Numerical Analysis
homework 05: Conjugate Gradient Methods

Due on Tuesday, April 4, 2017

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1 Introduction

To solve a such linear system:

$$Ax = b \tag{1}$$

We have already use **LU Decomposition**, **Jacobi**, **Gauss-Seidel** and **Symmetric Gauss-Seidel** to solve it. However, in the previous homework, we found that the three iterative methods are slower than **LU Decomposition**. As a result, to solve the system more faster, **Conjugate Gradient Descend Method** was introduced. **Conjugate Gradient** can solve Equation 1 faster a lot than **LU Decomposition**.

1.1 Resistor Network

To evaluate the performance of algorithm, we will build several simple resistor networks(shown in Figure 1) to test the accuracy and efficiency.

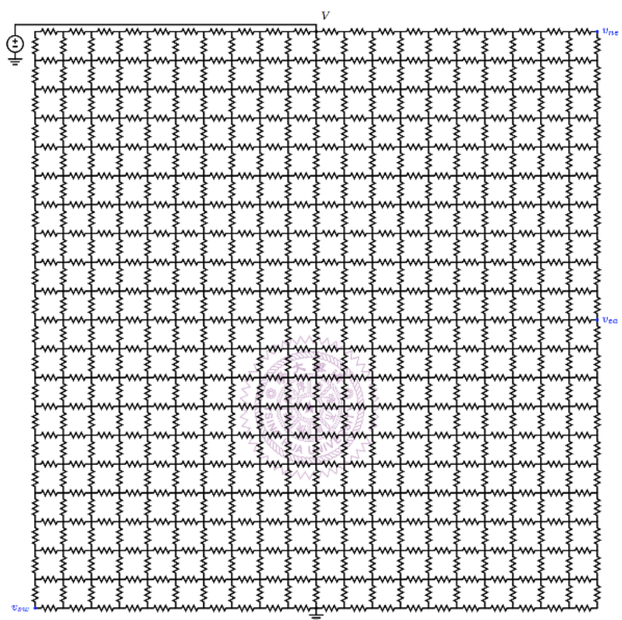

Figure 1: Simple Resistor Network

For **Confugate Gradient Methods**, the error between iteration is defined as Equation 2:

$$Error = \sqrt{\frac{r^T r}{n}} \tag{2}$$

2 Implementation

Algorithm 1 Conjugate Gradient Methods

$p^{(0)} = r^{(0)} = b - Ax$
for each iteration k $\in$ {0, maxIter-1} **do**
  $\alpha_k = \frac{(p^k)^T r^{(k)}}{(p^k)^T A p^k}$
  $x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$
  $r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)}$
  $\beta_k = \frac{(p^{(k)})^T A r^{(k+1)}}{(p^{(k)})^T A p^{(k)}}$
  $p^{(k+1)} = r^{(k+1)} - \beta_k p^{(k)}$
  if $\sqrt{\frac{(r^{(k+1)})^T r^{(k+1)}}{n}} < tol$ **then**
    break
  end if
end for

2.1 Complexity

For each iteration of Algorithm 1, the most time-consuming part is the multiplication of Matrix and Vector, which need a double for-loop. As a result, this is a $\boldsymbol{O(n^2)}$ problem.

3 Discussion

In this section, we will discuss the following issues:

1. Why is Conjugate Gradient better than LU?
2. Efficiency of Jacobi, Guass-Seidel, Symmetric Gauss-Seidel and Conjugate Gradient.
3. Error of each iteration.

3.1 Conjugate Gradient vs LU Decomposition

In the Discussion of homework03, we know that the complexity of LU Decomposition is $O(n^3)$, while each iteration of Conjugate Gradient is $O(n^2)$. As a result, if the procedure of solving linear system needs small numbers of iteration, Conjugate Gradient can provide a much faster method to solve Equation 1.
To compare the efficiency, I run several experiments for LU and CG. Table 1 shows the result of LU, because of LU is too slow when N is large, so the runtime after N > 2601 is predicted by $O(n^3)$ and the three corner voltage value is provided by another classmate in this couse. For Conjugate Gradient method, Table 2 shows the detail result to achieve 10^{-7} error compared with LU Decomposition.

N	9	25	121	441	1681	2601	6561	10201
Vne	0.75	0.7	0.648693	0.622178	0.603088	0.598084	0.588931	no data
Vea	0.5	0.5	0.5	0.5	0.5	0.5	0.5	no data
Vsw	0.25	0.3	0.351307	0.377822	0.396912	0.401916	0.411069	no data
Runtime(s)	0.003	0.003	0.007	0.2	9.948	36.1	580(predict)	2180(predict)

Table 1: Experiment result of LU Decomposition

N	441	1681	2601	3721	6561	10201
# of Resistor	20	40	50	60	80	100
Iterations	75	148	184	221	293	365
Vne	0.622178	0.603088	0.598084	0.594327	0.588931	0.585141
Vea	0.5	0.5	0.5	0.5	0.5	0.5
Vsw	0.377822	0.396912	0.401916	0.405673	0.411069	0.414859
Runtime(s)	0.036	0.8	2.38048	5.96049	23.9501	73.0905
iter_avg	0.00048	0.0054054054	0.0129373913	0.026970543	0.0817409556	0.2002479452

Table 2: Experiment result of Conjugate Gradient

Figure 2 shows **Runtime** in Table 1 and 2 vs N in log-scale. In this figure, we can see a significant efficiency gap between CG and LU. As a result, Conjugate Gradient is a better method for solving Simple Resistor Network.

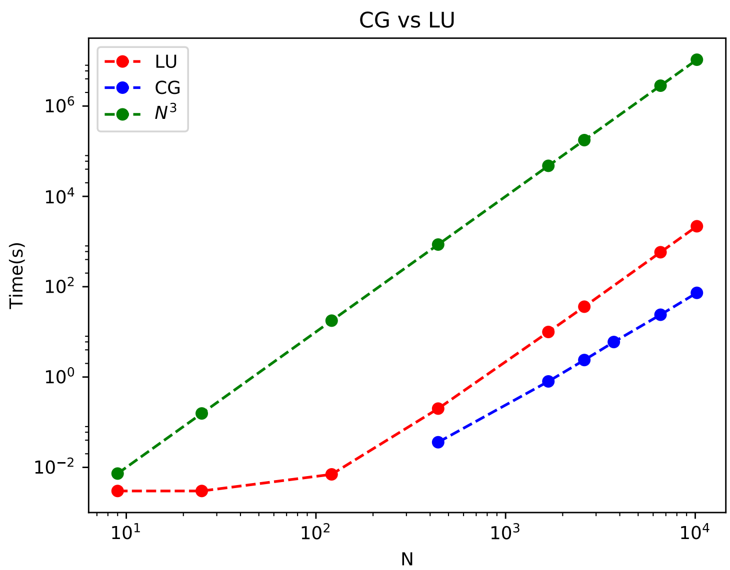

Figure 2: Comparison of CG and LU

3.2 Iterative Methods Comparison

To do comparison between **Jacobi**, **Gauss-Seidel**, **Symmetric Gauss-Seidel** and **Conjugate Gradient**, the most straightforward way is to compare the number of iteration. Table 3 show the result of iteration number.

N	9	25	121	441	1681
Jacobi	62	310	2404	10892	47614
Gauss-Seidel	32	155	1202	5441	23779
SGS	23	90	628	2773	12007
CG	4	10	37	75	148

Table 3: Iteration number of 4 algorithm

Figure 3 shows the result with log-scale.

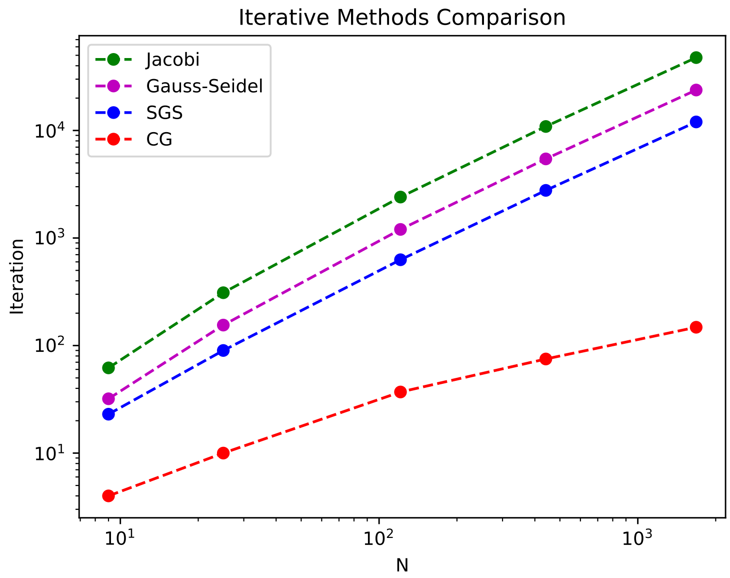

Figure 3: Comparison of 3 iterative methods

In Figure 3 we can see a dramatical drop of iteration number in Conjugate Gradient Method, which means CG can converge faster than the others. As a result, **Conjugate Gradient Method** is the best method for solving Simple Resistor Network.

3.3 Iteration Error

To see the trend of error, we can simply plot error of groundtruth(LU) and CG in each iteration. Figure 4 shows the result.

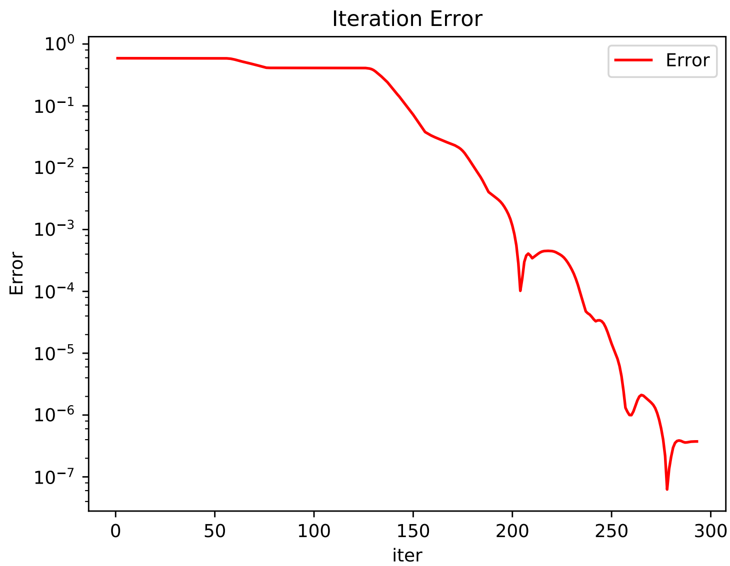

Figure 4: Iteration Error of Conjugate Gradient

In the begining, the error is almost a constant. After about 100 times of iteration, the error starts to drop with significant slope. This is because it was searching the right direction of convergence in the begining. After 100 iteration, it found the right direction and the converge speed increase a lot after that. About the time at 200 iteration, we can see the error starts to increase. This is because it adjusts the origin direction with an new one and make error increase in the begining. After several iteration, the error drop rate starts to increase again and so on.

3.4 Average Iteration Runtime

From Table 2, we can try to plot **iter_avg** vs N. Figure 5 shows the result with log-scale.

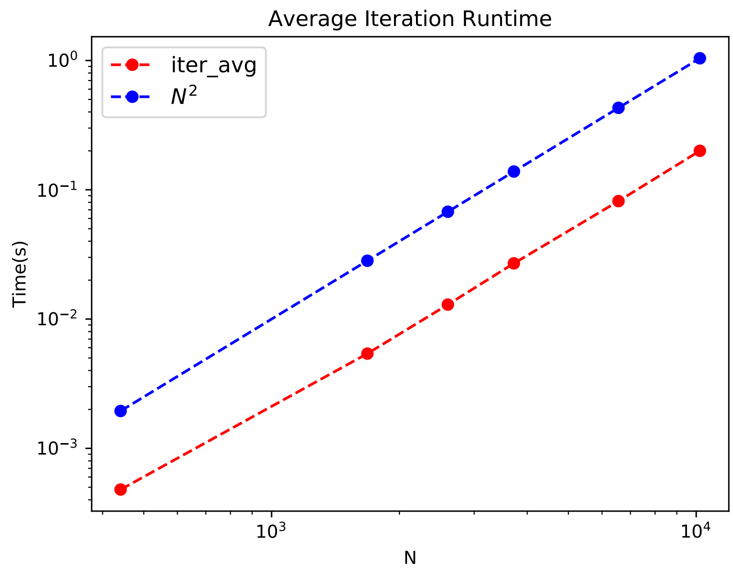

Figure 5: Iteration Runtime of Conjugate Gradient Method

From Figure 5, we can find that the slope of CG is same as N^2. As a result, the complexity of each iteration of Conjugate Gradient Method is $O(n^2)$, which satisfies my analysis in Section 2.1.

4 Conclusion

As we discuss in Section 3.1 and 3.2, **Conjugate Gradient** is faster than both **LU Decomposition** and the other three iteratiove methods. Therefore, the best method to solve a Simple Resistor Network is **Conjugate Gradient Method**.