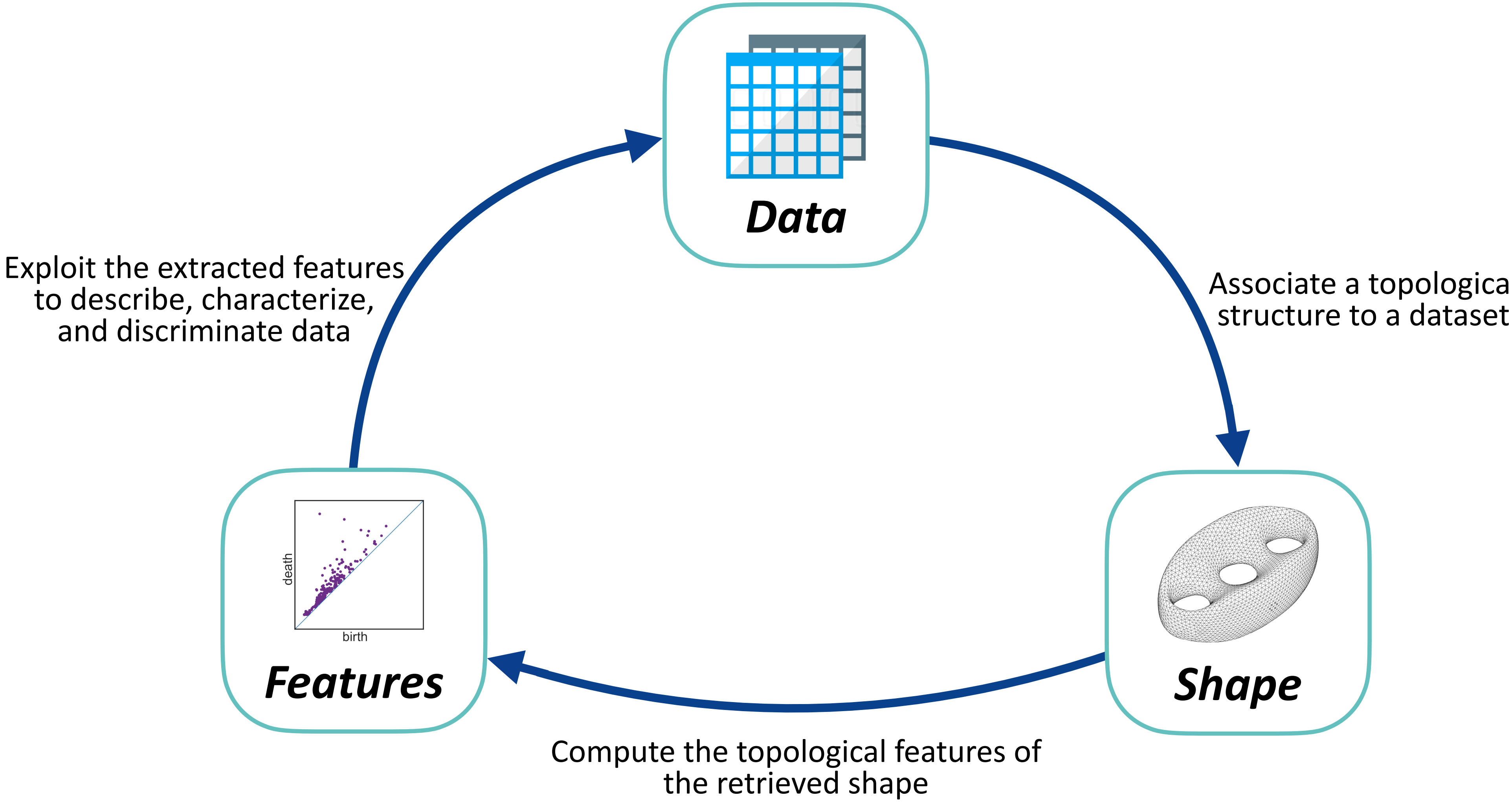
Topological Data Analysis

Kernels & Networks

Ulderico Fugacci

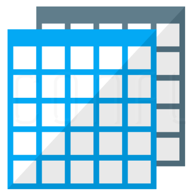
CNR - IMATI

Topological Data Analysis

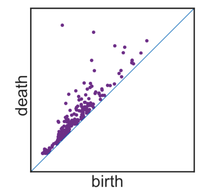


Kernels & Networks

Topological Data Analysis allows for assigning to (almost) ***any dataset*** a collection of features representing a ***topological summary*** of the input data



Data



Features

Goal:

Today, we address two main questions:

- ***Is this information immediately suitable for statistics and machine learning?***
- ***Do homology cycles provide a meaningful and descriptive information?***

Kernels & Networks

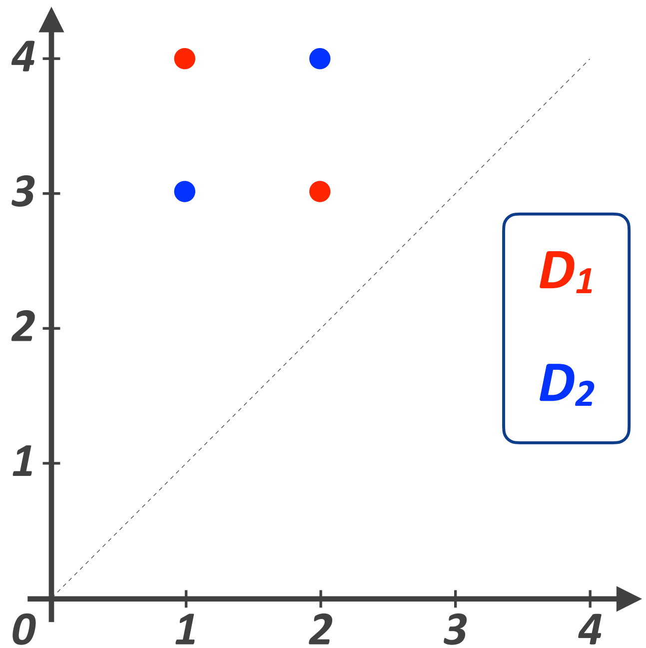
- *Kernels for Persistent Homology*
- *Persistence and Complex Networks*

Kernels & Networks

- ***Kernels*** ***for Persistent Homology***
- *Persistence and Complex Networks*

Kernels for Persistent Homology

A Naive Example:



Mean *of persistence diagrams is* ***not unique***

Kernels for Persistent Homology

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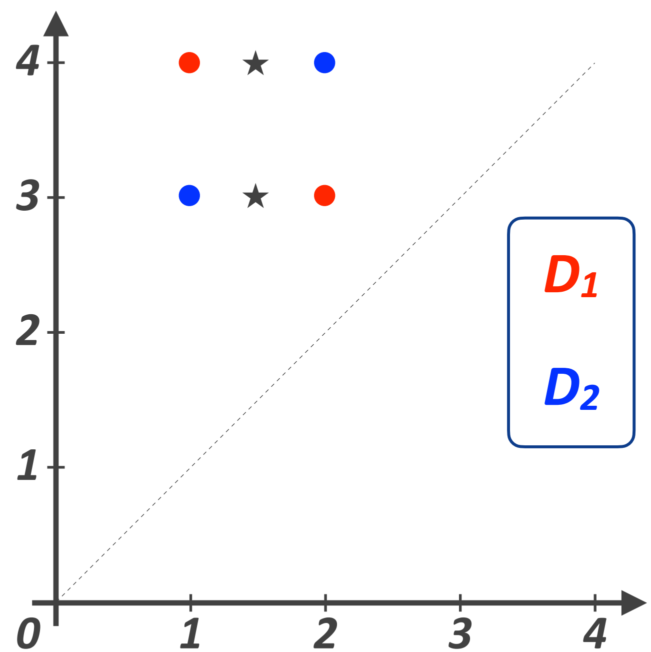

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Kernels for Persistent Homology

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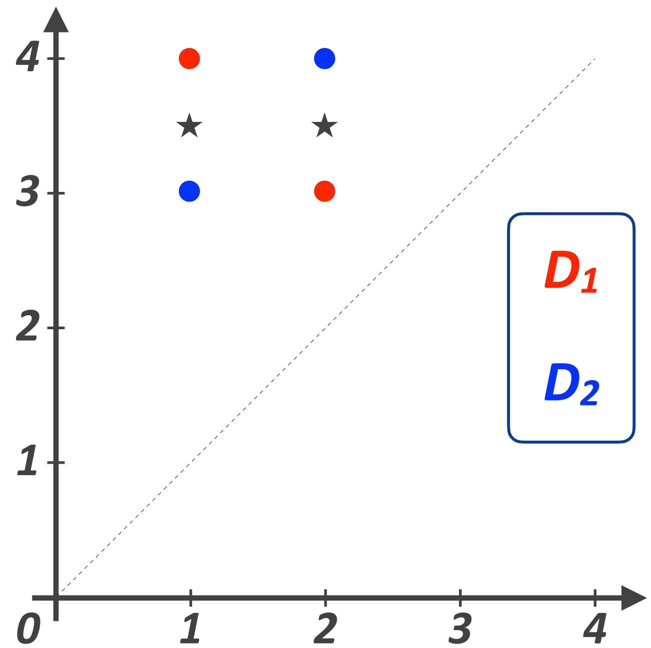
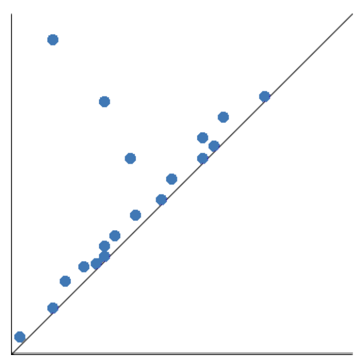
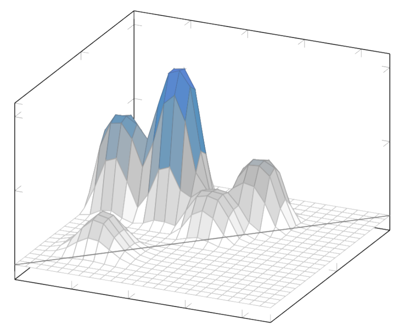


Mean** of persistence diagrams is **not unique

Kernels for Persistent Homology

Adopted Strategy:





Represent persistence diagrams as elements of a ***Hilbert space***

Kernels for Persistent Homology

Definitions:

*A **Hilbert space H** is*

*a **real or complex vector space** endowed with an **inner product***

$\langle\, .\, ,\, .\, \rangle : H \times H \longrightarrow \mathbb{R}$ such that, with respect to the distance induced by $\langle\, .\, ,\, .\, \rangle$,

*H is a **complete metric space***

*Recall that, a metric space H is called **complete** if*

every Cauchy sequence in H converges in H

Kernels for Persistent Homology

Example:

The space ***L^2*** of ***square-integrable functions*** on $\mathbb{R}^2$ is a Hilbert space

- $$\|f\|_{L^2} := \left(\int_{\mathbb{R}^2} |f|^2 d\mu\right)^{\frac{1}{2}} < +\infty$$
- $$\langle f, g\rangle_{L^2} := \int_{\mathbb{R}^2} f \cdot g \; d\mu$$

Kernels for Persistent Homology

Kernel Trick:

$$
\begin{tikzcd}
\Omega \times \Omega \arrow[rr, "k"] \arrow[d, "\phi \times \phi"'] & & \mathbb{R} \\
H \times H \arrow[urr, "\langle \cdot , \cdot \rangle"'] & &
\end{tikzcd}
$$

Definition:

*A **kernel k** for an input space Ω is a map $k\colon \Omega \times \Omega \longrightarrow \mathbb{R}$ such that there exist a **Hilbert space H** and a **feature map** $\phi\colon \Omega \longrightarrow H$ for which*

$$k(X, Y) = \langle \phi(X) , \phi(Y) \rangle$$

Kernels for Persistent Homology

Pseudo-Distance:

A kernel k: Ω × Ω ⟶ ℝ implicitly induces on Ω

a ***pseudo-distance*** ***d_k: Ω × Ω*** ⟶ ℝ defined, for each X, Y ∈ Ω, as

$$d_k(X,Y) := \|\phi(X) - \phi(Y)\|_H = \Big(k(X,X) + k(Y,Y) - 2k(X,Y)\Big)^{1/2}$$

Stability:

A kernel k: Ω × Ω ⟶ ℝ is ***stable*** *w.r.t a distance d in Ω*

if there is a constant C > 0 such that, for all X, Y ∈ Ω,

$$d_k(X,Y) \leq C \cdot d(X,Y)$$

Kernels for Persistent Homology

Our Goal:

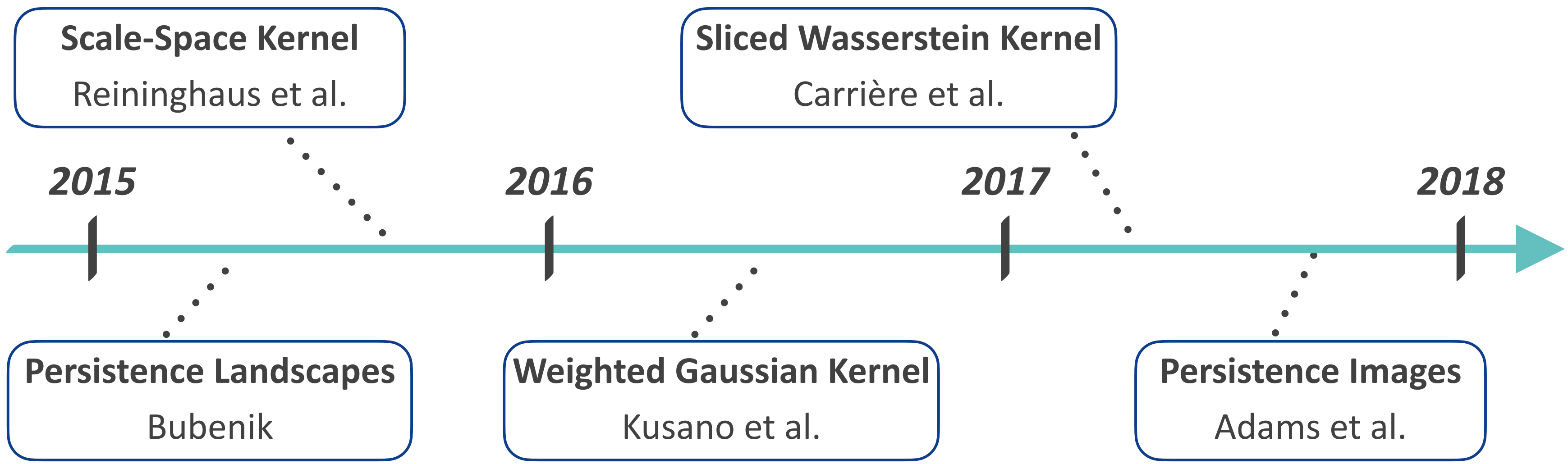
Defining a ***kernel*** for the set Ω of finite ***persistence diagrams***:

- ***Stable***
- ***Easy to be computed***
- Possible endowed with an ***explicit feature map*** ϕ***: Ω*** $\longrightarrow$ ***H***

The idea of a kernel for persistence diagrams has

- *Originally* ***born in the '90s*** *(see [Donatini et al. 1998; Ferri et al. 1998])*
- *Spread in the literature and* ***widely adopted in applications*** ***just recently***

Kernels for Persistent Homology

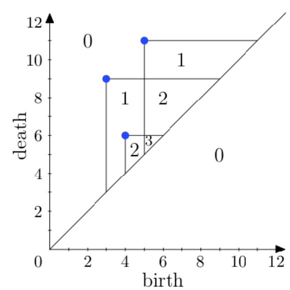


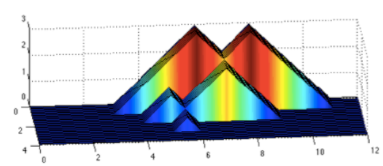
Kernels for Persistent Homology

2015 **2016** **2017** **2018**

Persistence Landscapes

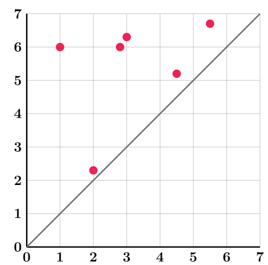
Bubenik

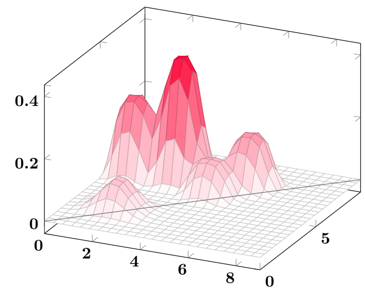




Kernels for Persistent Homology

Scale-Space Kernel
Reininghaus et al.





2015 ***2016*** ***2017*** ***2018***

Persistence Landscapes
Bubenik

Kernels for Persistent Homology

Scale-Space Kernel

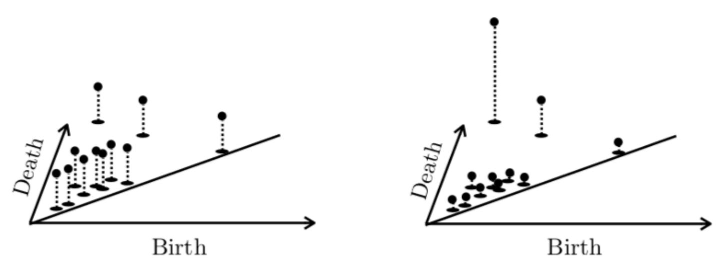
Reininghaus et al.

2015 *2016* *2017* *2018*

Persisten

Weighted Gaussian Kernel

Kusano et al.



Kernels for Persistent Homology

$$k_{SW}(D,E) := \exp\left(-\frac{d_{SW}(D,E)}{2\sigma^2}\right)$$

Sliced Wasserstein Kernel

Carrière et al.

Scale

Reini

2015

2016

2017

2018

Persistence Landscapes

Bubenik

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Kernels for Persistent Homology

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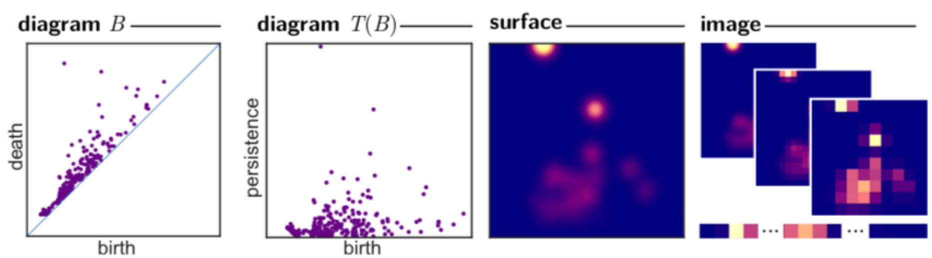
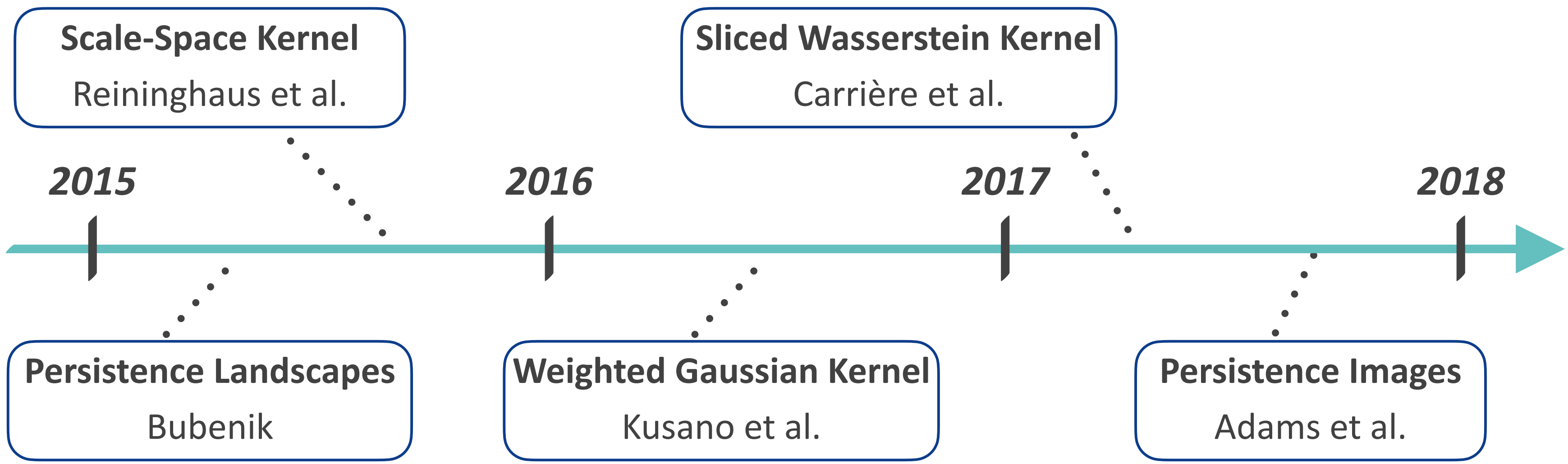
2015 2016 2017 2018

Persi



Persistence Images
Adams et al.

Kernels for Persistent Homology



Kernels for Persistent Homology

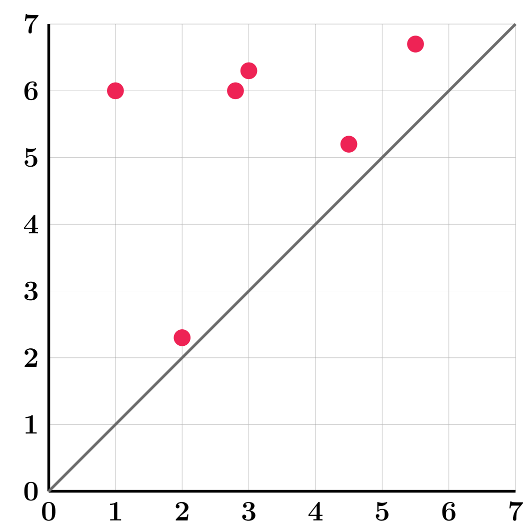
Scale-Space Kernel:

Given a finite persistence diagram D, we consider the solution

$\phi : \Delta^{+} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ of the following ***heat diffusion problem***:

- having a Dirichlet ***boundary condition*** on the diagonal
- setting as an ***initial condition*** a sum of Dirac deltas

$$\begin{cases} \Delta_p \phi = \partial_\sigma \phi & \text{in } \Delta^{+} \times \mathbb{R}_{>0} \\ \phi = 0 & \text{on } \partial\Delta^{+} \times \mathbb{R}_{\geq 0} \\ \phi = \sum_{q \in D} \boldsymbol{\delta}_q & \text{on } \Delta^{+} \times \{0\} \end{cases}$$



Kernels for Persistent Homology

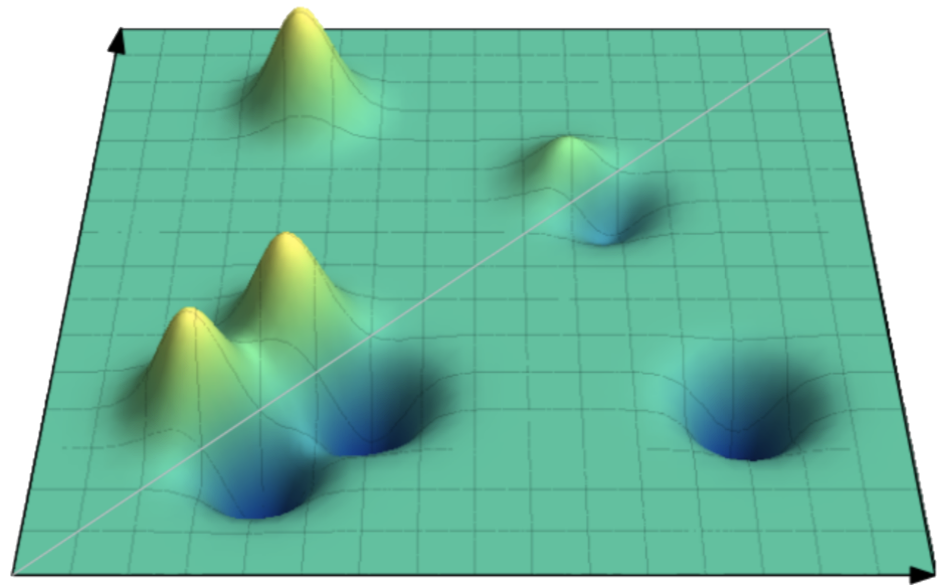
Scale-Space Kernel:

A *solution* is found by:

- ***extending*** Δ^+ to $\mathbb{R}^2$
- ***replacing*** the initial condition with

$$\phi = \Sigma_{q \in D} \boldsymbol{\delta}_q - \boldsymbol{\delta}_{q'} \qquad \text{on } \mathbb{R}^2 \times \{0\}$$

where, if $q=(a,b)$, then $q'=(b,a)$



Solution:

$$\phi_\sigma(p) = \frac{1}{4\pi\sigma} \sum_{q \in D} \Big(\exp\big(-\frac{\|p-q\|^2}{4\sigma}\big) - \exp\big(-\frac{\|p-q'\|^2}{4\sigma}\big)\Big)$$

Kernels for Persistent Homology

Scale-Space Kernel:

Stability Theorem:

Given two finite persistence diagrams D, E, we have that

$$\|\phi_\sigma(D) - \phi_\sigma(E)\|_{L^2} \leq \frac{1}{2\sqrt{\pi}\sigma} d_{W,1}(D,E)$$

where, for $r \geq 1$, the ***r-Wasserstein distance*** *is defined as*

$$d_{W,r}(D,E) := \left(\inf_\gamma \sum_{p \in D} \|p - \gamma(p)\|_\infty^r \right)^{1/r}$$

with γ running over all bijections from D to E

Kernels for Persistent Homology

Definitions:

A kernel k for the set Ω of finite persistence diagrams is called:

- ***additive*** if, for all D, E, F ∈ Ω, $k(D \cup E, F) = k(D, F) + k(E, F)$
- ***trivial*** if, for all D, E ∈ Ω, $k(D, E) = 0$

Theorem:

*Any **non-trivial additive** kernel k for the set Ω*

*is **not stable** with respect to $d_{W,r}$ for any $1 < r \leq \infty$*

(Notice that $d_{W,\infty} = d_B$)

Kernels for Persistent Homology

Sliced Wasserstein Kernel:

A standard way to construct a kernel is to exponentiate the negative of an Euclidean distance

$$k_\sigma(X,Y) := \exp\left(-\frac{\|X-Y\|^2}{2\sigma^2}\right)$$

Theorem:

$$k_\sigma(X,Y) := \exp\left(-\frac{f(X,Y)}{2\sigma^2}\right)$$

defines a ***valid kernel*** *for all $\sigma > 0$* ***if and only if*** *f is* ***conditionally negative definite*** *function*

I.e., for any $n \in \mathbb{N}$, for any $X_1, \dots, X_n \in \Omega$, and for any $a_1, \dots, a_n \in \mathbb{R}$ such that $\Sigma_i a_i = 0$,

one has $\Sigma_{i,j} a_i a_j f(X_i, X_j) \leq 0$

Kernels for Persistent Homology

Sliced Wasserstein Kernel:

Issue:

*None of the already introduced distances (and neither their squares) between persistence diagrams **is conditionally negative definite***

Solution:

*The **Sliced Wasserstein distance** $\mathbf{d}_{sw}$ is specifically designed to be **conditionally negative definite***

*Based on it, one can define the **Slice Wasserstein kernel** $\mathbf{k}_{sw}$ as*

$$k_{SW}(D,E) := \exp\left(-\frac{d_{SW}(D,E)}{2\sigma^2}\right)$$

Kernels & Networks

- *Kernels for Persistent Homology*
- ***Persistence and Complex Networks***

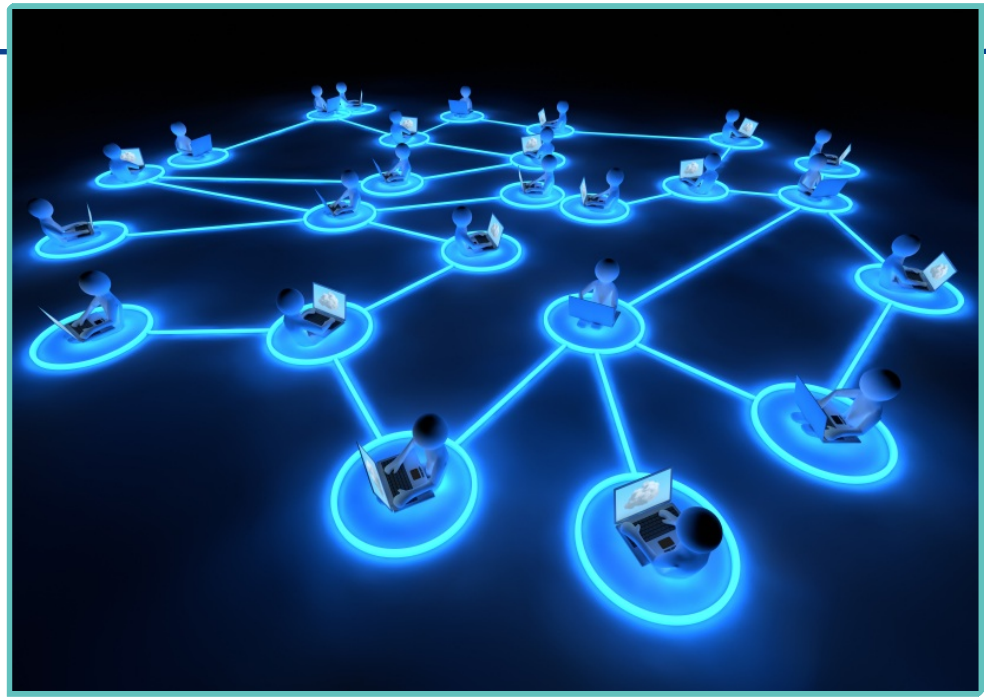
Persistence and Complex Networks

Networks:

*A **network** is a complex system consisting of **individuals or entities connected by specific ties** such as friendship, common interest, and shared knowledge*

E.g.

- ***Social*** Networks
- ***Sensor*** Networks
- ***Biological*** Networks
- ***Collaborative*** Networks
- …



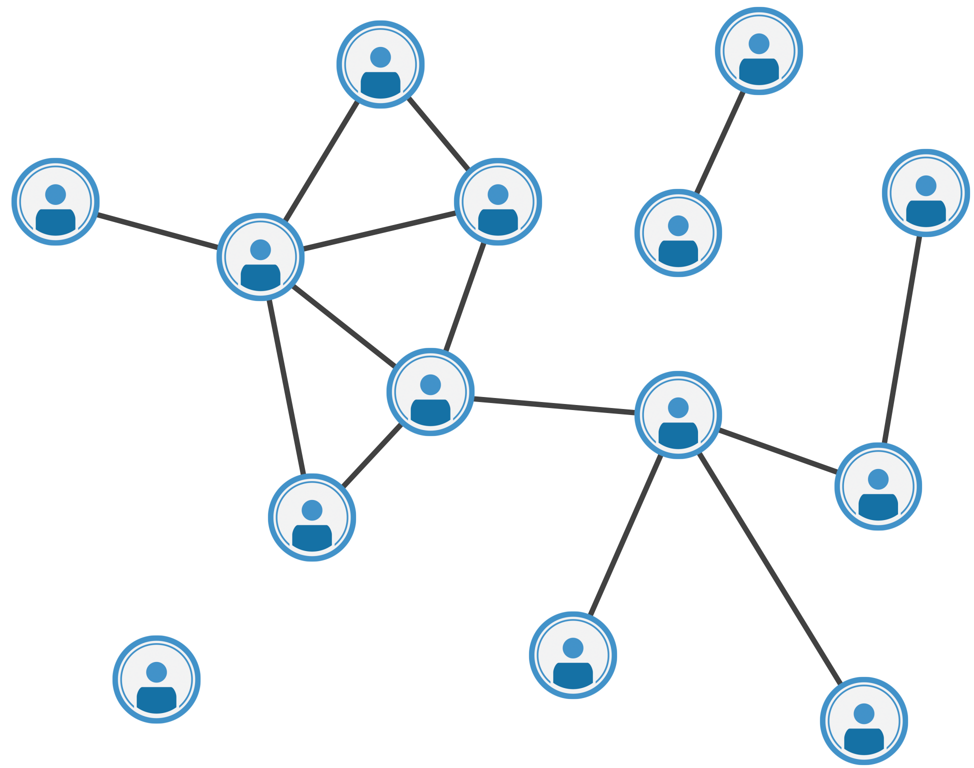
Persistence and Complex Networks

Representation:

A network can be represented by a ***graph*** ***G = (V, E)*** such that:

Nodes of V ⟺ *Entities or individuals*

Arcs of E ⟺ *Ties between entities*



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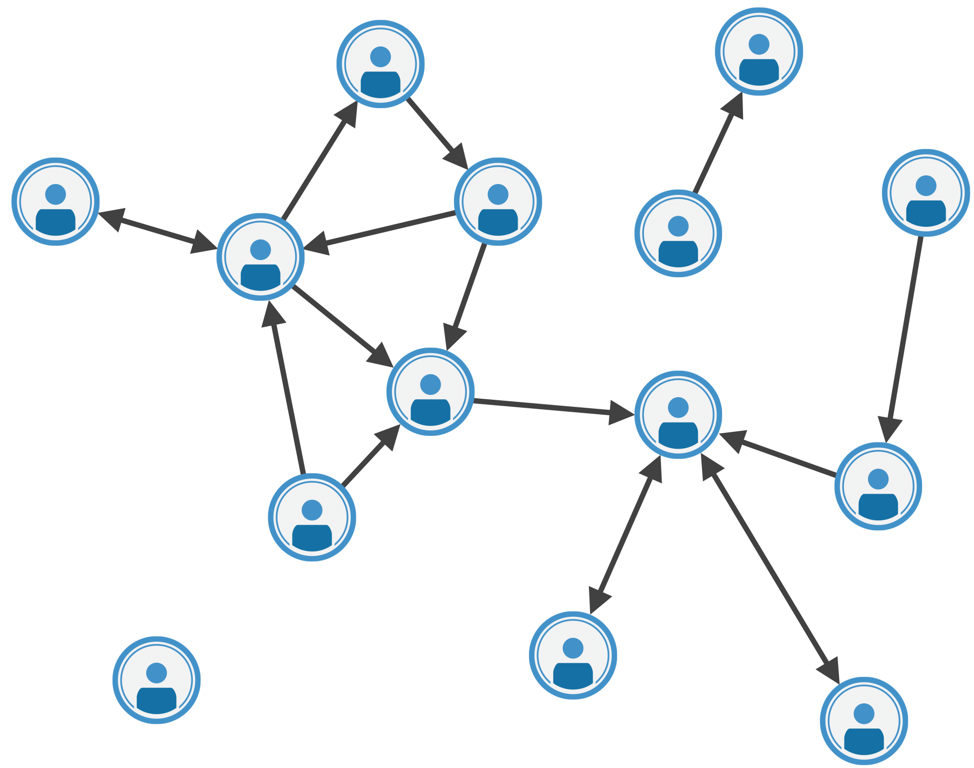
Arcs can be:

- ***Directed***
- ***Weighted***



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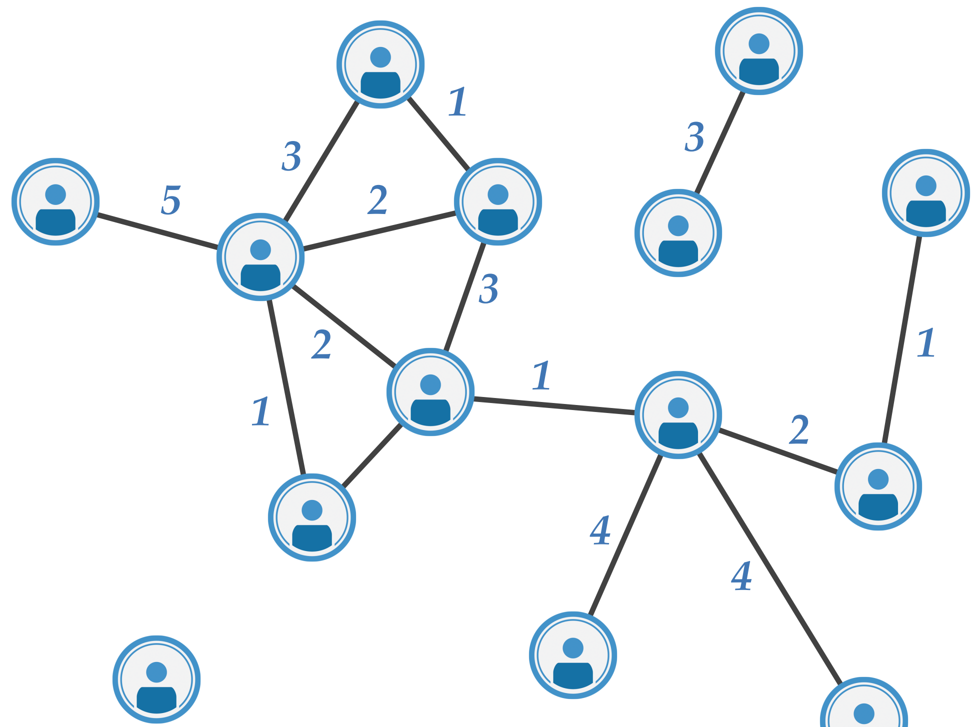

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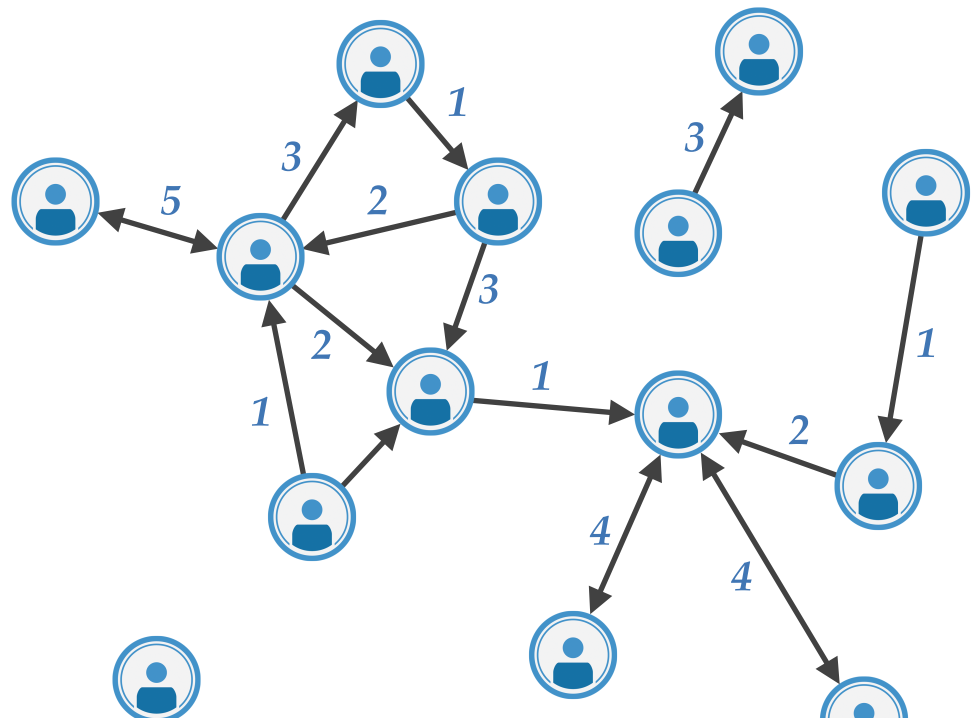
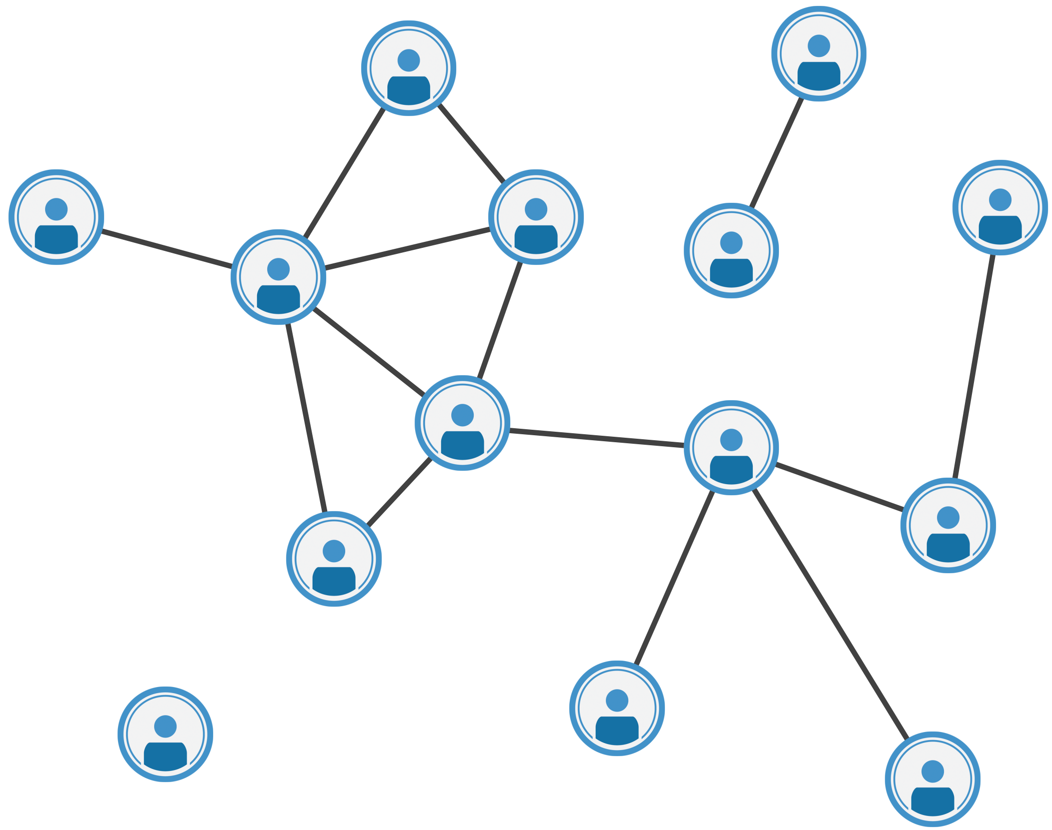


Persistence and Complex Networks

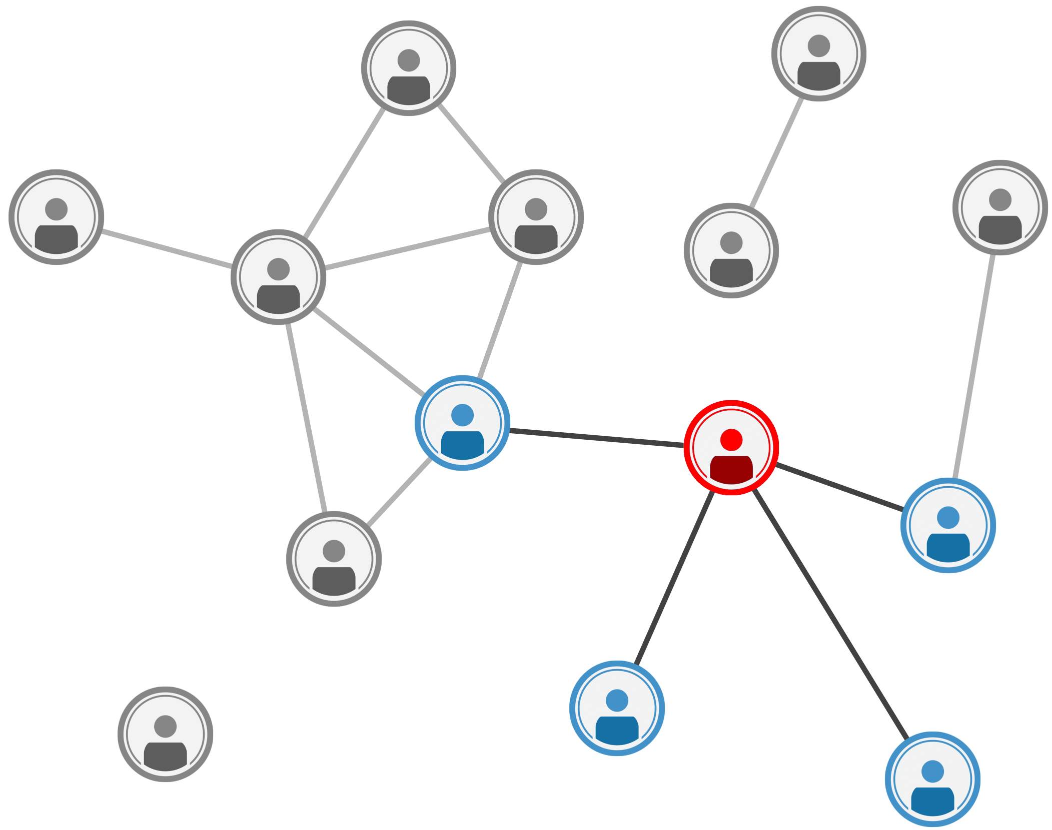
A Two-Level Analysis:



Persistence and Complex Networks

A Two-Level Analysis:

- ***Egocentric***
- *Sociocentric*



Persistence and Complex Networks

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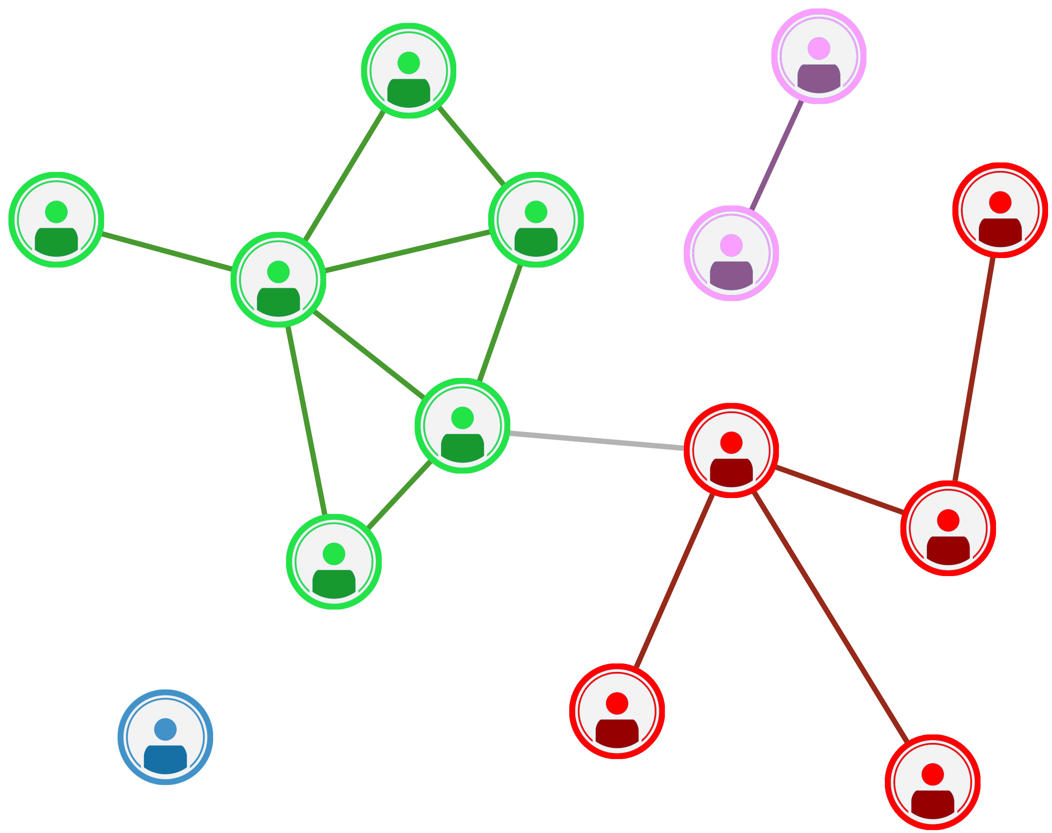
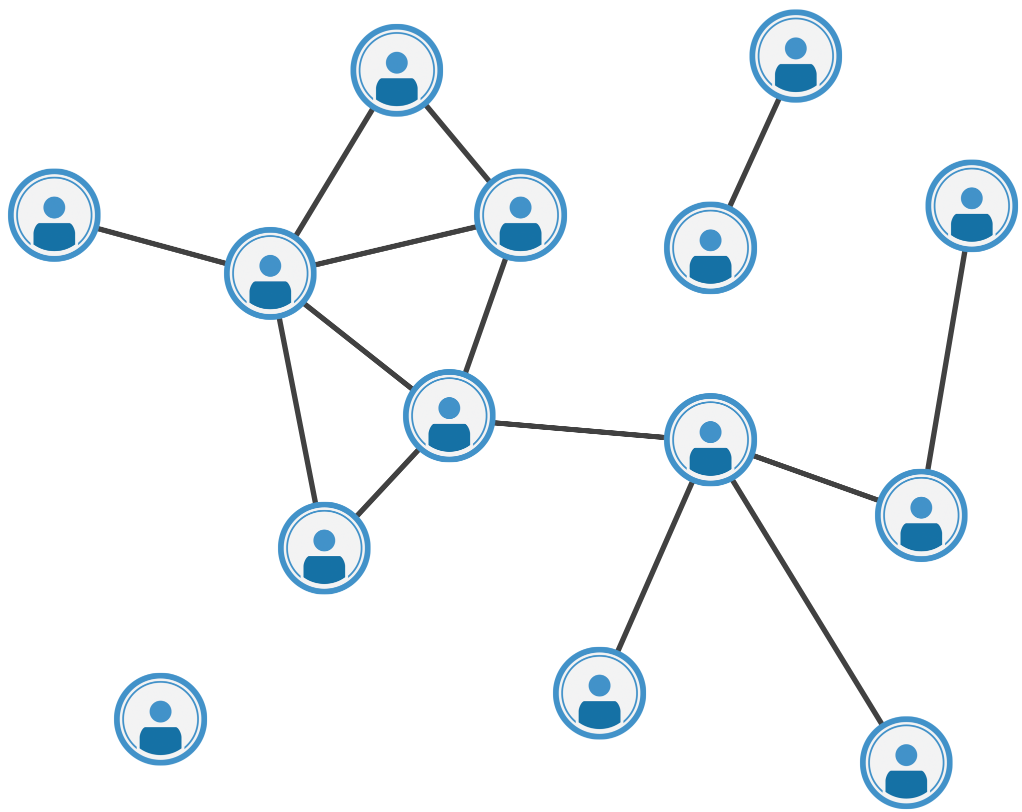


Persistence and Complex Networks

Who is the most important individual?



Persistence and Complex Networks

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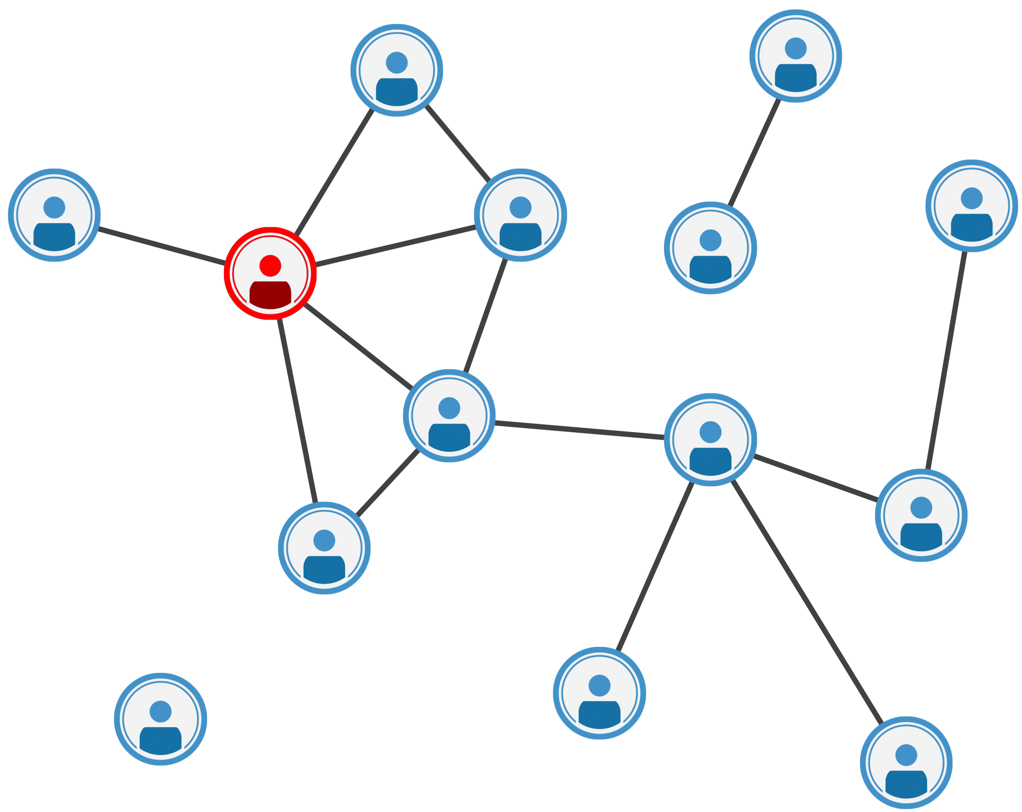

Persistence and Complex Networks

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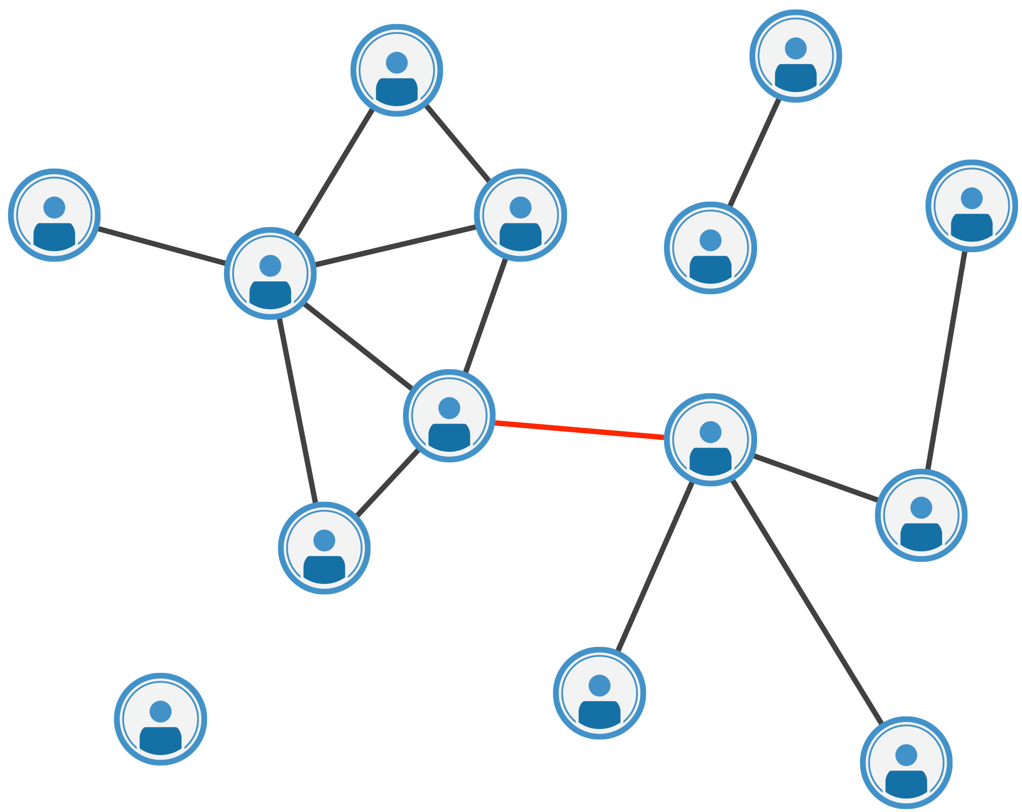

Persistence and Complex Networks

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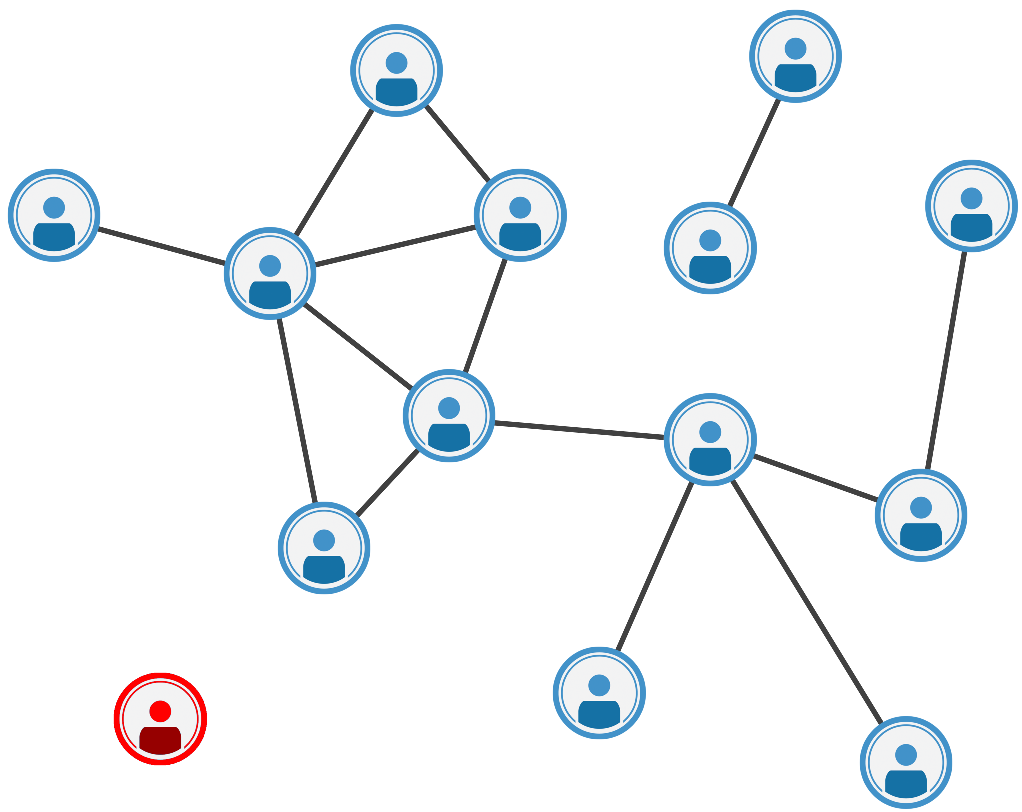


Persistence and Complex Networks

Centrality Measures:

Different criteria to underline ***different roles***:

- *Key players*
- *Brokers*
- *Bridges*
- *Isolated*
- *...*

Definition:

*A **centrality measure** is a function $F : V \longrightarrow \mathbb{R}$ assigning to each node a **"centrality"** value:*

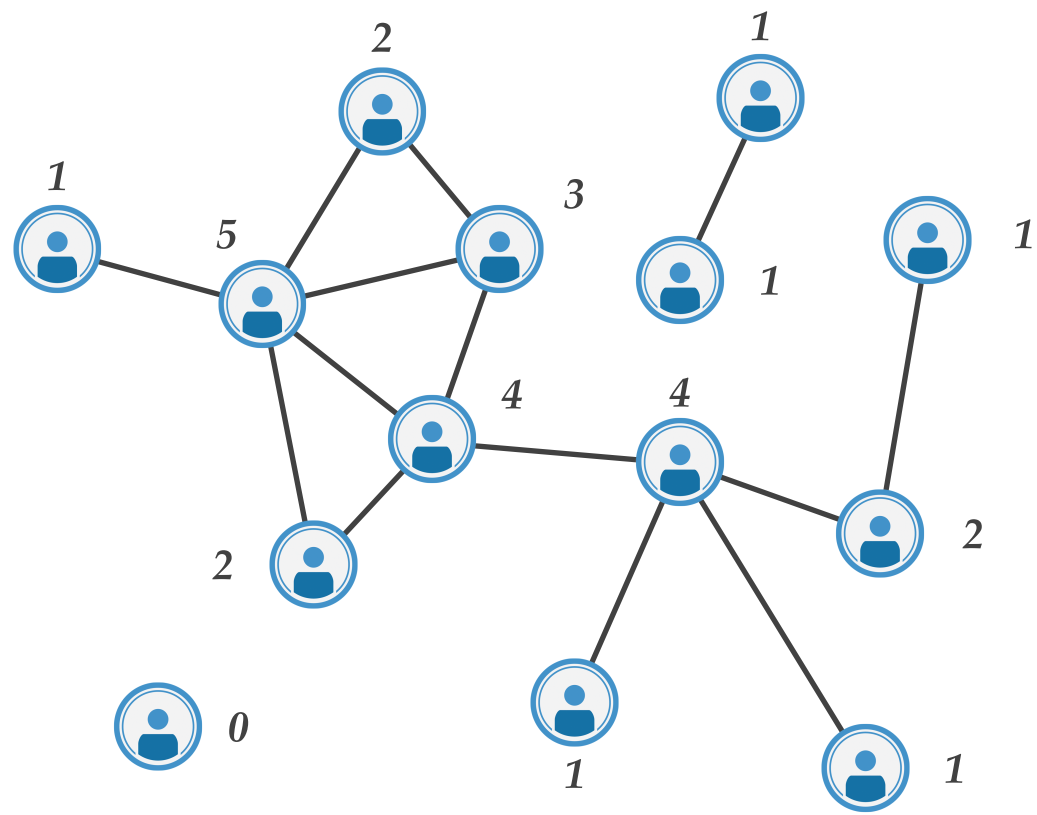
- ***Degree** centrality*
- ***Betweenness** centrality*
- ***Closeness** centrality*
- ***Eigenvector** centrality*
- ***Erdös** distance*

Persistence and Complex Networks

Degree Centrality:

Given a node v of G = (V, E),

$$D(v) := \#\{u \in V \,|\, (u,v) \in E\}$$

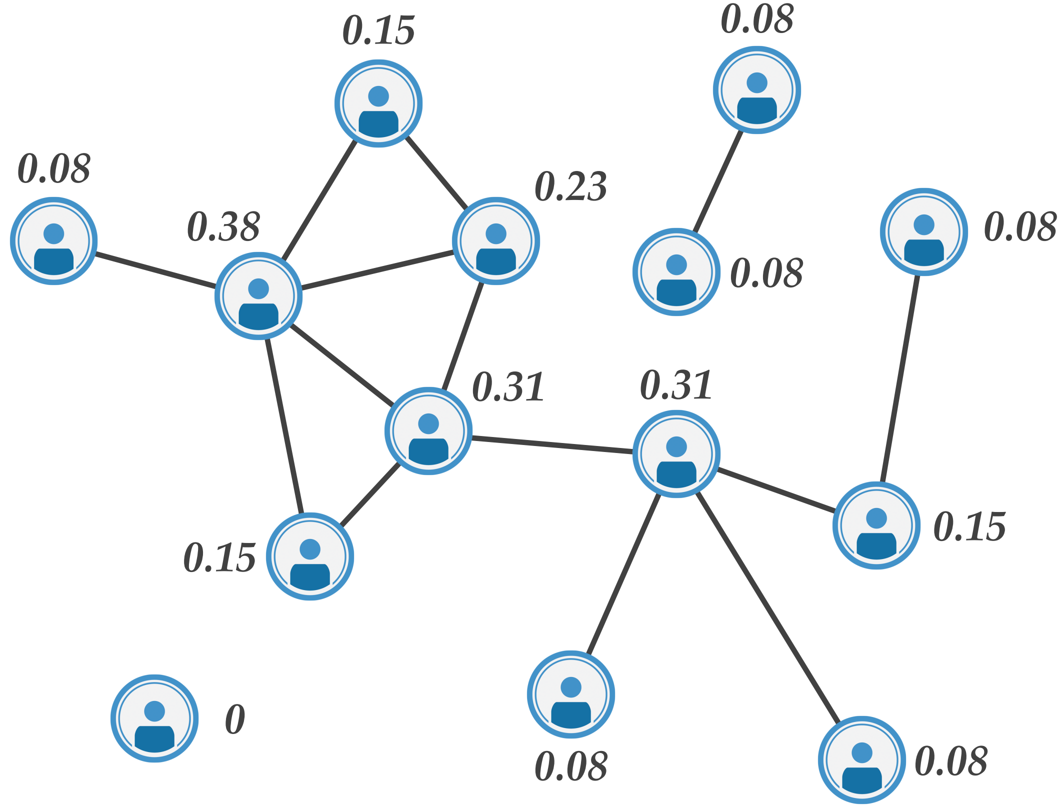


Persistence and Complex Networks

Degree Centrality:

Given a node v of G = (V, E),

$$D(v) := \frac{\#\{u \in V \mid (u,v) \in E\}}{\#V - 1}$$



Persistence and Complex Networks

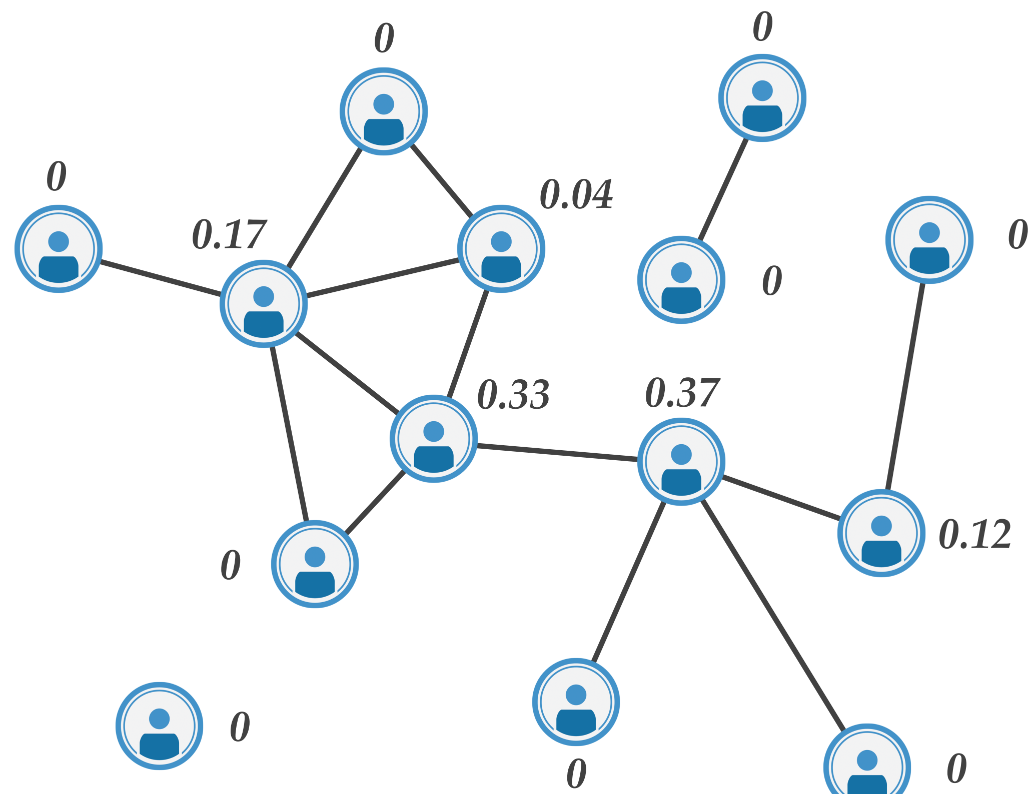
Betweenness Centrality:

Given a node v of G = (V, E),

$$B(v) := \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

where:

- σ_{st} is the number of ***shortest paths from s to t***
- $\sigma_{st}(v)$ is the number of those paths ***passing through v***

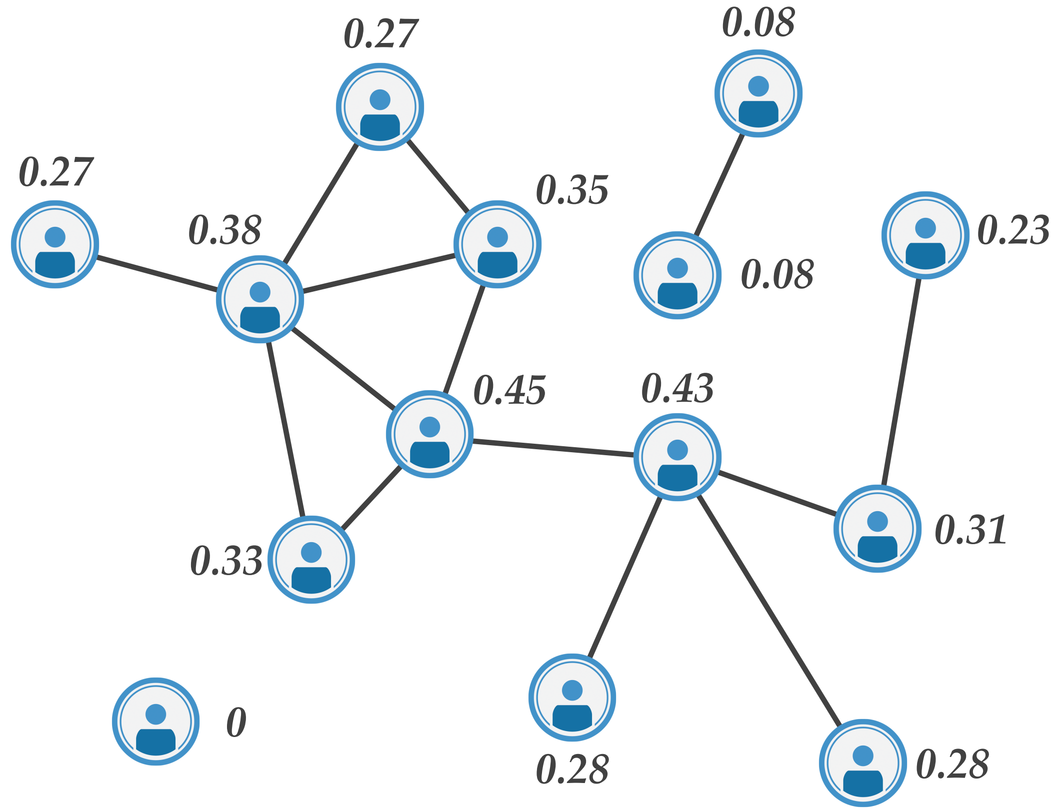


Persistence and Complex Networks

Closeness Centrality:

Given a node v of G = (V, E),

$$C(v) := \frac{\#V - 1}{\sum_{u \in V} d(u, v)}$$



Persistence and Complex Networks

Eigenvector Centrality:

Given a node v of G = (V, E),

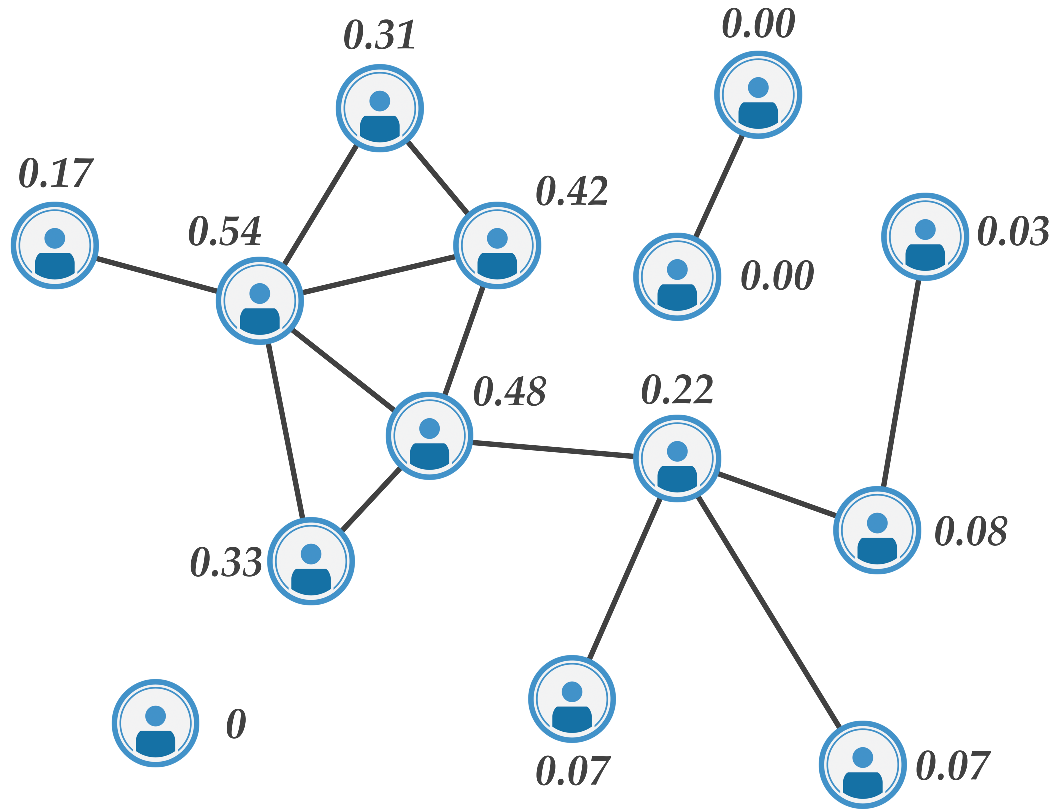
$$\boxed{x_v := \frac{1}{\lambda} \sum_{u \in V} A_{uv}\, x_u}$$

where λ is constant and

$$A_{uv} := \begin{cases} 1 & if (u,v) \in E \\ 0 & otherwise \end{cases}$$

I.e. the v^{th} entry of the eigenvector of

$$A\,x = \lambda\,x$$



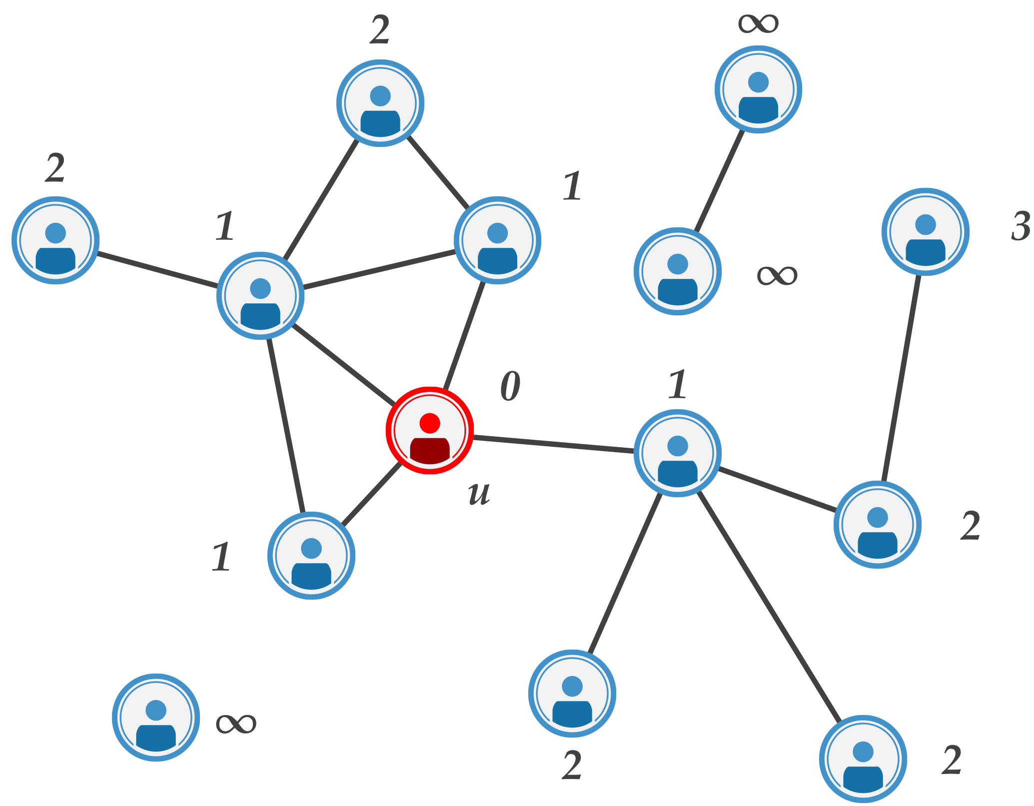
x* > *0 implies λ must be the largest eigenvalue of *A* and ***x*** the corresponding eigenvector

Persistence and Complex Networks

Erdös Distance:

Given two nodes u, v of G = (V, E),

$$E_u(v) := d(u, v)$$



Named after ***Paul Erdös***,

- one of the most prolific mathematicians of the 20th century

Persistence and Complex Networks

Centrality Measures:

A centrality measure for any query!

Degree	***How many individuals can v reach directly?***
Betweenness	***How likely is v to be the most direct route between two individuals?***
Closeness	***How fast can v reach everyone in the network?***
Eigenvector	***How well is v connected to other well-connected individuals?***
Erdös	***How far is v from a specific individual?***

Persistence and Complex Networks

Sociocentric Networks:

- ***Structural Metrics***:
  - *Average of a Centrality Measure*
  - *Diameter*
  - *Density*
  - *Transitivity*
  - ...
- ***Community Decompositions***:
  - *Atomic Communities*
  - *Clustering Techniques*

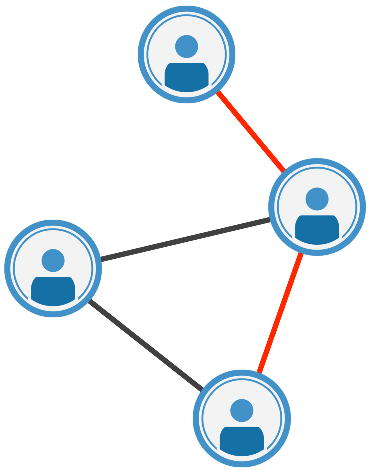
Persistence and Complex Networks

Structural Metrics:

How far are two individuals at most?

Diameter:

The longest shortest path between any two nodes



Diameter(G) = 2

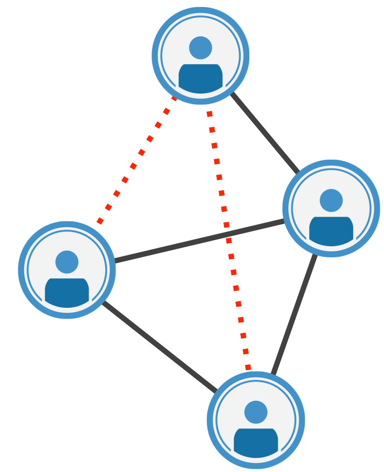
Persistence and Complex Networks

Structural Metrics:

How close is G to being an "everyone knows everyone" network?

Density:

$$\frac{\textit{Number of edges of } G}{\textit{Number of all possible edges}}$$



Density(G) = 4/6 = 0.67

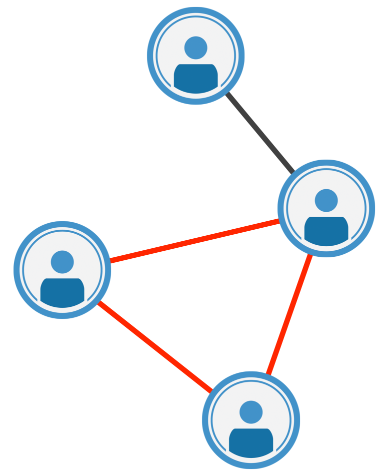
Persistence and Complex Networks

Structural Metrics:

How likely are two individuals connected to an individual v connected to each other?

Transitivity:

$$\frac{\textit{Number of closed triplets of nodes}}{\textit{Number of connected triplets}}$$



Transitivity(G) = 1/3 = 0.33

Persistence and Complex Networks

Community Decompositions:

- ***Atomic Communities:***
  - ***Clique***
  - ***n-Clique***
  - ***n-Clan***
  - ***n-Club***
  - ***k-Plex***
  - ***k-Core***
  - ...

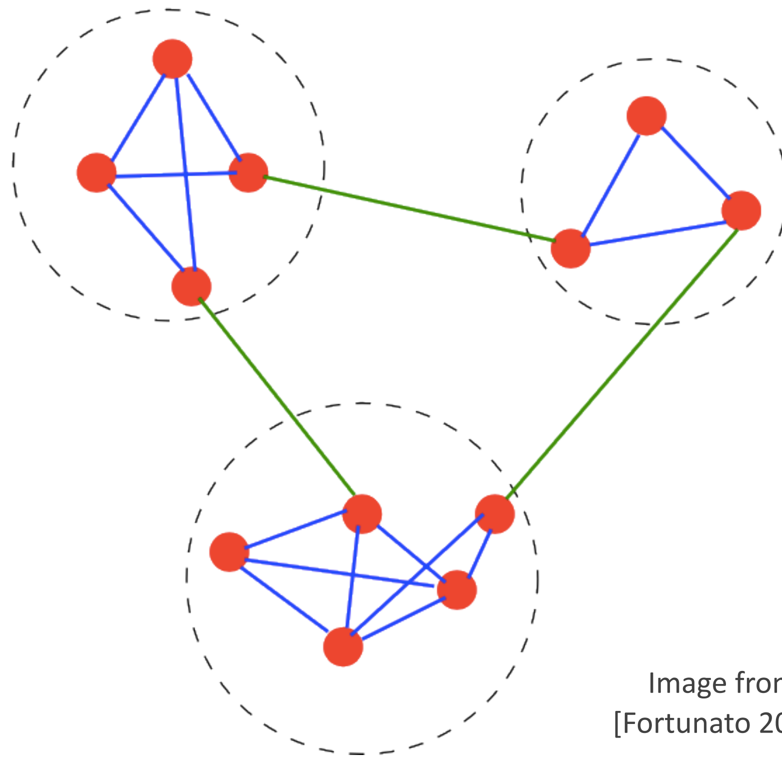


Image from
[Fortunato 2009]

Persistence and Complex Networks

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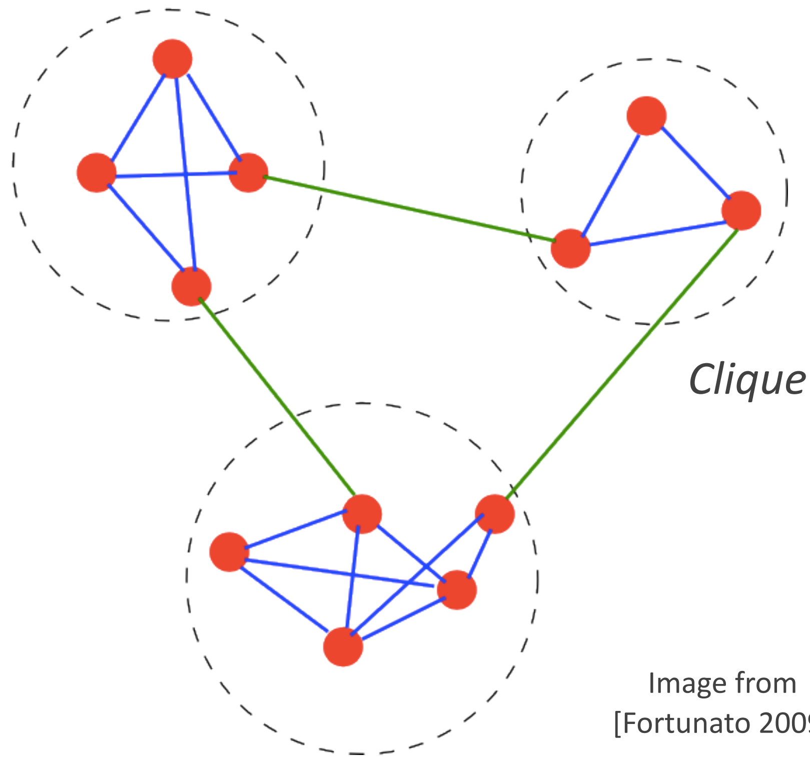

Image from [Fortunato 2009]

Clique:

A maximal subgraph whose nodes are all adjacent to each other

Persistence and Complex Networks

Community Decompositions:

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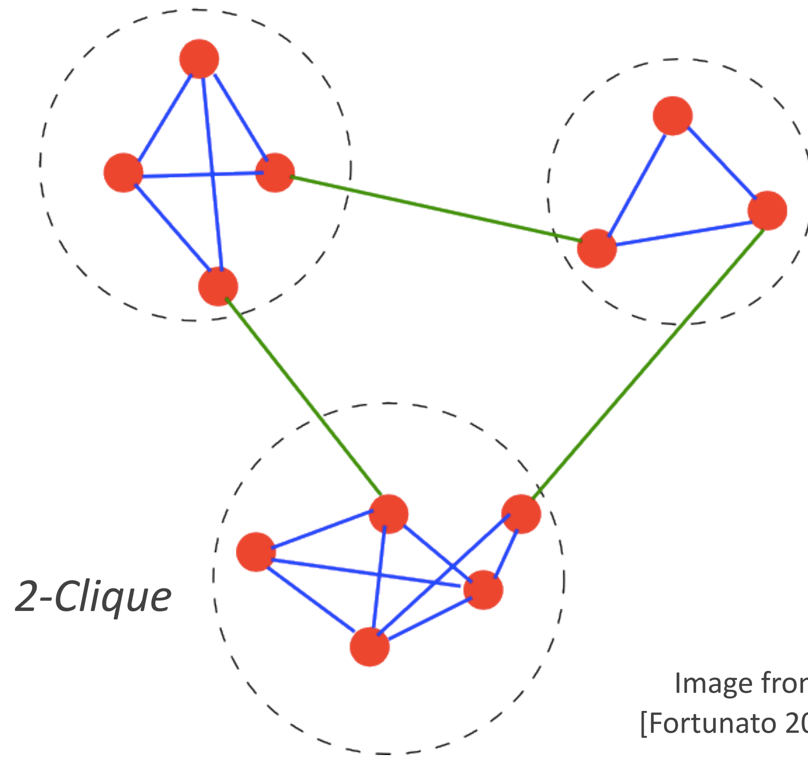

Image from [Fortunato 2009]

n-Clique:

A maximal subgraph such that the distance of each pair of its nodes is not greater than n

Persistence and Complex Networks

Community Decompositions:

- ***Atomic Communities:***
  - ***Clique***
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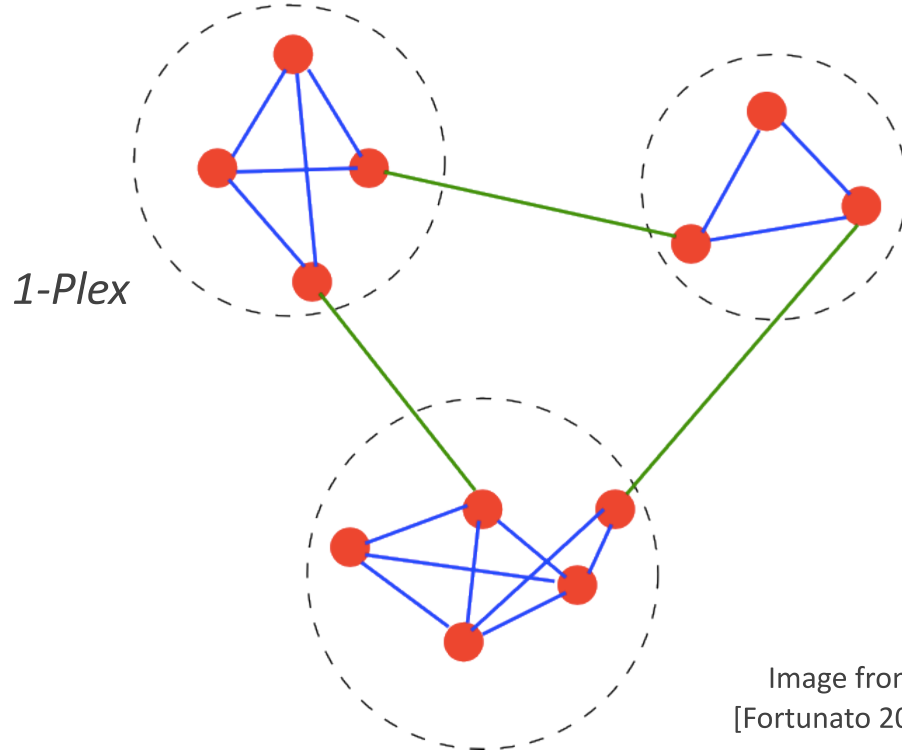


Image from [Fortunato 2009]

k-Plex:

A maximal subgraph in which each node is adjacent to all other nodes of the subgraph except at most k of them

Persistence and Complex Networks

Clustering Techniques:

Agglomerative (bottom-up)	approach based on	Centrality Measures	
		Atomic Communities	
Divisive (top-down)		Quality Functions	

Persistence and Complex Networks

Clustering Techniques:

Agglomerative (bottom-up)

Divisive (top-down)

approach based on

Centrality Measures

Atomic Communities

Quality Functions

Girvan-Newman Algorithm:

Iterated removal of the edge with largest ***betweenness centrality***

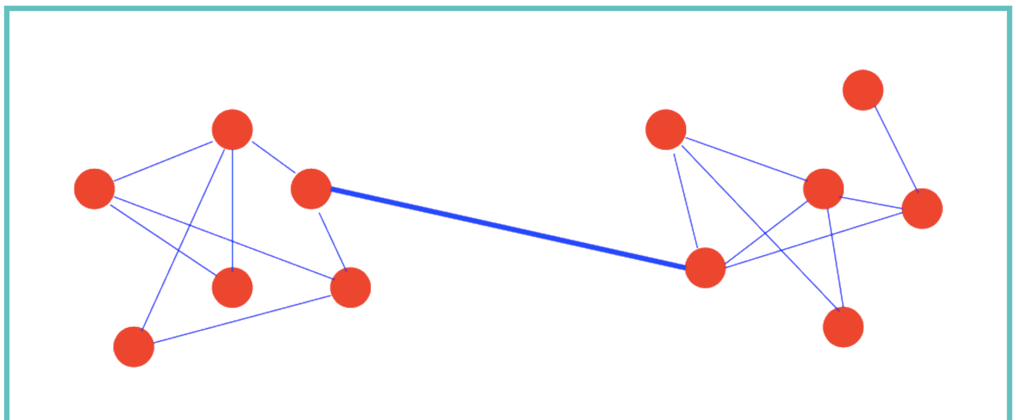


Image from [Fortunato 2009]

Persistence and Complex Networks

Clustering Techniques:

Agglomerative (bottom-up)

Divisive (top-down)

approach based on

Centrality Measures

Atomic Communities

Quality Functions

Clique Percolation:

k-adjacency: two cliques of size *k* are *k*-adjacent if they share *k-1* nodes

k-clique community: maximal union of cliques of size *k* pairwise connected by a sequence of *k*-adjacent cliques

Decomposition in *k*-clique communities

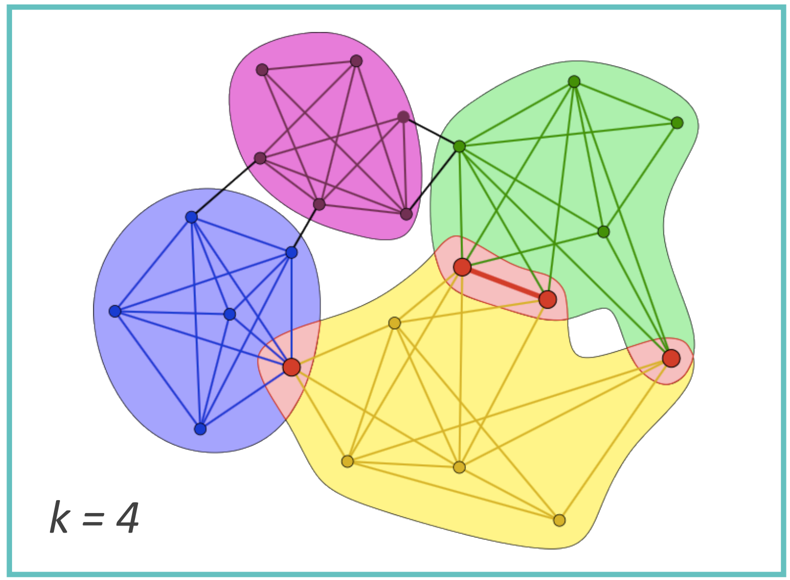


Image from [Palla et al. 2005]

Persistence and Complex Networks

Clustering Techniques:

Agglomerative (bottom-up)

Divisive (top-down)

approach based on

Centrality Measures

Atomic Communities

Quality Functions

Modularity-based Algorithm:

Modularity: measure for clustering quality

Iterated aggregation of communities of nodes whose merging ***increases modularity***

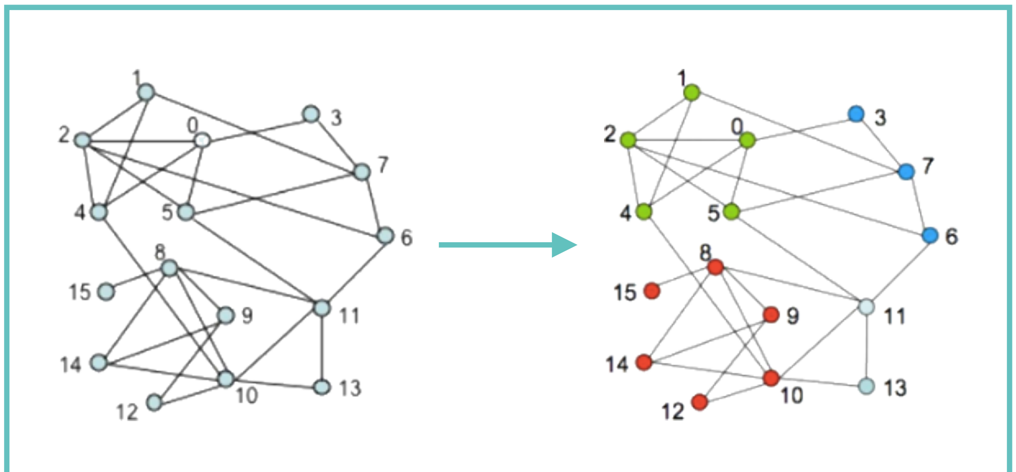


Image from [Blondel et al. 2008]

Persistence and Complex Networks

Several Application based on Persistent Homology:

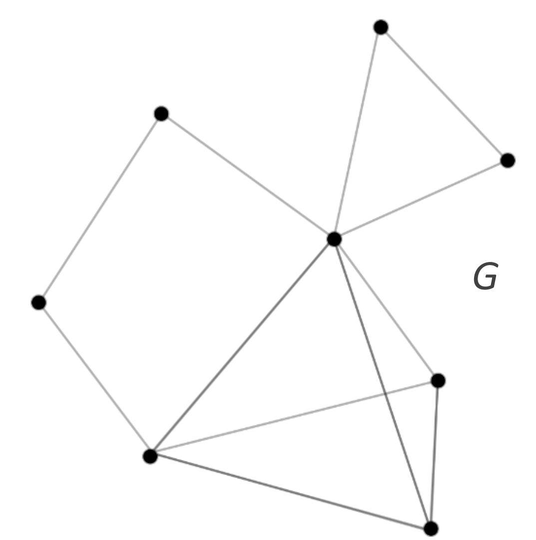
- ***Sensor*** Networks [De Silva 2013]
- ***Brain*** Networks [Lee et al. 2012]
- ***Collaborative/Co-occurence*** Networks [Carstens et al. 2013; Rieck et al. 2016]
- ***Geolocalized*** Networks [Fellegara et al. 2016]
- …

Simplicial Complex Representation:

A network is represented through:

- Simplicial complex ***Flag(G)*** induced by *G*

Simplices of Flag(G) ⟷ *Cliques of G*



Persistence and Complex Networks

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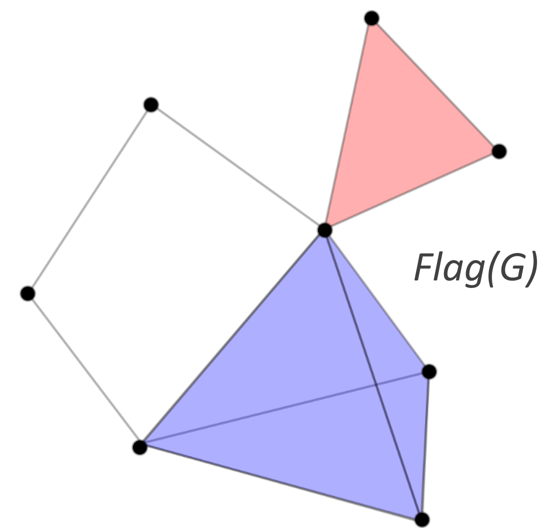
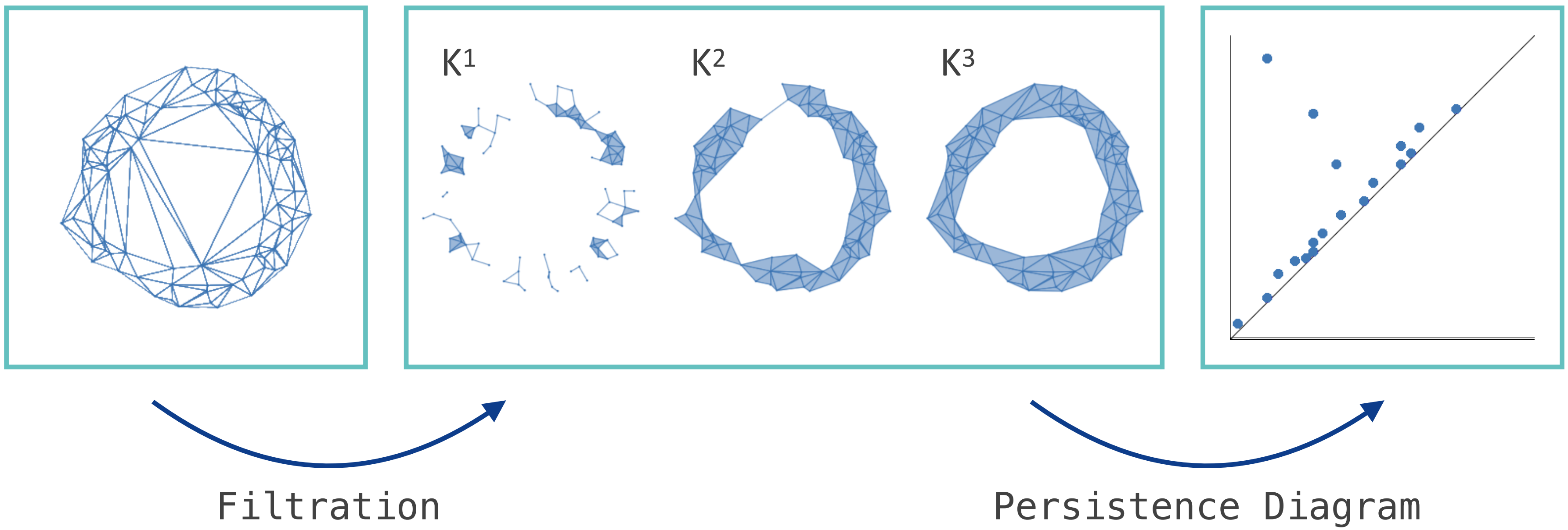


Persistence and Complex Networks

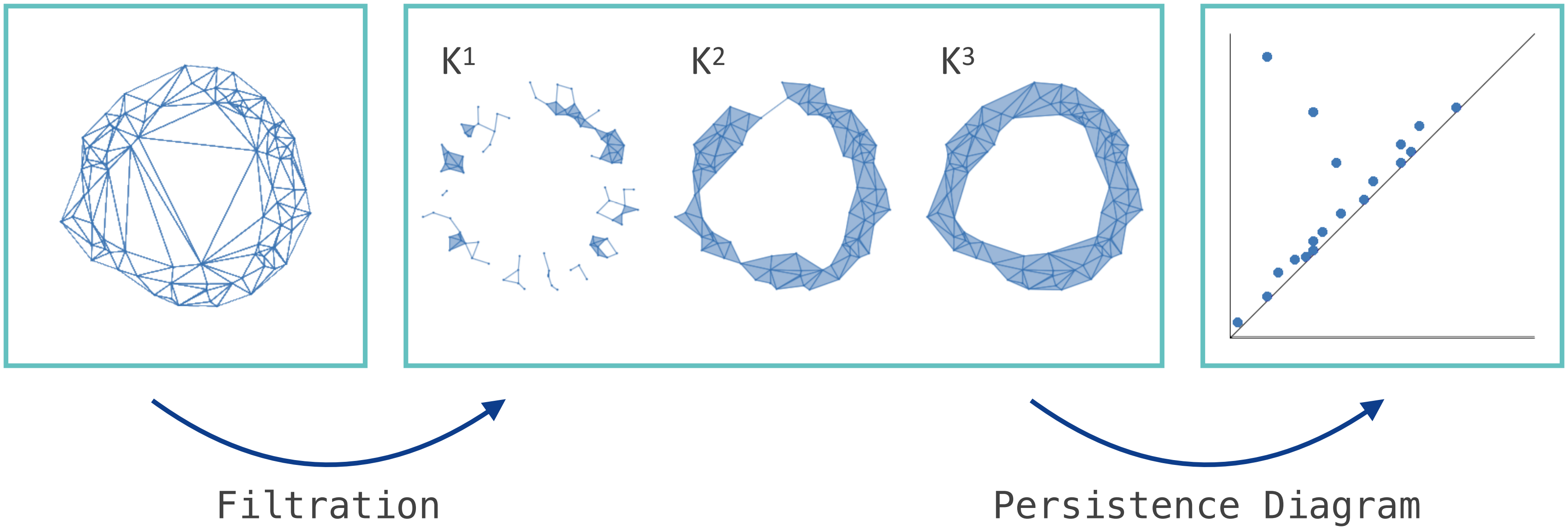


Topological summaries have proven to be particularly effective to ***distinguish shapes***

but

It is hard to obtain a ***meaningful interpretation*** for homological cycles

Persistence and Complex Networks



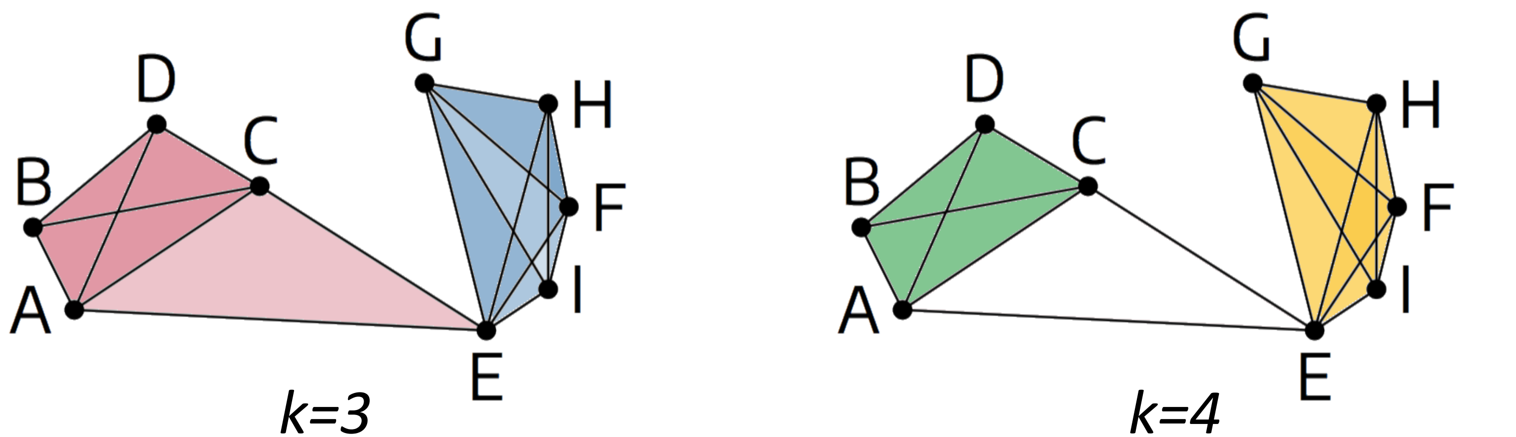
What if …

we study the persistence of something more meaningful than homological cycles?

Persistence and Complex Networks

k-Clique Community:

A ***maximal union*** of *k*-cliques ***pairwise connected*** by a ***sequence of k-adjacent cliques***

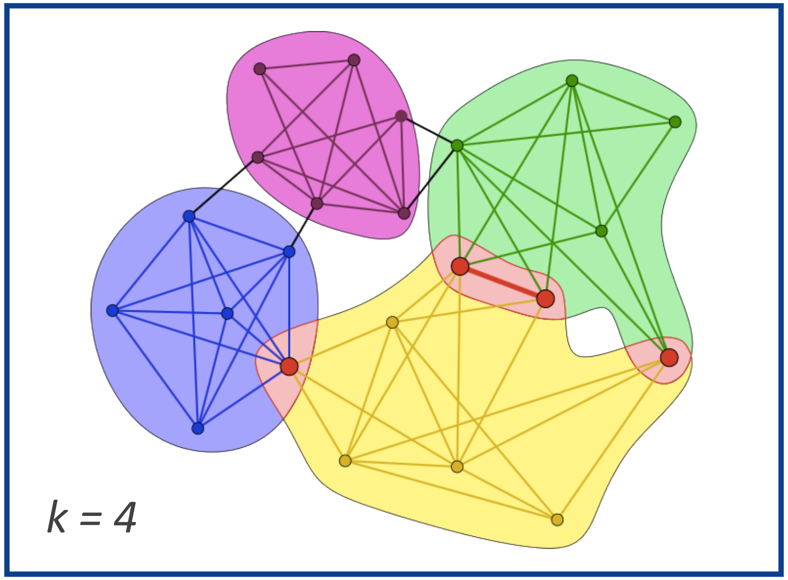


k-Adjacency:

Two *k*-cliques are ***k-adjacent*** if they ***share k-1 nodes***

Persistence and Complex Networks

k-Clique Community Decomposition:

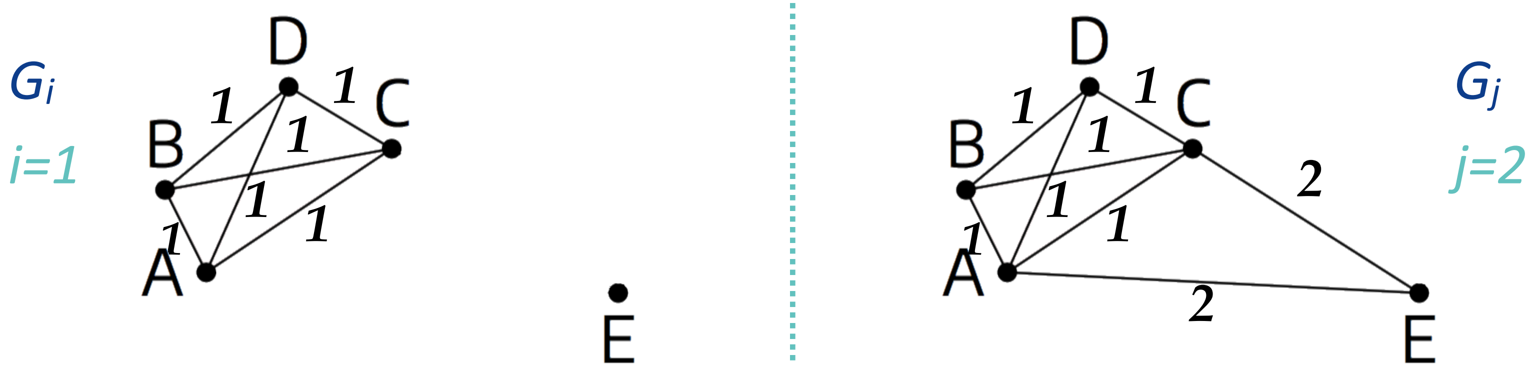


- Reveal ***highly connected*** communities
- Allow ***overlaps***
- Have a ***hierarchical structure***

Persistence and Complex Networks

Clique Communities and Weighted Networks:

Given a weighted network ***G*** and two threshold values ***i*** < ***j***,

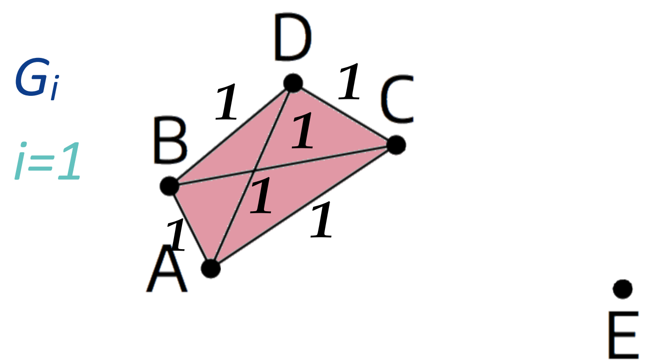


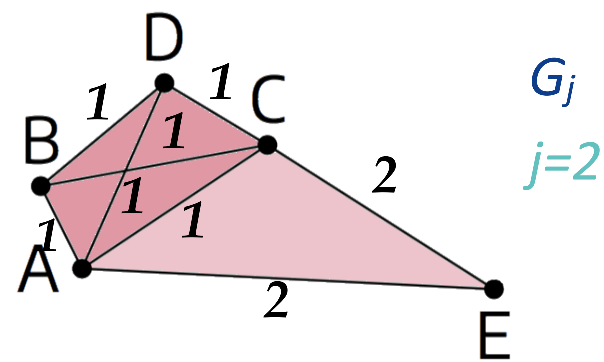
G_i is contained in G_j

Persistence and Complex Networks

Clique Communities and Weighted Networks:

Given a weighted network ***G*** and two threshold values ***i*** < ***j***,





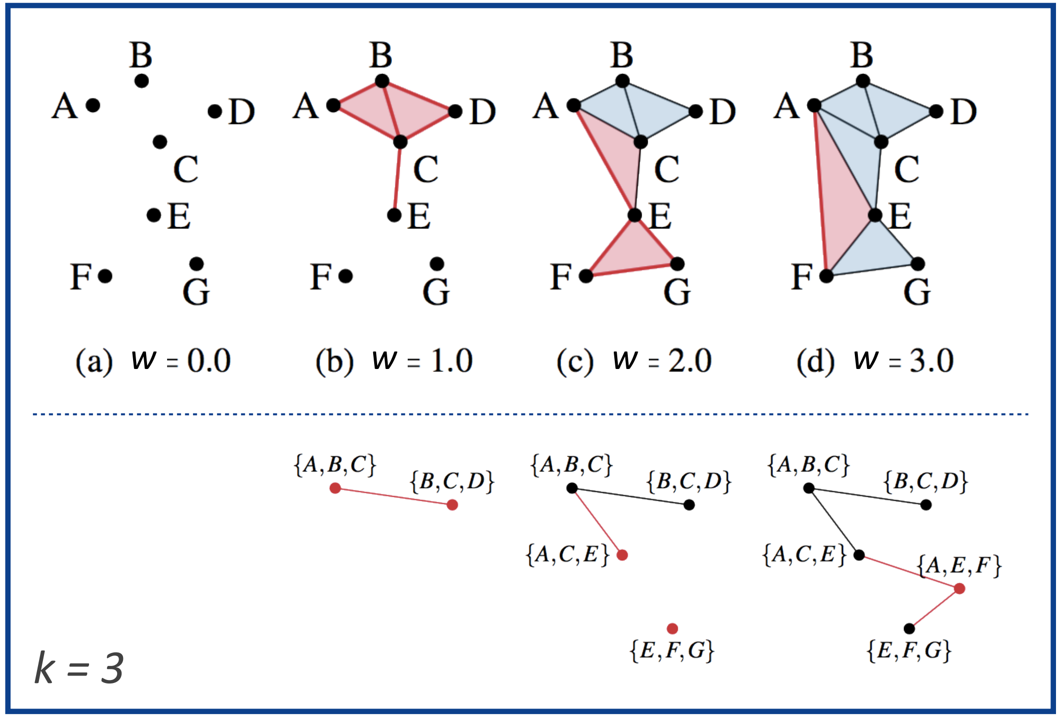
Each k-clique community of G_i is ***contained*** in ***exactly one*** k-clique community of G_j

Persistence and Complex Networks

Clique Community Persistence:

Fixing a value for k and varying the edge-weight threshold, the persistence of **k-clique communities** of G can be tracked by:

- Building a sequence of *k*-dual graphs:
  - ***vertices ↔ k-cliques***
  - ***edges ↔ adjacent k-cliques***
- Tracking the ***connected components*** of the sequence of *k*-dual graphs



Persistence and Complex Networks

Clique Community Persistence:

The presented approach allows for designing tools for:

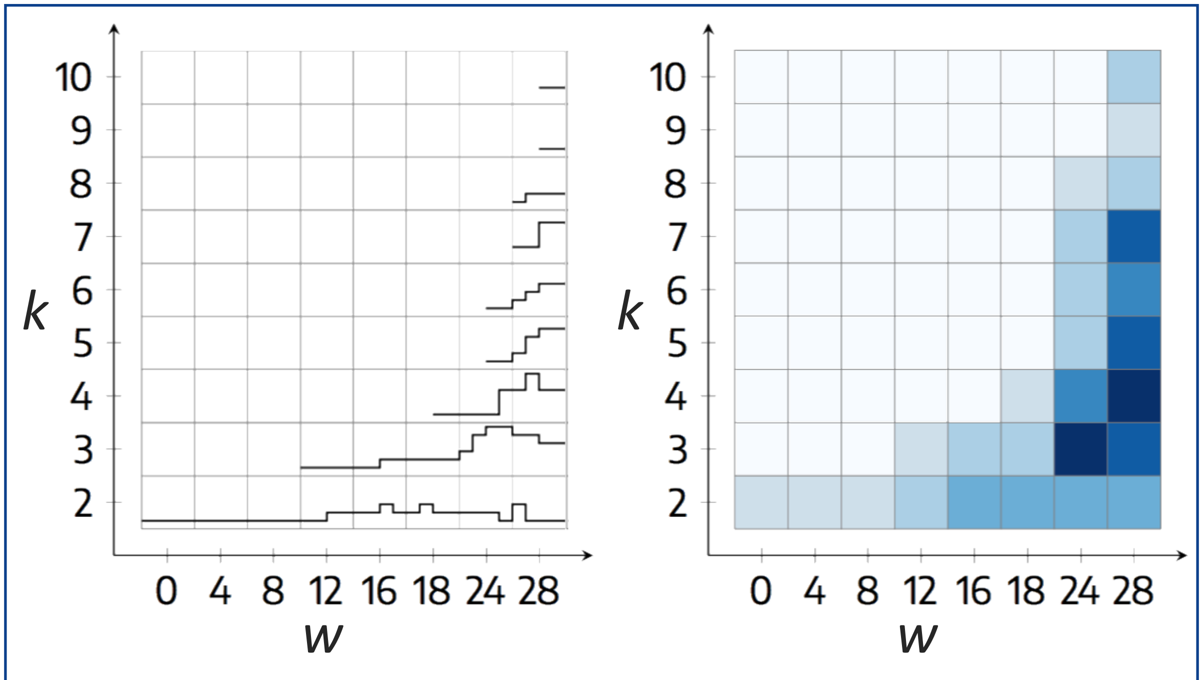
- ***Network Comparison***
  - ***Comparison Measures***
    - ***Persistence Indicator Function (PIF)***
    - ***PIF-based distance***
  - Clique Community ***Centrality Measure***
- ***Single Network Analysis***
  - ***Interactive Visualization Tool*** based on Nested Graphs

Persistence and Complex Networks

Persistence Indicator Function: Defined as the function $f_k : \mathbb{R} \longrightarrow \mathbb{N}$

assigning:

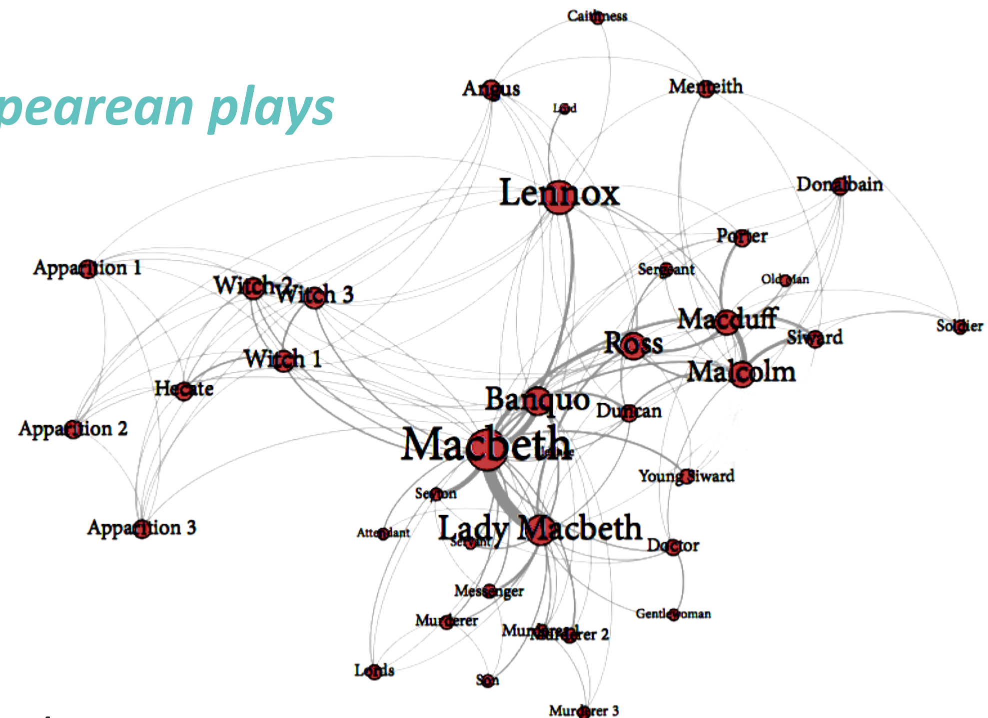
w $\longmapsto$ ***# k-cliques communities "alive" at threshold w***



Persistence and Complex Networks

Persistence Indicator Function:

- ***Co-occurrence networks*** of ***Shakespearean plays***
  - *37 plays considered*

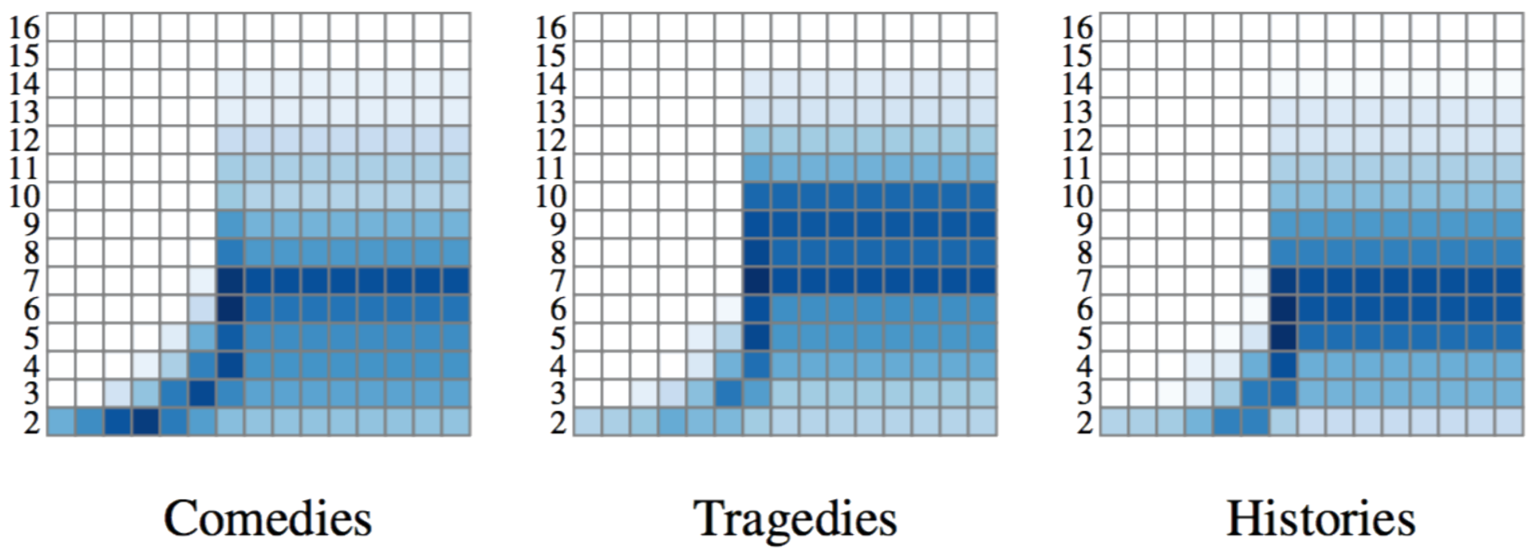


- In each network:
  - ***nodes*** ↔ ***characters*** *of the play*
  - ***edges*** ↔ *characters appearing in the* ***same scene***
  - ***edge weight*** ↔ *inverse of the* ***number of interactions***

Persistence and Complex Networks

Persistence Indicator Function:

PIF enables a comparison of structural differences between groups of plays



Persistence and Complex Networks

PIF-Based Distance:

Given two persistence indicator functions f and g,

PIF-based distance is defined to be the L^p distance between f and g:

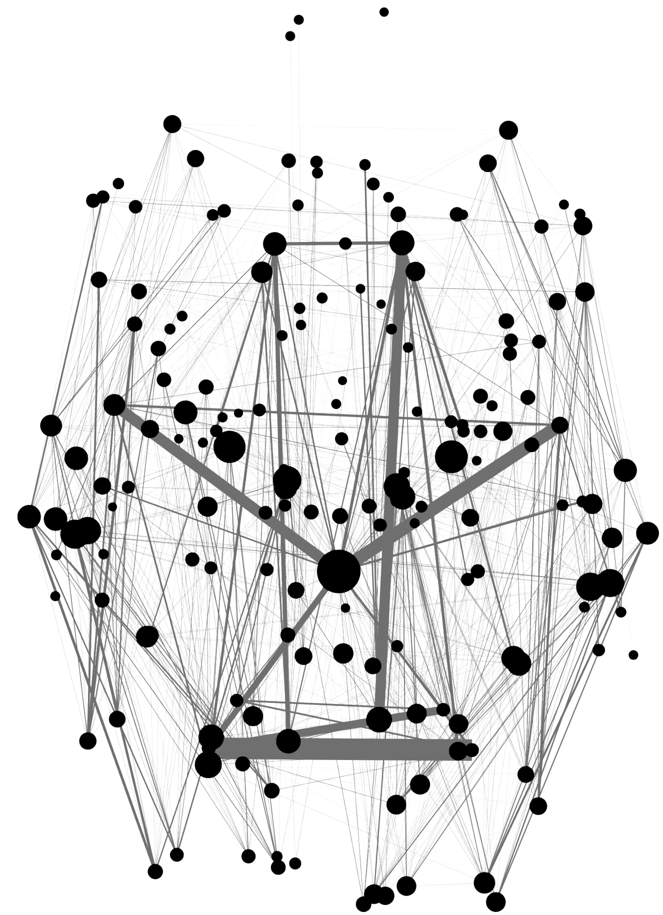
$$dist(f,g) = \left(\int_{\mathbb{R}} |f(x) - g(x)|^p \, dx \right)^{\frac{1}{p}}$$

- Quantifies dissimilarities between PIFs
- Easier to be computed than Wasserstein and bottleneck distances
- Highly correlated to Wasserstein distance

Persistence and Complex Networks

PIF-Based Distance:

- ***Biological networks*** representing variants of ***human brain connectivity***
    - *9 instances considered*

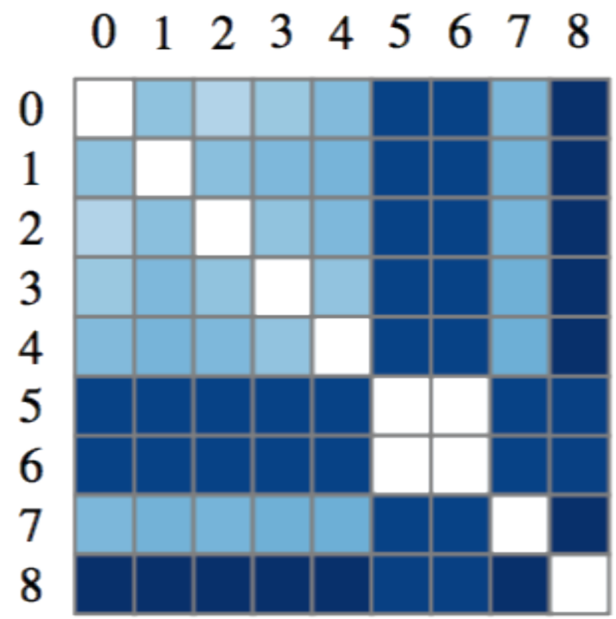


- In each network:
    - ***nodes*** ⟷ ***brain areas***
    - ***edges*** ⟷ ***fibers connecting different areas***

Persistence and Complex Networks

PIF-Based Distance:

Variant	Density	Diam. (weighted)	Avg. degree (weighted)
0	0.125	4 (60.0)	21.21 (2300.3)
1	0.124	4 (60.0)	21.06 (2296.0)
2	0.124	4 (60.0)	21.13 (2295.2)
3	0.124	4 (60.0)	21.16 (2282.0)
4	0.124	4 (60.0)	21.15 (2279.3)
5	0.125	4 (60.0)	21.19 (2264.0)
6	0.125	4 (60.0)	21.19 (2264.0)
7	0.124	4 (60.0)	21.16 (2279.6)
8	0.125	4 (60.0)	21.20 (2257.5)



PIF-based distance reveals differences between networks that common graph measures are incapable of detecting

Persistence and Complex Networks

Clique Community Centrality:

Clique community centrality of a node *v* is defined as

$$centrality(v) = \sum_{C \ni v} pers(C)$$

where:

- C is any clique community containing *v*
- ***pers(C)*** is the "lifespan" of C

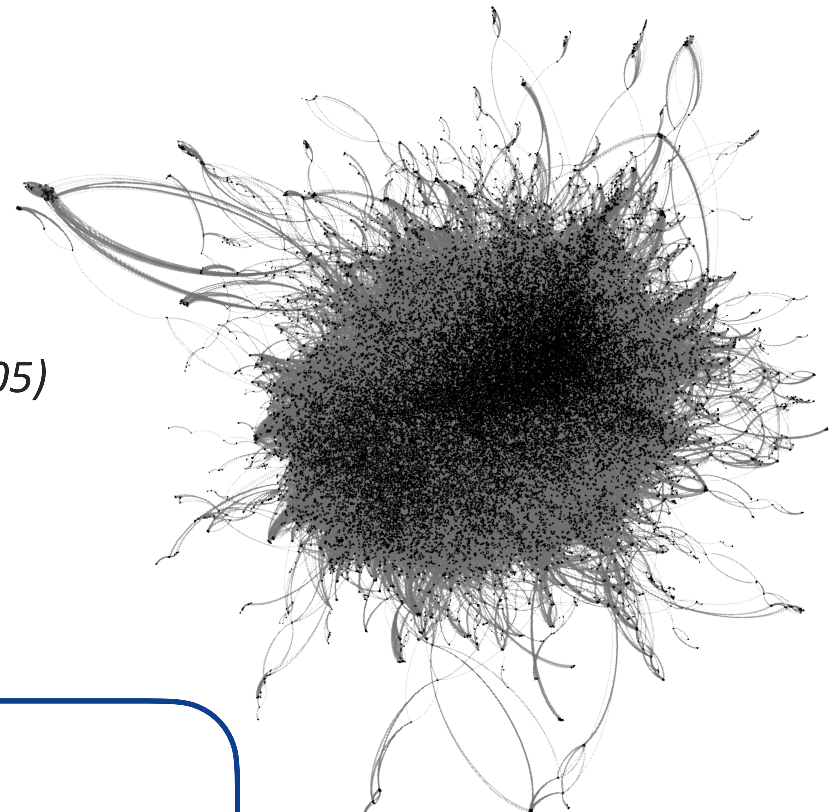
*Nodes belonging to **high-persistence communities** are identified as **relevant***

Persistence and Complex Networks

Clique Community Centrality:

- ***Collaborative networks*** describing ***scientist co-authorship of the "Condensed Matter"*** arXiv category
  - *3 snapshots in time considered (1999, 2003, 2005)*

- Network sizes:
  - *16K - 40K nodes*
  - *47K - 175K edges*



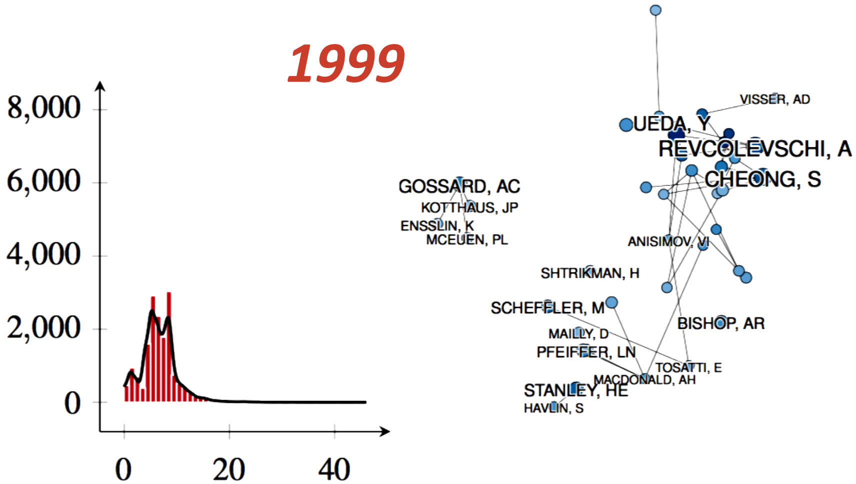
Clique community centrality allows for

- *evaluating the **evolution of network connectivity***
- *filtering away the **less relevant nodes***

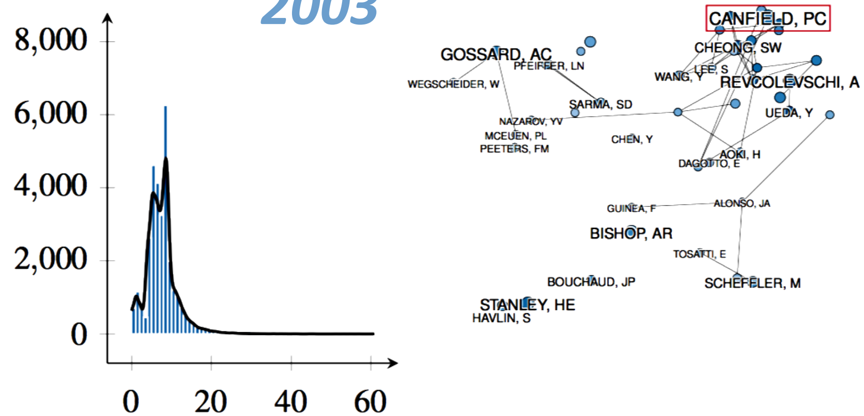
Persistence and Complex Networks

Clique Community Centrality:

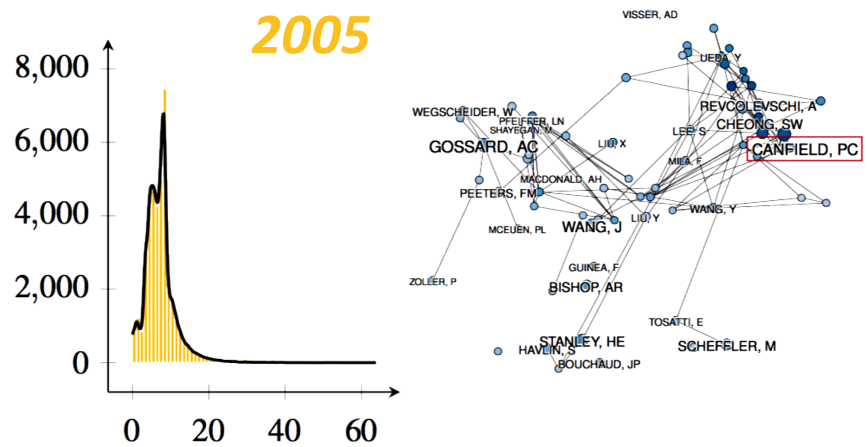
1999



2003



2005



Density estimates of the clique community centrality values

Persistence and Complex Networks

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The presented approach allows for designing tools for:

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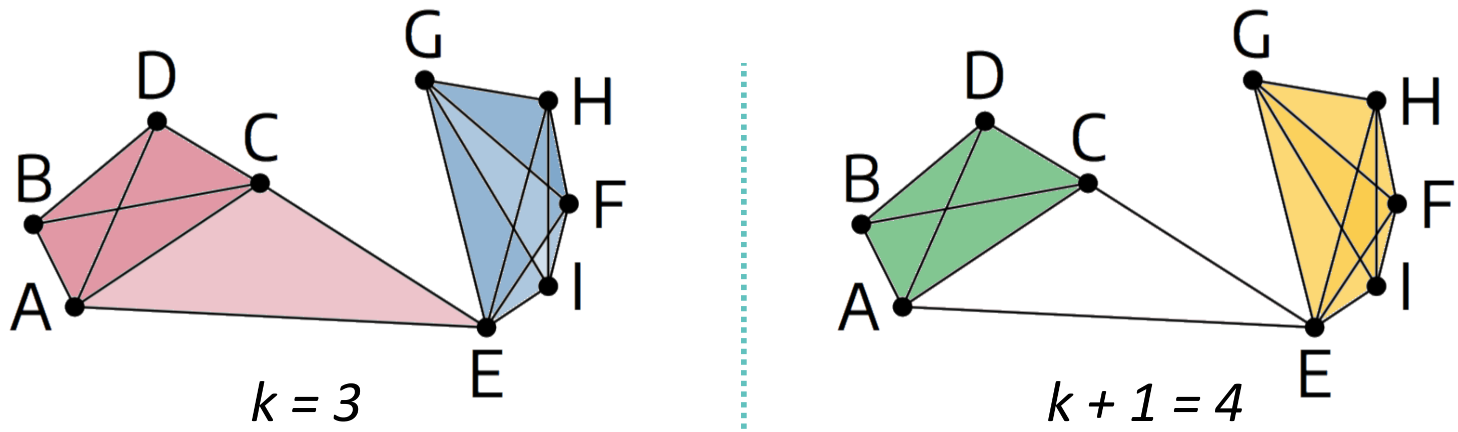
  - ***Interactive Visualization Tool*** based on Nested Graphs

Persistence and Complex Networks

Clique Communities and Multiple k-Values:

Given a weighted network ***G*** and any threshold value ***i***,

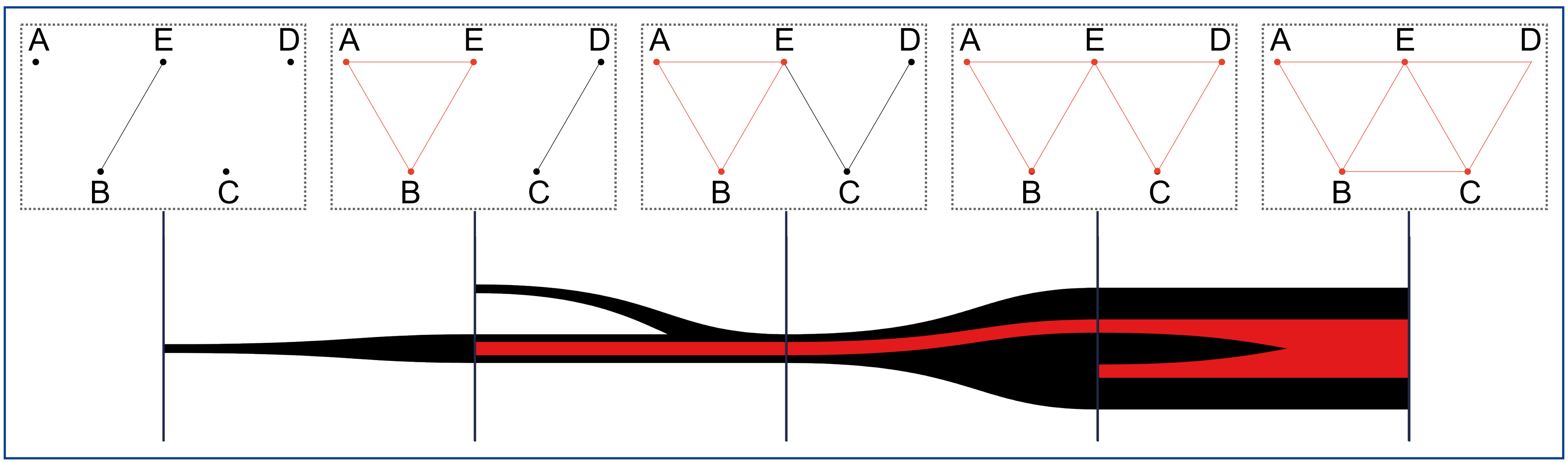


Each *(k+1)*-clique community of G_i is **contained** in ***exactly one*** *k*-clique community of G_i

Persistence and Complex Networks

Nested Graph:

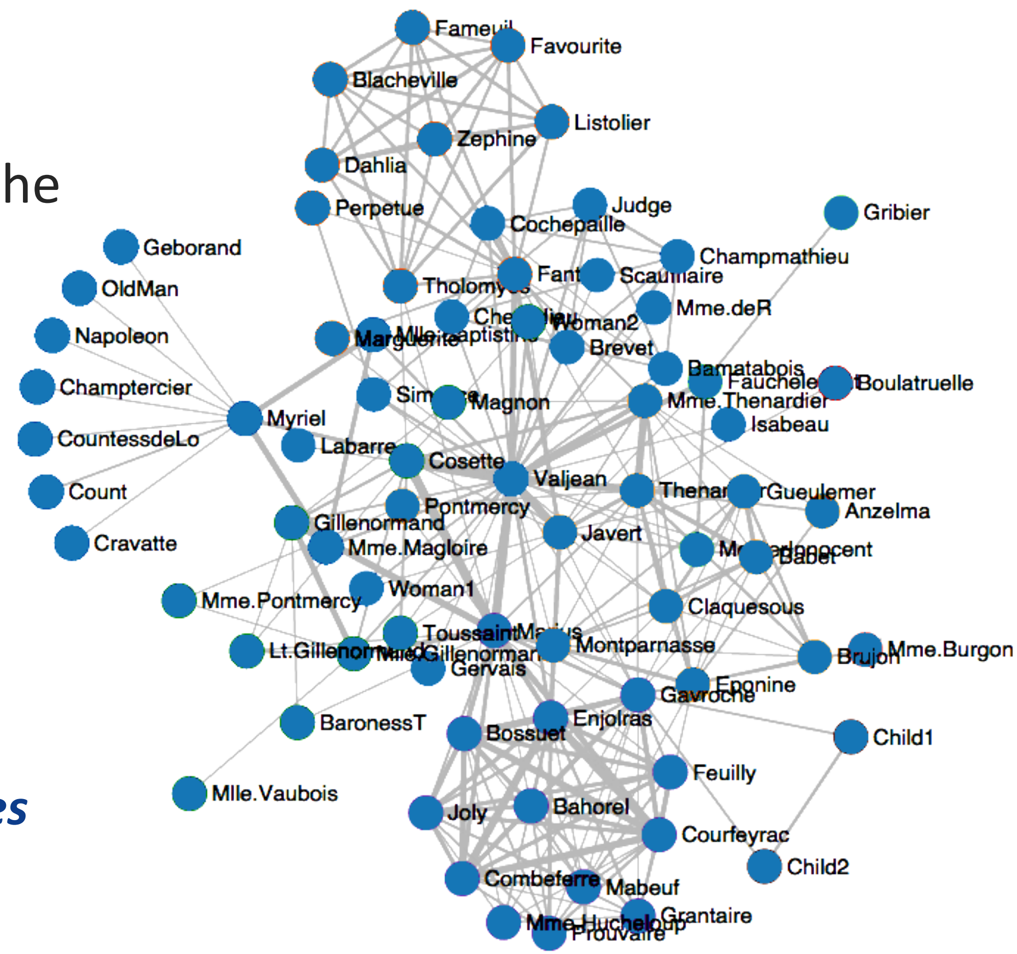
- Originally defined for connected components in scalar fields [Lukasczyk et al. 2017]
- Illustrates ***evolutions across two parameters***



Persistence and Complex Networks

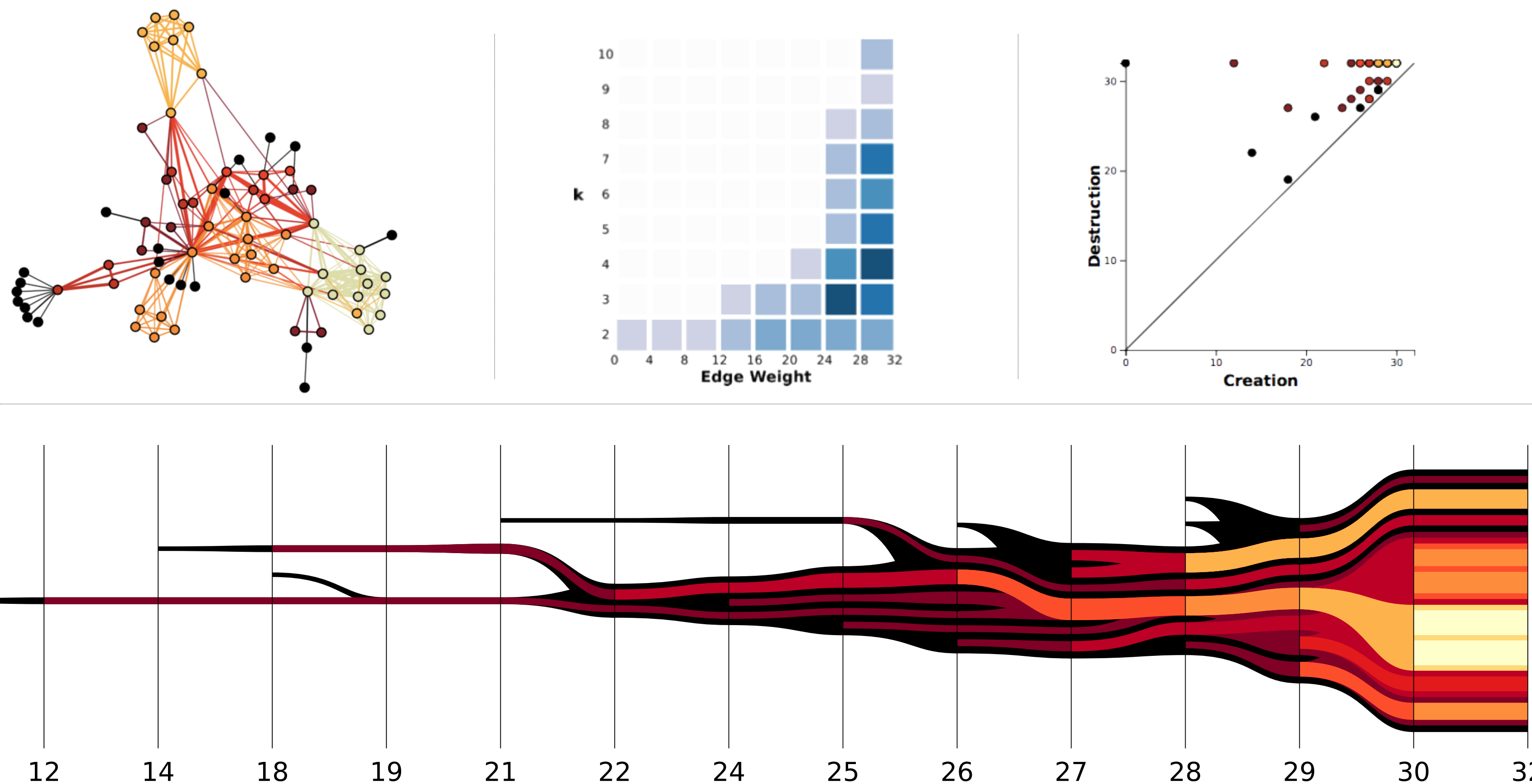
Nested Graph:

- ***Co-occurrence network*** between the characters in **Victor Hugo's novel** ***"Les Misérables"***
  - *77 nodes*
  - *254 edges*



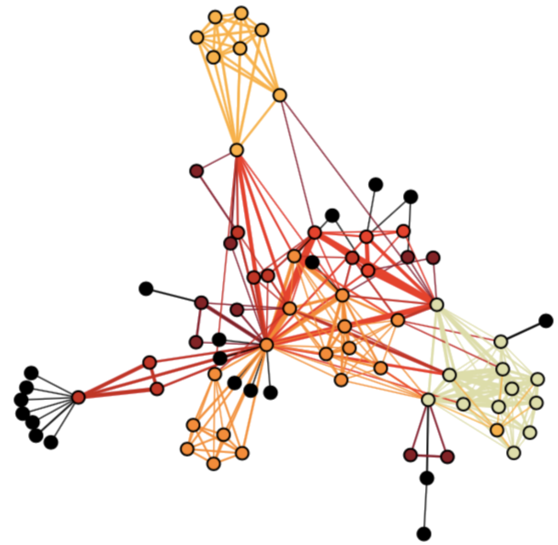
- ***edge weight*** $\leftrightarrow$ ***1 / # co-occurrences***

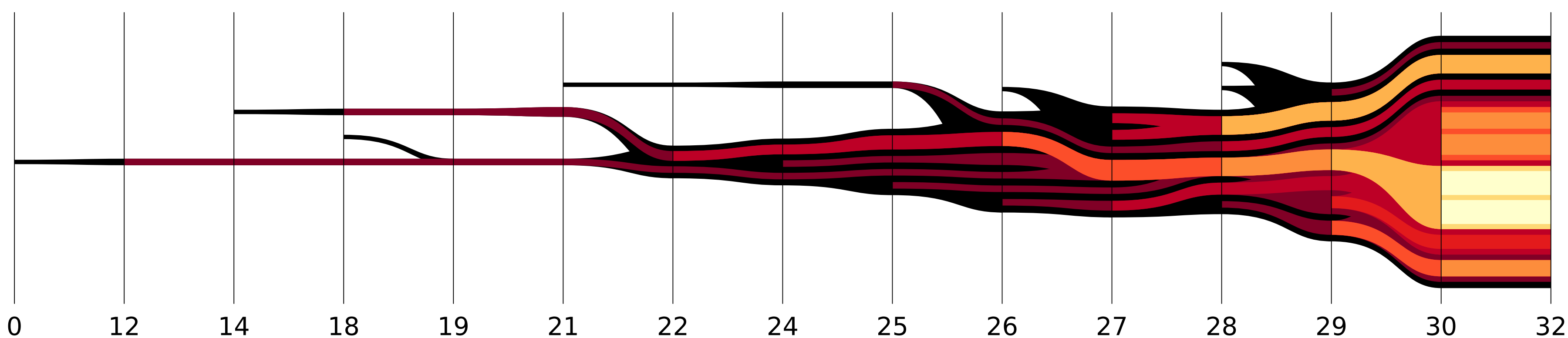
Persistence and Complex Networks



Persistence and Complex Networks

Nested-based visualization tool allows the user for

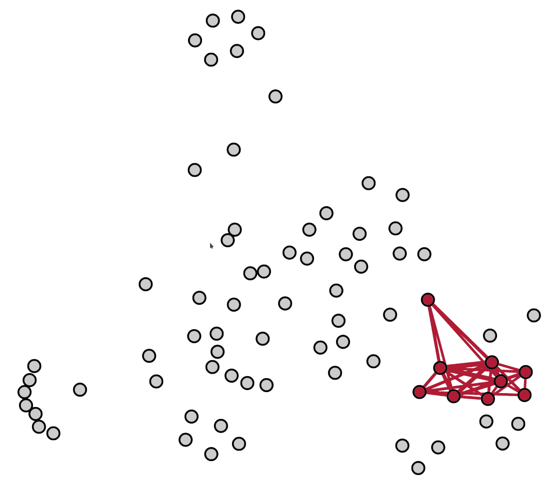




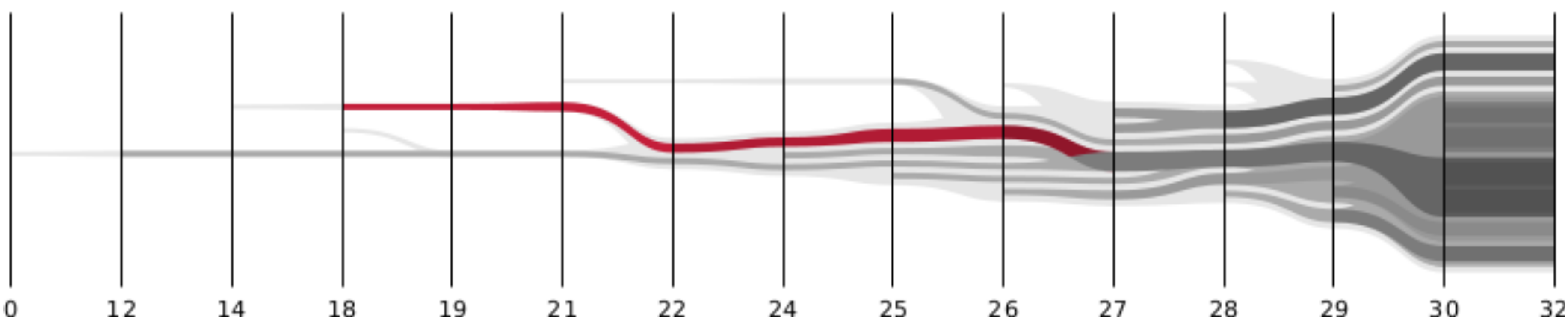
Persistence and Complex Networks

Nested-based visualization tool allows the user for

- ***focusing on*** *the evolution of a* ***specific clique community***
- ***selecting*** *and* ***interactivity exploring different edge weights*** *and* ***clique degrees***



while the force-directed graph layout and the nested graph ***change accordingly***



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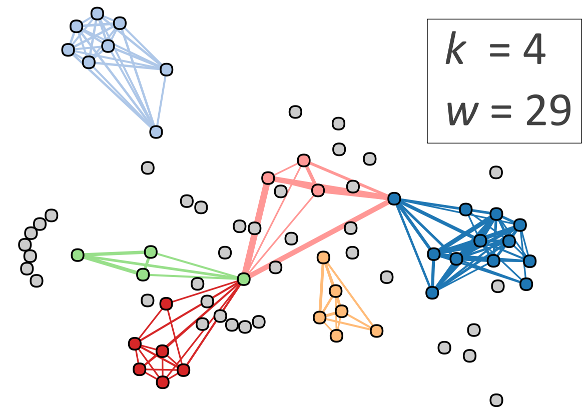

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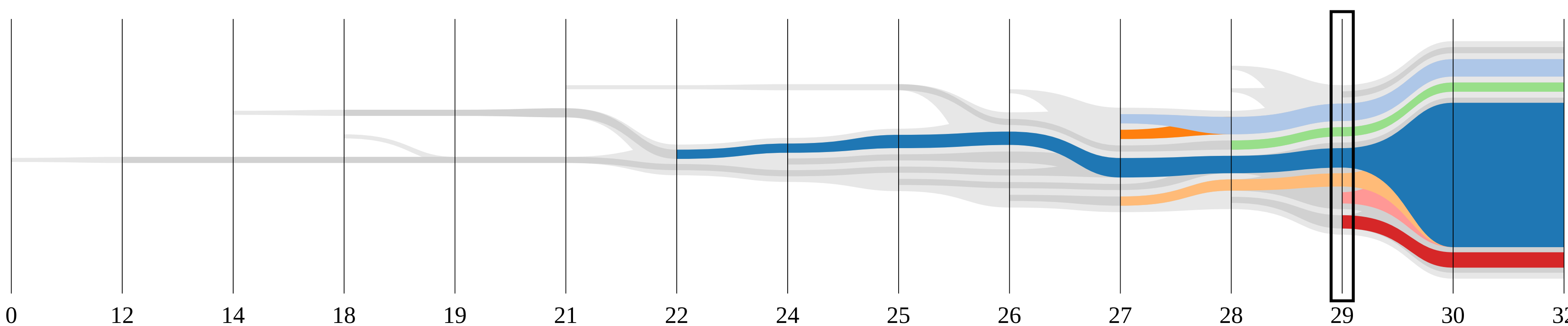

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- ***selecting*** *and* ***interactivity exploring*** ***different edge weights*** *and* ***clique degrees***

while the force-directed graph layout and the nested graph ***change accordingly***

Intuitively:

edge-weight *variation*	↔	*reveal the* ***core part of a community***
clique-degree *variation*	↔	*analyze* ***community according to different granularities***

Bibliography

General References:

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