C24 Dynamical Systems, Examples Sheet C24|1

Michaelmas Term 2018

(1)

1. $\dot{x}(t) = -x(t) + t$ (1)

(a) Instantaneous equilibrium:

$x = t \Rightarrow \dot{x} = 0$ hence 'equilibrium'

But time moves:

Transform to autonomous system by adding second state:

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = 1$$

$\Rightarrow$ No equilibrium for any finite x.

(b) solution of homogeneous equation

$\dot{y}(t) = -y(t)$ is $y(t) = e^{-t}$

Set $x(t) = y(t) \cdot z(t)$ and substitute in (1):

$$\dot{x}(t) = \underbrace{\dot{y}(t)}_{-y(t)} z(t) + y(t)\dot{z}(t) = -x(t) + y(t)\dot{z}(t)$$

$\Rightarrow y(t)\dot{z}(t) = t \qquad \Rightarrow \dot{z}(t) = (y(t))^{-1} \cdot t$

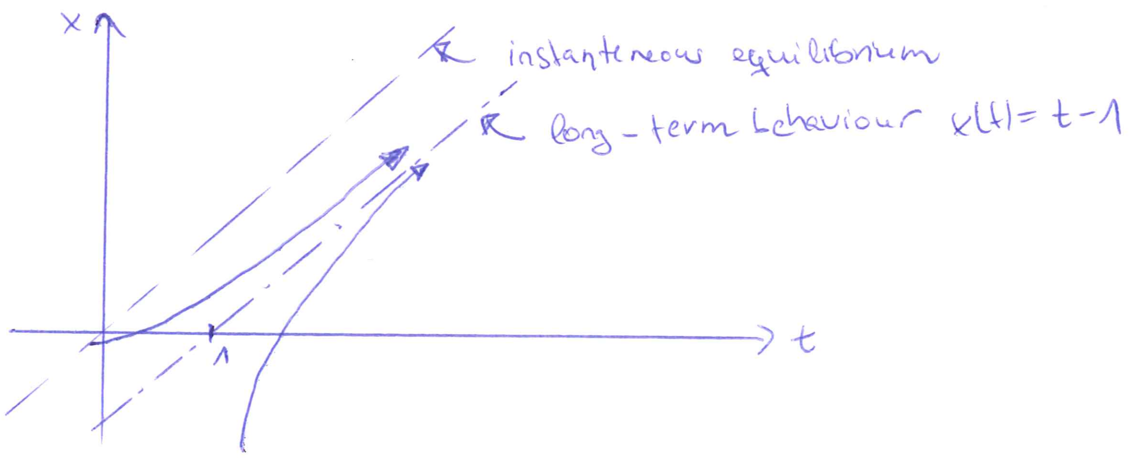
$\Rightarrow \dot{z}(t) = te^{t} \quad \Rightarrow z(t) = \int_{t_0}^{t} s e^{s} ds = te^{t} - e^{t} + c$

$\Rightarrow x(t) = e^{-t}(te^{t} - e^{t} + c) = t - 1 + ce^{-t}$

$x_0 = x(0) = -1 + c \quad \Rightarrow \quad c = x_0 + 1$

$\Rightarrow x(t) = t - 1 + (x_0 + 1)e^{-t}$

(c)



2. $$\left.\begin{aligned} x_{n+1} &= \lambda x_n + y_n \\ y_{n+1} &= \mu y_n \end{aligned}\right\} \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$z_{n+1} = \begin{pmatrix} \lambda & 1 \\ 0 & \mu \end{pmatrix} z_n . \qquad z_n = A^n z_0$$

Eigenvalues / vectors of A: $\lambda, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\mu, \begin{pmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{pmatrix}$ for $\mu \neq \lambda$

for $\mu = \lambda$: λ (repeated), $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (degenerate, only one eigenvector)

If A is diagonalisable,

$$A = \underbrace{\begin{pmatrix} 1 & \frac{1}{\mu-\lambda} \\ 0 & 1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1 & \frac{-1}{\mu-\lambda} \\ 0 & 1 \end{pmatrix}}_{V^{-1}}$$

Note that $A^n = V \Lambda^n V^{-1}$. With $w_n = V^{-1} z_n \Rightarrow w_n = \Lambda^n w_0$

(a) $|\lambda|, |\mu| > 1$: unstable equilibrium (node), $E^s = \phi$, $E^c = \phi$, $E^u = \mathbb{R}^2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{pmatrix} \right\}$

$\mu \neq \lambda$: two invariant directions
$\mu = \lambda$: one invariant direction

(b) $|\lambda|, |\mu| < 1$: asymp. stable equilibrium (node), $E^s = \mathbb{R}^2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{pmatrix} \right\}$, $E^c = \phi$, $E^u = \phi$

$\mu \neq \lambda$: two invariant directions
$\mu = \lambda$: one invariant direction

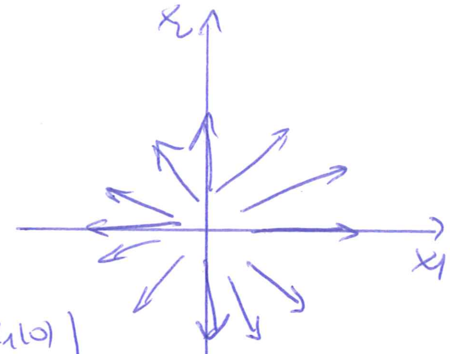
(c) $|\lambda| > 1$, $|\mu| < 1$: unstable 'saddle' : $E^u = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, $E^s = \text{span}\left\{ \begin{pmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{pmatrix} \right\}$, $E^c = \phi$

($\Rightarrow \lambda \neq \mu$) two invariant directions

(d) $|\lambda| = 1$, $|\mu| > 1$: unstable equilibrium $E^s = \phi$, $E^u = \text{span}\left\{ \begin{pmatrix} \frac{1}{\mu-\lambda} \\ 1 \end{pmatrix} \right\}$, $E^c = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$(0,0)$ not necessary isolated:
for $\lambda = 1$: vector components along $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are fixed
$\Rightarrow$ continuum of equilibrium points along $c \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

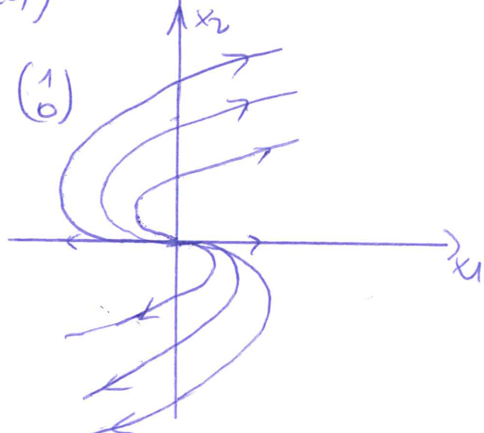
3. (a) (i) $\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$, $x_1(t) = e^t x_1(0)$
$x_2(t) = e^t x_2(0)$



$E^u = \mathbb{R}^2$, $E^s = \phi$, $E^c = \phi$

unstable 'star'

(ii) $\dot{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x$, $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$



Eigenvector corresponding to eigenvalue 1 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

At $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \dot{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E^s = \phi$, $E^u = \mathbb{R}^2$, $E^c = \phi$

(iii)

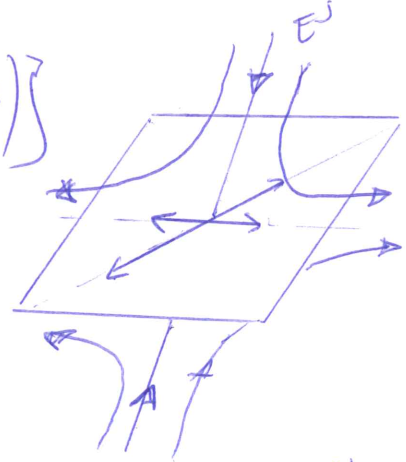
$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 0 & 0 & 2/3 \\ 0 & 1 & -2/3 \\ 1 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -0.5 & 0 & 1 \\ 1 & 1 & 0 \\ 1.5 & 0 & 0 \end{pmatrix} x$$

$$\text{So } x(t) = e^{At} x(0) = \begin{pmatrix} 0 & 0 & 2/3 \\ 0 & 1 & -2/3 \\ 1 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} -0.5e^{-t} & 0 & e^{-t} \\ e^{2t} & e^{2t} & 0 \\ 1.5e^t & 0 & 0 \end{pmatrix} x(0)$$

$$= \begin{pmatrix} e^t & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 \\ -0.5e^{-t} + 0.5e^t & 0 & e^{-t} \end{pmatrix} x(0)$$

$E^s = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$, $E^u = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}\right\}$

$E^c = \phi$

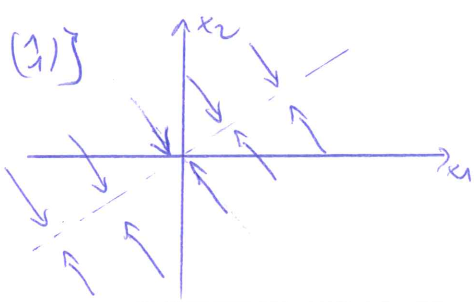


(iv) $\dot{x} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{pmatrix} x$

$$x(t) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{pmatrix} x(0)$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0.5e^{-2t} & -0.5e^{-2t} \\ 0.5 & 0.5 \end{pmatrix} x(0) = \begin{pmatrix} 0.5(e^{-2t}+1) & 0.5(-e^{-2t}+1) \\ 0.5(-e^{-2t}+1) & 0.5(e^{-2t}+1) \end{pmatrix} x(0)$$

$E^s = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$, $E^u = \phi$, $E^c = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$



3.(b) $e^{\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}} = I + \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{pmatrix} + \dots$

$$= \begin{pmatrix} I - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots & -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} \dots \\ \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(c) $\dot{x} = Ax$, $Av_i = \lambda_i v_i$ $(*)$

$x(0) = c_1 v_1 + \dots + c_n v_n$

$x(t) = e^{At} x(0) = \left(I + At + \frac{A^2}{2!}t^2 + \dots + \frac{A^k t^k}{k!} + \dots\right) x(0)$

$$\overset{(*)}{=} c_1 v_1 + \dots c_n v_n$$
$$+ c_1 \lambda_1 t v_1 + \dots + c_n \lambda_n t v_n$$
$$+ c_1 \frac{(\lambda_1 t)^2}{2!} v_1 + \dots + c_n \frac{(\lambda_n t)^2}{2!} v_n$$
$$\vdots$$
$$+ c_1 \frac{(\lambda_1 t)^k}{k!} v_1 + \dots + c_n \frac{(\lambda_n t)^k}{k!} v_n + \dots$$

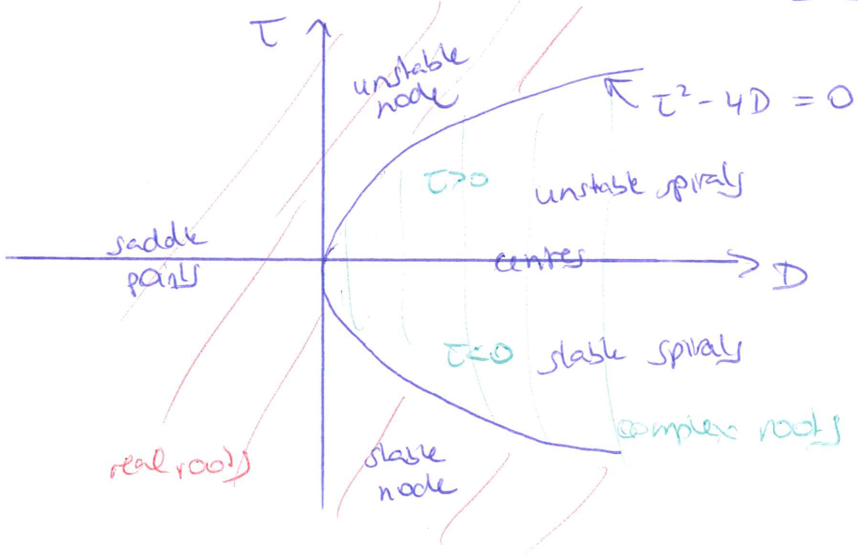
$$= c_1 e^{\lambda_1 t} v_1 + \dots + c_1 e^{\lambda_n t} v_n$$

(d) $\dot{x} = Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x$

eigenvalues: $0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - cb$

$\Rightarrow \lambda^2 - \underbrace{(a+d)}_{=\tau}\lambda + \underbrace{ad-bc}_{=D} = 0 \Rightarrow \lambda_{1,2} = \frac{\tau}{2} \pm \sqrt{\frac{\tau^2}{4} - D}$

$$= \frac{\tau \pm \sqrt{\tau^2 - 4D}}{2}$$



4) (a) $\dot{x}_1 = x_1(3-x_1-x_2) = 3x_1 - x_1^2 - x_1x_2$

$\dot{x}_2 = x_2(x_1-1) \qquad = x_1x_2 - x_2$

Equilibria: $(0,0), (1,2), (3,0)$

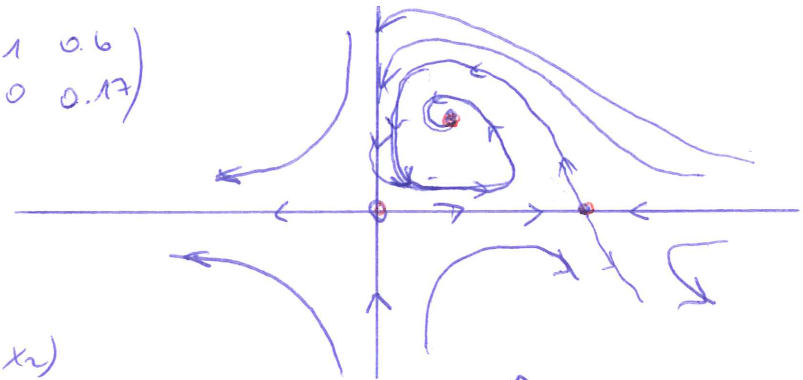
Jacobian
$$J = \begin{pmatrix} 3-2x_1-x_2 & -x_1 \\ x_2 & x_1-1 \end{pmatrix}$$

Equ: $(0,0)$ $\quad J|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ saddle

$(1,2)$ $\quad J|_{(1,2)} = \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}$ spiral (stable) counterclockwise

$(3,0)$ $\quad J|_{(3,0)} = \begin{pmatrix} -3 & -3 \\ 0 & 2 \end{pmatrix}$ saddle

$$= \begin{pmatrix} 1 & -0.51 \\ 0 & 0.86 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0.6 \\ 0 & 0.17 \end{pmatrix}$$



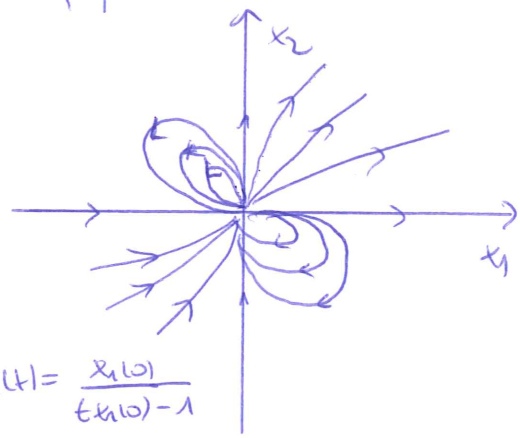
(b) $\dot{x}_1 = x_1^2 + x_1x_2 = x_1(x_1+x_2)$

$\dot{x}_2 = 0.5x_2^2 + x_1x_2 = x_2(x_1+0.5x_2)$

Equ. at $(0,0)$, linearisation gives $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ centre

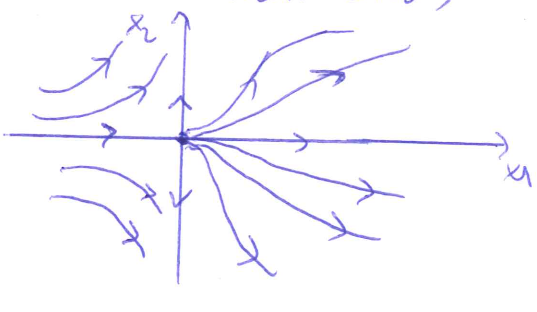
(use Matlab)



(c) $\dot{x}_1 = x_1^2 \Rightarrow \frac{dx_1}{x_1^2} = dt \Rightarrow -\frac{1}{x_1} = t - \frac{1}{x_1(0)} \Rightarrow x_1(t) = \frac{x_1(0)}{t x_1(0) - 1}$

$\dot{x}_2 = x_2 \quad \Rightarrow x_2(t) = e^t x_2(0)$

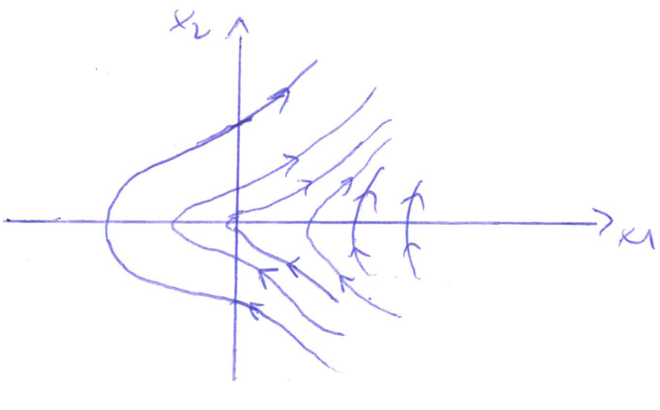
$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ unstable



(d) $\dot{x}_1 = x_2$

$\dot{x}_2 = x_1^2$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$



5. (a) Since $r^2 = x_1^2 + x_2^2$

Differentiating gives $r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$

$$\Rightarrow \boxed{\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}}$$

Similarly, $\tan\theta = \frac{x_2}{x_1}$

$$\Rightarrow (1+\tan^2\theta)\dot{\theta} = \frac{-\dot{x}_1 x_2 + \dot{x}_2 x_1}{x_1^2}$$

$$\Rightarrow \boxed{\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2\left(1+\frac{x_2^2}{x_1^2}\right)} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2+x_2^2} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}}$$

(b) $\dot{x}_1 = -x_2 + a\, x_1(x_1^2+x_2^2)$

$\dot{x}_2 = x_1 + a\, x_2(x_1^2+x_2^2)$

$$\dot{r} = \frac{-x_1x_2 + a x_1^2(x_1^2+x_2^2) + x_1x_2 + a x_2^2(x_1^2+x_2^2)}{r}$$

$$= \frac{a r^2(x_1^2+x_2^2)}{r} = a r^3 \Rightarrow \boxed{\dot{r} = a r^3}$$

$$\dot{\theta} = \frac{x_1^2 + a x_1x_2(x_1^2+x_2^2) + x_2^2 - a x_1x_2(x_1^2+x_2^2)}{x_1^2+x_2^2}$$

$$\Rightarrow \boxed{\dot{\theta} = 1}$$

$\Rightarrow$
- $a = 0 \Rightarrow$ centre (nonlinear)
- $a < 0 \Rightarrow$ stable spiral
- $a > 0 \Rightarrow$ unstable spiral

6. a) $\ddot{x} = f(x)$

Let $x_1 = x$, $x_2 = \dot{x}$

Then $\dot{x}_1 = x_2$

$\dot{x}_2 = f(x_1)$

(b) Consider $\dot{x}_1 x_1 + \dot{x}_2 x_2 = x_1 x_2 + x_2 f(x_1)$

$= x_2 (x_1 + f(x_1))$

$= \dot{x}_1 (x_1 + f(x_1))$

$\Rightarrow \dot{x}_2 x_2 = \dot{x}_1 f(x_1)$

$\Rightarrow -f(x_1)\dot{x}_1 + x_2 \dot{x}_2 = 0$

integration yields: $\underbrace{-\int_{x_1^0}^{x_1} f(s)\,ds + \frac{x_2^2}{2}}_{= V(x_1,x_2)} = \text{const}$

Note that we can write

$f(x_1) = -\frac{d}{dx_1}\phi(x_1)$ with $\phi(x_1) = -\int_{x_1^0}^{x_1} f(s)\,ds$

$\Rightarrow f$ is conservative force $\Leftrightarrow \ddot{x} = f(x)$ is conservative system

(c) $\left.\begin{array}{l} \dot{z}_1 = -z_2 - z_2^3 \\ \dot{z}_2 = z_1 \end{array}\right\} \xrightarrow[z_1 = x_2]{z_2 = x_1} \left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_1^3 \end{array}\right.$

Therefore $\dot{x}_2 x_2 = \dot{x}_1(-x_1 - x_1^3)$

$\Rightarrow \frac{d}{dt}\left(\frac{x_2^2}{2}\right) = -\frac{d}{dt}\left(\frac{x_1^2}{2}\right) - \frac{d}{dt}\left(\frac{x_1^4}{4}\right)$

$\Rightarrow \underbrace{\frac{x_2^2}{2} + \frac{x_1^2}{2} + \frac{x_1^4}{4}}_{= V(x_1,x_2)} = \text{const}$

7. (a) $\dot{x} = -y - x(x^2+y^2)$

$\dot{y} = x - y(x^2+y^2)$

Consider $V(x,y) = \frac{1}{2}(x^2+y^2)$

$$\dot{V}(x,y) = x\dot{x} + y\dot{y} = -xy - x^2(x^2+y^2) + xy - y^2(x^2+y^2)$$
$$= -(x^2+y^2)^2$$

$$\left.\begin{array}{l} V(0,0)=0 \\ V>0 \\ \dot{V}<0 \end{array}\right\} \Rightarrow$$ Equilibrium is asymptotically stable [global as V is radially unbounded and $V>0$, $\dot{V}<0$ hold for the whole $\mathbb{R}^2$]

(b) Consider $\bar{V}(x,y) = 2y^2 - 2x^2 + x^4 + 1$ (because $V(\pm1,0) = -1$)

$$\frac{d\bar{V}}{dt} = \frac{\partial \bar{V}}{\partial x}\dot{x} + \frac{\partial \bar{V}}{\partial y}\dot{y} = (-4x+4x^3)\dot{x} + 4y\dot{y}$$
$$= (-4x+4x^3)y + 4y(x - x^3 - \gamma y)$$
$$= 4(-x+x^3)y + 4y(x-x^3) - 4\gamma y^2$$
$$= -4\gamma y^2$$

$\bar{V}(\pm1,0) = 0$

$\dot{\bar{V}} \leq 0$

Is $\bar{V}>0$ around $(\pm1,0)$? $\left.\begin{array}{l} \frac{\partial \bar{V}}{\partial x} = -4x+4x^3 \\ \frac{\partial \bar{V}}{\partial y} = 4y \end{array}\right\}$ critical points $(\pm1,0)$

$$H = \begin{pmatrix} \frac{\partial^2 \bar{V}}{\partial x^2} & \frac{\partial^2 \bar{V}}{\partial y \partial x} \\ \frac{\partial^2 \bar{V}}{\partial x \partial y} & \frac{\partial^2 \bar{V}}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -4+12x^2 & 0 \\ 0 & 4 \end{pmatrix}$$

Hence, at $(\pm1,0)$, $H>0 \Rightarrow$ minima

$\Rightarrow$ stable as long as $\gamma > 0$

7. (c) $\dot{x}_1 = -x_1 + 2x_2^3 - 2x_2^4$ (0,0) is equilibrium

$\dot{x}_2 = -x_1 - x_2 + x_1 x_2$

$V(x_1,x_2) = x_1^{\alpha_1} + k\, x_2^{\alpha_2}$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= \alpha_1 x_1^{\alpha_1-1}(-x_1+2x_2^3-2x_2^4) + k\alpha_2 x_2^{\alpha_2-1}(-x_1-x_2+x_1x_2)$$

$$= -\alpha_1 x_1^{\alpha_1} - k\alpha_2 x_2^{\alpha_2} + 2\alpha_1 x_1^{\alpha_1-1}x_2^3 + k\alpha_2 x_2^{\alpha_2}x_1$$
$$- 2\alpha_1 x_1^{\alpha_1-1}x_2^4 - k\alpha_2 x_2^{\alpha_2-1}x_1$$

(set $\alpha_1 = 2$, $\alpha_2 = 4$, $k=1$)

$$= -2x_1^2 - 4x_2^4 + \cancel{4x_1x_2^3} + \cancel{4x_2^4x_1} - \cancel{4x_1x_2^4} - \cancel{4x_2^3x_1}$$

< 0 for $(x_1,x_2) \neq (0,0)$

$\Rightarrow$ asymptotic stability

$\Rightarrow V(x_1,x_2) = x_1^2 + x_2^4$

(global, as V is radially unbounded and $\begin{cases} V>0 \\ \dot{V}<0 \end{cases}$ hold for whole $\mathbb{R}^2 \setminus \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$

8. (a) $\dot{x} = 2\cos x + \cos y$

$\dot{y} = 2\cos y + \cos x$

$$\left.\begin{array}{l} y \mapsto -\tilde{y} \\ x \mapsto -\tilde{x} \\ t \mapsto -\hat{t} \end{array}\right\} \qquad \begin{array}{l} \dot{\tilde{x}} = 2\cos\tilde{x} + \cos\tilde{y} \\ \dot{\tilde{y}} = 2\cos\tilde{y} + \cos\tilde{x} \end{array}$$

hence system is reversible (symmetry about origin)

Equilibrium: $\cos x = \cos y = 0 \Rightarrow x = k\pi + \frac{\pi}{2}$
$y = m\pi + \frac{\pi}{2}$

Jacobian $J = -\begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}$

Consider equilibrium $(x,y) = (\frac{\pi}{2}, \frac{\pi}{2})$

$J(x,y) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$, eigenvalues at $-3, -1$
sink

Since $\exists$ an attracting equilibrium, system is not conservative.

(b)

$\dot{x}_1 = \sin x_2$

$\dot{x}_2 = x_1 \cos x_2$

Indeed gradient system with $V = x_1 \sin x_2$

$$\dot{x}_1 = \frac{\partial V}{\partial x_1} = \sin x_2$$
$$\dot{x}_2 = \frac{\partial V}{\partial x_2} = x_1 \cos x_2$$

Hamiltonian system: $\dot{x}_1 = x_1 \cos x_2$
$\dot{x}_2 = -\sin x_2$

with Hamiltonian $H = x_1 \sin x_2$ and

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_1 \cos x_2$$
$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\sin x_2$$