Unit 9.2 Finite Element Approach

Numerical Analysis

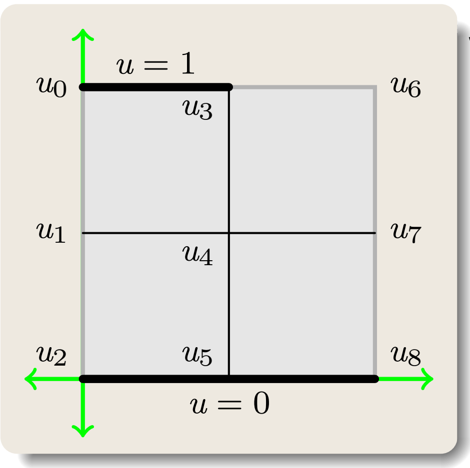
EE/NTHU

Jun. 7, 2017

A 2-D Boundary Value Problem

- Given the Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{9.1.1}$$



with the boundary condtions

$$u = \begin{cases} 0, & 0 \le x \le 1, \quad y = 0, \\ 1, & 0 \le x \le 0.5, \quad y = 1, \end{cases}$$

$$\frac{\partial u}{\partial x} = \begin{cases} 0, & x = 0, \quad 0 \le y \le 1, \\ 0, & x = 1, \quad 0 \le y \le 1, \end{cases}$$

$$\frac{\partial u}{\partial y} = 0, \quad 0.5 \le x \le 1, \quad y = 1.$$

and the grid shown on the left, Finite Difference (FD) discretization on u_4 results in the following equation.

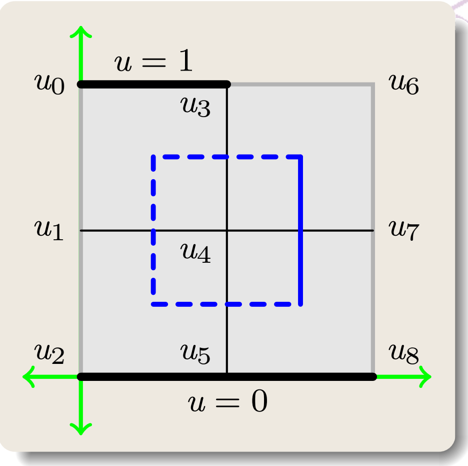
$$\frac{1}{h^2}(u_7 + u_1 + u_3 + u_5 - 4u_4) = 0. \tag{9.1.2}$$

Integral Form

- The Laplace's equation has a dual, integral, form (Gauss' Law)

$$\oint_S \nabla u \; \mathrm{d}s = 0, \tag{9.1.3}$$

where the integral is carried out for an enclosed path S over the normal field ∇u.



- For u_4, the integration over the solid line segment is

$$\nabla u \times h = \frac{u_7 - u_4}{h} \times h = u_7 - u_4. \tag{9.1.4}$$

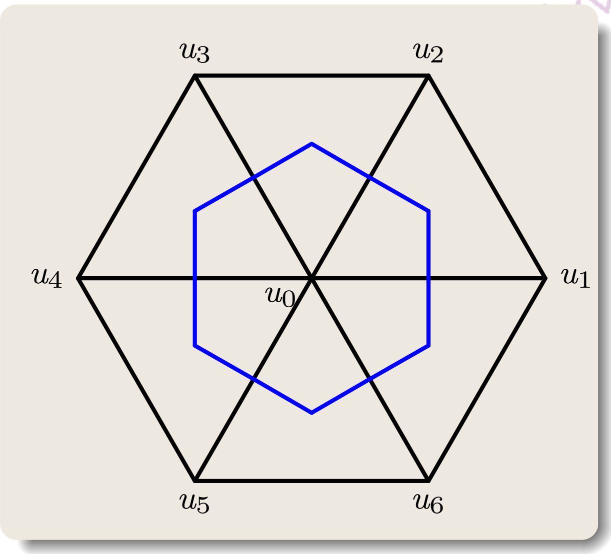
- Thus the path integral is

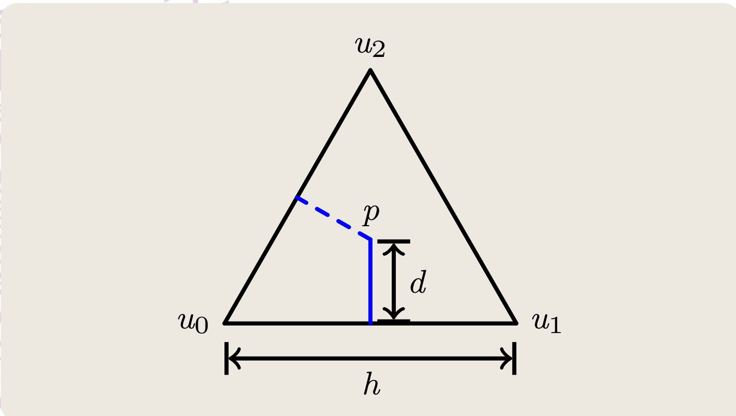
$$\oint_S \nabla u \; \mathrm{d}s = u_7 + u_1 + u_3 + u_5 - 4u_4 = 0. \tag{9.1.5}$$

- Since $h^2 \neq 0$, this equation is the equivalent to Eq. (9.1.2).

Finite Element Approach

- The finite element approach divides the domain into triangles.





p is the circumcenter of the triangle.

- The integral for the segment is

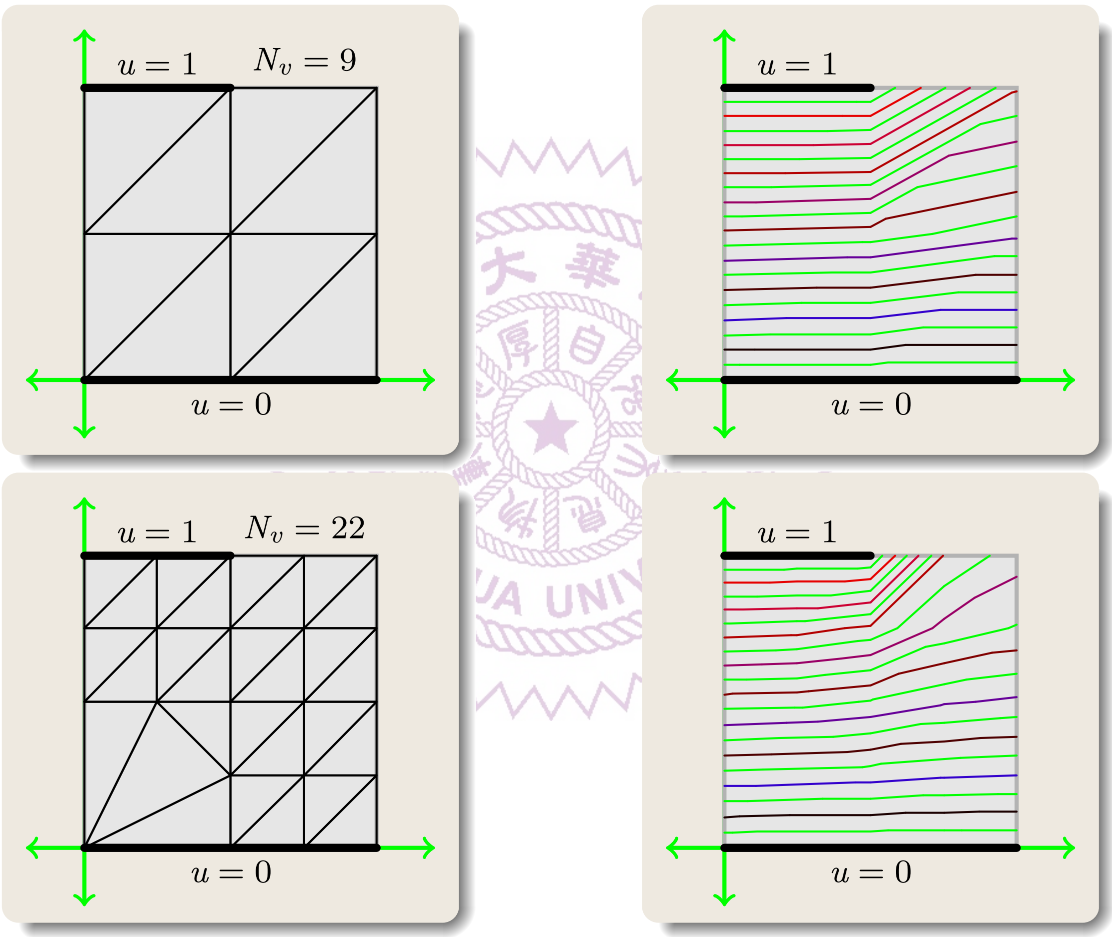
$$\frac{d}{h}(u_1 - u_0) \tag{9.1.6}$$

- The path integral for u_0 follows the blue line.

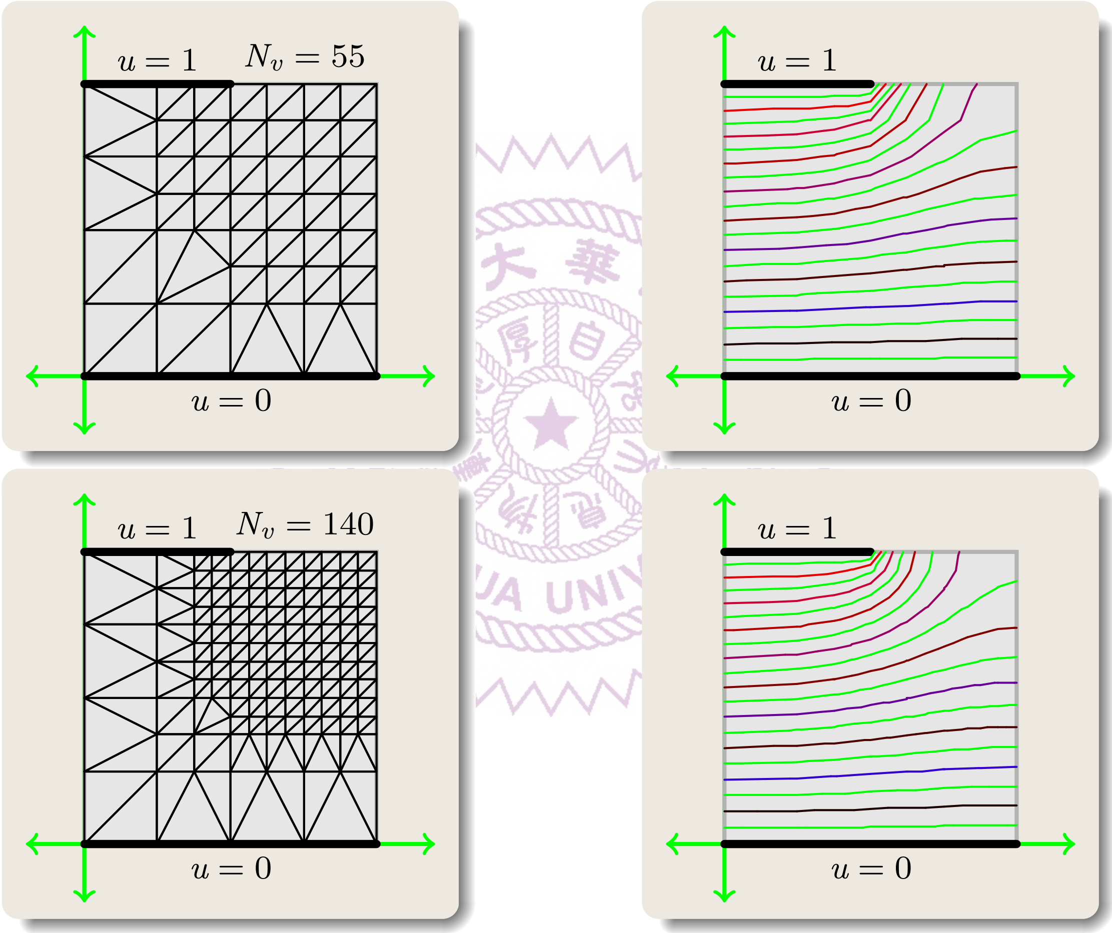
Finite Element Approach

- For each triangle the circumcenter can be calculated since the position of the three vertices are known.
- Thus, the length of each line segments can also be calculated.
- The contribution of each segment to the path integral can be summed to each vertex.
- After all triangles are processed, the linear system is formed to solved for all node variables.
- Note that for finite element approach, the circumcenter of all the triangles should reside inside of the triangle.
  - If there are obtuse triangles, which have the circumcenter outside of the triangle, the path integral cannot be properly performed.
- Other types of triangles can all be handled in finite element approach.

Grids for Finite Element Approach



Grids for Finite Element Approach



Grids for Finite Element Approach, II

- Triangles are more flexible to form various grid to solve the Gauss' Law, or Laplace's Equation.
  - Variable grid spacing easily formed.
  - More flexible boundary contours.
- Grid generation, however, can be more complicated.

Summary

- Finite element approach transform the differential equation into integral equation.
- Then assume the solution is a (lower degree) function in each element
- Assemble all the equations into a linear system to solve for the solution
- The basic element can be any simple shape
  - Triangle is easy to handle and conform to different domain shapes
- Element size can affect the solution accuracy and the computation time
  - A trade off to be considered
- Grid generation may need special considerations.