

Lecture 2: Equilibria and stability

- An equilibrium is where the function in the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a zero solution, i.e. $\mathbf{x}^* \in \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}^*) = 0$.
- There may be many solutions to the equation $\mathbf{f}(\mathbf{x}^*) = 0$, but each is characterised by $\mathbf{f}(\mathbf{x}^*) = 0 \Rightarrow \dot{\mathbf{x}} = 0$, i.e. \mathbf{x} does not change. Equilibrium points are sometimes be called 'fixed points'.
- For a **linear** system with non-zero eigenvalues there is only one solution to $\dot{\mathbf{x}} = \mathbf{Ax}$, i.e. $\mathbf{x} = 0$. A nonlinear system can have many non-zero equilibria.

Maps and equilibria

- In a similar way to differential equations, the equilibria of **maps** are the fixed points of the update equations, i.e.

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k) \text{ such that } \mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}^*$$

- In the case of differential equations, the equilibria are special points on a **flow** in phase space, whereas maps jump around without a flow – this property makes maps hard to understand and visualise.

Flows and equilibria

- The solution of the non-linear differential equation in phase space can be thought of as a flow in an n -dimensional phase space, like a liquid. $\mathbf{f}(\mathbf{x})$ can be thought of as a vector field describing the flow or ‘fluid velocity’.
- Flows can end or start at equilibria, or can circulate around equilibria
- The stability of the flow near an equilibrium is an important property and something we will study today

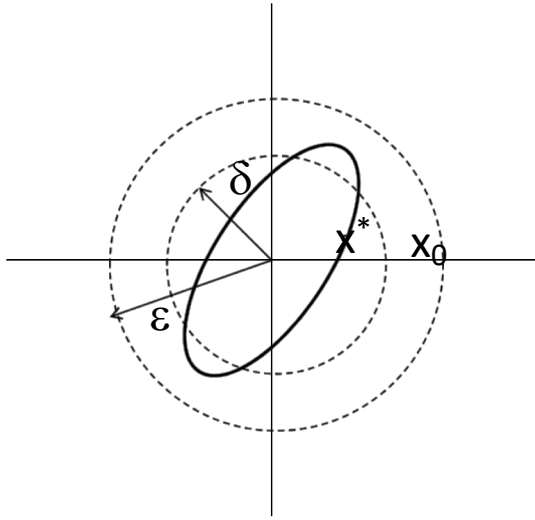
Stability of flow equilibria

- **Definition:** An equilibrium point \mathbf{x}^* is said to be **stable** if, given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that all solutions $\mathbf{y}(t)$ satisfy

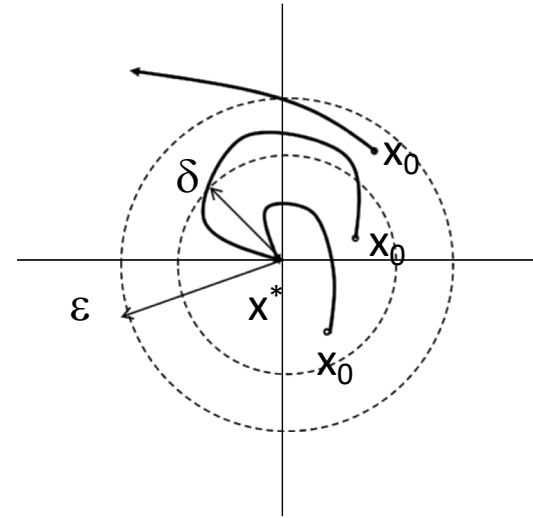
$$|\mathbf{y}(t) - \mathbf{x}^*| < \epsilon \text{ for all } t \geq 0 \text{ whenever } |\mathbf{y}(0) - \mathbf{x}^*| < \delta$$

- Otherwise it is **unstable**
- **Definition:** An equilibrium point \mathbf{x}^* is said to be **asymptotically stable** if it is stable and there is a real number $b > 0$ such that

$$\lim_{t \rightarrow \infty} |\mathbf{y}(t) - \mathbf{x}^*| = 0 \text{ whenever } |\mathbf{y}(0) - \mathbf{x}^*| < b$$



A stable equilibrium



An asymptotically stable equilibrium

- The solution cannot escape from a stable equilibrium
- The solution ends up 'going down the hole' if it starts close enough to an asymptotically stable equilibrium

Epsilon-delta arguments

- The arguments in the definitions are in the form of a game
 - I give you a positive number ε that I am free to choose
 - You respond with another number δ that satisfies some condition
 - If you can always find a number δ then you ‘win’
- Many proofs in mathematics are based on this terminology

Exponential stability

- **Definition:** An equilibrium point \mathbf{x}^* is said to be exponentially stable if \mathbf{x}^* is asymptotically stable and there exist finite $\alpha, \beta, \delta > 0$ such that if $|\mathbf{y}(0) - \mathbf{x}^*| < \delta$ then

$$|\mathbf{y}(t) - \mathbf{x}^*| \leq \alpha e^{-\beta t} |\mathbf{y}(0) - \mathbf{x}^*| \text{ for all } t \geq 0$$

- The above is a statement about the **rate** of convergence of the solution to the equilibrium point – how ‘fast’ it ‘goes down the hole’.

Flows in 2x2 linear systems

- We will be studying the flows around equilibrium points using their local linearisations
- Each flow has a topology (shape) that falls into one of a number of distinct categories
- These flows can often be continuously distorted into the flows that solve the non-linear differential equations – so the topologies of a number of exemplar linear systems can provide families to which the non-linear solutions belong

The uncoupled case

$$\dot{x}_1 = \alpha_1 x_1$$

$$\dot{x}_2 = \alpha_2 x_2$$

Solution:

$$x_1(t) = e^{\alpha_1 t} x_1(0)$$

$$x_2(t) = e^{\alpha_2 t} x_2(0)$$

These are parametric equations of curves in phase space

The equation of any such curve is of the form

$$x_1 = c x_2^{\alpha_1/\alpha_2}$$

The uncoupled case

$$\begin{aligned}x_1(t) &= e^{\alpha_1 t} x_1(0) \\ x_2(t) &= e^{\alpha_2 t} x_2(0)\end{aligned}$$

- The equilibrium point is the origin (it is a linear autonomous system)
- If $\alpha_1 < 0$ and $\alpha_2 < 0$, then the equilibrium is asymptotically (exponentially) stable
- If $\alpha_1 > 0$ or $\alpha_2 > 0$ the origin is unstable

Coupled 2x2 linear systems

- The general form

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2\end{aligned}$$

Or

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ \mathbf{A} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\end{aligned}$$

Solution

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$$

The eigenvalues

The eigenvalues of \mathbf{A} satisfy

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

or

$$\lambda^2 - \text{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

The trace $\tau = \text{tr}(\mathbf{A})$ of a matrix \mathbf{A} is the sum of its eigenvalues:

$$\lambda_1 + \lambda_2 = \tau$$

the determinant $D = \det(\mathbf{A})$ of \mathbf{A} is the product of its eigenvalues:

$$\lambda_1 \lambda_2 = D$$

Solving using eigenvectors and values

- If the eigenvalues are real and distinct, then are two independent eigenvectors and we can re-write the initial condition as

$$\mathbf{x}(0) = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$$

Now work through the solution:

$$\begin{aligned}\mathbf{x}(t) &= e^{t\mathbf{A}}\mathbf{x}(0) \\ &= \left(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \dots \right) (c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) \\ &= c_1 e^{t\lambda_1} \mathbf{v}_1 + c_2 e^{t\lambda_2} \mathbf{v}_2\end{aligned}$$

(we have assumed neither eigenvalue is zero)

Some characteristics

- If $\operatorname{Re}(\lambda) < 0$, then the component along the corresponding eigenvector decays to zero
- If $\operatorname{Re}(\lambda) > 0$, then the component along the corresponding eigenvector grows without bound
- If $\lambda = 0$, then the component along the corresponding eigenvector is constant
- If $\operatorname{Im}(\lambda) \neq 0$, then the solution spirals around the origin
- If $\operatorname{Im}(\lambda) = 0$, then the solution does not spiral around the origin
- If λ is real, the solution will tend towards the eigenvector with the dominant eigenvalue

Coordinate transformation

- We next discuss the characteristic shapes of trajectories in 2-D phase space using a coordinate transformation so that the matrix \mathbf{A} is in a standard form:
 - Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, define new coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$
 - The new equations of motion are then $\dot{\mathbf{y}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{y}$
 - If the eigenvalues are real and distinct and \mathbf{V} is the eigenvector matrix then $\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{y}$
 - Study the motion in the new coordinates. Motion in the original coordinates is obtained by multiplying $\mathbf{y}(t)$ by \mathbf{V}
- The transformed matrix $\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is called the '**normal form**'

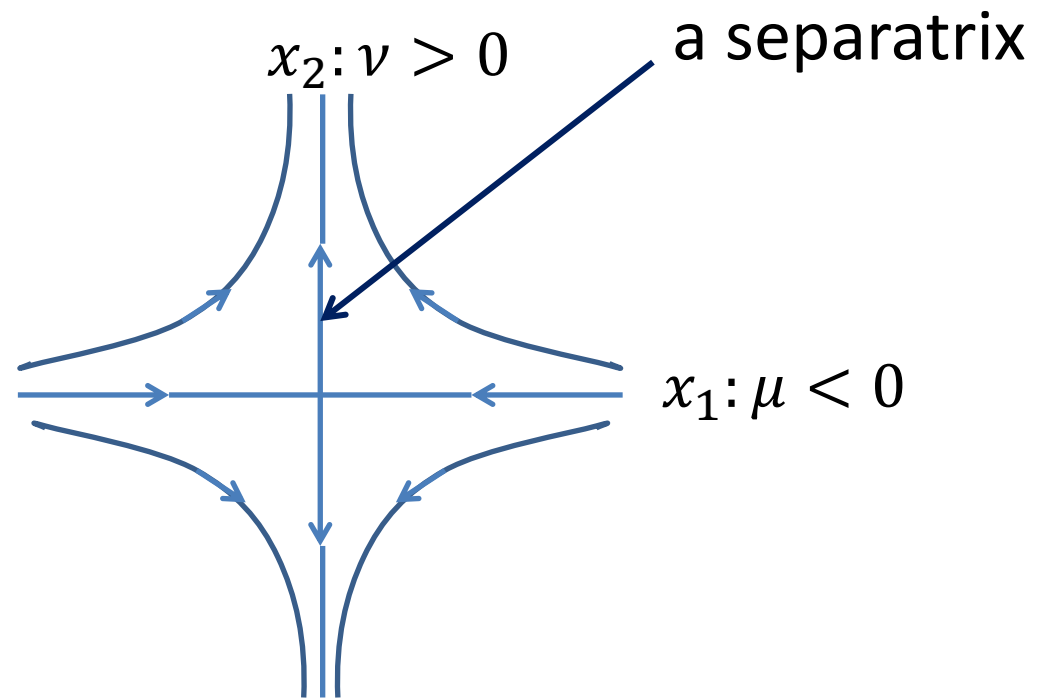
The Saddle normal form

- If the eigenvalues are real and non-zero and $\lambda_1\lambda_2 < 0$, then the equilibrium is unstable. In the transformed coordinates:

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix}$$

- There are four asymptotes that approach the origin, two are for $t \rightarrow \infty$ and two are for $t \rightarrow -\infty$
- These four non-zero trajectories are called **separatrices**

The Saddle shape



The Stable Node

- If the eigenvalues are real and both negative $\lambda_1 < 0, \lambda_2 < 0$, then
 - the eigenvalues are distinct:

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix}$$

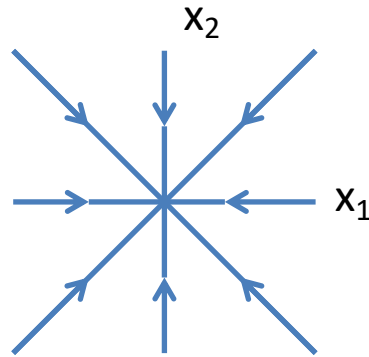
- or \mathbf{A} is diagonal: $\mathbf{A} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$
 - or the eigenvalues are equal, but there is only one eigenvector (the degenerate case):

$$\tilde{\mathbf{V}}^{-1}\mathbf{A}\tilde{\mathbf{V}} = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}$$

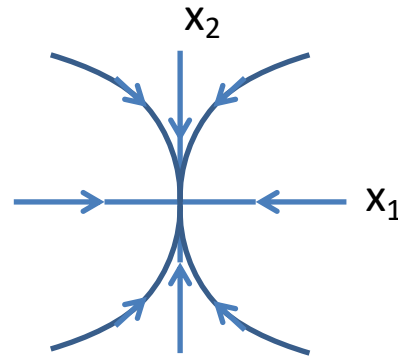
Here $\tilde{\mathbf{V}}$ consists of ‘generalized eigenvectors’ (see Perko Section 1.3)

Stable and unstable node shapes

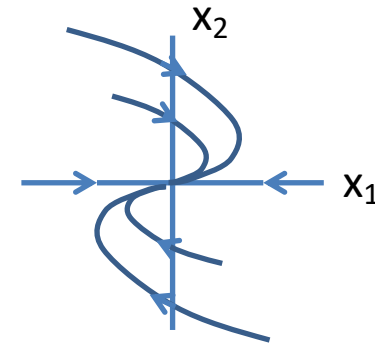
- Stable:



Diagonal matrix: $\mathbf{A} = \mu \mathbf{I}$



Generic case $\mu < \nu$



Degenerate case

- Unstable: Both eigenvalues are positive. The same shapes as above, but reverse the directions of the arrows.

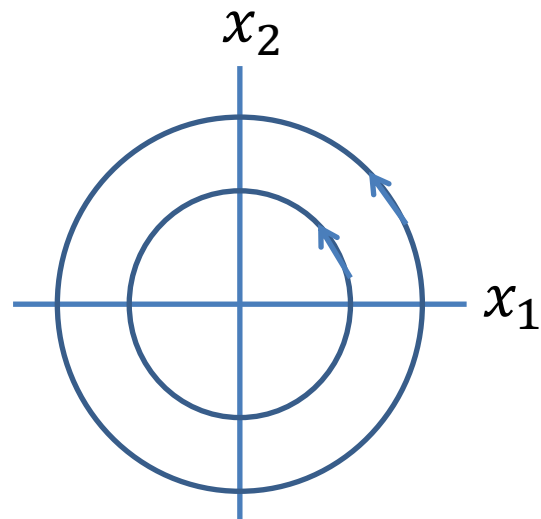
The Centre Normal Form

- If the eigenvalues are purely imaginary (the real part is zero and the imaginary part is not zero). The equilibrium is neutrally stable
- There are no real eigenvectors and so use the construction from Lecture 1 to find the coordinate transformation matrix

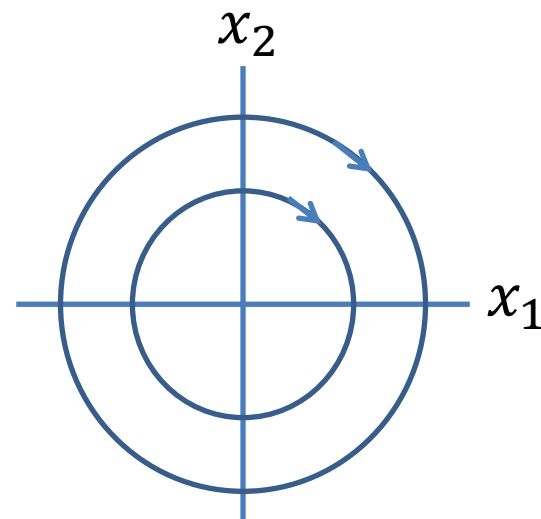
$$\tilde{\mathbf{V}}^{-1} \mathbf{A} \tilde{\mathbf{V}} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

- Matrix $\tilde{\mathbf{V}}$ is constructed from the real and imaginary parts of the complex eigenvectors

Centre Normal form Phase Space Portrait



$$b > 0$$



$$b < 0$$

Stable and unstable spirals (foci)

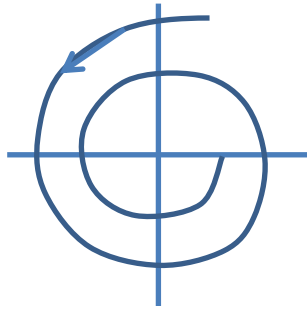
The normal form of the matrix \mathbf{A} is:

$$\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
$$\lambda_1 = a + jb, \quad \lambda_2 = a - jb,$$

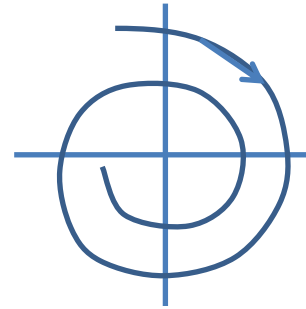
- $a < 0$: the spiral is stable
- $a > 0$: the spiral is unstable
- $b > 0$: the spiral is anti-clockwise
- $b < 0$: the spiral is clockwise

Spirals or foci

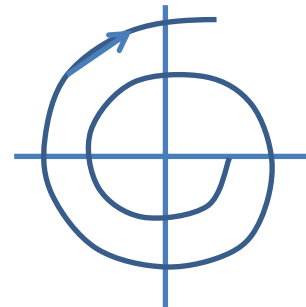
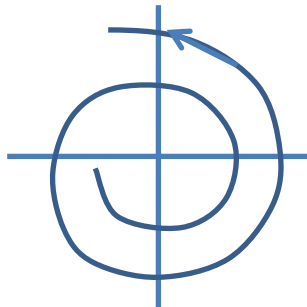
$$b > 0$$



$$b < 0$$



Stable focus



Unstable focus

The Phase Space equations so far

- We have drawn the diagrams of the solutions of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$:

$$\text{- } \dot{\mathbf{x}} = \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix} \mathbf{x} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} e^{\mu t} & 0 \\ 0 & e^{\nu t} \end{bmatrix} \mathbf{x}_0$$

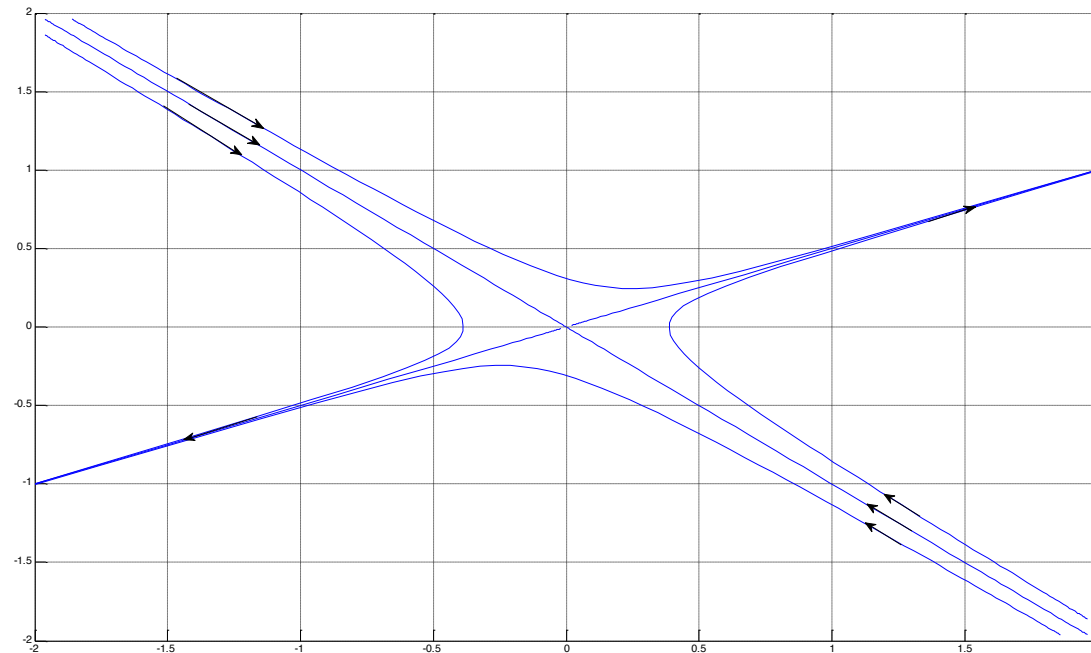
$$\text{- } \dot{\mathbf{x}} = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix} \mathbf{x} \Rightarrow \mathbf{x}(t) = e^{\mu t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$$

$$\text{- } \dot{\mathbf{x}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{x} \Rightarrow \mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0$$

Changing coordinates

- Consider $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{x}$
- Characteristic equation: $\lambda^2 + \lambda - 2 = 0$, eigenvalues: 1, -2
eigenvectors: $[2 \ 1]^\top$ and $[-1 \ 1]^\top$.
- Thus $\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$ i.e. $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$
- Define new coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$ defining new axes along $[2 \ 1]^\top$ ($\lambda=1$ and thus unstable) and $[-1 \ 1]^\top$ ($\lambda=-2$ and thus stable) .

$$\begin{aligned}\mathbf{x}(t) &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}_0 \\ &= \begin{bmatrix} 2/3 e^t + 1/3 e^{-2t} & 2/3 e^t - 2/3 e^{-2t} \\ 1/3 e^t - 1/3 e^{-2t} & 1/3 e^t + 2/3 e^{-2t} \end{bmatrix} \mathbf{x}_0\end{aligned}$$



Nonlinear systems

Consider the solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ near to where $\mathbf{f}(\mathbf{x}^*) = 0$.

Let $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$, assume \mathbf{f} is differentiable and note that the equilibrium point does not change with time. Taylor expansion of f_j about \mathbf{x}^* :

$$f_j(\mathbf{x}^* + \mathbf{w}) = f_j(\mathbf{x}^*) + \sum_{k=1}^n \frac{\partial f_j}{\partial x_k} w_k + \mathcal{O}(|\mathbf{w}|^2)$$

For the entire vector of functions \mathbf{f} , also noting that $\mathbf{f}(\mathbf{x}^*) = 0$:

$$\dot{\mathbf{x}} = \dot{\mathbf{w}} = D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w} + \mathcal{O}(|\mathbf{w}|^2)$$

$D\mathbf{f}$ is the Jacobian whose (i, j) th element is $\frac{\partial f_i}{\partial x_j}$

Small perturbation model

Close to the equilibrium $\dot{\mathbf{w}} \cong D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w}$ and is approximated by the linear system

$$\dot{\mathbf{w}} = D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w} = \mathbf{A}\mathbf{w}$$

We consider the stability of this linear system near to the origin ($\mathbf{w}=0$). Note we then have

$$\mathbf{w}(t) = e^{\mathbf{A}t}\mathbf{w}(0) = e^{D\mathbf{f}(\mathbf{x}^*)t}\mathbf{w}(0)$$

The solution is asymptotically stable if all the eigenvalues of \mathbf{A} have negative real parts.

Hyperbolic equilibria

- **Definition** (Hyperbolic equilibrium): Let \mathbf{x}^* be an equilibrium of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then \mathbf{x}^* is called a hyperbolic fixed point if none of the eigenvalues of $D\mathbf{f}(\mathbf{x}^*)$ real part equal to 0.
- **Theorem:** Suppose \mathbf{x}^* is a hyperbolic fixed point and all the real parts of the eigenvalues are negative. Then the equilibrium solution $\mathbf{x} = \mathbf{x}^*$ is asymptotically stable.

The properties of hyperbolic equilibria

- If the eigenvalues of the local linearisation all have negative real parts, then we have asymptotic stability of the non-linear system at that equilibrium. Note we say nothing about how big the region of stability is – this depends on your number δ that you responded with during the ‘game’.
- The properties of the solution of the non-linear system near to a hyperbolic equilibrium point are similar in shape to the linearised system – they are topologically identical (covered in Lecture 3).
- If any of the eigenvalues have zero real part, further investigation is needed.

The Duffing Oscillator example

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^3 - \gamma y, \quad \gamma \geq 0\end{aligned}$$

Equilibria: $(x^*, y^*) = (0,0)$ and $(\pm 1,0)$

$$\text{Jacobian: } D\mathbf{f} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -\gamma \end{bmatrix}$$

At $(0,0)$: $\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4}}{2}$ so the equilibrium point is unstable

At $(\pm 1,0)$: $\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 8}}{2}$ so for $\gamma > 0$ we have asymptotic stability

but for $\gamma = 0$ we have a centre in this case the local behaviour of the linearization and the nonlinear system are different!