“Persistent Homology” Summer School - Rabat

Persistent Homology in Complex Network Analysis

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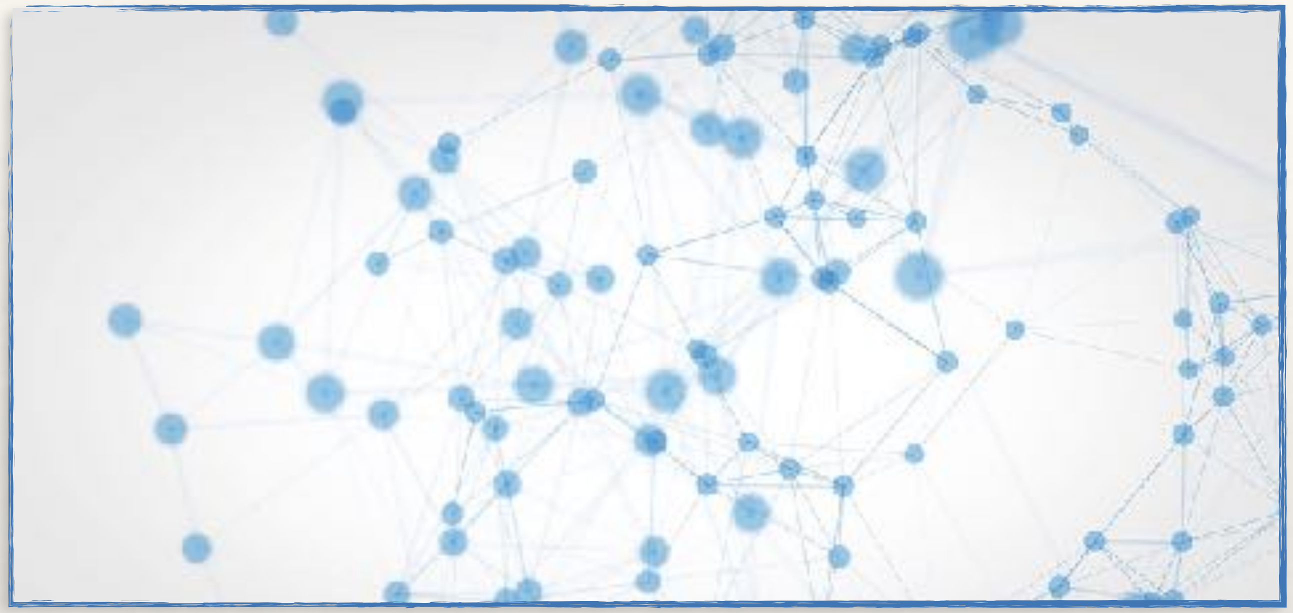
July 7, 2017

Anything has Shape



"Data has shape and
shape has meaning"

Gunnar Carlsson

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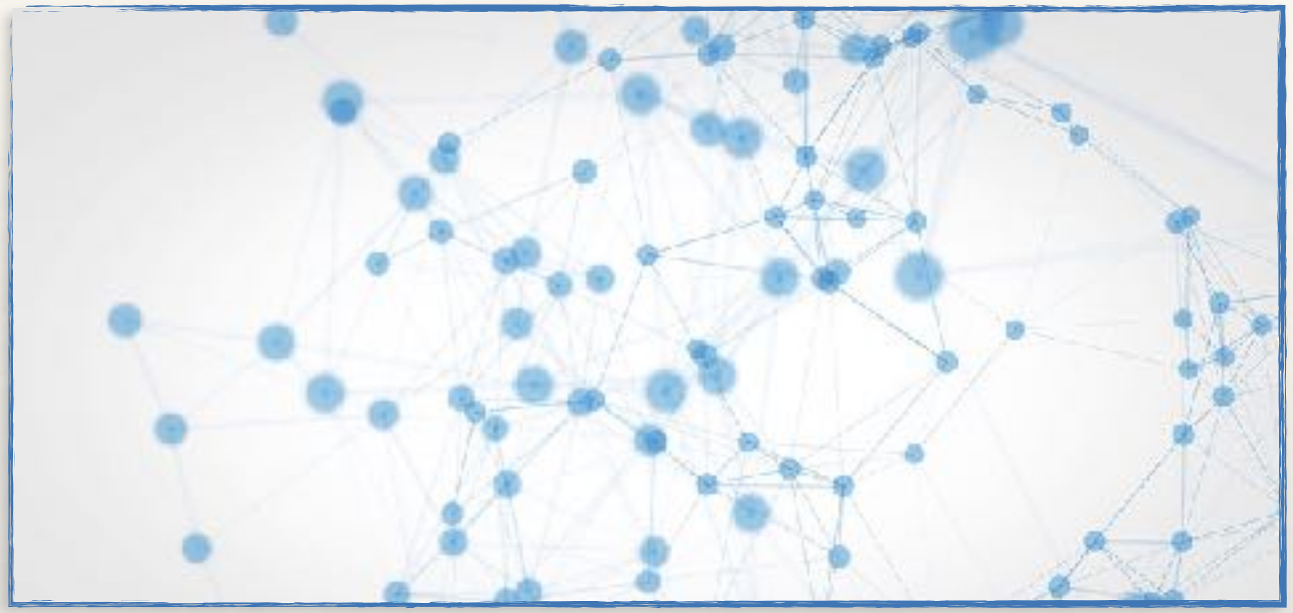

"Data has shape and
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Persistent Homology:

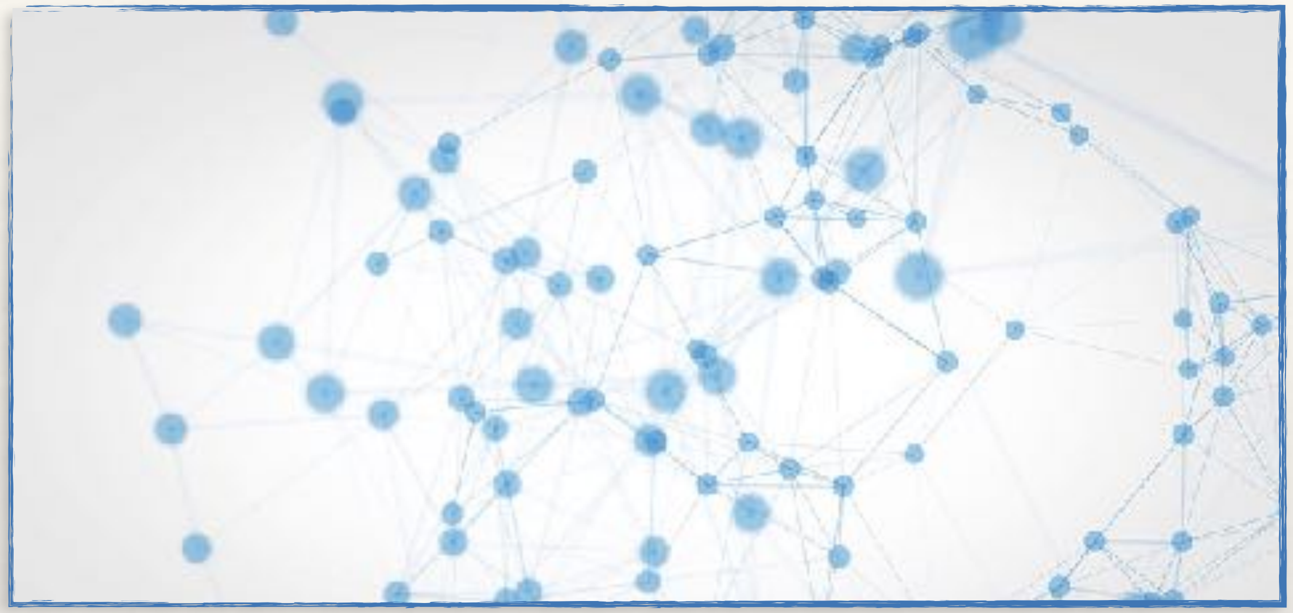
From:

Shape Analysis

To:

Neuroscience
Chemistry
Geography
Shape Analysis
Biophysics
Biology
Network Analysis
Oncology
…

Anything has Shape



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Complex Networks

Definition:

A ***network*** is a **complex system** consisting of **individuals** or **entities** connected by specific **ties** such as

- ***Personal Relationship***
- ***Shared Knowledge***
- …

References:
M. Newman, *Networks: An Introduction*, 2010
J. Scott, *Social Network Analysis*, 2017

Complex Networks

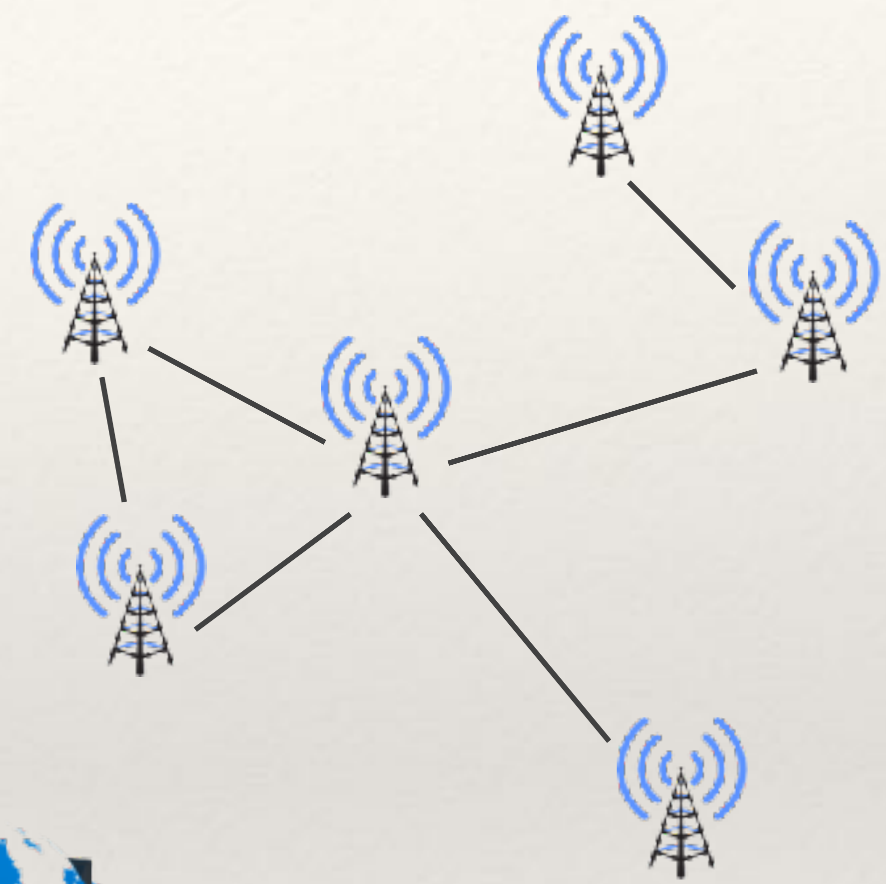
A Bunch of Examples:

- ***Social** Networks*

Complex Networks

A Bunch of Examples:

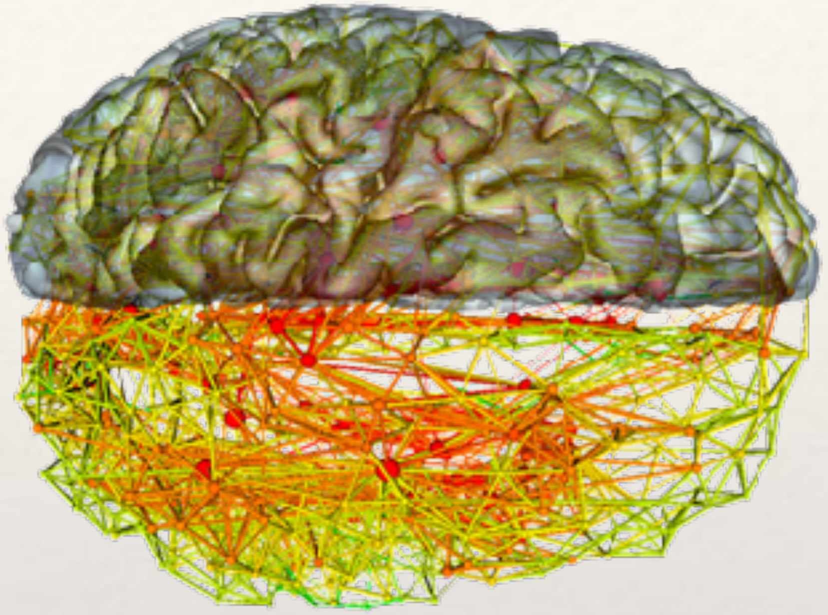
- ***Social*** *Networks*
- ***Sensor*** *Networks*

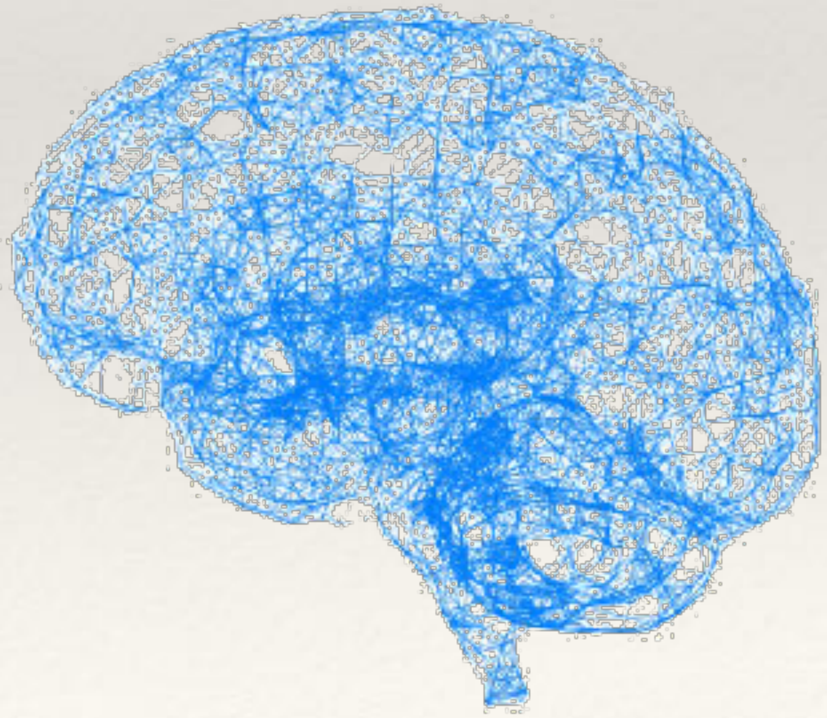


Complex Networks

A Bunch of Examples:

- ***Social*** *Networks*
- ***Sensor*** *Networks*
- ***Biological*** *Networks*





Complex Networks

A Bunch of Examples:

- ***Social*** *Networks*
- ***Sensor*** *Networks*
- ***Biological*** *Networks*
- ***Collaborative*** *Networks*

Complex Networks

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- …

Outline

Brief Introduction to Complex Network Analysis

Outline

Brief Introduction to
Complex Network
Analysis

Persistence-based
Network Analysis

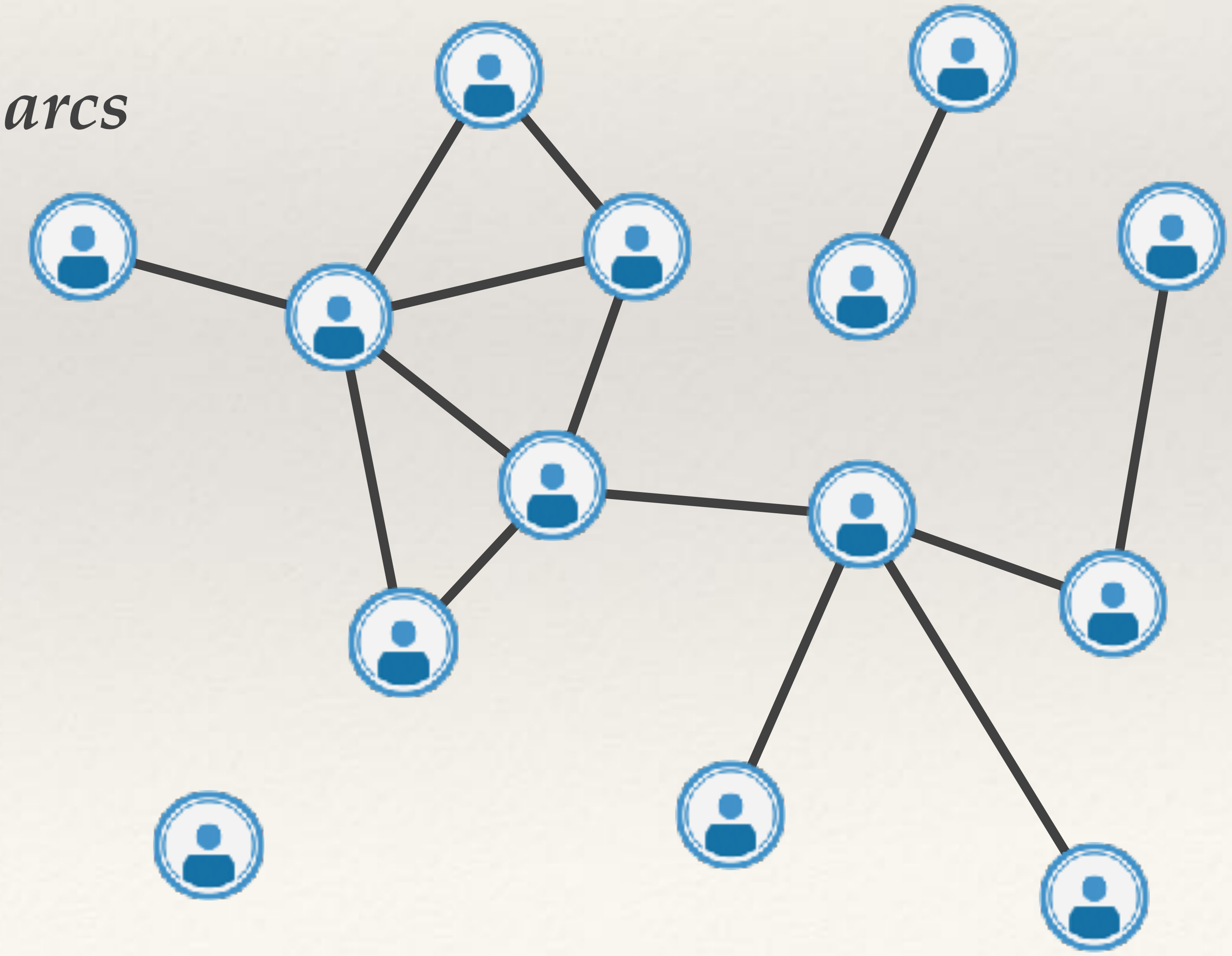
Network Analysis

Representation:

A network can be represented by a **graph** ***G=(V, E)*** such that:

individuals ⟺ *nodes*

ties ⟺ *arcs*



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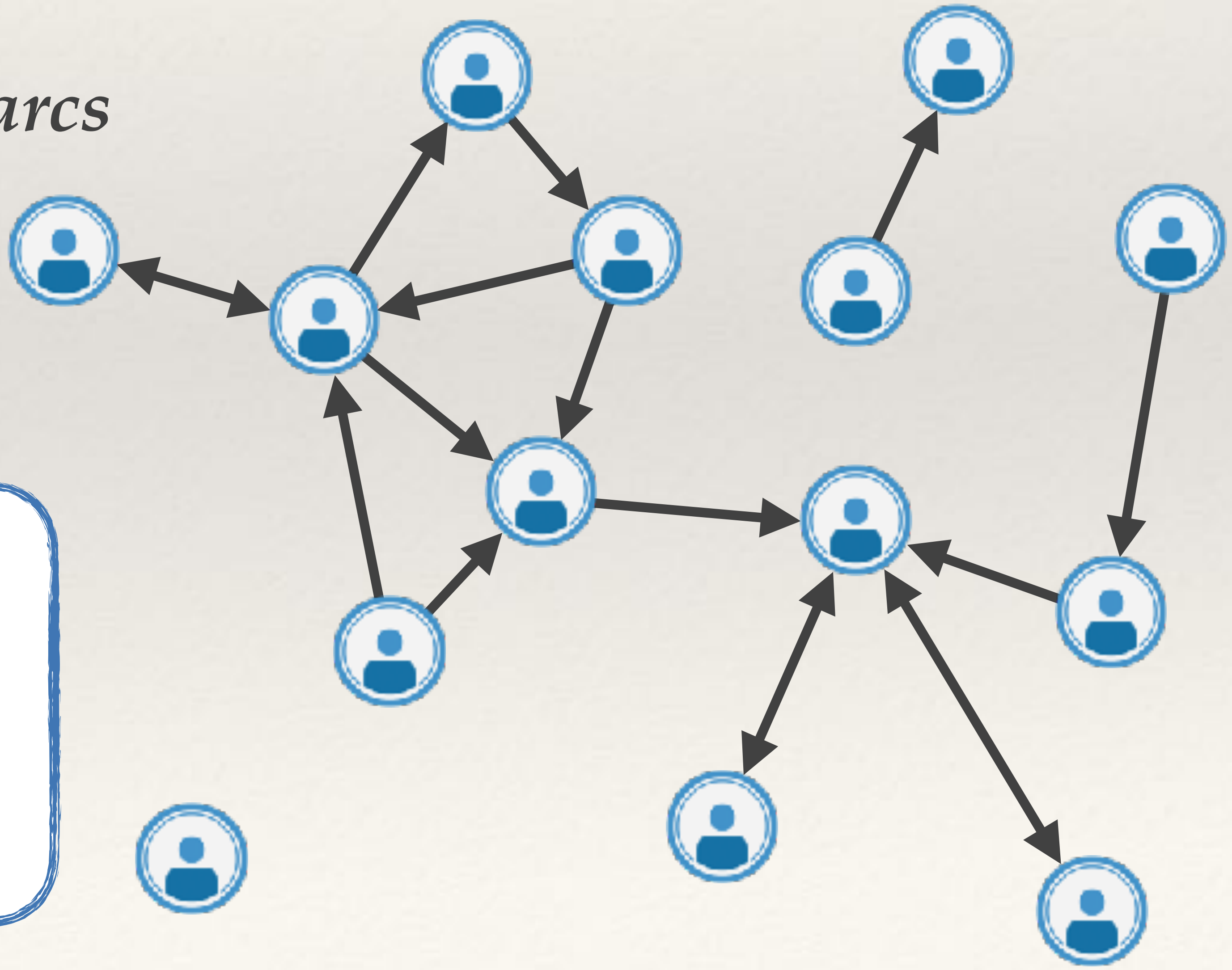

Arcs can be:

- ***directed***
- ***weighted***

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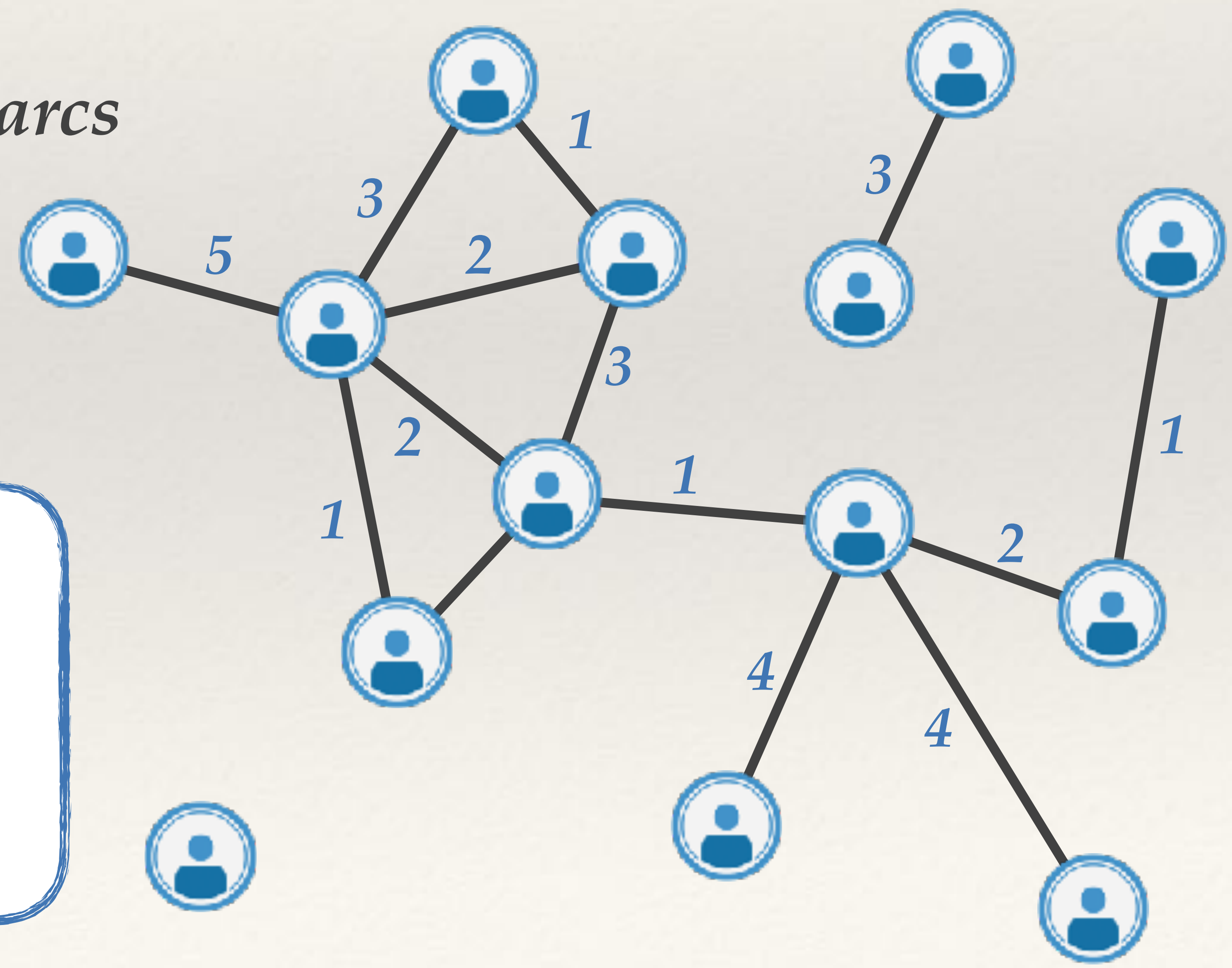

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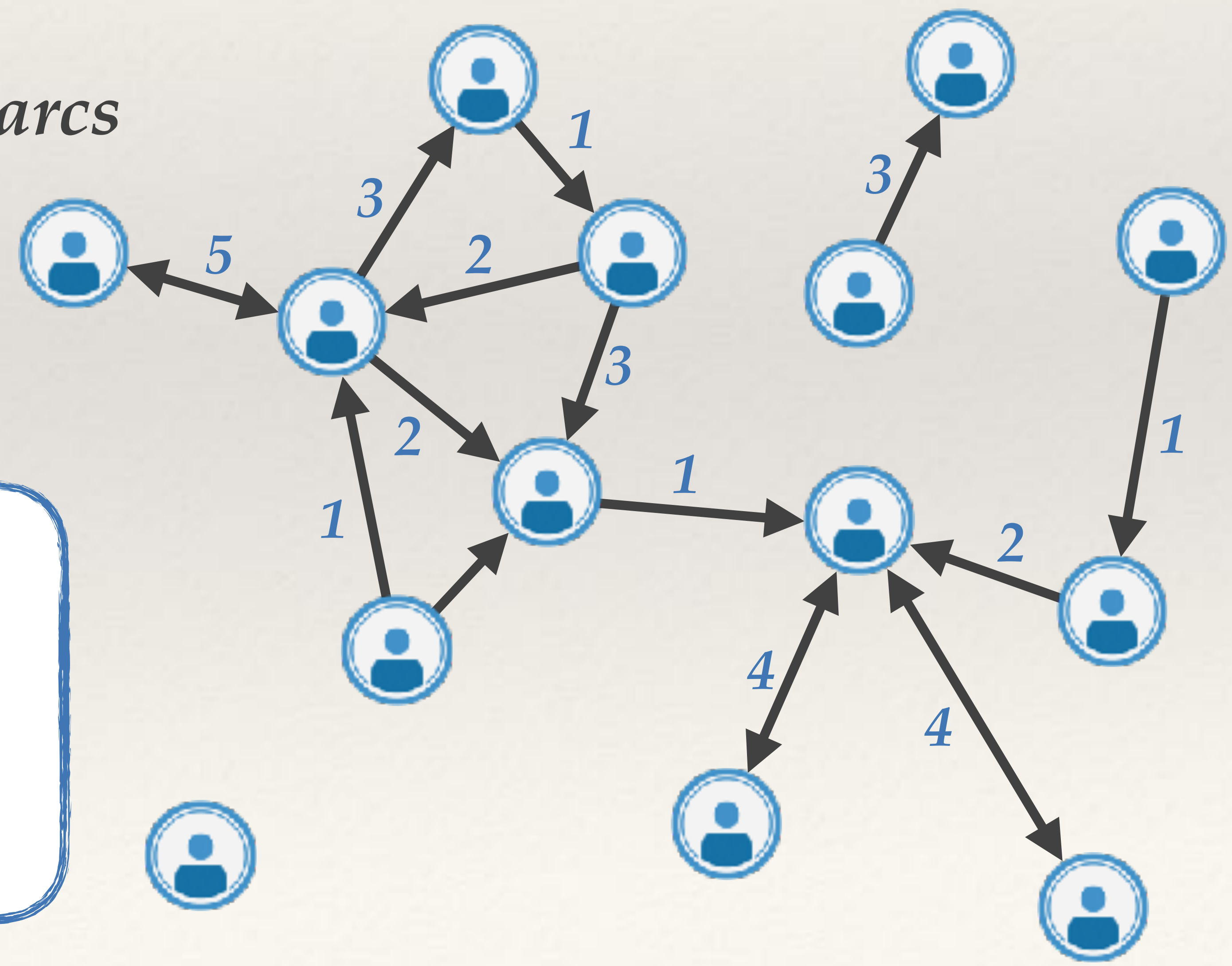

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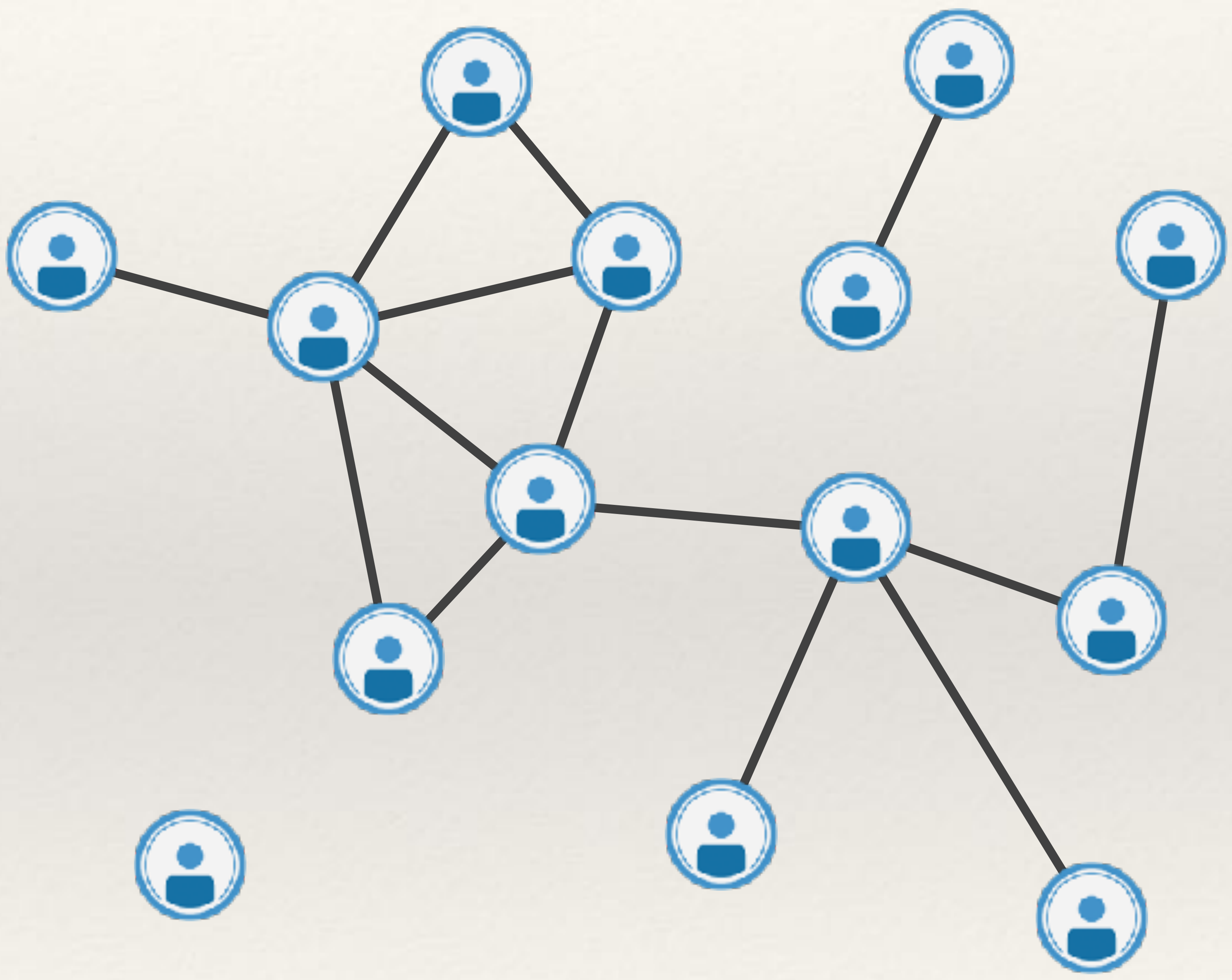
Network Analysis

A Two-level Analysis:



Network Analysis

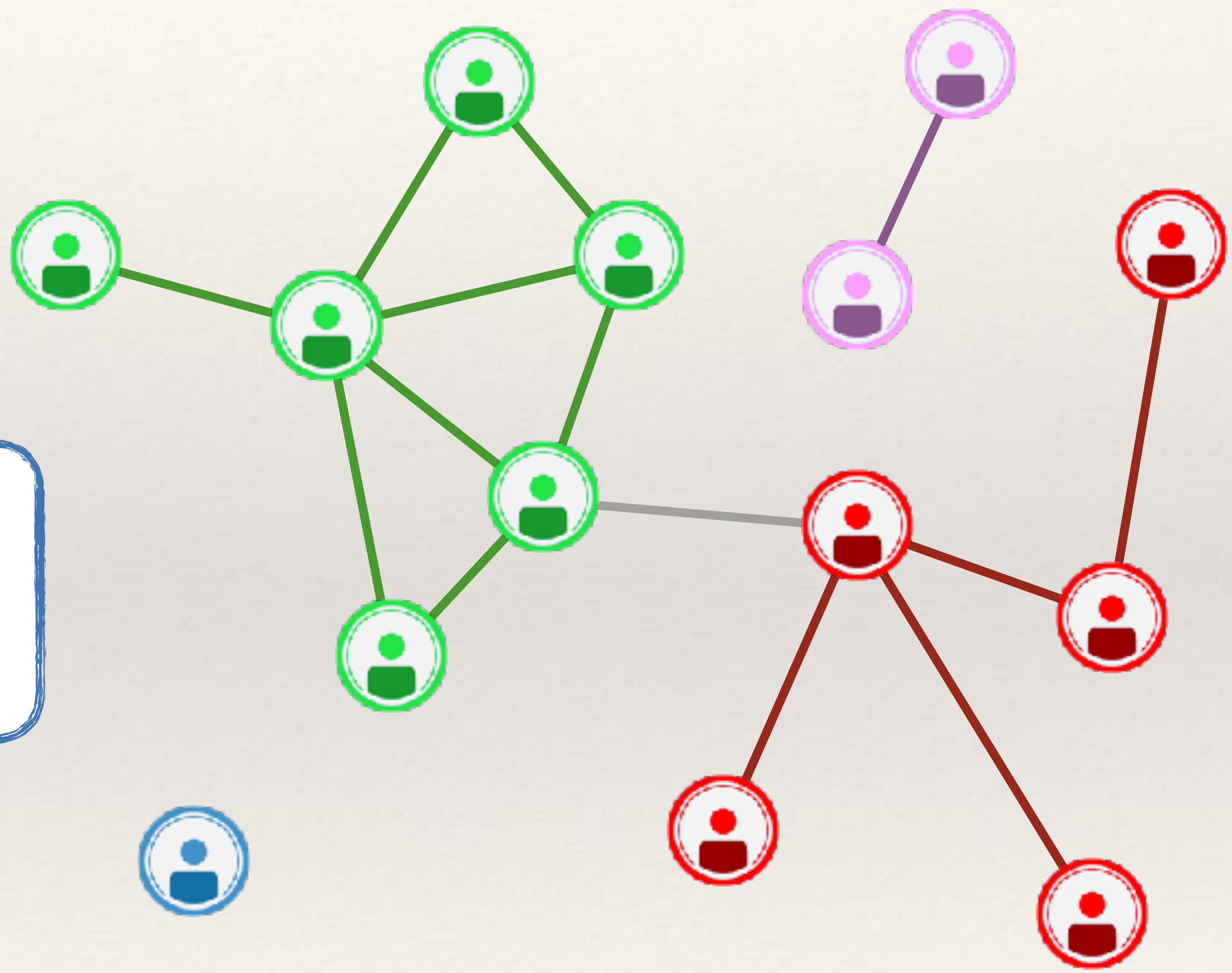
A Two-level Analysis:

- *Egocentric*
- *Sociocentric*

Network Analysis

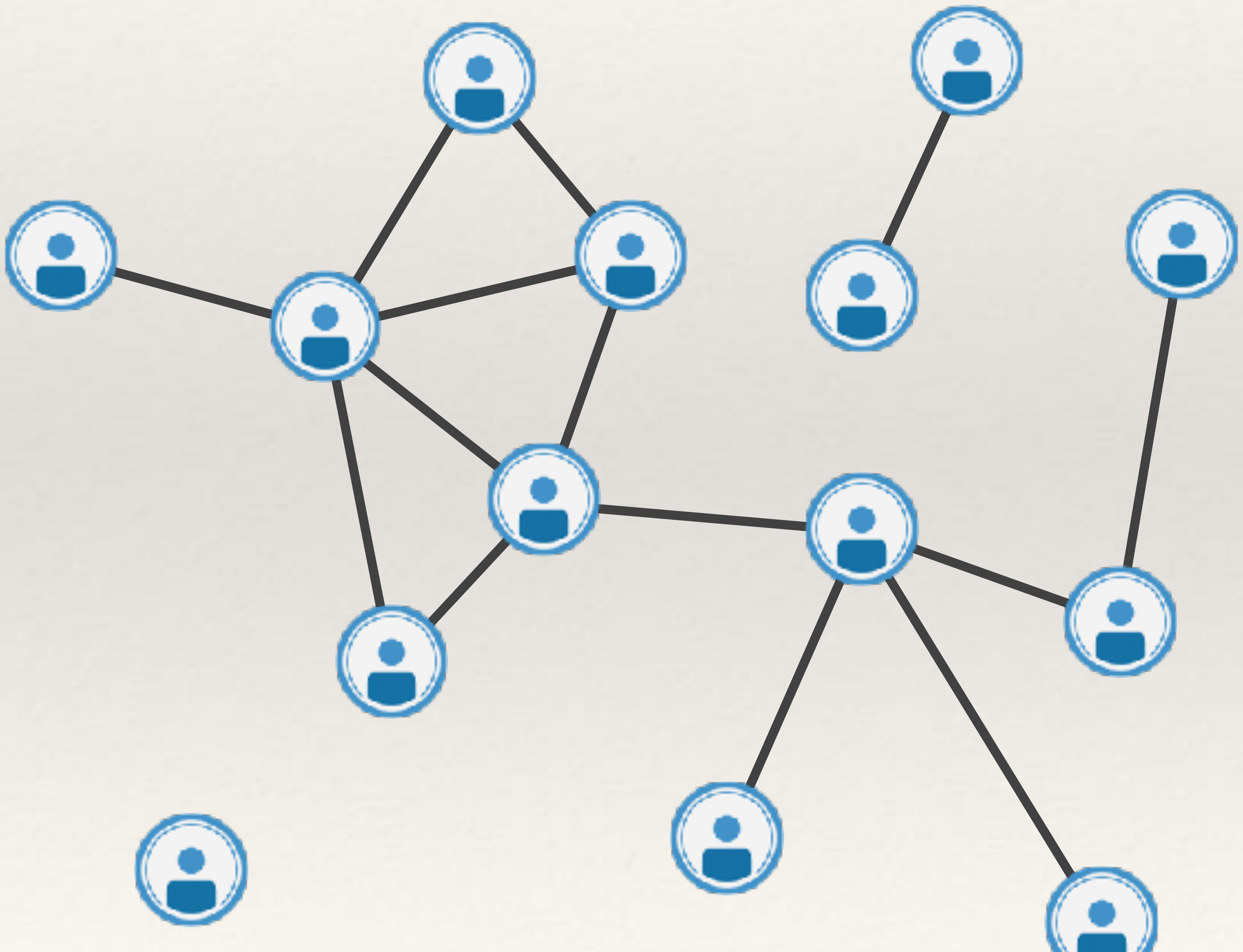
A Two-level Analysis:

- *Egocentric*
- ***Sociocentric***



Identifying Key Players

What is the most important individual?



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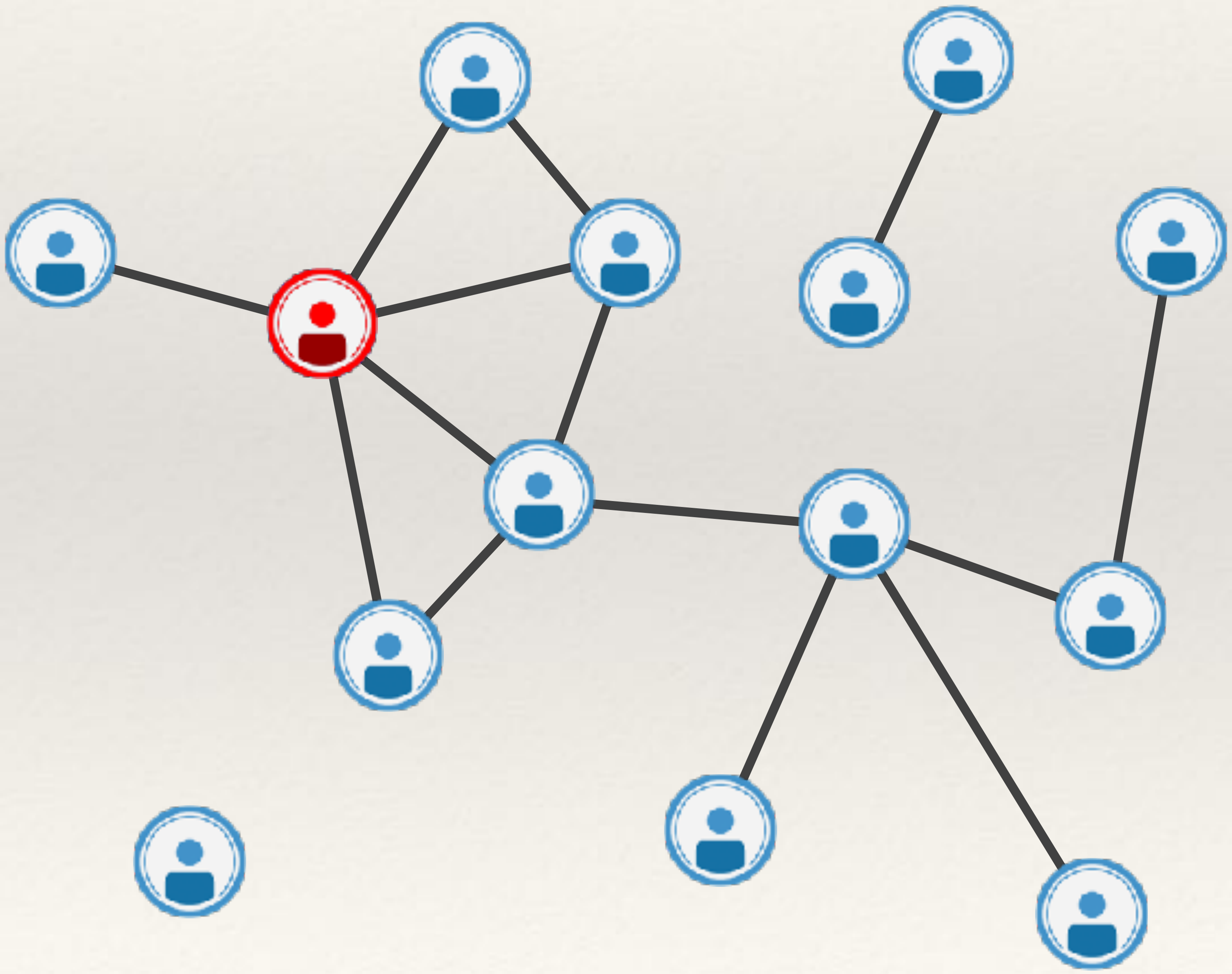

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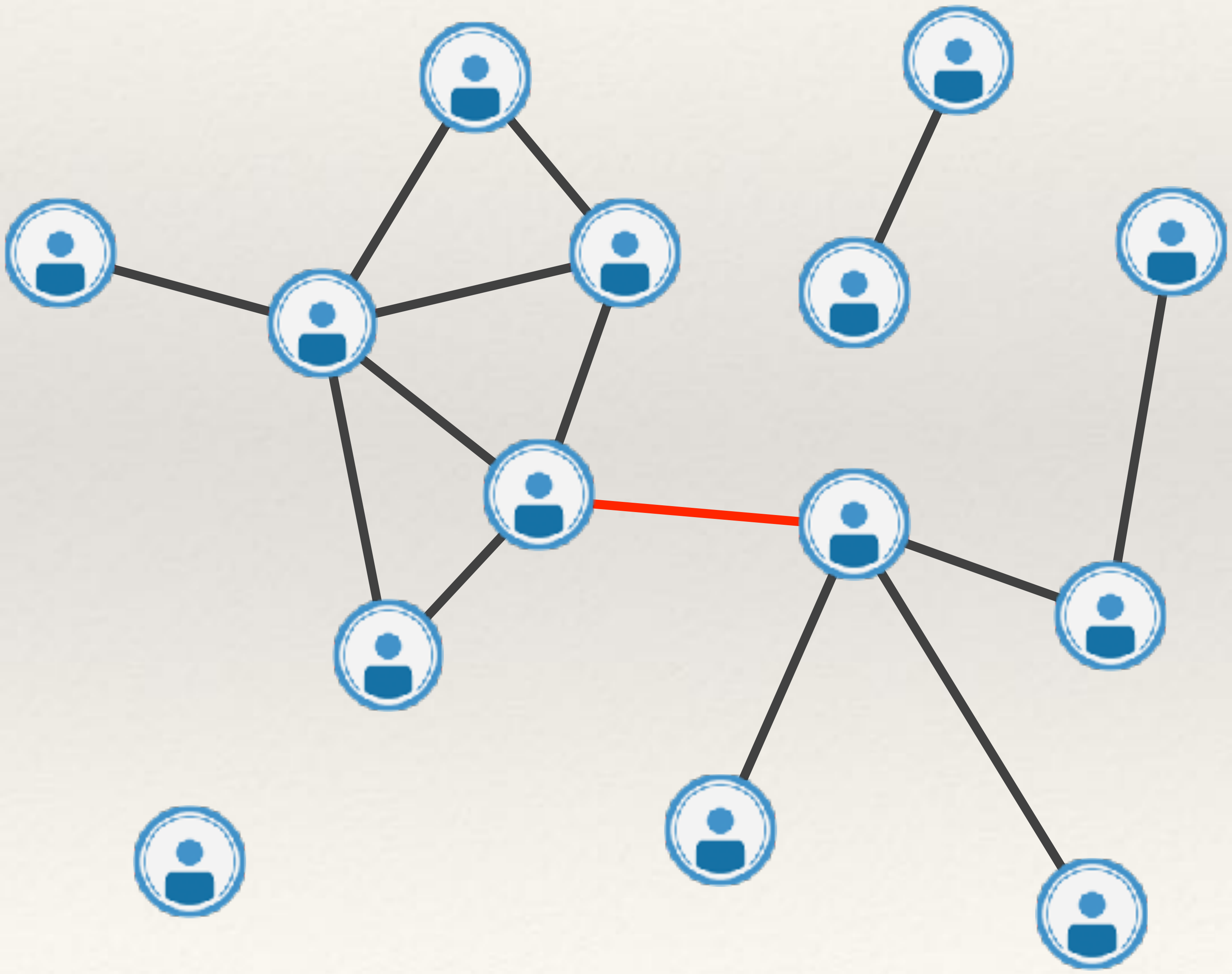

Identifying Key Players

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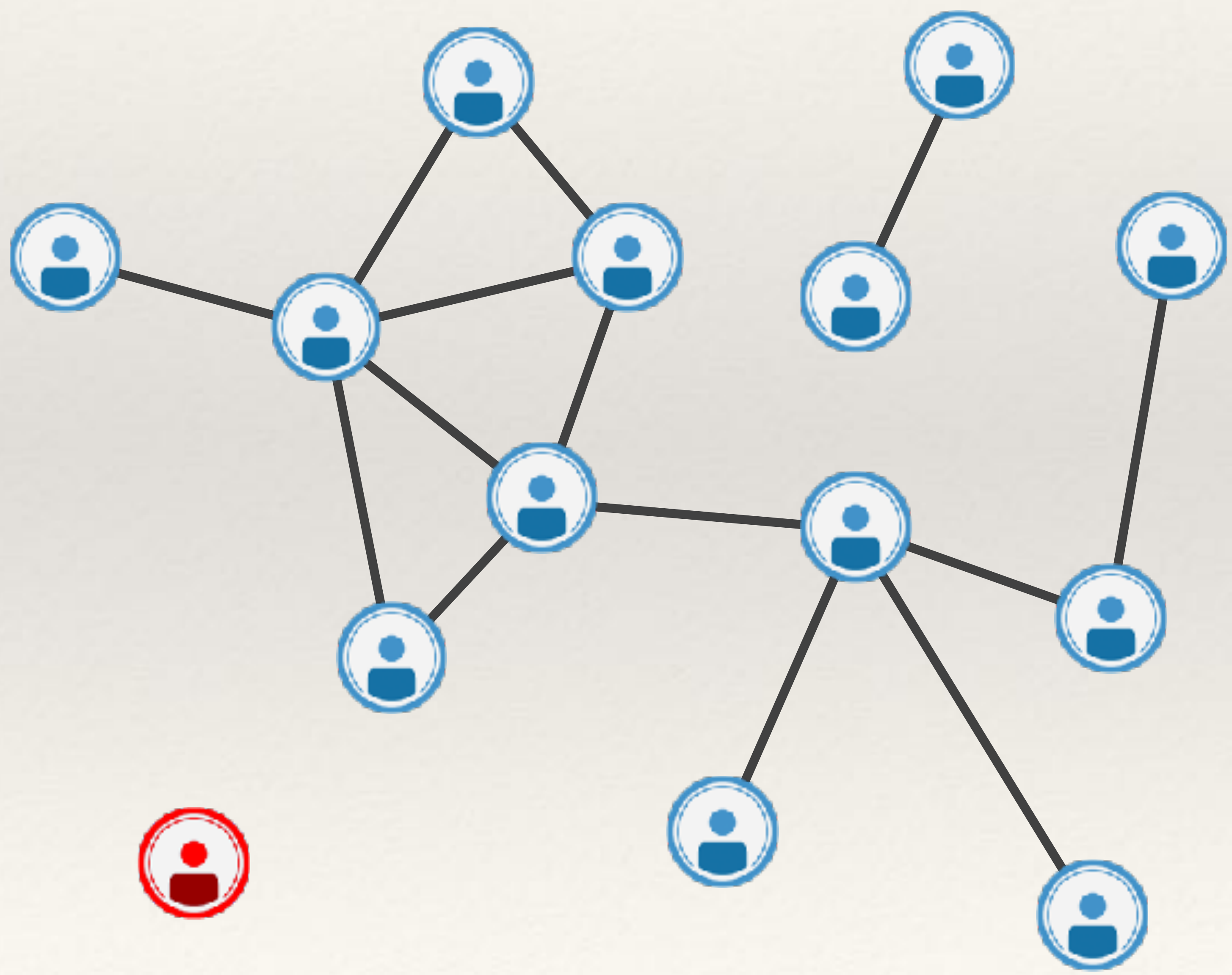


Identifying Key Players

Centrality Measures:

Different criteria to underline **different roles**:

- *Key players*
- *Brokers*
- *Bridges*
- *Isolated*
- …

A function $F : V \longrightarrow R$ assigning to each node a ***"centrality"*** value:

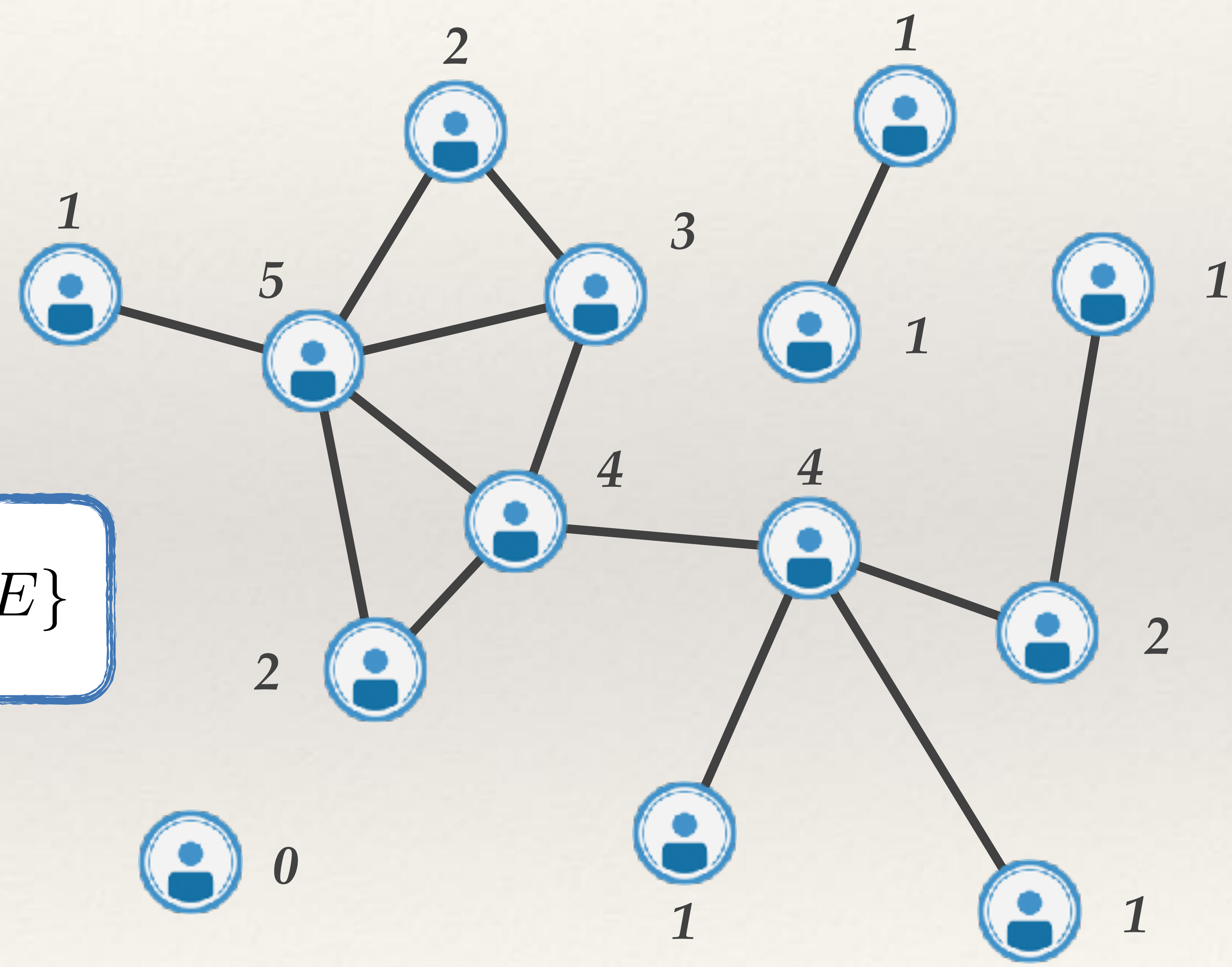
- ***Degree*** *centrality*
- ***Betweenness*** *centrality*
- ***Closeness*** *centrality*
- ***Eigenvector*** *centrality*
- ***Erdös*** *distance*

Identifying Key Players

Degree Centrality:

Given a node v of G=(V, E),

$$D(v) := \#\{u \in V \mid (u,v) \in E\}$$

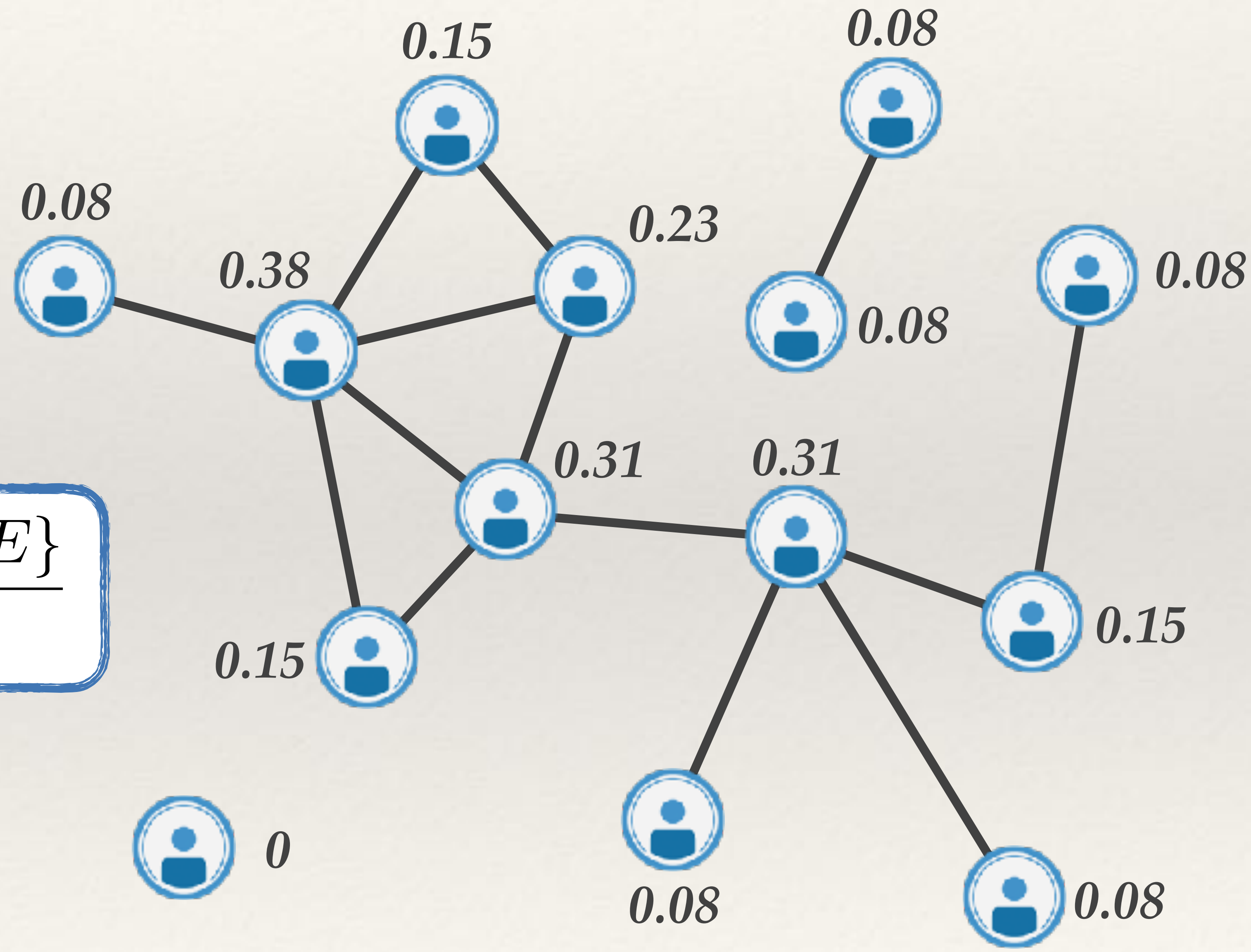


Identifying Key Players

Degree Centrality:

Given a node v of $G=(V, E)$,

$$D(v) := \frac{\#\{u \in V \mid (u,v) \in E\}}{\#V - 1}$$



Identifying Key Players

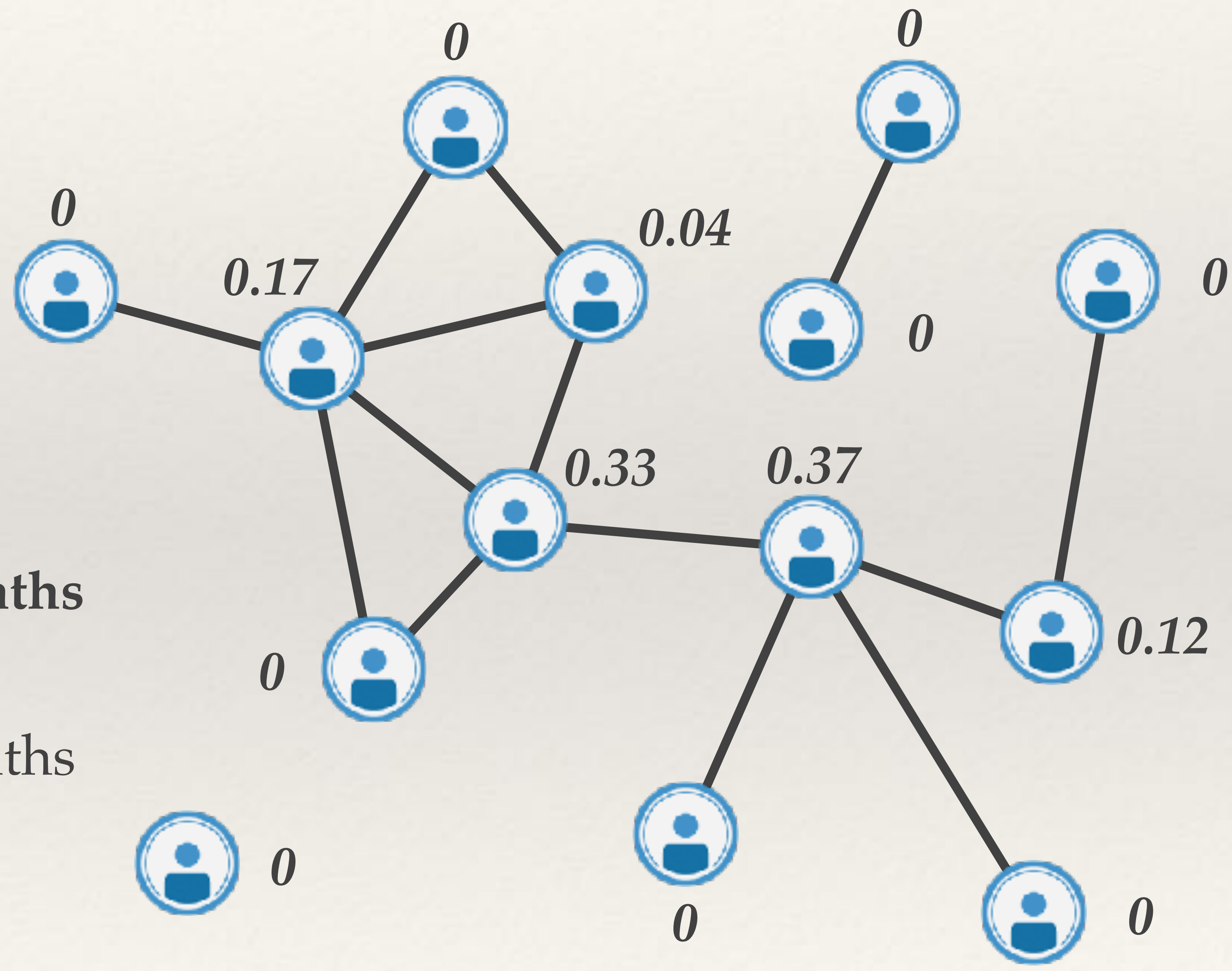
Betweenness Centrality:

Given a node v of $G=(V, E)$,

$$B(v) := \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

where:

- σ_{st} is the number of **shortest paths from s to t**
- $\sigma_{st}(v)$ is the number of those paths **passing through v**

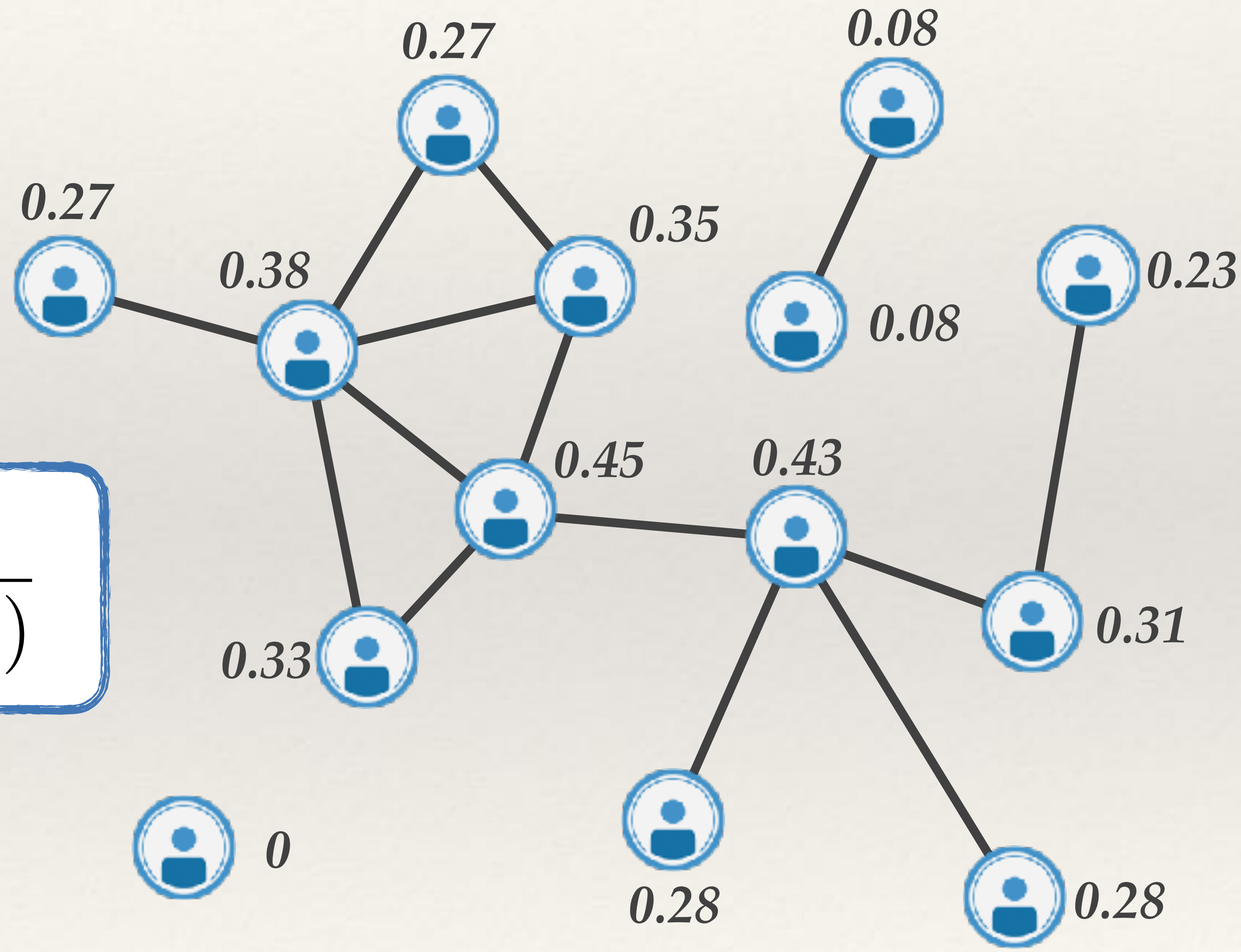


Identifying Key Players

Closeness Centrality:

Given a node v of $G=(V, E)$,

$$C(v) := \frac{\#V - 1}{\sum_{u \in V} d(u,v)}$$



Identifying Key Players

Eigenvector Centrality:

Given a node v of $G=(V, E)$,

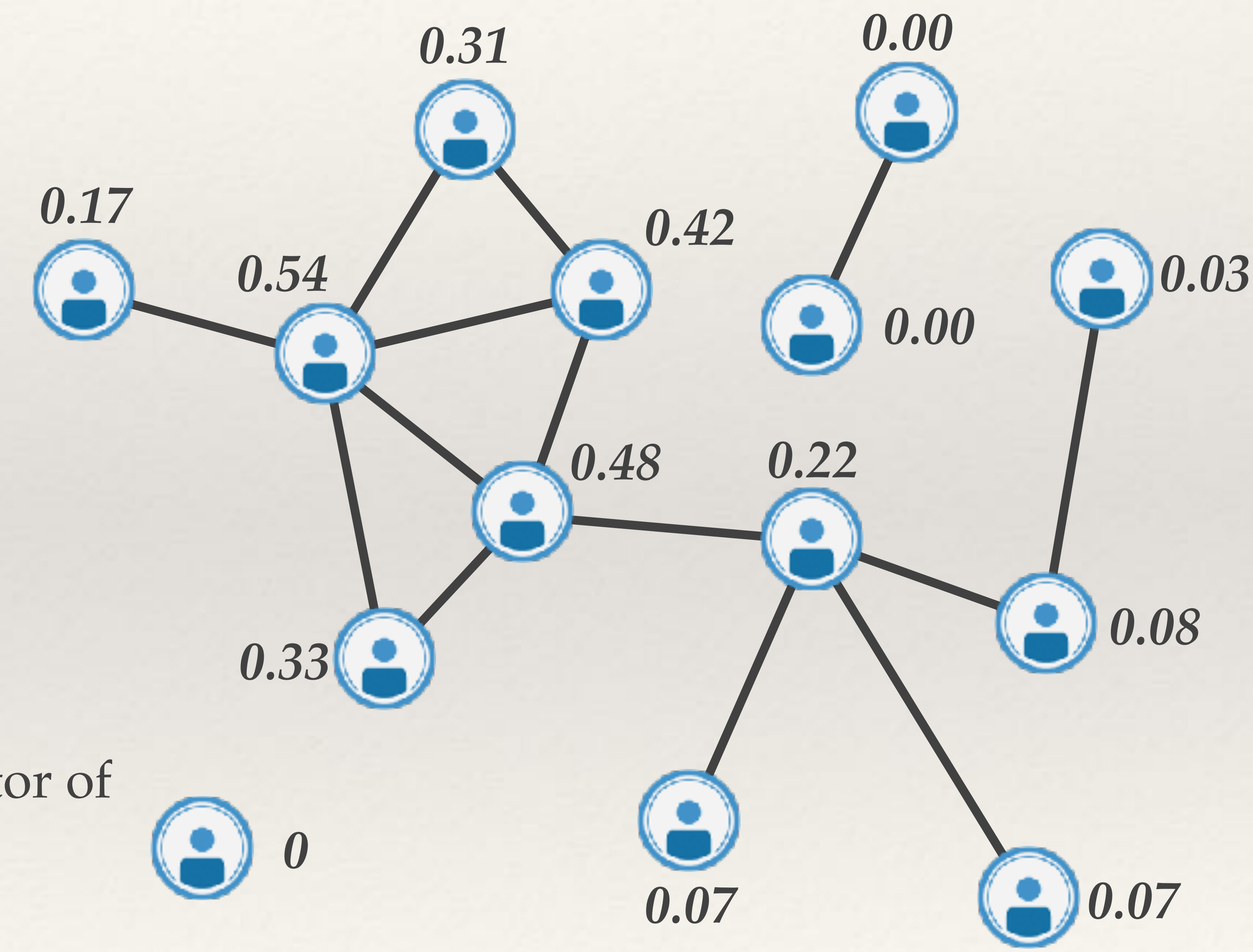
$$x_v := \frac{1}{\lambda} \sum_{u \in V} A_{uv} \, x_u$$

where λ is constant and

$$A_{uv} := \begin{cases} 1 & if(u,v) \in E \\ 0 & otherwise \end{cases}$$

i.e., the v^{th} entry of the eigenvector of

$$\boldsymbol{Ax} = \boldsymbol{\lambda x}$$



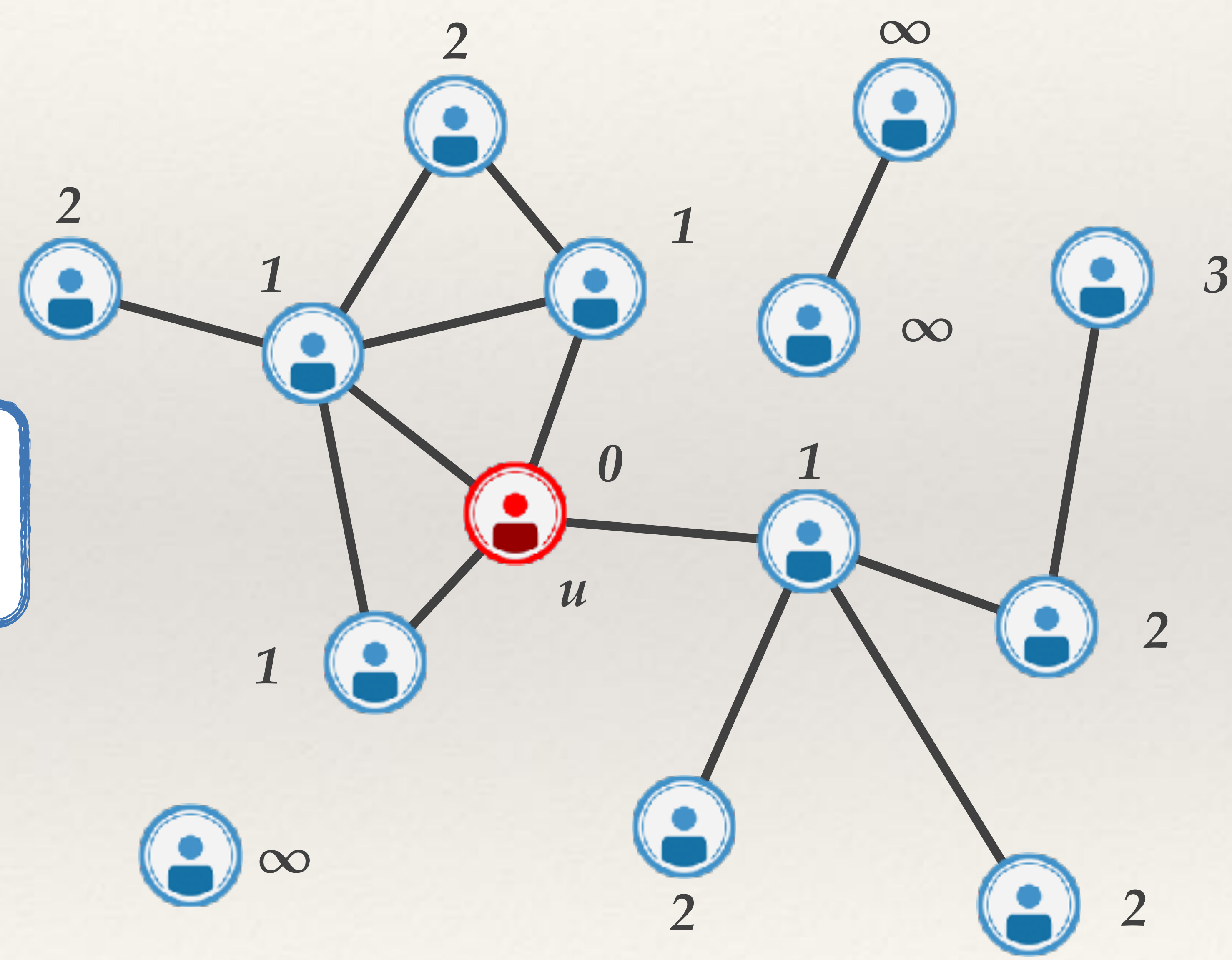
x*>*0 implies λ must be the largest eigenvalue of A and $\boldsymbol{x}$ the corresponding eigenvector

Identifying Key Players

Erdös Distance:

Given two nodes u, v of $G=(V, E)$,

$$E_u(v) := d(u, v)$$



Named after ***Paul Erdös***,

- one of the most prolific mathematicians of the 20th century

Identifying Key Players

Centrality Measures:

A centrality measure for ***any query***

Degree	***How many individuals can v reach directly?***
Betweenness	***How likely is v to be the most direct route between two individuals?***
Closeness	***How fast can v reach everyone in the network?***
Eigenvector	***How well is v connected to other well-connected individuals?***
Erdös	***How far is v from a specific individual?***

Structural Analysis

Sociocentric Networks:

- **Structural Metrics:**
  - *Average of a Centrality Measure*
  - *Diameter*
  - *Density*
  - *Transitivity*
  - …
- **Community Decomposition:**
  - *Atomic Communities*
  - *Clustering Techniques*

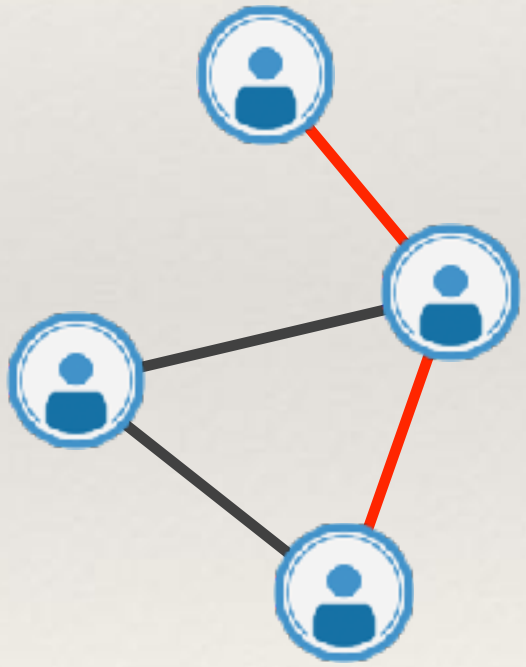
Structural Analysis

Structural Metrics:

- *How far are two individuals at most?*

Diameter:

The longest shortest path between any two nodes



$Diameter(G) = 2$

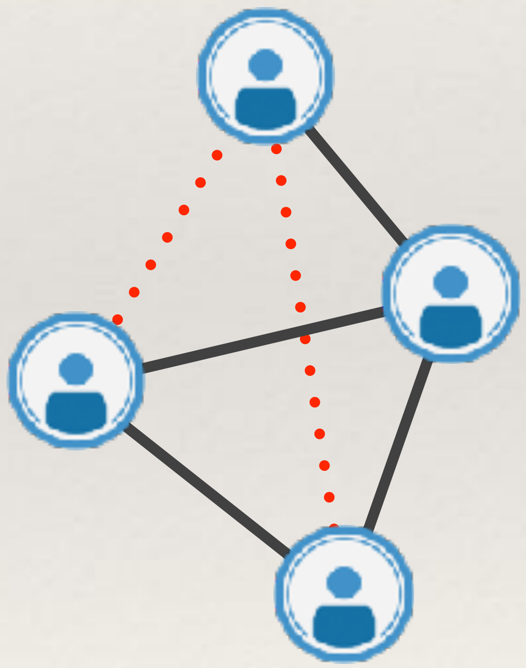
Structural Analysis

Structural Metrics:

- *How close is G to being an "everyone knows everyone" network?*

Density:

$$\frac{\textit{Number of edges of G}}{\textit{Number of all possible edges}}$$



Density(G) = 4/6= 0.67

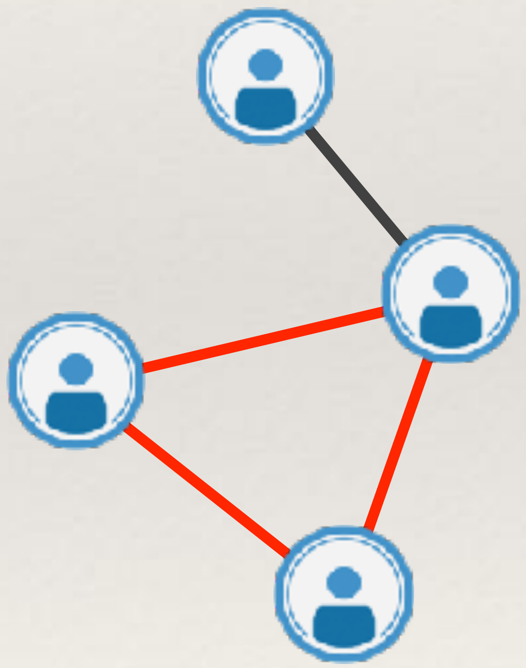
Structural Analysis

Structural Metrics:

- *How likely are two individuals connected to an individual v to be connected to each other?*

Transitivity:

$$\frac{\textit{Number of closed triplets of nodes}}{\textit{Number of connected triplets}}$$



Transitivity(G) = 1/3 = 0.33

Structural Analysis

Community Decomposition:

- *Atomic Communities:*
  - *Clique*
  - *n-Clique*
  - *n-Clan*
  - *n-Club*
  - *k-Plex*
  - *k-Core*
  - …

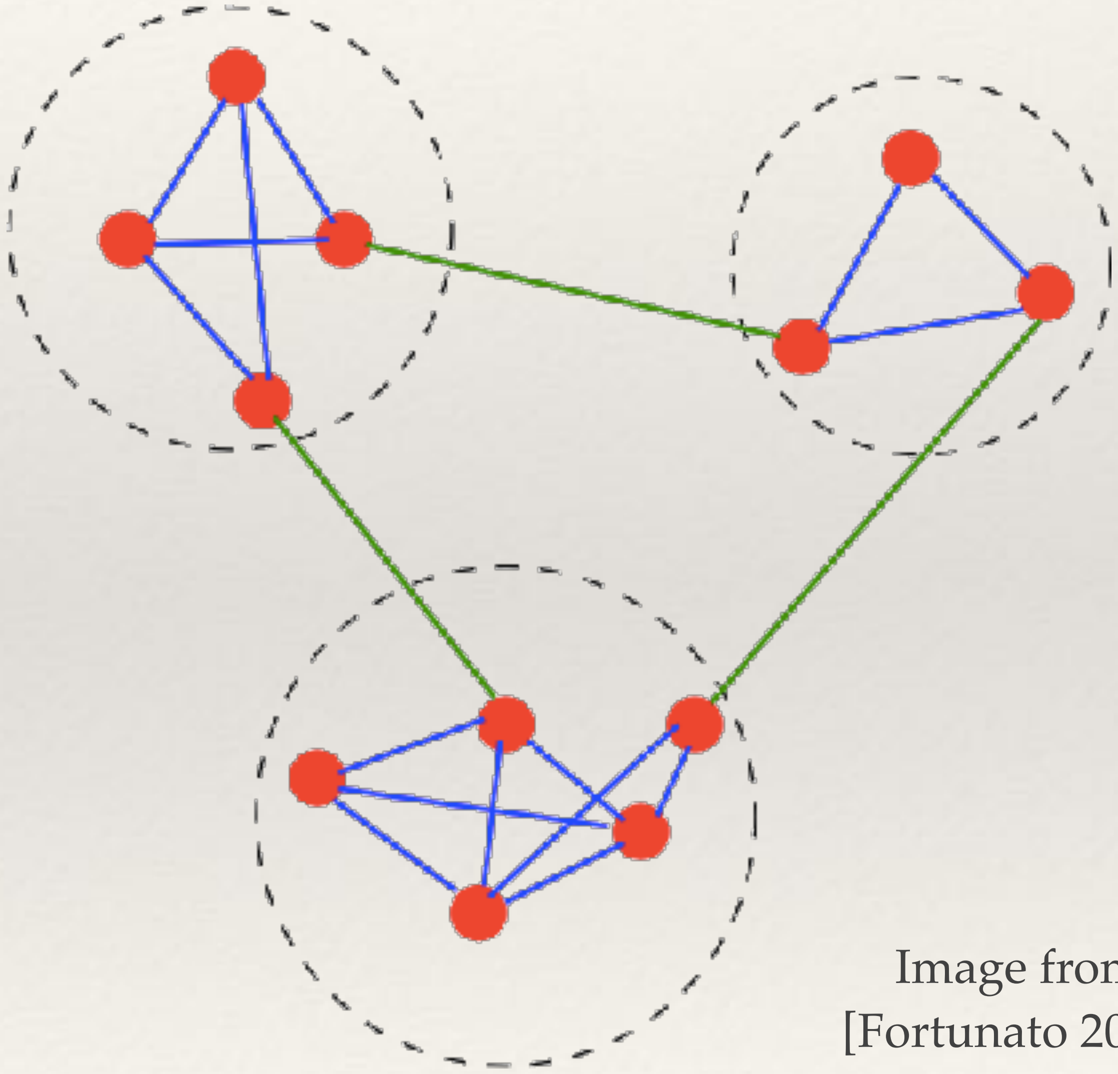


Image from [Fortunato 2009]

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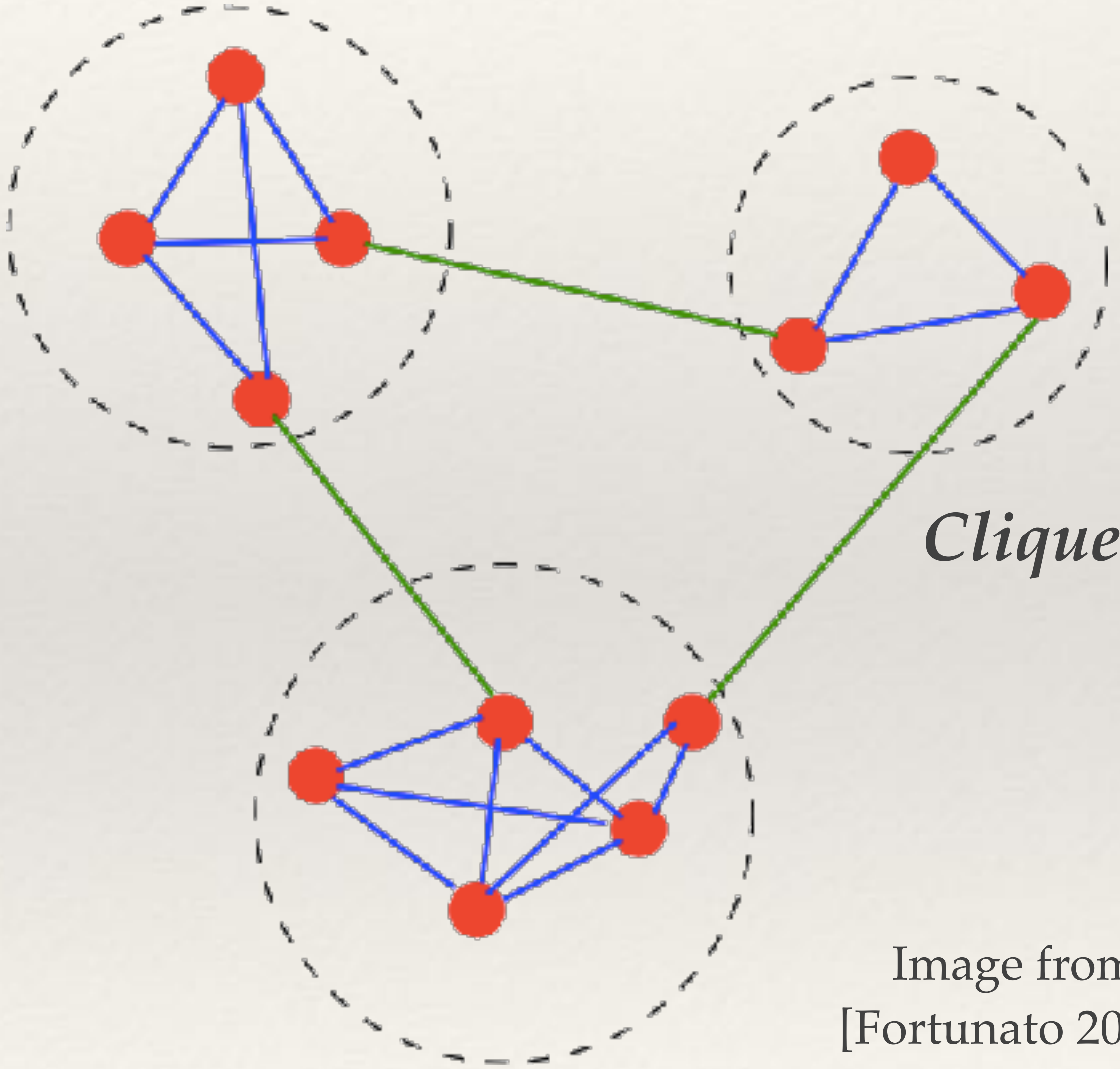

Image from [Fortunato 2009]

Clique:

maximal subgraph whose nodes are all adjacent to each other

Structural Analysis

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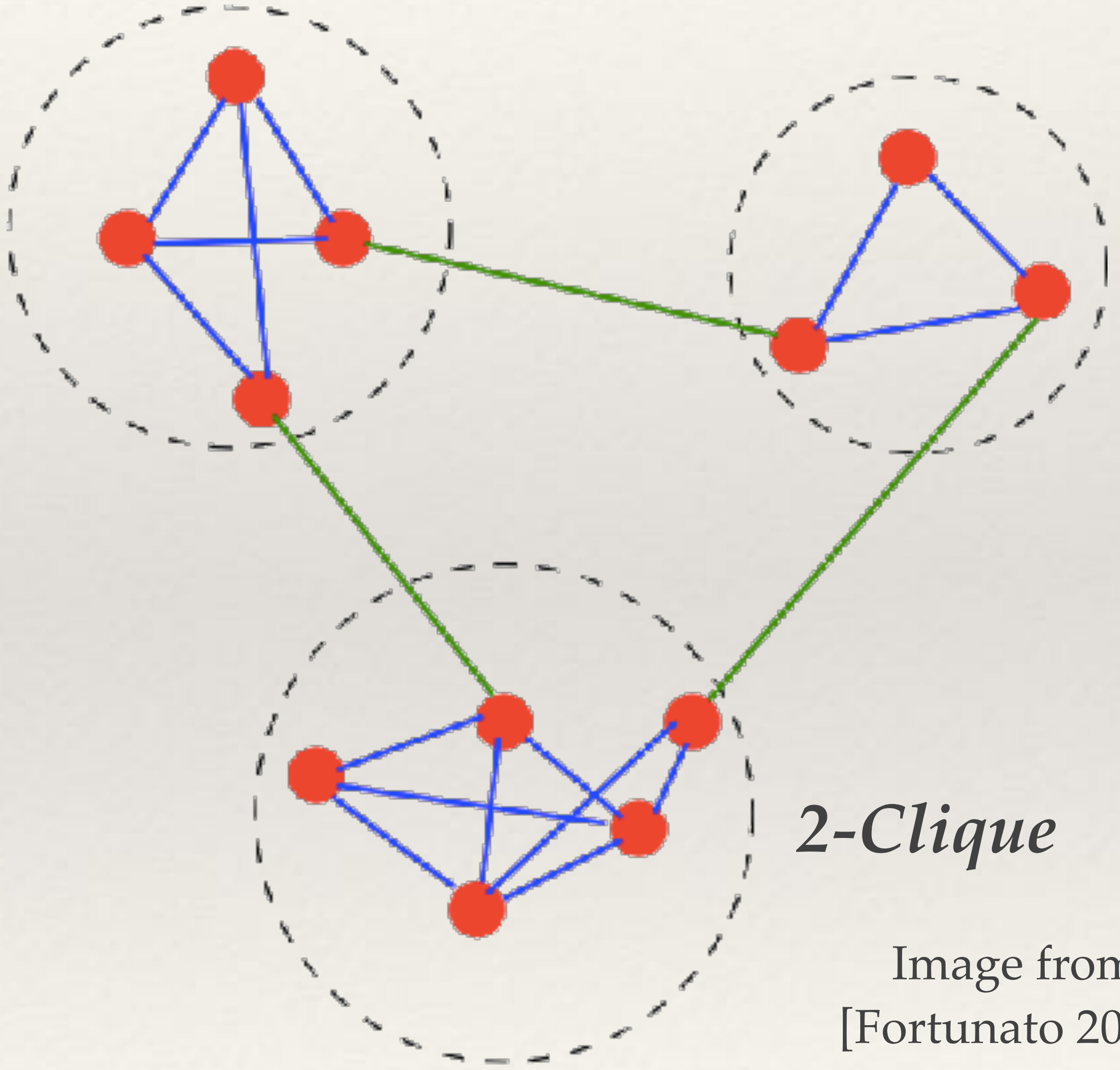

2-Clique

Image from
[Fortunato 2009]

***n*-Clique:**

maximal subgraph such that the distance of each pair of its nodes is not greater than n

Structural Analysis

Community Decomposition:

- *Atomic Communities:*
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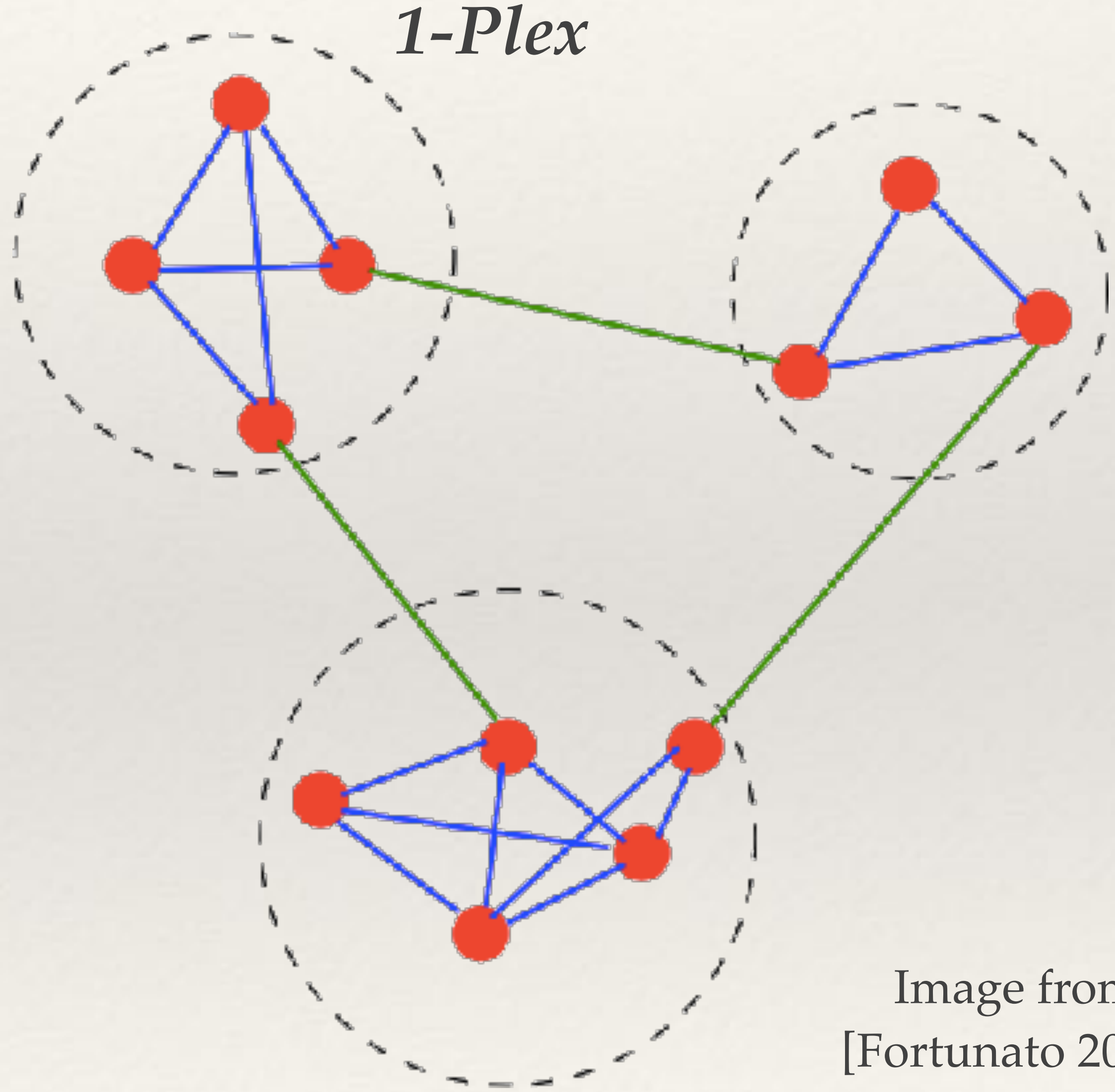

Image from [Fortunato 2009]

***k*-Plex:**

maximal subgraph in which each node is adjacent to all other nodes of the subgraph except at most *k* of them

Structural Analysis

Clustering Techniques:

Agglomerative (bottom-up)
Divisive (top-dow)

approach based on

- Centrality Measures
- Atomic Communities
- Quality Functions

Structural Analysis

Clustering Techniques:

Agglomerative (bottom-up)

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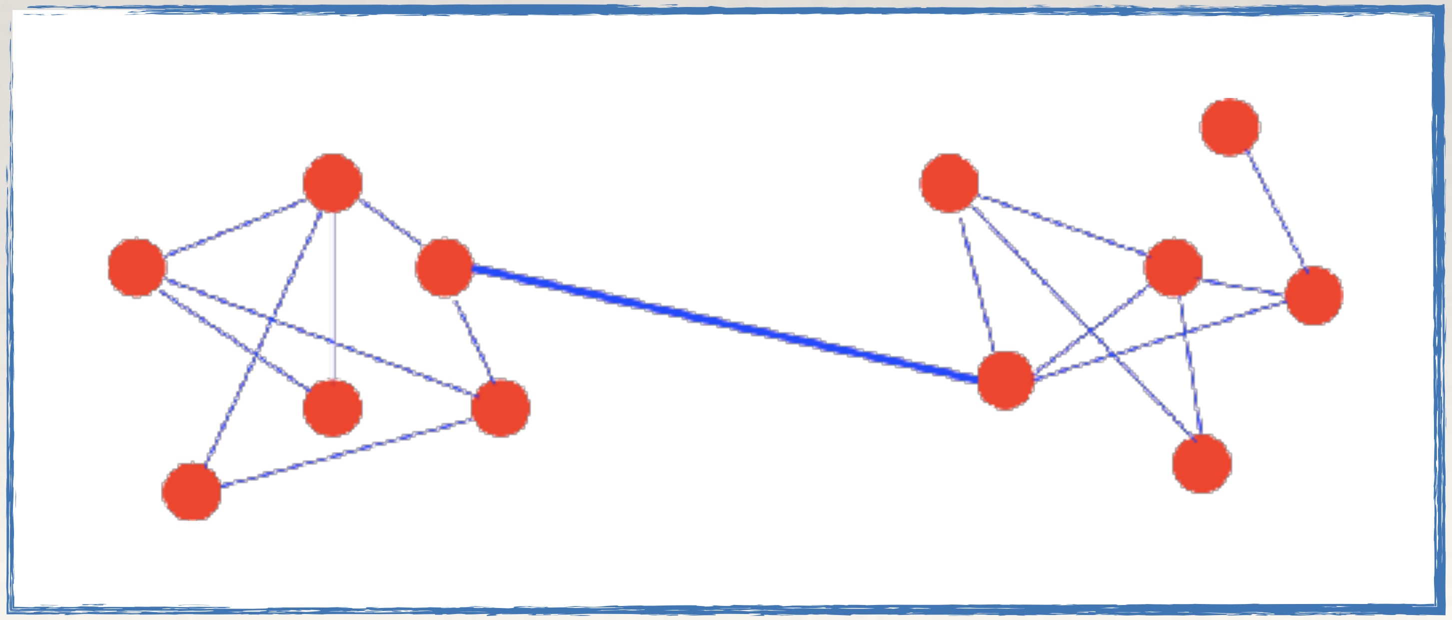
Quality Functions

Girvan-Newman Algorithm:

Iterated removal of the edge with largest ***betweenness centrality***



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Quality Functions

Clique Percolation:

k-adjacency: two clique of size *k* are *k*-adjacent if they share *k-1* nodes

k-clique community: maximal union of cliques of size *k* pairwise connected by a sequence of *k*-adjacent cliques

Decomposition in *k*-clique communities

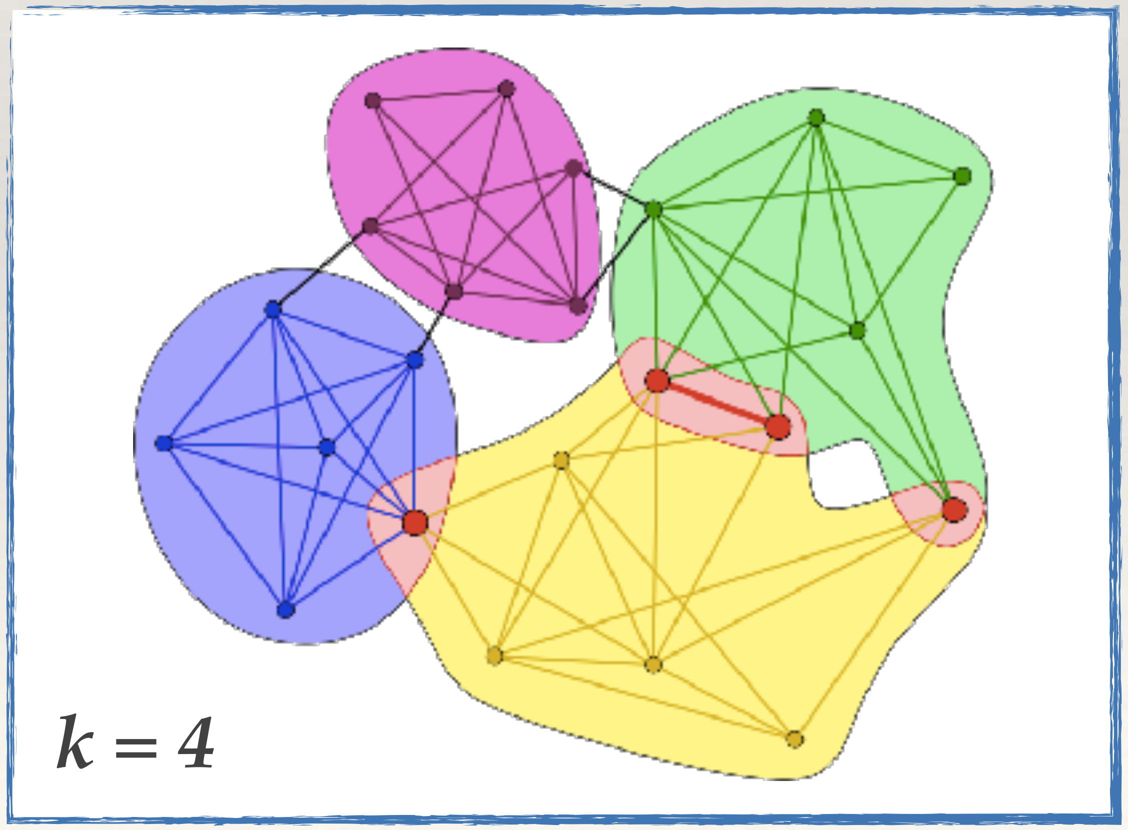

Image from [Palla et al. 2005]

Structural Analysis

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Atomic Communities

Quality Functions

Modularity-based Algorithm:

Modularity: measure for clustering quality

Iterated aggregation of communities of nodes whose merging *increases modularity*

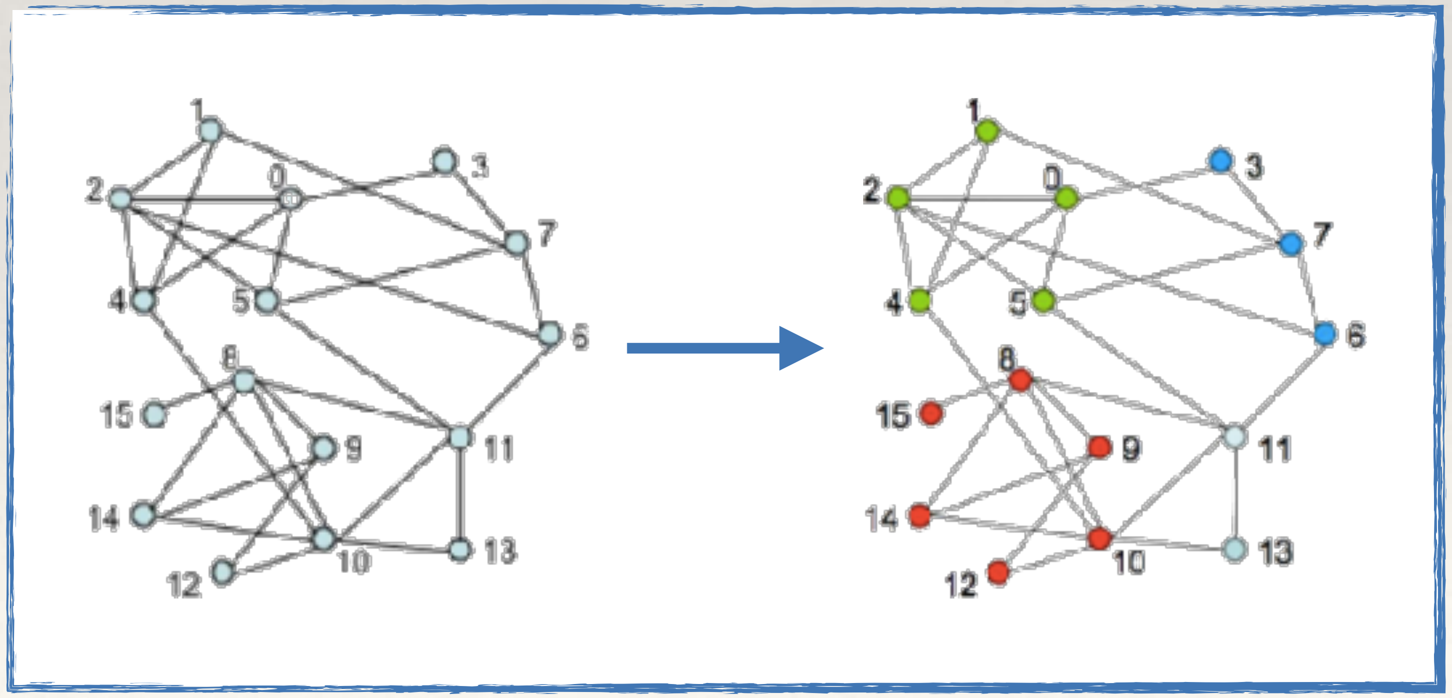

Image from [Blondel et al. 2008]

Outline

Brief Introduction to Complex Network Analysis

Persistence-based Network Analysis

Persistence-based Network Analysis

Several Application based on Persistent Homology:

- *Sensor* Networks [De Silva 2013]
- *Brain* Networks [Lee et al. 2012]
- *Collaborative/Co-occurence* Networks [Carstens et al. 2013; Rieck et al. 2016]
- *Geolocalized* Networks [Fellegara et al. 2016]
- …

Simplicial Complex Representation:

A network is represented through:

- Simplicial complex *Flag(G)* induced by G
  - *simplices of Flag(G)* ⟷ *cliques of G*

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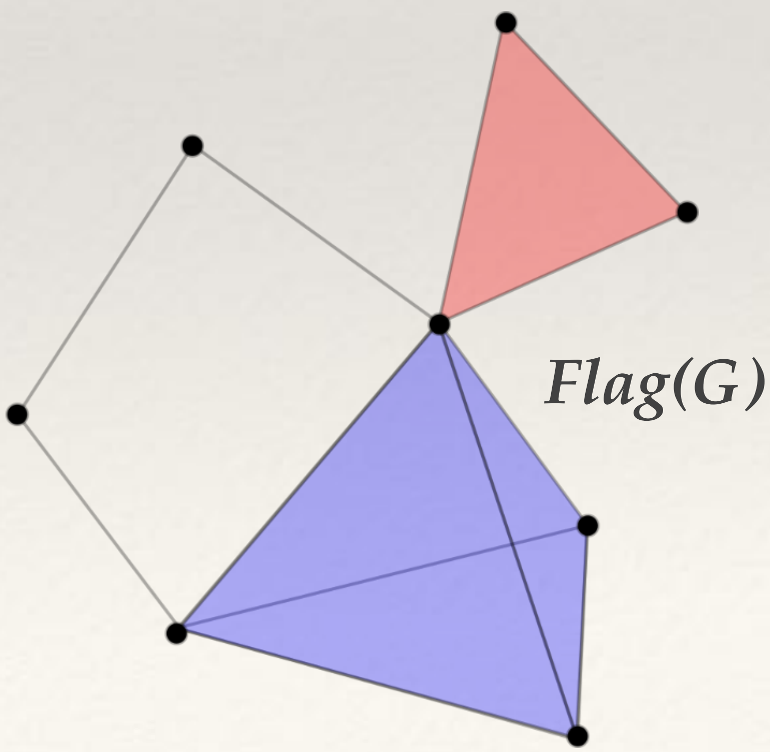
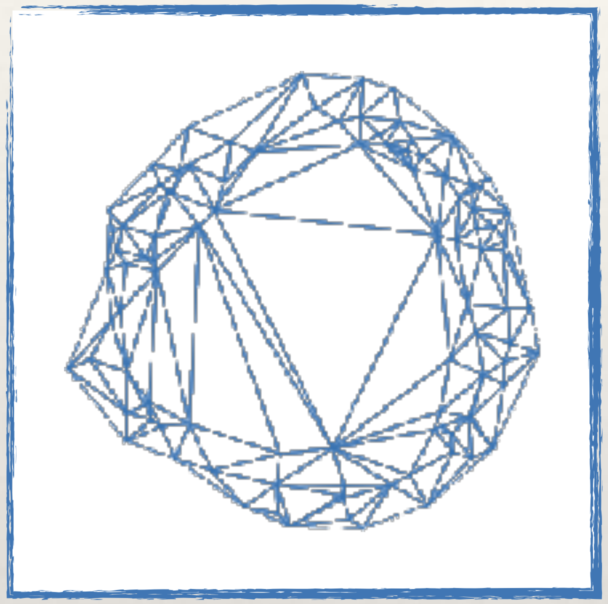
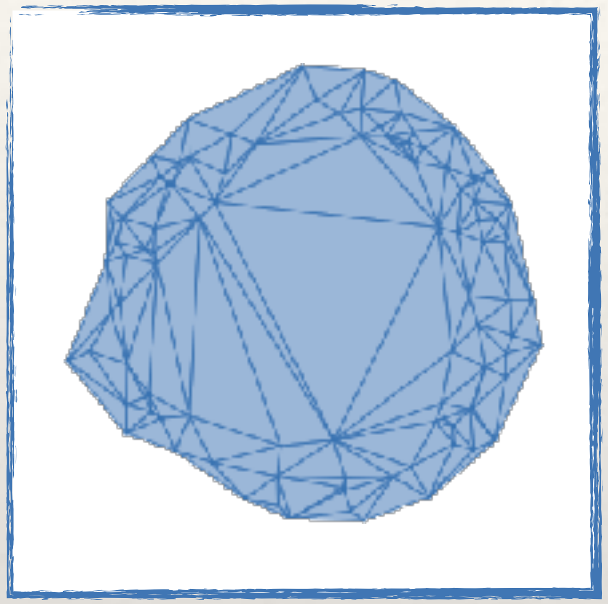
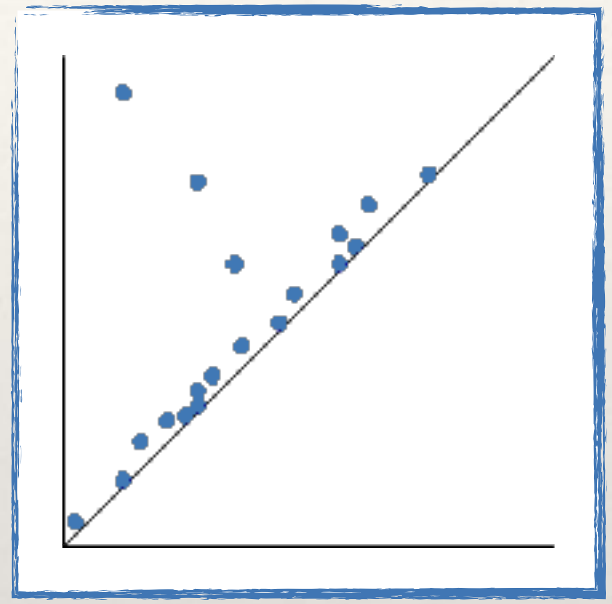


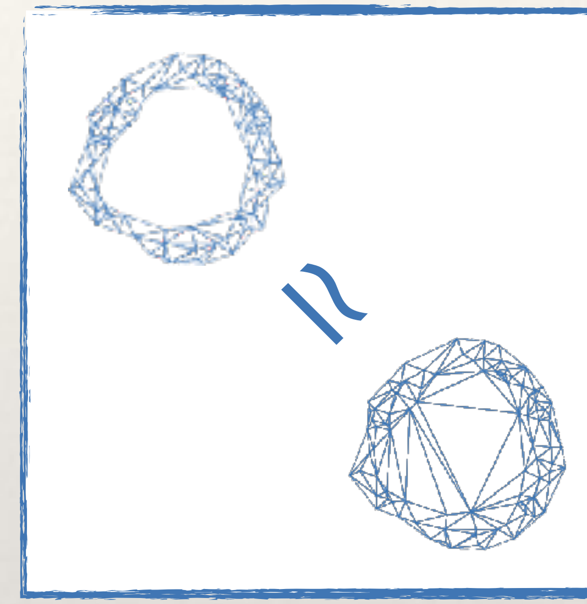
Persistence-based Network Analysis

A Common Pipeline in TDA:









Complex Construction *PH_k module* *Distance Computation*

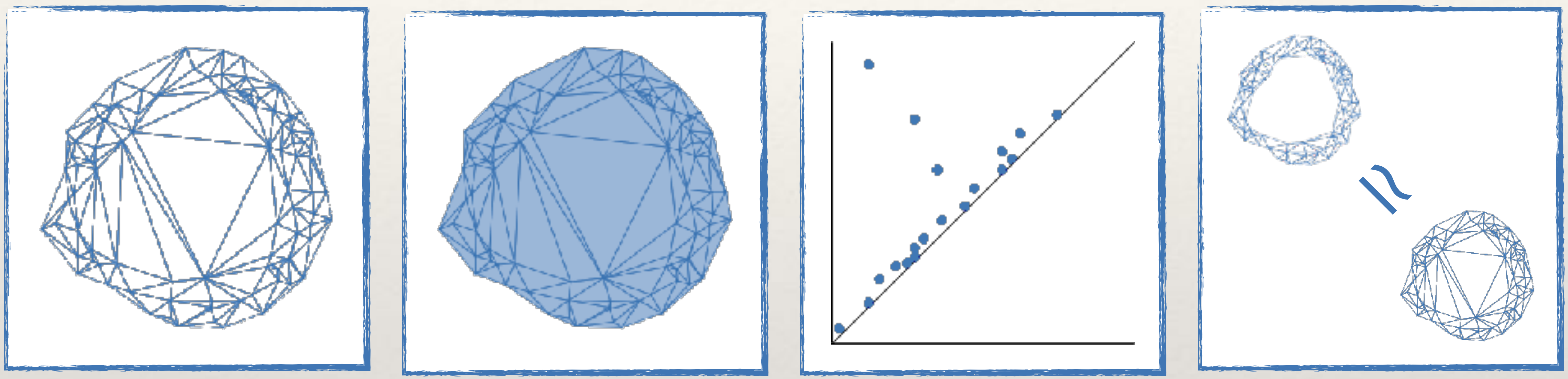
Topological Summaries have proven to be particularly effective to **distinguish shapes**

but

It's still hard to give a **meaningful interpretation** of what homological cycles represent

Persistence-based Network Analysis

A Common Pipeline in TDA:



Complex Construction *PH_k module* *Distance Computation*

What if …

we replace the PH_k module with something having a more natural interpretation?

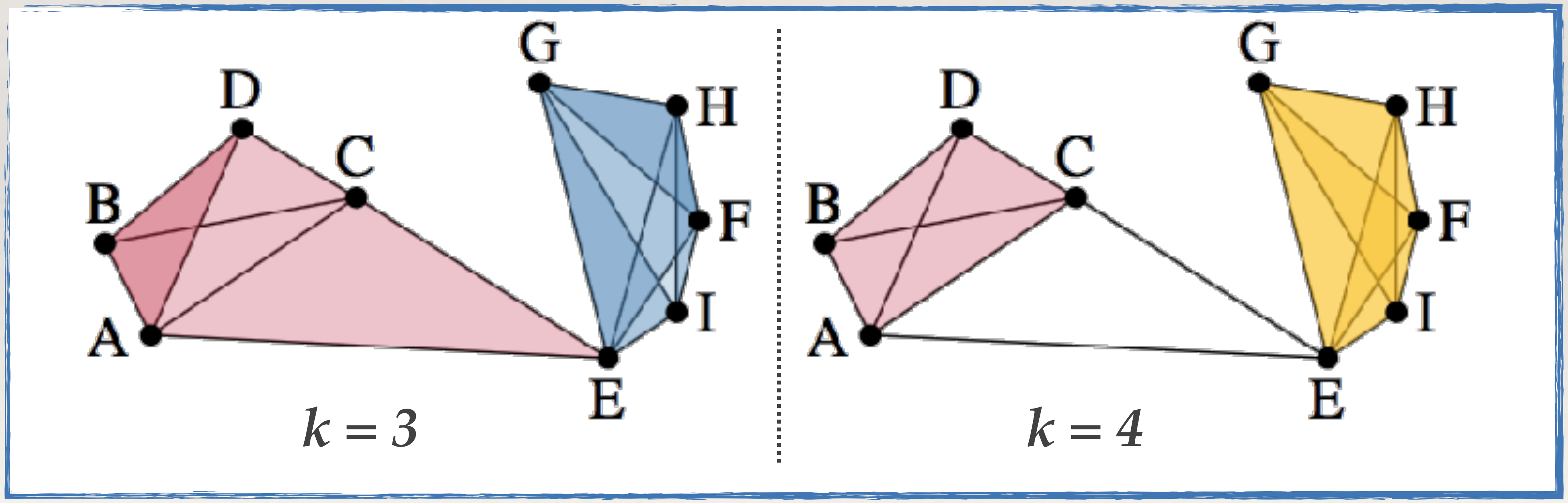
Persistence-based Network Analysis

k-Clique Communities:

Given a network $G=(V, E)$,

***Clique of size k*:**

a complete subgraph of k vertices of G ⇔ *a (k-1)-simplex of Flag(G)*



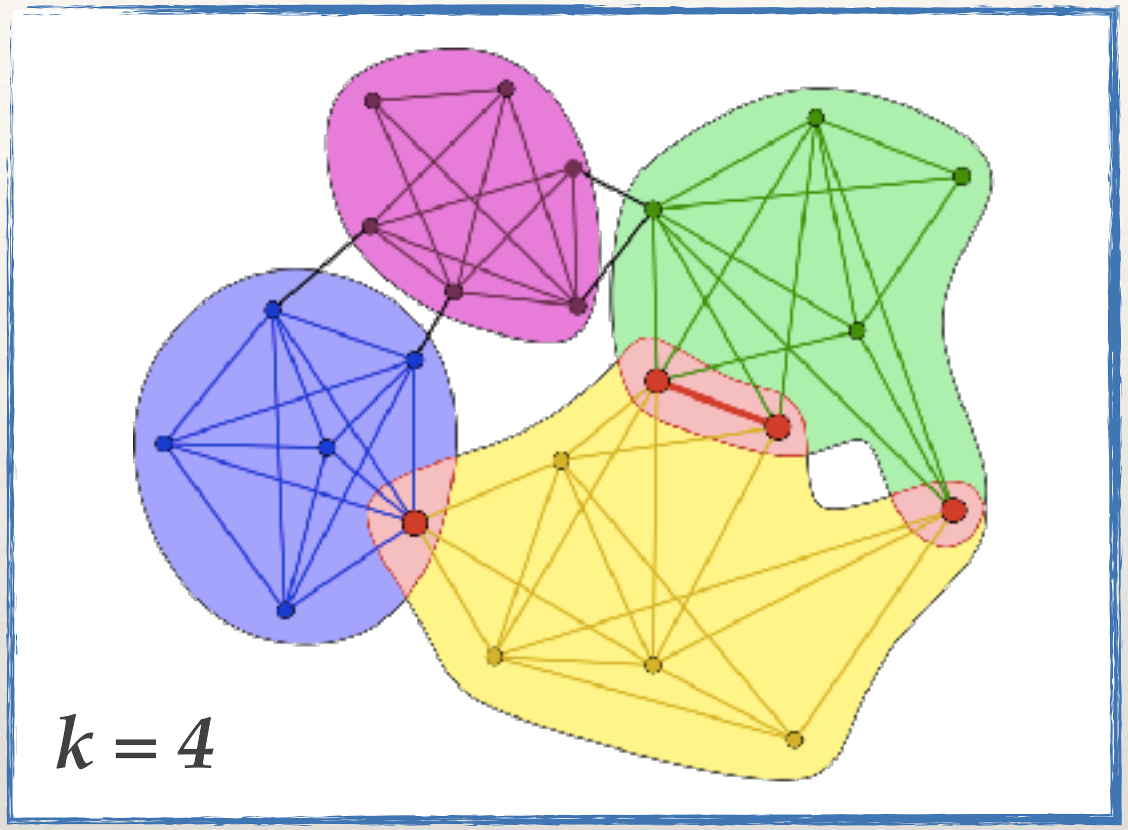
***k-adjacency*:** two clique of size k are k-adjacent if they share k-1 nodes

***k-clique community*:** maximal union of cliques of size k pairwise connected by a sequence of k-adjacent cliques

Persistence-based Network Analysis

k-Clique Community Decomposition:

- *Reveal **Highly Connected** Communities*
- *Allow **Overlaps***
- *Have a **Hierarchical Structure***

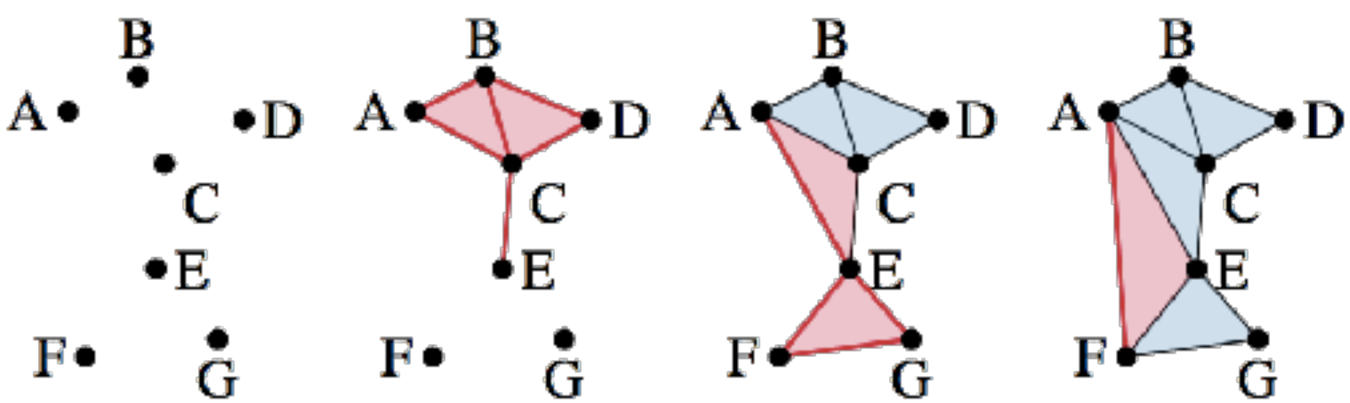


- *Difficult to be Computed*
- *What is the right k?*
- *How to Manage Weighted Networks?*

Persistence-based Network Analysis

Clique Community Persistence:

Use **Persistent Homology** *to keep track of the evolution of these communities*

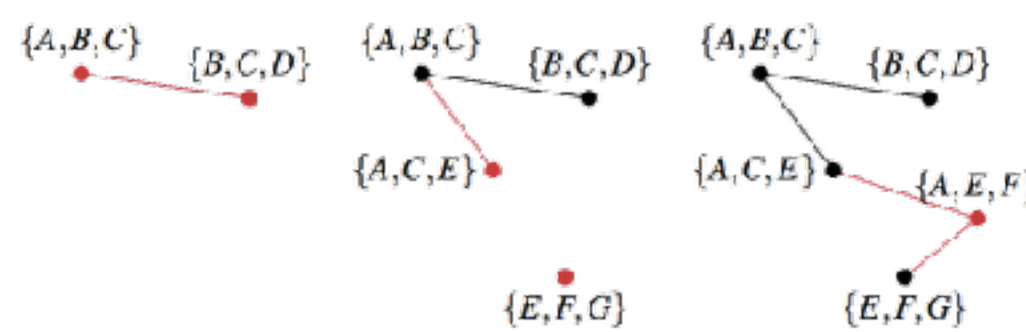


(a) $\varepsilon = 0.0$ (b) $\varepsilon = 1.0$ (c) $\varepsilon = 2.0$ (d) $\varepsilon = 3.0$

k-clique communities of G

0-homology of G_k

(its connected components)



(a) $\varepsilon = 0.0$ (b) $\varepsilon = 1.0$ (c) $\varepsilon = 2.0$ (d) $\varepsilon = 3.0$

Persistence-based Network Analysis

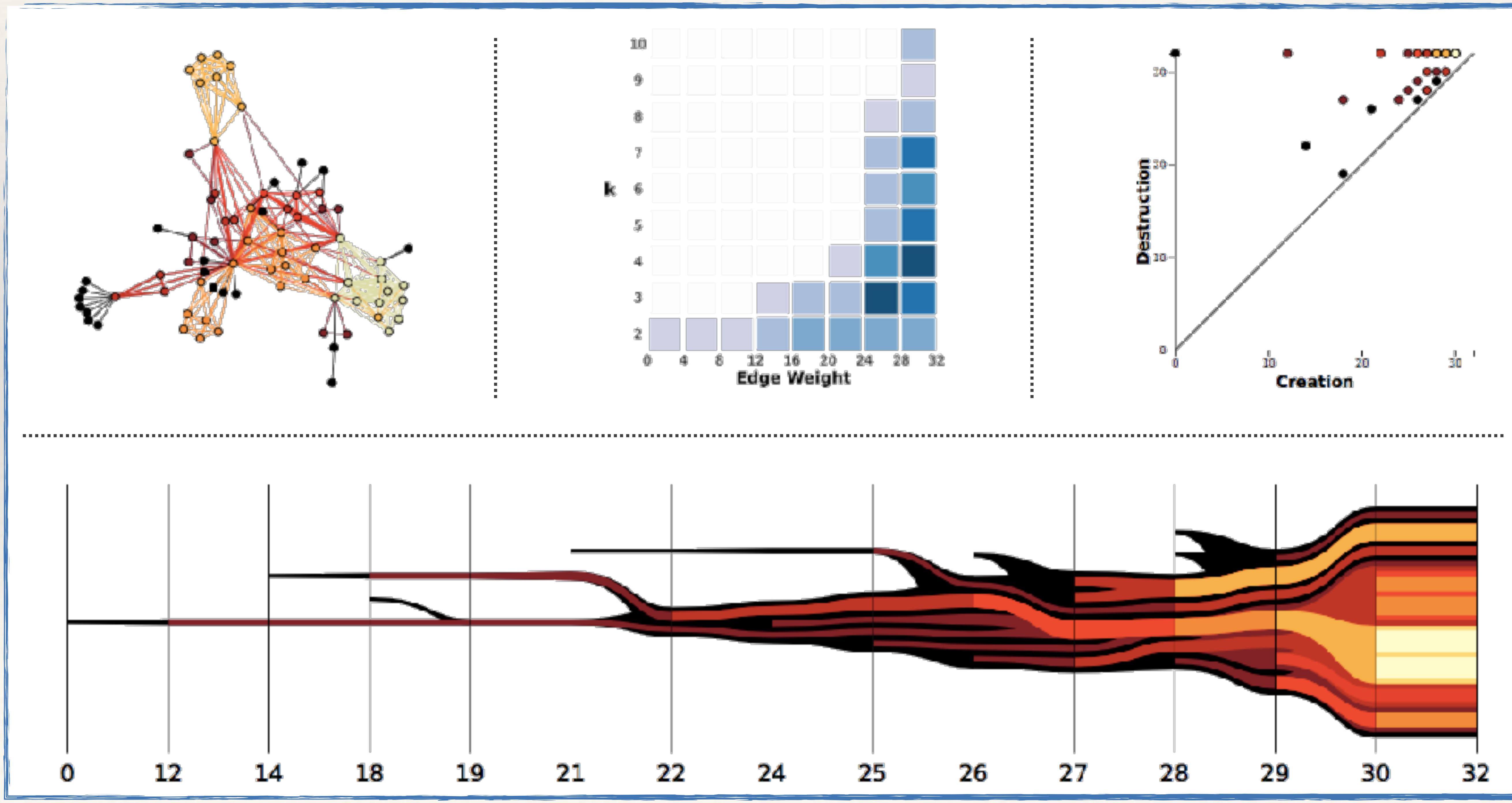
Clique Community Persistence:

The presented approach allows for designing:

- **Nested Graph Visualization**
- Definition of a new **Centrality Measure**
- Introduction of a **Persistence Indicator Function**
- …

Persistence-based Network Analysis

Nested Graph Visualization:



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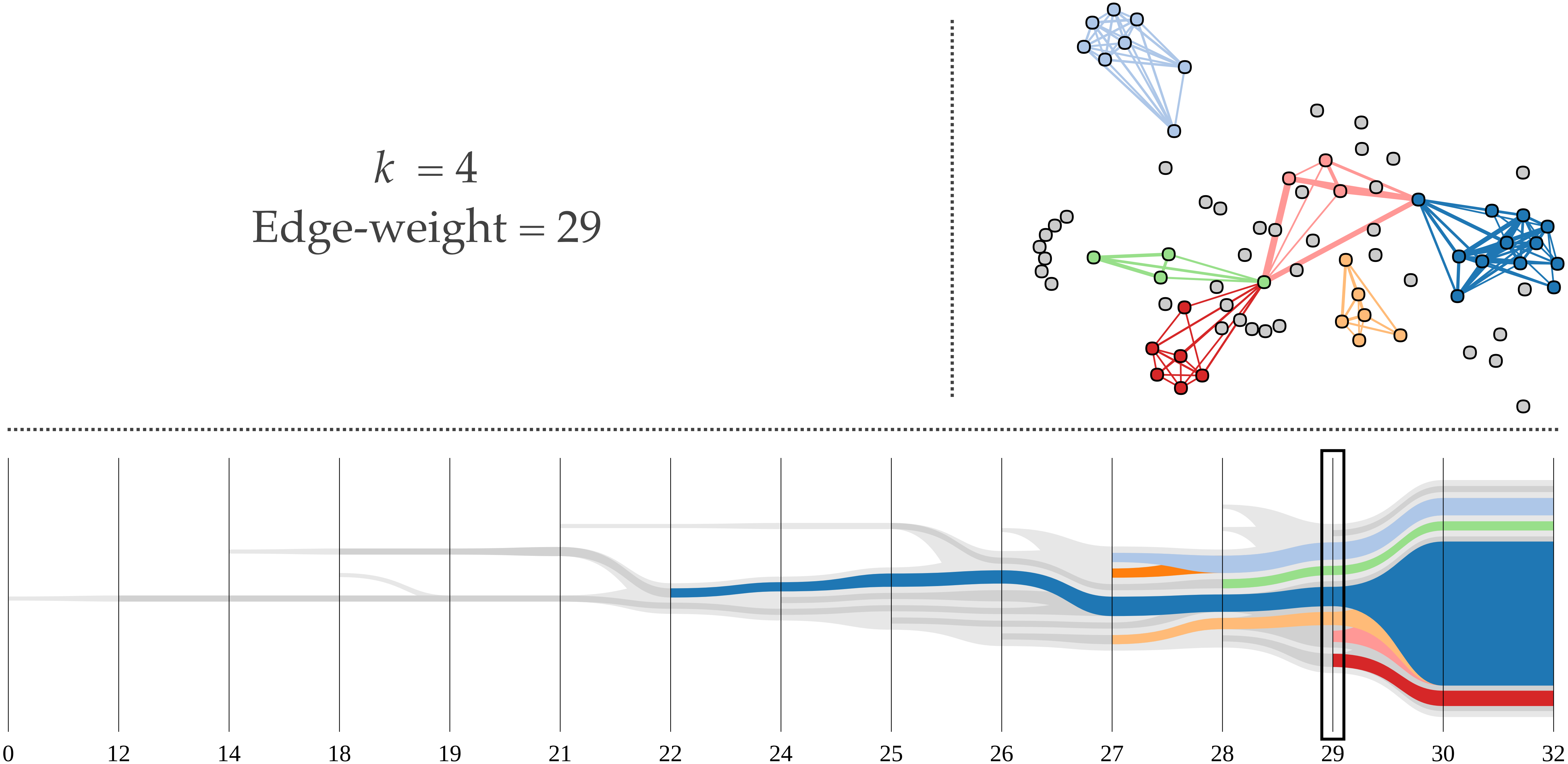
Nested Graph Visualization:

$k = 4$

Edge-weight = 29



Outline

Brief Introduction to Complex Network Analysis

Persistence-based Network Analysis

Thank you

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