

# *Persistent Homology for Shape Comparison*

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CNR - IMATI



# *Persistent Homology for Shape Comparison*

- ◆ *Quick Overview on TDA*
- ◆ *Case Study*

# Persistent Homology for Shape Comparison

- ◆ **Quick Overview on TDA**
- ◆ *Case Study*

# Topological Data Analysis

*Topological Data Analysis (TDA) aims at describing, characterizing, and discriminating data on the basis of their shape*

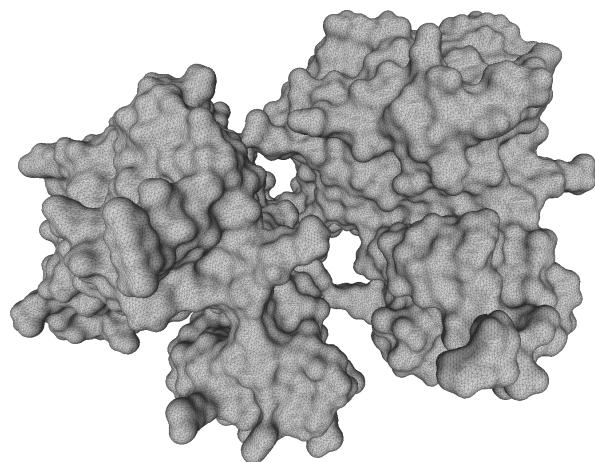
# Topological Data Analysis

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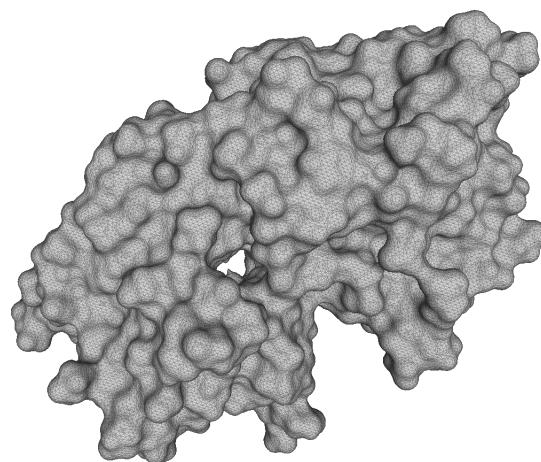
**Example:**

Consider a dataset consisting of *molecular surfaces* and pick two of them, X and Y

X



Y

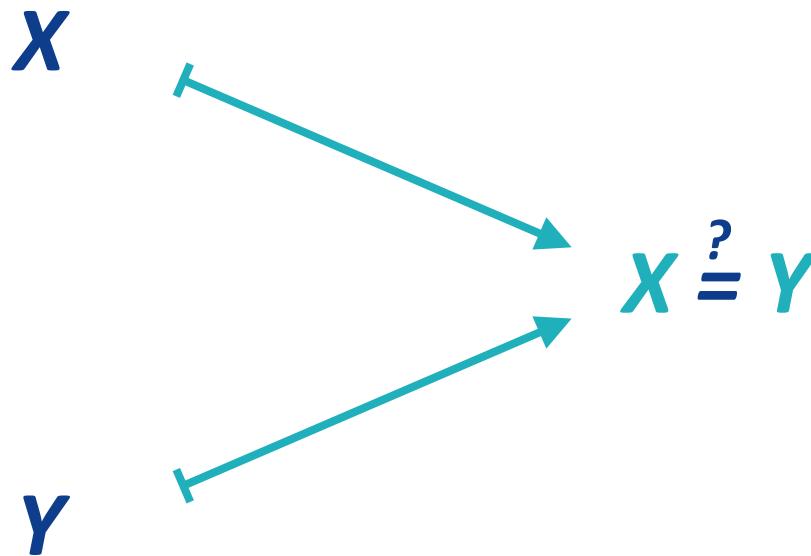


*Do X and Y represent the same molecule?*

# Are $X$ , $Y$ equal?

*Topological  
Space*

*Equality  
Check*



# Do $X$ and $Y$ have the same shape?

Topological  
Space

Algebraic  
Structure

Equality  
Check

$X$        $\longmapsto$

$Shape(X)$

$Y$        $\longmapsto$

$Shape(Y)$

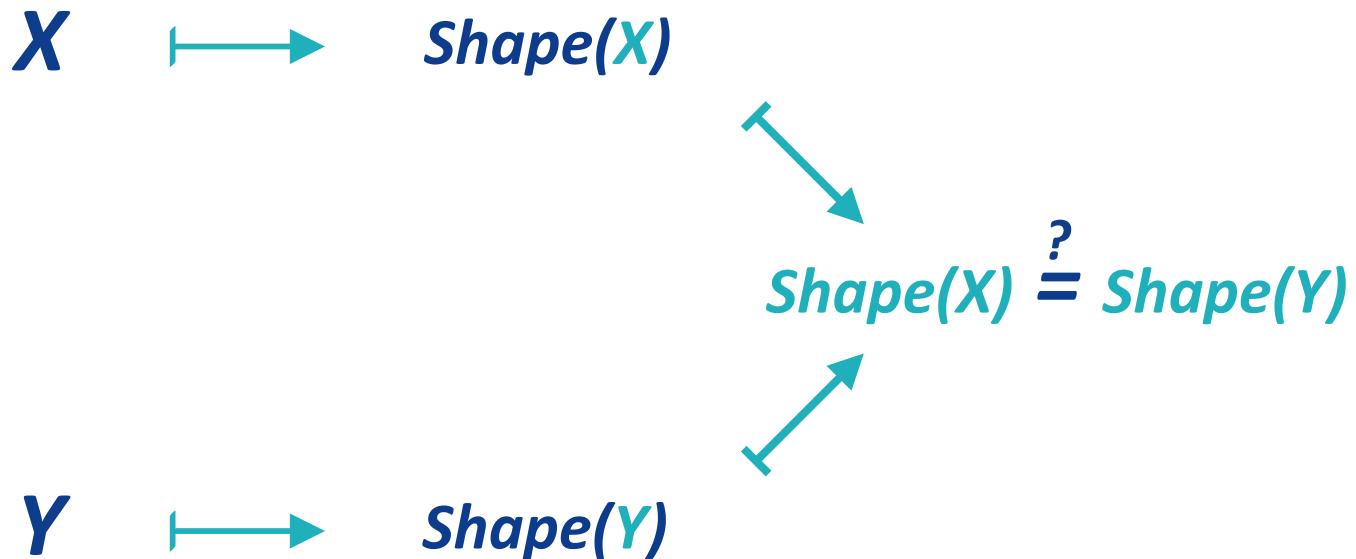
$Shape(X) \stackrel{?}{=} Shape(Y)$

# Do $X$ and $Y$ have the same shape?

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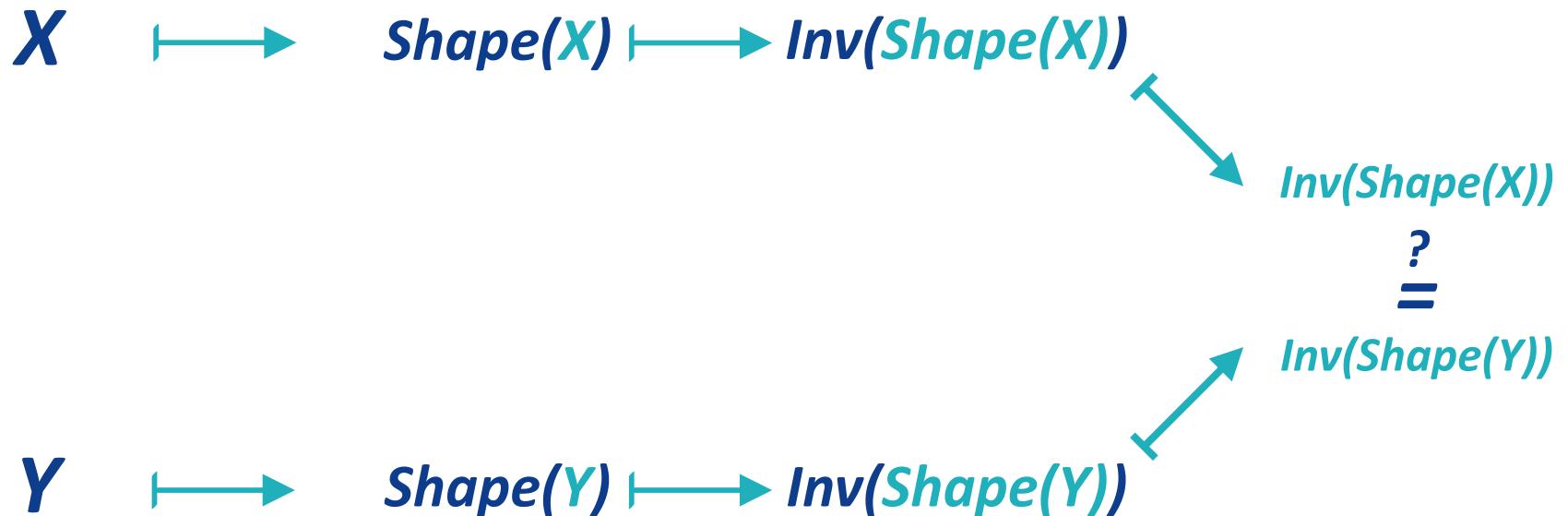
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# Homology

Given a topological space  $X$  and a field  $\mathbb{F}$ , the *homology of  $X$*  is a *topological invariant*

*intuition*

*detecting the “holes” of  $X$*

*capturing the independent non-bounding cycles of  $X$*

*measuring how far the chain complex associated with  $X$  is from being exact*

*formalism*

# Homology

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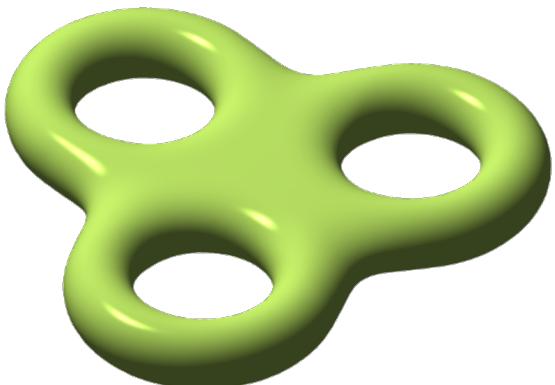
↑  
*intuition*

*detecting the “holes” of  $X$*

↓  
*formalism*

*capturing the independent non-bounding cycles of  $X$*

*measuring how far the chain complex associated with  $X$  is from being exact*



$$H_i(X; \mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{for } i = 0 \\ \mathbb{F}^6 & \text{for } i = 1 \\ \mathbb{F} & \text{for } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

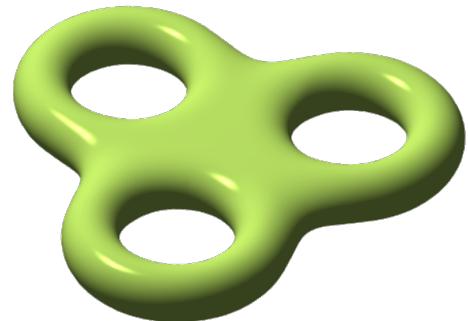
# Homology

**Theorem:**

Any homology group of  $X$  with coefficients in  $\mathbb{F}$  can be expressed as

$$H_i(X; \mathbb{F}) \cong \mathbb{F}^{\beta_i}$$

where  $\beta_i$  is called the  **$i^{th}$  Betti number** of  $X$



$$\beta_i = \begin{cases} 1 & \text{for } i = 0 \\ 6 & \text{for } i = 1 \\ 1 & \text{for } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

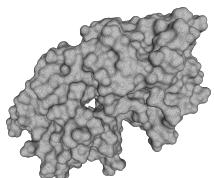
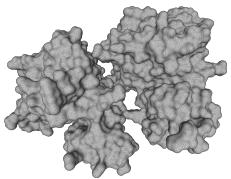
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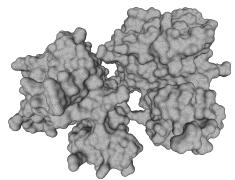
# Do $X$ and $Y$ have the same shape?

Topological Space

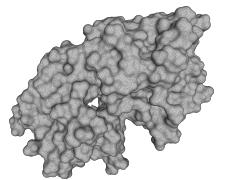
Algebraic Structure

Algebraic Invariant

Equality Check



$$H_i(X; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } i = 0 \\ (\mathbb{Z}_2)^4 & \text{for } i = 1 \\ \mathbb{Z}_2 & \text{for } i = 2 \end{cases}$$



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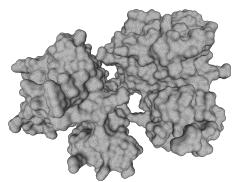
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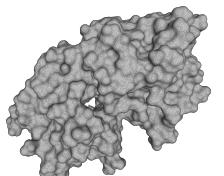
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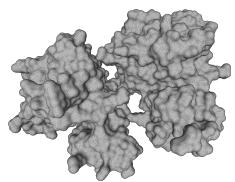
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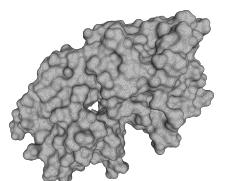


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$$(1, 4, 1)$$



$X \neq Y$



$$H_i(Y; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } i = 0 \\ (\mathbb{Z}_2)^2 & \text{for } i = 1 \\ \mathbb{Z}_2 & \text{for } i = 2 \end{cases}$$

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# ***Homology Fails***

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*Do they have the same shape?*



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In Practice?

In Theory?

# Homology Fails

*Do they have the same shape?*



In Practice?

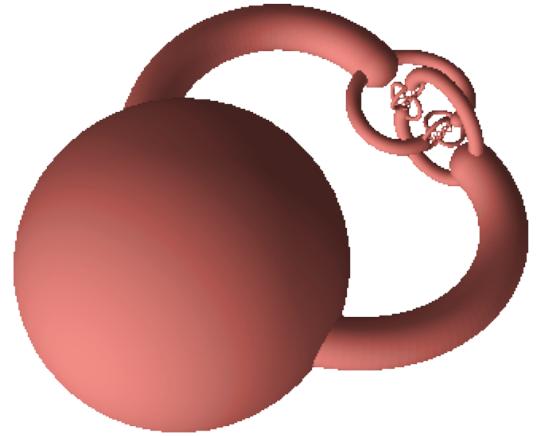
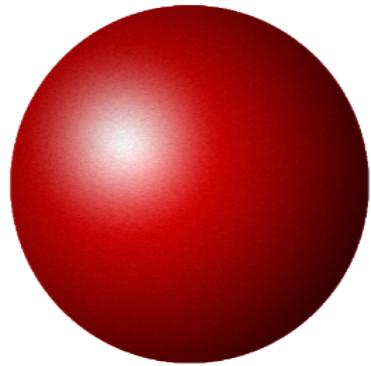


In Theory?



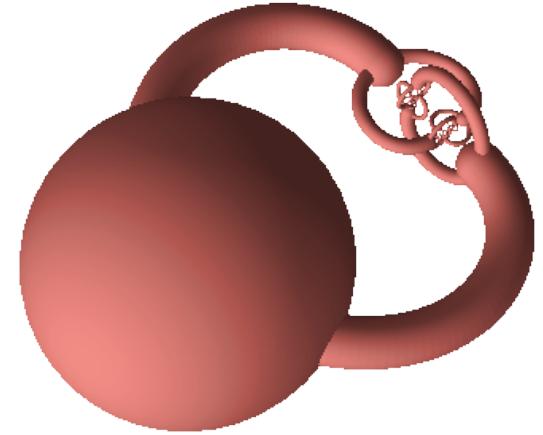
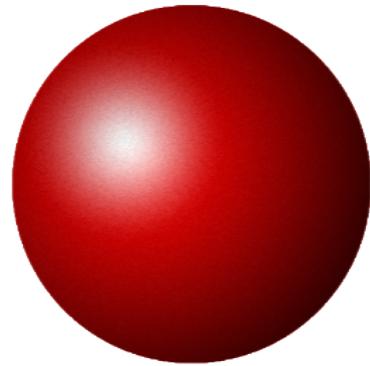
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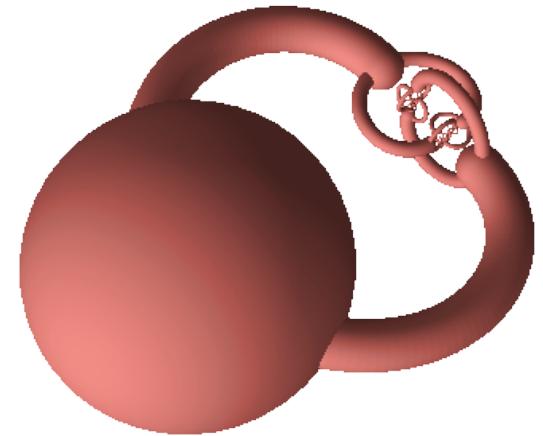
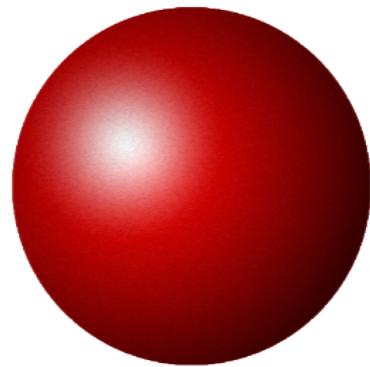


In Practice?

In Theory?

# Homology Fails

*Do they have the same shape?*



In Practice?



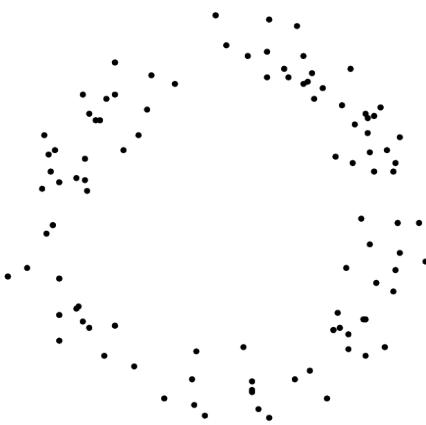
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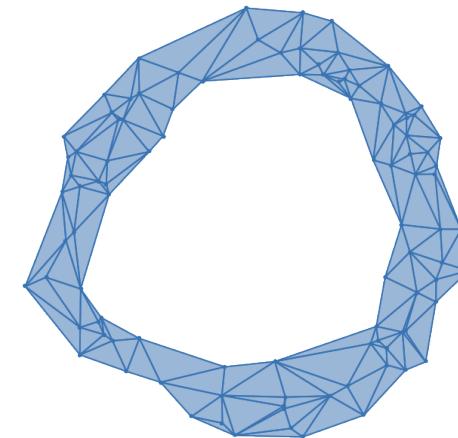
# Homology Fails

*Which is the shape of a given data?*

We would like to retrieve the “*actual*” homological information of a data



*Point Cloud Dataset*



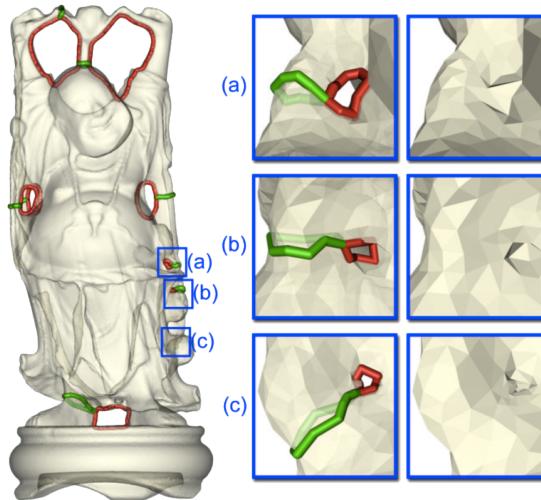
*Topological Nature of  
the “Underlying” Shape*

Image from [Bauer 2015]

# Homology Fails

*Which is the shape of a given data?*

We would like to retrieve the “*actual*” homological information of a data



*Noisy Dataset*



*Relevant Homological  
Information*

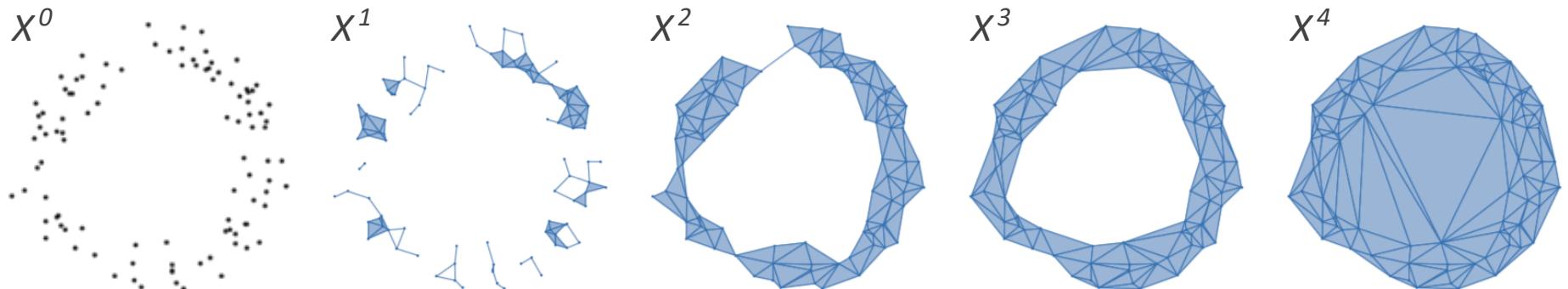
Image from [Dey et al. 2008]

# *The Solution?*

# The Solution? Persistent Homology

*In a Nutshell:*

*Persistent homology allows for  
describing the homology changes of an evolving object*



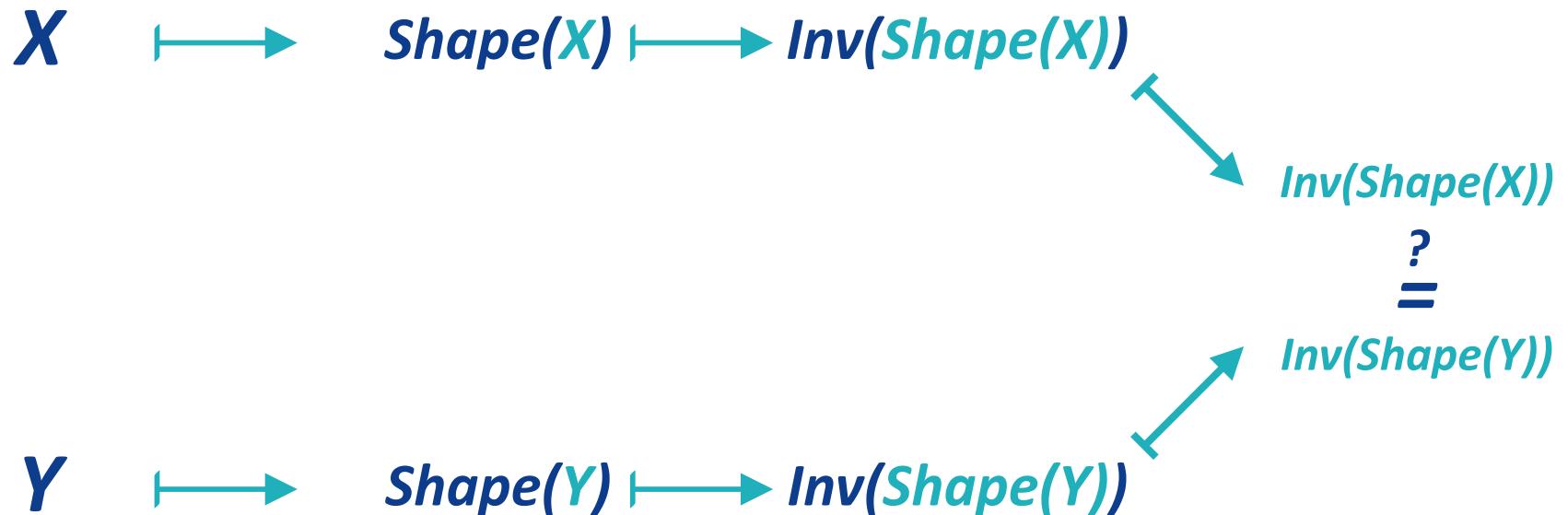
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Topological  
Space

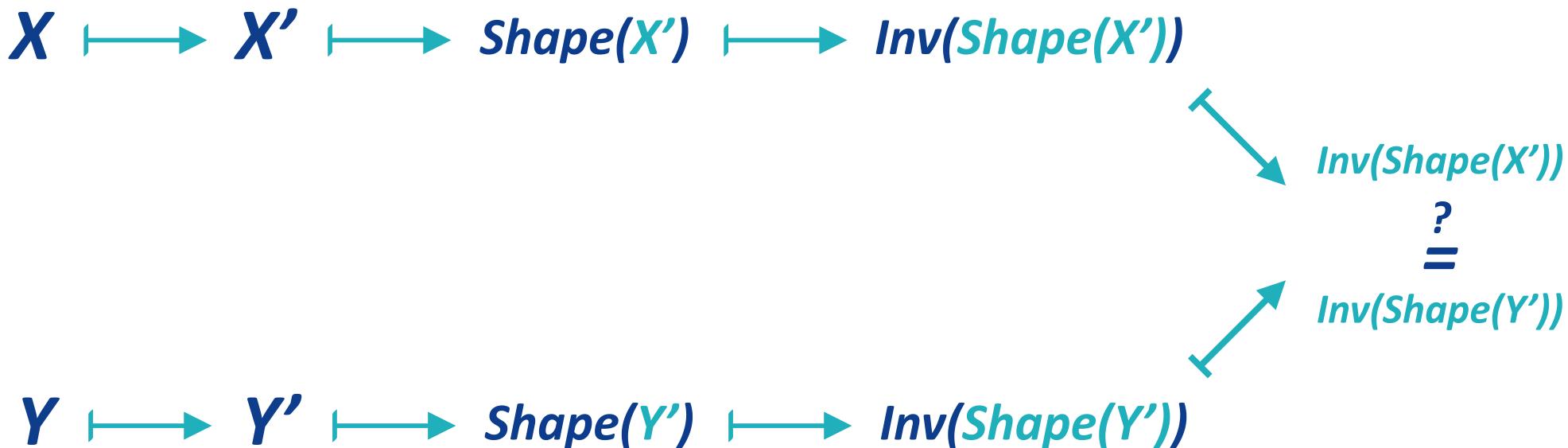
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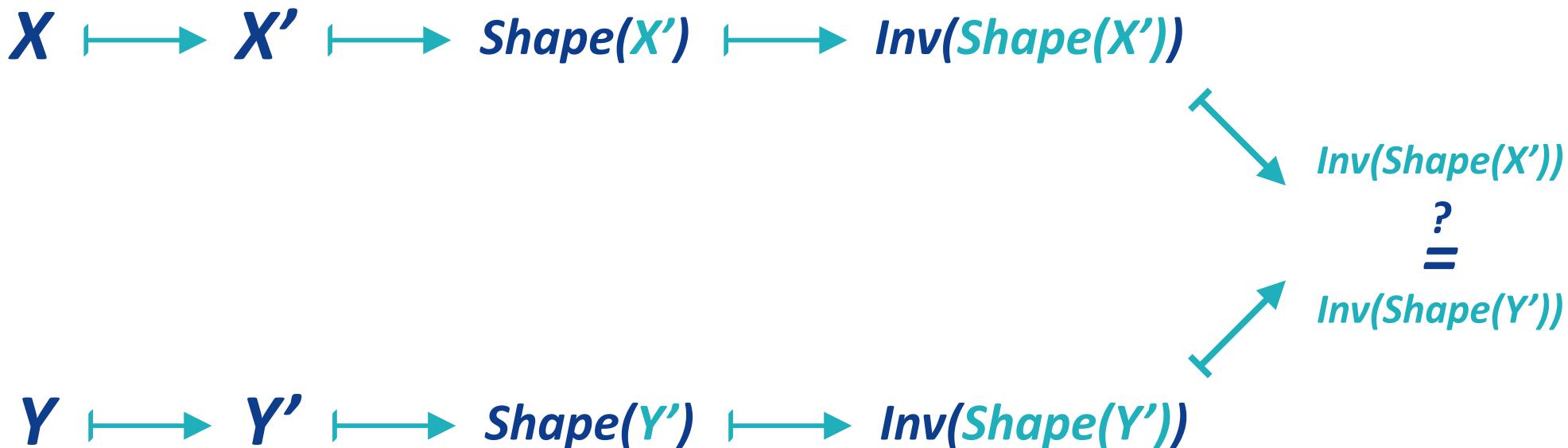
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# Do $X$ and $Y$ have the same shape?



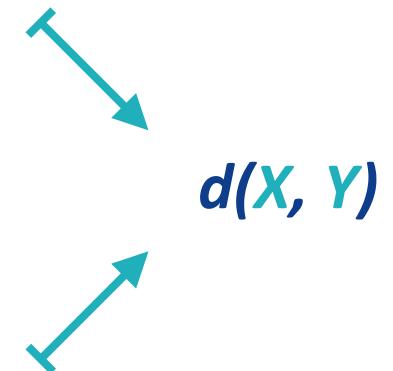
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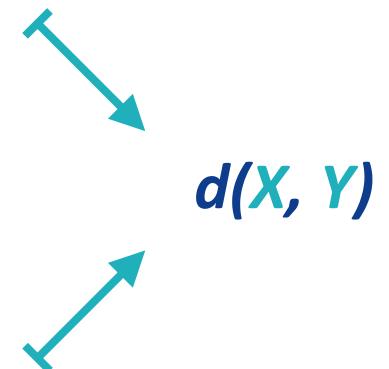
# Do $X$ and $Y$ have a *similar* shape?


$$X \xrightarrow{\quad} X' \xrightarrow{\quad} \text{Shape}(X') \xrightarrow{\quad} \text{Inv}(\text{Shape}(X'))$$
$$d(\text{Inv}(\text{Shape}(X')), \text{Inv}(\text{Shape}(Y')))$$
$$Y \xrightarrow{\quad} Y' \xrightarrow{\quad} \text{Shape}(Y') \xrightarrow{\quad} \text{Inv}(\text{Shape}(Y'))$$


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$$\begin{array}{c} d(X, Y) \\ \swarrow \quad \searrow \\ \end{array}$$

The entire procedure has to be *stable*  
i.e. *robust to noise and small perturbations*

# Do $X$ and $Y$ have a similar shape?



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$$\begin{array}{c} d(X, Y) \\ \swarrow \quad \searrow \\ \end{array}$$

In other words,  
*Similar Data  $\Rightarrow$  Similar Algebraic Invariants*

# Persistent Homology

***From an Input Data  $X$  to a Topological Representation  $X'$ :***

Let  $X$  be a

We set  $\mathbf{X}' := \{ X^p \}_{p \in \mathbb{R}}$ , where, chosen a value  $p \in \mathbb{R}$ ,  $X^p$  is defined as

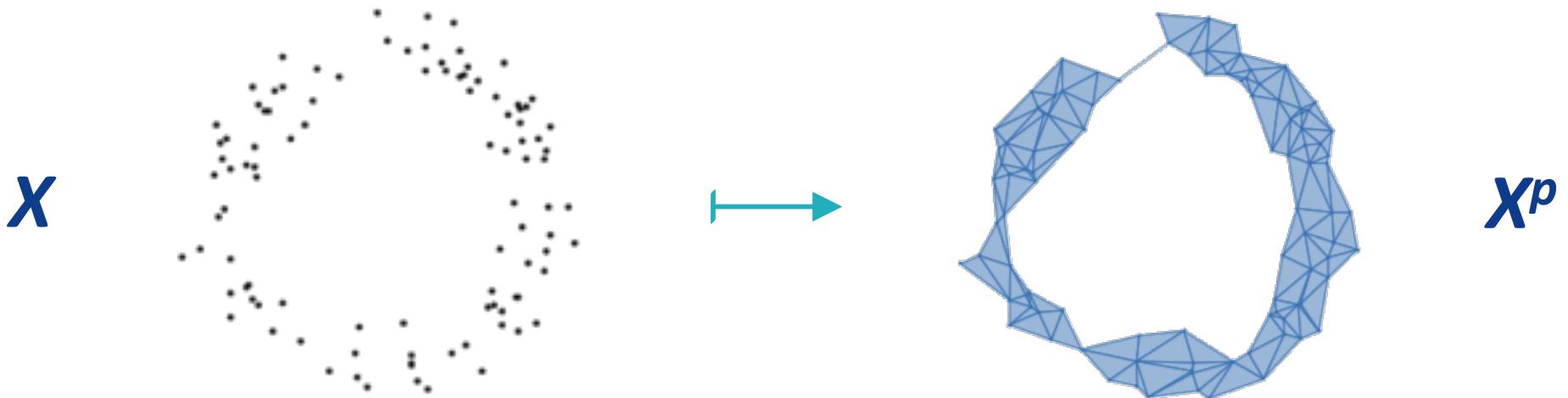
# Persistent Homology

**From an Input Data  $X$  to a Topological Representation  $X'$ :**

Let  $X$  be a **point cloud**  $V \subseteq \mathbb{R}^d$

We set  $X' := \{X^p\}_{p \in \mathbb{R}}$ , where, chosen a value  $p \in \mathbb{R}$ ,  $X^p$  is defined as

$$\{\sigma \subseteq V \mid d(u, v) \leq p, \forall u, v \in \sigma\}$$



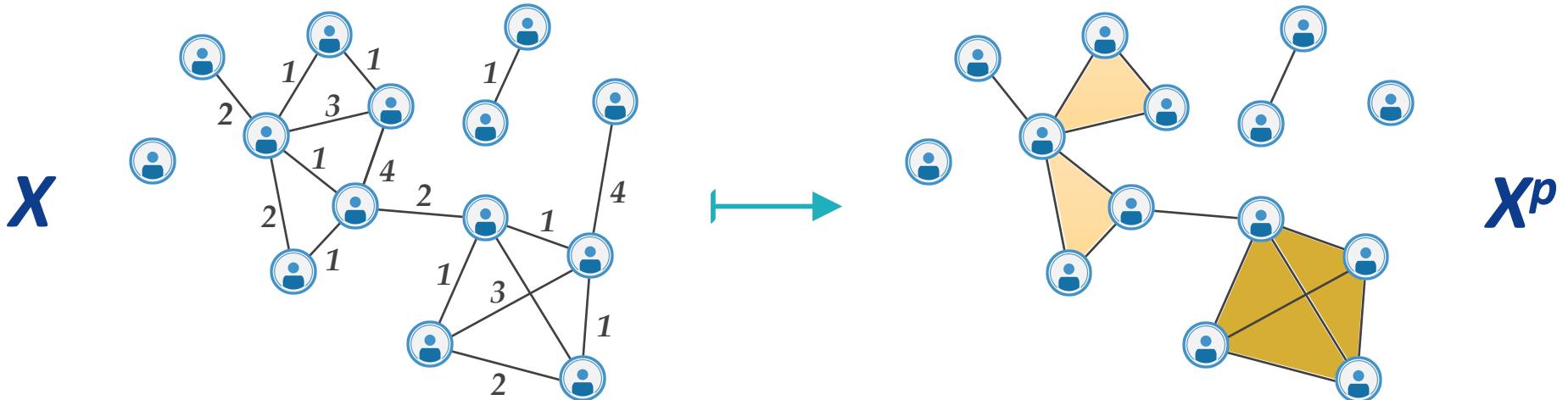
# Persistent Homology

**From an Input Data  $X$  to a Topological Representation  $X'$ :**

Let  $X$  be a **weighted graph**  $G := (V, E, w: E \rightarrow \mathbb{R})$

We set  $X' := \{X^p\}_{p \in \mathbb{R}}$ , where, chosen a value  $p \in \mathbb{R}$ ,  $X^p$  is defined as

$$\{\sigma \subseteq V \mid w(u, v) \leq p, \forall u, v \in \sigma\}$$



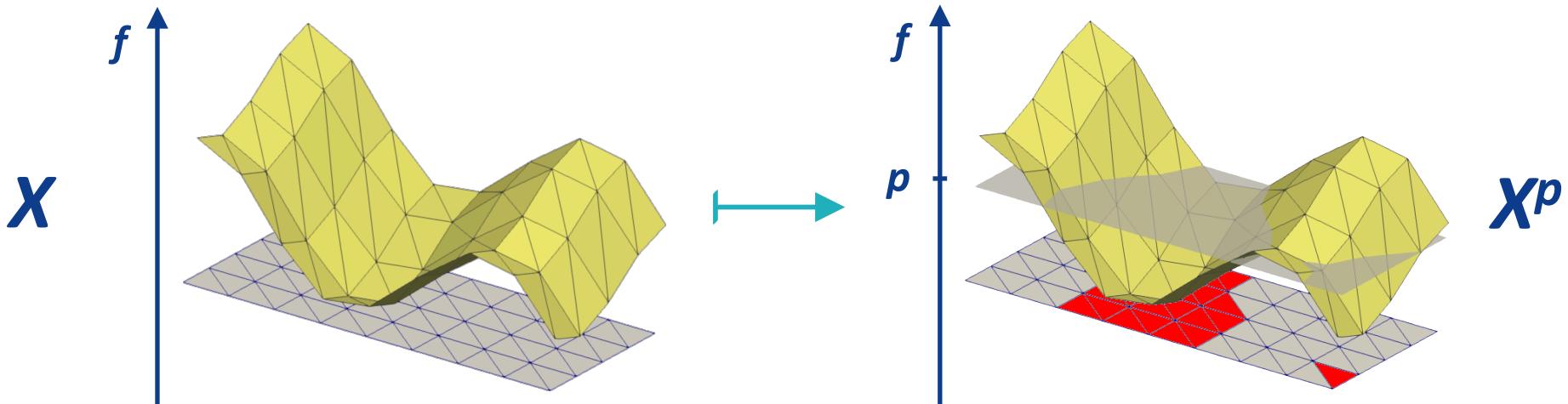
# Persistent Homology

**From an Input Data  $X$  to a Topological Representation  $X'$ :**

Let  $X$  be a **function**  $f: D \rightarrow \mathbb{R}$

We set  $X' := \{X^p\}_{p \in \mathbb{R}}$ , where, chosen a value  $p \in \mathbb{R}$ ,  $X^p$  is defined as

$$\{ \sigma \in D \mid f(\sigma) \leq p \}$$



# Persistent Homology

**From an *Input Data X* to a *Topological Representation X'*:**

*Independently from the construction procedure,  
the collection of topological spaces  $\{X^p\}_{p \in \mathbb{R}}$ , called **filtration** of X,  
satisfies that, for any  $p, q \in \mathbb{R}$  such that  $p \leq q$ ,*

$$X^p \subseteq X^q$$

# Persistent Homology

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**Working Assumption:**

*We can always pretend that parameter  $p$  varies over  $\mathbb{N}$*

# Persistent Homology

**From a Topological Representation  $X'$  to an Algebraic Structure  $\text{Shape}(X')$ :**

Given a filtration  $X' := \{X^p\}_{p \in \mathbb{N}}$ , a value  $i \in \mathbb{N}$ , and a field  $\mathbb{F}$ , the  $i^{\text{th}}$  persistence module  $M$  of  $X'$  over  $\mathbb{F}$  is defined as the **finitely generated graded  $\mathbb{F}[x]$ -module**

$$M := \bigoplus_{p \in \mathbb{N}} M_p$$

where:

- ◆  $M_p := H_i(X^p; \mathbb{F})$ , the set of **homogeneous elements of grade  $p$**
- ◆ The **action  $x^{q-p} h$  over an element  $h$  of grade  $p$**  is defined as  $\mu_{i,p,q}(h)$ , where:
  - ❖  $\mu_{i,p,q}(h) : H_i(X^p; \mathbb{F}) \rightarrow H_i(X^q; \mathbb{F})$  is the linear map induced by the inclusion  $X^p \subseteq X^q$

# Persistent Homology

*From an Algebraic Structure  $\text{Shape}(X')$  to an Algebraic Invariant  $\text{Inv}(\text{Shape}(X'))$ :*

*Theorem (structure for finitely generated graded modules over a PID):*

Any persistence module  $M$  can be expressed as

$$M \cong \bigoplus_{k=1}^n \mathbb{F}[x](-r_k) \oplus \bigoplus_{j=1}^m \left( \mathbb{F}[x]/(x^{q_j-p_j}) \right) (-p_j)$$

So,  $M$  is completely determined by the collection of values  $r_k$  and of pairs  $(p_j, q_j)$

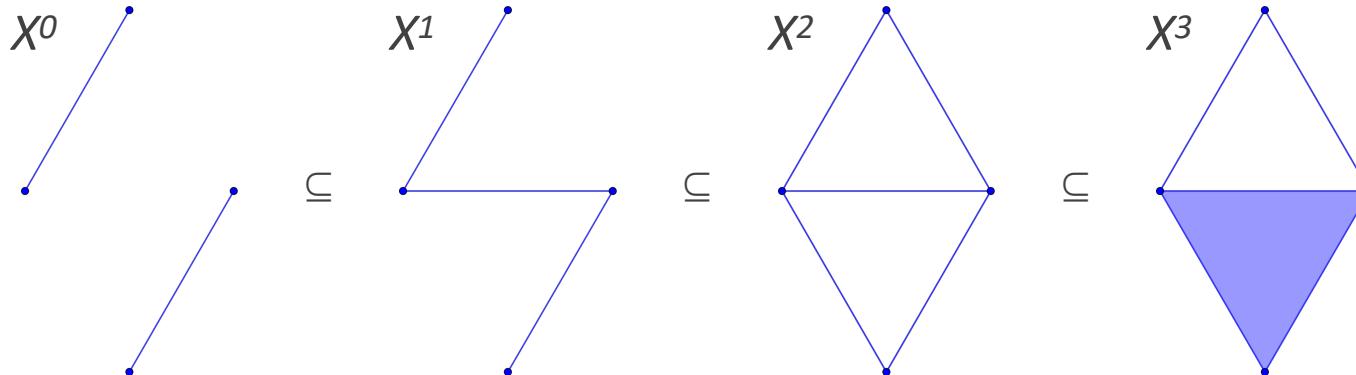
Such descriptors are typically expressed as pairs, called **persistence pairs** of  $M$ , of

the kind  $(r_k, \infty)$  and  $(p_j, q_j)$

# Persistent Homology

*Intuitively:*

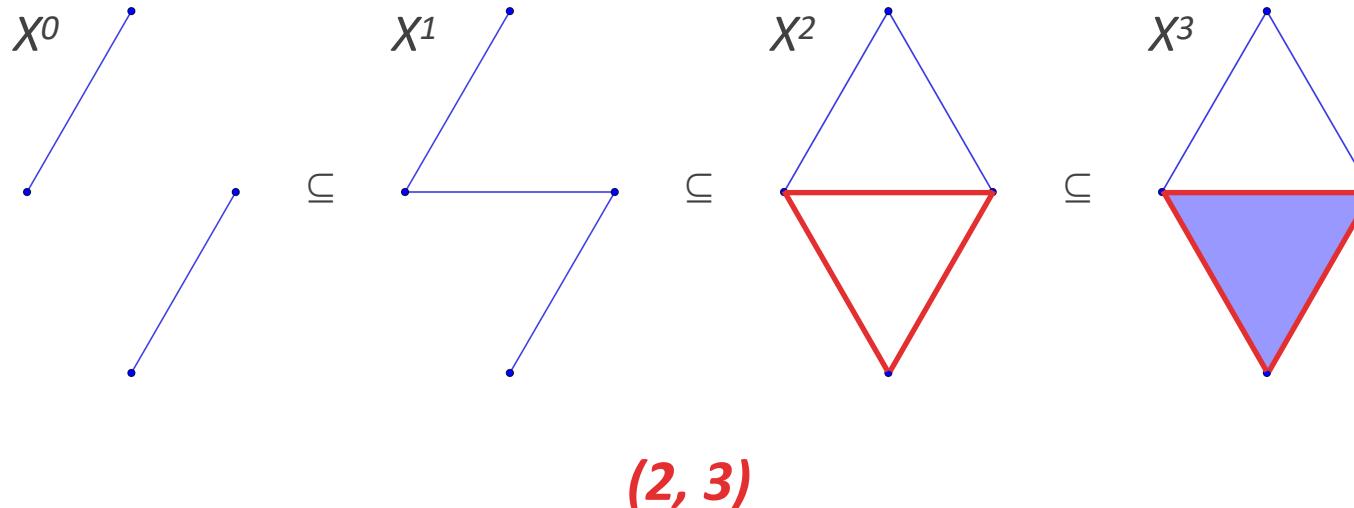
Given a filtration  $X' := \{X^p\}_{p \in \mathbb{N}}$ , a **persistence pair**  $(p, q) \in \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$  with  $p < q$  represents a **homological class** that is **born at step p** and **dies at step q**



# Persistent Homology

*Intuitively:*

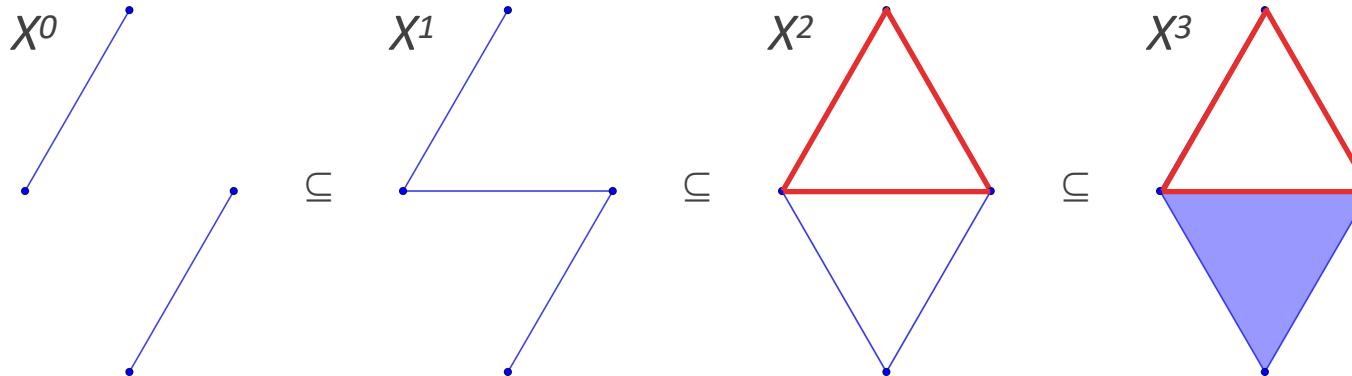
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# Persistent Homology

*Intuitively:*

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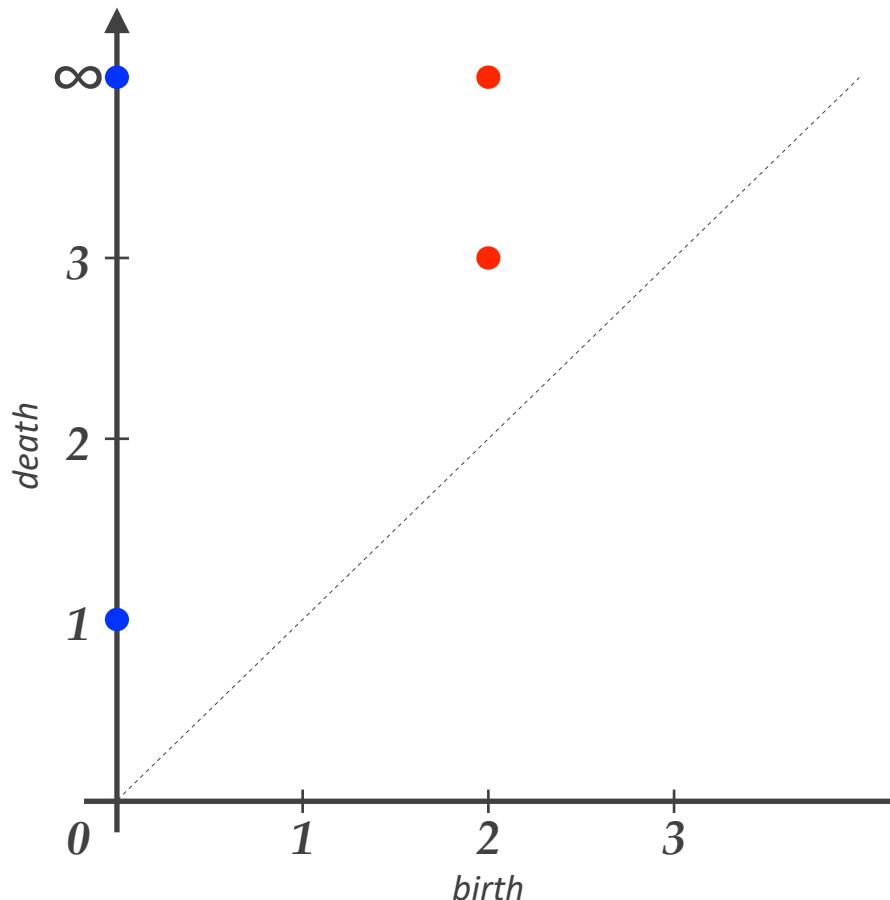


**(2,  $\infty$ ) essential pair**

# Persistent Homology

## Persistence Diagrams:

Persistence pairs are represented as *points* in  $\mathbb{R} \times (\mathbb{R} \cup \{\infty\})$



$H_0$	$(0, 1)$	$H_1$	$(2, 3)$
	$(0, \infty)$		$(2, \infty)$

Formally, a persistence diagram is a *multi-set*  
i.e. points are endowed with *multiplicity*

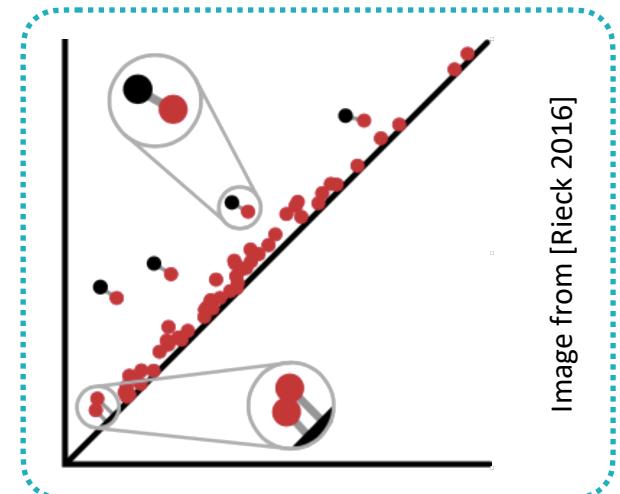
# Persistent Homology

**From an Algebraic Invariant  $\text{Inv}(\text{Shape}(X'))$  to a Similarity Check  $d(X, Y)$ :**

Given two persistence diagrams  $D_X$  and  $D_Y$ , their **bottleneck distance  $d_B$**  is defined as

$$d_B(D_X, D_Y) := \inf_{\gamma} \left\{ \sup_{x \in D_X} \left\{ \|x - \gamma(x)\|_{\infty} \right\} \right\}$$

where  $\gamma$  ranges over all bijections from  $D_X$  to  $D_Y$



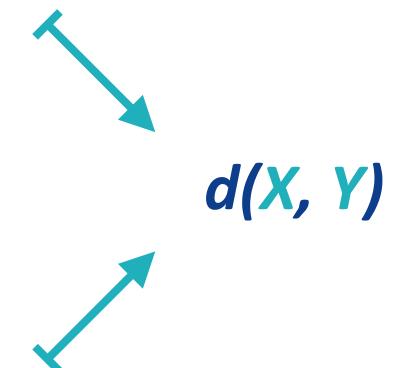
# Do $X$ and $Y$ have a similar shape?

*Persistent Homology Pipeline:*



$X \longleftrightarrow X' \longleftrightarrow Shape(X') \longleftrightarrow Inv(Shape(X'))$

$Y \longleftrightarrow Y' \longleftrightarrow Shape(Y') \longleftrightarrow Inv(Shape(Y'))$

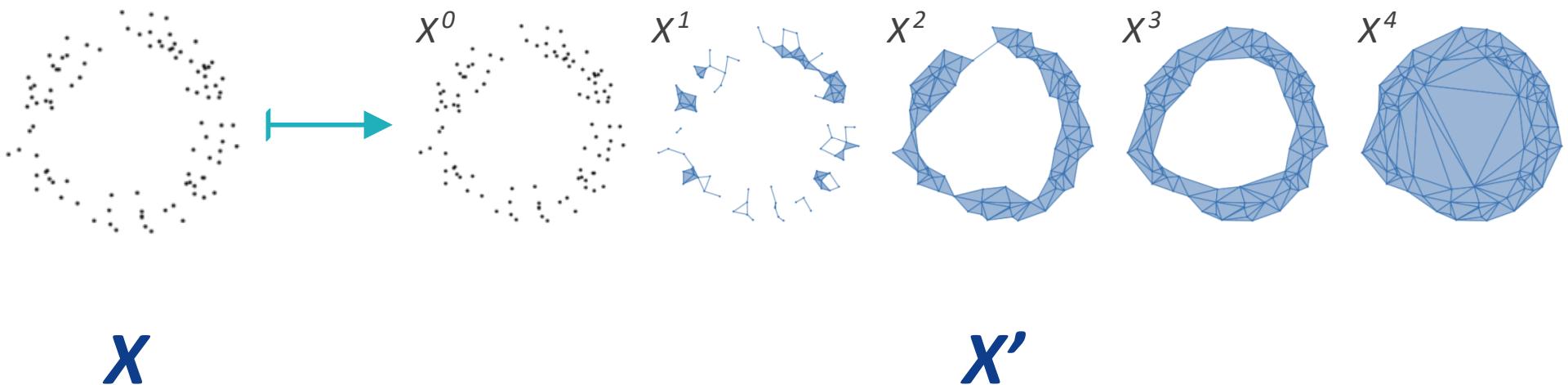


# *Do $X$ and $Y$ have a similar shape?*

*Persistent Homology Pipeline:*

*Input  
Data*

*Topological  
Representation*

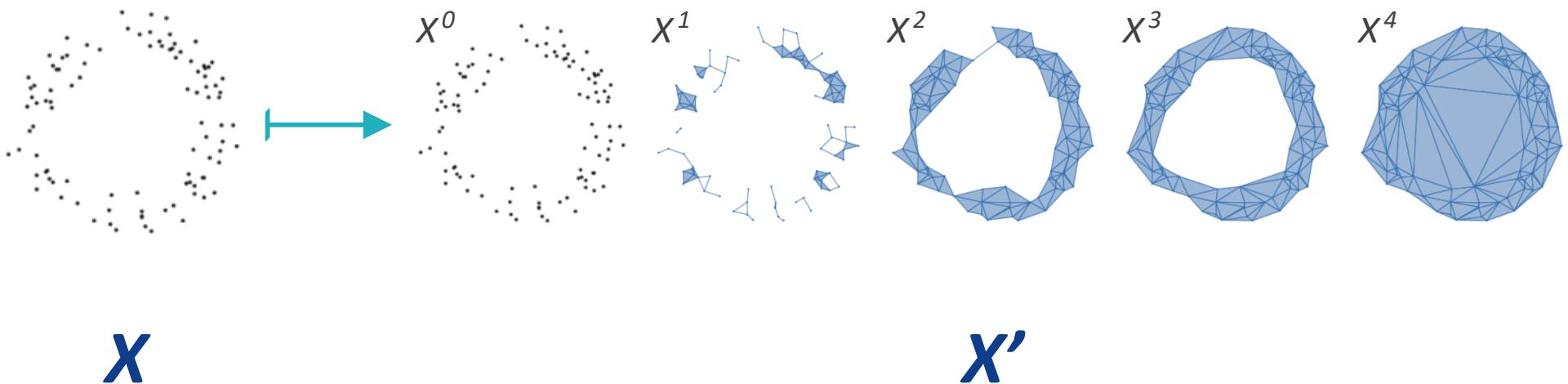


# *Do $X$ and $Y$ have a similar shape?*

*Persistent Homology Pipeline:*

*Input  
Data*

*Topological  
Representation*

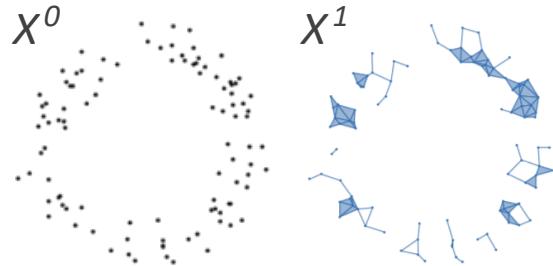


# Do $X$ and $Y$ have a similar shape?

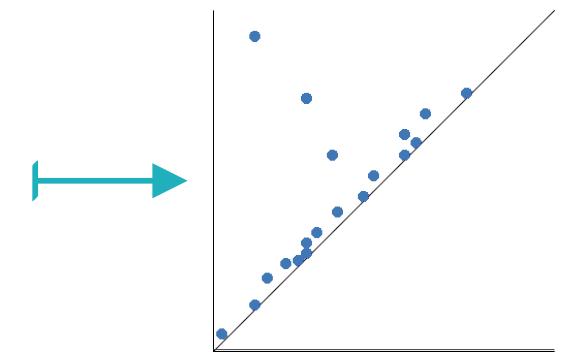
**Persistent Homology Pipeline:**

**Topological  
Representation**

**Algebraic  
Invariant**



**$X'$**



**$\text{Inv}(\text{Shape}(X'))$**

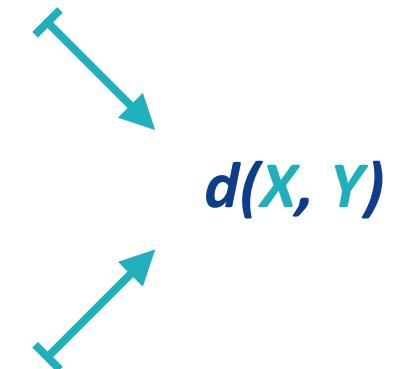
# Do $X$ and $Y$ have a similar shape?

*Persistent Homology Pipeline:*



$X \longleftrightarrow X' \longleftrightarrow Shape(X') \longleftrightarrow Inv(Shape(X'))$

$Y \longleftrightarrow Y' \longleftrightarrow Shape(Y') \longleftrightarrow Inv(Shape(Y'))$



# Do $X$ and $Y$ have a similar shape?

*AI-Oriented Alternative:*

*Input  
Data*

*Topological  
Representation*

*Algebraic  
Structure*

*Feature  
Vector*

*Learning  
Process*

$$X \longleftrightarrow X' \longleftrightarrow \text{Shape}(X') \longleftrightarrow \phi(\text{Shape}(X'))$$

$$Y \longleftrightarrow Y' \longleftrightarrow \text{Shape}(Y') \longleftrightarrow \phi(\text{Shape}(Y'))$$

$$k(X, Y)$$

# Do $X$ and $Y$ have a similar shape?

*AI-Oriented Alternative:*

*Input  
Data*

*Topological  
Representation*

*Algebraic  
Structure*

*Feature  
Vector*

*Learning  
Process*

$$X \longleftrightarrow X' \longleftrightarrow \text{Shape}(X') \longleftrightarrow \phi(\text{Shape}(X'))$$

$$Y \longleftrightarrow Y' \longleftrightarrow \text{Shape}(Y') \longleftrightarrow \phi(\text{Shape}(Y'))$$

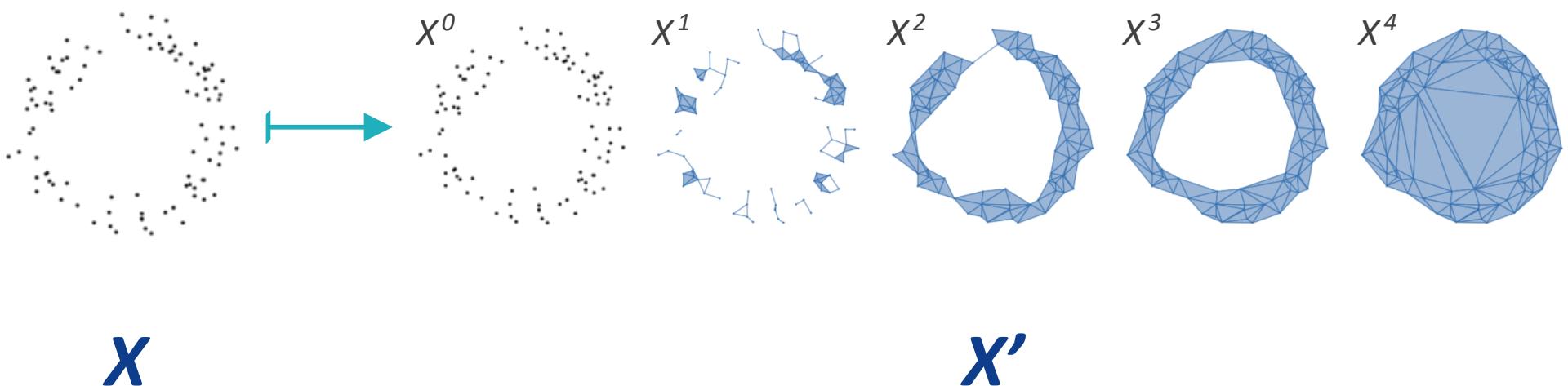
$$k(X, Y)$$

# *Do $X$ and $Y$ have a similar shape?*

*AI-Oriented Alternative:*

*Input  
Data*

*Topological  
Representation*

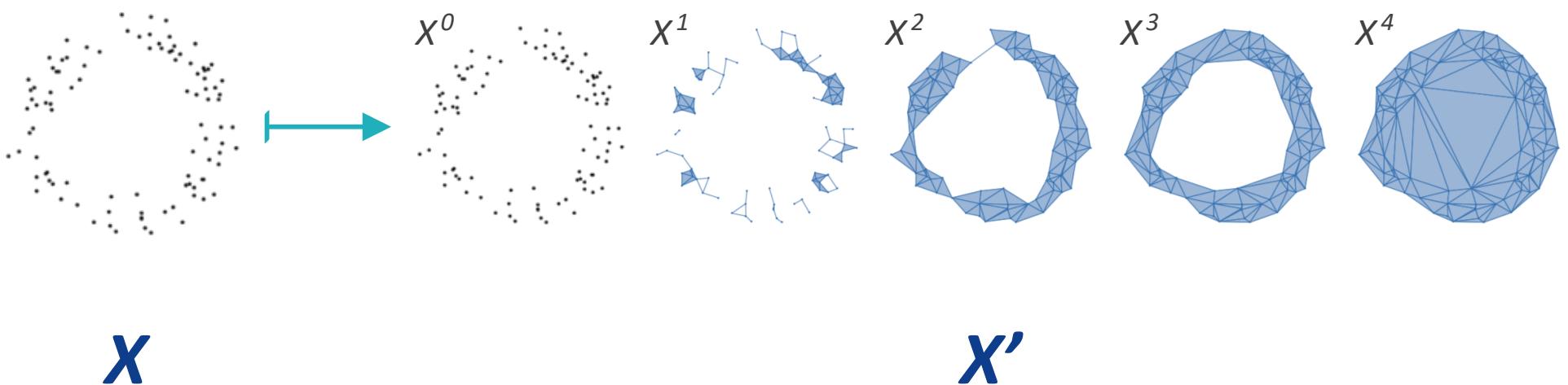


# *Do $X$ and $Y$ have a similar shape?*

*AI-Oriented Alternative:*

*Input  
Data*

*Topological  
Representation*

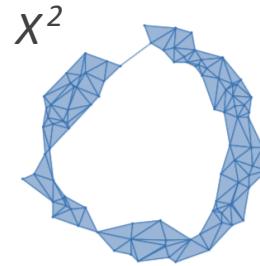
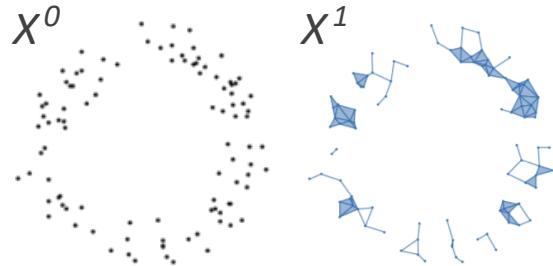


# Do $X$ and $Y$ have a similar shape?

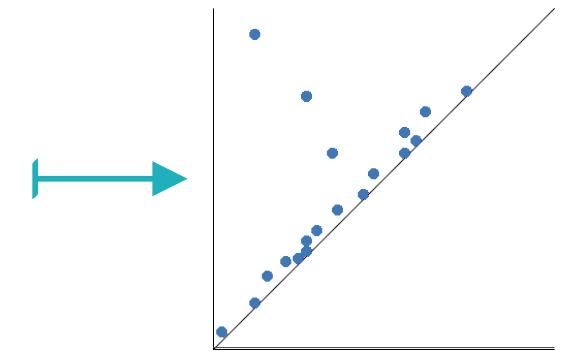
AI-Oriented Alternative:

Topological  
Representation

Algebraic  
Invariant



$X'$



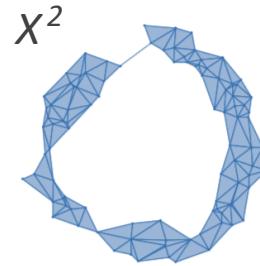
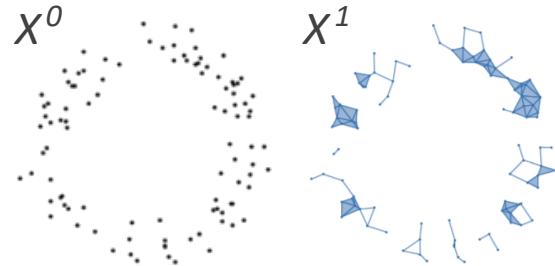
$\text{Inv}(\text{Shape}(X'))$

# Do $X$ and $Y$ have a similar shape?

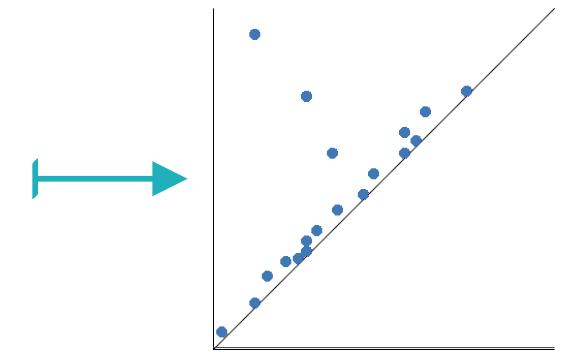
AI-Oriented Alternative:

Topological  
Representation

Algebraic  
Invariant



$X'$



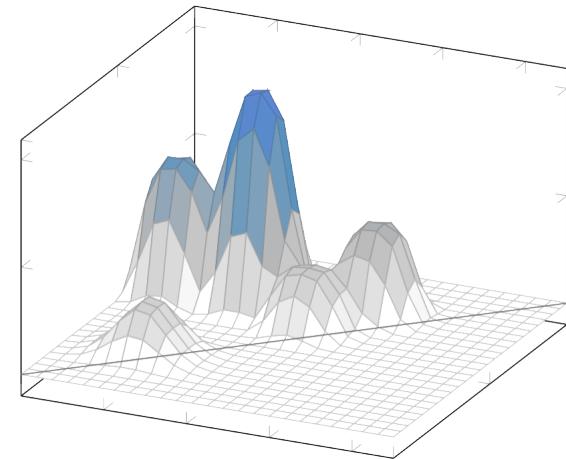
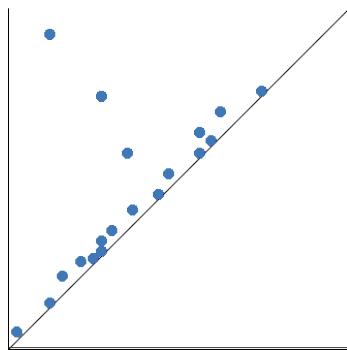
$\text{Inv}(\text{Shape}(X'))$

# *Do $X$ and $Y$ have a similar shape?*

*AI-Oriented Alternative:*

*Algebraic  
Invariant*

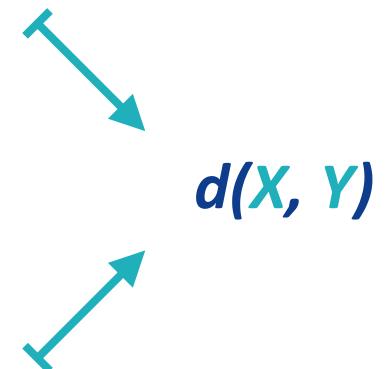
*Feature  
Vector*



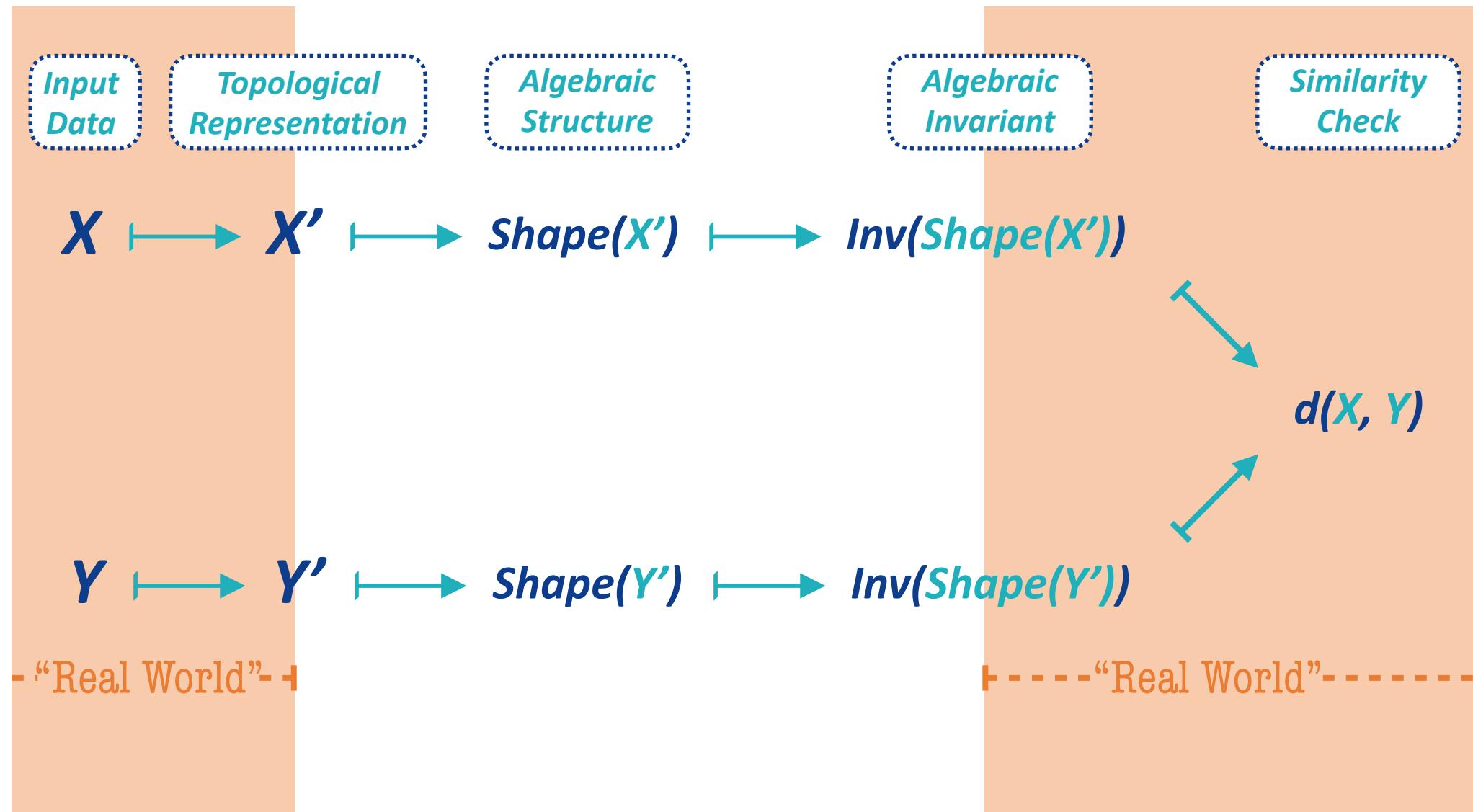
$\text{Inv}(\text{Shape}(X'))$

$\phi(\text{Shape}(X'))$

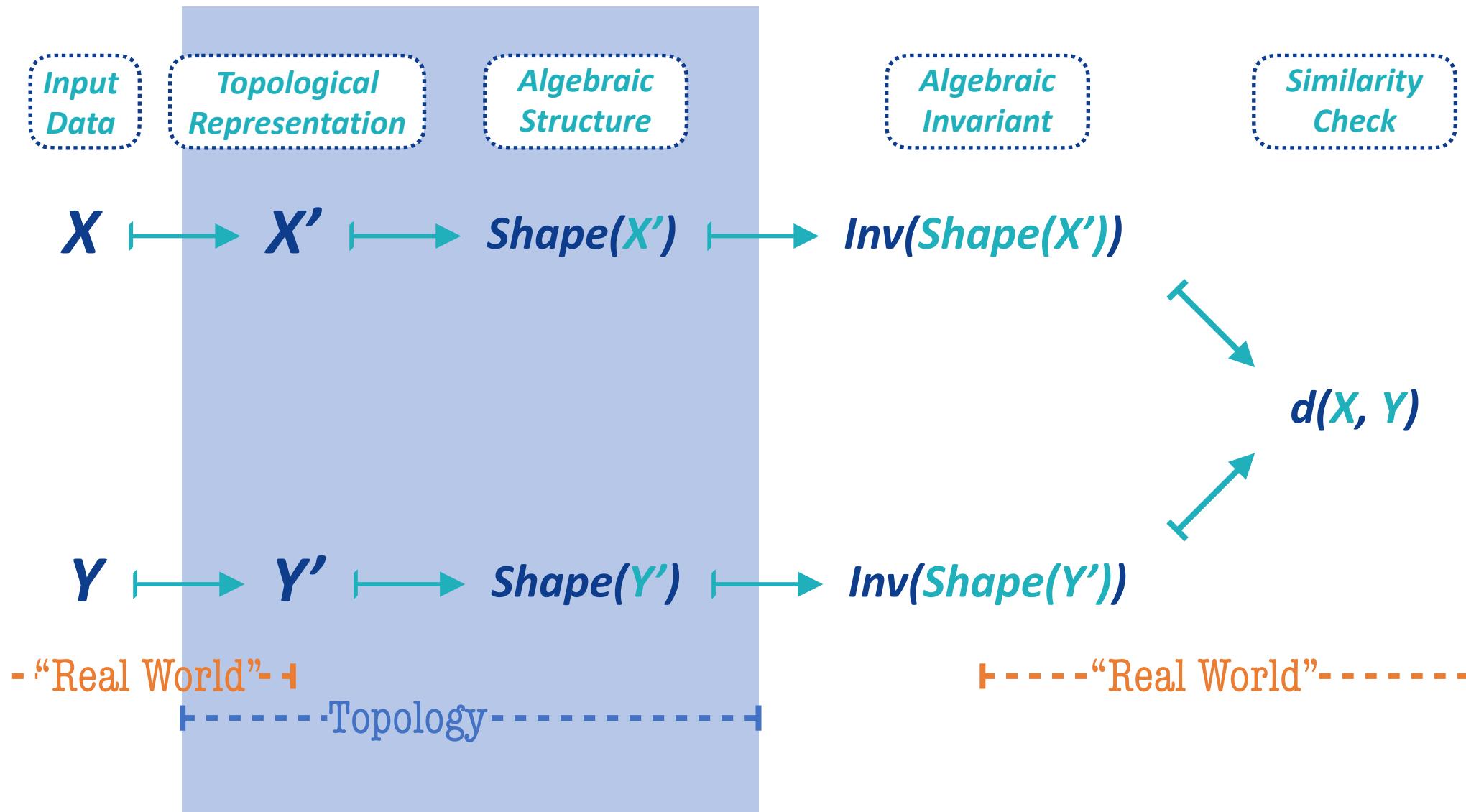
# Do $X$ and $Y$ have a similar shape?


$$X \xrightarrow{\quad} X' \xrightarrow{\quad} \text{Shape}(X') \xrightarrow{\quad} \text{Inv}(\text{Shape}(X'))$$
$$Y \xrightarrow{\quad} Y' \xrightarrow{\quad} \text{Shape}(Y') \xrightarrow{\quad} \text{Inv}(\text{Shape}(Y'))$$


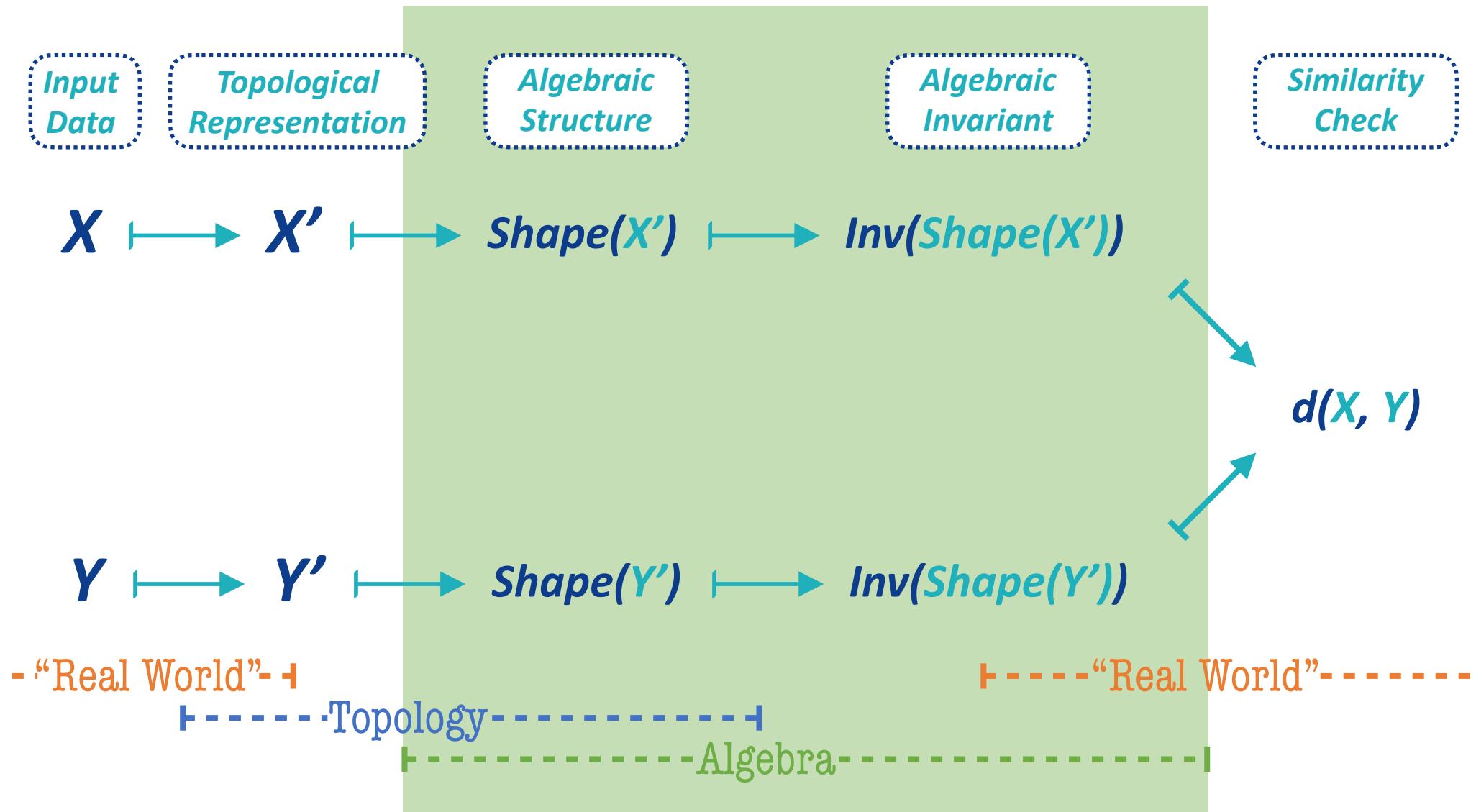
# Do $X$ and $Y$ have a similar shape?



# Do $X$ and $Y$ have a similar shape?



# Do $X$ and $Y$ have a similar shape?



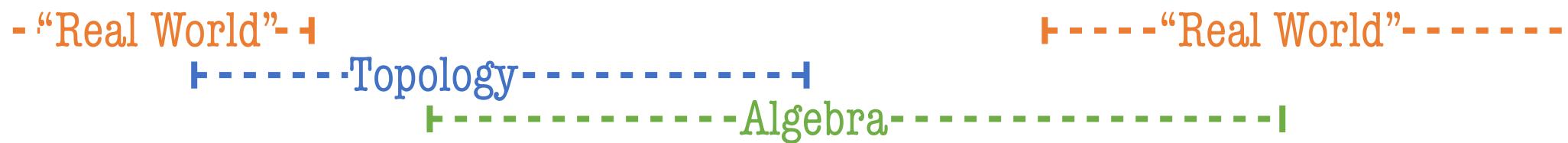
# Do $X$ and $Y$ have a similar shape?



$$X \xrightarrow{\quad} X' \xrightarrow{\quad} \text{Shape}(X') \xrightarrow{\quad} \text{Inv}(\text{Shape}(X'))$$

$$\begin{array}{c} d(X, Y) \\ \longleftrightarrow \\ \begin{matrix} & & & \\ & & & \\ & & & \end{matrix} \\ d(X, Y) \end{matrix}$$

$$Y \xrightarrow{\quad} Y' \xrightarrow{\quad} \text{Shape}(Y') \xrightarrow{\quad} \text{Inv}(\text{Shape}(Y'))$$



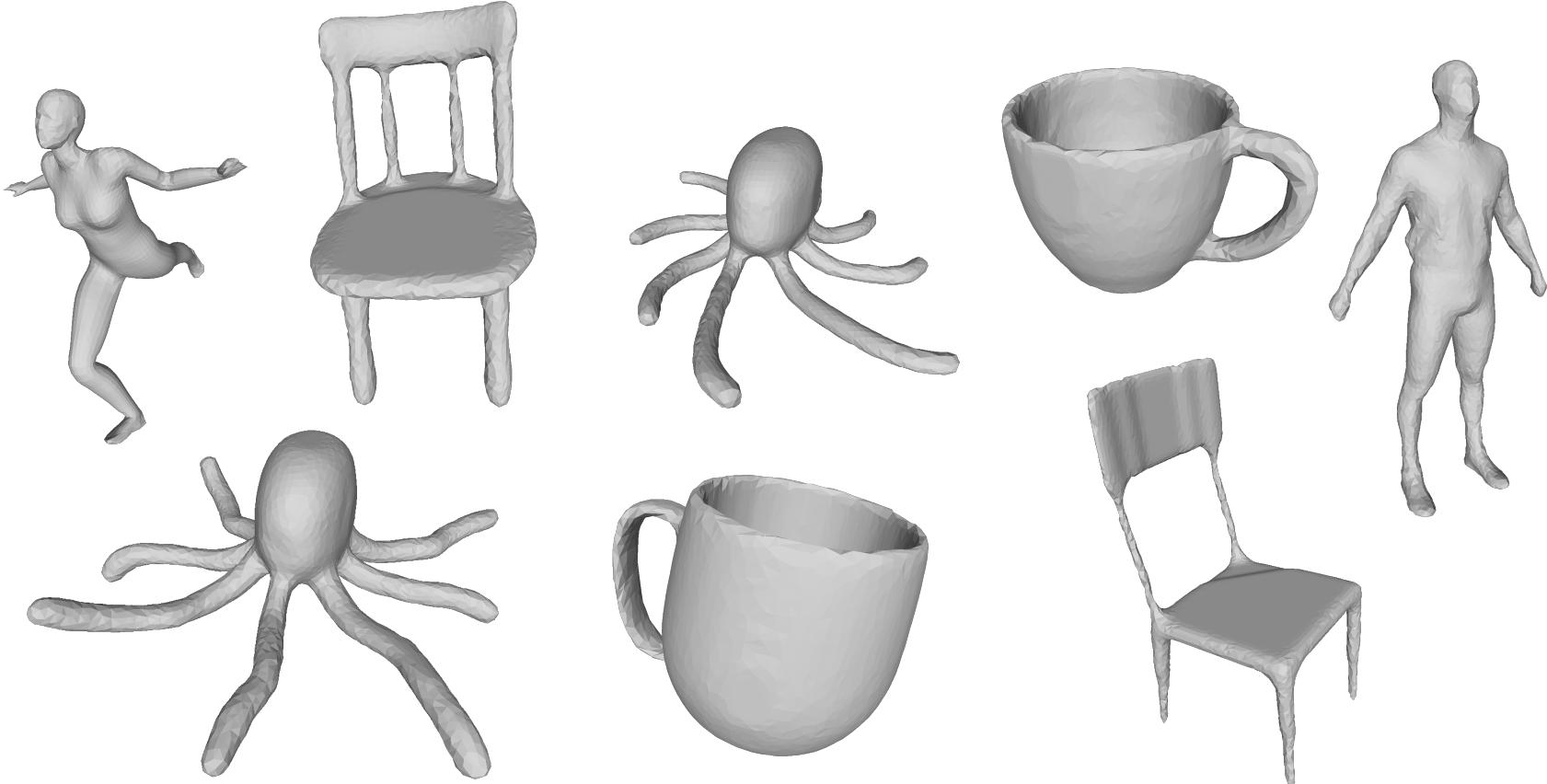
# **Persistent Homology for Shape Comparison**

- ◆ *Quick Overview on TDA*
  
- ◆ ***Case Study***

# Case Study

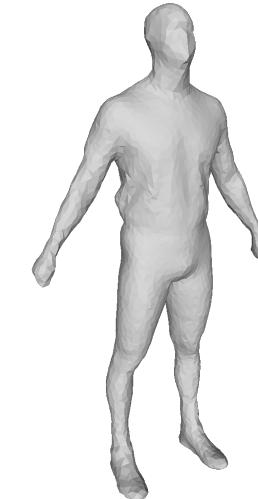
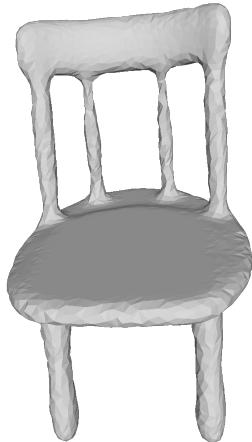
**Goal:**

*Given a dataset of 80 surfaces, classify them on the basis of their shape*



# Case Study

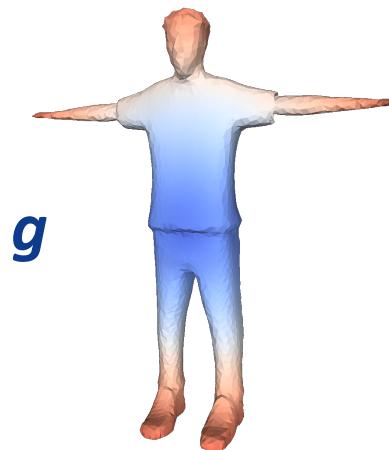
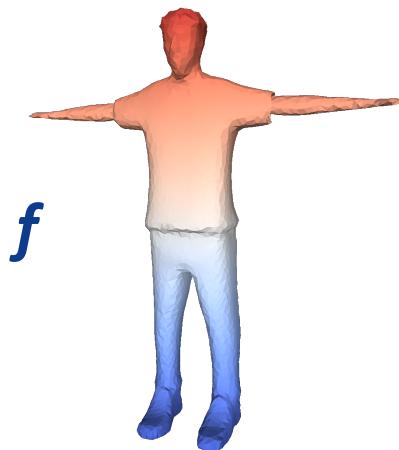
*Homology is not enough*



# Case Study

*In order to adopt **persistent homology**,  
we need to filter each surface through a function:*

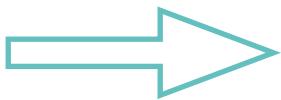
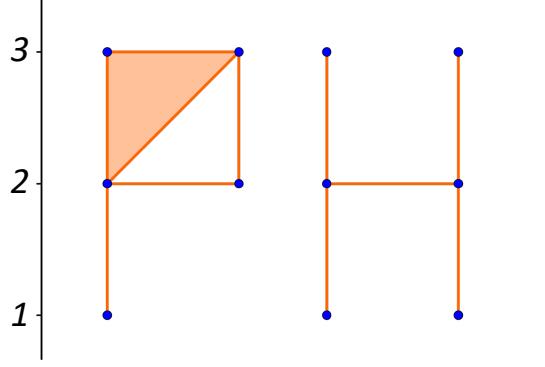
- ◆  $f(x, y, z) := z$
- ◆  $g(x, y, z) := \text{distance of } (x, y, z) \text{ from the model barycentre}$



# Persistent Homology Computation

## Standard Algorithm:

From:



[Zomorodian & Carlsson 2005]

To:

$[1, 2]$

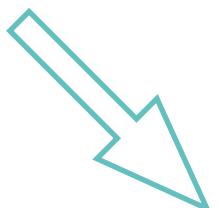
$H_0$

$[1, \infty)$

$H_1$

$[3, \infty)$

$[1, \infty)$



$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1																							
2																							
3																							
4																							
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17																							
18																							
19																							
20																							
21																							
22																							
23																							
low								4	6	7	5	3					13	14	15	16	22		



Compute a **reduced boundary matrix** for  $K^f$  from which easily read the persistence pairs

# Persistent Homology Computation

Given a filtered simplicial complex, let us consider its *filtering function*  $f$ :

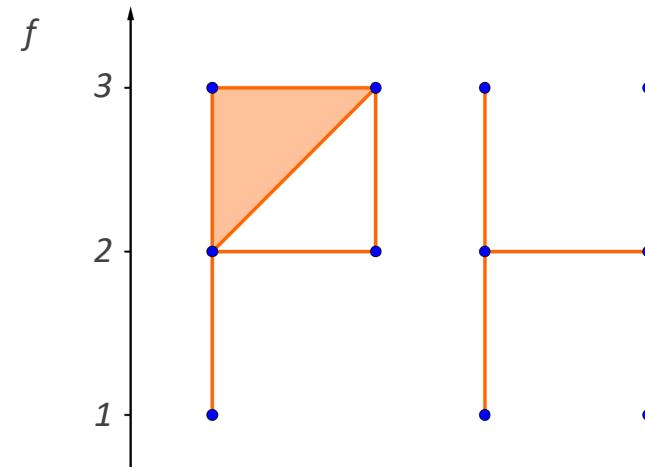
$$f(\sigma) := \min \{ p \mid \sigma \in K^p \}$$

Conversely,  $K^p := \{ \sigma \in K \mid f(\sigma) \leq p \}$

**Total Ordering on  $K^f$ :**

A sequence  $\sigma_1, \sigma_2, \dots, \sigma_n$  of the simplices of  $K^f$  such that:

- ◆ if  $f(\sigma_i) < f(\sigma_j)$ , then  $i < j$
- ◆ if  $\sigma_i$  is a proper face of  $\sigma_j$ , then  $i < j$



# Persistent Homology Computation

Given a filtered simplicial complex, let us consider its *filtering function*  $f$ :

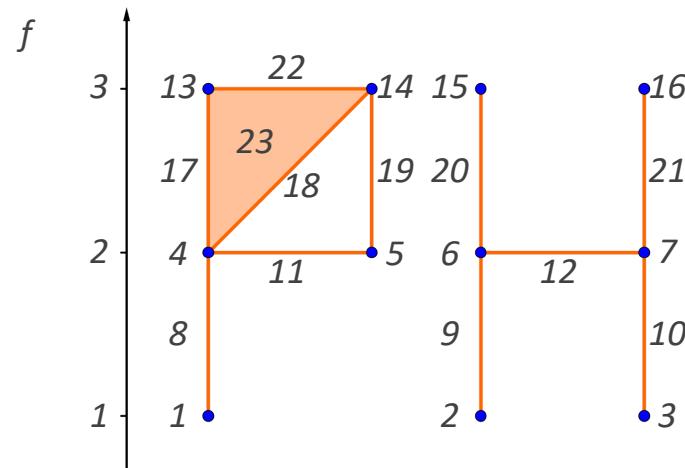
$$f(\sigma) := \min \{ p \mid \sigma \in K^p \}$$

Conversely,  $K^p := \{ \sigma \in K \mid f(\sigma) \leq p \}$

**A Possible Choice:**

Set  $\sigma < \sigma'$  if:

- ◆  $f(\sigma) < f(\sigma')$
- ◆  $f(\sigma) = f(\sigma')$  and  $\dim(\sigma) < \dim(\sigma')$
- ◆  $f(\sigma) = f(\sigma')$ ,  $\dim(\sigma) = \dim(\sigma')$ , and  $\sigma$  precedes  $\sigma'$  w.r.t. the *lexicographic order* of their vertices

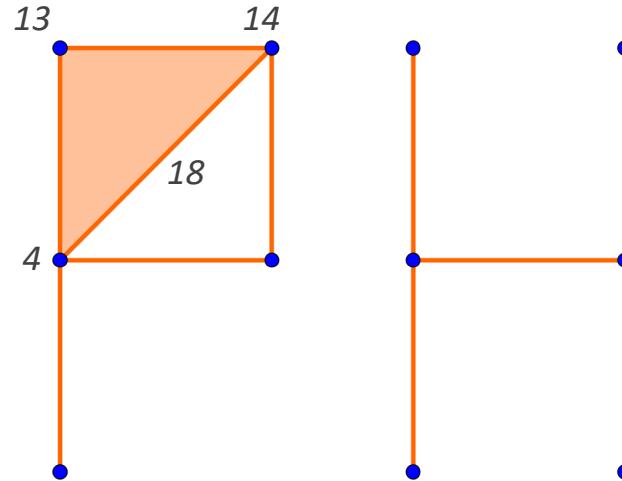


# Persistent Homology Computation

## Boundary Matrix:

A square matrix  $\mathbf{M}$  of size  $n \times n$  defined by

$$M_{i,j} := \begin{cases} 1 & \text{if } \sigma_i \text{ is a face of } \sigma_j \text{ s.t. } \dim(\sigma_i) = \dim(\sigma_j) - 1 \\ 0 & \text{otherwise} \end{cases}$$



E.g.

- ◆  $M_{4,18} = 1$
- ◆  $M_{14,18} = 1$
- ◆  $M_{13,18} = 0$

# Persistent Homology Computation

## **Reduced Matrix:**

Given a non-null column  $j$  of a boundary matrix  $M$ ,

$$\text{low}(j) := \max \{ i \mid M_{i,j} \neq 0 \}$$

A matrix  $R$  is called **reduced** if, for each pair of non-null columns  $j_1, j_2$ ,

$$\text{low}(j_1) \neq \text{low}(j_2)$$

**Equivalently**, if low function is **injective** on its domain of definition

# Persistent Homology Computation

$i \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
1								1																
2									1															
3										1														
4										1		1						1	1					
5											1									1				
6											1		1								1			
7											1		1									1		
8																								
9																								
10																								
11																								
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13																	1					1		
14																		1	1				1	
15																			1					
16																				1				
17																							1	
18																							1	
19																								
20																								
21																								
22																							1	
23																								
low								4	6	7	5	7						13	14	14	15	16	14	22

$low(10) = 7 = low(12)$



$M$  is not reduced

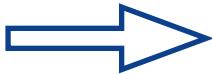
# Persistent Homology Computation

## Reduction Algorithm:

```
Matrix  $R = M$ 
for  $j = 1, \dots, n$  do
    while  $\exists j' < j$  with  $\text{low}(j') = \text{low}(j)$  do
         $R.\text{column}(j) = R.\text{column}(j) + R.\text{column}(j')$ 
    endwhile
endfor
return  $R$ 
```

## Time Complexity:

At most  $n^2$  column additions



$O(n^3)$  in the worst case

$i \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
1								1																
2									1															
3										1														
4							1				1							1	1					
5											1									1				
6									1			1									1			
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19																								
20																								
21																								
22																							1	
23																								
low								4	6	7	5	7						13	14	14	15	16	14	22

Initialize  $\mathbf{R}$  to  $\mathbf{M}$ , where

$\mathbf{M}$  is the **boundary matrix** of  $K^f$

expressed according with a **total ordering** of its simplices

$i \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1				1										
3										1		1											
4								1			1							1	1				
5												1											
6									1												1		
7										1												1	
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19																							
20																							
21																							
22																						1	
23																							
low								4	6	7	5	3						13	14		15	16	22

The algorithm returns the above **reduced matrix  $R$**

# Persistent Homology Computation

## Retrieving Persistence Pairs:

- ◆ For each  $i = 1, \dots, n$ ,  
if there exists  $j$  such that  $\text{low}(j) = i$    $[i, j]$  is a pair for  $R$
- ◆ Once every  $i$  has been parsed,  
if  $i$  is an **unpaired** value   $[i, \infty)$  is a pair for  $R$

From pairs of  $R$  to the “**actual**” persistence pairs of  $K^f$ :

$[i, j]$  corresponds to  $[f(\sigma_i), f(\sigma_j)]$

( homological degree =  $\dim(\sigma_i)$  )

$[i, \infty)$  corresponds to  $[f(\sigma_i), \infty)$

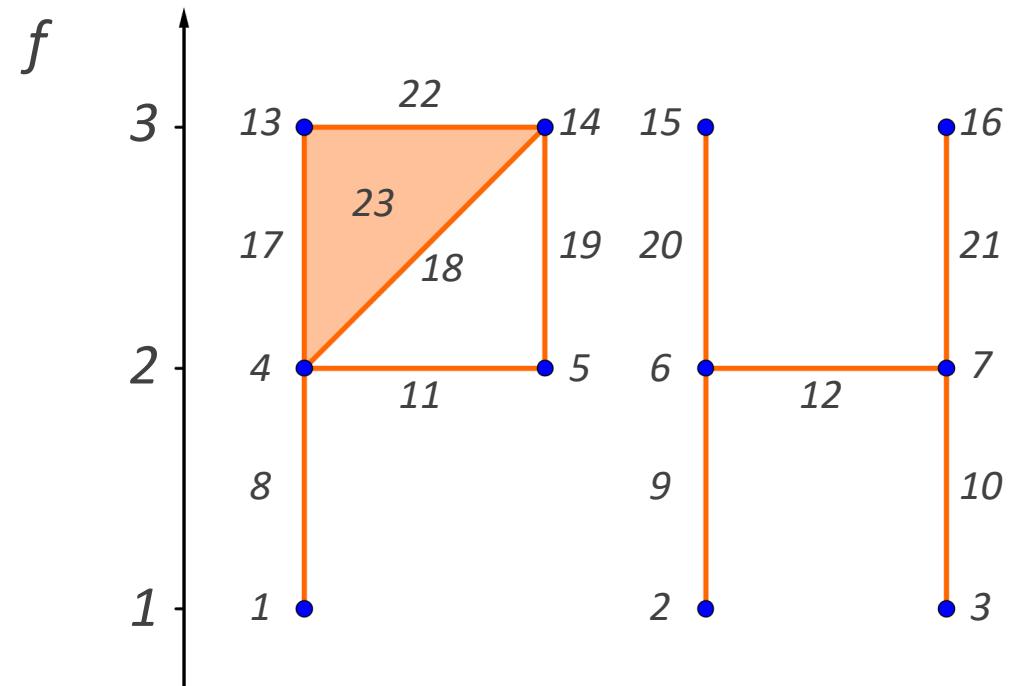
# Persistent Homology Computation

$H_0$   
 $[1, \infty)$   
 $[2, \infty)$   
 $[3, 12]$   
 $[4, 8]$   
 $[5, 11]$   
 $[6, 9]$   
 $[7, 10]$   
 $[13, 17]$   
 $[14, 18]$   
 $[15, 20]$   
 $[16, 21]$   
  
 $H_1$   
 $[19, \infty)$   
 $[22, 23]$

$i \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23		
1								1																	
2									1				1												
3										1			1												
4										1			1						1	1					
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19																									
20																									
21																									
22																							1		
23																									
low								4	6	7	5	3						13	14		15	16		22	

# Persistent Homology Computation

	$H_0$
$[1, \infty)$	$[1, \infty)$
$[2, \infty)$	$[1, \infty)$
$[3, 12]$	$[1, 2]$
$[4, 8]$	$[2, 2]$
$[5, 11]$	$[2, 2]$
$[6, 9]$	$[2, 2]$
$[7, 10]$	$[2, 2]$
$[13, 17]$	$[3, 3]$
$[14, 18]$	$[3, 3]$
$[15, 20]$	$[3, 3]$
$[16, 21]$	$[3, 3]$



$H_1$      $[19, \infty)$      $[3, \infty)$   
 $[22, 23]$      $\rightarrow$      $[3, 3]$

# Software Packages

*Several software packages for computing persistent homology have been developed:*

- ◆ *javaPlex*
  - ◆ *jHoles*
  - ◆ *Dionysus*
  - ◆ *Perseus*
  - ◆ *PHAT*
  - ◆ *DIPHA*
  - ◆ *Gudhi*
  - ◆ *SimpPers*
  - ◆ *Ripser*
  - ◆ *giotto-tda*
  - ◆ *TDAstats*
- ◆ ...

# Software Packages

## **PHAT:**

- ◆ **Language:**
  - ❖ *C++* (with *Python* bindings)
- ◆ **Algorithms:**
  - ❖ *Standard, Dual, Twist, Chunk, Spectral Sequences*
- ◆ **Coefficient Fields:**
  - ❖  $\mathbb{Z}_2$
- ◆ **Homology:**
  - ❖ *Simplicial, Cubical*
- ◆ **Accepted Inputs:**
  - ❖ *Boundary Matrices*
- ◆ **Computed Filtrations:**
  - ❖ -
- ◆ **Visualizations:**
  - ❖ -
- ◆ **Additional Features:**
  - ❖ -

# Software Packages

## *Ripser:*

- ◆ **Language:**
  - ❖ C++ (with Python bindings)
- ◆ **Algorithms:**
  - ❖ Dual, Twist
- ◆ **Coefficient Fields:**
  - ❖  $\mathbb{Z}_p$
- ◆ **Homology:**
  - ❖ Simplicial
- ◆ **Accepted Inputs:**
  - ❖ Point Clouds, Distance Matrices

- ◆ **Computed Filtrations:**
  - ❖ Vietoris-Rips and Čech complexes, Alpha-Shapes, Lower Star of Cubical complexes
- ◆ **Visualizations:**
  - ❖ Persistence Diagrams (through Persim: Persistence Images)
- ◆ **Additional Features:**
  - ❖ Representative Cocycles (through Persim: Bottleneck distance, modified Gromov–Hausdorff distance, Sliced Wasserstein kernel, Heat kernel)