Riemann Surfaces

Winter Term 2018/19

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Version of October 29, 2018

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1 Introduction

What is a Riemann Surface?

A Riemann Surface is not a surface with a Riemannian metric, but it is a one-dimensional complex manifold.

1.1 Definition. A *manifold* is a topological space which is Hausdorff, second-countable and locally Euclidean together with an atlas $\mathcal{A}$ of charts $\alpha = (\mathrm{U}_\alpha, \varphi_\alpha)$ consisting of an open set U_α and a map

$$\varphi_\alpha \colon \mathrm{U}_\alpha \to \mathbb{R}^n$$

which is homeomorphic onto its image $\varphi_\alpha[\mathrm{U}_\alpha] \subseteq_{\text{open}} \mathbb{R}^n$ each and such that $M = \bigcup_\alpha \mathrm{U}_\alpha$ is covered.

When the *transition maps*

$$\mathrm{t}_{\alpha,\beta} \coloneqq \varphi_\beta \circ {\varphi_\alpha}^{-1} \colon \varphi_\alpha[\mathrm{U}_\alpha \cap \mathrm{U}_\beta] \to \varphi_\beta[\mathrm{U}_\alpha \cap \mathrm{U}_\beta]$$

are all in a class $C \subseteq \{f \colon U \to \mathbb{R}^n \mid U \subseteq_{\text{open}} \mathbb{R}^n\}$, we call $(M, \mathcal{A})$ a C-manifold.

By identifying $\mathbb{R}^2 = \mathbb{C}$ and setting

$$\mathcal{H} \coloneqq \{f \colon U \to \mathbb{C} \mid U \subseteq_{\text{open}} \mathbb{C},\ f \text{ holomorphic}\}$$

we obtain the definition of a Riemann Surface.

1.2 Definition. A *Riemann Surface* is a $\mathcal{H}$-manifold.

Since we have $\mathcal{H} \subseteq \{f \in \mathrm{C}^\infty(U, \mathbb{R}^2) \mid U \subseteq_{\text{open}} \mathbb{R}^2\}$, any Riemann Surface is a smooth two-dimensional real manifold, i.e. a surface.

Why is a Riemann Surface?

We give a few, increasingly complex[1] examples of Riemann Surfaces.

[1] pun intended

1.3 Example. Most likely the easiest and most well-known constructions of Riemann Surfaces are the following four.

- $\mathbb{C}$ is a Riemann Surface.
- Any open subset $U \subseteq_{\text{open}} M$ of a Riemann Surface M is a Riemann Surface itself.
- The Riemann Sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$ is a Riemann Surface. Topologically, it is the Alexandroff compactification of $\mathbb{C}$.
- Suitable quotients of Riemann Surfaces are Riemann Surfaces, e.g. the Torus $\mathbb{C}/(\mathbb{Z} + \mathrm{i}\mathbb{Z})$.

1.4 Example. We can associate Riemann Surfaces to holomorphic functions. For example, we would like to define the natural logarithm of any $z = r\mathrm{e}^{\mathrm{i}\theta} \neq 0$ with $r > 0$ and $\theta \in \mathbb{R}$ by

$$\ln(z) = \ln(r\mathrm{e}^{\mathrm{i}\theta}) = \ln(r) + \ln(\mathrm{e}^{\mathrm{i}\theta}) = \ln(r) + \mathrm{i}\theta,$$

However, there is a problem: For any $z \neq 0$, the argument θ is only well-defined modulo 2π. Choosing any system of representatives to obtain a total function $\ln\colon \mathbb{C}^* \to \mathbb{C}$ is neither natural nor will it ever lead to a continuous function.

This can be fixed in two ways. For one, we can think of the logarithm as being multi-valued, obtaining

$$\ln(\mathrm{i}) = \mathrm{i}\frac{\pi}{2} + 2\pi\mathrm{i}\mathbb{Z} = \left\{\frac{1}{2}\pi\mathrm{i}, \frac{5}{2}\pi\mathrm{i}, -\frac{3}{2}\pi\mathrm{i}, \frac{9}{2}\pi\mathrm{i}, -\frac{7}{2}\pi\mathrm{i}, \dots\right\}.$$

For the other, we need Riemann Surfaces to “unfold the complex plane”.

1.5 Example. Projective non-singular curves are Riemann Surfaces. For example, we can take the polynomial equation

$$y^2 = x^3 + ax + b$$

with parameters a and b and study the set of solutions $S \subseteq \mathbb{C}^2 \subseteq \mathbb{C}\mathrm{P}^2$, where the latter is a holomorphic manifold of complex dimension 2. In the language of algebraic geometry, S is an algebraic subvariety of $\mathbb{C}\mathrm{P}^2$ and whenever S has no singularities, it is a Riemann Surface.

How is a Riemann Surface?

To conclude our initial tour of Riemann Surfaces, we consider alternative ways to look at or define this class of spaces. A Riemann Surface is a real surface with some extra structure. There are various way to characterize this extra structure:

As defined The holomorphic structure can be given with an atlas.

Algebraically Any compact Riemann Surface corresponds to a non-singular projective curve.

Conformal Two Riemannian metrics are considered conformal it they only differ by a (pointwise) rescaling or – equivalently – if they measure the same angles. A Riemann Surface can be equivalently given as a real surface together with a conformal class of Riemann metrics.

Hyperbolic Using the Poincaré uniformization theorem, it is possible to show that “nearly all” Riemann surfaces are hyperbolic, i.e. have negative Gauß curvature, which means that they are quotients of the hyperbolic space $\mathbb{H}^2$.

2 Review of Complex Analysis

We will take some time to review complex analysis in one variable.

Holomorphic functions

Let $f \colon U \to \mathbb{C}$ be a function defined on an open set $U \subseteq_{\text{open}} \mathbb{C}$ and $z_0 \in U$ a point. The f is *complex differentiable* at z_0 iff the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0)$$

exists. Decomposing $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)$ and identifying $\mathbb{C} = \mathbb{R}^2$ gives us that any complex differentiable function f is real differentiable and satisfies

$$\mathrm{d}_{z_0} f(h) = f'(z_0) \cdot h.$$

2.1 Proposition. *A function $f \colon U \to \mathbb{C}$ is complex differentiable at $z_0 \in U$ iff $f \colon U \to \mathbb{R}^2$ is real differentiable at z_0 with $\mathrm{d}_{z_0} f \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ being $\mathbb{C}$-linear.*

2.2 Definition. A function $f \colon U \to \mathbb{C}$ is *holomorphic* iff f is complex differentiable at every point in U.

Thinking of U as an open subset of $\mathbb{R}^2$, we can define $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$. Since $\frac{\partial f}{\partial \bar{z}}$ is equivalent to the Cauchy-Riemann equations, we obtain another characterization of holomorphy.

2.3 Proposition. *A function $f \colon U \to \mathbb{C}$ is holomorphic iff f is real differentiable and $\frac{\partial f}{\partial \bar{z}} = 0$ vanishes.*

Another characterization arises from the notion of an (oriented) angle.

2.4 Definition. A function is called *conformal* iff it preserves oriented angles.

2.5 Proposition. *A function is holomorphic iff it is conformal.*

Power series

Cauchy theory

A 1-*form* is a function of the form $\alpha = a(x, y)\mathrm{d}x + b(x, y)\mathrm{d}y$ for functions $a, b\colon U \to \mathbb{C}$ or, equivalently, a function of the form $\alpha = u(z)\mathrm{d}z + v(z)\mathrm{d}\bar{z}$ with $u, v\colon U \to \mathbb{C}$, $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - \mathrm{i}\mathrm{d}y$.

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