Numerical Analysis
homework 06: Matrix Condition Numbers

Due on Tuesday, April 11, 2017

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1 Introduction

In this homework, we will implement the Power Method algorithm to find certain eigenvalues of a given matrix A.

1.1 Termination Condition

To calculate error of each iteration, we will use four kind of error fomula:

1. $\epsilon_1 = |V^{(k+1)} - V^k|$
2. $\epsilon_2 = \|q^{(k+1)} - q^k\|_2$
3. $\epsilon_3 = \|r^{(k+1)}\|$
4. $\epsilon_4 = \frac{\|r^{(k+1)}\|}{|(W^k)^T q^k|}$

where $r^k = Aq^k - V^k q^k$ and $W^k = \frac{(q^k)^T A}{\|(q^k)^T A\|_2}$. In this project, we need to use the four error to test Power Method and find out which error we prefer.

1.2 Power Method

We will implement three Power Method algorithm to find eigenvalue.

1. **Power Method**(to find largest eigenvalue).
2. **Inverse Power Method**(to find smallest eigenvalue).
3. **Inverse Power Method with Shift**(to find eigenvalue closest to ω).

1.3 Condition Numbers

Condition Numbers is defined as

$$k = \frac{\lambda_1}{\lambda_n}$$

We need to find the condition numbers of the following resistor network.

1. 2×2 resistor network
2. 4×4 resistor network
3. 10×10 resistor network
4. 20×20 resistor network
5. 40×40 resistor network
6. 50×50 resistor network

2 Implementation

Algorithm 1 **Power Method**

for each k ∈ {1, ..., maxIter} **do**
  $q^{(k+1)} = \frac{Aq^k}{\|Aq^k\|_2}$
  $V^{(k+1)} = (q^k)^T Aq^k$
  if $error < tol$ **then**
    break
  end if
end for

Algorithm 2 **Inverse Power Method**

for each k ∈ {1, ..., maxIter} **do**
  $Az^k = q^{(k-1)}$
  $q^k = \frac{z^k}{\|z^k\|_2}$
  $V^k = (q^k)^T Aq^k$
  if $error < tol$ **then**
    break
  end if
end for

Algorithm 3 **Inverse Power Method with Shift**

for each k ∈ {1, ..., maxIter} **do**
  $(A - \omega I)z^k = q^{(k-1)}$
  $q^k = \frac{z^k}{\|z^k\|_2}$
  $V^k = (q^k)^T Aq^k$
  if $error < tol$ **then**
    break
  end if
end for

2.1 Complexity

For each iteration of the three Power Method, the most time-consuming part is Matrix x Vector, which is a $\boldsymbol{O(n^2)}$ problem. However, because I use LU Decomposition for Inverse Power Method and Inverse Power Method with Shift, so their overall runtime should be influenced by $\boldsymbol{O(n^3)}$.

3 Discussion of Termination Condition

In this section, we will discuss which termination condition is the best. Table 1 shows average iteration time of a 20 x 20 resostor network.

	ϵ_1	ϵ_2	ϵ_3	ϵ_4
# of iter	299	795	683	683
λ	0.0795492	0.0795492	0.0795492	0.0795492
Runtime	0.073175	0.163965	0.149694	0.303234
iter_avg	2.45E-04	2.06E-04	2.19E-04	4.44E-04

Table 1: Average iteration time of 20 x 20 resistor network

In Table 1, ϵ_4 has the largest iter_avg, while ϵ_2 has the smallest. For more detailed observation, I plot the error vs iter as shown in Figure 1.

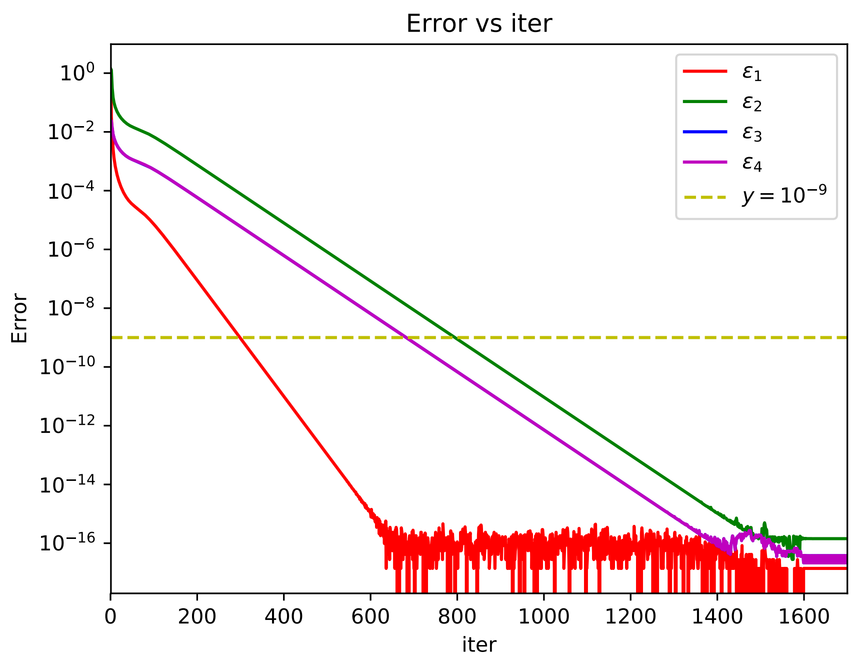

Figure 1: Error vs iter of 20 x 20 resistor network

In Figure 1, ϵ_3 and ϵ_4 almost overlap together and we can find that ϵ_1 has the smallest number of iteration to converge. In addition, because the purpose of Power Method is to find the eigenvalue, so I think we just need to check the convergence of lambda. As a result, I think **ϵ_1 is the best**.

4 Discussion of Power Method

In this section, we will determine the condition number of several resistor network and discuss the complexity of the three Power Method. Table 2, 3 and 4 show the detailed result of experiment with different resistor number.

resistor num	2	4	10	20	40	50
N	9	25	121	441	1681	2601
# of iter	27	21	91	299	984	1439
runtime(s)	3.70E-05	7.10E-05	1.68E-03	7.04E-02	2.98E+00	1.04E+01
iter_avg(s)	1.37E-06	3.38E-06	1.84E-05	2.36E-04	3.03E-03	7.22E-03
λ	0.00523607	0.0142445	0.0391638	0.0795492	0.159765	0.19981

Table 2: Result of Power Method

resistor num	2	4	10	20	40	50
N	9	25	121	441	1681	2601
# of iter	11	8	5	4	4	3
runtime(s)	2.40E-05	0.000153	0.004976	0.211479	12.4805	42.3624
iter_avg(s)	1.82E-06	6.00E-06	1.15E-04	1.46E-03	2.14E-02	5.91E-02
λ	0.000763932	0.000390179	0.000129496	5.42E-05	2.28E-05	1.73E-05

Table 3: Result of Inverse Power Method

resistor num	2	4	10	20	40	50
N	9	25	121	441	1681	2601
# of iter	11	7	5	3	4	4
runtime(s)	3.80E-05	1.59E-04	5.19E-03	2.17E-01	1.23E+01	4.31E+01
iter_avg(s)	2.00E-06	5.71E-06	1.15E-04	1.53E-03	2.54E-02	5.65E-02
λ	0.000763932	0.000390179	0.000129496	5.42E-05	2.28E-05	1.73E-05

Table 4: Result of Inverse Power Method with Shift(mu = 5E-05)

Because Condition Number is $\frac{\lambda_1}{\lambda_n}$, so we can get Condition Number of the 6 resistor networks from the tables above, as shown in Table 5.

resistor num	2	4	10	20	40	50
N	9	25	121	441	1681	2601
condition num	6.8541	36.5077	302.432	1467.21	7016.84	11574.3

Table 5: Condition Number of the 6 resistor networks

4.1 Complexity

From Table 2, 3 and 4, we can plot **iter_avg vs N** and **overall runtime vs N** as shown in Figure 2 and 3.

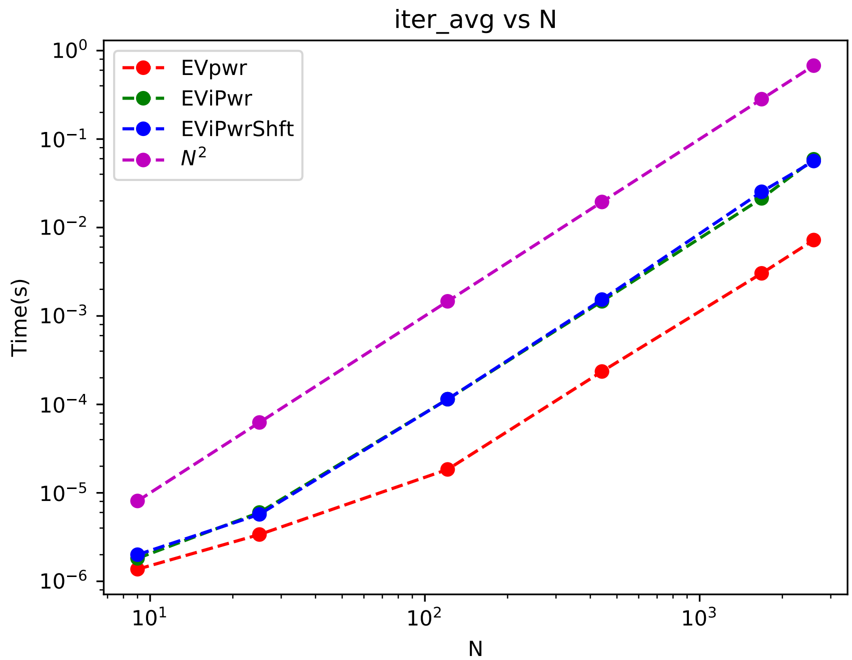

Figure 2: iter_avg vs N

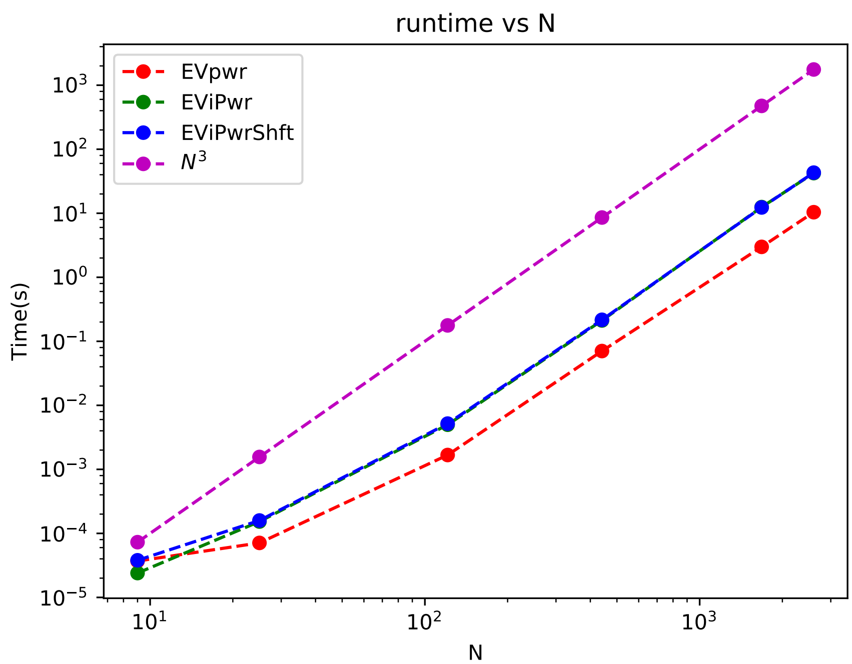

Figure 3: runtime vs N

In Figure 2 and 3, EViPwr and EViPwrShft almost overlap together because they have almost same speed. From Figure 2, we can find that the slope of each iteration of the three Power Method is same as N^2, which means each iteration is a $O(n^2)$ problem. From Figure 3, we can find that the slope of runtime of Inverse Power Method and Inverse Power Method with Shift is almost same as N^3, this is because the overall runtime will be affected by **LU Decomposition**, which is $O(n^3)$ problem. For EVpwr, it seems that EVpwr also have N^3 slope, but I think this will vary from our initial guess. If we have a initial guess close to answer then the iteration number will drop and change the slope.