

## Unit 8.2. Multistep Methods

Numerical Analysis

May 15, 2017

EE/NTHU

### Polynomial Approximation

- We assume the solution of the dynamic equation  $x(t)$  can be approximated by a polynomial of degree  $p$  as

$$x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_p t^p. \quad (8.2.1)$$

- Furthermore, we assume the dynamic equation is solved with a fixed time step,  $h$ . Thus,  $x(t)$  is evaluated at  $t = nh$  and we can write

$$x_n = x(nh).$$

- Since the dynamic equation is

$$\frac{dx(t)}{dt} = f(x, t).$$

We have

$$f(x, t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots + p a_p t^{p-1}.$$

- $f(x, t)$  is also evaluated at  $t = nh$ , we can write

$$f_n = f(x(nh), nh).$$

## 2nd Order Approximation – Trapezoidal Rule

- Approximate  $x(t)$  by

$$x(t) = a_0 + a_1 t + a_2 t^2.$$

- Then

$$f(x, t) = a_1 + 2a_2 t$$

$$f_n = f(x(nh), nh) = a_1 + 2a_2 nh$$

$$f_{n+1} = a_1 + 2a_2(n+1)h$$

$$f_{n+1} - f_n = 2a_2 h$$

- And

$$x_n = a_0 + a_1 nh + a_2 n^2 h^2$$

$$x_{n+1} = a_0 + a_1(n+1)h + a_2(n+1)^2 h^2$$

$$= a_0 + a_1(n+1)h + a_2(n^2 + 2n + 1)h^2$$

$$= x_n + a_1 h + a_2(2n+1)h^2$$

$$= x_n + f_n h + a_2 h^2$$

$$= x_n + f_n h + h(f_{n+1} - f_n)/2$$

$$= x_n + h(f_{n+1} + f_n)/2$$

- This is the Trapezoidal rule

## 2nd Order Forward Integration

- Approximate  $x(t)$  by

$$x(t) = a_0 + a_1 t + a_2 t^2.$$

- Then

$$f(x, t) = a_1 + 2a_2 t$$

$$f_n = f(x(nh), nh) = a_1 + 2a_2 nh$$

$$f_{n-1} = a_1 + 2a_2(n-1)h$$

$$f_n - f_{n-1} = 2a_2 h$$

- And

$$x_n = a_0 + a_1 nh + a_2 n^2 h^2$$

$$x_{n+1} = a_0 + a_1(n+1)h + a_2(n+1)^2 h^2$$

$$= a_0 + a_1(n+1)h + a_2(n^2 + 2n + 1)h^2$$

$$= x_n + a_1 h + a_2(2n+1)h^2$$

$$= x_n + f_n h + a_2 h^2$$

$$= x_n + f_n h + h(f_n - f_{n-1})/2$$

$$= x_n + h(3f_n - f_{n-1})/2$$

- 2nd order forward integration formula

## 2nd Order Forward Integration, II

- Trapezoidal rule

$$x_{n+1} = x_n + h \frac{f_{n+1} + f_n}{2}$$

- 2nd order forward integration method

$$x_{n+1} = x_n + h \frac{3f_n - f_{n-1}}{2}$$

- Nodal equation:

$$\frac{dv_1}{dt} = \frac{V - v_1}{RC}$$

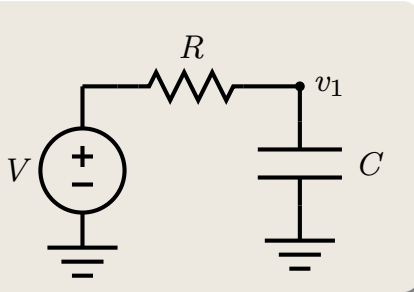
- Let  $x = v_1$  and  $f = \frac{V - v_1}{RC} = \frac{V - x}{RC}$

$$x_{n+1} = x_n + (3V_n - 3x_n - V_{n-1} + x_{n-1}) \frac{h}{2RC}$$

$$= \left(1 - \frac{3h}{2RC}\right) x_n + \frac{h}{2RC} x_{n-1} + (3V_n - V_{n-1}) \frac{h}{2RC}$$

Analytical solution:

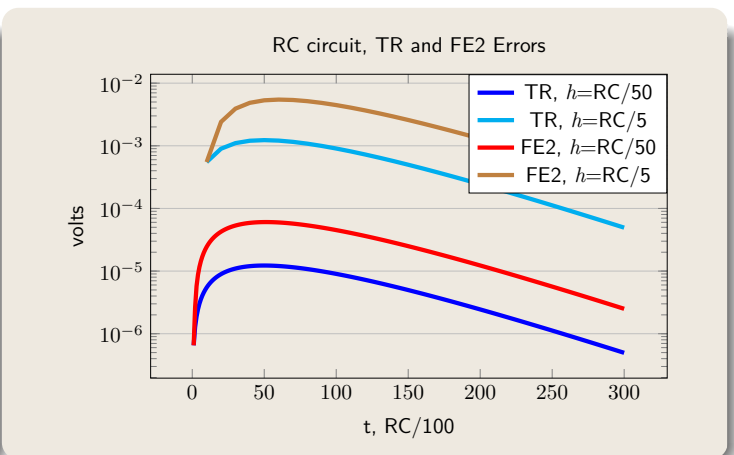
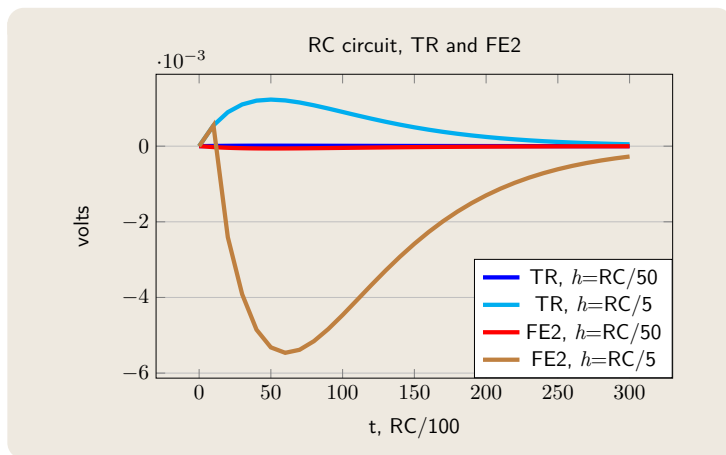
$$v_1(t) = 1 - \exp\left(\frac{-t}{RC}\right)$$



$$V(t) = 1, \quad t \geq 0,$$

$$v_1(0) = 0.$$

## 2nd Order Forward Integration, III



- Both trapezoidal and 2nd order forward integration methods produce accurate results.
  - Especially for small  $h$ .
  - No error accumulation is observed.
- 2nd order forward integration has larger errors than trapezoidal method.

# Local Truncation Error – Trapezoidal Rule

- In trapezoidal rule, we have the following approximations

$$x_n = a_0 + a_1 nh + a_2 n^2 h^2$$

$$f_n = a_1 + 2a_2 nh$$

- And the integration formula is

$$x_{n+1} = x_n + h \cdot \frac{f_{n+1} + f_n}{2}$$

- Error in  $x(t)$  is dominated by  $t^3$  term

$$x_n = a_0 + a_1 nh + a_2 n^2 h^2 + a_3 n^3 h^3$$

$$x_{n+1} = a_0 + a_1(n+1)h + a_2(n+1)^2 h^2 + a_3(n+1)^3 h^3 \quad (8.2.2)$$

$$= a_0 + a_1(n+1)h + a_2(n^2 + 2n + 1)h^2 + a_3(n^3 + 3n^2 + 3n + 1)h^3 \quad (8.2.3)$$

$$\begin{aligned} x_n + h \cdot \frac{f_{n+1} + f_n}{2} &= a_0 + a_1 nh + a_2 n^2 h^2 + a_3 n^3 h^3 \\ &+ h \cdot \frac{a_1 + 2a_2(n+1)h + 3a_3(n+1)^2 h^2 + a_1 + 2a_2 nh + 3a_3 n^2 h^2}{2} \end{aligned} \quad (8.2.4)$$

- Thus

$$x_n + h \cdot \frac{f_{n+1} + f_n}{2} - x_{n+1} = \frac{a_3 h^3}{2} \quad (8.2.5)$$

- Local truncation error for trapezoidal rule is  $\frac{a_3 h^3}{2}$ .

## Local Truncation Error – Trapezoidal Rule, II

- For trapezoidal rule, we assume  $x(t)$  is a second order polynomial of  $t$ .
- If that is the case, we get the exact solution.
- However, if  $x(t)$  is a higher order polynomial then the local truncation error is

$$LTE = \frac{a_3 h^3}{2} \quad (8.2.6)$$

- Note that

$$x(t) = x_0 + x'_0 t + \frac{x''_0}{2} t^2 + \frac{x'''_0}{3!} t^3 + \cdots + \frac{1}{p!} \cdot \frac{d^p x}{dt^p} t^p \quad (8.2.7)$$

- Thus,  $a_3 = \frac{x'''_0}{6}$

- And, the local truncation error for trapezoidal rule is

$$LTE = \frac{h^3}{12} x'''_0. \quad (8.2.8)$$



# Local Truncation Error – 2nd Order Forward Integration

- In 2nd order forward integration method, we approximate

$$x_n = a_0 + a_1 nh + a_2 n^2 h^2$$

$$f_n = a_1 + 2a_2 nh$$

- Integration formula:  $x_{n+1} = x_n + h \cdot \frac{3f_n - f_{n-1}}{2}$ .

- Error in  $x(t)$  is dominated by the  $t^3$  term

$$x_n = a_0 + a_1 nh + a_2 n^2 h^2 + a_3 n^3 h^3$$

$$x_{n+1} = a_0 + a_1(n+1)h + a_2(n^2 + 2n + 1)h^2 + a_3(n^3 + 3n^2 + 3n + 1)h^3 \quad (8.2.9)$$

$$\begin{aligned} x_n + h \cdot \frac{3f_n - f_{n-1}}{2} &= a_0 + a_1 nh + a_2 n^2 h^2 + a_3 n^3 h^3 \\ &+ h \cdot \frac{3a_1 + 6a_2 nh + 9a_3 n^2 h^2 - a_1 - 2a_2(n-1)h - 3a_3(n-1)^2 h^2}{2}. \end{aligned} \quad (8.2.10)$$

- Thus,

$$x_n + h \cdot \frac{f_{n+1} + f_n}{2} - x_{n+1} = -5h^3 \frac{a_3}{2}$$

- And

$$LTE = -5a_3 \frac{h^3}{2} = \frac{-5}{12} x''' h^3. \quad (8.2.11)$$

# Local Truncation Error – Forward Euler

- In forward Euler method, we approximate

$$x_n = a_0 + a_1 nh$$

$$f_n = a_1$$

- Integration formula:  $x_{n+1} = x_n + h \cdot f_n$ .

- Error in  $x(t)$  is dominated by the  $t^2$  term

$$x_n = a_0 + a_1 nh + a_2 n^2 h^2$$

$$x_{n+1} = a_0 + a_1(n+1)h + a_2(n^2 + 2n + 1)h^2 \quad (8.2.12)$$

$$x_n + h \cdot f_n = a_0 + a_1 nh + a_2 n^2 h^2 + h(a_1 + 2a_2 nh) \quad (8.2.13)$$

- Thus,

$$x_n + h \cdot f_n - x_{n+1} = -a_2 h^2.$$

- And

$$LTE = -a_2 h^2 = \frac{-1}{2} x'' h^2. \quad (8.2.14)$$

# Local Truncation Error – Backward Euler

- In backward Euler method, we approximate

$$x_n = a_0 + a_1 nh$$

$$f_n = a_1$$

- Integration formula:  $x_{n+1} = x_n + h \cdot f_{n+1}$ .

- Error in  $x(t)$  is dominated by the  $t^2$  term

$$\begin{aligned} x_n &= a_0 + a_1 nh + a_2 n^2 h^2 \\ x_{n+1} &= a_0 + a_1(n+1)h + a_2(n^2 + 2n + 1)h^2 \end{aligned} \quad (8.2.15)$$

$$x_n + h \cdot f_{n+1} = a_0 + a_1 nh + a_2 n^2 h^2 + h(a_1 + 2a_2(n+1)h) \quad (8.2.16)$$

- Thus,

$$x_n + h \cdot f_{n+1} - x_{n+1} = a_2 h^2.$$

- And

$$LTE = a_2 h^2 = \frac{1}{2} x'' h^2. \quad (8.2.17)$$

## Local Truncation Errors

- Local truncation errors for the integration methods discussed so far

$$\text{Forward Euler: } -\frac{1}{2} x'' h^2$$

$$\text{Backward Euler: } \frac{1}{2} x'' h^2$$

$$\text{Trapezoidal rule: } \frac{1}{12} x''' h^3$$

$$\text{2nd order forward integration: } -\frac{5}{12} x''' h^3$$

- Euler methods have the same  $LTE$  but with opposite signs.
- Trapezoidal method, indeed, has the smallest  $LTE$ , so far.
- As  $h \rightarrow 0$ ,  $LTE \rightarrow 0$ , thus all these methods can be very accurate.
  - Trapezoidal rule may be more efficient.

## 3rd Order Integration – Forward Method

- 3rd order approximation:  $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ .

$$f_n = a_1 + 2a_2 nh + 3a_3 n^2 h^2$$

$$f_{n-1} = a_1 + 2a_2(n-1)h + 3a_3(n-1)^2 h^2$$

$$f_{n-2} = a_1 + 2a_2(n-2)h + 3a_3(n-2)^2 h^2$$

$$f_n - f_{n-1} = 2a_2 h + 3a_3(2n-1)h^2$$

$$f_{n-1} - f_{n-2} = 2a_2 h + 3a_3(2n-3)h^2$$

$$f_n - 2f_{n-1} + f_{n-2} = 6a_3 h^2$$

$$x_{n+1} = a_0 + a_1(n+1)h + a_2(n+1)^2 h^2 + a_3(n+1)^3 h^3$$

$$= a_0 + a_1(n+1)h + a_2(n^2 + 2n + 1)h^2 + a_3(n^3 + 3n^2 + 3n + 1)h^3$$

$$= x_n + a_1 h + a_2(2n+1)h^2 + a_3(3n^2 + 3n + 1)h^3$$

$$= x_n + hf_n + a_2 h^2 + a_3(3n+1)h^3$$

$$= x_n + hf_n + h(f_n - f_{n-1} + 3a_3 h^2)/2 + a_3 h^3$$

$$= x_n + hf_n + h(f_n - f_{n-1})/2 + 5a_3 h^3/2$$

$$= x_n + hf_n + h(f_n - f_{n-1})/2 + h(f_n - 2f_{n-1} + f_{n-2}) \cdot 5/12$$

$$= x_n + h(23f_n - 16f_{n-1} + 5f_{n-2})/12$$

## 3rd Order Integration – Backward Method

- 3rd order approximation:  $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ .

$$f_n = a_1 + 2a_2 nh + 3a_3 n^2 h^2$$

$$f_{n-1} = a_1 + 2a_2(n-1)h + 3a_3(n-1)^2 h^2$$

$$f_{n+1} = a_1 + 2a_2(n+1)h + 3a_3(n+1)^2 h^2$$

$$f_{n+1} - f_n = 2a_2 h + 3a_3(2n+1)h^2$$

$$f_n - f_{n-1} = 2a_2 h + 3a_3(2n-1)h^2$$

$$f_n - 2f_{n-1} + f_{n-2} = 6a_3 h^2$$

$$x_{n+1} = a_0 + a_1(n+1)h + a_2(n+1)^2 h^2 + a_3(n+1)^3 h^3$$

$$= x_n + a_1 h + a_2(2n+1)h^2 + a_3(3n^2 + 3n + 1)h^3$$

$$= x_n + hf_n + a_2 h^2 + a_3(3n+1)h^3$$

$$= x_n + hf_n + h(f_{n+1} - f_n - 3a_3 h^2)/2 + a_3 h^3$$

$$= x_n + hf_n + h(f_{n+1} - f_n)/2 - a_3 h^3/2$$

$$= x_n + hf_n + h(f_{n+1} - f_n)/2 - h(f_{n+1} - 2f_n + f_{n-1})/12$$

$$= x_n + h(5f_{n+1} + 8f_n - f_{n-1})/12$$

# 3rd Order Integration Methods

- Thus, the 3rd order integration methods with constant steps are
- 3rd order forward integration

$$x_{n+1} = x_n + h \cdot \frac{23f_n - 16f_{n-1} + 5f_{n-2}}{12} \quad (8.2.18)$$

- To carry out the integration, the value of the derivatives of the previous three time points are needed.
- Integration is straightforward since  $f_{n+1}$  is not required.
- 3rd order backward integration

$$x_{n+1} = x_n + h \cdot \frac{5f_{n+1} + 8f_n - f_{n-1}}{12} \quad (8.2.19)$$

- To carry out the integration, the value of the derivatives of the previous two time points are needed.
- $f_{n+1}$ , which is a function of  $x_{n+1}$ , is needed and solved for.
- If the system is nonlinear, Newton's method is usually required.
- Stamps can be derived to form the linear system.

## General Approach – Backward Integration

- To find the  $k$ 'th order backward integration formula, assume

$$x_{n+1} = x_n + h(b_0f_{n+1} + b_1f_n + b_2f_{n-1} + b_3f_{n-2} \cdots + b_{k-1}f_{n-k+2})$$

- For convenience, let  $n = 0$  and  $h = 1$ .

$$x_1 = x_0 + b_0f_1 + b_1f_0 + b_2f_{-1} + b_3f_{-2} \cdots + b_{k-1}f_{-k+2} \quad (8.2.20)$$

- If  $x$  is a polynomial of degree less than  $k$ , then the above equation is exact.
- If  $x$  is order 1, then

$$x = a_0 + a_1t$$

$$f = x' = a_1$$

This equation holds for any value of  $t$ , then  $\forall k, f_k = a_1$ . And we have

$$x_0 = a_0,$$

$$x_1 = a_0 + a_1$$

$$= x_0 + b_0f_1 + b_1f_0 + b_2f_{-1} + b_3f_{-2} \cdots + b_{k-1}f_{-k+2}$$

$$= a_0 + a_1(b_0 + b_1 + b_2 + b_3 \cdots + b_{k-1})$$

Or,

$$b_0 + b_1 + b_2 + \cdots + b_{k-1} = 1. \quad (8.2.21)$$



## General Approach – Backward Integration, II

- If  $x$  is of order 2

$$x = a_0 + a_1 t + a_2 t^2$$

$$x_1 = a_0 + a_1 + a_2$$

$$x_0 = a_0$$

$$f = a_1 + 2a_2 t$$

- And we have

$$\begin{aligned}x_1 &= a_0 + a_1 + a_2 \\&= x_0 + b_0 f_1 + b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} \cdots + b_{k-1} f_{-k+2} \\&= a_0 + b_0(a_1 + 2a_2) + b_1 a_1 + b_2(a_1 - 2a_2) + \cdots + b_{k-1}(a_1 - 2(k-2)a_2) \\&= a_0 + a_1(b_0 + b_1 + b_2 + \cdots + b_{k-1}) + 2a_2(b_0 - b_2 - \cdots - (k-2)b_{k-1}) \\&= a_0 + a_1 + 2a_2(b_0 - b_2 - \cdots - (k-2)b_{k-1})\end{aligned}$$

- Thus,

$$b_0 - b_2 - 2b_3 - \cdots - (k-2)b_{k-1} = \frac{1}{2}. \quad (8.2.22)$$

## General Approach – Backward Integration, III

- If  $x$  is of order 3

$$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$x_1 = a_0 + a_1 + a_2 + a_3$$

$$x_0 = a_0$$

$$f = a_1 + 2a_2 t + 3a_3 t^2$$

- And we have

$$\begin{aligned}x_1 &= a_0 + a_1 + a_2 + a_3 \\&= x_0 + b_0 f_1 + b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} \cdots + b_{k-1} f_{-k+2} \\&= a_0 + b_0(a_1 + 2a_2 + 3a_3) + b_1 a_1 + b_2(a_1 - 2a_2 + 3a_3) + \cdots \\&\quad + b_{k-1}(a_1 - 2(k-2)a_2 + 3(k-2)^2 a_3) \\&= a_0 + a_1(b_0 + b_1 + b_2 + \cdots + b_{k-1}) + 2a_2(b_0 - b_2 - \cdots - (k-2)b_{k-1}) \\&\quad + 3a_3(b_0 + b_2 + 4b_3 + \cdots + (k-2)^2 b_{k-1}) \\&= a_0 + a_1 + a_2 + 3a_3(b_0 + b_2 + 4b_3 + \cdots + (k-2)^2 b_{k-1})\end{aligned}$$

- Thus,

$$b_0 + b_2 + 4b_3 + \cdots + (k-2)^2 b_{k-1} = \frac{1}{3}. \quad (8.2.23)$$

## General Approach – Backward Integration, IV

- Combining all  $k$  constraints, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & -1 & -2 & \cdots & -(k-2) \\ 1 & 0 & 1 & 4 & \cdots & (k-2)^2 \\ 1 & 0 & -1 & -8 & \cdots & -(k-2)^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & (-1)^{k-1} & (-2)^{k-1} & \cdots & (-k+2)^{k-1} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{k-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ \vdots \\ 1/k \end{bmatrix} \quad (8.2.24)$$

- Coefficients  $b_0, b_1, b_2, \dots, b_{k-1}$ , can be solved from the above equation and we have a general  $k$ 'th order backward integration formula

$$x_1 = x_0 + h(b_0 f_1 + b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} + \cdots + b_{k-1} f_{-k+2})$$

- Or

$$x(t+h) = x(t) + h(b_0 f(t+h) + b_1 f(t) + b_2 f(t-h) + b_3 f(t-2h) + \cdots + b_{k-1} f(t-(k-2)h)).$$

## General Approach – Backward Integration, V

- 2nd order backward integration formula

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$
$$b_0 = 1/2, \quad b_1 = 1/2.$$

- Thus, the 2nd order backward integration formula is

$$x(t+h) = x(t) + h(f(t+h) + f(t))/2.$$

- 3rd order backward integration formula

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$$
$$b_0 = 5/12, \quad b_1 = 8/12, \quad b_2 = -1/12.$$

- Thus, the 3rd order backward integration formula is

$$x(t+h) = x(t) + h(5f(t+h) + 8f(t) - f(t-h))/12.$$

- Higher order formulas can be found in the same way.

# Multi-Step Backward Integration Formulas

- Adams-Moulton's formulas
- 4th order backward integration formula

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & -1 & -8 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix}$$

$$b_0 = 9/24, \quad b_1 = 19/24, \quad b_2 = -5/24, \quad b_3 = 1/24.$$

- Thus, the 4th order backward integration formula is

$$x(t+h) = x(t) + h(9f(t+h) + 19f(t) - 5f(t-h) + f(t-2h))/24.$$

## Multi-Step Backward Integration Formulas, II

- 5th order backward integration formula

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 & -3 \\ 1 & 0 & 1 & 4 & 9 \\ 1 & 0 & -1 & -8 & -27 \\ 1 & 0 & 1 & 16 & 81 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix}$$

$$b_0 = 251/720, \quad b_1 = 646/720, \quad b_2 = -264/720, \quad b_3 = 106/720, \quad b_4 = -19/720.$$

- Thus, the 5th order backward integration formula is

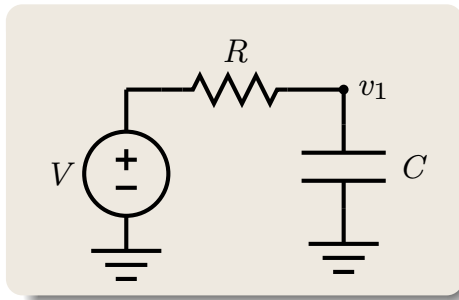
$$x(t+h) = x(t) + h(251f(t+h) + 646f(t) - 264f(t-h) + 106f(t-2h) - 19f(t-3h))/720.$$

# Multi-Step Backward Integration Formulas, III

- For the simple RC circuit, we have

$$\frac{dx}{dt} = \frac{V - x}{RC}.$$

- Using fixed step backward integration method

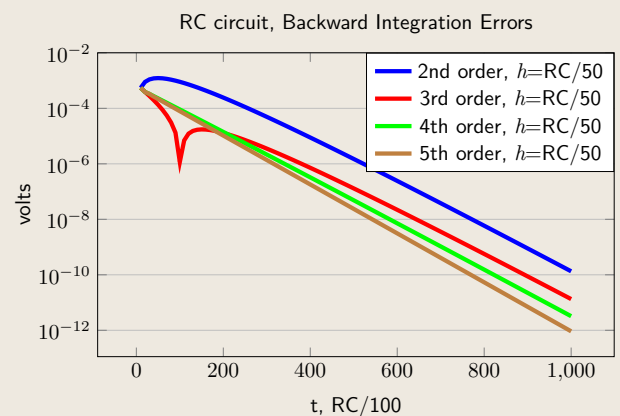
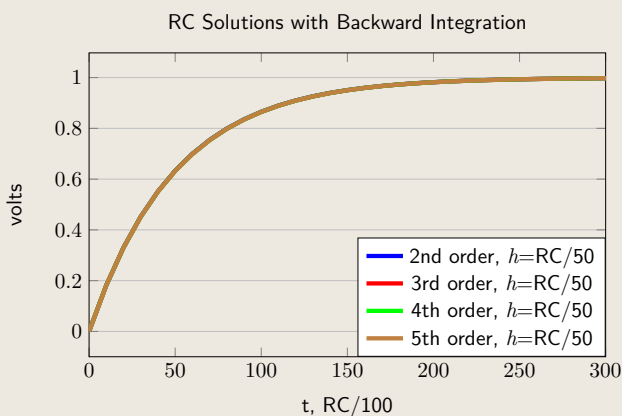


$$\begin{aligned} x(t+h) &= x(t) + \frac{h}{RC} \left( b_0(V - x(t+h)) + b_1(V - x(t)) \right. \\ &\quad \left. + \cdots + b_{k-1}(V - x(t - (k-2)h)) \right) \\ &= x(t) + \frac{hV}{RC} - \frac{h}{RC} \left( b_0x(t+h) + b_1x(t) \right. \\ &\quad \left. + \cdots + b_{k-1}x(t - (k-2)h) \right) \end{aligned}$$

- Or

$$\begin{aligned} x(t+h) &= \left(1 + \frac{hb_0}{RC}\right)^{-1} \left( \frac{hV}{RC} + x(t) \left(1 - \frac{hb_1}{RC}\right) \right. \\ &\quad \left. - \frac{h}{RC} \left( b_2x(t-h) + \cdots + b_{k-1}x(t - (k-2)h) \right) \right). \end{aligned}$$

# Multi-Step Backward Integration Formulas, IV



- Higher order integration methods can deliver higher accuracy.
- A  $k$ 'th order integration method needs  $k - 1$  previous time points.
  - For example, 3rd order integration method needs two past time points
    - The first two time steps cannot be used
  - It is common to use lower order methods for the first few time points, and then switch to higher order ones.



- $k'$ th order backward integration formula

$$x_{n+1} = x_n + h(b_0 f_{n+1} + b_1 f_n + b_2 f_{n-1} + b_3 f_{n-2} + \cdots + b_{k-1} f_{n-k+2}).$$

where  $x$  is assumed to be a  $k'$ th order polynomial.

- In that case, the integration formula is exact, i.e., no errors. Otherwise, local truncation error is dominated by  $t^{k+1}$  term.

$$\begin{aligned} x(t) &= a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + a_{k+1} t^{k+1} \\ x(t+h) &= a_0 + a_1(t+h) + a_2(t+h)^2 + \cdots + a_k(t+h)^k + a_{k+1}(t+h)^{k+1} \\ f(t) &= a_1 + 2a_2 t + 3a_3 t^2 + \cdots + k a_k t^{k-1} + (k+1)a_{k+1} t^k. \end{aligned}$$

- Let  $t = 0$  and  $h = 1$

$$\begin{aligned} x_1 &= a_0 + a_1 + a_2 + \cdots + a_k + a_{k+1} \\ f_j &= a_1 + 2a_2 j + 3a_3 j^2 + \cdots + k a_k j^{k-1} + (k+1)a_{k+1} j^k \\ x_1 &= x_0 + h(b_0 f_1 + b_1 f_0 + b_2 f_{-1} + \cdots + b_{k-1} f_{-k+2}) \end{aligned}$$

## LTE for Backward Integration Methods, II

- In deriving the backward integration formulas, we have matched the coefficients up to the  $k'$ th term.
- Thus, we need to consider  $a_{k+1}$  term only.
- Coefficients for  $a_{k+1}$  term in  $x_0 + h(b_0 f_1 + \cdots + b_{k-1} f_{-k+2}) - x_1$  is

$$\begin{aligned} RHS: & \quad 1 \\ LHS: & \quad b_0(k+1) + b_1 \cdot 0 + b_2(k+1)(-1)^k + b_3(k+1)(-2)^k + \cdots \\ & \quad + b_{k-1}(k+1)(-k+2)^k \end{aligned}$$

- Thus,

$$LTE = \left( (k+1) \left( b_0 + b_2(-1)^k + b_3(-2)^k + \cdots + b_{k-1}(-k+2)^k \right) - 1 \right) a_{k+1} h^{k+1}.$$

$$LTE = \left( (k+1) \left( b_0 + b_2(-1)^k + b_3(-2)^k + \cdots + b_{k-1}(-k+2)^k \right) - 1 \right) a_{k+1} h^{k+1}.$$

- 2nd order integration

$$b_0 = 1/2, \quad b_1 = 1/2, \\ LTE = \left( 3 \frac{1}{2} - 1 \right) a_3 h^3 = \frac{1}{2} a_3 h^3.$$

- 3rd order integration

$$b_0 = 5/12, \quad b_1 = 8/12, \quad b_2 = -1/12 \\ LTE = \left( 4 \left( \frac{5}{12} - \frac{(-1)^3}{12} \right) - 1 \right) a_4 h^4 = a_4 h^4.$$

- 4th order integration

$$b_0 = 9/24, \quad b_1 = 19/24, \quad b_2 = -5/24, \quad b_3 = 1/24, \\ LET = \left( 5 \left( \frac{9}{24} - (-1)^4 \frac{5}{24} + (-2)^4 \frac{1}{24} \right) - 1 \right) a_5 h^5 = \frac{19}{6} \times a_5 h^5.$$

- Note:  $a_{k+1} = \frac{1}{(k+1)!} \frac{d^{k+1}x}{dt^{k+1}}.$

## General Approach – Forward Integration

- To find the  $k$ 'th order forward integration formula, assume

$$x_{n+1} = x_n + h(b_1 f_n + b_2 f_{n-1} + b_3 f_{n-2} \cdots + b_k f_{n-k+1})$$

- For convenience, let  $n = 0$  and  $h = 1$ .

$$x_1 = x_0 + b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} \cdots + b_k f_{-k+1} \quad (8.2.25)$$

- If  $x$  is a polynomial of degree less than  $k$ , then the above equation is exact.
- If  $x$  is order 1, then

$$x = a_0 + a_1 t \\ f = x' = a_1$$

This equation holds for any value of  $t$ , then  $\forall k, f_k = a_1$ . And we have

$$\begin{aligned} x_0 &= a_0, \\ x_1 &= a_0 + a_1 \\ &= x_0 + b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} \cdots + b_k f_{-k+1} \\ &= a_0 + a_1 (b_1 + b_2 + b_3 \cdots + b_k) \end{aligned}$$

Or,

$$b_1 + b_2 + \cdots + b_k = 1. \quad (8.2.26)$$

## General Approach – Forward Integration, II

- If  $x$  is of order 2

$$x = a_0 + a_1 t + a_2 t^2$$

$$x_1 = a_0 + a_1 + a_2$$

$$x_0 = a_0$$

$$f = a_1 + 2a_2 t$$

- And we have

$$\begin{aligned}x_1 &= a_0 + a_1 + a_2 \\&= x_0 + b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} \cdots + b_k f_{-k+1} \\&= a_0 + b_1 a_1 + b_2(a_1 - 2a_2) + b_3(a_1 - 4a_2) + \cdots + b_k(a_1 - 2(k-1)a_2) \\&= a_0 + a_1(b_1 + b_2 + \cdots + b_k) + 2a_2(-b_2 - 2b_3 - \cdots - (k-1)b_k) \\&= a_0 + a_1 + 2a_2(-b_2 - 2b_3 - \cdots - (k-1)b_k)\end{aligned}$$

- Thus,

$$-b_2 - 2b_3 - \cdots - (k-1)b_k = \frac{1}{2}. \quad (8.2.27)$$

## General Approach – Forward Integration, III

- If  $x$  is of order 3

$$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$x_1 = a_0 + a_1 + a_2 + a_3$$

$$x_0 = a_0$$

$$f = a_1 + 2a_2 t + 3a_3 t^2$$

- And we have

$$\begin{aligned}x_1 &= a_0 + a_1 + a_2 + a_3 \\&= x_0 + b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} \cdots + b_k f_{-k+1} \\&= a_0 + b_1 a_1 + b_2(a_1 - 2a_2 + 3a_3) + b_3(a_1 - 2 \cdot 2a_2 + 3 \cdot 4a_3) + \cdots \\&\quad + b_k(a_1 - 2(k-1)a_2 + 3(k-1)^2 a_3) \\&= a_0 + a_1(b_1 + b_2 + b_3 + \cdots + b_k) + 2a_2(-b_2 - 2b_3 - \cdots - (k-1)b_k) \\&\quad + 3a_3(b_2 + 4b_3 + \cdots + (k-1)^2 b_k) \\&= a_0 + a_1 + a_2 + 3a_3(b_2 + 4b_3 + \cdots + (k-1)^2 b_k)\end{aligned}$$

- Thus,

$$b_2 + 4b_3 + \cdots + (k-1)^2 b_k = \frac{1}{3}. \quad (8.2.28)$$

## General Approach – Forward Integration, IV

- Combining all  $k$  constraints, we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & -2 & -3 & \cdots & -(k-1) \\ 0 & 1 & 4 & 9 & \cdots & (k-1)^2 \\ 0 & -1 & -8 & -27 & \cdots & -(k-1)^3 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & (-1)^{k-1} & (-2)^{k-1} & (-3)^{k-1} & \cdots & (-k+1)^{k-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_k \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ \vdots \\ 1/k \end{bmatrix} \quad (8.2.29)$$

- Coefficients  $b_1, b_2, b_3, \dots, b_k$ , can be solved from the above equation and we have a general  $k$ 'th order forward integration formula

$$x_1 = x_0 + h(b_1 f_0 + b_2 f_{-1} + b_3 f_{-2} + \cdots + b_k f_{-k+1})$$

- Or

$$x(t+h) = x(t) + h(b_1 f(t) + b_2 f(t-h) + b_3 f(t-2h) + \cdots + b_k f(t-(k-1)h)).$$

## General Approach – Forward Integration, V

- 2nd order forward integration formula

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$$b_1 = 3/2, \quad b_2 = -1/2.$$

- Thus, the 2nd order forward integration formula is

$$x(t+h) = x(t) + h(3f(t) - f(t-h))/2.$$

- 3rd order forward integration formula

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$$

$$b_1 = 23/12, \quad b_2 = -16/12, \quad b_3 = 5/12.$$

- Thus, the 3rd order forward integration formula is

$$x(t+h) = x(t) + h(23f(t) - 16f(t-h) + 5f(t-2h))/12.$$



# Multi-Step Forward Integration Formulas

- Adams-Bashforth's formulas
- 4th order forward integration formula

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 4 & 9 \\ 0 & -1 & -8 & -27 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix}$$

$$b_1 = 55/24, \quad b_2 = -59/24, \quad b_3 = 37/24, \quad b_4 = -9/24.$$

- Thus, the 4th order forward integration formula is

$$x(t+h) = x(t) + h(55f(t) - 59f(t-h) + 37f(t-2h) - 9f(t-3h))/24.$$

## Multi-Step Forward Integration Formulas, II

- 5th order forward integration formula

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 4 & 9 & 16 \\ 0 & -1 & -8 & -27 & -64 \\ 0 & 1 & 16 & 81 & 256 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix}$$

$$b_1 = 1901/720, \quad b_2 = -2774/720, \quad b_3 = 2616/720, \quad b_4 = -1274/720, \quad b_5 = 251/720.$$

- Thus, the 5th order forward integration formula is

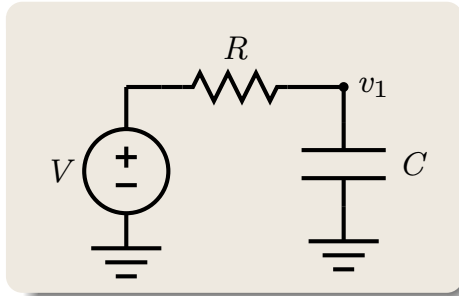
$$x(t+h) = x(t) + h(1901f(t) - 2774f(t-h) + 2616f(t-2h) - 1274f(t-3h) + 251f(t-4h))/720.$$

# Multi-Step Forward Integration Formulas, III

- For the simple RC circuit, we have

$$\frac{dx}{dt} = \frac{V - x}{RC}.$$

- Using fixed step forward integration method



$$\begin{aligned} x(t+h) &= x(t) + \frac{h}{RC} \left( b_1(V - x(t)) + b_2(V - x(t-h)) \right. \\ &\quad \left. + \cdots + b_k(V - x(t - (k-1)h)) \right) \\ &= x(t) + \frac{hV}{RC} - \frac{h}{RC} \left( b_1x(t) + b_2x(t-h) \right. \\ &\quad \left. + \cdots + b_kx(t - (k-1)h) \right) \end{aligned}$$

- Or

$$x(t+h) = \frac{hV}{RC} + x(t)(1 - \frac{hb_1}{RC}) - \frac{h}{RC} (b_2x(t-h) + \cdots + b_kx(t - (k-1)h)).$$

## LTE for Forward Integration Methods

- $k'$ 'th order forward integration formula

$$x_{n+1} = x_n + h(b_1f_n + b_2f_{n-1} + b_3f_{n-2} + \cdots + b_kf_{n-k+1}).$$

where  $x$  is assumed to be a  $k'$ 'th order polynomial.

- In that case, the integration formula is exact, i.e., no errors. Otherwise, local truncation error is dominated by  $t^{k+1}$  term.

$$\begin{aligned} x(t) &= a_0 + a_1t + a_2t^2 + \cdots + a_kt^k + a_{k+1}t^{k+1} \\ x(t+h) &= a_0 + a_1(t+h) + a_2(t+h)^2 + \cdots + a_k(t+h)^k + a_{k+1}(t+h)^{k+1} \\ f(t) &= a_1 + 2a_2t + 3a_3t^2 + \cdots + ka_kt^{k-1} + (k+1)a_{k+1}t^k. \end{aligned}$$

- Let  $t = 0$  and  $h = 1$

$$\begin{aligned} x_1 &= a_0 + a_1 + a_2 + \cdots + a_k + a_{k+1} \\ f_j &= a_1 + 2a_2j + 3a_3j^2 + \cdots + ka_kj^{k-1} + (k+1)a_{k+1}j^k \\ x_1 &= x_0 + h(b_1f_0 + b_2f_{-1} + b_3f_{-2} + \cdots + b_kf_{-k+1}) \end{aligned}$$

## LTE for Forward Integration Methods, II

- In deriving the forward integration formulas, we have matched the coefficients up to the  $k$ 'th term.
- Thus, we need to consider  $a_{k+1}$  term only.
- Coefficients for  $a_{k+1}$  term

$$LHS : 1$$

$$RHS : b_1 \cdot 0 + b_2(k+1)(-1)^k + b_3(k+1)(-2)^k + \dots \\ + b_k(k+1)(-k+1)^k$$

- Thus,

$$LTE = \left( (k+1) \left( b_2(-1)^k + b_3(-2)^k + \dots + b_k(-k+1)^k \right) - 1 \right) a_{k+1} h^{k+1}.$$

## LTE for Forward Integration Methods, III

$$LTE = \left( (k+1) \left( b_2(-1)^k + b_3(-2)^k + \dots + b_k(-k+1)^k \right) - 1 \right) a_{k+1} h^{k+1}.$$

- 2nd order integration

$$b_1 = 3/2, \quad b_2 = -1/2,$$

$$LTE = \left( 3 \frac{-1}{2} - 1 \right) a_3 h^3 = -\frac{5}{2} a_3 h^3.$$

- 3rd order integration

$$b_1 = 23/12, \quad b_2 = -16/12, \quad b_3 = 5/12$$

$$LTE = \left( 4 \left( \frac{-16}{12} - \frac{5 \times (-2)^3}{12} \right) - 1 \right) a_4 h^4 = -9 a_4 h^4.$$

- 4th order integration

$$b_1 = 55/24, \quad b_2 = -59/24, \quad b_3 = 37/24, \quad b_4 = -9/24,$$

$$LET = \left( 5 \left( \frac{-59}{24} \times (-1)^4 + \frac{37}{24} \times (-2)^4 - \frac{9}{24} \times (-3)^4 \right) - 1 \right) a_5 h^5 = \frac{-251}{6} \times a_5 h^5.$$

- Note:  $a_{k+1} = \frac{1}{(k+1)!} \frac{d^{k+1}x}{dt^{k+1}}.$

- LTE for backward and forward integration formulas

Order	Backward integration	Forward integration
2nd order	$\frac{1}{2} a_3 h^3$	$-\frac{5}{2} a_3 h^3$
3rd order	$a_4 h^4$	$-9 a_4 h^4$
4th order	$\frac{19}{6} a_5 h^5$	$-\frac{251}{6} a_5 h^5$

- Note:  $a_k = \frac{1}{k!} \frac{d^k x}{dt^k}$ .
- Backward integration methods are usually more accurate.
  - The order of error is the same but the coefficients are very different.
- But forward integration maybe easier to solve for.

## Summary

- Polynomial approximation and dynamic equation solutions
- 3rd order backward integration formula
- 3rd order forward integration formula
- General approach in deriving integration formulas and their local truncation errors
  - Backward and forward integration methods
  - Higher order integration methods tends to have smaller local truncation errors.