C24: Dynamical Systems

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Lecture 1: Introduction

- We will study the **topological properties** of solutions of ordinary differential equations **without** solving them

- Essentially about the geometry of the paths describing the time evolution of solutions and how such paths can be thought of as lying on ‘surfaces’

- Intimately associated with the idea of State (or Phase Space) – how solutions are related to a ‘state’

Course summary

1. Introduction to dynamical systems
2. Phase space and equilibria
3. The stable, unstable and centre subspaces
4. Lyapunov functions, gradient and Hamiltonian systems. Vector fields possessing an integral
5. Invariance. La Salle's theorem. Domain of attraction
6. Limit sets, attractors, orbits, limit cycles, Poincaré maps
7. Saddle-node, transcritical, pitchfork and Hopf bifurcations
8. Logistic map, fractals and Chaos. Lorenz equations

C24: Dynamical Systems

- 8 lectures:
  - 11am on Thurs & Fri, weeks 5-8, LR2
- Examples class 1 (lectures 1-4):
  - Thu, week 8, 2-3pm or 3-4pm, LR4
  - Fri, week 8, 2-3pm or 3-4pm, LR5
- Examples class 2 (lectures 5-8):
  - Mon & Tue, week 1 Hilary 2019
- Revision class in Trinity
- Lecture notes + slides available on WebLearn & markcannon.github.io

Examples of Dynamical Systems

- Single species growth: the logistic equation

$$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right)$$

x: population at time t

$b > 0$: birth rate

K: carrying capacity.

Solution is lengthy! (see lecture notes):

$$x(t) = \frac{cKe^{bt}}{K + ce^{bt}}$$

Examples of Dynamical Systems

- What does the analytic solution tell us? Is it very informative?
- What happens if $x(0) = 0$. What does this mean?
- What happens when $t \rightarrow \infty$? (does $x \rightarrow K$?)
- Can we analyse the solution properties **without** solving the equation?

- Try to introduce geometry into the problem.

Phase space

- Any nth order differential equation in a single unknown variable $x(t)$ can be written as n coupled first order differential equations in n unknown variables $x_1(t), x_2(t)\ \ldots.\, x_n(t)$.

- Each variable defines a coordinate in phase space.

- Solutions are curves (or **trajectories**) in phase space determined by the initial conditions.

Names of Phase Spaces

- If $n = 1$ we have a Phase Line.
- If $n = 2$ we have a Phase Plane.
- If $n > 2$ we have a general Phase Space.

- Collections of similar trajectories can form surfaces, sometimes called solution manifolds (a fancy name for a smooth surface).

- You encountered phase space in P1 ‘Mathematical Modelling’!

The single species revisited

- There are special points in Phase Space where the solution remains stationary, i.e. $\frac{dx}{dt} = 0$.
- The single species equation is first order, $n = 1$, and the special points are when

$$bx\left(1 - \frac{x}{K}\right) = 0 \;\Leftrightarrow x = 0 \; or \; x = K$$

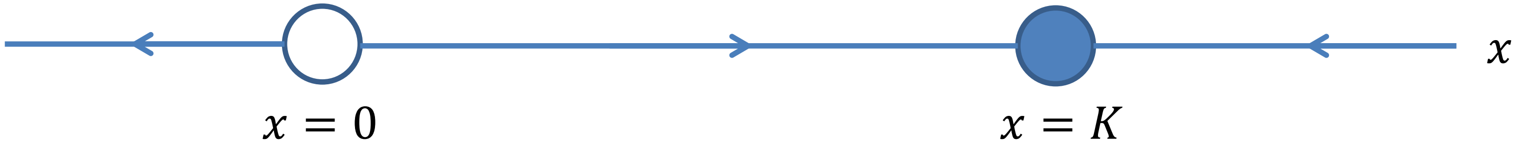
- These special points are called **equilibria**.
- As $n = 1$, the solution trajectories lie on a single Phase Line.

Consider $\frac{dx}{dt}\left(= bx\left(1-\frac{x}{K}\right)\right)$ as a function of x:

$$x < 0 \Rightarrow \frac{dx}{dt} < 0$$

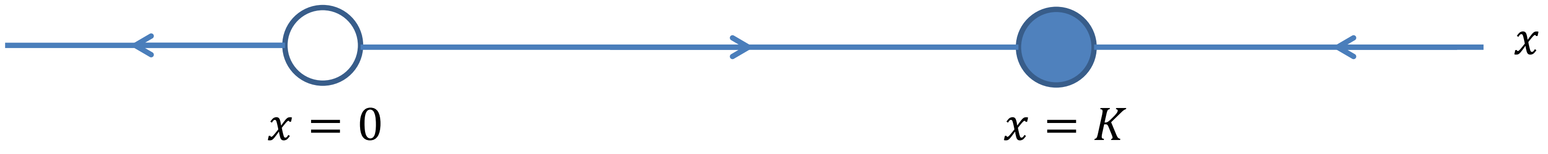
$$0 < x < K \Rightarrow \frac{dx}{dt} > 0$$

$$x > K \Rightarrow \frac{dx}{dt} < 0$$

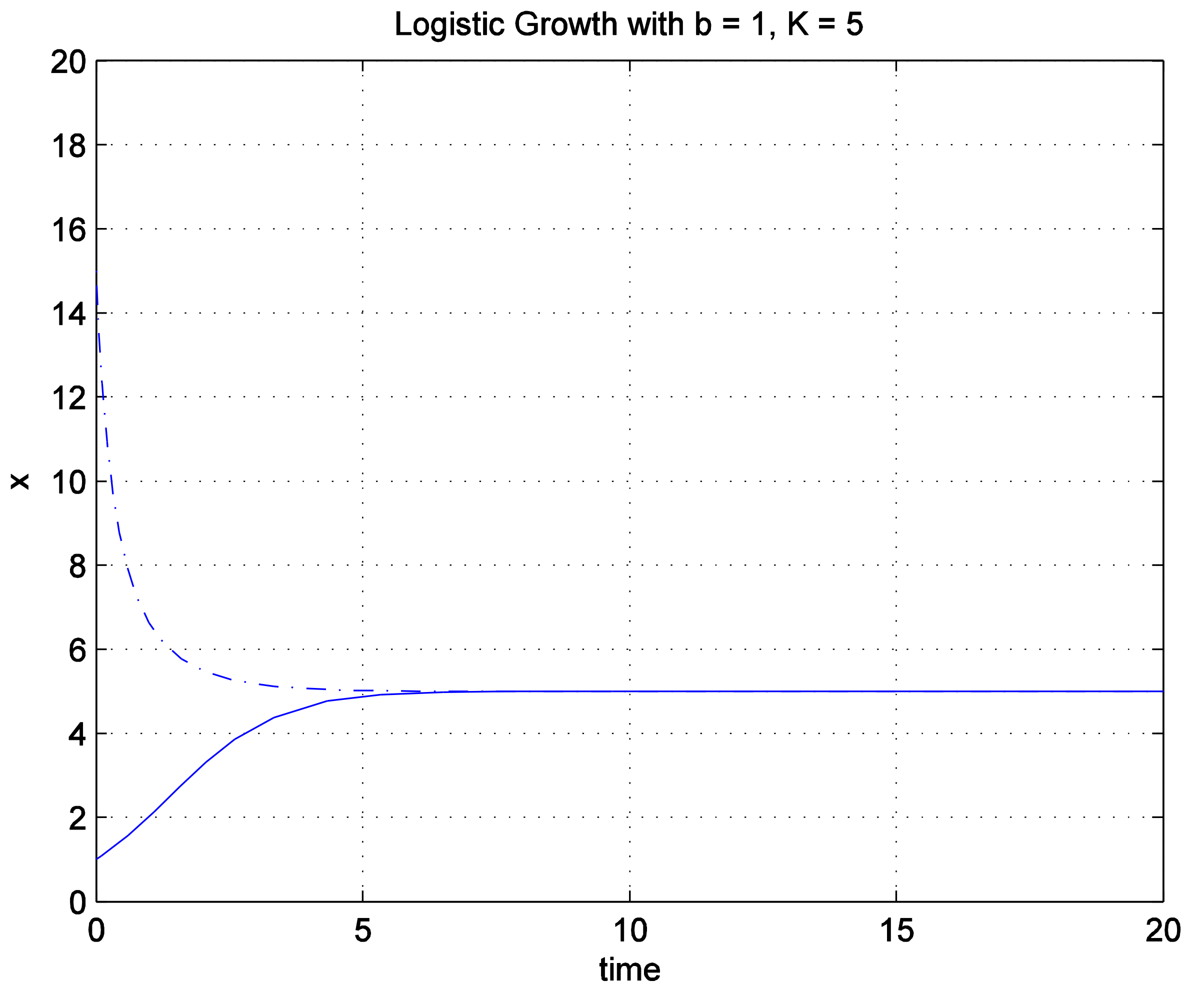


The resulting phase portrait

Stable and unstable equilibria

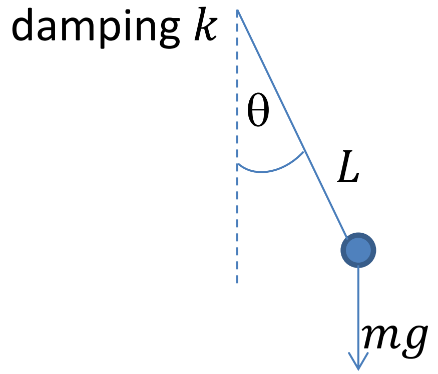


- All points near to $x = 0$ move away from this equilibrium – it is unstable.
- All points near to $x = K$ more towards this equilibrium – it is stable.
- It is not possible to go from $x < K$ to $x > K$ without $x = K$ at some point – when it stops! Thus there is no overshoot at $x = K$.



The solution as a function of time

The damped simple pendulum



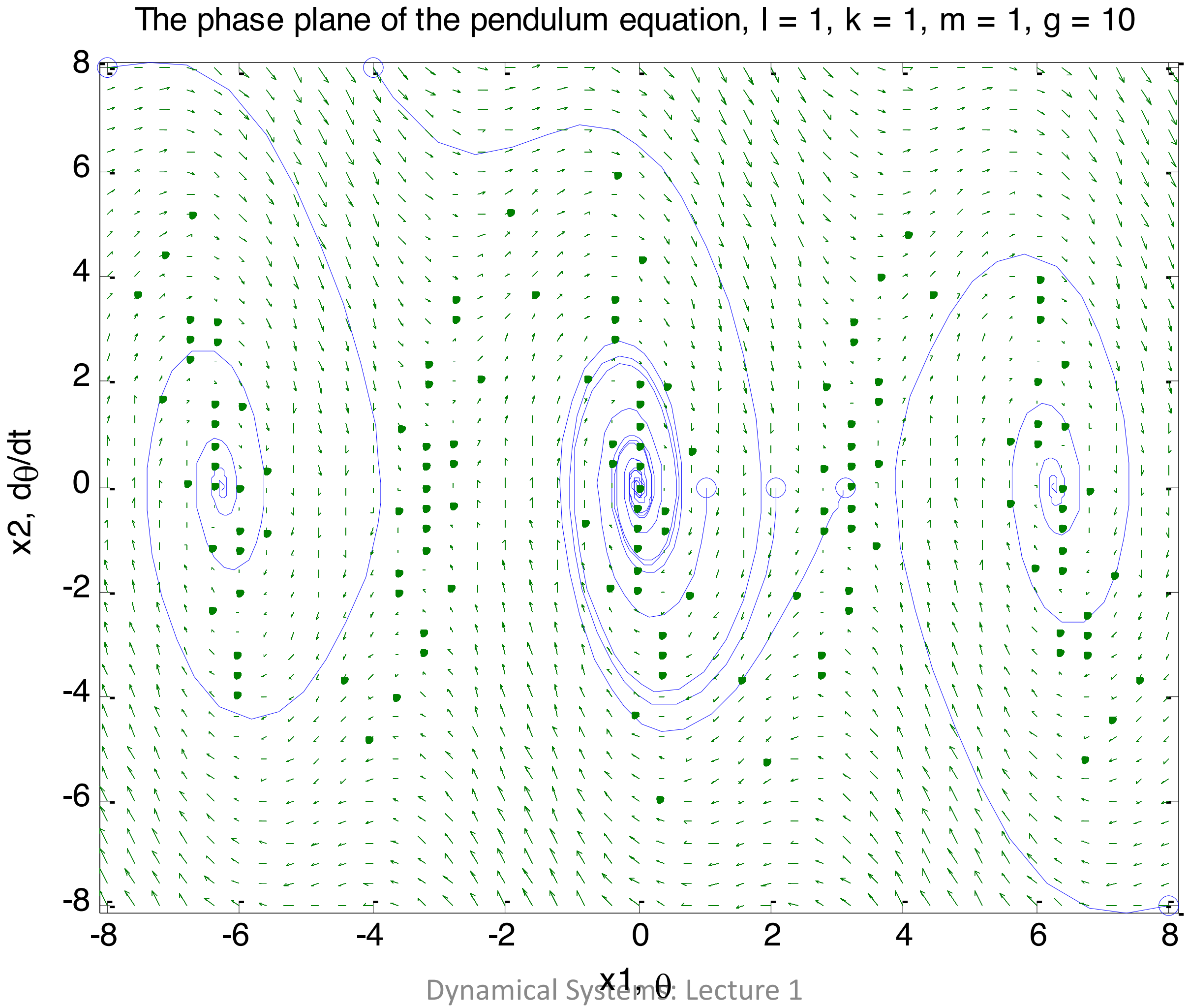
$$mL\ddot{\theta} = -mg\sin\theta - kL\dot{\theta}$$

There are two states: let $x_1 = \theta$ and $x_2 = \frac{dx_1}{dt}$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{g\sin x_1}{L} - \frac{k}{m}x_2$$

The equilibria (both $\frac{dx_1}{dt} = 0$ and $\frac{dx_2}{dt} = 0$) are when $x_2 = 0$ and $\sin x_1 = 0$



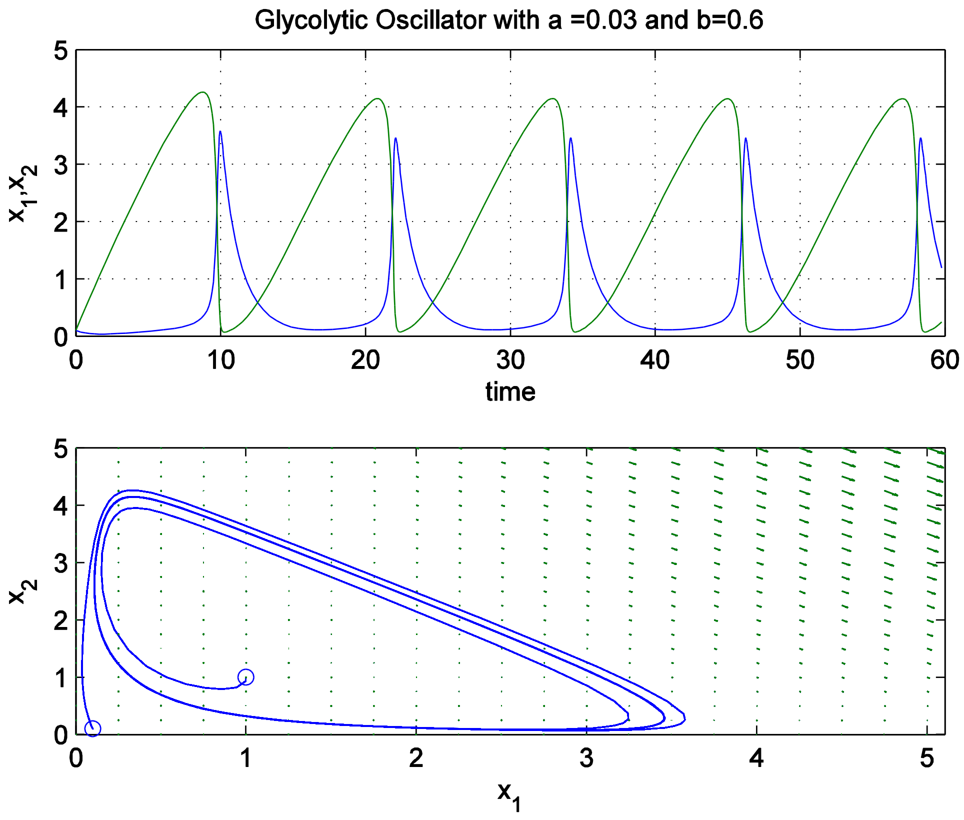
Glycolytic oscillations

- Involves turning glucose into energy compounds such as ATP within a cell:

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

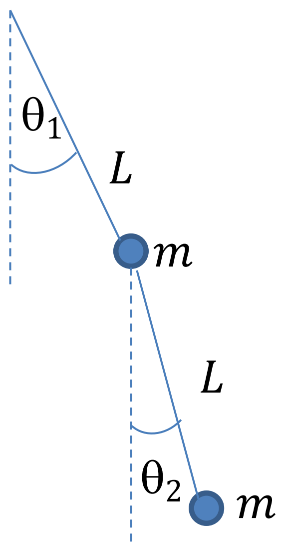
variables x and y are concentrations of two intermediaries

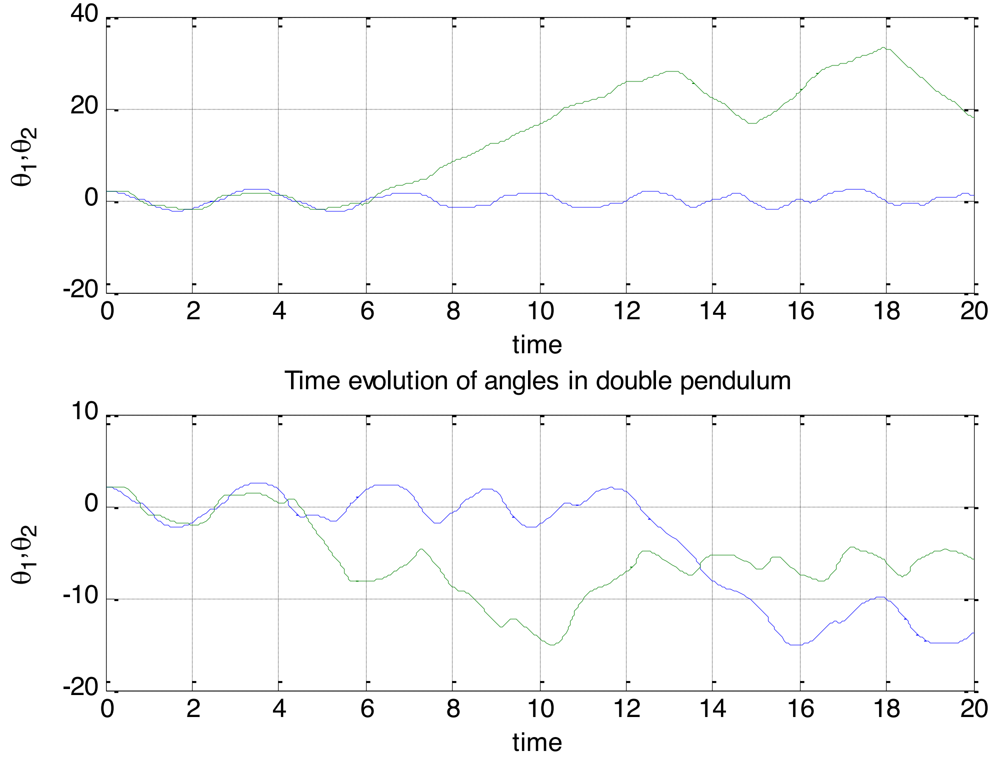
Glycolytic oscillations



System behaviour for one parameter set but with two different initial conditions

The double pendulum





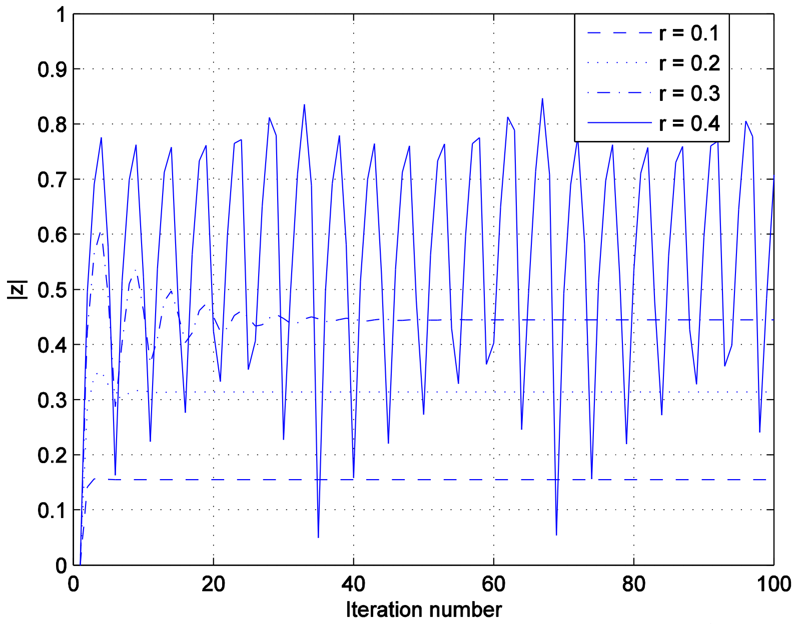
The two responses start at $x_1 = x_2 = 2\pi/3$, zero velocity, but the bottom adds 0.01 radians to x_1 for its initial condition. Why are they so different?

The Mandelbrot set

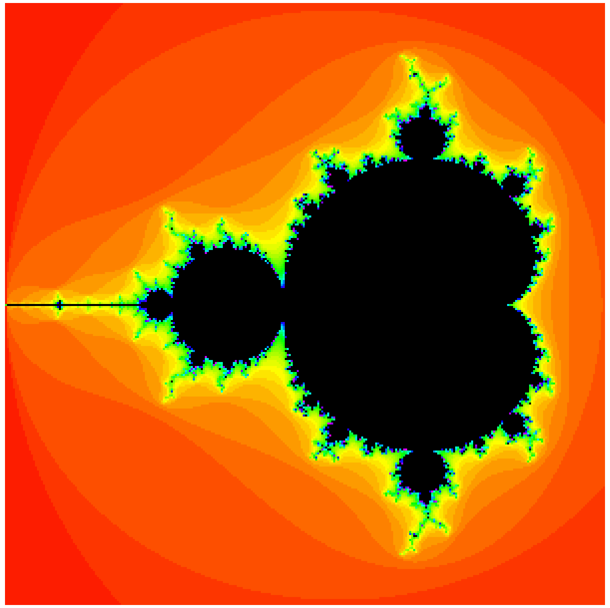
- An iterative equation:

$$z_{k+1} = z_k^2 + c$$

z_k and c are complex. If $z_0 = 0$, for which values of c does $|z_k|$ remain bounded?



effect of varying r with $c = r(1 + j)$



the set for general complex c

Our Strategy

- We will be studying equilibrium points of differential equations.
- The **nature** of equilibria are largely defined by their local linearisations.
- We then study the geometry and topology (connectedness) of regions around equilibria in phase space.
- We reason about the nature of the flows through these regions.
- To begin we need to understand the geometry of local linearisations – revisit eigenvalues and eigenvectors of matrices.

Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ square matrix mapping vectors from $\mathbb{R}^n$ to $\mathbb{R}^n$. Eigenvalues and eigenvectors of $\mathbf{A}$ satisfy

$$\mathbf{Av} = \lambda\mathbf{v}$$

- Eigenvalues λ_i are found by solving the characteristic equation
- Complex λ_i are come in complex conjugate pairs
- If real and distinct then there is a complete independent set of eigenvectors $\mathbf{v}_i$ (one for each eigenvalue)

Eigenvalues and Eigenvectors

If there is a complete set of eigenvectors, then they span the vector space $\mathbb{R}^n$

This means that ANY vector $\mathbf{x}$ in $\mathbb{R}^n$ can be expressed as a weighted sum of eigenvectors:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

If the eigenvalues are not distinct, there may not be a complete set of eigenvectors. (See Perko Chapter 1 on how to deal with this)

Matrix diagonalisation

- If a real matrix $\mathbf{A}$ has n distinct real eigenvalues, then there is a complete set of real eigenvectors that span the vector space $\mathbb{R}^n$

- The matrix $\mathbf{A}$ is then directly diagonalizable

$$\mathbf{AV} = \mathbf{V\Lambda} \Longrightarrow \mathbf{A} = \mathbf{V\Lambda V^{-1}}$$

- $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues, **V** is a matrix of eigenvectors.

Complex eigenvalues

- If a matrix has complex eigenvalues then its eigenvectors are complex, i.e. it cannot be diagonalized using matrices of real numbers
- For a 2×2 real matrix $\mathbf{A}$:

$$\lambda = a + jb, \bar{\lambda} = a - jb$$

$$\mathbf{v} = \mathbf{u} + j\mathbf{w}, \bar{\mathbf{v}} = \mathbf{u} - j\mathbf{w}$$

- Let **V** = [**w**, **u**], then

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Example from the notes

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\lambda = 2 + j, \qquad \bar{\lambda} = 2 - j$$
$$\mathbf{v} = [1 + j \quad 1]^T, \qquad \bar{\mathbf{v}} = [1 - j \quad 1]^T$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 + j & 1 - j \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 + j & 0 \\ 0 & 2 - j \end{bmatrix} \begin{bmatrix} 1 + j & 1 - j \\ 1 & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

This is a standard form for complex eigenvalues

Linear Autonomous Systems

- A system of first order linear differential equations can be written in vector form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Define

$$e^{\mathbf{A}} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

Then

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(\mathbf{0})$$

Computing the matrix exponential

- Use eigenvalues! If the eigenvalues are real and distinct
$$e^{\mathbf{A}} = \mathbf{V}\mathrm{diag}\{e^{\lambda_i}\}\mathbf{V}^{-1}$$

If the eigenvalues are complex, use the previous expansion
$$e^{\mathbf{A}} = \mathbf{V}e^{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}\mathbf{V}^{-1} = \mathbf{V}\begin{bmatrix} e^{a}\cos b & -e^{a}\sin b \\ e^{a}\sin b & e^{a}\cos b \end{bmatrix}\mathbf{V}^{-1}$$

(see Perko 'Differential equations and dynamical systems' Sec.1.5)

Representing dynamical systems

- Ordinary differential equations can be represented as n **coupled** first order differential equations
- Each of the n unknowns is called a 'state', $x_i(t) \in \mathbb{R}$
  $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector
- $\dot{x}_i = f_i(x_1, \quad \cdots \quad x_n)$, where each f_i maps $\mathbf{x}$ to a real number x_i
  $f_i: \mathbb{R}^n \to \mathbb{R}$
- If f_i is defined on a subset of $\mathbb{R}^n$ (its **domain**), $D \subseteq \mathbb{R}^n$, then
  $f_i: D \to \mathbb{R}$ (e.g. $\sqrt{x}$ is only real for $x \geq 0$, so $D = \{x: x \geq 0\}$)
- **f** is the vector with ith element f_i, i.e. **f**: $\mathbb{R}^n \to \mathbb{R}^n$ or **f**: $D \to \mathbb{R}^n$

Representing dynamical systems

The set of non-linear differential equations may now be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

This is an autonomous equation as it does not depend on time explicitly. The equation is linear if $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ and $\mathbf{A}$ is an $n \times n$ real matrix

Non-autonomous systems are of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

and are not part of the course (there is one example in the first sheet as a warning of their difficulty)

Representing dynamical systems

Parameters: $\mathbf{f}$ may depend on a parameter vector $\boldsymbol{\mu} \in \mathbb{R}^p$ where p does not necessarily equal n. For example, the equations may be those of motion dependent on a single mass and then $p = 1$. We then write

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

Maps or difference equations are not differential equations, but represent recurrence relations such as

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k; \boldsymbol{\mu})$$

These are written as

$$\mathbf{x} \mapsto \mathbf{g}(\mathbf{x}; \boldsymbol{\mu})$$

This is similar to the representation for register transfers in digital logic

Representing dynamical systems

A solution of a differential equation is a map from the time interval $t \in (\alpha, \beta)$ to the space $\mathbb{R}^n$, passing through the initial condition $\mathbf{x}_0$ at $t = 0$:

$$\mathbf{x}: (\alpha, \beta) \to \mathbb{R}^n \text{ such that } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(\mathrm{t}); \boldsymbol{\mu}) \text{ and } \mathbf{x}(0) = \mathbf{x}_0$$

Note that $\alpha < 0$ and $\beta > 0$ if $\mathbf{x}_0$ is at $t = 0$.

We will not solve such equations – we will look at the **geometry** of the solutions.

Existence and uniqueness of solutions

- Does a solution **exist?** Is it **unique?**
- The study of existence and uniqueness is highly technical – typically part of a typical maths degree
- The lecture notes describe an aside (outside the course!) considering existence (Lipschitz continuity) and dependence of convergence on initial conditions and parameters (Gronwall's lemma) – see Perko sections 2.2 & 2.3