Solutions

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Answer Q.1. ESL 5.4 : Recall that for natural cubic spline, the function is linear beyond the boundary knots. So we first consider the case where $X < \xi_1$, then we have

$$f(X) = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3$$

We need $\beta_2 = 0$ and $\beta_3 = 0$ to get linearity. Then we consider the case where $X > \xi_K$, then we have

$$\begin{aligned} f(X) &= \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \sum_{k=1}^{K} \theta_k (X - \xi_k)^3 \\ &= \beta_0 + \beta_1 X + X^3 \sum_{k=1}^{K} \theta_k - 3X^2 \sum_{k=1}^{K} \theta_k \xi_k + 3X \sum_{k=1}^{K} \theta_k \xi_k^2 - \sum_{k=1}^{k} \theta_k \xi_k^3 \end{aligned}$$

In order to get linearity, we conclude that $\sum_{k=1}^{K} \theta_k = 0$ and $\sum_{k=1}^{K} \theta_k \xi_k = 0$. To derive a formula like 5.4 and 5.5, suppose that f(X) can written in the form of $\sum_{k=1}^{K} \alpha_k N_k(X)$ and we need to find an expression of α_k. It is obvious that $\alpha_1 = 1$ and $\alpha_2 = X$, then we calculate

$$\begin{aligned} \sum_{k=1}^{K-2} \alpha_k N_{k+2}(X) &= \sum_{k=1}^{K-2} (\xi_K - \xi_k)\theta_k [\frac{(X-\xi_k)_+^3 - (X-\xi_K)_+^3}{\xi_K - \xi_k} - \frac{(X-\xi_{K-1})_+^3 - (X-\xi_K)_+^3}{\xi_k - \xi_{K-1}}] \\ &= \sum_{k=1}^{K-2} \theta_k (X-\xi_k)_+^3 - (X-\xi_K)_+^3 \sum_{k=1}^{K-2} \theta_k - (\frac{1}{\xi_K - \xi_{K-1}}) \\ &(\xi_K \sum_{k=1}^{K-2} \theta_k - \sum_{k=1}^{K-2} \theta_k \xi_k)[(X-\xi_{K-1})_+^3 - (X-\xi_K)_+^3] \\ &= \sum_{k=1}^{K-2} \theta_k (X-\xi_k)_+^3 + (\theta_{K-1} + \theta_K)(X-\xi_K)_+^3 \\ &- \frac{-\xi_K \theta_{K-1} - \xi_K \theta_K + \xi_{K-1}\theta_{K-1} + \theta_K \xi_K}{\xi_K - \xi_{K-1}} [(X-\xi_{K-1})_+^3 - (X-\xi_K)_+^3] \\ &= \sum_{k=1}^{K-2} \theta_k (X-\xi_k)_+^3 + (\theta_{K-1} + \theta_K)(X-\xi_K)_+^3 + \theta_{K-1}[(X-\xi_{K-1})_+^3 - (X-\xi_K)_+^3] \\ &= \sum_{k=1}^{K-2} \theta_k (X-\xi_k)_+^3 + \theta_{K-1}(X-\xi_{K-1})_+^3 + \theta_K (X-\xi_K)_+^3 \\ &= \sum_{k=1}^{K} \theta_k (X-\xi_k)_+^3 \end{aligned}$$

This is the exact result we want.

Answer Q.2. ESL 5.13 Recall that we have

$$\hat{f}_\lambda(x_i) = \sum_{j=1}^{n} S_\lambda(i,j)y_j$$

and we use $\hat{f}_\lambda^{-i}(x_i)$ to denote the case when i-th observation is taken out. Notice that

$$(1 - S_\lambda(i,i))\hat{f}_\lambda^{-1}(x_i) = \sum_{i \neq j} S_\lambda(i,j)y_j$$

Combine these two functions together we have

$$\hat{f}_\lambda^{-i}(x_i) = \hat{f}_\lambda^{i}(x_i) + S_\lambda(i,i)\hat{f}_\lambda^{-i}(x_i) - S_\lambda(i,i)y_i$$

Rearrange gives us

$$y_i - \hat{f}_\lambda^{-i}(x_i) = \frac{y_i - \hat{f}_\lambda^{i}(x_i)}{1 - S_\lambda(i,i)}$$

Summing both sides gives us the formula of CV.