

**C24: Dynamical Systems****Example Sheet C24/2****Michaelmas 2018**

You are expected to refer to textbooks as well as the lecture notes when attempting the questions on this sheet. Feel free to use MATLAB.

**1. Asymptotic Behaviour**

(a). Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2 - x_2^2)x_2\end{aligned}$$

- (i). Is the origin stable?
- (ii). Show that this system has a limit cycle and find it.
- (iii). Show that all trajectories not starting from the origin converge to the limit cycle.

(b). Show that the  $(0, 0)$  equilibrium of

$$\begin{aligned}\dot{x} &= -x^3 + 2y^3 \\ \dot{y} &= -2xy^2\end{aligned}$$

is asymptotically stable, using the candidate Lyapunov function  $V(x, y) = \frac{1}{2}(x^2 + y^2)$ .

**2. Constructing a Trapping Region**

Consider the system

$$\dot{x}_1 = -x_1 + ax_2 + x_1^2 x_2$$

$$\dot{x}_2 = b - ax_2 - x_1^2 x_2$$

where  $a, b > 0$ .

- (a). Find the stability properties of the equilibrium at  $(b, b/(a + b^2))$ . When is this unstable?
- (b). Show that the region defined by  $x_1 \geq 0$ ,  $x_2 \geq 0$ ,  $x_2 \leq b/a$  and  $x_2 \leq -x_1 + (b/a + b)$  is a trapping region.
- (c). Use Poincaré Bendixson to show that this system can admit a limit cycle.

### 3. Nonexistence of Periodic Orbits

- (a). Show that the system

$$\dot{x}_1 = x_1(2 - x_1 - x_2)$$

$$\dot{x}_2 = x_2(4x_1 - x_1^2 - 3)$$

has no closed orbits in the first quadrant using Dulac's theorem with  $B = 1/(x_1^a x_2^b)$  for some  $a, b$ . Find the four equilibria, classify their stability and sketch the phase portrait.

- (b). Use the Dulac function  $B(x, y) = be^{-2\beta x}$  to show that the system

$$\dot{x} = y$$

$$\dot{y} = -ax - by + \alpha x^2 + \beta y^2$$

has no limit cycle on the plane.

(c). Consider the system

$$\begin{aligned}\dot{x}_1 &= -\varepsilon \left( \frac{x_1^3}{3} - x_1 + x_2 \right), \quad \varepsilon > 0 \\ \dot{x}_2 &= -x_1\end{aligned}$$

Show that there is no limit cycle within the strip  $|x_1| < 1$ , using Bendixson's theorem.

#### 4. Index Theory

(a). Using Index theory where possible, and Dulac-Bendixson otherwise, show that the following system

$$\begin{aligned}\dot{x}_1 &= x_1(4 - x_2 - x_1^2) \\ \dot{x}_2 &= x_2(x_1 - 1)\end{aligned}$$

does not have any closed orbits.

(b). Show that the system

$$\begin{aligned}\dot{x}_1 &= x_1 e^{-x_1} \\ \dot{x}_2 &= 1 + x_1 + x_2^2\end{aligned}$$

does not have any closed trajectories.

(c). A system has three closed trajectories,  $C_1$ ,  $C_2$  and  $C_3$ , all counterclockwise, with  $C_2$  and  $C_3$  enclosed by  $C_1$ .  $C_2$  does not enclose  $C_3$  and vice-versa. Show that there is at least one fixed point enclosed by  $C_1$  but not by  $C_2$  and  $C_3$ .

(d). Determine the index of the zero equilibrium of

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{y} &= -y\end{aligned}$$

(You may plot the vector field in MATLAB.)

(e). Determine the index of the zero equilibrium of

$$\dot{x} = x^2 - y^2$$

$$\dot{y} = 2xy$$

(You may plot the vector field in MATLAB.)

## 5. One-dimensional bifurcations

For the following systems, classify the number and type of bifurcations:

(a).  $\dot{x} = \mu - x - e^{-x}$ .

(b).  $\dot{x} = -x + \mu \tanh x$ .

(c).  $\dot{x} = \mu x + x^4$ .

## 6. Hopf bifurcations

Show that the following system

$$\dot{x}_1 = \mu x_1 - x_2 + \sigma x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = x_1 + \mu x_2 + \sigma x_2(x_1^2 + x_2^2)$$

undergoes a Hopf bifurcation at  $(0, 0)$  when  $\mu = 0$  and that this bifurcation is

(a). Supercritical if  $\sigma = -1$ .

(b). Subcritical if  $\sigma = 1$ .

(c). Degenerate if  $\sigma = 0$ .

## 7. Real Quadratic Map

Consider the system

$$x \mapsto x^2 + c$$

- (a). Find all equilibria as a function of  $c$ .
- (b). Considering  $c$  as a bifurcation parameter, find values of  $c$  where bifurcations occur.
- (c). For which values of  $c$  does this system admit a stable 2-cycle?
- (d). Draw an orbit diagram for this map. You may use MATLAB.

## 8. The Rikitake system

Consider the Rikitake system

$$\dot{x} = -\mu x + zy$$

$$\dot{y} = -\mu y + (z - a)x$$

$$\dot{z} = 1 - xy$$

- (a). Show that the system is dissipative.
- (b). Find the fixed points and show that they can be written in the form  $x^* = \pm k$ ,  $y^* = \pm 1/k$  and  $z^* = \mu k^2$  where  $\mu(k^2 - k^{-2}) = a$ .
- (c). Classify the stability of the fixed points.
- (d). Simulate this system with parameters  $a = 5$ ,  $\mu = 2$  using MATLAB.

## Some Answers/Hints

1(b). Hint: Use a Lyapunov function of the form  $V(x, y) = \frac{1}{2}(x^2 + y^2)$ .

3(a).  $a = b = 1$ .

4(a). Hint: In one of the quadrants you may have to use Dulac's theorem (see also 3(a)).

4(d). Index = 0.

5(a). A saddle-node bifurcation at  $\mu = 1$ .

5(b). A pitchfork bifurcation at  $\mu = 1$ .

6. Hint: Change to polar coordinates.

7(c). Hint: Consider the two-hop map,  $x_{k+2} = f(f(x_k))$ .

8(a).  $\nabla \cdot \mathbf{f} = -2\mu$ , hence the system is dissipative.