Unit 8. Ordinary Differential Equations

Numerical Analysis

May 10, 2017

EE/NTHU

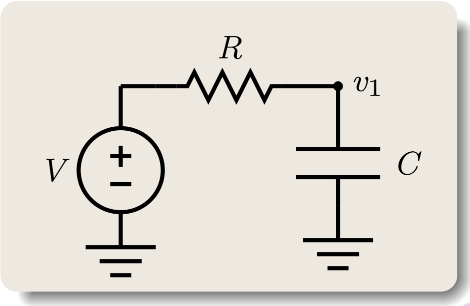
Introduction

- In this unit, we are solving the ordinary differential equation (ODE)

$$\frac{dx}{dt} = f(t, x), \tag{8.1.1}$$

where x is a function of t and with the conditions $t \in [t_0, t_f]$ and $x(t_0) = x_0$.
  - t_f can approach infinite.
- Since the value of x_0 needs to be known and we solve for $t > t_0$, this type of problems is also known as initial value problem (IVP).
- Problems of this type are abundant in our world.
  - In SPICE, this is the transient analysis.

Simple RC Circuit



- A simple example, to solve for the RC network with

$$V(t) = 1, \qquad t \geq 0,$$
$$v_1(0) = 0.$$

- Analytical solution

$$v_1(t) = 1 - \exp(-\frac{t}{RC}).$$

- Nodal analysis at node v_1 (KCL)

$$\frac{v_1 - V}{R} + C\frac{dv_1}{dt} = 0.$$

- Or

$$\frac{dv_1}{dt} = \frac{V - v_1}{RC}. \tag{8.1.2}$$

- This equation has the same form as Eq. (8.1.1), with $x = v_1$ and $f(x,t) = (V - v_1)/RC$.
  - Note that f depends on x as well, and t is implicit.
  - In some applications, f can be explicit function of t as well.

Simple RC Circuit, II

- Assuming v_1 is differentiable,

$$\frac{dv_1}{dt} = \frac{v_1(t+h) - v_1(t)}{h} \qquad \text{as } h \to 0.$$

- Substitute into Eq. (8.1.2),

$$\frac{v_1(t+h) - v_1(t)}{h} = \frac{V(t) - v_1(t)}{RC}$$
$$v_1(t+h) = v_1(t) + h \cdot \frac{V(t) - v_1(t)}{RC}$$

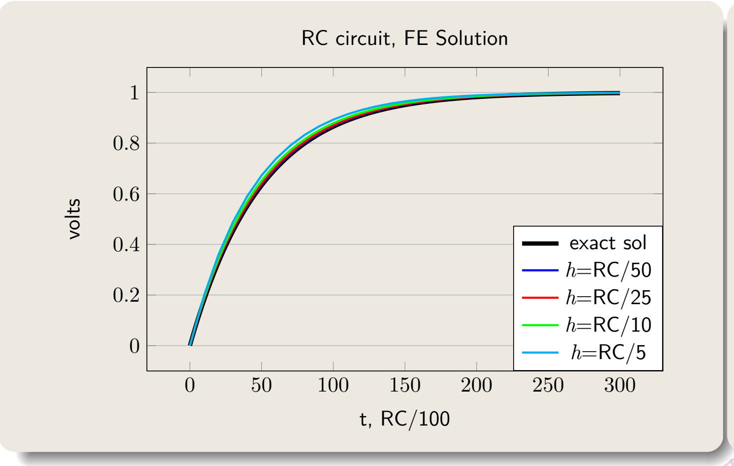
- Giving $V(t), t \geq 0$, and $v_1(0)$ then we can find $v_1(t)$, $t \geq 0$.
- Let $y = \dfrac{h}{RC}$ then

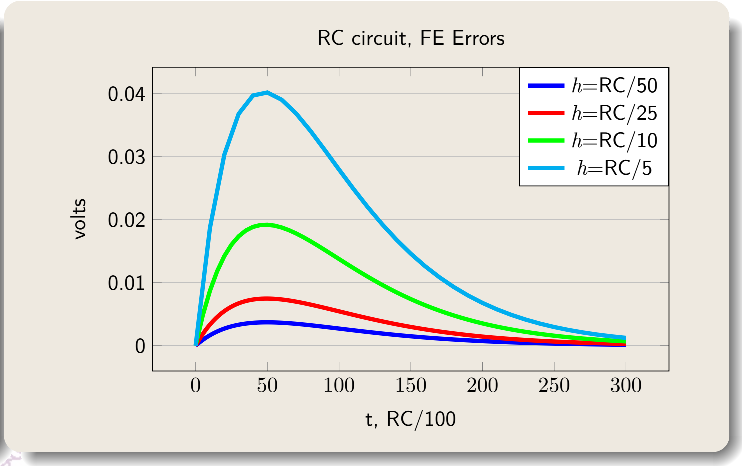
$$v_1(t+h) = (1-y)v_1(t) + y \cdot V(t) \tag{8.1.3}$$

- And

$$\begin{aligned} v_1(0) &= 0, \\ v_1(h) &= y, \\ v_1(2h) &= (1-y)y + y = (2-y)y, \\ v_1(3h) &= (1-y)(2-y)y + y = (3 - 3y + y^2)y, \\ &\cdots \end{aligned}$$

Forward Euler Method



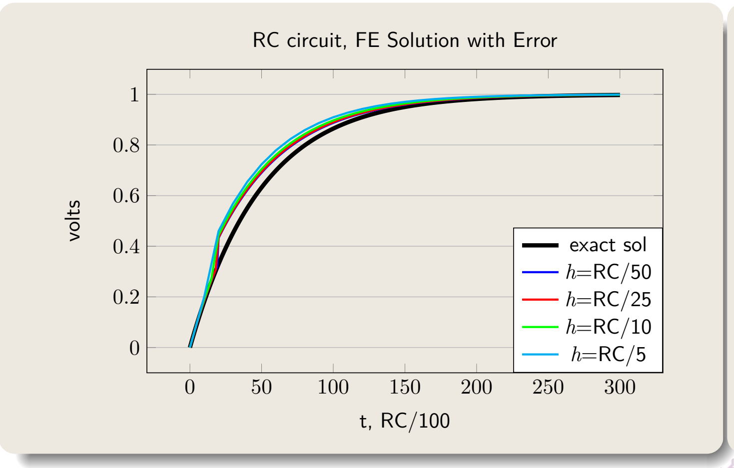


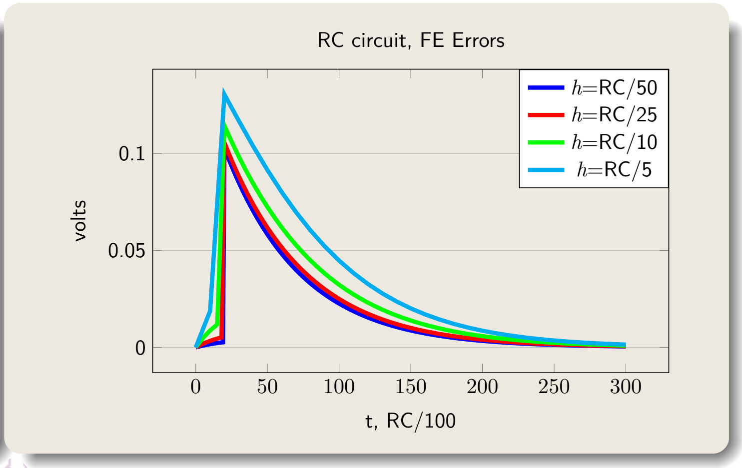
- In general, Eq. (8.1.1) can be solved by

$$x(t+h) = x(t) + h \cdot f(t) \tag{8.1.4}$$

  This is the Forward Euler method.
- For the simple RC network example, it can be observed that the Forward Euler method produces accurate solution.
  - even for relative large h.
- Of course, smaller h produces more accurate solution.

Forward Euler Method, II





- An error is intentionally inserted at $t = 0.2 \cdot RC$ when carrying out Forward Euler method.
- The error gradually decreases as t increases
- Error does not accumulated in Forward Euler method.
- If the initial solution, or the solution at any time point, is erroneous, the solution for large t can still be accurate.
- Error damping is also a function of h.

Backward Euler Method

- Equation (8.1.1) can also be solved using the following equation.

$$\frac{x(t+h)-x(t)}{h}=f(t+h,x(t+h)).$$

And, hence

$$x(t+h)=x(t)+h\cdot f(t+h,x(t+h)). \tag{8.1.5}$$

- This is the Backward Euler method.
- The solution to the simple RC circuit can be written as

$$v_1(t+h)=v_1(t)+h\cdot\frac{V(t+h)-v_1(t+h)}{RC}$$
$$(1+\frac{h}{RC})v_1(t+h)=v_1(t)+\frac{h}{RC}V(t+h)$$

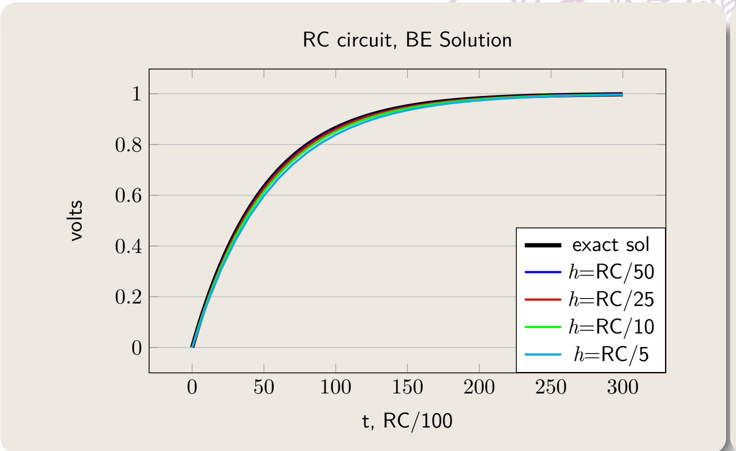
Let $y=\dfrac{h}{RC}$ then

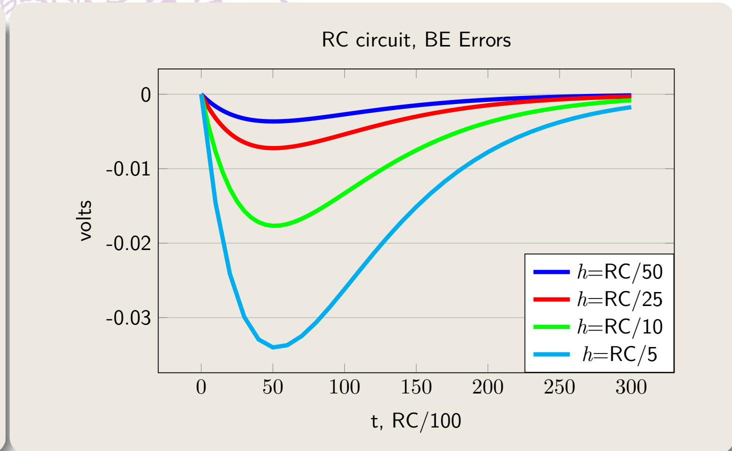
$$v_1(t+h)=\frac{1}{1+y}v_1(t)+\frac{y}{1+y}V(t+h). \tag{8.1.6}$$

Backward Euler Method, II

- And

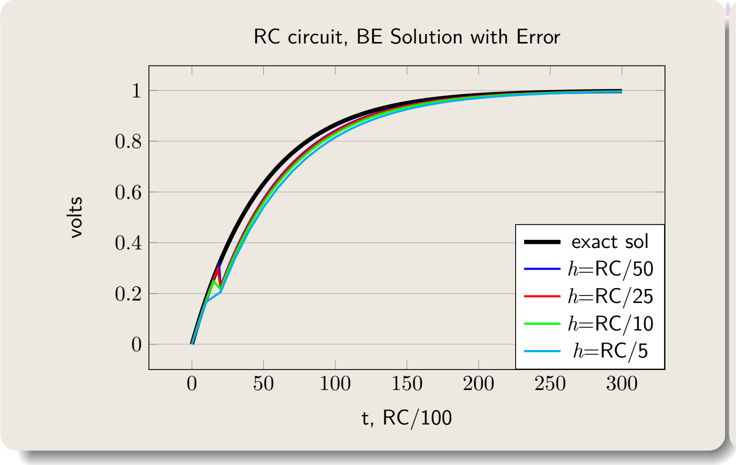
$$v_1(0)=0$$
$$v_1(h)=\frac{y}{1+y}$$
$$v_1(2h)=\frac{y}{1+y}\left(\frac{1}{1+y}+1\right)$$
$$v_1(3h)=\frac{y}{1+y}\left(\frac{1}{(1+y)^2}+\frac{1}{1+y}+1\right)$$
$$\cdots$$

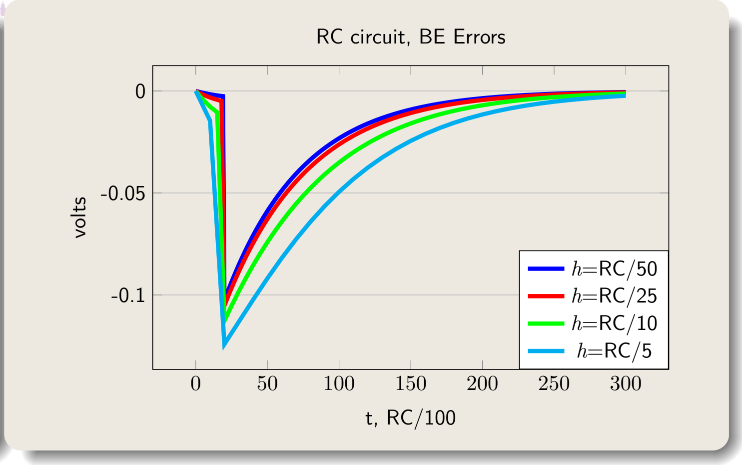




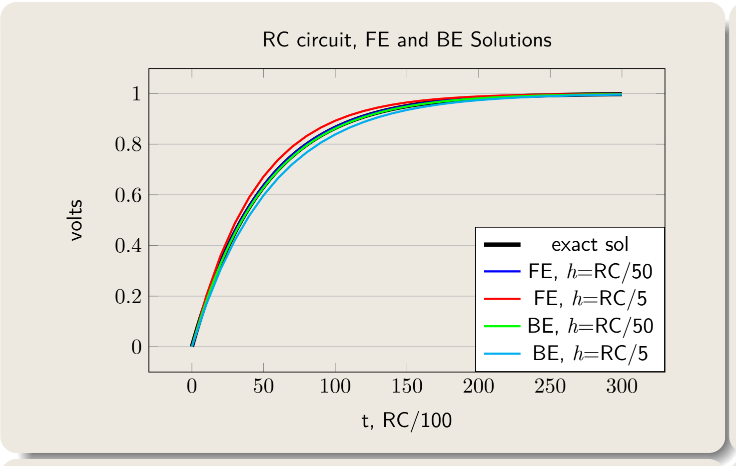
Backward Euler Method, III

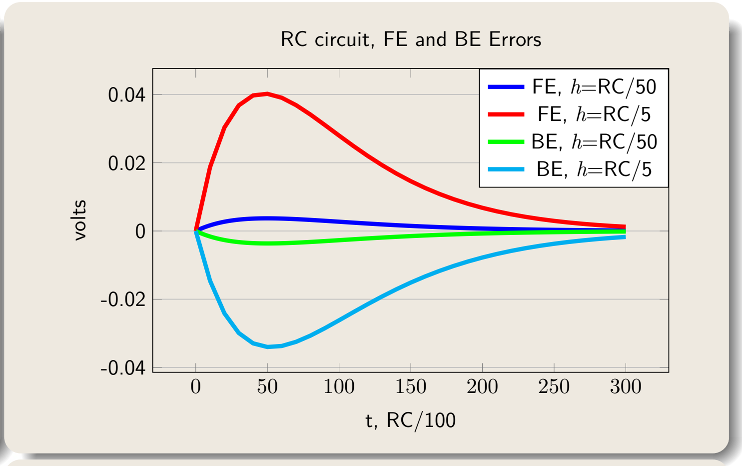
- Backward Euler method produces accurate results as well
- Even an error of -0.1 volts is introduced intentionally at $t = 0.2RC$
- Error damps out - no error accumulation
- Backward Euler method appears to be a little more accurate than Forward Euler method.

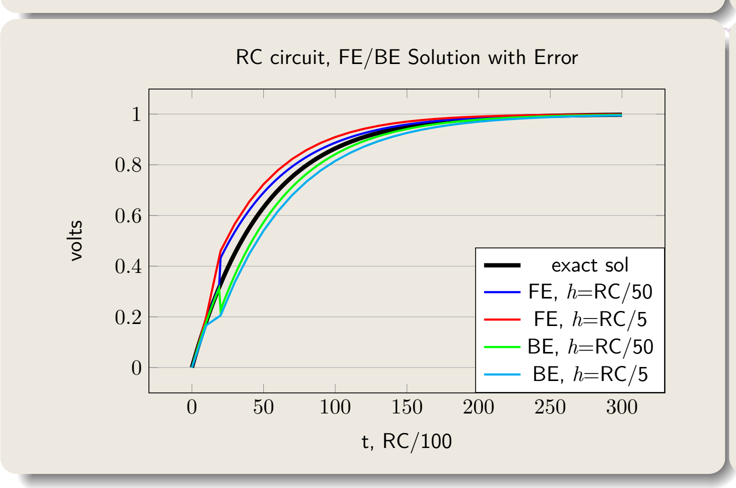


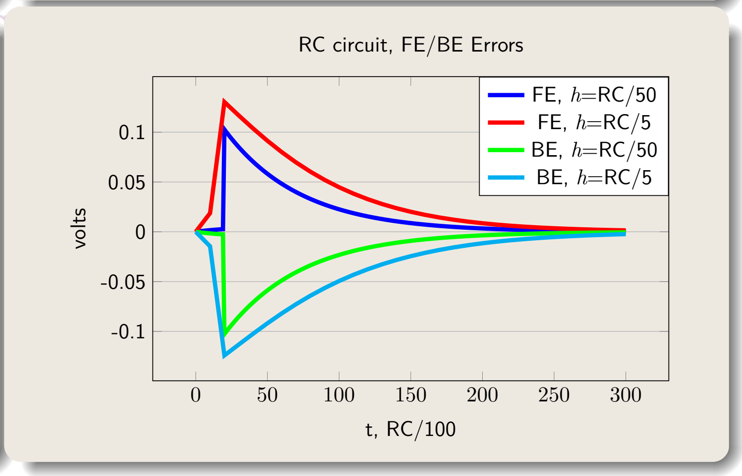


Backward Euler Method, IV









First Order Solution Methods

- To solve the ordinary differential equation

$$\frac{dx(t)}{dt} = f(t)$$

- Forward Euler method

$$\frac{x(t+h) - x(t)}{h} = f(t)$$

- Backward Euler method

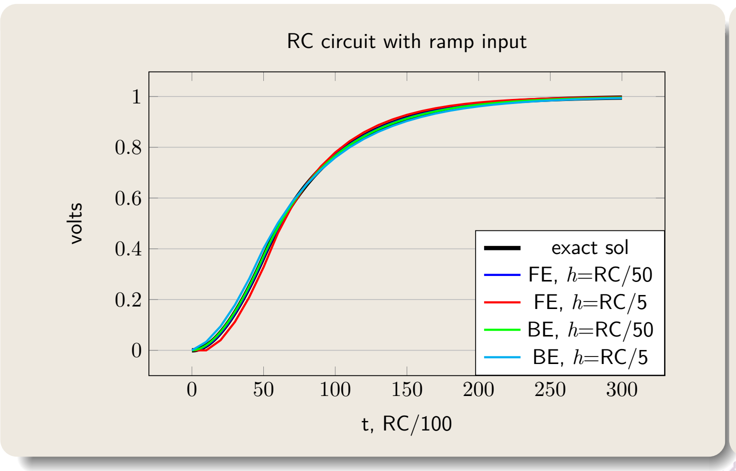
$$\frac{x(t+h) - x(t)}{h} = f(t+h)$$

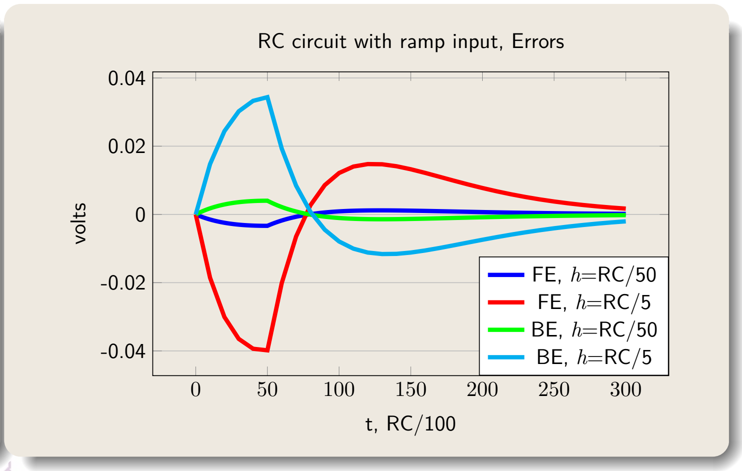
- In the simple RC circuit example, these two methods do not make much difference.
- Let the voltage source waveform of the simple RC circuit be

$$\begin{aligned} V(t) &= t/RC, & 0 \le t \le RC, \\ &= 1, & t \ge RC. \end{aligned}$$

- Forward Euler: $v_1(t+h) = (1-y)v_1(t) + yV(t)$.
- Backward Euler: $v_1(t+h) = (v_1(t) + yV(t+h))/(1+y)$.
  - $y = h/RC$.

First Order Solution Methods, II





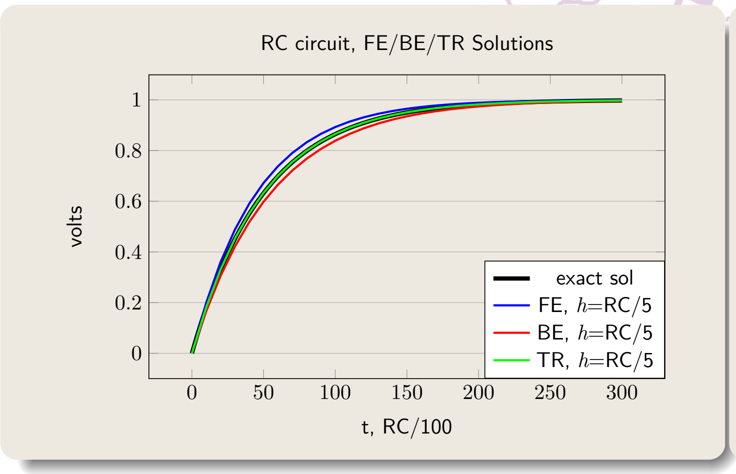
- Both methods produce accurate solutions.
- Backward Euler appears to be more accurate.
- Any input voltages can be solved.
- No error accumulation.
- Good accuracy even with relative large time steps.

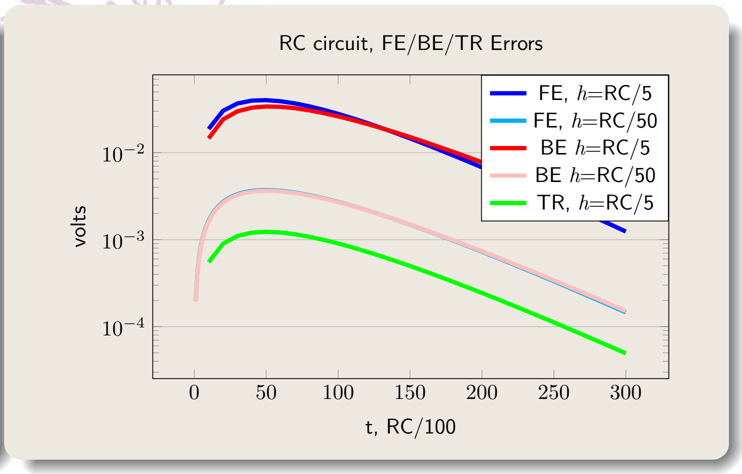
Trapezoidal Rule

- To solve the ODE
$$\frac{dx(t)}{dt} = f(x(t), t)$$
Note that
$$x(t) = x(t_0) + \int_{t=t_0}^{t} f(x(\tau), \tau)\, d\tau$$
- Both Forward Euler and Backward Euler methods are composite integration formula with zero'th order quadrature
- Trapezoidal rule can be more accurate and it is expressed as
$$x(t) = x(t_0) + h \cdot \frac{f(x(t+h), t+h) + f(x(t), t)}{2}.$$
- For the RC network
$$\frac{dv_1}{dt} = \frac{V(t) - v_1(t)}{RC}$$
Thus,
$$v_1(t+h) = v_1(t) + h \cdot \frac{V(t+h) - v_1(t+h) + V(t) - v_1(t)}{2RC}$$

Trapezoidal Rule, II

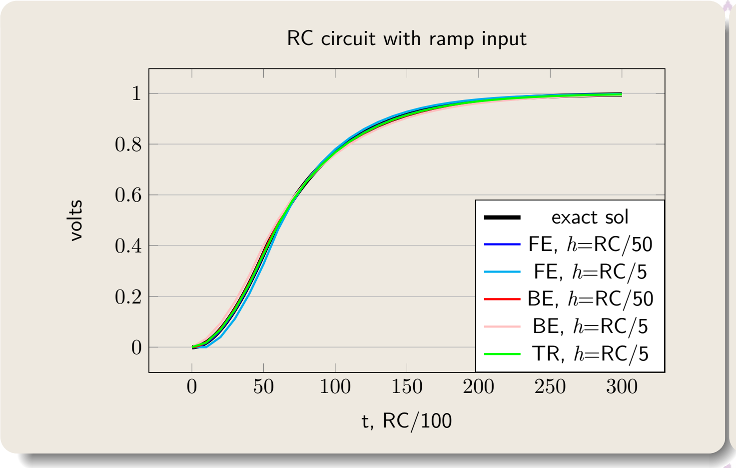
- Let $y = \dfrac{h}{RC}$, then
$$(1 + 0.5y)v_1(t+h) = (1 - 0.5y)v_1(t) + 0.5y\,(V(t+h) + V(t))$$
$$v_1(t+h) = \frac{1 - 0.5y}{1 + 0.5y} v_1(t) + \frac{0.5y}{1 + 0.5y}(V(t+h) + V(t))$$

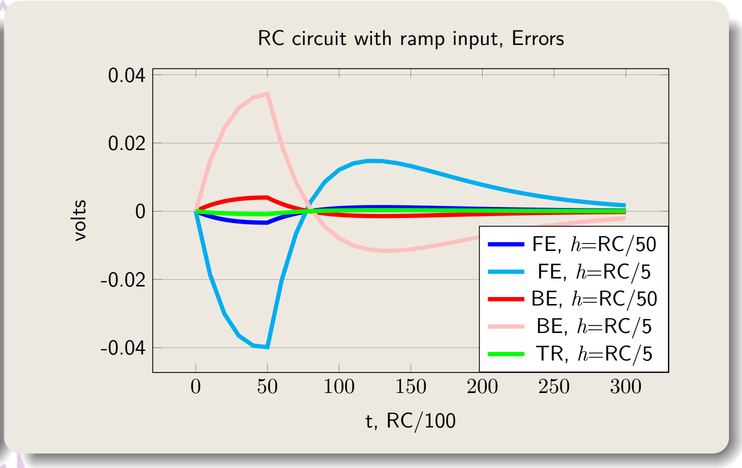




- For RC network with step input, trapezoidal method with large time step is vary accurate.
  - More accurate than Forward Euler or Backward Euler with 10 times small time step.

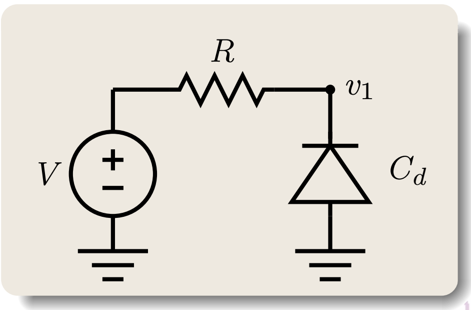
Trapezoidal Rule, III





- For RC network with ramp input, trapezoidal rule is still more accurate than Forward Euler or Backward Euler with larger time steps.

Nonlinear Dynamic Equation

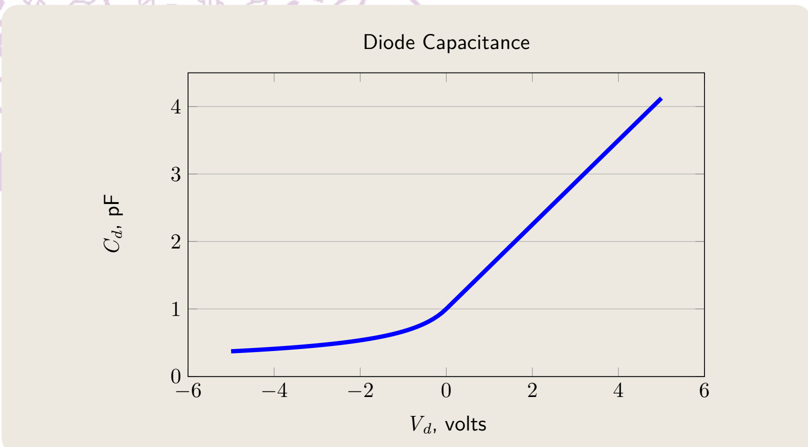


- The diode capacitance is a nonlinear function of the voltage.

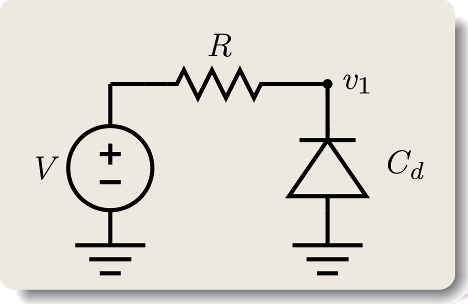
$$C_d = C_J\left(1 - \frac{V_d}{\phi_B}\right)^{-M}, \qquad V_d < 0,$$
$$= C_J\left(1 + \frac{MV_d}{\phi_B}\right), \qquad V_d \geq 0.$$

V_d: voltage across diode,
C_J: junction capacitance at $V_d = 0$ volts,
M: junction grading coefficient,
ϕ_B: junction contact potential.

$$R = 50 \text{ K}\Omega,$$
$$C_J = 1 \text{ pF},$$
$$M = 0.5,$$
$$\phi_B = 0.8.$$



Nonlinear Dynamic Equation, II



$$R = 50\ \text{K}\Omega,$$
$$C_J = 1\ \text{pF},$$
$$M = 0.5,$$
$$\phi_B = 0.8,$$
$$V(t) = 1, \quad t \geq 0,$$
$$v_1(0) = 0\ \text{volts}.$$

- Ignoring diode off current for the time being
- Nodal equation for v_1

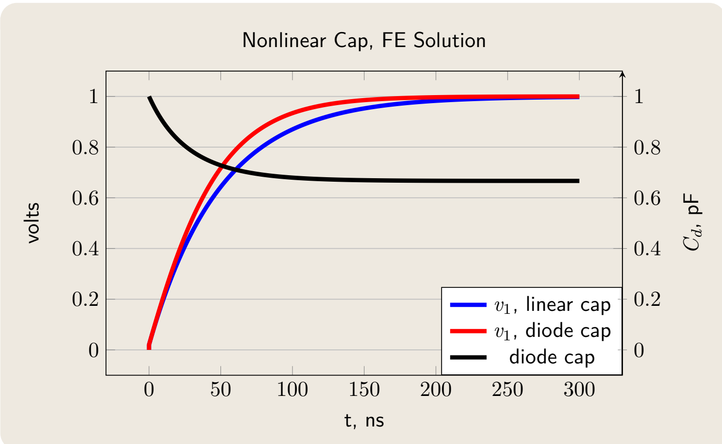
$$C_d \frac{dv_1}{dt} + \frac{v_1 - V}{R} = 0$$
$$\frac{dv_1}{dt} = \frac{V - v_1}{RC_d}$$

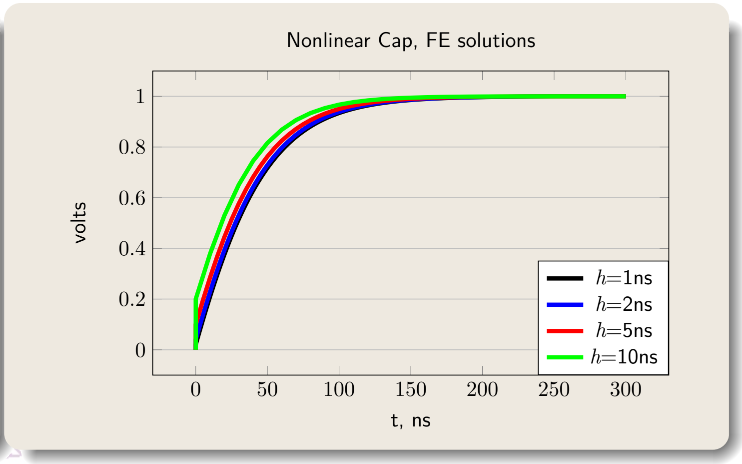
- Apply forward Euler method

$$\frac{v_1(t+h) - v_1(t)}{h} = \frac{V(t) - v_1(t)}{RC_d(-v_1(t))}$$
$$v_1(t+h) = v_1(t) + h \cdot \frac{V(t) - v_1(t)}{RC_d(-v_1(t))}$$

- The same equation as the linear cap case, except C_d is a function of v_1 now
- Since the right-hand side is evaluated at time t, $v_1(t+h)$ can be easily calculated.
- The advantage of forward Euler method.

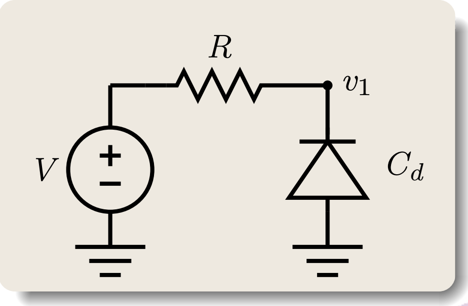
Nonlinear Dynamic Equation, III





- Forward Euler method is effective in solving nonlinear dynamic equation.
- Since diode is in reverse bias region, the capacitance decreases and faster voltage ramp up is observed.
- Different time steps can still be exploited for speed-accuracy trade off.

NDE, Backward Euler Solution



$$R = 50\ \text{K}\Omega,$$
$$C_J = 1\ \text{pF},$$
$$M = 0.5,$$
$$\phi_B = 0.8,$$
$$V(t) = 1, \quad t \geq 0,$$
$$v_1(0) = 0\ \text{volts}.$$

- Nodal equation for v_1

$$\frac{dv_1}{dt} = \frac{V - v_1}{RC_d}$$

- Backward Euler method:

$$\frac{v_1(t+h) - v_1(t)}{h} = \frac{V(t+h) - v_1(t+h)}{RC_d},$$

$$(1 + \frac{h}{RC_d})v_1(t+h) - v_1(t) - \frac{h}{RC_d}V(t+h) = 0. \tag{8.1.7}$$

This equation is nonlinear and can be solved by Newton's method.

$$v_1^{(k+1)}(t+h) = v_1^{(k)}(t+h) - \frac{f(v_1(t+h))}{\partial f(v_1(t+h))/\partial v_1(t+h)}. \tag{8.1.8}$$

NDE, Backward Euler Solution, II

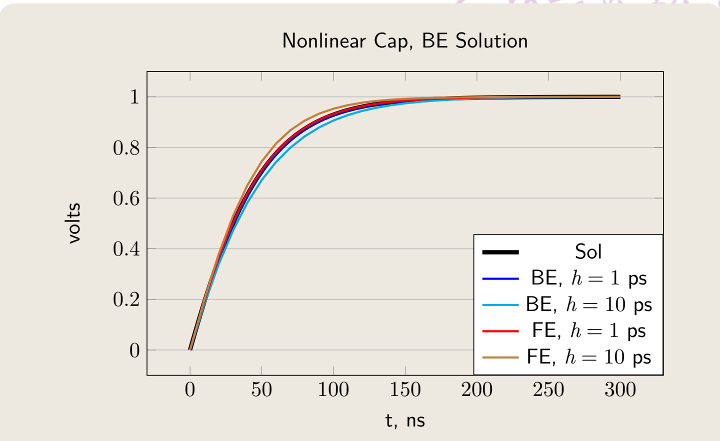
- In Eq. (8.1.8)

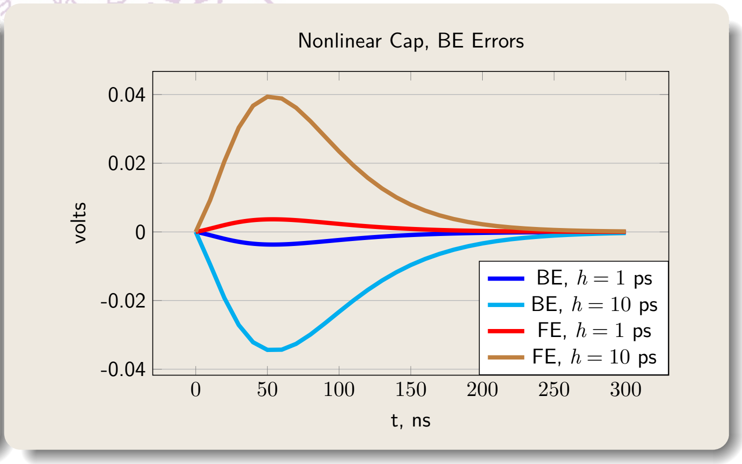
$$f(v_1(t+h)) = (1 + \frac{h}{RC_d})v_1(t+h) - v_1(t) - \frac{h}{RC_d}V(t+h), \tag{8.1.9}$$

and

$$\frac{\partial f(v_1(t+h))}{\partial v_1(t+h)} = 1 + \frac{h}{RC_d}, \tag{8.1.10}$$

where C_d should be evaluated at $v_1(t+h)$.





NDE, Trapezoidal Rule Solution

- Apply trapezoidal rule to the nodal equation of the nonlinear diode capacitor circuit

$$\frac{v_1(t+h)-v_1(t)}{h}=\frac{V(t+h)-v_1(t+h)}{2RC_d(t+h)}+\frac{V(t)-v_1(t)}{2RC_d(t)}.$$

Again, apply Newton's method to solve this nonlinear equation with

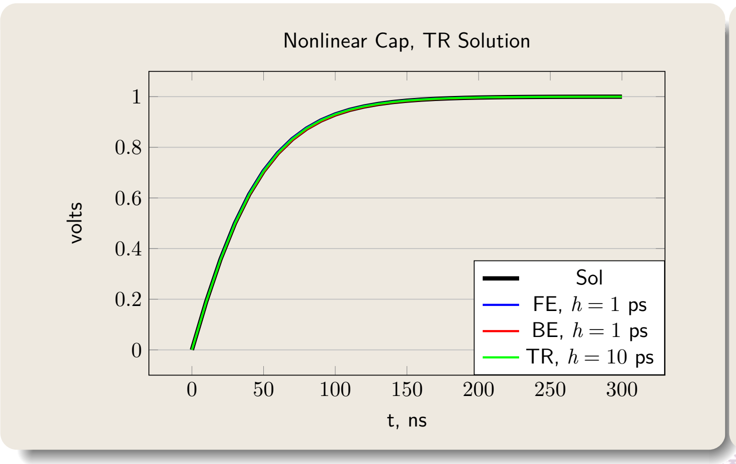
$$\begin{aligned}f(v_1(t+h)) =&(1+\frac{h}{2RC_d(t+h)})v_1(t+h)-(1-\frac{h}{2RC_d(t)})v_1(t)\\&-\frac{h}{2RC_d(t+h)}V(t+h)-\frac{h}{2RC_d(t)}V(t),\end{aligned}$$

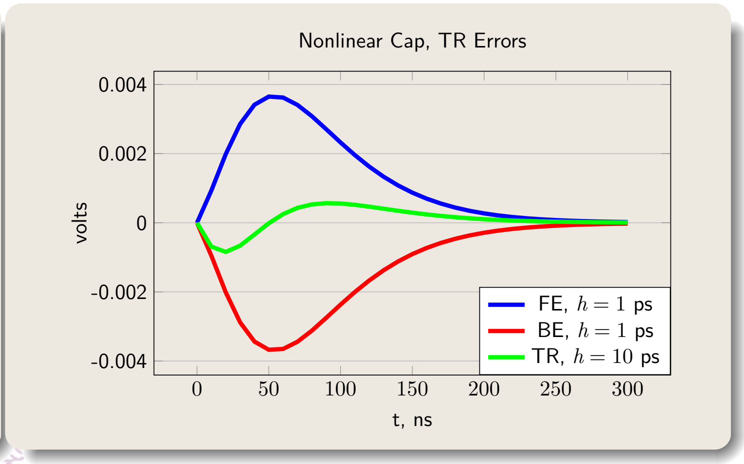
$$\frac{\partial f(v_1(t+h))}{\partial v_1(t+h)}=1+\frac{h}{2RC_d(t+h)}.$$

and iterate

$$v_1^{(k+1)}(t+h)=v_1^{(k)}(t+h)-\left(\frac{\partial f(v_1(t+h))}{\partial v_1(t+h)}\right)^{-1}f(v_1(t+h)).$$

NDE, Trapezoidal Rule Solution, II

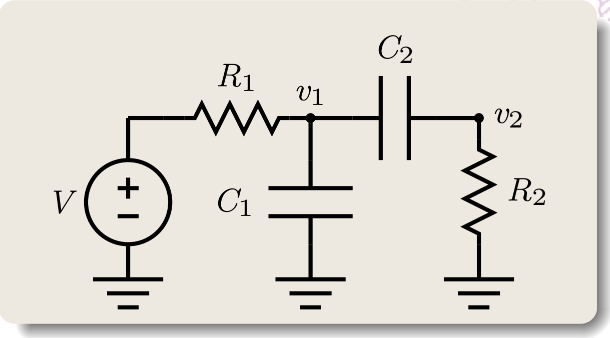




- Nonlinear dynamic equations can be solved using Newton's method and forward Euler, backward Euler or trapezoidal rule
- Trapezoidal rule has more complex formulation but with higher accurate solutions.
  - Higher accuracy with the same time step,
  - Or, with similar accuracy but larger time steps
- Newton's method needs good initial guess
  - In solving time point $t+h$, the solutions at t can be used as initial guess.

Solving Dynamic Systems

- The forward Euler, backward Euler and trapezoidal rule methods can be applied to dynamic systems that have more than one variables.
- For example a two-stage RC ladder network.



- Given initial conditions: $v_1(0)$, $v_2(0)$ and power supply $V(t)$, for $t \geq 0$, to find $v_1(t)$, $v_2(t)$, $t > 0$.
- This linear dynamic system can be solved using any of the integration methods developed above.
- Applying KCL at node v_2

$$C_2 \frac{d(v_2 - v_1)}{dt} + \frac{v_2}{R_2} = 0. \tag{8.1.11}$$

Using backward Euler method and assuming we know $v_1(t)$ and $v_2(t)$ to solve for $v_1(t+h)$, $v_2(t+h)$.

Solving Dynamic Systems, II

- Backward Euler approximates $\dfrac{dx}{dt} = f(x, t)$ by

$$\frac{x(t+h) - x(t)}{h} = f(x(t+h), t+h).$$

- Eq. (8.1.11) can be rewritten as

$$\frac{C_2}{h}\big(v_2(t+h) - v_1(t+h) - v_2(t) + v_1(t)\big) + \frac{v_2(t+h)}{R_2} = 0.$$

Since $v_1(t)$ and $v_2(t)$ are already known, it can be rewritten as

$$\frac{C_2}{h}\big(v_2(t+h) - v_1(t+h)\big) + \frac{v_2(t+h)}{R_2} = \frac{C_2}{h}\big(v_2(t) - v_1(t)\big). \tag{8.1.12}$$

- Similarly for v_1

$$\frac{v_1 - V}{R_1} + C_1 \frac{dv_1}{dt} + C_2 \frac{d(v_1 - v_2)}{dt} = 0.$$

And, with backward Euler

$$\frac{v_1(t+h) - V(t+h)}{R_1} + \frac{C_1}{h} v_1(t+h) + \frac{C_2}{h}\big(v_1(t+h) - v_2(t+h)\big) = \frac{C_1}{h} v_1(t) + \frac{C_2}{h}\big(v_1(t) - v_2(t)\big). \tag{8.1.13}$$

Solving Dynamic Systems, III

- Merging Eqs (8.1.12) and (8.1.13) and arrange in matrix-vector form

$$\begin{bmatrix} \frac{1}{R_1}+\frac{C_1}{h}+\frac{C_2}{h} & -\frac{C_2}{h} \\ -\frac{C_2}{h} & \frac{1}{R_2}+\frac{C_2}{h} \end{bmatrix} \begin{bmatrix} v_1(t+h) \\ v_2(t+h) \end{bmatrix} = \begin{bmatrix} \frac{C_1}{h} v_1(t) + \frac{C_2}{h}\Big(v_1(t)-v_2(t)\Big) \\ \frac{C_2}{h}\Big(v_2(t)-v_1(t)\Big) \end{bmatrix} \tag{8.1.14}$$

- Note that the stamps for a resistor, R_k, connecting nodes i and j are

$$A_{ii}=A_{ii}+\frac{1}{R_k},\quad A_{ij}=A_{ij}-\frac{1}{R_k},\quad A_{jj}=A_{jj}+\frac{1}{R_k},\quad A_{ji}=A_{ji}-\frac{1}{R_k}. \tag{8.1.15}$$

- In a similar way, we can define the stamps for a capacitor, C_k, connect nodes, i and j, to be

$$\begin{aligned} A_{ii}&=A_{ii}+\frac{C_k}{h}, & A_{ij}&=A_{ij}-\frac{C_k}{h}, & b_i&=b_i+\frac{C_k}{h}\Big(v_i(t)-v_j(t)\Big), \\ A_{jj}&=A_{jj}+\frac{C_k}{h}, & A_{ji}&=A_{ji}-\frac{C_k}{h}, & b_j&=b_j+\frac{C_k}{h}\Big(v_j(t)-v_i(t)\Big). \end{aligned} \tag{8.1.16}$$

  when the backward Euler method is used to solve the circuit.
- Using stamping method, we can formulate and simulate RC circuit effectively.

Solving Dynamic Systems, IV

- When using backward Euler method to solve the dynamic circuits, the stamps of a capacitor, C_k, connecting nodes i and j, can also be derived as the following.
- KCL requires the total current leaving a node to be zero. And the current of the capacitor is

$$C_k\frac{d(v_i-v_j)}{dt}=I_c$$

  Using the backward Euler method

$$C_k\frac{v_i(t+h)-v_j(t+h)-v_i(t)+v_j(t)}{h}=I_c(t+h)$$

$$\frac{C_k}{h}\Big(v_i(t+h)-v_j(t+h)-v_i(t)+v_j(t)\Big)=I_c(t+h)$$

  Since $\frac{C_k}{h}\Big(v_i(t)-v_j(t)\Big)$ is a known quantity, it should be added to the right-hand side of the equation. Thus, the stamps are

$$\begin{aligned} A_{ii}&=A_{ii}+\frac{C_k}{h} & A_{ij}&=A_{ij}-\frac{C_k}{h} & b_i&=b_i+\frac{C_k}{h}\Big(v_i(t)-v_j(t)\Big) \\ A_{jj}&=A_{jj}+\frac{C_k}{h} & A_{ji}&=A_{ji}-\frac{C_k}{h} & b_j&=b_j+\frac{C_k}{h}\Big(v_j(t)-v_i(t)\Big) \end{aligned}$$

Solving Dynamic Systems, V

- The forward Euler method, which does not have $I_c(t+h)$ term in the formula and, thus, cannot formulate stamps.
- If the trapezoidal rule is applied, Eq. (8.1.11) should be written as

$$C_k \frac{v_i(t+h) - v_j(t+h) - v_i(t) + v_j(t)}{h} = \frac{I_c(t+h) + I_c(t)}{2}$$

And, thus the current through the capacitor at time $t+h$ is

$$I_c(t+h) = \frac{2C_k}{h}\Big(v_i(t+h) - v_j(t+h) - v_i(t) + v_j(t)\Big) - I_c(t). \tag{8.1.17}$$

This current should be added to the matrix equation, and thus the stamps are

$$\begin{aligned} A_{ii} &= A_{ii} + \frac{2C_k}{h} & A_{ij} &= A_{ij} - \frac{2C_k}{h} & b_i &= b_i + \frac{2C_k}{h}\Big(v_i(t) - v_j(t)\Big) + I_c(t) \\ A_{jj} &= A_{jj} + \frac{2C_k}{h} & A_{ji} &= A_{ji} - \frac{2C_k}{h} & b_j &= b_j + \frac{2C_k}{h}\Big(v_j(t) - v_i(t)\Big) + I_c(t) \end{aligned} \tag{8.1.18}$$

where $I(t)$ is the current through the capacitor at time t.
- When using trapezoidal rule, the capacitor current of the previous time step needs to be used and it can be calculated using Eq. (8.1.17).
- At $t = 0$, DC condition is assumed and $I_c(0) = 0$.

Theoretical Results

- It is assumed

$$\mathbf{x}' = f(t, \mathbf{x}) \tag{8.1.19}$$

is a system of n ordinary differential equations, and

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{8.1.20}$$

Theorem 8.1.1.

Let f be defined and continuous on $\mathbf{S} = \{(t, \mathbf{x}), a \le t \le b, \mathbf{x} \in \mathbb{R}^n\}$, a and b are finite. Furthermore, let there be a constant L such that

$$\|f(t, \mathbf{x}_1) - f(t, \mathbf{x}_2)\| \le L\|\mathbf{x_1} - \mathbf{x_2}\| \tag{8.1.21}$$

for all $t \in [a, b]$ and all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ (Lipschitz condition). Then for every $t_0 \in [a, b]$ and every $\mathbf{x}_0 \in \mathbb{R}^n$ there is exactly one function $\mathbf{x}(t)$ such that

(a) $\mathbf{x}(t)$ is continuous and continuously differentiable for $t \in [a, b]$;
(b) $\mathbf{x}'(t) = f(t, \mathbf{x}(t))$ for $t \in [a, b]$;
(c) $\mathbf{x}(t_0) = \mathbf{x}_0$.

Theoretical Results, II

Theorem 8.1.2.

Let the function $\mathbf{f}: \mathbf{S} \to \mathbb{R}^n$ be continuous on $\mathbf{S} = \{(t, \mathbf{x}), a \le t \le b, \mathbf{x} \in \mathbb{R}^n\}$ and satisfy the Lipschitz condition

$$\|f(t, \mathbf{x}_1) - f(t, \mathbf{x}_2)\| \le L\|\mathbf{x}_1 - \mathbf{x}_2\|$$

for all $(t, \mathbf{x}_1), (t, \mathbf{x}_2) \in \mathbf{S}$. Let $a \le t_0 \le b$. Then for the solution $\mathbf{X}(t, \mathbf{s})$ of the initial value problem

$$\mathbf{x}' = f(t, \mathbf{x}), \qquad \mathbf{x}(t_0, \mathbf{s}) = \mathbf{s} \tag{8.1.22}$$

there holds the estimate

$$\|\mathbf{x}(t, \mathbf{s}_1) - \mathbf{x}(t, \mathbf{s}_2)\| \le e^{L|t-t_0|}\|\mathbf{s}_1 - \mathbf{s}_2\| \tag{8.1.23}$$

for $a \le t \le b$.

Theoretical Results, III

Theorem 8.1.3.

If in addition to assumption of the preceding theorem the Jacobian matrix $\mathbf{J}_\mathbf{x} = [\partial f_i/\partial x_j]$ exists on $\mathbf{S}$ and is continuous and bounded,

$$\|\mathbf{J}_\mathbf{x}\| \le L \qquad \text{for } (t, \mathbf{x}) \in \mathbf{S},$$

then the solution $\mathbf{x}(t, \mathbf{s})$ of $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$, $\mathbf{x}(t_0, \mathbf{s}) = \mathbf{s}$, is continuously differentiable for all $t \in [a, b]$ and all $\mathbf{s} \in \mathbb{R}^n$. The derivative

$$\mathbf{Z}(t, \mathbf{x}) = [\partial x_i(t, \mathbf{s})/\partial s_j], \tag{8.1.24}$$

is the solution of the initial value problem

$$\mathbf{Z}' = \mathbf{J}_\mathbf{x}\mathbf{Z}, \qquad \mathbf{Z}(t_0, \mathbf{s}) = \mathbf{I}. \tag{8.1.25}$$

Note that all entities in Eq. (8.1.25) are $n \times n$ matrices, and can be obtained by differentiating with respect to s the original system of equations.

$$\mathbf{x}' = f(t, \mathbf{x}(t, \mathbf{s})), \qquad \mathbf{x}(t_0, \mathbf{s}) = \mathbf{s}.$$

Theoretical Results, IV

- Suppose Eq. (8.1.25) is rewritten as

$$\mathbf{Z}' = \mathbf{T}(t)\mathbf{Z}, \qquad \mathbf{Z}(a) = \mathbf{I}. \tag{8.1.26}$$

Theorem 8.1.4.

If $\mathbf{T}(t)$ is continuous on $[a, b]$, and let $k(t) = \|\mathbf{T}(t)\|$, then the solution $\mathbf{Z}(t)$ of Eq. (8.1.26) satisfies

$$\|\mathbf{Z}(t) - \mathbf{I}\| \leq \exp\left(\int_a^b k(t)\,dt\right) - 1, \qquad t \geq a. \tag{8.1.27}$$

- This is the extended version of Theorem (8.1.2).
- The solution of the initial value problem depends on the initial condition and grows exponentially with the independent variable t.

Summary

- Ordinary differential equation
  - Initial value problem
- Forward Euler Method
  - RC network example
- Backward Euler method
  - RC network with ramp input
- Trapezoidal rule
- Nonlinear dynamic equations
- Capacitor stamps
  - Backward Euler method
  - Trapezoidal rule method