Unit 3 Iterative Solutions to Linear Systems

Numerical Analysis

Mar. 13, 2017

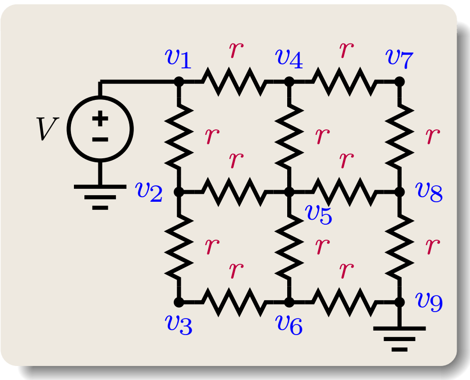
EE/NTHU

A Resistor Network

- Let $g = 1/r$, then the system of equations for the resistor network can be formulated as

$$\begin{bmatrix} 1 & & & & & & & & \\ -g & 3g & -g & & -g & & & & \\ & -g & 2g & & & -g & & & \\ -g & & & 3g & -g & & -g & & \\ & -g & & -g & 4g & -g & & -g & \\ & & -g & & -g & 3g & & & -g \\ & & & -g & & & 2g & -g & \\ & & & & -g & & -g & 3g & -g \\ & & & & & & & & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.1.1}$$

- Since $v_1 = V$ and $v_9 = 0$, it can be reformulated as



$$\begin{bmatrix} 3g & -g & & -g & & & \\ -g & 2g & & & -g & & \\ & & 3g & -g & & -g & \\ -g & & -g & 4g & -g & & -g \\ & -g & & -g & 3g & & \\ & & -g & & & 2g & -g \\ & & & -g & & -g & 3g \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} gV \\ 0 \\ gV \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.1.2}$$

A Resistor Network, II

- Note the sparsity of the matrices for both Eqs. (3.1.1) and (3.1.2).
- Eq. (3.1.2) has two fewer variables and lower matrix dimension.
- Matrix of Eq. (3.1.2) is symmetric.
- The structure of the matrix is not dependent on the values of the resistors
- If the resistor mesh is large, the forms of the matrices remain the same.
  - Sparsity and symmetry.
- Gaussian elimination or any direct solution methods applicable to find the solution.
- Fill-ins will be created that decrease the sparsity of the matrices.
- Iterative solution methods can be effective in solving this kind of matrices.
  - Can provide higher degree of parallelism in solving the system of equations.

Jacobi Method

- For the resistor network, any node voltage, v_i, must satisfy Kirchoff current law.

$$\sum_j g_{ij}(v_i - v_j) = 0. \tag{3.1.3}$$

The summation is carried out for any node j that is connected to node i through a resistor, $r_{ij} = 1/g_{ij}$.

- Assembly all the node variables, then the system of equations (3.1.1) or (3.1.2) is formed.
- Eq. (3.1.3) can be reformulated as

$$\begin{aligned} \sum_j g_{ij} v_i &= \sum_j g_{ij} v_j \\ G_i v_i &= \sum_j g_{ij} v_j \\ v_i &= \sum_j \frac{g_{ij}}{G_i} v_j \end{aligned} \tag{3.1.4}$$

where $G_i = \sum_j g_{ij}$ and the summation is carried out where j is connected to node i through a resistor.

Jacobi Method, II

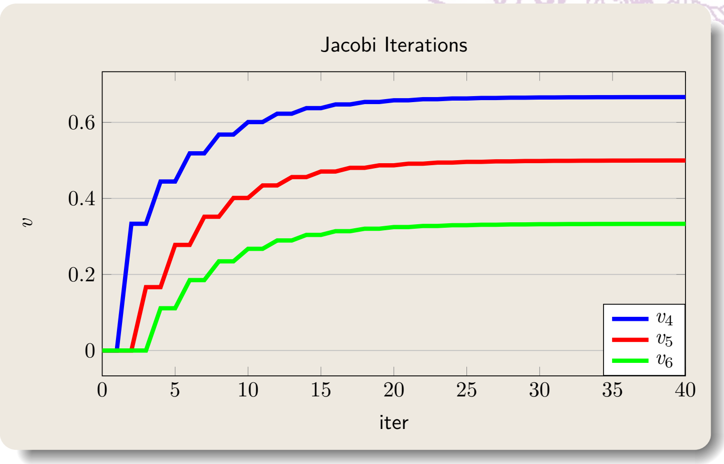
- Compare Eqs. (3.1.2) and (3.1.4), we found that (3.1.4) can be formed by letting $G_i = a_{ii}$, the diagonal matrix element, and moving all off-diagonal elements to the right hand side of the equations.
- For the resistor network, we have

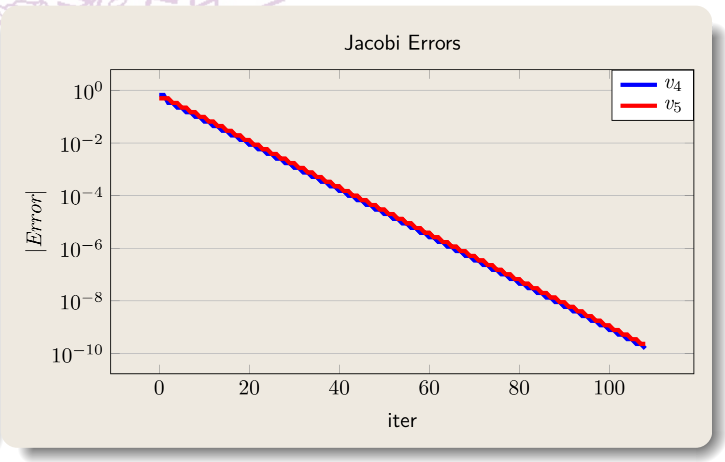
$$
\begin{aligned}
v_2 &= 1/3(V + v_3 + v_5) \\
v_3 &= 1/2(v_2 + v_6) \\
v_4 &= 1/3(V + v_5 + v_7) \\
v_5 &= 1/4(v_2 + v_4 + v_6 + v_8) \\
v_6 &= 1/3(v_3 + v_5) \\
v_7 &= 1/2(v_4 + v_8) \\
v_8 &= 1/3(v_5 + v_7)
\end{aligned}
\qquad
\begin{aligned}
v_2^{(k+1)} &= 1/3(V + v_3^{(k)} + v_5^{(k)}) \\
v_3^{(k+1)} &= 1/2(v_2^{(k)} + v_6^{(k)}) \\
v_4^{(k+1)} &= 1/3(V + v_5^{(k)} + v_7^{(k)}) \\
v_5^{(k+1)} &= 1/4(v_2^{(k)} + v_4^{(k)} + v_6^{(k)} + v_8^{(k)}) \\
v_6^{(k+1)} &= 1/3(v_3^{(k)} + v_5^{(k)}) \\
v_7^{(k+1)} &= 1/2(v_4^{(k)} + v_8^{(k)}) \\
v_8^{(k+1)} &= 1/3(v_5^{(k)} + v_7^{(k)})
\end{aligned}
$$

- Jacobi method assumes an initial guess $v^{(0)}$ for the node's voltages, and applies it to the right hand side to find the new values of the solution, $v^{(k+1)}$.
- This process is repeated until a converged solution is found.

Jacobi Method, III

- The Jacobi method is shown to be convergent for the resistor network problem.
- The convergent rate is shown to be constant.
  - It takes 100 iterations to reach $\epsilon < 1e-9$.
- Each iteration takes NZ' multiplications, where NZ' is the number of off-diagonal elements in the system matrix, and n divisions.
- This method takes full advantage of the sparsity of the matrix.





Jacobi Method, IV

- Given a linear system $\mathbf{Ax} = \mathbf{b}$, it can also be written as

$$\sum_{i=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, \ldots, n. \tag{3.1.5}$$

Assuming $a_{ii} \neq 0$, $i = 1, \ldots, n$, it can be reformulated as

$$x_i = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij}x_j \right], \qquad i = 1, \ldots, n. \tag{3.1.6}$$

Given an initial guess $\mathbf{x}^{(0)}$, then the Jacobi method is to compute new solution at iteration k by

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij}x_j^{(k)} \right], \qquad i = 1, \ldots, n. \tag{3.1.7}$$

Jacobi Method, V

- In matrix form, we write

$$\mathbf{A} = -\mathbf{E} + \mathbf{D} - \mathbf{F}, \tag{3.1.8}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.1.9}$$

where $\mathbf{E}$ is a strictly lower triangular matrix, $\mathbf{D}$ is a diagonal matrix and $\mathbf{F}$ is a strictly upper triangular matrix. Then Jacobi iteration is

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1} \left[\mathbf{b} + (\mathbf{E} + \mathbf{F})\mathbf{x}^{(k)} \right]. \tag{3.1.10}$$

Over-Relaxation Method (JOR)

- The Jacobi method can be generalized as

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^{(k)} \right] + (1-\omega) x_i^{(k)}, \qquad i = 1, \dots, n. \tag{3.1.11}$$

  Or in matrix form

$$\mathbf{x}^{(k+1)} = \omega \mathbf{D}^{-1} \left[\mathbf{b} + (\mathbf{E}+\mathbf{F}) \mathbf{x}^{(k)} \right] + (1-\omega) \mathbf{x}^{(k)}. \tag{3.1.12}$$

- Define the residue at kth iteration as

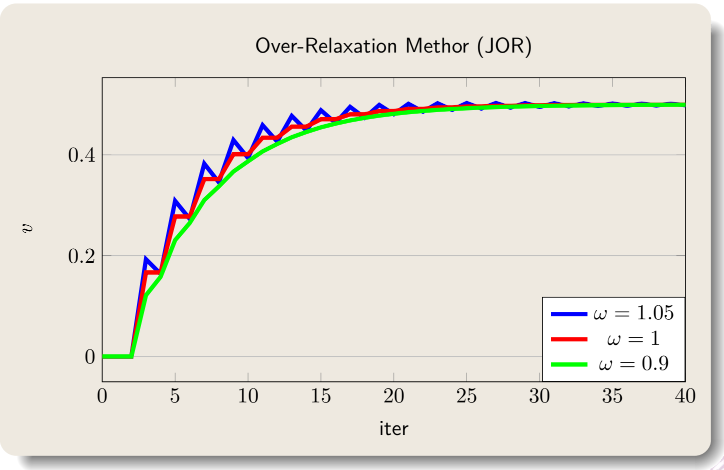
$$\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}, \tag{3.1.13}$$

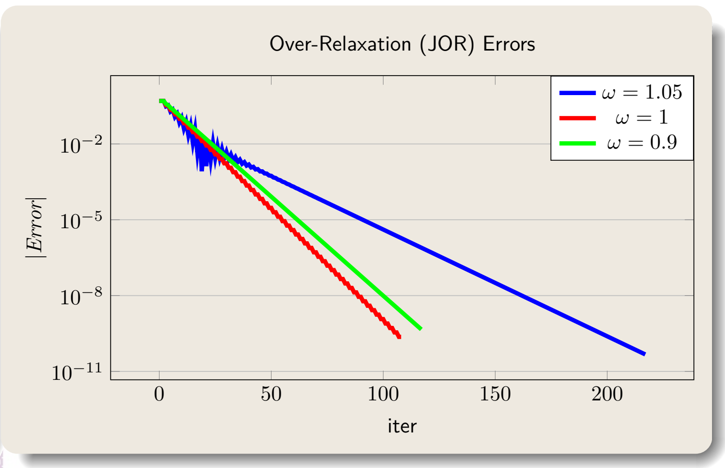
  then

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} + \omega \mathbf{D}^{-1} (\mathbf{r}^{(k)} + \mathbf{A}\mathbf{x}^{(k)} + (\mathbf{E}+\mathbf{F})\mathbf{x}^{(k)} - \mathbf{D}\mathbf{x}^{(k)}) \\ &= \mathbf{x}^{(k)} + \omega \mathbf{D}^{-1} \mathbf{r}^{(k)}. \end{aligned} \tag{3.1.14}$$

- When $\omega = 1$ then it reduces to the Jacobi iteration.
- When $\omega < 1$ it is called under-relaxation.
- When $\omega > 1$ it is called over-relaxation.

Over-Relaxation Method (JOR), II





- In Jacobi iterations we see the solution converges by increasing the node voltages toward the solutions.
- Over-relaxation method predicts the solution with larger amount of voltage increases.
- For the resistor network problem, the over-relaxation method does not appear to improve the convergence rate.

Gauss-Seidel Method

- Note that in Jacobi iterations, some variables are updated before others. These updated values can be used in the current iteration to speed up the convergence.

$$
\begin{aligned}
v_2 &= 1/3(V + v_3 + v_5) \\
v_3 &= 1/2(v_2 + v_6) \\
v_4 &= 1/3(V + v_5 + v_7) \\
v_5 &= 1/4(v_2 + v_4 + v_6 + v_8) \\
v_6 &= 1/3(v_3 + v_5) \\
v_7 &= 1/2(v_4 + v_8) \\
v_8 &= 1/3(v_5 + v_7)
\end{aligned}
\qquad
\begin{aligned}
v_2^{(k+1)} &= 1/3(V + v_3^{(k)} + v_5^{(k)}) \\
v_3^{(k+1)} &= 1/2(v_2^{(k+1)} + v_6^{(k)}) \\
v_4^{(k+1)} &= 1/3(V + v_5^{(k)} + v_7^{(k)}) \\
v_5^{(k+1)} &= 1/4(v_2^{(k+1)} + v_4^{(k+1)} + v_6^{(k)} + v_8^{(k)}) \\
v_6^{(k+1)} &= 1/3(v_3^{(k+1)} + v_5^{(k+1)}) \\
v_7^{(k+1)} &= 1/2(v_4^{(k+1)} + v_8^{(k)}) \\
v_8^{(k+1)} &= 1/3(v_5^{(k+1)} + v_7^{(k+1)})
\end{aligned}
$$

- Thus, Gauss-Seidel method has the same iteration process as Jacobi method, and it uses updated values whenever possible.
- Convergence rate is expected to be faster.

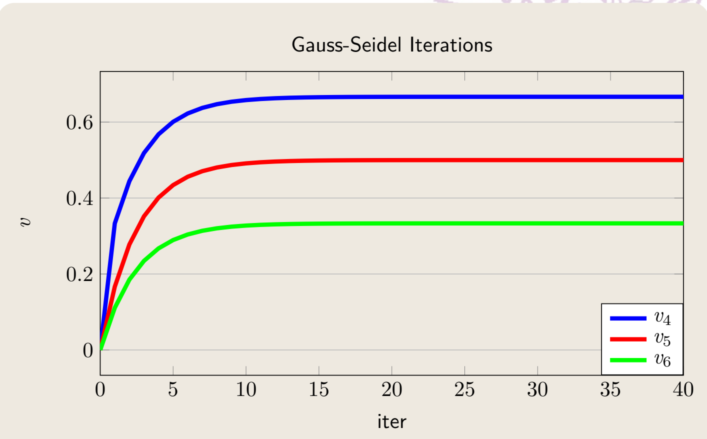
Gauss-Seidel Method, II

- Comparing to the Jacobi method, Eq. (3.1.7), Gauss-Seidel method can be formulated as

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right], \qquad i = 1, \ldots, n. \qquad (3.1.15)$$

Or using matrix form, as compared to Eq. (3.1.10),

$$
\begin{aligned}
\mathbf{D}\mathbf{x}^{(k+1)} &= \mathbf{b} + \mathbf{E}\mathbf{x}^{(k+1)} + \mathbf{F}\mathbf{x}^{(k)}, \\
(\mathbf{D} - \mathbf{E})\mathbf{x}^{(k+1)} &= \mathbf{b} + \mathbf{F}\mathbf{x}^{(k)}, \\
\mathbf{x}^{(k+1)} &= (\mathbf{D} - \mathbf{E})^{-1}(\mathbf{b} + \mathbf{F}\mathbf{x}^{(k)}).
\end{aligned} \qquad (3.1.16)
$$



Successive Over-Relaxation Method (SOR)

- The Gauss-Seidel appears to have better convergence rate than the Jacobi method.
- The Gauss-Seidel method can also be generalized as the following successive over-relaxation method.

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right] + (1-\omega) x_i^{(k)}, \qquad i = 1, \dots, n. \tag{3.1.17}$$

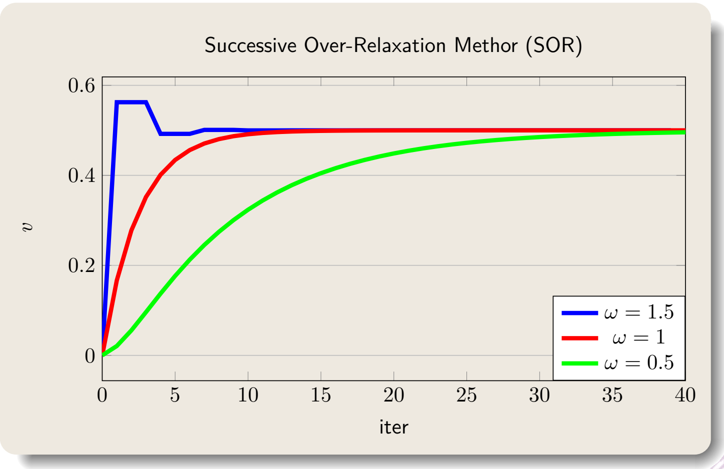
- Or in matrix form

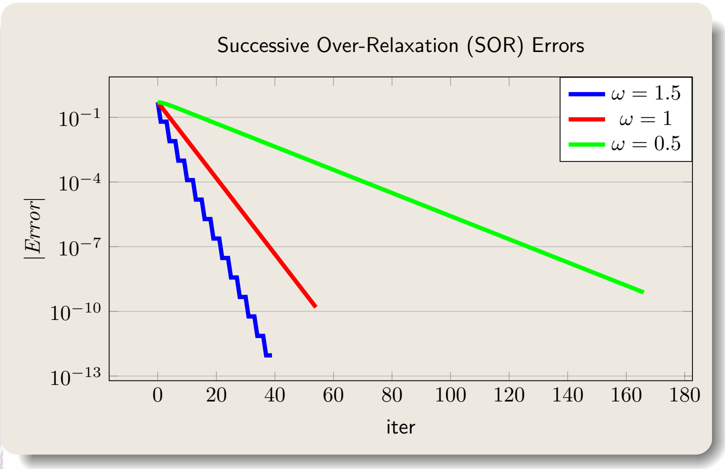
$$\begin{aligned}
(\mathbf{D} - \omega \mathbf{E}) \mathbf{x}^{(k+1)} &= \omega \mathbf{b} + [\omega \mathbf{F} + (1-\omega)\mathbf{D}] \mathbf{x}^{(k)} \\
&= \omega [\mathbf{A} \mathbf{x}^{(k)} + \mathbf{r}^{(k)}] + [\omega \mathbf{F} + (1-\omega)\mathbf{D}] \mathbf{x}^{(k)} \\
&= \omega \mathbf{r}^{(k)} + [\omega(-\mathbf{E} + \mathbf{D} - \mathbf{F}) + \omega \mathbf{F} + \mathbf{D} - \omega \mathbf{D}] \mathbf{x}^{(k)} \\
&= \omega \mathbf{r}^{(k)} + (\mathbf{D} - \omega \mathbf{E}) \mathbf{x}^{(k)}
\end{aligned} \tag{3.1.18}$$

Thus

$$\begin{aligned}
\mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} + \omega (\mathbf{D} - \omega \mathbf{E})^{-1} \mathbf{r}^{(k)} \\
&= \mathbf{x}^{(k)} + \left(\frac{1}{\omega} \mathbf{D} - \mathbf{E} \right)^{-1} \mathbf{r}^{(k)}.
\end{aligned} \tag{3.1.19}$$

Successive Over-Relaxation Method (SOR), II





- The successive over-relaxation method can improve the convergence rate.
- For the resistor network problem
  - $\omega < 1$ has lower convergence rate,
  - $\omega = 1.5$ has higher convergence rate,
  - when ω is too large, it does not converge.
- Implementation using Eq. (3.1.17) is easier than Eq. (3.1.19).

Iterative Methods

- In this Chapter, we solve the following linear system using iterative method

$$\mathbf{Ax} = \mathbf{b}. \tag{3.1.20}$$

- A general iterative method is to form a sequence

$$\begin{aligned} \mathbf{x}^{(0)} &= \mathbf{f}_0(\mathbf{A}, \mathbf{b}), \\ \mathbf{x}^{(k+1)} &= \mathbf{f}_{k+1}(\mathbf{x}^{(k)}, \mathbf{x}^{(k-1)}, \ldots, \mathbf{x}^{(n-m)}, \mathbf{A}, \mathbf{b}), \text{ for } n \geq m, \end{aligned} \tag{3.1.21}$$

such that $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$, and $\mathbf{x}$ satisfies Eq. (3.1.20).

- The number of steps that $\mathbf{x}^{(k+1)}$ depends on is called the order of the method.
- If $\mathbf{f}_k$ is independent of k, then the method is called stationary, otherwise it is called nonstationary.
- If $\mathbf{f}_k$ depends linearly on $\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(m)}$, then the method is called linear, otherwise it is called nonlinear.
- In this section, our methods are stationary linear iterative methods of the first order that have the following form, given $\mathbf{x}^{(0)}$

$$\mathbf{x}^{(k+1)} = \mathbf{Bx}^{(k)} + \mathbf{f}, \qquad k \geq 0, \tag{3.1.22}$$

where $\mathbf{B}$ is an $n \times n$ matrix and is called the iteration matrix.

Iterative Methods, II

- Note that Eq. (3.1.22) needs to converge to the solution of Eq. (3.1.20). Thus, when converge

$$\begin{aligned} \mathbf{x} &= \mathbf{Bx} + \mathbf{f} \\ (\mathbf{I} - \mathbf{B})\mathbf{x} &= \mathbf{f} \\ \mathbf{x} &= (\mathbf{I} - \mathbf{B})^{-1}\mathbf{f} = \mathbf{A}^{-1}\mathbf{b} \\ \text{Thus,} \qquad \mathbf{f} &= (\mathbf{I} - \mathbf{B})\mathbf{A}^{-1}\mathbf{b}. \end{aligned}$$

Definition 3.1.1.

A stationary linear iterative method of the equation form (3.1.22) is said to be consistent with (3.1.20) if it satisfies

$$\mathbf{f} = (\mathbf{I} - \mathbf{B})\mathbf{A}^{-1}\mathbf{b}. \tag{3.1.23}$$

Definition 3.1.2.

The error at the k-th iteration is given by

$$\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}. \tag{3.1.24}$$

If the iterative method is convergent then $\lim_{k\to\infty} \mathbf{e}^{(k)} = \mathbf{0}$ for any choice of initial guess $\mathbf{x}^{(0)}$.

Iterative Methods, III

Theorem 3.1.3.

Given a consistent iteration method as

$$\mathbf{x}^{(k+1)} = \mathbf{B}\mathbf{x}^{(k)} + \mathbf{f}, \qquad k \geq 0,$$

then the sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to the solution of

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

for any initial guess $\mathbf{x}^{(0)}$ if and only if $\rho(\mathbf{B}) < 1$.

Definition 3.1.4.

Let $\mathbf{B}$ be the iteration matrix, then

1. $\|\mathbf{B}^m\|$ is the convergence factor after m steps of the iteration;
2. $\|\mathbf{B}^m\|^{1/m}$ is the average convergence factor after m steps;
3. $R_m(\mathbf{B}) = -\dfrac{1}{m} \log \|\mathbf{B}^m\|$ is the average convergence rate after m steps;
4. $R(\mathbf{B}) = \lim_{k \to \infty} R_k(\mathbf{B}) = -\log \rho(\mathbf{B})$ is the asymptotic convergence rate.

- The convergence behavior is determined by the matrix $\mathbf{B}$.

Iterative Methods, IV

- A general approach to solve the linear system of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is to split matrix $\mathbf{A}$ additively to the form $\mathbf{A} = \mathbf{P} - \mathbf{N}$ with $\mathbf{P}$ nonsingular. Matrix $\mathbf{P}$ is called preconditioning matrix or preconditioner. Then the iteration is

$$\mathbf{P}\mathbf{x}^{(k+1)} = \mathbf{N}\mathbf{x}^{(k)} + \mathbf{b}, \qquad k \geq 0. \tag{3.1.25}$$

Comparing to Eq. (3.1.22), we have $\mathbf{B} = \mathbf{P}^{-1}\mathbf{N}$ and $\mathbf{f} = \mathbf{P}^{-1}\mathbf{b}$.

Definition 3.1.5.

Given an iterative method as Eq. (3.1.22) to solve the linear equation (3.1.20), the residue at iteration k is

$$\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}. \tag{3.1.26}$$

- Substitute Eq. (3.1.26) into Eq. (3.1.25), we have

$$\begin{aligned} \mathbf{P}\mathbf{x}^{(k+1)} &= \mathbf{N}\mathbf{x}^{(k)} + \mathbf{r}^{(k)} + \mathbf{A}\mathbf{x}^{(k)} \\ &= \mathbf{N}\mathbf{x}^{(k)} + \mathbf{r}^{(k)} + \mathbf{P}\mathbf{x}^{(k)} - \mathbf{N}\mathbf{x}^{(k)} \\ &= \mathbf{P}\mathbf{x}^{(k)} + \mathbf{r}^{(k)} \end{aligned}$$

Thus, we have

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{r}^{(k)}. \tag{3.1.27}$$

Iterative Methods, V

Theorem 3.1.6.

Let $\mathbf{A} = \mathbf{P} - \mathbf{N}$, with $\mathbf{A}$ and $\mathbf{P}$ symmetric and positive definite. If the matrix $2\mathbf{P} - \mathbf{A}$ is positive definite, then the iterative method defined by (3.1.25) or (3.1.27) is convergent for any choice of initial guess $\mathbf{x}^{(0)}$ and

$$\rho(\mathbf{B}) = \|\mathbf{B}\|_{\mathbf{A}} = \|\mathbf{B}\|_{\mathbf{P}} < 1.$$

Moreover, the convergence of the iteration is monotone with respect to the norms $\|\cdot\|_{\mathbf{P}}$ and $\|\cdot\|_{\mathbf{A}}$.

Theorem 3.1.7.

Let $\mathbf{A} = \mathbf{P} - \mathbf{N}$ with $\mathbf{A}$ being symmetric and positive definite. If the matrix $\mathbf{P} + \mathbf{P}^T - \mathbf{A}$ is positive definite, then $\mathbf{P}$ is invertible, the iterative method defined by (3.1.25) or (3.1.27) is monotonically convergent with respect to norm $\|\cdot\|_{\mathbf{A}}$ and $\rho(\mathbf{B}) \leq \|\mathbf{B}\|_{\mathbf{A}} < 1$.

Iterative Methods, VI

- In Jacobi method, the matrix $\mathbf{A}$ is splatted as

$$\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{F},$$

where $\mathbf{D}$ is the diagonal matrix, $\mathbf{E}$ is the strictly lower triangle matrix and $\mathbf{F}$ is the strictly upper triangle matrix. And the iteration is

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\left[\mathbf{b} + (\mathbf{E}+\mathbf{F})\mathbf{x}^{(k)}\right]$$
$$\mathbf{D}\mathbf{x}^{(k+1)} = \mathbf{b} + (\mathbf{E}+\mathbf{F})\mathbf{x}^{(k)}$$

Compared to Eq. (3.1.25), we have

$$\mathbf{P} = \mathbf{D}, \mathbf{N} = \mathbf{E} + \mathbf{F},$$

and the iterative matrix of Jacobi method is

$$\mathbf{B}_J = \mathbf{D}^{-1}(\mathbf{E}+\mathbf{F}) = \mathbf{I} - \mathbf{D}^{-1}\mathbf{A}. \tag{3.1.28}$$

- In the over-relaxation method,

$$\mathbf{x}^{(k+1)} = \omega\mathbf{D}^{-1}\left[\mathbf{b} + (\mathbf{E}+\mathbf{F})\mathbf{x}^{(k)}\right] + (1-\omega)\mathbf{x}^{(k)},$$

and the iteration matrix is

$$\mathbf{B}_{J_\omega} = \omega\mathbf{B}_J + (1-\omega)\mathbf{I}. \tag{3.1.29}$$

Iterative Methods, VII

- The Gauss-Seidel method has

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\left[\mathbf{b} + \mathbf{E}\mathbf{x}^{(k+1)} + \mathbf{F}\mathbf{x}^{(k)}\right],$$

thus, the splitting of $\mathbf{A}$ is $\mathbf{P} = \mathbf{D} - \mathbf{E}, \mathbf{N} = \mathbf{F}$, and the iteration matrix is

$$\mathbf{B}_{GS} = (\mathbf{D} - \mathbf{E})^{-1}\mathbf{F}. \tag{3.1.30}$$

- The successive over-relaxation method has

$$(\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{E})\mathbf{x}^{(k+1)} = \left[(1-\omega)\mathbf{I} + \omega\mathbf{D}^{-1}\mathbf{F}\right]\mathbf{x}^{(k)} + \omega\mathbf{D}^{-1}\mathbf{b}, \tag{3.1.31}$$

and the iteration matrix is

$$\mathbf{B}(\omega) = (\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{E})^{-1}\left[(1-\omega)\mathbf{I} + \omega\mathbf{D}^{-1}\mathbf{F}\right]. \tag{3.1.32}$$

Iterative Methods, VIII

Theorem 3.1.8.

If $\mathbf{A}$ is a strictly diagonally dominant matrix by rows, i.e., $|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|, i = 1,\ldots,n,$ then the Jacobi and Gauss-Seidel methods are convergent.

Theorem 3.1.9.

If $\mathbf{A}$ and $2\mathbf{D} - \mathbf{A}$ are symmetric and positive definite matrices, then the Jacobi is convergent and $\rho(\mathbf{B}_J) = \|\mathbf{B}_J\|_{\mathbf{A}} = \|\mathbf{B}_J\|_{\mathbf{D}}$.

Theorem 3.1.10.

If $\mathbf{A}$ is symmetric positive definite, then the JOR method is convergent if

$$0 < \omega < \frac{2}{\rho(\mathbf{D}^{-1}\mathbf{A})}. \tag{3.1.33}$$

Iterative Methods, IX

Theorem 3.1.11.

If $\mathbf{A}$ is symmetric positive definite, the Gauss-Seidel method is monotonically convergent with respect to the norm $\|\cdot\|_{\mathbf{A}}$.

- If $\mathbf{A}$ is tridiagonal (or block tridiagonal), then it can be shown that

$$\rho(\mathbf{B}_{GS}) = \rho^2(\mathbf{B}_J). \tag{3.1.34}$$

Thus, both methods converge or diverge at the same time. If they converge, then the Gauss-Seidel method converge faster than Jacobi method, and the asymptotic convergence rate of the Gauss-Seidel method is twice than that of Jacobi method.

Theorem 3.1.12.

If the Jacobi method is convergent, then the JOR method converges if $0 < \omega \leq 1$.

Iterative Methods, X

Theorem 3.1.13. (Ostrowski)

If $\mathbf{A}$ is symmetric and positive definite, then the SOR method is convergent if and only of $0 < \omega < 2$. Moreover, its convergence is monotone with respect to $\|\cdot\|_{\mathbf{A}}$.

- If $\mathbf{A}$ is strictly diagonally dominant by rows, then SOR method converges if $0 < \omega \leq 1$.

Theorem 3.1.14.

If the matrix $\mathbf{A}$ enjoys the A-property and if $\mathbf{B}_j$ has real eigenvalues, then the SOR method converges for any choice of $\mathbf{x}^{(0)}$ if and only if $\rho(\mathbf{B}_J) < 1$ and $0 < \omega < 2$. Moreover,

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2(\mathbf{B}_J)}} \tag{3.1.35}$$

and the corresponding asymptotic convergence factor is

$$\rho(\mathbf{B}(\omega_{opt})) = \frac{1 - \sqrt{1 - \rho^2(\mathbf{B}_J)}}{1 + \sqrt{1 - \rho^2(\mathbf{B}_J)}}. \tag{3.1.36}$$

Iterative Methods, XI

Definition 3.1.15. A-property

A consistently ordered matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ (that is, a matrix such that $\alpha\mathbf{D}^{-1}\mathbf{E} + \alpha^{-1}\mathbf{D}^{-1}\mathbf{F}$, for $\alpha \neq 0$, has eigenvalues that do not depend on α, where $\mathbf{M} = \mathbf{D} - \mathbf{E} - \mathbf{F}$, $\mathbf{D}$ is a diagonal matrix, $\mathbf{E}$ and $\mathbf{F}$ are strictly lower and upper triangular matrices, respectively) enjoys the A-property if it can be partitioned into the 2×2 block form

$$\mathbf{M} = \begin{bmatrix} \mathbf{D}_1' & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{D}_2' \end{bmatrix},$$

where $\mathbf{D}_1'$ and $\mathbf{D}_2'$ are diagonal matrices.

Symmetric Gauss-Seidel Method

- In the Gauss-Seidel method, the values of the updated variables can be used immediately provided those variables are ordered first.
- The variables are updated only in one direction.
- The symmetric Gauss-Seidel method adds another step in each iteration that updates each variable in backward order.
- Using the resistor network example,

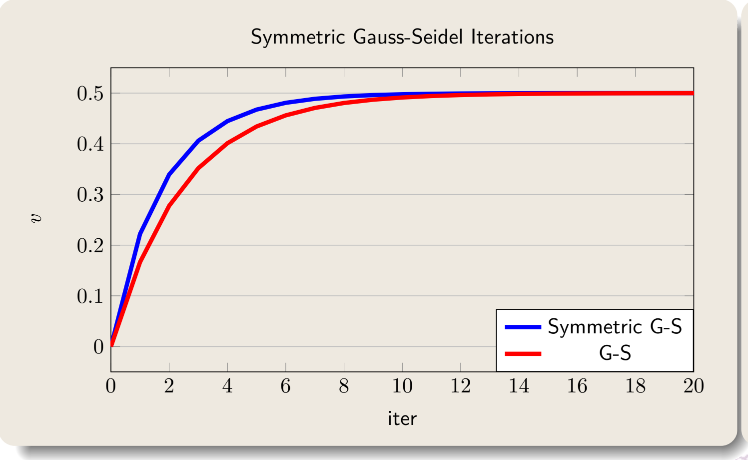
- Forward Gauss-Seidel

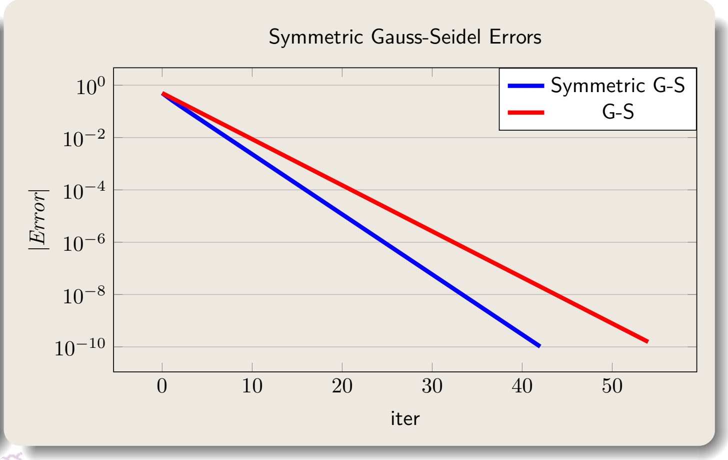
$$\begin{aligned}
v_2^{(k+1/2)} &= 1/3(V + v_3^{(k)} + v_5^{(k)}) \\
v_3^{(k+1/2)} &= 1/2(v_2^{(k+1/2)} + v_6^{(k)}) \\
v_4^{(k+1/2)} &= 1/3(V + v_5^{(k)} + v_7^{(k)}) \\
v_5^{(k+1/2)} &= 1/4(v_2^{(k+1/2)} + v_4^{(k+1/2)} + v_6^{(k)} + v_8^{(k)}) \\
v_6^{(k+1/2)} &= 1/3(v_3^{(k+1/2)} + v_5^{(k+1/2)}) \\
v_7^{(k+1/2)} &= 1/2(v_4^{(k+1/2)} + v_8^{(k)}) \\
v_8^{(k+1/2)} &= 1/3(v_5^{(k+1/2)} + v_7^{(k+1/2)})
\end{aligned}$$

- Backward Gauss-Seidel

$$\begin{aligned}
v_8^{(k+1)} &= 1/3(v_5^{(k+1/2)} + v_7^{(k+1/2)}) \\
v_7^{(k+1)} &= 1/2(v_4^{(k+1/2)} + v_8^{(k+1)}) \\
v_6^{(k+1)} &= 1/3(v_3^{(k+1/2)} + v_5^{(k+1/2)}) \\
v_5^{(k+1)} &= 1/4(v_2^{(k+1/2)} + v_4^{(k+1/2)} + v_6^{(k+1)} + v_8^{(k+1)}) \\
v_4^{(k+1)} &= 1/3(V + v_5^{(k+1)} + v_7^{(k+1)}) \\
v_3^{(k+1)} &= 1/2(v_2^{(k+1/2)} + v_6^{(k+1)}) \\
v_2^{(k+1)} &= 1/3(V + v_3^{(k+1)} + v_5^{(k+1)})
\end{aligned}$$

Symmetric Gauss-Seidel Method, II





- By combining forward and backward Gauss-Seidel method, symmetric Gauss-Seidel method improves the convergence rate.
- The symmetric Gauss-Seidel method appears to converge faster then the Gauss-Seidel method.
- But, each iteration of the symmetric Gauss-Seidel method needs more effort.

Symmetric Gauss-Seidel Method, III

- The forward Gauss-Seidel method is

$$\mathbf{D}\mathbf{x}^{(k+1/2)} = \mathbf{b} + \mathbf{E}\mathbf{x}^{(k+1/2)} + \mathbf{F}\mathbf{x}^{(k)}$$
$$\mathbf{x}^{(k+1/2)} = (\mathbf{D}-\mathbf{E})^{-1}(\mathbf{b} + \mathbf{F}\mathbf{x}^{(k)}). \quad (3.1.37)$$

- The backward Gauss-Seidel method is then

$$\mathbf{D}\mathbf{x}^{(k+1)} = \mathbf{b} + \mathbf{E}\mathbf{x}^{(k+1/2)} + \mathbf{F}\mathbf{x}^{(k+1)}$$
$$\mathbf{x}^{(k+1)} = (\mathbf{D}-\mathbf{F})^{-1}(\mathbf{b} + \mathbf{E}\mathbf{x}^{(k+1/2)}). \quad (3.1.38)$$

- Combining these two equations, we have

$$\mathbf{x}^{(k+1)} = (\mathbf{D}-\mathbf{F})^{-1}\mathbf{E}(\mathbf{D}-\mathbf{E})^{-1}\mathbf{F}\mathbf{x}^{(k)} + (\mathbf{D}-\mathbf{F})^{-1}[\mathbf{E}(\mathbf{D}-\mathbf{E})^{-1} + \mathbf{I}]\mathbf{b}. \quad (3.1.39)$$

Thus, the iterative matrix is

$$\mathbf{B}_{SGS} = (\mathbf{D}-\mathbf{F})^{-1}\mathbf{E}(\mathbf{D}-\mathbf{E})^{-1}\mathbf{F}. \quad (3.1.40)$$

And

$$\mathbf{f}_{SGS} = (\mathbf{D}-\mathbf{F})^{-1}[\mathbf{E}(\mathbf{D}-\mathbf{E})^{-1} + \mathbf{I}]\mathbf{b}. \quad (3.1.41)$$

And the preconditioning matrix is

$$\mathbf{P}_{SGS} = \{(\mathbf{D}-\mathbf{F})^{-1}[\mathbf{E}(\mathbf{D}-\mathbf{E})^{-1} + \mathbf{I}]\}^{-1}$$
$$= (\mathbf{D}-\mathbf{E})\mathbf{D}^{-1}(\mathbf{D}-\mathbf{F}). \quad (3.1.42)$$

Symmetric Gauss-Seidel Method, IV

Theorem 3.1.16.

If $\mathbf{A}$ is a symmetric positive definite matrix, the symmetric Gauss-Seidel method is convergent, and , moreover, $\mathbf{B}_{SGS}$ is symmetric positive definite.

- Note that in Eq. (3.1.18), the forward SOR is

$$(\mathbf{D}-\omega\mathbf{E})\mathbf{x}^{(k+1)} = [\omega\mathbf{F}+(1-\omega)\mathbf{D}]\mathbf{x}^{(k)} + \omega\mathbf{b}.$$

- The backward SOR is then

$$(\mathbf{D}-\omega\mathbf{F})\mathbf{x}^{(k+1)} = [\omega\mathbf{E}+(1-\omega)\mathbf{D}]\mathbf{x}^{(k)} + \omega\mathbf{b}.$$

- Combine these two equations, the symmetric SOR (SSOR) method is

$$\mathbf{x}^{(k+1)} = \mathbf{B}_S(\omega)\mathbf{x}^{(k)} + \mathbf{b}_\omega,$$

where

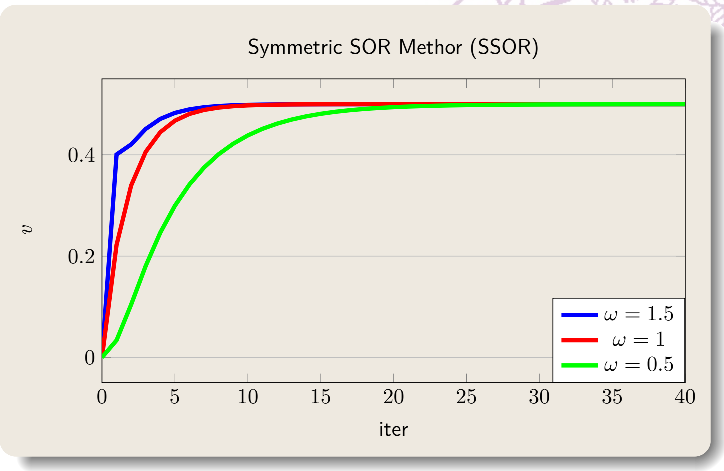
$$\begin{aligned}\mathbf{B}_S(\omega) &= (\mathbf{D}-\omega\mathbf{F})^{-1}(\omega\mathbf{E}+(1-\omega)\mathbf{D})(\mathbf{D}-\omega\mathbf{E})^{-1}(\omega\mathbf{F}+(1-\omega)\mathbf{D}),\\ \mathbf{b}_\omega &= \omega(2-\omega)(\mathbf{D}-\omega\mathbf{F})^{-1}\mathbf{D}(\mathbf{D}-\omega\mathbf{E})^{-1}\mathbf{b}.\end{aligned}$$

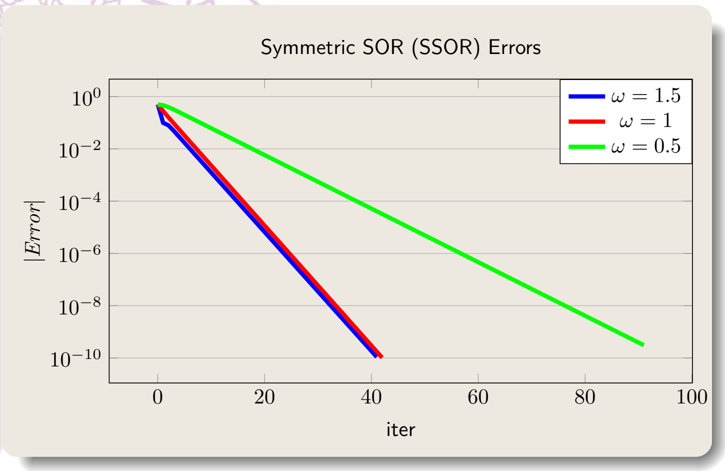
Symmetric Gauss-Seidel Method, IV

- And the preconditioning matrix is

$$\mathbf{P}_{SSOR}(\omega) = \left(\frac{1}{\omega}\mathbf{D}-\mathbf{E}\right)\frac{\omega}{2-\omega}\mathbf{D}^{-1}\left(\frac{1}{\omega}\mathbf{D}-\mathbf{F}\right). \qquad (3.1.43)$$

- If $\mathbf{A}$ is symmetric and positive definite, the SSOR method is convergent if $0<\omega<2$.
- Typically, the SSOR method with the optimal ω converges slower than the corresponding SOR method. But, $\rho(\mathbf{B}_S(\omega))$ is less sensitive to ω.
- Also, each iteration is SSOR method needs more efforts.





Summary

- Resistor network example
- Jacobi's method
- Over-relaxation method
- Gauss-Seidel Method
- Successive over-relaxtion method
- Mathematical treatments
- Symmetric Gauss-Seidel method