Unit 9 Boundary Value Problems

Numerical Analysis

EE/NTHU

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Example Problem

- Example of a two-point boundary problem

$$-u''(x) = f(x), \qquad 0 < x < 1, \tag{9.1.1}$$

$$u(0) = u(1) = 0. \tag{9.1.2}$$

- This problem has the solution, $u \in C^2([0,1])$ and

$$u(x) = c_1 + c_2 x - \int_0^x F(s)\, ds, \tag{9.1.3}$$

where c_1 and c_2 are integration constants and

$$F(s) = \int_0^s f(t)\, dt. \tag{9.1.4}$$

- Using integration by parts,

$$\int_0^x F(s)\, ds = [sF(s)]_0^x - \int_0^x sF'(s)\, ds = \int_0^x (x-s) f(s)\, ds. \tag{9.1.5}$$

Green's Function

- The integration constants c_1 and c_2 can be determined by the boundary conditions.
- The condition $u(0) = 0$ implies $c_1 = 0$.
- And to satisfies $u(1) = 0$

$$c_2 = \int_0^1 (1 - s)f(s)\, ds \tag{9.1.6}$$

- Therefore, we have

$$\begin{aligned} u(x) &= x\int_0^1 (1-s)f(s)\, ds - \int_0^x (x-s)f(s)\, ds \\ &= \int_0^x x(1-s)f(s)\, ds + \int_x^1 x(1-s)f(s)\, ds - \int_0^x (x-s)f(s)\, ds \\ &= \int_0^x s(1-x)f(s)\, ds + \int_x^1 x(1-s)f(s)\, ds = \int_0^1 G(x,s)f(s)\, ds \end{aligned} \tag{9.1.7}$$

where $G(x, s)$ is the Green's function and

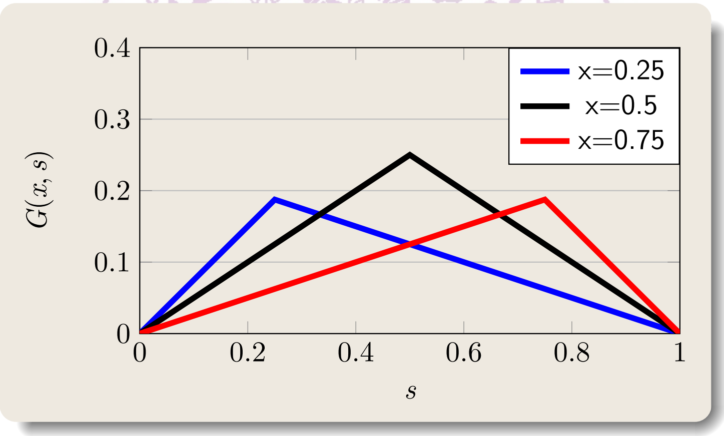
$$G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \le s < x, \\ x(1-s) & \text{if } x \le s \le 1. \end{cases} \tag{9.1.8}$$

Properties of the Green's Function

- The solution to the two-point boundary value problem of Eqs (9.1.1), (9.1.2)

$$u(x) = \int_0^1 G(x,s)f(s)\, ds. \tag{9.1.9}$$

- $G(x, s)$ is a piecewise linear function of s given fixed x, and vice versa.
- $G(x, s)$ is continuous and symmetric (i.e., $G(x, s) = G(s, x)$ for $x, s \in [0, 1]$).
- $G(x, s)$ is nonnegative and $G(x, s) = 0$, $x = 0, 1$, or $s = 0, 1$.
- $\int_0^1 G(x,s)\, ds = \frac{1}{2}x(1-x)$.



Properties of the Green's Function, II

- If $f(x) \in C^0([0,1])$ then there is a unique solution $u \in C^2([0,1])$ for the boundary value problem (9.1.1), (9.1.2);
- It can be further generalized that if $f(x) \in C^m([0,1])$ for some $m \geq 0$, then $u \in C^{m+2}([0,1])$.
- (Monotonicity Property) If $f \in C^0([0,1])$ is a nonnegative function, then u is also nonnegative.
- (Maximum Principle) If $f \in C^0([0,1])$, then

$$\|u\|_\infty \leq \frac{1}{8}\|f\|_\infty, \tag{9.1.10}$$

where $\|u\|_\infty = \max_{0 \leq x \leq 1} |u(x)|$ is the maximum norm.
This is due to

$$|u(x)| \leq \int_0^1 G(x,s)|f(s)|\, ds \leq \|f\|_\infty \int_0^1 G(x,s)\, ds = \frac{1}{2}x(1-x)\|f\|_\infty.$$

Solution using Green's Function

- The solution to the two-point boundary value problem of Eqs. (9.1.1) and (9.1.2) using Green's function approach is Eq. (9.1.9)

$$u(x) = \int_0^1 G(x,s)f(s)\, ds.$$

where

$$G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \leq s < x, \\ x(1-s) & \text{if } x \leq s \leq 1. \end{cases}$$

- It is an integration problem
  - Given $x \in [0,1]$, $u(x)$ can be found by integrating the product function
  - The integration methods introduced in Unit 6 can be applied
- The properties of the solution $u(x)$ are still valid even if other approaches, such as finite difference or finite element, are used to find $u(x)$.

Finite Difference Approximation

- In finite difference approximation we solve $u(x), 0 \le x \le 1$, on a set of grid points, $\{x_j\}_{j=0}^{n}$, where $n \ge 2$ is an integer and $h = \frac{1}{n}$ is the grid spacing.
- The following notations are used

$$\begin{array}{ll} u_j = u(x_j) = u(j \times h), & j = 0, \cdots, n. \\ f_j = f(x_j) = f(j \times h), & j = 1, \cdots, n-1. \end{array} \tag{9.1.11}$$

- Using the second order centered finite difference in place of $u''(x_j)$, we have

$$-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f(x_j), \qquad j = 1, \cdots, n-1 \tag{9.1.12}$$

and $u_0 = u_n = 0$.
- And,

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 \\ & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_{n-1} \end{bmatrix}. \tag{9.1.13}$$

Finite Difference Approximation, II

- Let $\mathbf{u}^T = \{u_1, u_2, \cdots, u_{n-1}\}$ be the unknown vector and $\mathbf{f}^T = \{f_1, f_2, \cdots, f_{n-1}\}$ be the right-hand side vector, then the above equation can be written as

$$\mathbf{A}_{fd}\mathbf{u} = \mathbf{f}, \tag{9.1.14}$$

where $\mathbf{A}_{fd}$ is the symmetric $(n-1) \times (n-1)$ finite difference matrix defined as

$$\mathbf{A}_{fd} = h^{-2}\text{tridiag}_{n-1}(-1, 2, -1). \tag{9.1.15}$$

- Note that $\mathbf{A}_{fd}$ is diagonally dominant by rows, and since

$$\mathbf{x}^T\mathbf{A}_{fd}\mathbf{x} = h^{-2}\left[x_1^2 + x_{n-1}^2 + \sum_{i=1}^{n-2}(x_i - x_{i+1})^2\right] \tag{9.1.16}$$

for any $\mathbf{x} \in \mathbb{R}^{n-1}$, thus $\mathbf{A}_{fd}$ is also positive definite.
- Thus, the finite difference system (9.1.12) or (9.1.13) has a unique solution.
- It can be shown that $\mathbf{A}_{fd}$ is an M-matrix and that $\mathbf{u}$ is nonnegative if $\mathbf{f}$ is nonnegative.
  - This property is known as the discrete maximum principle.

Finite Difference Approximation, III

- Note that by Taylor series expansion

$$u_{j+1} = u(x_j + h) = u(x_j) + hu'(x_j) + \frac{h^2}{2}u''(x_j) + \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(iv)}(x_j) + \cdots$$
$$u_{j-1} = u(x_j - h) = u(x_j) - hu'(x_j) + \frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(iv)}(x_j) - \cdots$$

then

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = u''(x_j) + \frac{h^2}{12}u^{(iv)}(x_j) + \cdots \tag{9.1.17}$$

- Compare Eqs. (9.1.1) and (9.1.12) the local truncation error of Eq. (9.1.12) is

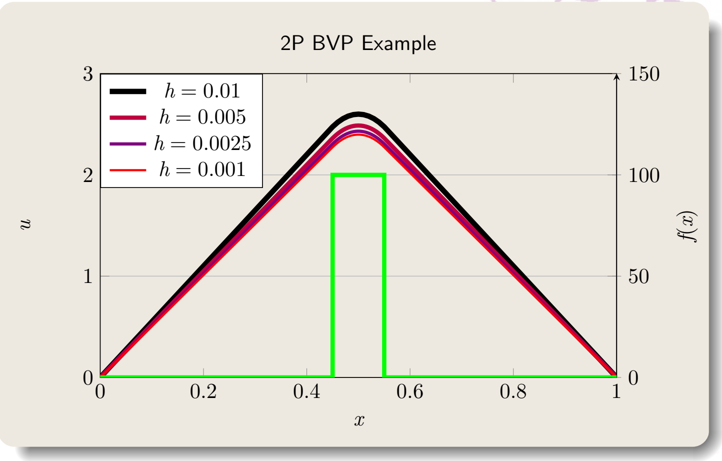
$$\tau_h(x_j) = \frac{h^2}{12}u^{(iv)}(x_j) \tag{9.1.18}$$

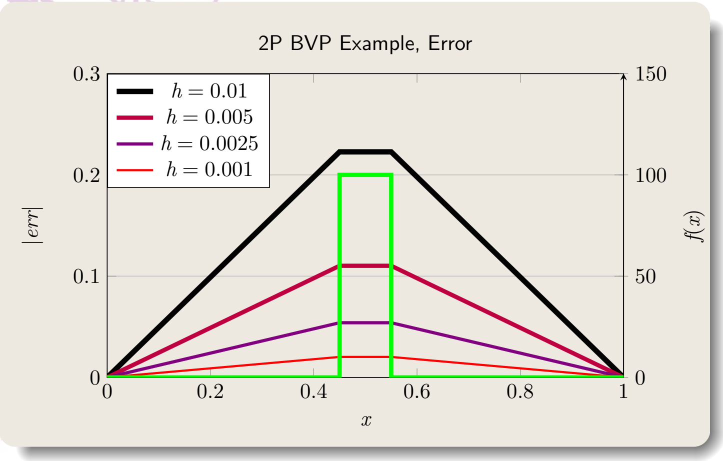
  - Solution accuracy is proportional to h^2, where h is the grid spacing
- Thus, finite difference approach to solve the two-point boundary value problem is very efficient
  - Solving a tridiagonal system ($\mathcal{O}(n)$)
  - Accuracy improves quadratically with grid spacing, h

Example

- Example, solving for

$$-u''(x) = f(x), \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$





- The finite difference approach results in a tridiagonal system and can be solved efficiently
- Larger number of grid points, smaller grid spacing h, results in a more accurate solution.

Change of Boundary Condition

- In the case the boundary condition is $u_0 = a$ and $u_n = b$ then we add $u_0 = a$ and $u_n = b$ to Eq. (9.1.12) and the linear system becomes

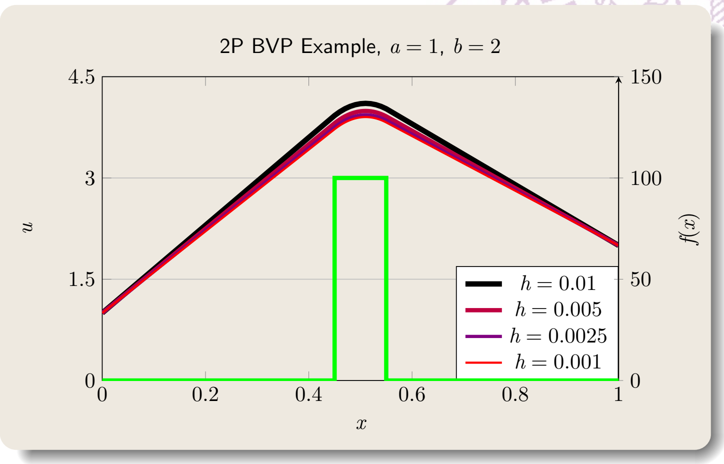
$$\frac{1}{h^2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & & -1 & 2 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} a/h^2 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_{n-1} \\ b/h^2 \end{bmatrix}. \tag{9.1.19}$$

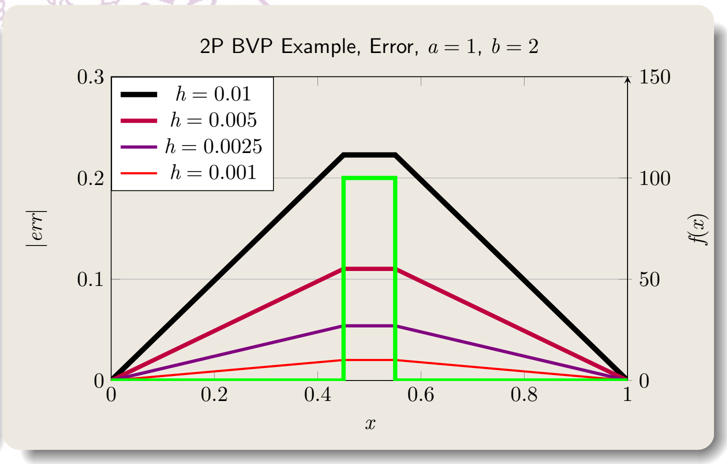
Change of Boundary Condition, II

- Or,

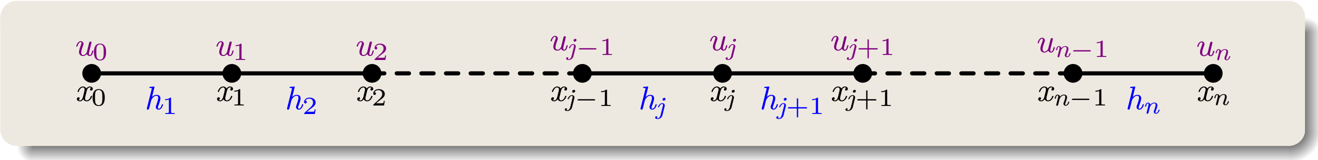
$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & & -1 & 2 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 + a/h^2 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_{n-1} + b/h^2 \end{bmatrix}. \tag{9.1.20}$$

- This linear system is still tridiagonal and can be solved efficiently and similar errors can be obtained.





Variable Grid Spacing



- Suppose the grid spacing is not a constant, then the two first order finite differences for $u(x_{j+1})$ and $u(x_j)$ are

$$u'(x_{j+1}) = \frac{u(x_{j+1}) - u(x_j)}{h_{j+1}}$$

$$u'(x_j) = \frac{u(x_j) - u(x_{j-1})}{h_j}$$

Then

$$\begin{aligned}
u''(x_j) &= \frac{u'(x_{j+1}) - u'(x_j)}{(h_{j+1} + h_j)/2} \\
&= \frac{2\Big(h_j u(x_{j+1}) - (h_j + h_{j+1}) u(x_j) + h_{j+1} u(x_{j-1})\Big)}{h_j h_{j+1}(h_j + h_{j+1})} \\
&= 2\left(\frac{u(x_{j-1})}{h_j(h_j + h_{j+1})} - \frac{u(x_j)}{h_j h_{j+1}} + \frac{u_(x_{j+1})}{h_{j+1}(h_j + h_{j+1})}\right)
\end{aligned}$$

Variable Grid Spacing, II

Therefore, we have

$$-2\left(\frac{u(x_{j-1})}{h_j(h_j + h_{j+1})} - \frac{u(x_j)}{h_j h_{j+1}} + \frac{u(x_{j+1})}{h_{j+1}(h_j + h_{j+1})}\right) = f(x_j) \qquad j = 1, \cdots, n-1. \quad (9.1.21)$$

- Note that if $h_j = h_{j+1}$ then the above equation reduces to Eq. (9.1.12).
- Again, Taylor series expansion yields

$$u(x_{j+1}) = u(x_j) + h_{j+1} u'(x_j) + \frac{h_{j+1}^2}{2} u''(x_j) + \frac{h_{j+1}^3}{6} u'''(x_j) + \frac{h_{j+1}^4}{24} u^{(iv)}(x_j) + \cdots$$

$$u(x_{j-1}) = u(x_j) - h_j u'(x_j) + \frac{h_j^2}{2} u''(x_j) - \frac{h_j^3}{6} u'''(x_j) + \frac{h_j^4}{24} u^{(iv)}(x_j) + \cdots$$

Substitude into Eq. (9.1.21), we have

$$\begin{aligned}
&-2\left(\frac{u(x_{j-1})}{h_j(h_j + h_{j+1})} - \frac{u(x_j)}{h_j h_{j+1}} + \frac{u(x_{j+1})}{h_{j+1}(h_j + h_{j+1})}\right) \\
&\quad = -u''(x_j) - \frac{h_{j+1} - h_j}{3} u'''(x_j) - \frac{h_{j+1}^2 - h_j h_{j+1} + h_{j+1}^2}{12} u^{(iv)} + \cdots
\end{aligned}$$

Variable Grid Spacing, III

- The local truncation error for the variable grid spacing case is

$$\tau(x_j) = \frac{h_{j+1} - h_j}{3} u'''(x_j) + \frac{h_{j+1}^2 - h_j h_{j+1} + h_{j+1}^2}{12} u^{(iv)} \tag{9.1.22}$$

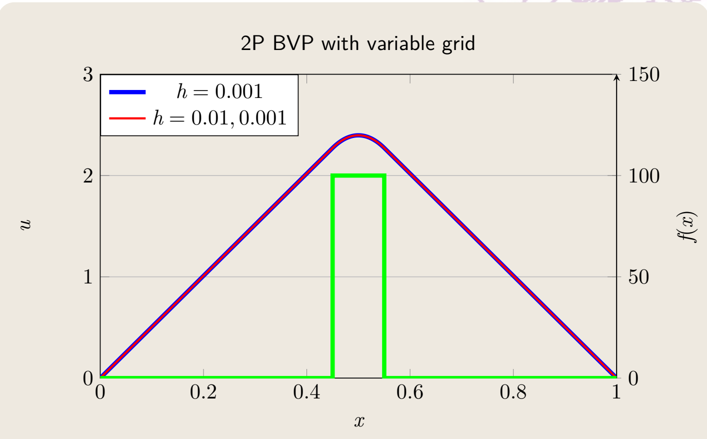
- Note that it reduces to Eq. (9.1.18) if $h_{j+1} = h_j$.
- When $h_j \neq h_{j+1}$ the local truncation error is dominated by

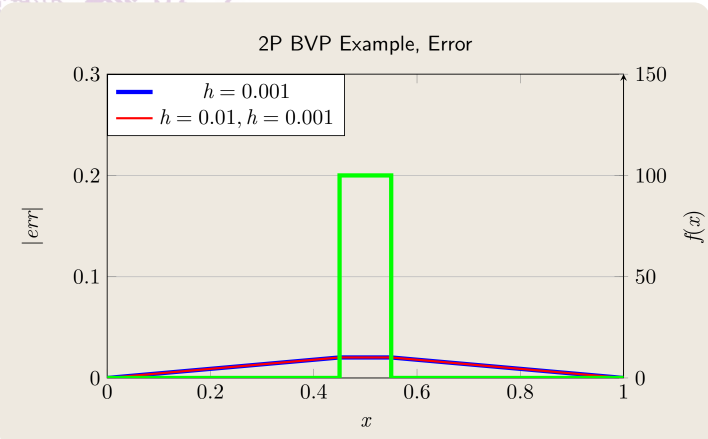
$$\frac{h_{j+1} - h_j}{3} u'''(x_j)$$

  - Larger local truncation error for variable grid spacing
- Strategy to choose grid points that $h_j \neq h_{j+1}$
  - $|u'''(x_j)| \ll 1$
  - Or equivalently, $|f'(x_j)| \ll 1$
  - Other grids should maintain $h_j = h_{j+1}$

Variable Grid Spacing, IV

- Example
  - Constant grid spacing, $N = 1001$, $h = 0.001$ for $x \in [0, 1]$
  - Variable grid spacing, $N = 281$,
    - $h = 0.01$, $x \in [0, 0.4]$ or $x \in [0.6, 1.0]$
    - $h = 0.001$, $x \in [0.4, 0.6]$





- Using variable grid spacing properly can
  - Maintain the same accuracy with better efficiency
  - Or increase accuracy with the same efficiency

2-D Boundary Value Problem

- An example of two-dimensional boundary value problem
  Let Ω be a two-dimensional bounded domain with boundary $\partial\Omega$

$$-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y) \qquad \text{in } \Omega, \tag{9.1.23}$$

  And with Dirichlet boundary condition,

$$u(x,y) = v(x,y) \qquad \text{on } \partial\Omega. \tag{9.1.24}$$

  Or with Neumann boundary condition,

$$\frac{\partial u(x,y)}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{9.1.25}$$

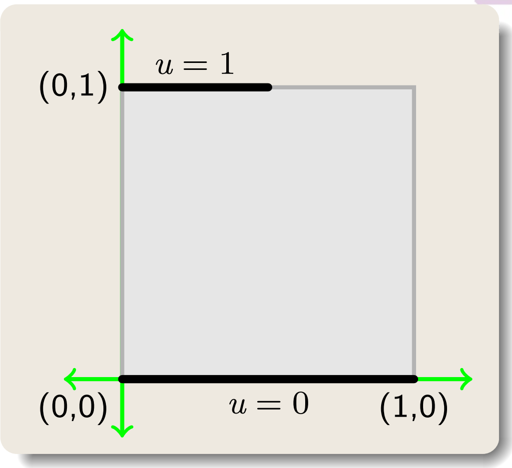
  where n is normal to $\partial\Omega$.
- An example of bounded domain is the unit square:

$$\Omega = \{(x,y) | 0 \le x, y \le 1\}, \tag{9.1.26}$$

$$\begin{aligned}\partial\Omega =& \{(x,y)|x=0, 0 \le y \le 1\} \cup \{(x,y)|x=1, 0 \le y \le 1\} \\ & \cup \{(x,y)|y=0, 0 \le x \le 1\} \cup \{(x,y)|y=1, 0 \le x \le 1\}.\end{aligned} \tag{9.1.27}$$

A 2-D Boundary Value Problem

- A 2-D boundary value problem



- Interior points, Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

- Boundary conditions

$$u = \begin{cases} 0, & 0 \le x \le 1, \quad y = 0, \\ 1, & 0 \le x \le 0.5, \quad y = 1, \end{cases}$$

$$\frac{\partial u}{\partial x} = \begin{cases} 0, & x = 0, \quad 0 \le y \le 1, \\ 0, & x = 1, \quad 0 \le y \le 1, \end{cases}$$

$$\frac{\partial u}{\partial y} = 0, \qquad 0.5 \le x \le 1, \quad y = 1.$$

A 2-D Boundary Value Problem, II

- Assuming we solve for $u(x, y)$ on a set of equally spaced grid points, $\{(x_i, y_j)|0 \le i, j \le n\}$, where $n \ge 2$ and $h = \dfrac{1}{n}$.
- Note that

$$u(x_{i+1}, y_j) = u(x_i, y_j) + h\frac{\partial u(x_i, y_j)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u(x_i, y_j)}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u(x_i, y_j)}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u(x_i, y_j)}{\partial x^4} + \cdots$$

$$u(x_{i-1}, y_j) = u(x_i, y_j) - h\frac{\partial u(x_i, y_j)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u(x_i, y_j)}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u(x_i, y_j)}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u(x_i, y_j)}{\partial x^4} - \cdots$$

$$u(x_i, y_{j+1}) = u(x_i, y_j) + h\frac{\partial u(x_i, y_j)}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u(x_i, y_j)}{\partial y^2} + \frac{h^3}{6}\frac{\partial^3 u(x_i, y_j)}{\partial y^3} + \frac{h^4}{24}\frac{\partial^4 u(x_i, y_j)}{\partial y^4} + \cdots$$

$$u(x_i, y_{j-1}) = u(x_i, y_j) - h\frac{\partial u(x_i, y_j)}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u(x_i, y_j)}{\partial y^2} - \frac{h^3}{6}\frac{\partial^3 u(x_i, y_j)}{\partial y^3} + \frac{h^4}{24}\frac{\partial^4 u(x_i, y_j)}{\partial y^4} - \cdots$$

Thus,

$$\begin{aligned} &u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j) \\ &= h^2\left(\frac{\partial^2 u(x_i, y_j)}{\partial x^2} + \frac{\partial^2 u(x_i, y_j)}{\partial y^2}\right) + \frac{h^4}{12}\left(\frac{\partial^4 u(x_i, y_j)}{\partial x^4} + \frac{\partial^4 u(x_i, y_j)}{\partial y^4}\right) + \cdots \end{aligned}$$

Or,

$$\begin{aligned} &\frac{1}{h^2}\Big(u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j)\Big) \\ &= \frac{\partial^2 u(x_i, y_j)}{\partial x^2} + \frac{\partial^2 u(x_i, y_j)}{\partial y^2} + \frac{h^2}{12}\left(\frac{\partial^4 u(x_i, y_j)}{\partial x^4} + \frac{\partial^4 u(x_i, y_j)}{\partial y^4}\right) + \cdots \end{aligned} \quad (9.1.28)$$

A 2-D Boundary Value Problem, III

- Therefore, for all the interior points the Laplace's equation can be solved by

$$\frac{1}{h^2}\Big(u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j)\Big) = 0, \quad 1 \le i, j \le n-1, \quad (9.1.29)$$

with local truncation of

$$\tau(x_i, y_j) = \frac{h^2}{12}\left(\frac{\partial^4 u(x_i, y_j)}{\partial x^4} + \frac{\partial^4 u(x_i, y_j)}{\partial y^4}\right) \quad (9.1.30)$$

- Dirichlet boundary condition can be set by

$$u(x_i, y_j) = \begin{cases} 0, & 0 \le x_i \le 1, \quad y_j = 0, \\ 1, & 0 \le x_i \le 0.5, \quad y_j = 1, \end{cases} \quad (9.1.31)$$

A 2-D Boundary Value Problem, IV

- For Neumann boundary condition, i.e., $\dfrac{\partial u(x_i, y_j)}{\partial x} = 0$, at $x_i = 0$ or $x_i = 1$,

$$\frac{u(x_{i+1}, y_j) - u(x_i, y_j)}{h} = \frac{u(x_i, y_j) - u(x_{i-1}, y_j)}{h} = 0.$$

For $x_i = 0$, $u(x_{i-1}, y_j)$ is undefined, we can write

$$2u(x_i, y_j) - 2u(x_{i+1}, y_j) = 0.$$

Combine with $\dfrac{\partial^2 u(x_i, y_j)}{\partial y^2}$ term, we have

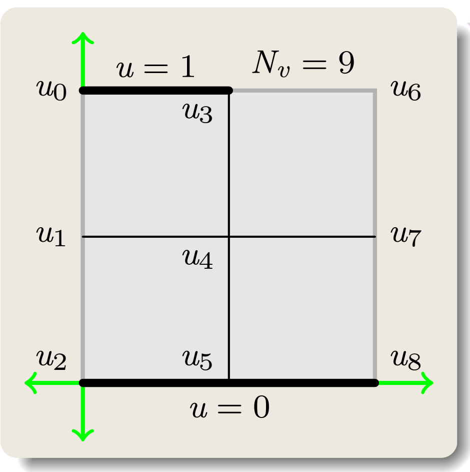
$$4u(x_i, y_j) - 2u(x_{i+1}, y_j) - u(x_i, y_{j+1}) - u(x_i, y_{j-1}) = 0, \qquad (9.1.32)$$

- Similarly, for $x_i = 1$ we have

$$4u(x_i, y_j) - 2u(x_{i-1}, y_j) - u(x_i, y_{j+1}) - u(x_i, y_{j-1}) = 0. \qquad (9.1.33)$$

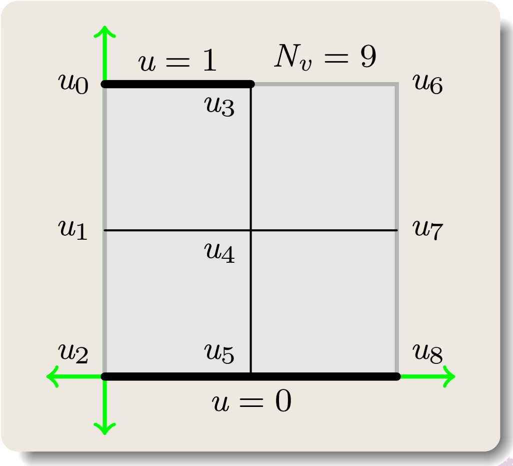
A 2-D Boundary Value Problem, V

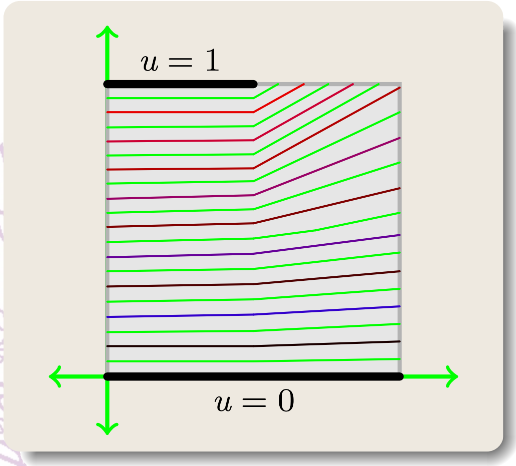
- Suppose a set of grid is built as shown, then the linear system can be formed to find the solution at each grid node.

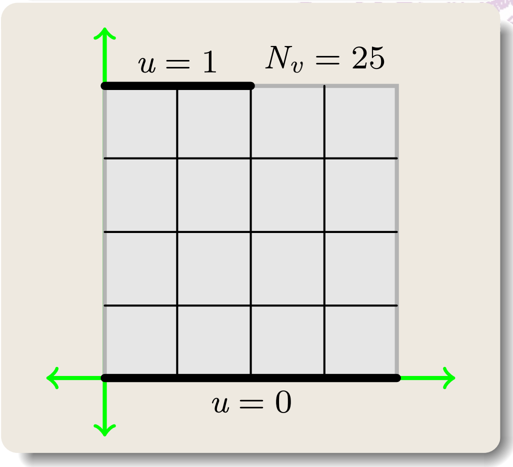


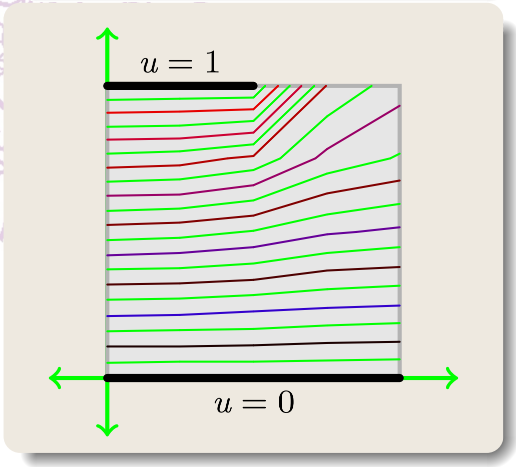
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A 2-D Boundary Value Problem, VI

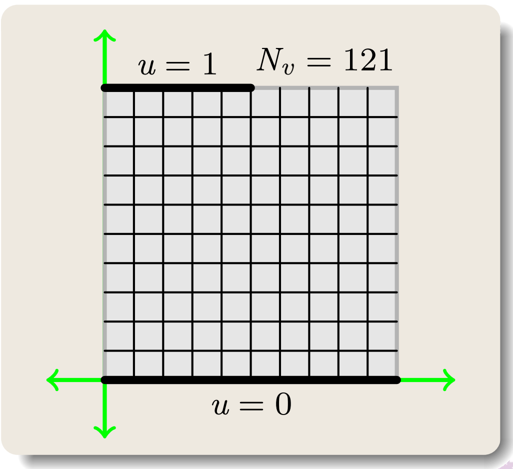


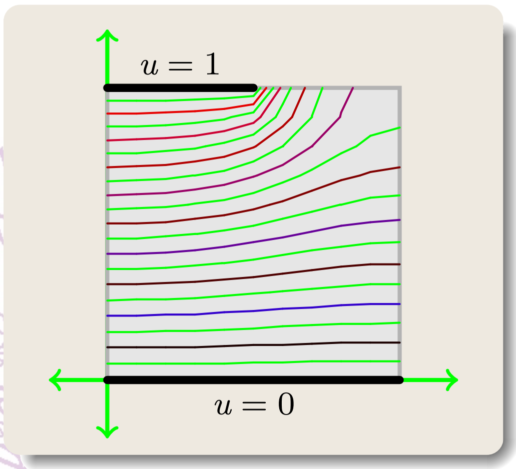


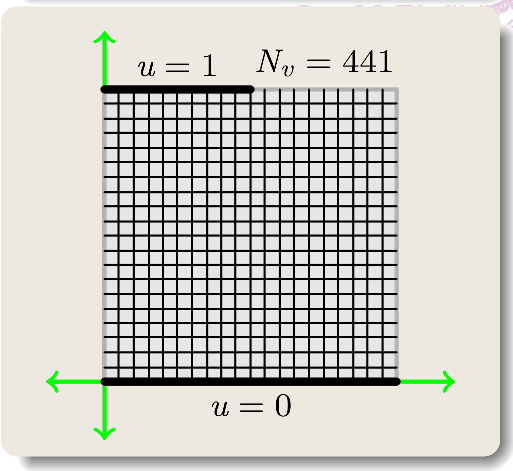


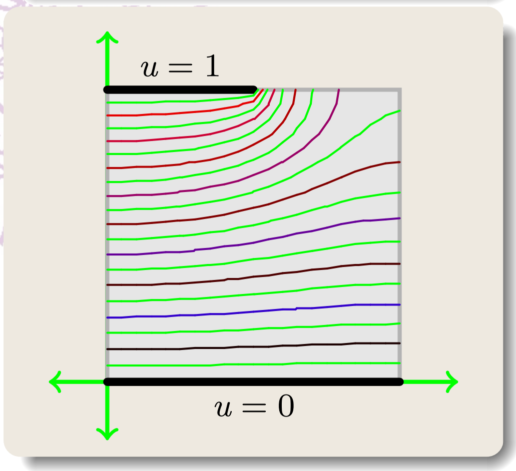


A 2-D Boundary Value Problem, VII









2-D Boundary Value Problem

- Given a bounded domain with proper boundary conditions, the Laplace's equation can be solved.
  - Finer grids tend to give more accurate solutions.
  - Local truncation error at each interior grid point is a good indicator for solution accuracy.
    - $\mathcal{O}(h^4)$ for equal spacing grids.
    - Variable grid spacing can still be adopted for better solution efficiency.
- Finite difference approach can be used to solve general partial differential equations.
  - Rectangular grid system is required.
- Finite element method can be used for non-rectangular domains.

Summary

- Two-point boundary value problem.
- Green's function.
- Finite difference approximation.
  - Non-zero boundary condition.
  - Variable grid spacing.
- 2-D boundary value problem.
  - Solving Laplace's equation.