Extended Application of Burnside's Lemma to Higher Dimension using Matrix and Linear Transformation

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概 要


```
module Burnside where
```

針對清大課程 Burnside's Lemma 去從線性轉換與矩陣、向量的角度去做延伸。令嘗試將此想法推廣到更高維度、非正多面體與較複雜之正多面體。

目錄

How to Read

0.1 Defining Notations

0.1.1 Vector

All vectors within this document (no matter if it's in Haskell code or TeX equations) refers to column vectors (even if it's written in style of row vector).

0.1.2 Matrix

Within this document, we'll often use matrix as a denotation for an ordered-set of vectors.

$$[v_1, \ldots, v_k] = \begin{bmatrix} \begin{pmatrix} v_{1,1} \\ \vdots \\ v_{1,n} \end{pmatrix} & \ldots & \begin{pmatrix} v_{k,1} \\ \vdots \\ v_{k,n} \end{pmatrix} \end{bmatrix} = \begin{bmatrix} v_{1,1} & \ldots & v_{k,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \ldots & v_{k,n} \end{bmatrix}$$

$k =$amount of vectors, $n =$vectors' dimension

1 Brief Introduction to Matrix and Linear Transformation

So what does matrix in linear transformation means? Well, for any $n \times n$ matrix, you could think of each column as representing each axis' unit vector's position after the transformation. So for example:

$$\text{Let } T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{1}$$

$$\text{means, shifting } \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ to} \tag{2}$$

$$\hat{i}' = \begin{bmatrix} a \\ d \\ g \end{bmatrix}, \hat{j}' = \begin{bmatrix} b \\ e \\ h \end{bmatrix}, \hat{k}' = \begin{bmatrix} c \\ f \\ i \end{bmatrix} \tag{3}$$

Hence by these definitions we could conclude that:

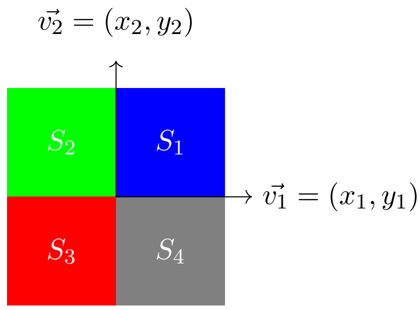
{data}

定理 1 (最少資料量)
For any object within n-Dimensional space, by specifying n-Surfaces' location could unique identify an object's orientation.

2 Describing Object's State

To descibe an object's current orientation, we'll use a set of vectors (which is collected into a matrix). For example, a 2×2 square could be denoted by two 2-D vectors (aka a 2×2 matrix) like following:

$$M = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \tag{4}$$



in which we could clearly see by giving two 2-D vectors, we could rigidly define our 4 different squadron (in counterclockwise fashion).
Now we can apply transformations onto these vectors. But what could be counted as a valid transformation?

定義 1.1 (Valid Transformation)
A transformation is consider valid if it only suffles the order of column vectors within the given matrix.

```haskell
validVectors :: Matrix -> Matrix -> Bool
--              sample    subject   result
validVectors (Matrix s) (Matrix x) = (sort s') == (sort x')
                                     where s' = nub s
                                           x' = nub x
```

3 Describing Operations

4 Properties

當我們在談到旋轉一物體的時候我們可以將其總結成：

$$\mathbb{T} = \{T \mid T \cdot x \in \mathbb{X}, x \in \mathbb{X}\} \tag{5}$$

其中 $\mathbb{T}$ 代表所有合法的 $n \times n$ 矩陣使得作用於 $\mathbb{X}$ 中任意元件會得結果 $T \cdot x = x' \ni x, x' \in \mathbb{X}$。那也就是換成講義裡的記號 $|G| = \#\mathbb{T}$。

4.1 Calculate $\#\mathbb{T}$

還記得在定理??中所說的最少資料量嗎？現在對我們的 n 維物體來標示各種 orientation 的話，舉 $\mathbb{R}^3$ 物體為例，可靠標記其中三個面的方位即可。所以（為了後續計算方便）我們就永遠選相鄰的三個邊，以數對 (A, B, C) 表示[1]。那現在令所有可能的數對之集合：

$$\mathbb{P} = \{(D, E, F) \mid \angle DOE = \angle AOB, \angle DOF = \angle AOC, \angle EOF = \angle BOC\} \tag{6}$$

其中我們叫這保角性質我們所選之數對 (A, B, C) 的 Structure，而單純允許旋轉而不允許映射的情況下，這是一個該被確保的 Structure。另外須注意這 Structure 是有方向之分的，通常 A, B, C 採逆時鐘排列（從原點向外指，右手方向）。
那現在定義完 $\mathbb{P}$，要計算旋轉方式就簡單多了。首先我們知道 $\forall P \in \mathbb{P}, \exists T \in \mathbb{T} \ni T \cdot (A, B, C) = P$（其中 $T \cdot (A, B, C) = (TA, TB, TC)$ 所以 $\#\mathbb{P} = \#\mathbb{T} = |G|$。而至於 $\mathbb{P}$ 則可用排列組合推出，對立方體舉例：

$$\begin{gathered}
\text{先任意選一面 } A \\
\text{再選一面相鄰 } A \text{ 的面 } B \\
\text{最後再選唯一一個在這兩面逆時鐘方向的面 } C \\
\text{得 } \#\mathbb{P} = C^6_1 \times C^4_1 \times C^1_1 = 24
\end{gathered}$$

對正四面體亦同：

$$\begin{gathered}
\text{先任意選一面 } A \\
\text{再選一面相鄰 } A \text{ 的面 } B \\
\text{最後再選唯一一個在這兩面逆時鐘方向的面 } C \\
\text{得 } \#\mathbb{P} = C^4_1 \times C^3_1 \times C^1_1 = 12
\end{gathered}$$

或者嘗試將它 Generalized 對任意正多面體：

$$|G| = \text{（面數）} \times \text{（一面的邊數、一面相鄰的面數）} \tag{7}$$

[1] 本文中所有表示面之數，皆為向量，且皆由原點指向該面之重心

4.2 Calculate $\mathbb{T}$

那算完他的數量，我們有沒有從它回推 $\mathbb{T}$ 裡面的內容呢？有的（至少用電腦算式簡單的），我們甚至能算出他的組數（那個 k^n 的 n）一樣拿立方體舉例：

$$\text{令所有面之向量之列表 } L = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{8a}$$

$$\text{舉例取轉換 } T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8b}$$

$$\text{得 } TL = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{8c}$$

其中我們可看到，縱列 1∼4 向右位移了一格，而 5,6 不變。因此可得對變換 T（z 軸左手旋轉 90 度）可分成三組，也就是對 L, TL 做 disjoint set 後數他的數量。
就此我們就可以算出他的塗色可能性了。

```
disjointSetNum :: Matrix -> Matrix -> Int
disjointSetNum (Matrix a) (Matrix b) = sum $ map fromEnum (map (uncurry (==)) (zip a b))
```

5 Extend Research Topics（未寫）

5.1 2x2 魔術方塊

The way I model 2x2 Rubik's Cube is by first giving a position vector p_i and then a facing vector f_i which tells you which direction is the first face facing.

$$P = \begin{bmatrix} p_1 & \dots & p_8 \end{bmatrix}, \text{ each component of } p_i \text{ is either 1 or -1} \tag{9a}$$

$$F = \begin{bmatrix} f_1 & \dots & f_8 \end{bmatrix}, \ f_i \text{ is one of the 6 different (directional) unit vector} \tag{9b}$$

And know with (p_i, f_i) we could denote any block we want, where we can then give a list of 3 colours (order-sensitive) which will be coloured counterclockwise.

```
data Rubiks2x2 = Rubiks2x2 Matrix Matrix
```

5.1.1 Proof of Completeness

A Matrix

```
newtype Matrix = Matrix [[Int]]
vector :: [Int] -> Matrix
vector a = Matrix [a]
stripMatrix :: Matrix -> [[Int]]
stripMatrix (Matrix x) = x
```

```
instance Show Matrix where
    show (Matrix [x:xs])
                    | null xs = "[" ++ show x ++ "]"
                    | otherwise = "[" ++ show x ++ "]\n" ++ show (Matrix [xs])
    show (Matrix rows)
             | length (head rows) > 1  = "[" ++ (intercalate ", " (map (show.head) rows)) ++ "]\n"
                                  ++ show (Matrix (map tail rows))
             | otherwise = "[" ++ (intercalate ", " (map (show.head) rows)) ++ "]"
```

```
a = Matrix [[1,2],[2,0]]
b = Matrix [[2,4],[2,0]]
```

```
concatM :: Matrix -> Matrix -> Matrix
concatM (Matrix x) (Matrix y) = Matrix $ x ++ y
diagFlip' :: [[a]] -> [[a]]
diagFlip' xs
             | length (head xs) > 1 = map head xs : (diagFlip' (map tail xs))
             | otherwise = [map head xs]
diagFlip :: Matrix -> Matrix
diagFlip (Matrix xs) = Matrix $ diagFlip' xs
```

```
instance Num Matrix where
    (Matrix [xs]) + (Matrix [ys]) = Matrix [zipWith (+) xs ys]
    (Matrix (x:xs)) + (Matrix (y:ys)) = concatM (Matrix [zipWith (+) x y]) ((Matrix xs)+(Matrix ys))
    (Matrix []) + a@(Matrix _) = a
    a@(Matrix _) + (Matrix []) = a
    x * (Matrix ys) = Matrix $ map f ys
                    where f y = map (sum.(zipWith (*)) y) xs'
                          (Matrix xs') = diagFlip x
```

B Sage Graphics

B.1 2x2 Rubik's Cube

```
def colorRect3D(x, c, l, f): # f: 0 (xy), 1 (xz), 2 (yz)
```

```
    x = vector(x)
    if f == 0:
        sv1 = vector((-1, 0, 0))
        sv2 = vector((0, -1, 0))
    elif f == 1:
        sv1 = vector((-1, 0, 0))
        sv2 = vector((0, 0, -1))
    elif f == 2:
        sv1 = vector((0, -1, 0))
        sv2 = vector((0, 0, -1))
    Gph = Graphics()
    Gph += polygon3d([x, x+sv1, x+sv1+sv2, x+sv2])
    return Gph

def colorBlock(p, f, c):
    Gph = Graphics()
    baseVector = p * 0.5
    for cl in c:
        if f[2]!=0:
            Gph += colorRect3D(baseVector, c, 1, 0)
    return Gph

def plotRubiks2x2(P, F, cs):
    Gph = Graphics()
    for p, f, c in zip(P, F, cs):
        Gph += colorBlock(p, f, c)
    # save3D(Gph) # Comment out if you don't want auto-export
    return Gph

def save3D(g, n="plot"):
    filename = "/tmp/"+n+".html"
    g.save(filename)
    os.system("sed -i 's/\/usr\/share/..\/usr\/share/g' "+filename)
    print("Plot3D saved to: "+filename)
```

LICENSE

Codes

Documentation