@dartmouth.edu)



# Important Distributions

Discrete Uniform Distribution

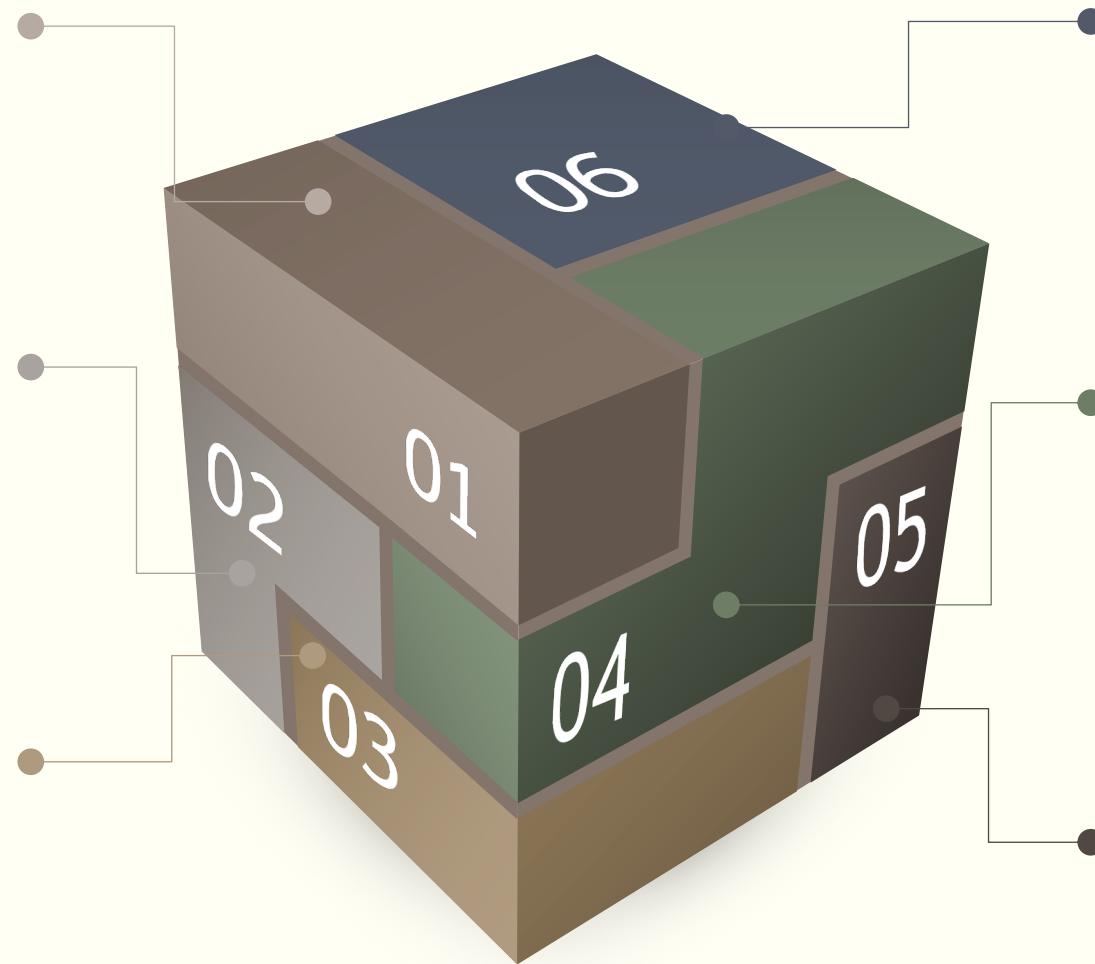
$$m(\omega) = \frac{1}{n}$$

Binomial Distribution

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

Geometric Distribution

$$P(T = n) = q^{n-1} p$$



Hypergeometric Distribution

$$h(N, k, n, x) = \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}}$$

Negative Binomial Distribution

$$u(x, k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

## Binomial

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k} \quad E(X) = np$$

## Geometric

$$P(T = n) = q^{n-1} p \quad E(X) = \frac{1}{p}$$

## Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad E(X) = \lambda$$

## Negative binomial

$$u(x, k, p) = \binom{x-1}{k-1} p^k q^{x-k} \quad E(X) = k \frac{q}{p}$$

## Hypergeometric

$$h(N, k, n, x) = \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}} \quad E(X) = n \frac{k}{N}$$

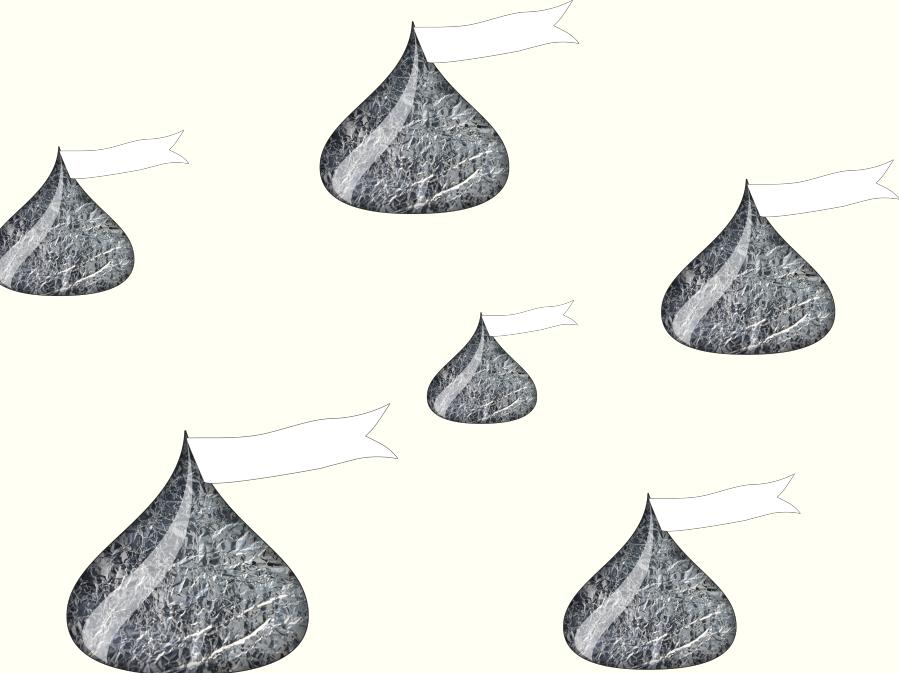
# How Much Does a Hershey Kiss Weight?

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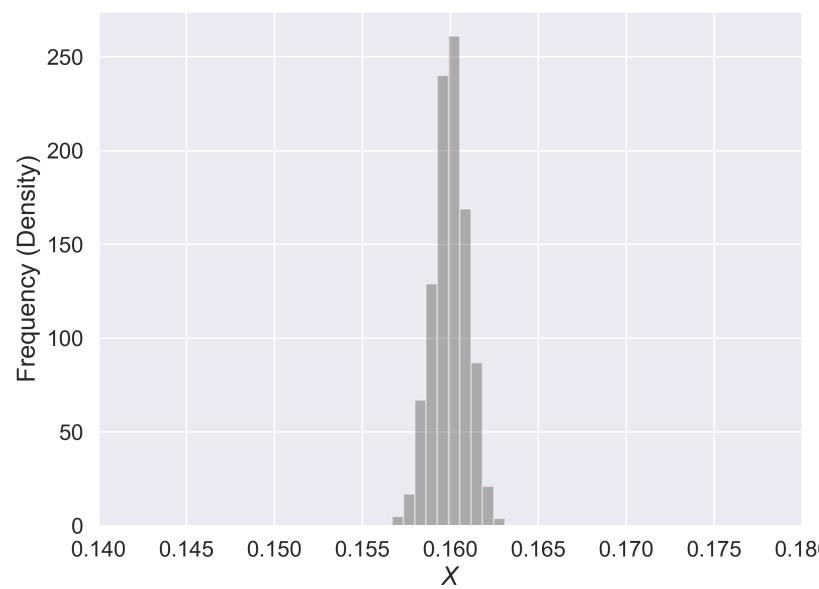
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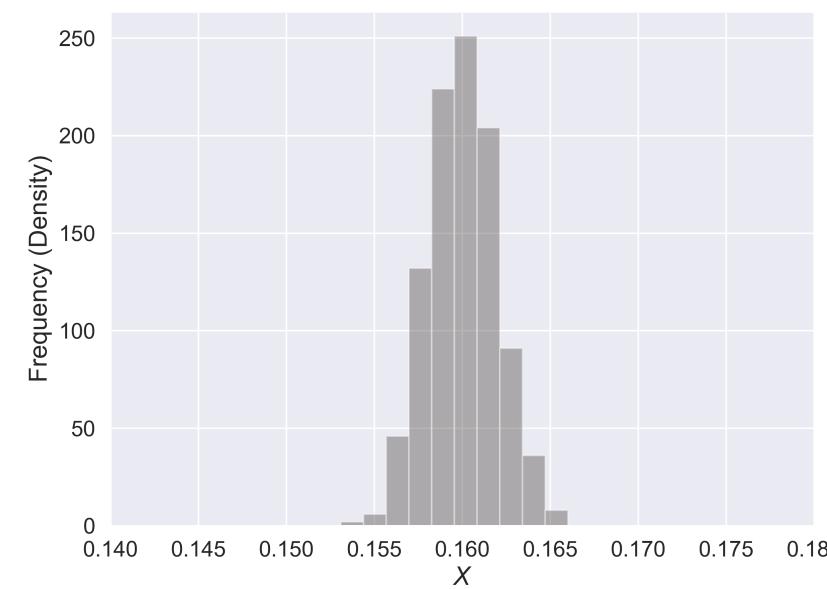
- A single standard Hershey's Kiss weighs 0.16 ounces.



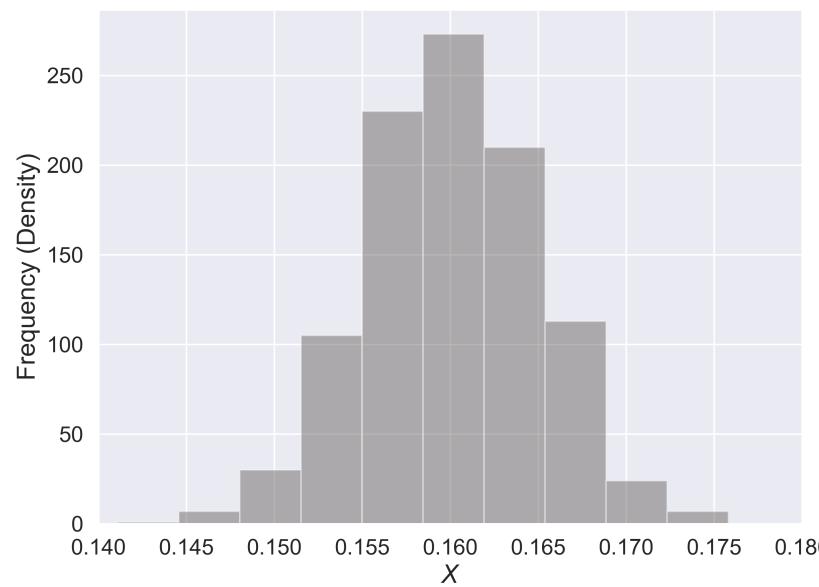
Grade A,  $\sigma = 0.001$



Grade B,  $\sigma = 0.002$



Grade C,  $\sigma = 0.005$



Same  $\mu$ , different  $\sigma$ .

How to measure the quality of products?

?

# Home Made Versus Factory Made



# Variance of Discrete Random Variables

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- Let  $X$  be a numerically-valued random variable with expected value  $\mu = E(X)$ . Then the variance of  $X$ , denoted by  $V(X)$ , is

$$V(X) = E((X - \mu)^2).$$

Expected value  $E(X)$

$$\sum_{x \in \Omega} xm(x)$$

Variance  $V(X)$

$$\sum_{x \in \Omega} (x - \mu)^2 m(x)$$

Expected value  $E(\phi(X))$

$$\sum_{x \in \Omega} \phi(x)m(x)$$

# Variance of Discrete Random Variables

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- Let  $X$  be a numerically-valued random variable with expected value  $\mu = E(X)$ . Then the variance of  $X$ , denoted by  $V(X)$ , is

$$V(X) = E((X - \mu)^2).$$

- Standard deviation of  $X$ , denoted by  $D(X)$ , is  $D(X) = \sqrt{V(X)}$ .
- We often write  $\sigma$  for  $D(X)$  and  $\sigma^2$  for  $V(X)$ .

Variance  $V(X)$

$$\sum_{x \in \Omega} (x - \mu)^2 m(x)$$

# Variance of Discrete Random Variables

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- Let  $X$  be a numerically-valued random variable with expected value  $\mu = E(X)$ . Then the variance of  $X$ , denoted by  $V(X)$ , is

$$V(X) = E((X - \mu)^2).$$

Variance  $V(X)$

$$\sum_{x \in \Omega} (x - \mu)^2 m(x)$$

- If  $X$  is any random variable with  $\mu = E(X)$ , then

$$V(X) = E(X^2) - \mu^2.$$

$$\sum_{x \in \Omega} x^2 m(x) - \sum_{x \in \Omega} xm(x)$$

# Proof

Let  $X$  be a numerically-valued random variable with expected value  $\mu = E(X)$ .

$$V(X) = \sum_{x \in \Omega} (x - \mu)^2 m(x)$$

$$\sum_{x \in \Omega} x^2 m(x) = E(X^2)$$

$$\sum_{x \in \Omega} xm(x) = \mu$$

$$\sum_{x \in \Omega} m(x) = 1$$

$$V(X) = \sum_{x \in \Omega} (x - \mu)^2 m(x)$$

$$= \sum_{x \in \Omega} (x^2 - 2\mu x + \mu^2) m(x)$$
$$= \sum_{x \in \Omega} x^2 m(x) - 2\mu \sum_{x \in \Omega} xm(x) + \mu^2 \sum_{x \in \Omega} m(x)$$

$$V(X) = \sum_{x \in \Omega} x^2 m(x) - 2\mu \sum_{x \in \Omega} xm(x) + \mu^2 \sum_{x \in \Omega} m(x)$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

## Example 1

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Toss a coin

head or tail

1 or 0

$$m(x) = \frac{1}{2}$$

$$\mu = E(X) = \frac{1}{2}$$

$$\sigma^2 = V(X) = \dots$$



## Example 1

---

---

Toss a coin

head or tail

1 or 0

$$m(x) = \frac{1}{2}$$

$$\mu = E(X) = \frac{1}{2}$$

$$\sigma^2 = V(X) = \frac{1}{4}$$



## Example 2

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Roll a dice

1, 2, 3, 4, 5, or 6

$$m(x) = \frac{1}{6}$$

$$\mu = E(X) = \frac{7}{2}$$

$$\sigma^2 = V(X) = \dots$$



## Example 2

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Roll a dice

1, 2, 3, 4, 5, or 6

$$m(x) = \frac{1}{6}$$

$$\mu = E(X) = \frac{7}{2}$$

$$\sigma^2 = V(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$



## Quiz 9

What is the expected number of dice tosses needed to get two consecutive six's?

Number of tosses

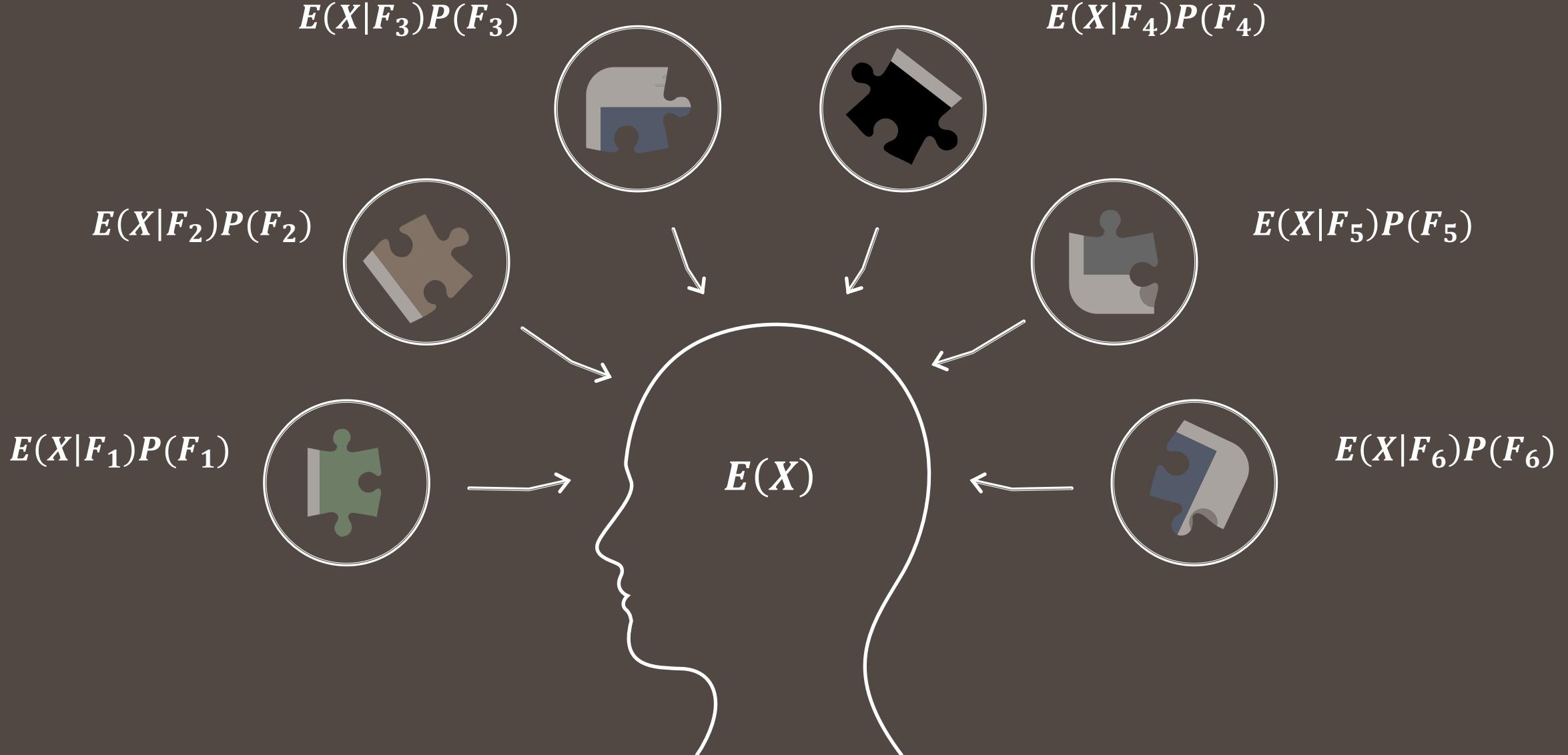
2, 3, 4, ...

$$E(X) = E(X|1)P(1) + E(X|2) + E(X|3)P(3) + E(X|4)P(4) + E(X|5)P(5) + E(X|6)P(6)$$

$$E(X) = \frac{5}{6}E(X|1) + \frac{1}{6}E(X|6) = \frac{5}{6}(1 + E(X)) + \frac{1}{6}\left(\frac{5}{6}E(X|61) + \frac{1}{6}E(X|66)\right)$$

$$E(X) = \frac{5}{6}(1 + E(X)) + \frac{1}{6}\left[\frac{5}{6}(2 + E(X)) + \frac{2}{6}\right]$$

# Is the distribution function a must for calculating expectation?



## Example 3

Consider the general Bernoulli trial process. As usual, we let  $X = 1$  if the outcome is a success and 0 if it is a failure.

### Bernoulli trial

$$m(x) = \begin{cases} p, & X = 1 \\ 1 - p, & X = 0 \end{cases}$$

### Expected value $E(X)$

$$\sum_{x \in \Omega} xm(x) = 1 \times p + 0 \times (1 - p) = p$$

### Variance $V(X)$

$$E(X^2) - \mu^2 = \dots$$

## Example 3

Consider the general Bernoulli trial process. As usual, we let  $X = 1$  if the outcome is a success and  $0$  if it is a failure.

### Bernoulli trial

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$$\sum_{x \in \Omega} xm(x) = 1 \times p + 0 \times (1 - p) = p$$

### Variance $V(X)$

$$E(X^2) - \mu^2 = p - p^2$$

## Binomial

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

$$E(X) = np, V(X) = npq$$

## Geometric

$$P(T = n) = q^{n-1} p$$

$$E(X) = \frac{1}{p}, V(X) = \frac{1-p}{p^2}$$

## Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$E(X) = \lambda, V(X) = \lambda$$

## Negative binomial

$$u(x, k, p) = \binom{x-1}{k-1} p^k q^{x-k}$$

$$E(X) = k \frac{q}{p}, V(X) = k \frac{q}{p^2}$$

## Hypergeometric

$$h(N, k, n, x) = \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}}$$

$$E(X) = n \frac{k}{N}, V(X) = \frac{nk(N-k)(N-n)}{N^2(N-1)}$$

# Binomial Distribution and Poisson Distribution

## Binomial Distribution

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

$$E(X) = np, V(X) = npq$$

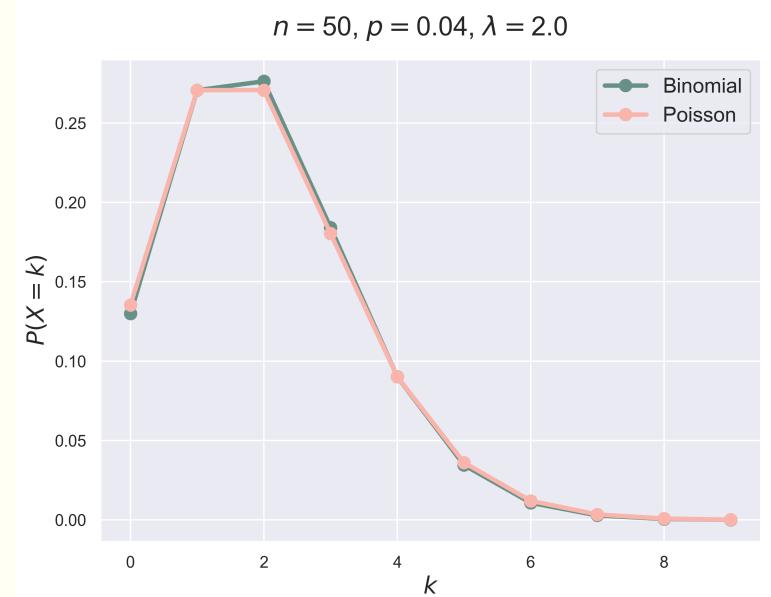
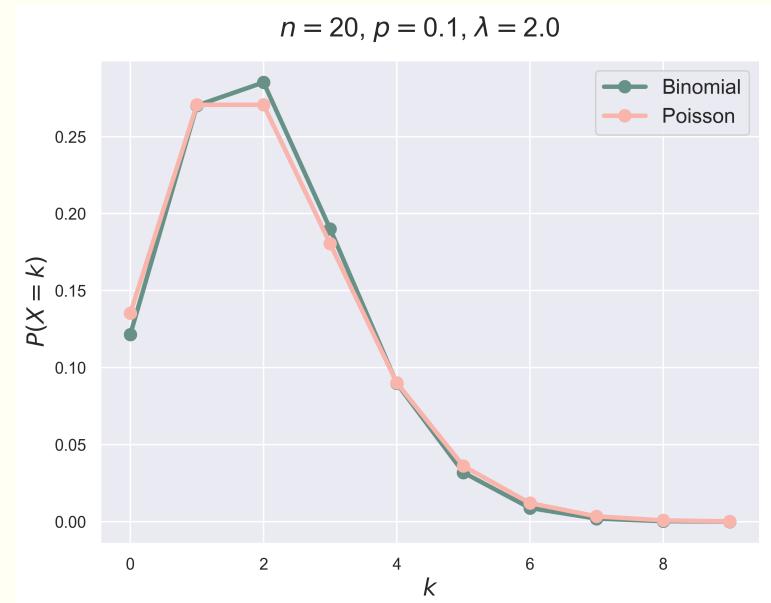
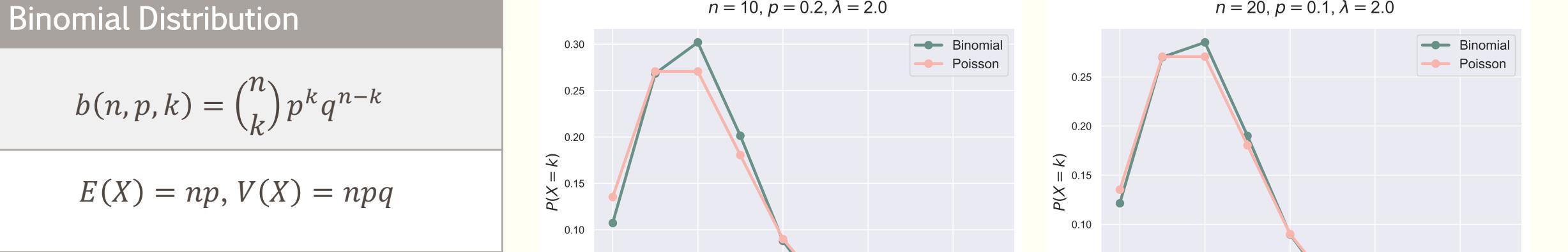
## Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$E(X) = \lambda, V(X) = \lambda$$

$$p = \frac{\lambda t}{n}, t = 1, n \rightarrow \infty, p \rightarrow 0$$

=



## Binomial

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k} \quad E(X) = np$$

### Variance $V(X)$

$$E(X^2) - \mu^2 = \sum_{x \in \Omega} x^2 m(x) - \mu^2$$

$$\begin{aligned} \sum_{x \in \Omega} x^2 m(x) &= \sum_{k=0}^n k^2 \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^n k^2 \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^n k^2 \frac{n!}{k! (n-k)!} p^k q^{n-k} \\ &= \sum_{k=1}^n npk \frac{(n-1)!}{(k-1)! (n-k)!} p^{k-1} q^{n-k} = np \sum_{k=1}^n (k-1+1) \binom{n-1}{k-1} p^{k-1} q^{n-k} \\ &= np \sum_{l=0}^{n-1} l \binom{n-1}{l} p^l q^{n-1-l} + np \sum_{l=0}^{n-1} \binom{n-1}{l} p^l q^{n-1-l} \end{aligned}$$

## Binomial

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k} \quad E(X) = np$$

### Variance $V(X)$

$$E(X^2) - \mu^2 = \sum_{x \in \Omega} x^2 m(x) - \mu^2$$

- $\sum_{x \in \Omega} x^2 m(x) = n(n-1)p^2 \sum_{l=1}^{n-1} \binom{n-2}{l-1} p^{l-1} q^{n-1-l} + np = n(n-1)p^2 + np$
- $\mu^2 = n^2 p^2$
- $\sum_{x \in \Omega} x^2 m(x) - \mu^2 = n(n-1)p^2 + np - n^2 p^2 = np(1-p)$

## Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad E(X) = \lambda$$

Variance  $V(X)$

$$E(X^2) - \mu^2 = \sum_{x \in \Omega} x^2 m(x) - \mu^2$$

$$\begin{aligned} \sum_{x \in \Omega} x^2 m(x) &= \sum_{k=0}^{+\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{+\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \sum_{k=1}^{+\infty} \lambda k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \sum_{k=1}^{+\infty} \lambda(k-1+1) \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\ &= \lambda \sum_{l=0}^{+\infty} l \frac{\lambda^l}{l!} e^{-\lambda} + \lambda \sum_{l=0}^{+\infty} \frac{\lambda^l}{l!} e^{-\lambda} \end{aligned}$$

## Poisson

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad E(X) = \lambda$$

Variance  $V(X)$

$$E(X^2) - \mu^2 = \sum_{x \in \Omega} x^2 m(x) - \mu^2$$

- $\sum_{x \in \Omega} x^2 m(x) = \lambda \sum_{l=0}^{+\infty} l \frac{\lambda^l}{l!} e^{-\lambda} + \lambda \sum_{l=0}^{+\infty} \frac{\lambda^l}{l!} e^{-\lambda} = \lambda^2 \sum_{l=1}^{+\infty} \frac{\lambda^{l-1}}{(l-1)!} e^{-\lambda} + \lambda = \lambda^2 + \lambda$
- $\mu^2 = \lambda^2$
- $\sum_{x \in \Omega} x^2 m(x) - \mu^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

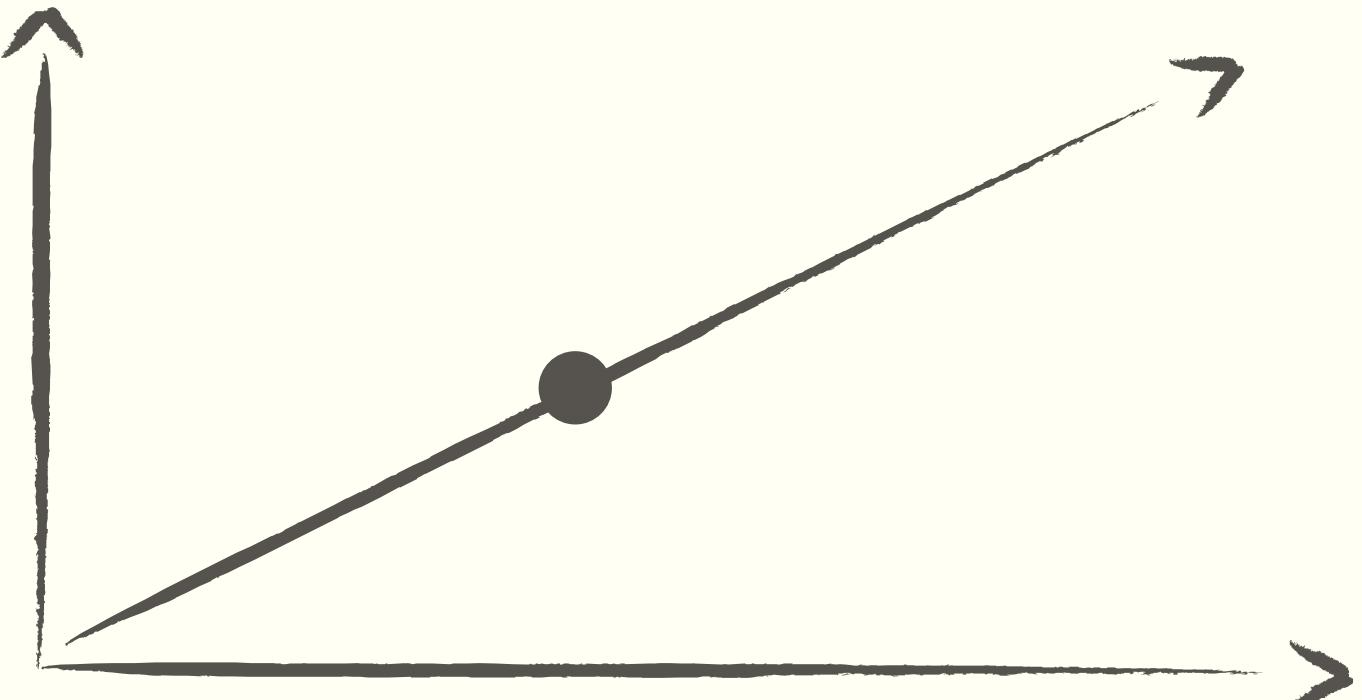
# Linearity

$$E(X + Y) = E(X) + E(Y)$$

$$E(cX) = cE(X).$$



$$E(aX + b) = aE(X) + b$$



# Properties of Variance

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- If  $X$  is any random variable and  $c$  is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

## Variance

$$\begin{aligned}V(X) &= E(X^2) - \mu^2 \\&= E(X^2) - [E(X)]^2\end{aligned}$$

$$V(cX) = E((cX)^2) - [E(cX)]^2$$

# Properties of Variance

---

---

- If  $X$  is any random variable and  $c$  is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

## Variance

$$V(X) = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$$

$$\begin{aligned}V(cX) &= E((cX)^2) - [E(cX)]^2 = E(c^2X^2) - [cE(X)]^2 \\&= c^2E(X^2) - c^2[E(X)]^2 = c^2(E(X^2) - [E(X)]^2) \\&= c^2V(X)\end{aligned}$$

# Properties of Variance

---

---

- If  $X$  is any random variable and  $c$  is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

## Variance

$$V(X) = E((X - \mu)^2) = E((X - E(X))^2)$$

$$V(X + c) = E((X + c - E(X + c))^2)$$

# Properties of Variance

---

---

- If  $X$  is any random variable and  $c$  is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

$$E(aX + b) = aE(X) + b$$

## Variance

$$V(X) = E((X - \mu)^2) = E((X - E(X))^2)$$

$$\begin{aligned}V(X + c) &= E((X + c - E(X + c))^2) = E((X + c - E(X) - c)^2) \\&= E\left((X - E(X))^2\right) \\&= V(X).\end{aligned}$$

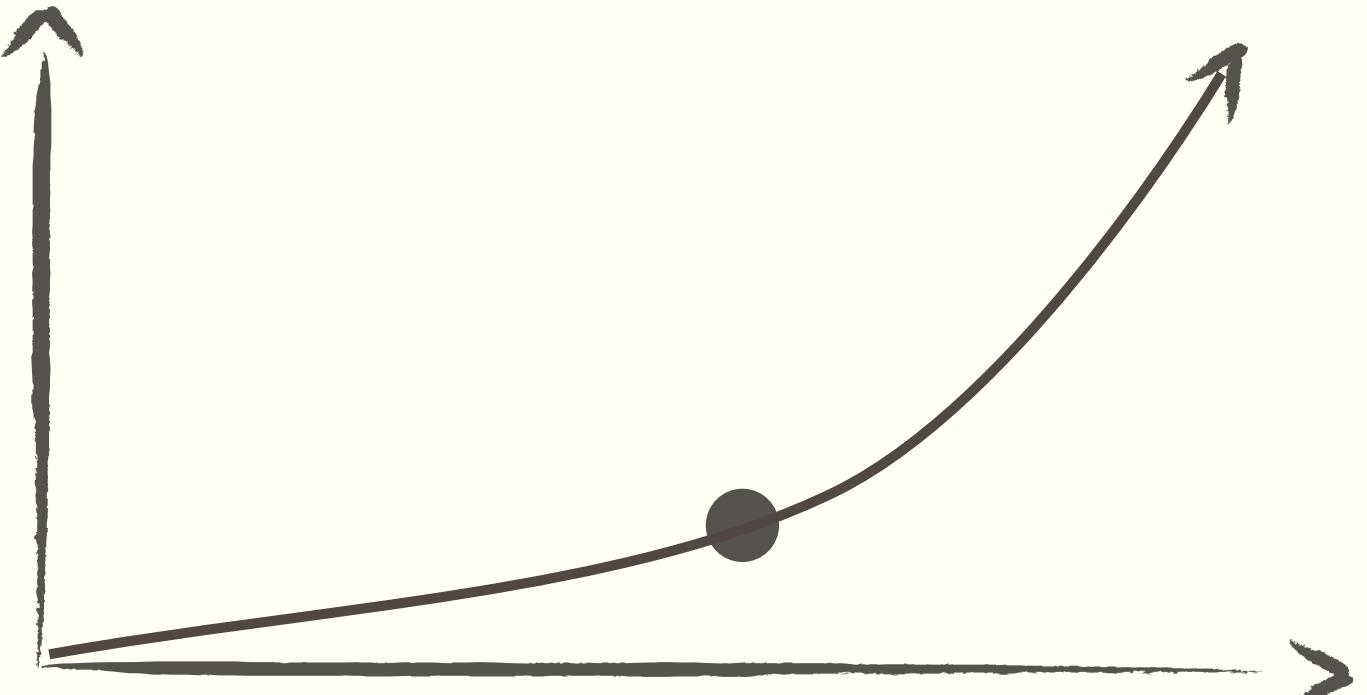
# Non-linearity

$$V(cX) = c^2V(X)$$

$$V(X + c) = V(X)$$



$$V(aX + b) = a^2V(X)$$



# When do we need independence?

$X + Y$

$XY$

Do not need

$$E(X + Y) = E(X) + E(Y)$$

...

...

$$V(X + Y) = V(X) + V(Y)$$

Need

$$E(XY) = E(X)E(Y)$$

...

# When do we need independence?

$X + Y$

$XY$

Do not need

$$E(X + Y) = E(X) + E(Y)$$

...

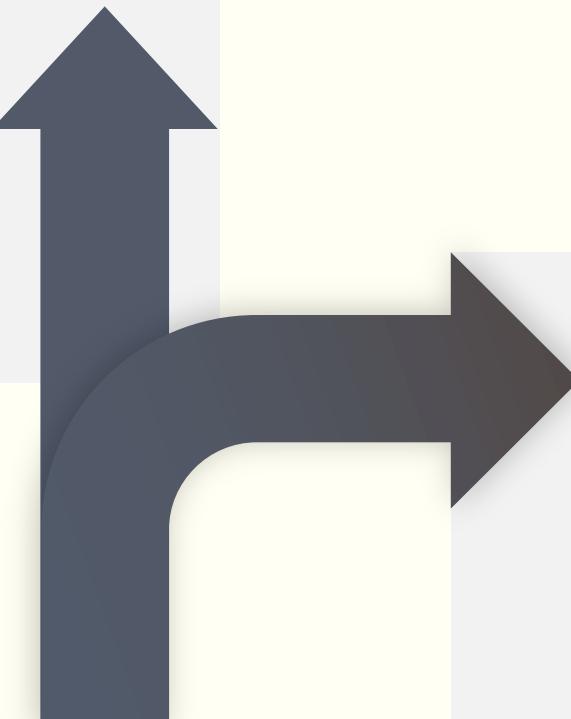
...

Need

$$E(XY) = E(X)E(Y)$$

$$V(X + Y) = V(X) + V(Y)$$

...



# Proof

Let  $X$  and  $Y$  be random variables with finite expected values.  
And  $X$  and  $Y$  are independent random variables.

$$V(X) = E(X^2) - \mu^2$$

$$E(X) = a$$

$$E(Y) = b$$

$$E(X + Y) = a + b$$

$$E(XY) = E(X)E(Y)$$

$$\begin{aligned} V(X + Y) &= E((X + Y)^2) - (a + b)^2 \\ &= E(X^2 + 2XY + Y^2) - (a + b)^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - a^2 - 2ab - b^2 \end{aligned}$$

$$\begin{aligned} V(X + Y) &= E(X^2) + 2E(X)E(Y) + E(Y^2) - a^2 - 2ab - b^2 \\ &= E(X^2) + 2ab + E(Y^2) - a^2 - 2ab - b^2 \\ &= E(X^2) - a^2 + E(Y^2) - b^2 \\ &= V(X) + V(Y) \end{aligned}$$

# Properties of Variance

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- If  $X$  is any random variable and  $c$  is any constant, then

$$V(cX) = c^2V(X), V(X + c) = V(X).$$

- Let  $X$  and  $Y$  be two **independent** random variables. Then

$$V(X + Y) = V(X) + V(Y).$$

- It can be shown that the variance of the sum of any number of mutually independent random variables is the sum of the individual variances.

# Properties of Variance

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- Let  $X_1, X_2, \dots, X_n$  be an independent trials process with  $E(X_j) = \mu$  and  $V(X_j) = \sigma^2$ .  
Let  $S_n = X_1 + X_2 + \dots + X_n$  be the sum, and  $A_n = \frac{S_n}{n}$  be the average. Then

$$V(cX) = c^2V(X)$$

independent

$$V(X + Y) = V(X) + V(Y).$$

$$E(S_n) = \dots$$

?

$$V(S_n) = \dots$$

?

# Properties of Variance

---

---

- Let  $X_1, X_2, \dots, X_n$  be an independent trials process with  $E(X_j) = \mu$  and  $V(X_j) = \sigma^2$ .  
Let  $S_n = X_1 + X_2 + \dots + X_n$  be the sum, and  $A_n = \frac{S_n}{n}$  be the average. Then

$$V(cX) = c^2V(X)$$

independent

$$V(X + Y) = V(X) + V(Y).$$

$$E(A_n) = \dots$$

?

$$E(S_n) = n\mu$$

=

$$V(A_n) = \dots, D(A_n) = \dots$$

?

$$V(S_n) = n\sigma^2$$

=

# Properties of Variance

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---

- Let  $X_1, X_2, \dots, X_n$  be an independent trials process with  $E(X_j) = \mu$  and  $V(X_j) = \sigma^2$ .  
Let  $S_n = X_1 + X_2 + \dots + X_n$  be the sum, and  $A_n = \frac{S_n}{n}$  be the average. Then

$$E(S_n) = n\mu$$

=

$$E(A_n) = \mu$$

=

$$V(S_n) = n\sigma^2$$

=

$$V(A_n) = \frac{\sigma^2}{n}, D(A_n) = \frac{\sigma}{\sqrt{n}}$$

=

# Example

## Bernoulli trial

$$m(x) = \begin{cases} p, & X = 1 \\ 1 - p, & X = 0 \end{cases}$$

$$E(X) = p$$

$$V(X) = p - p^2$$

## Binomial

n independent Bernoulli trials

$$E(S_n) = n\mu$$

=

$$V(S_n) = n\sigma^2$$

=

# Law of Large Numbers

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- Let  $X_1, X_2, \dots, X_n$  be an independent trials process with  $E(X_j) = \mu$  and  $V(X_j) = \sigma^2$ .
- Let  $S_n = X_1 + X_2 + \dots + X_n$  be the sum, and  $A_n = \frac{S_n}{n}$  be the average. Then

$$E(A_n) = \mu$$

$$=$$

$$n \rightarrow +\infty$$

$$V(A_n) \rightarrow \dots$$

$$V(A_n) = \frac{\sigma^2}{n}, D(A_n) = \frac{\sigma}{n}$$

$$=$$

$$D(A_n) \rightarrow \dots$$