MATH 20: PROBABILITY

Midterm 2

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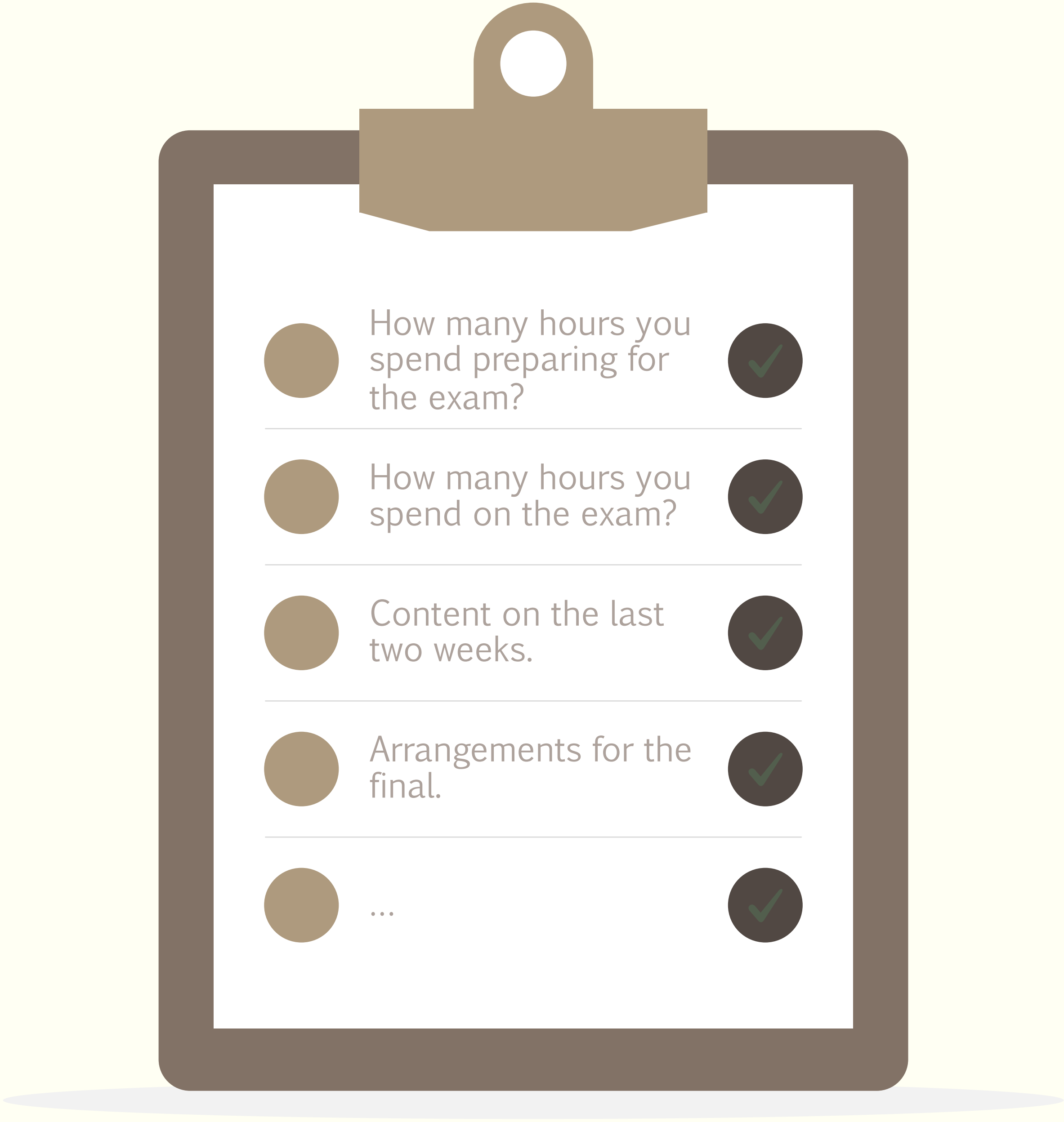
Exam Wrapper

for midterm 2



Problem 1: True or False

(b) False Let X be a random variable which can take on values $\{0, 1, 2, 3, \ldots\}$ (all non-negative integers). X can be uniformly distributed.

Hint: The sum of countable many zeros is still zero.

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Hint: The sum of countable many zeros is still zero.

?

Can X follow a continuous uniform distribution?

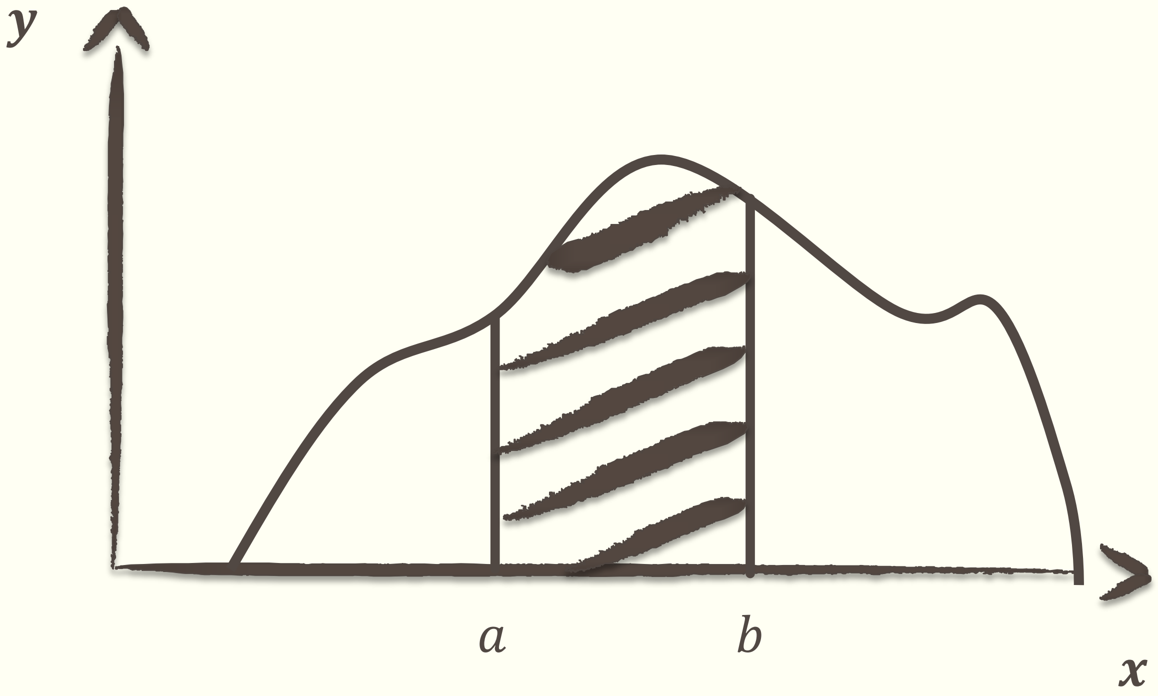
Density Functions of Continuous Random Variable

- Let X be a continuous real-valued random variable. A density function for X is a real-valued function f that satisfies

$$P(a \leq X \leq b) = \int_a^b f(x)dx, \text{ for } a, b \in \mathbb{R}.$$

- If E is a subset of $\mathbb{R}$, then $P(x \in E) = \int_E f(x)dx$.
- In particular, if E is an interval $[a, b]$, the probability that the outcome of the experiment falls in E is given by

$$P([a, b]) = \int_a^b f(x)dx.$$



(b) False Let X be a random variable which can take on values $\{0, 1, 2, 3, \ldots\}$ (all non-negative integers). X can be uniformly distributed.

Hint: The sum of countable many zeros is still zero.

?

Can X follow a continuous uniform distribution?

No

?

Can X follow a discrete uniform distribution?

$m(x) = \cdots$

(b) False Let X be a random variable which can take on values $\{0, 1, 2, 3, \ldots\}$ (all non-negative integers). X can be uniformly distributed.

Hint: The sum of countable many zeros is still zero.

$m(x) = 0$

?

Can X follow a discrete uniform distribution?

$m(x) > 0$

(b) False Let X be a random variable which can take on values $\{0, 1, 2, 3, \ldots\}$ (all non-negative integers). X can be uniformly distributed.

Hint: The sum of countable many zeros is still zero.

?

Can X follow a discrete uniform distribution?

$m(x) = 0$

$\sum_{i=0}^{+\infty} 0 = 0$

No

$m(x) > 0$

$\sum_{i=0}^{+\infty} c = +\infty$

Problem 1: True or False

(c) False For a numerically-valued random variable X, if $E(X^2) = E^2(X)$, X can take two different values.

Discrete variance $V(X)$

$$E(X^2) - \mu^2 = 0 =$$
$$= E\big((X-\mu)^2\big) = \sum_{x\in\Omega} (x-\mu)^2 m(x)\,.$$

Continuous variance $V(X)$

$$0 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx$$

$$X = \mu$$

Problem 2: Computation

(a) Assume that X is the outcome of a single Bernoulli trial where

$$m(x) = \begin{cases} p, & X = 1 \\ 1-p, & X = 0 \end{cases}.$$

Find $E(X^{2020})$.

3 pts

$$E(X^{2020}) = 1^{2020} \times p + 0^{2020} \times (1-p) = p.$$

(a) Assume that X is the outcome of a single Bernoulli trial where

$$m(x) = \begin{cases} p, & X = 1 \\ 1-p, & X = 0 \end{cases}.$$

Find $E(X^{2020})$.

3 pts

$$E(X^{2020}) = 1^{2020} \times p + 0^{2020} \times (1-p) = p.$$

?

$$E(X^{2020}) = E^{2020}(X)$$

The Product of Two Random Variables

- Let X and Y be independent real-valued continuous random variables with finite expected values. Then we have

$$E(XY) = E(X)E(Y).$$

- More generally, for n mutually independent random variables X_i, we have

$$E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2) \cdots E(X_n).$$

?

$$E(X^{2020}) = E^{2020}(X)$$

?

Are X and X independent?

- If X is any random variable and c is any constant, then
$$V(cX) = c^2 V(X),\ V(X+c) = V(X).$$
- Let X and Y be two independent random variables. Then
$$V(X+Y) = V(X) + V(Y).$$

$V(2X) = 4V(X)$?

$V(X+X) = 2V(X)$?

Problem 2: Computation

(b) Assume that X is Poisson distributed with parameter λ. Find $E(X^3)$.

5 pts

$$
\begin{aligned}
E(X^3) &= \sum_{k=0}^{+\infty} k^3 \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{+\infty} k^2 \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=1}^{+\infty} k^2 \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{k=1}^{+\infty} (k^2-1) \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} + \lambda \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{k=2}^{+\infty} (k^2-1) \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} + \lambda \sum_{l=0}^{+\infty} \frac{\lambda^{l}}{l!} e^{-\lambda} \\
&= \lambda \sum_{k=2}^{+\infty} (k+1) \frac{\lambda^{k-1}}{(k-2)!} e^{-\lambda} + \lambda = \lambda^2 \sum_{l=0}^{+\infty} (l+3) \frac{\lambda^{l}}{l!} e^{-\lambda} + \lambda \\
&= \lambda^2 \sum_{l=0}^{+\infty} l \frac{\lambda^{l}}{l!} e^{-\lambda} + 3\lambda^2 \sum_{l=0}^{+\infty} \frac{\lambda^{l}}{l!} e^{-\lambda} + \lambda \\
&= \lambda^3 + 3\lambda^2 + \lambda.
\end{aligned}
$$

!

Moment:

$E(X^n)$, where $n = 1, 2, 3, \cdots$

Problem 3: Proof

(c) $E(s^2) = \frac{n-1}{n}\sigma^2$.

5 pts

$$
\begin{aligned}
E(s^2) &= E[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2] = \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\bar{x})^2] = \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu+\mu-\bar{x})^2] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}((x_i-\mu)^2 - 2(x_i-\mu)(\bar{x}-\mu) + (\bar{x}-\mu)^2)] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2 - 2\sum_{i=1}^{n}(x_i-\mu)(\bar{x}-\mu) + \sum_{i=1}^{n}(\bar{x}-\mu)^2] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2 - 2\sum_{i=1}^{n}(x_i-\mu)(\bar{x}-\mu) + \sum_{i=1}^{n}(\bar{x}-\mu)^2] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2 - 2n(\bar{x}-\mu)^2 + n(\bar{x}-\mu)^2] = \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2] \\
&= \frac{1}{n}\sum_{i=1}^{n}E((X_i-\mu)^2) - E((\bar{x}-\mu)^2) = \frac{1}{n}\sum_{i=1}^{n}V(x_i) - V(\bar{x}) \\
&= \frac{n-1}{n}\sigma^2.
\end{aligned}
$$

$$x_i - \bar{x} =$$
$$(x_i - \mu) + (\mu - \bar{x})$$

(c) $E(s^2) = \frac{n-1}{n}\sigma^2$.

5 pts

$$
\begin{aligned}
E(s^2) &= E[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2] = \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\bar{x})^2] = \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu+\mu-\bar{x})^2] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}((x_i-\mu)^2-2(x_i-\mu)(\bar{x}-\mu)+(\bar{x}-\mu)^2)] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2-2\sum_{i=1}^{n}(x_i-\mu)(\bar{x}-\mu)+\sum_{i=1}^{n}(\bar{x}-\mu)^2] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2-2\sum_{i=1}^{n}(x_i-\mu)(\bar{x}-\mu)+\sum_{i=1}^{n}(\bar{x}-\mu)^2] \\
&= \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2-2n(\bar{x}-\mu)^2+n(\bar{x}-\mu)^2] = \frac{1}{n}E[\sum_{i=1}^{n}(x_i-\mu)^2-n(\bar{x}-\mu)^2] \\
&= \frac{1}{n}\sum_{i=1}^{n}E((X_i-\mu)^2)-E((\bar{x}-\mu)^2) = \frac{1}{n}\sum_{i=1}^{n}V(x_i)-V(\bar{x}) \\
&= \frac{n-1}{n}\sigma^2.
\end{aligned}
$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2.$$

$$E(s^2) = \frac{n-1}{n}\sigma^2$$

How to redefine s^2, so that $E(s^2) = \sigma^2$?

(d) Let X be a random variable with density function $f(x)$. Show, using elementary calculus, that the function

$$\phi(a) = E((X-a)^2)$$

takes its minimum value when $a = \mu(X)$, and in that case $\phi(a) = V(X)$.

5 pts

We know that $\phi(a) = E((X-a)^2) = \int_{-\infty}^{+\infty}(x-a)^2 f(x)dx$. Consider the derivative $\frac{d}{da}\phi(a)$:

$$\frac{d}{da}\phi(a) = \frac{d}{da}\int_{-\infty}^{+\infty}(x-a)^2 f(x)dx = -2\int_{-\infty}^{+\infty}(x-a)f(x)dx.$$

Furthermore the second-order derivative is $\frac{d^2}{da^2}\phi(a) = 2\int_{-\infty}^{+\infty} f(x)dx = 2 > 0$.

Therefore, $\phi(a)$ will take its minimum value when $\frac{d}{da}\phi(a) = 0$, that is,

$$\int_{-\infty}^{+\infty}(x-a)f(x)dx = \int_{-\infty}^{+\infty} xf(x)dx - a\int_{-\infty}^{+\infty} f(x)dx = \mu(X) - a = 0.$$

Now that we have $a = \mu(X)$, it follows that $\phi(a) = E((X-\mu(X))^2) = V(X)$.

Furthermore the second-order derivative is $\frac{d^2}{da^2}\phi(a) = 2\int_{-\infty}^{+\infty} f(x)dx = 2 > 0$.

!

Extreme:

$f'(x) = 0$

!

Maximum: $f''(x) < 0$

!

Minimum: $f''(x) > 0$

Problem 4: Manipulation

A number U is chosen at random in the interval $[0,1]$.

(a) Find the probability that $T = \sin(\pi U) < \frac{1}{2}$.

5 pts

Given that

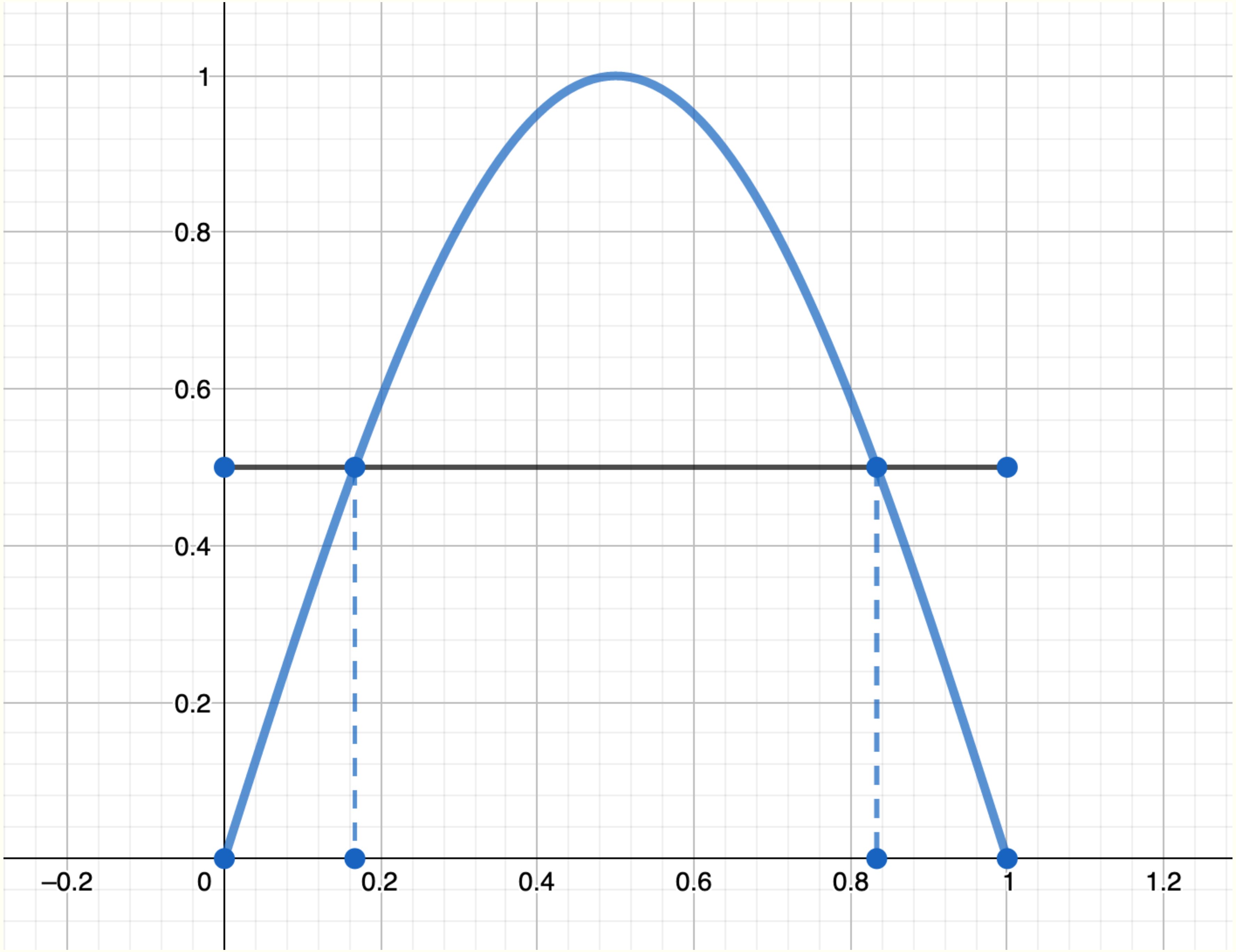
$$\sin(\frac{\pi}{6}) = \sin(\frac{5\pi}{6}) = \frac{1}{2},$$

the probability that $T = \sin(\pi U) < \frac{1}{2}$ is equivalent to the probability that

$$0 \leq \pi U < \frac{\pi}{6}, \qquad \text{or} \qquad \frac{5\pi}{6} < \pi U \leq \pi.$$

Therefore,

$$P(T < \frac{1}{2}) = P((0 < U < \frac{1}{6}) \cup (\frac{5}{6} < U < 1)) = \frac{1}{3}.$$



1
0.8
0.6
0.4
0.2
−0.2
0
0.2
0.4
0.6
0.8
1
1.2

(b) Let X be a random variable uniformly distributed over $[c, d]$. For what choice of a and b do we have $U = aX + b$?

5 pts

Consider the cumulative distribution function of $U = aX + b$. We have

$$F_U(u) = P(U \leq u) = P(aX + b \leq u) = P(X \leq \frac{u-b}{a}).$$

Since U is uniformly distributed on $[0, 1]$, $F_U(u) = u$, for $0 \leq u \leq 1$.

Also, as X is uniformly distributed over $[c, d]$, $F_X(x) = \frac{x-c}{d-c}$, for $c \leq x \leq d$.

Therefore, the original equation can be rewritten as

$$u = F_U(u) = P(X \leq \frac{u-b}{a}) = \frac{\frac{u-b}{a} - c}{d-c}.$$

That is,

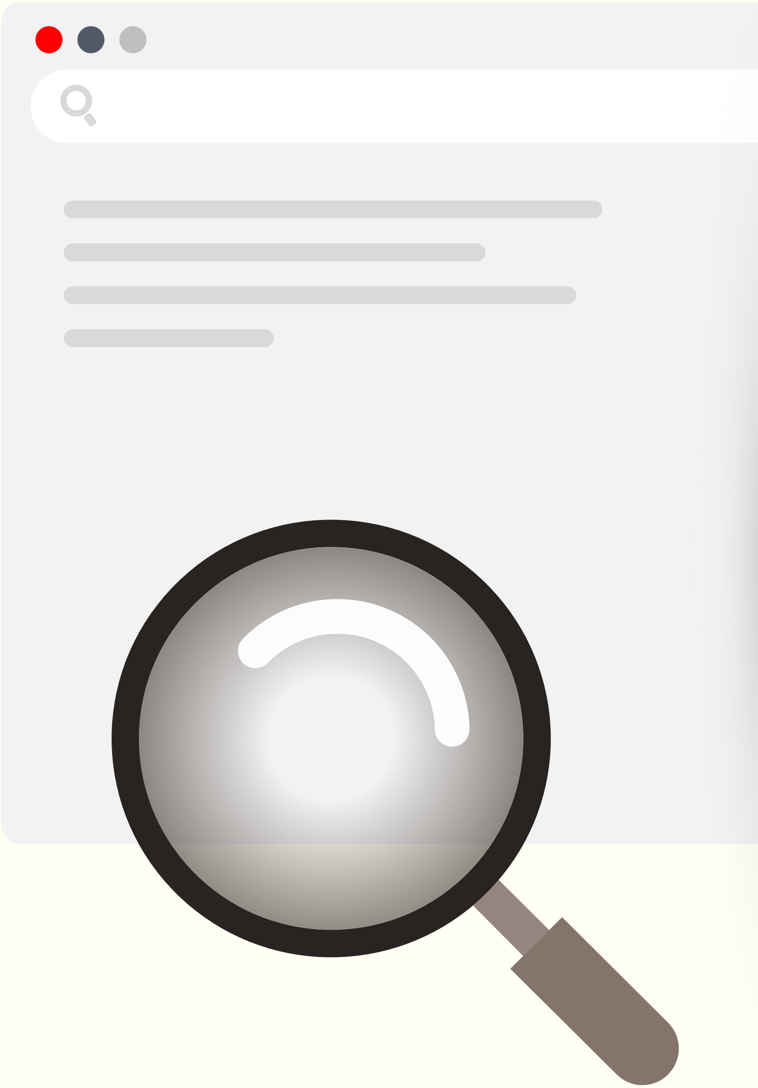
$$a(d-c)u = u - b - ac, \qquad \forall 0 \leq u \leq 1.$$

Comparing the coefficients, we get

$$a(d-c) = 1, \qquad b + ac = 0.$$

Simple calculation gives that

$$a = \frac{1}{d-c}, \qquad b = -\frac{c}{d-c}.$$



Calvin Atkeson, Max Telemaque

$U = aX + b$, $E(U) = aE(X) + b$, $V(U) = a^2V(X)$

U: uniform on $[0,1]$, $E(U) = \frac{1}{2}$, $V(U) = \frac{1}{12}$

X: uniform on $[c,d]$, $E(X) = \frac{c+d}{2}$, $V(X) = \frac{(d-c)^2}{12}$

Problem 5: Educational Attainment

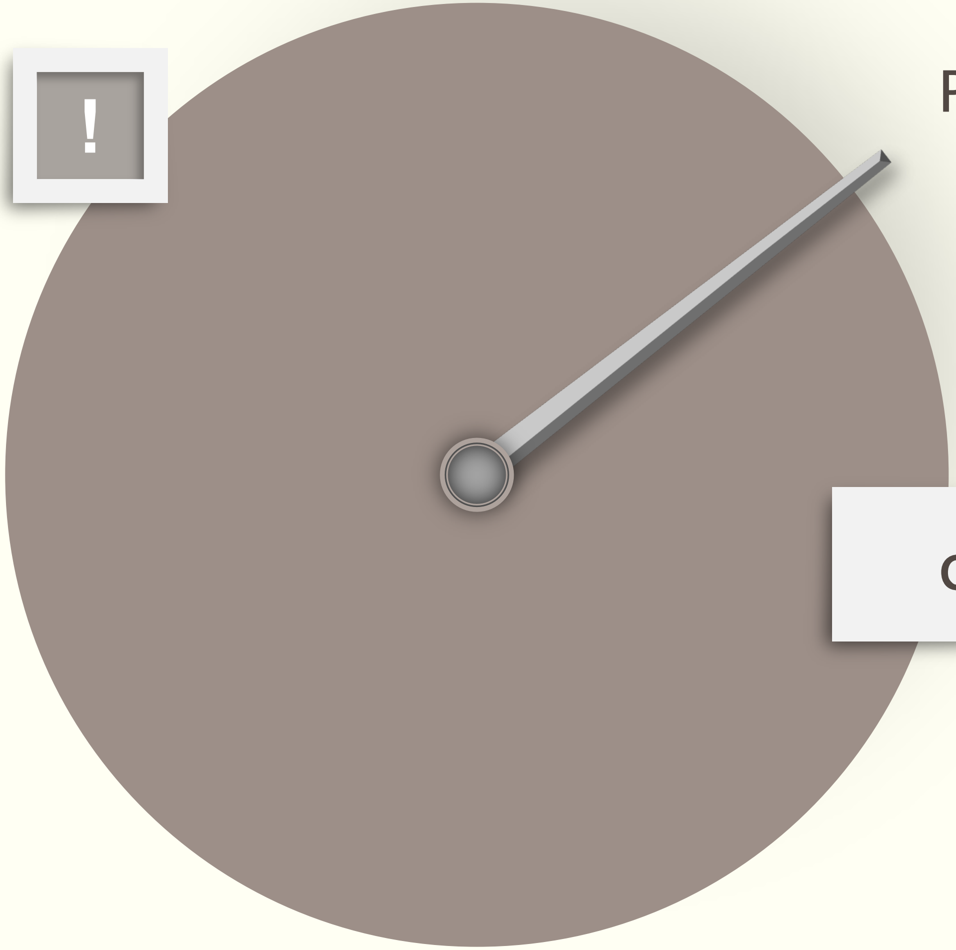
In August 2020, the Animal Bank released the Sharing Higher Education's Promise Report, highlighting the rising demand and supply of tertiary education. The Marmot Kingdom saw the fastest growth in its tertiary gross enrollment ratio (GER) during 2019 - 2020.

Back to 10 years ago, on the average, only 1 marmot in 1000 had a Bachelor degree.

Which one to use?

Binomial distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$



Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

two parameters !

n p

one parameter !

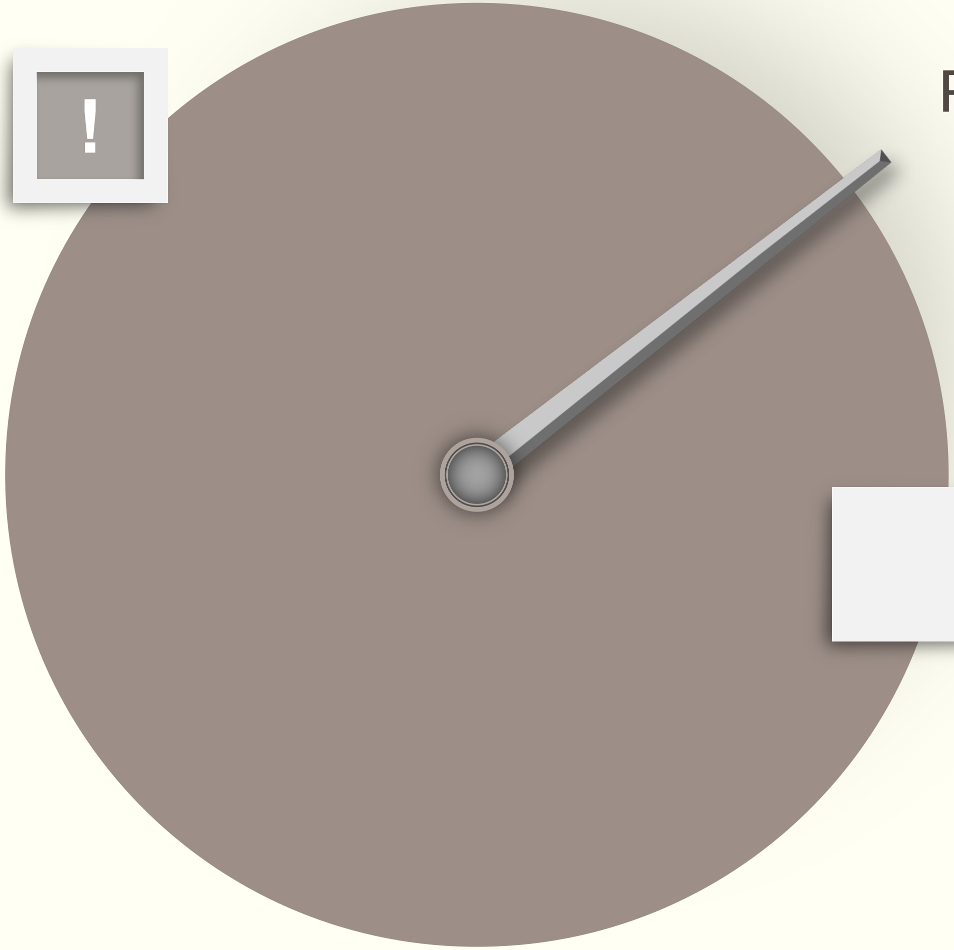
λ

Which one to use?

Binomial distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

$\boldsymbol{n} < +\infty$!



Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

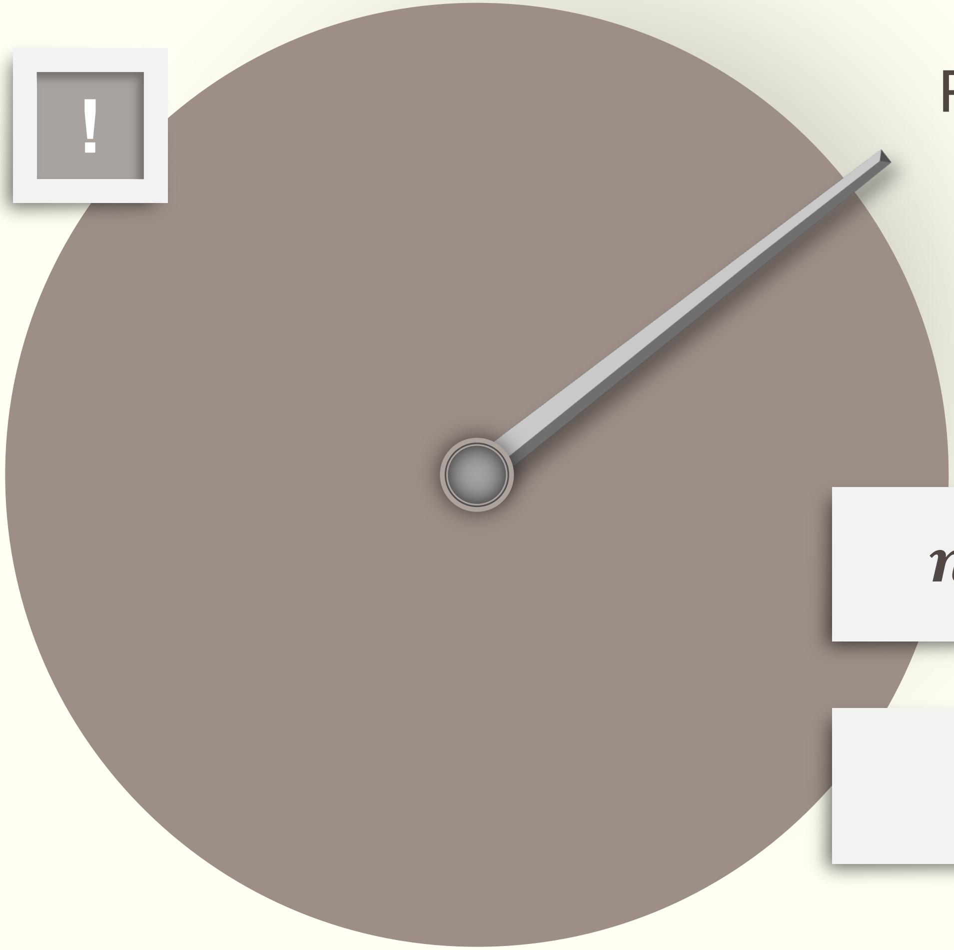
$\boldsymbol{n} \to +\infty$!

Which one to use?

Binomial distribution

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

$n = 2, 5, \cdots$!



Poisson Distribution

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$n = 50, 100, \cdots$!

$np = \lambda$ =

MATH 20 BABY PROBABILISTS

When you cannot explain something: use **Poisson distribution (Poisson process)!**

5 pts

Let X be the number of marmots in the county having a Bachelor degree. We use a Poisson distribution with parameter

$$\lambda = np = 10000 \times \frac{1}{1000} = 10.$$

Then,

$$P(X = 0) = e^{-\lambda} = e^{-10}.$$

5 pts

Let n be the number of marmots taking the survey and X be the number of them having a Bachelor degree. This time we use a Poisson distribution with parameter

$$\lambda = np = \frac{n}{1000}.$$

And

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda} = 1 - e^{-n/1000} > \frac{1}{2}.$$

Therefore,

$$n > 1000 \ln(2) \approx 693.1472.$$

Since n is an integer, $n_{\min} = 694$.

Problem 6: Cupid's Arrow

It's said that Cupid carries two kinds of arrows, one with a sharp golden point, and the other with a blunt tip of lead. A person wounded by the golden arrow is filled with uncontrollable desire, but the one struck by the lead feels aversion and desires only to flee.

When Apollo taunts Cupid as the lesser archer, Cupid shoots him with the golden arrow, but strikes the object of his desire, the nymph Daphne, with the lead.

The duration of an arrow is the length of time that particular arrow is effective. The duration of the golden arrow has an exponential density with average time 200 years. And that of the lead arrow also has an exponential density but with average time 500 years.

golden arrow

lead arrow

Homework 6

Problem 5

4 pts

Chapter 5.2 Exercise 34

Jones puts in two new lightbulbs: a 60 watt bulb and a 100 watt bulb. It is claimed that the lifetime of the 60 watt bulb has an exponential density with average lifetime 200 hours ($\lambda = 1/200$). The 100 watt bulb also has an exponential density but with average lifetime of only 100 hours ($\lambda = 1/100$). Jones wonders what is the probability that the 100 watt bulb will outlast the 60 watt bulb.

4 pts

Let X and Y denote the lifetime of the 60 watt bulb and the 100 watt bulb, respectively. We have

$$f_X(x) = \frac{1}{200}e^{-x/200}, \qquad f_Y(y) = \frac{1}{100}e^{-x/100},$$

and

$$F_X(x) = 1 - e^{-x/200}, \qquad F_Y(y) = 1 - e^{-y/100}.$$

What we are after is

$$\begin{aligned}
P(Y > X) &= \int_{x=0}^{\infty}\int_{y=x}^{\infty} f_X(x)f_Y(y)dydx \\
&= \int_{x=0}^{\infty} f_X(x)(1 - F_Y(x))dx \\
&= \int_{x=0}^{\infty} \frac{1}{200}e^{-x/200}e^{-x/100}dx \\
&= \int_{x=0}^{\infty} \frac{1}{200}e^{-3x/200}dx \\
&= \frac{1}{3}.
\end{aligned}$$

$$P(Y > X)$$

$$\int_{x=0}^{+\infty}\int_{y=x}^{+\infty} \quad dydx$$

4 pts

Let X and Y denote the lifetime of the 60 watt bulb and the 100 watt bulb, respectively. We have

$$f_X(x) = \frac{1}{200}e^{-x/200}, \qquad f_Y(y) = \frac{1}{100}e^{-x/100},$$

and

$$F_X(x) = 1 - e^{-x/200}, \qquad F_Y(y) = 1 - e^{-y/100}.$$

What we are after is

$$\begin{aligned} P(Y > X) &= \int_0^\infty P(Y > X \,|\, X = u) f_X(u) du \\ &= \int_0^\infty P(Y > u) f_X(u) du \\ &= \int_0^\infty (1 - F_y(u)) f_X(u) du \\ &= \int_0^\infty e^{-u/100} \frac{1}{200} e^{-u/200} du \\ &= \int_0^\infty \frac{1}{200} e^{-3u/200} du \\ &= \frac{1}{3}. \end{aligned}$$

7 pts

Let X be the duration of the golden arrow and Y be that of the lead arrow. We have

$$f(x) = \frac{1}{200}e^{-x/200}, \qquad g(y) = \frac{1}{500}e^{-x/500}.$$

What we are looking for is the probability that

$$\begin{aligned}
P(X > Y) &= \int_{x=0}^{+\infty} \int_{y=0}^{x} f(x)g(y)dydx = \int_{x=0}^{+\infty} f(x)G(x)dx \\
&= \int_{x=0}^{+\infty} \frac{1}{200}e^{-x/200}(1 - e^{-x/500})dx \\
&= \int_{x=0}^{+\infty} \frac{1}{200}e^{-x/200}dx - \int_{x=0}^{+\infty} \frac{1}{200}e^{-7x/1000}dx \\
&= 1 - \frac{5}{7} = \frac{2}{7}.
\end{aligned}$$

$$P(X > Y)$$

$$\int_{x=0}^{+\infty} \int_{y=0}^{x} \quad dydx$$

(b) himself will break free from the enchantment first and Daphne has to wait another 300 years or more before her arrow wears off?

8 pts

What we are now looking for is the probability that

$$\begin{aligned}
P(Y \geq X + 300) &= \int_{x=0}^{+\infty} \int_{y=x+300}^{+\infty} f(x)g(y)dydx \\
&= \int_{x=0}^{+\infty} f(x)[1 - G(x+300)]dx \\
&= \int_{x=0}^{+\infty} \frac{1}{200} e^{-x/200} e^{-(x+300)/500} dx \\
&= \int_{x=0}^{+\infty} \frac{1}{200} e^{-3/5} e^{-7x/1000} dx \\
&= \frac{5}{7} e^{-3/5}.
\end{aligned}$$

$$P(Y \geq X + 300)$$

$$\int_{x=0}^{+\infty} \int_{y=x+300}^{+\infty} \quad dydx$$

?

Daphne will break free from the enchantment first?

golden arrow: 200 years

$$\lambda_X = \frac{1}{200}$$

lead arrow: 500 years

$$\lambda_Y = \frac{1}{500}$$

$$P(X > Y) = \frac{2}{7}$$

$$\frac{\frac{1}{500}}{\frac{1}{200} + \frac{1}{500}} = \frac{2}{7}$$

$$P(X > Y) = \frac{\lambda_Y}{\lambda_X + \lambda_Y}$$

?

Apollo will break free from the enchantment first and Daphne has to wait another 300 years or more before her arrow wears off?

$$P(Y > X \;\cap\; Y \geq X + 300)$$

$$P(Y \geq X + 300) = \frac{5}{7} e^{-3/5}$$

$$\lambda_X = \frac{1}{200}$$

$$\lambda_Y = \frac{1}{500}$$

$$P(X > Y) = \frac{\lambda_Y}{\lambda_X + \lambda_Y} = \frac{2}{7}$$

$$P(Y > X) = \frac{\lambda_X}{\lambda_X + \lambda_Y} = \frac{5}{7}$$

$$P(Y \geq X + 300) = \frac{5}{7} e^{-3/5}$$

$$P(Y > y) = e^{-\lambda_Y y}$$

$$P(Y > 300) = e^{-3/5}$$

$$P(Y > X) = \frac{\lambda_X}{\lambda_X + \lambda_Y} = \frac{5}{7}$$

$$P(Y > 300) = e^{-3/5}$$

$$P(Y \geq X + 300) = \frac{5}{7} e^{-3/5}$$

$$P(Y \geq X + 300) = P(Y > 300) P(Y > X)$$

$$P(Y > X + 300 | Y > X) P(Y > X)$$

Memoryless Property

$$P(X > r + s | X > r) \\ = P(X > s)$$

Problem 7: Man with No Name: A fistful of Nuts

A game is played as follows: a participant choose a random number X uniformly from $[0,1]$. Then a sequence $Y_1, Y_2, \cdots$ of random numbers is chosen independently and uniformly from $[0,1]$. The game ends the first time that $Y_i > X$ and the participant is paid $(i-1)$ pecans.

(a) Let $P(N = n | X = x)$ be the probability that the payout will be $N = n$ pecans given $X = x$. Which amount would the participant be most likely to get given $X = x$?

(b) What is a fair entrance fee for this game?

Conditional Expectation

- If F is any event and X is a random variable with sample space $\Omega = \{x_1, x_2, \cdots\}$, then the conditional expectation given F is defined by

$$E(X|F) = \sum_j x_j P(X = x_j|F).$$

- Let X be a random variable with sample space Ω. If $F_1, F_2, \cdots, F_r$ are events such that $F_i \cap F_j = \emptyset$ for $i \neq j$ and $\Omega = \cup_j F_j$, then

$$E(X) = \sum_j E\left(X|F_j\right)P(F_j).$$

Conditional Expectation

- Conditional density

joint density

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

marginal density

- Conditional expected value

$$E(X|Y=y) = \int x f_{X|Y}(x|y)dx.$$

- Expected value

$$E(X) = \int E(X|Y=y) f_Y(y)dy.$$

EXAMPLE

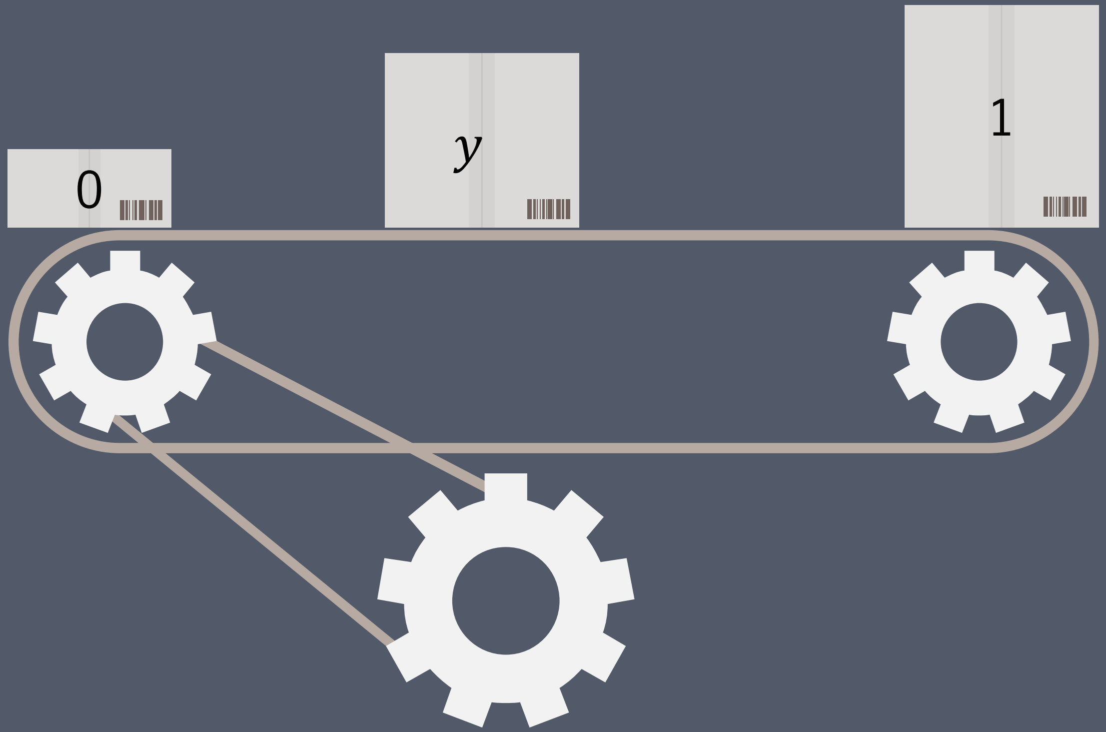
Farming Sim

Example

- A point Y is chosen at random from $[0, 1]$ uniformly. A second point X is then uniformly and randomly chosen from the interval $[0, Y]$. Find the expected value for X.

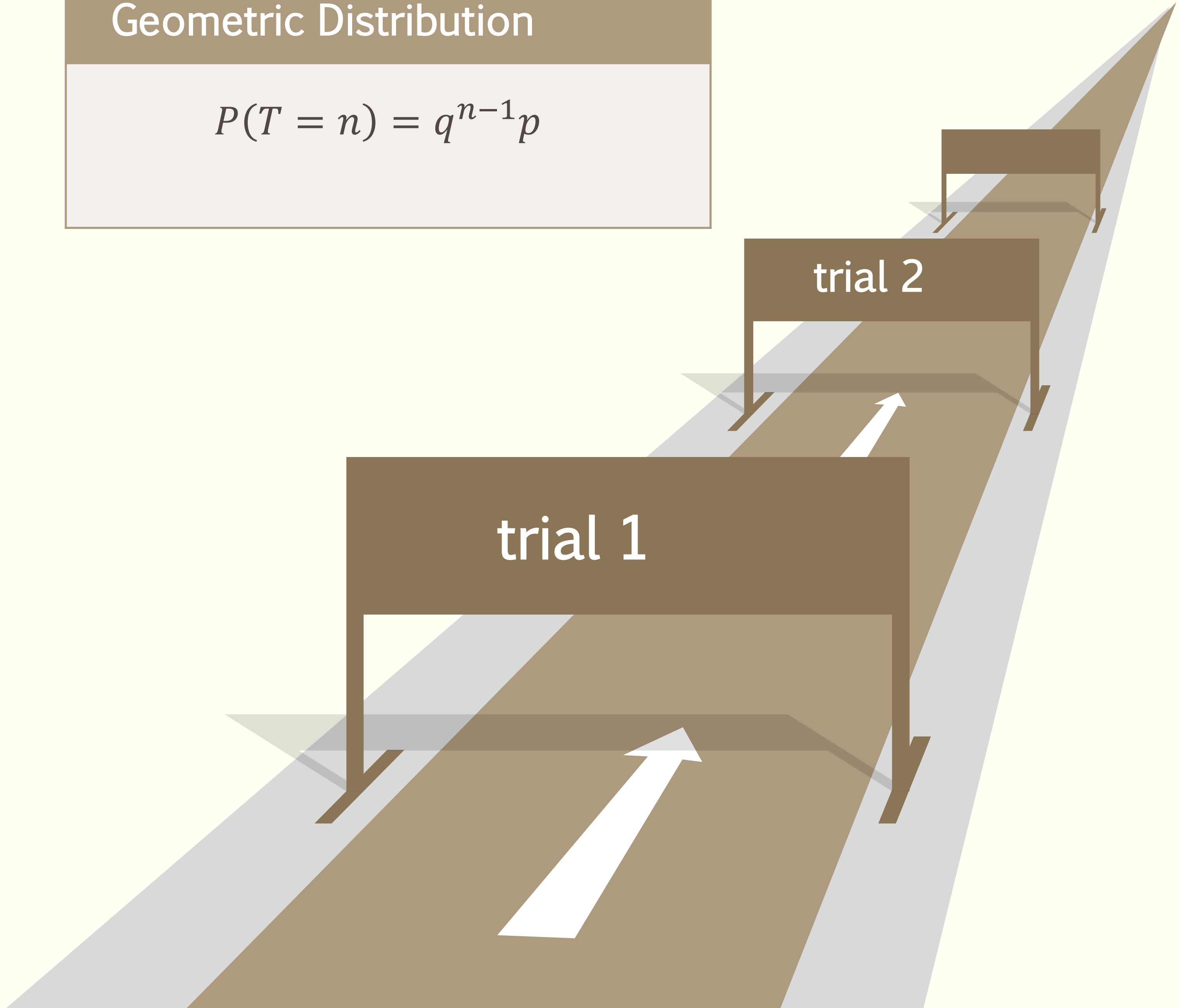
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,\ y)}{f_Y(y)}$$

$$E(X|Y = y) = \int x f_{X|Y}(x|y)dx$$



Geometric Distribution

$$P(T = n) = q^{n-1}p$$



two !

same !

first !

independent !

Geometric Distribution

$$P(T = n) = q^{n-1}p$$

Geometric

$$P(T = n) = q^{n-1}p \qquad E(X) = \frac{1}{p}$$

$$P(N = n) = x^n(1 - x)$$

(a) Let $P(N = n | X = x)$ be the probability that the payout will be $N = n$ pecans given $X = x$. Which amount would the participant be most likely to get given $X = x$?

Hint from Mr. Eyes Closed: Which amount would the participant **expect** to get given $X = x$?

7 pts

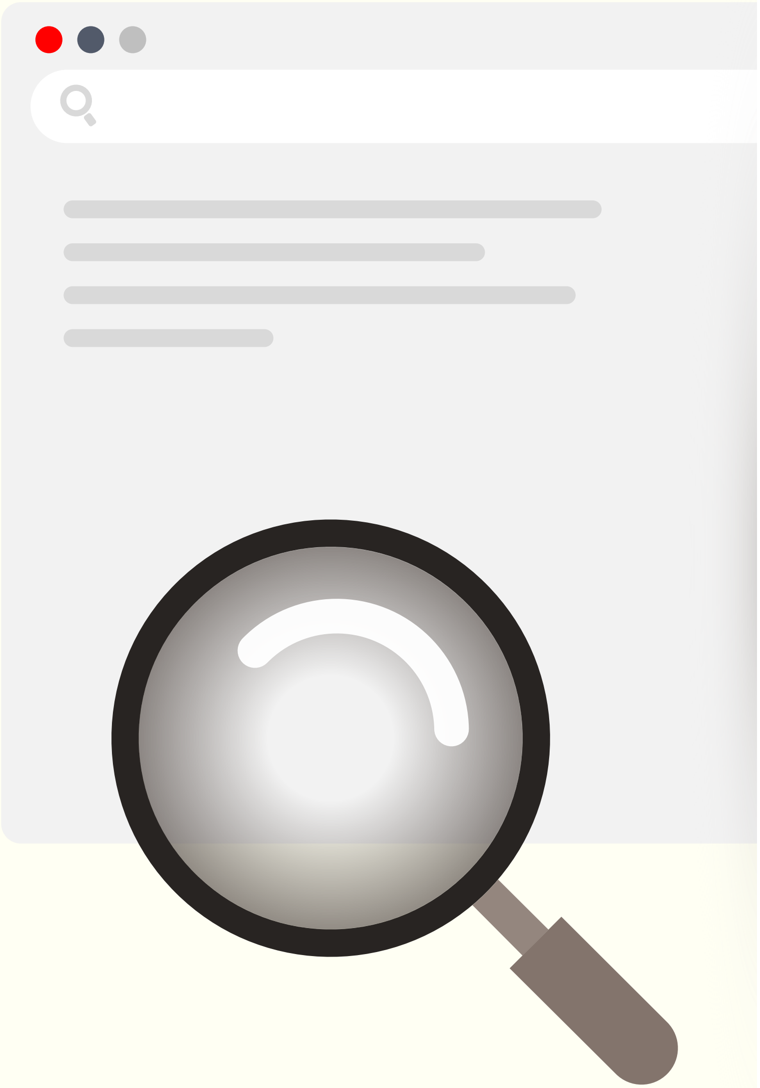
We know that

$$P(N = n | X = x) = x^n(1 - x),$$

which resembles a geometric distribution with $p = 1 - x$.

Therefore,

$$E(N | X = x) = \sum_{n=0}^{+\infty} n x^n (1 - x) = x \sum_{n=0}^{+\infty} n x^{n-1}(1 - x) = \frac{x}{1 - x}.$$



$$P(N = n|X = x) = x^n(1 - x)$$

$$E(N = n|X = x) = \int_0^1 xP(N = n|X = x)dx$$

$$E(N = n|X = x) = \sum_{n=0}^{+\infty} nP(N = n|X = x)$$

8 pts

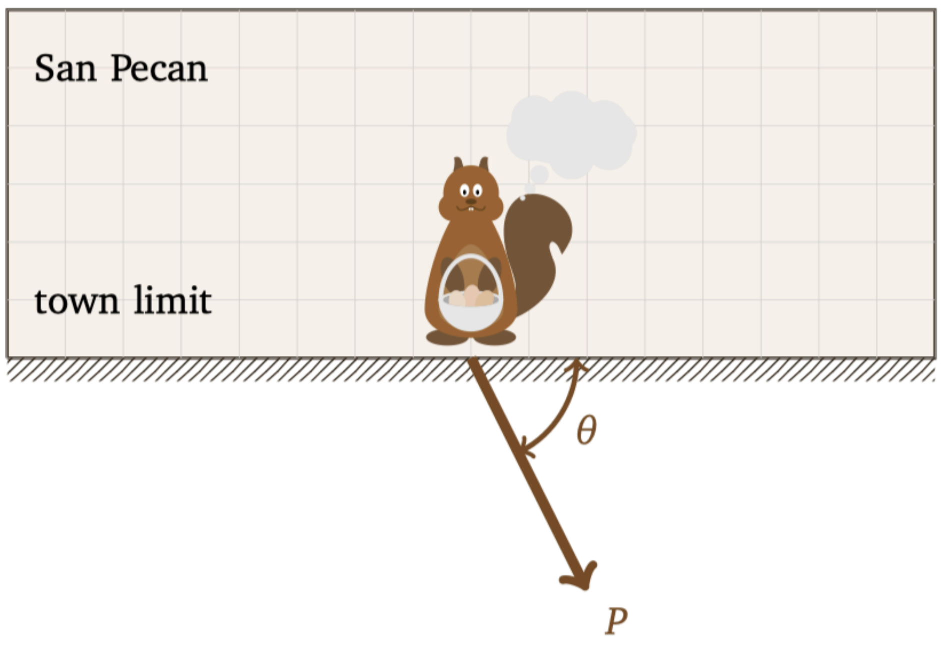
Notice that the density function of X is $f(x) = 1$. Hence,

$$
\begin{aligned}
E(N) &= \int_0^1 E(N|X = x)f(x)dx \\
&= \int_0^1 \frac{x}{1-x}dx \\
&= \int_0^1 (\frac{1}{1-x} - 1)dx \\
&= -\ln(1-x)|_0^1 \\
&= +\infty.
\end{aligned}
$$

Therefore, a participant is willing to pay any amount of pecans for this game.

Problem 8: Man with No Name: Out of San Pecan

Now Nutella is standing on the town limit. He chooses, at random, a direction which will lead him to the world, and walks 2 miles in that direction. Let P denote his position. What is the expected distance from P to the town limit?



Expectation of Functions of Random Variables

- If X is a real-valued random variable and if $\phi: R \to R$ is a continuous real-valued function with domain $[a, b]$, then

$$E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x) f(x) dx,$$

provided the integral exists.

Discrete expected value $\boldsymbol{E(\phi(X))}$	Continuous expected value $\boldsymbol{E(\phi(X))}$
$\sum_{x \in \Omega} \phi(x) m(x)$	$\int_{-\infty}^{+\infty} \phi(x) f(x) dx$

10 pts

Consider the random variable θ uniformly distributed on $[0, \frac{\pi}{2}]$. The density function is $f(\theta) = \frac{2}{\pi}$.

We know that the distance from P to the town limit is

$$\phi(\theta) = 2\sin(\theta).$$

Therefore, the expected distance is

$$\begin{aligned} E(\phi(\theta)) &= \int_0^{\frac{\pi}{2}} \phi(\theta) f(\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} 2\sin(\theta)\frac{2}{\pi} d\theta \\ &= -\frac{4}{\pi}\cos(\theta)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{4}{\pi}. \end{aligned}$$