Lecture 8: Chaos

1-Dimensional discrete systems can be chaotic but continuous 1-D systems cannot. So we start with discrete maps:

$$x_{k+1} = F(x_k)$$

- A fixed point x^* satisfies

$$x^* = F(x^*)$$

- Linearise the Map around x^*:

$$x^* + w_{k+1} = F(x^*) + DF(x^*)w_k + \cdots$$

Analyse the Jacobian

$$w_{k+1} = DF(x^*)w_k$$

The unit circle

- The stability of x^* depends on whether the eigenvalues of the Jacobian $DF(x^*)$ lie inside the unit circle (see e.g. Examples Sheet 1, q2)

- For the 1-D case this constraint reduces to
$$|DF(x^*)| < 1$$

- The same reasoning about stable, unstable and centre manifolds as in the case of continuous systems applies here.

Cobweb diagrams

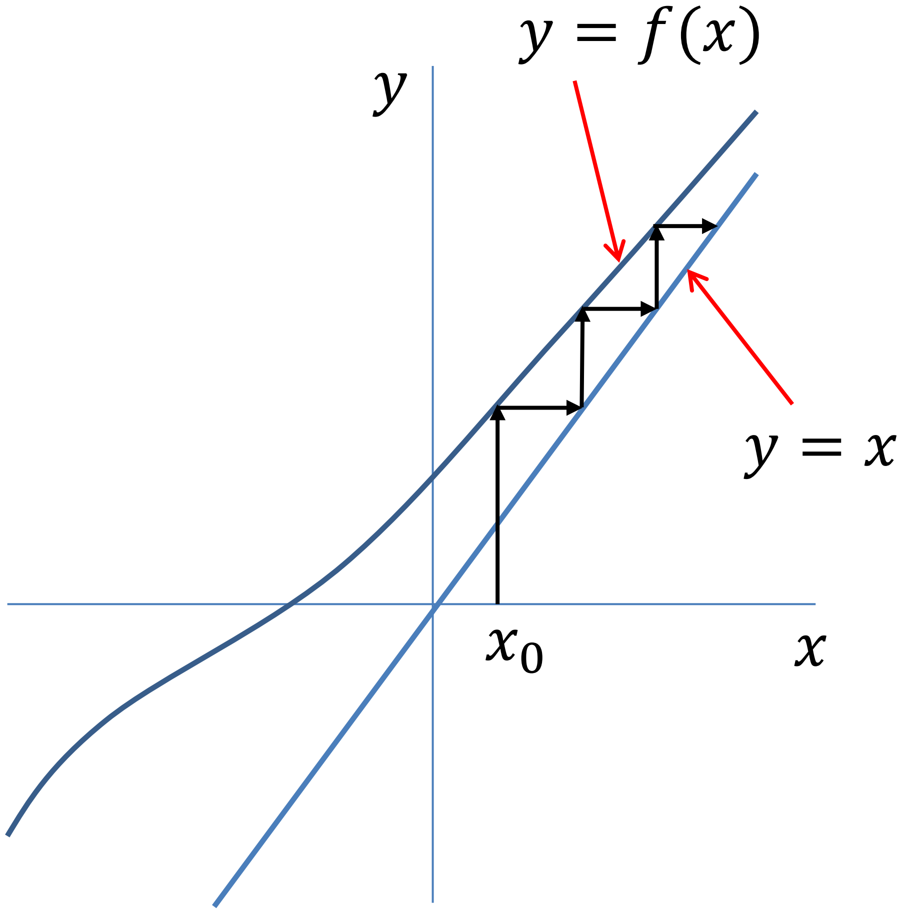
Consider the 1-D map

$$x_{k+1} = f(x_k)$$

To find x_1 given x_0:

- plot x_0 on the x-axis
- draw a vertical line to $y = f(x_0)$
- draw a horizontal line to $x_1 = y$ to set $x_1 = f(x_0)$

Repeat to find x_2, x_3 ...



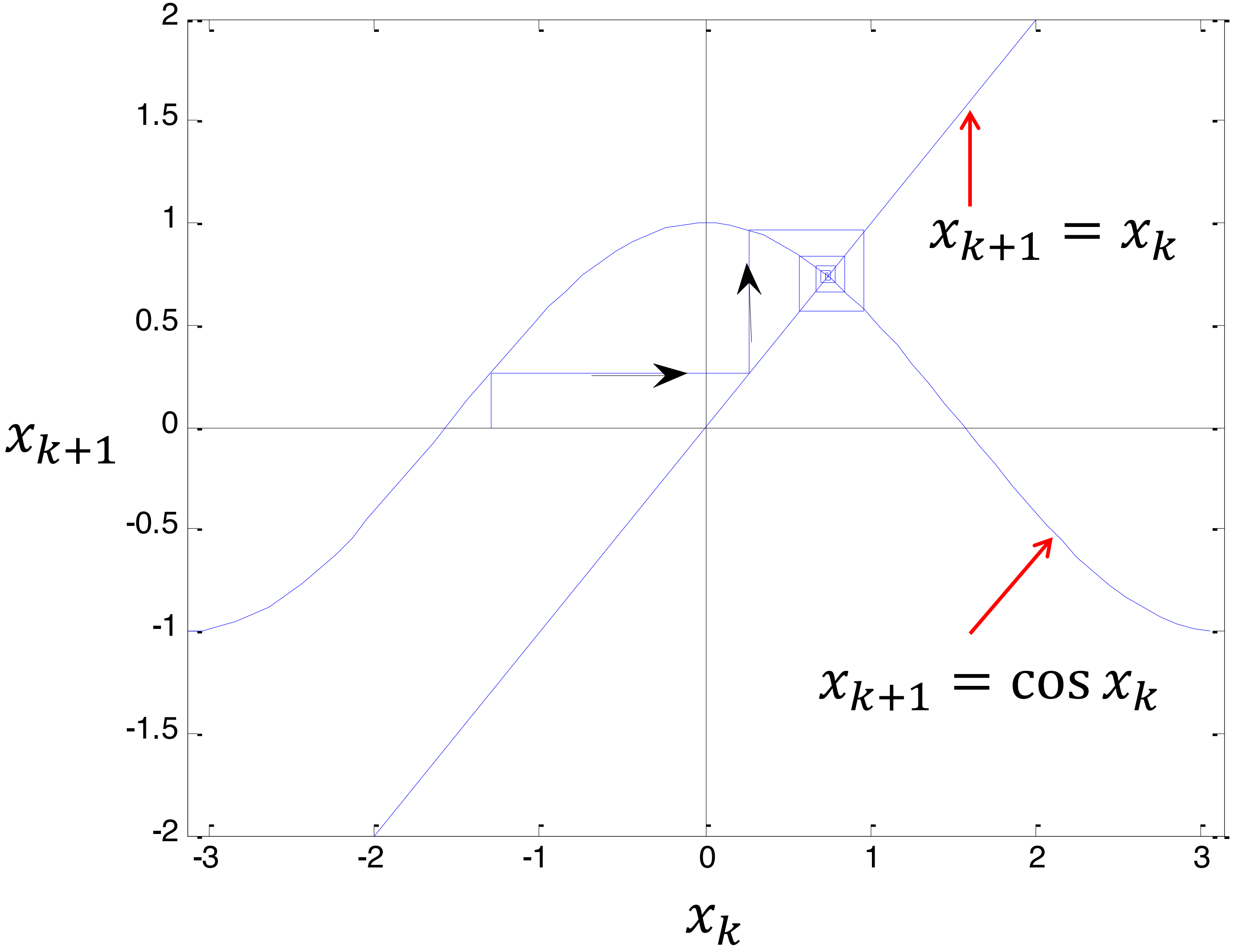
Cosine map

Cosine map:

$$x_{k+1} \longmapsto \cos(x_k)$$

has fixed point

$$x^* = 0.739$$



Logistic map

- Model of e.g. population dynamics:

$$x_{k+1} = rx_k(1 - x_k)$$

  Equilibria $x^* = 0$ and $x^* = (1 – 1/r)$

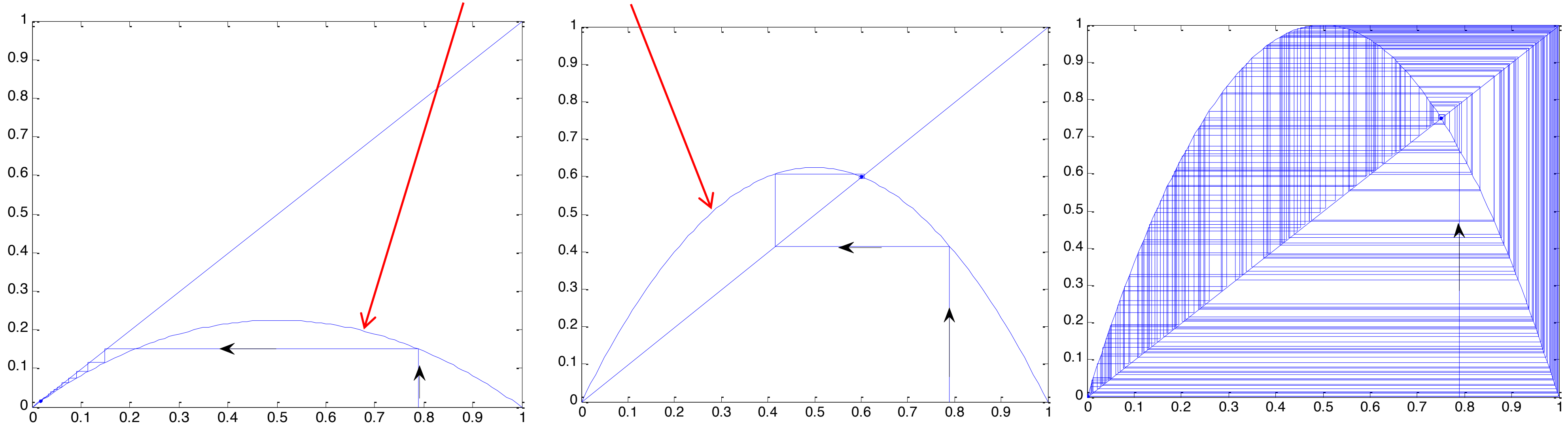
- Linearise:

$$w_{k+1} = r(1 - 2x^*)w_k$$

  - If $r < 1$, then $x^* = 0$ is stable and populations go extinct.
  - If $r > 1$, then $x^* = 0$ is unstable and populations grow – but what happens to the other equilibrium?

Cobwebs for the logistic map

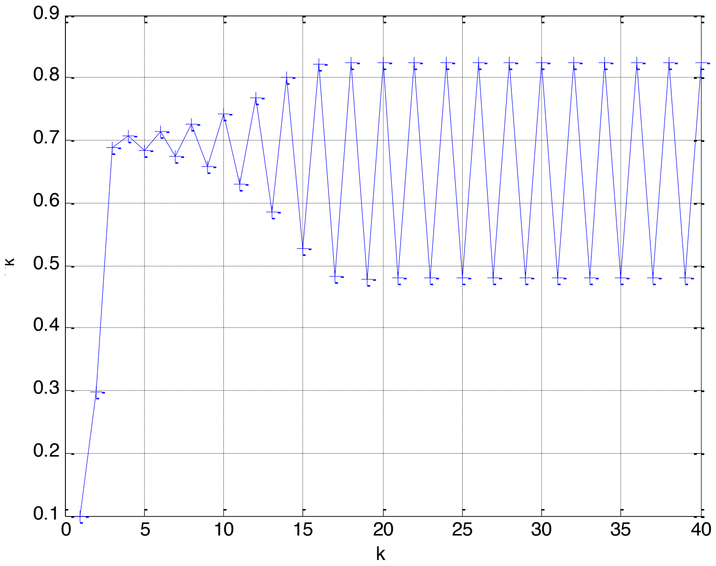
$rx_k(1 - x_k)$



$r = 0.9$ $r = 2.5$ $r = 4.0$

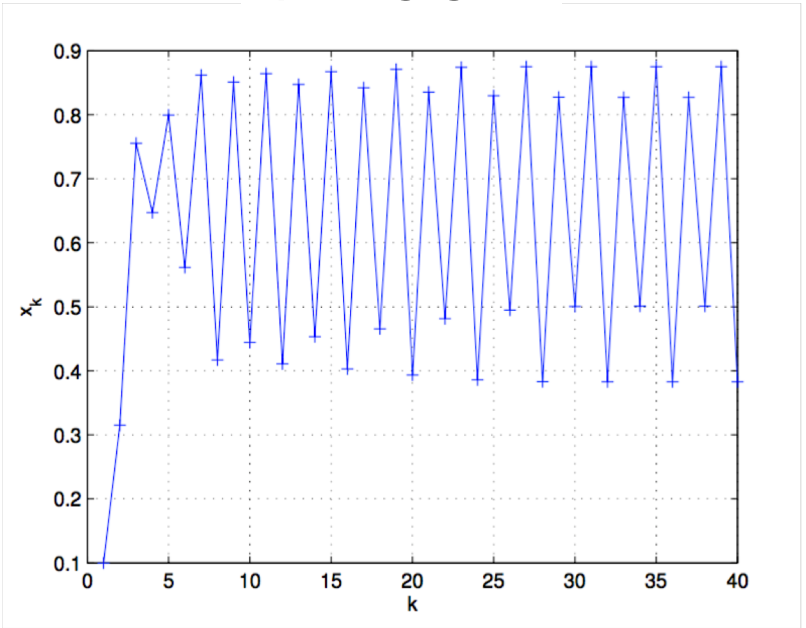
Solution becomes non-periodic for $r = 3.59$

$r = 3.3$



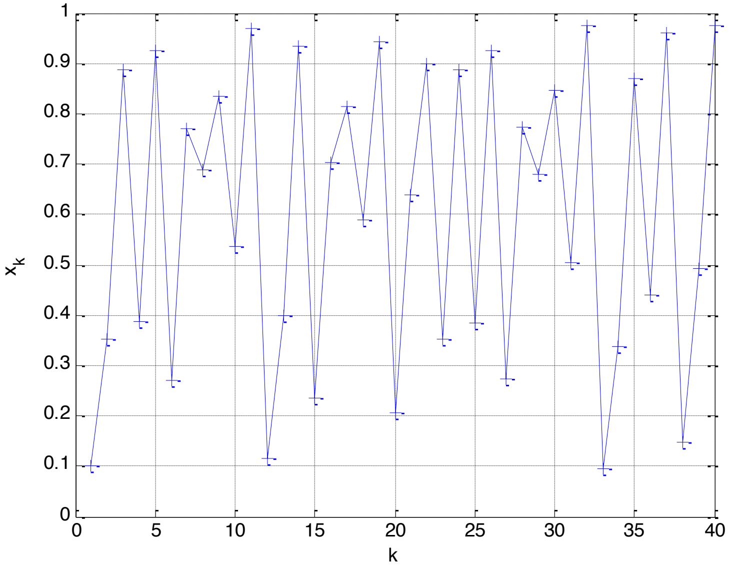
periodic

$r = 3.5$



double-periodic

$r = 3.9$



non-periodic

Logistic map simulations

Behaviour of the logistic map

- For $1 < r < 3$ the system is stable around $(1 - 1/r)$

- For $r > 3$ oscillations appear, with period doubling as r increases at well-defined values of r.

- The rate of doubling increases as r approaches 3.5699… and x is then no longer periodic.

- As r increases, chaotic behaviour ensues with brief windows of periodic behaviour for certain ranges of r.

Analysing periodicity

Consider

$$x_{k+2} \longmapsto f\big(f(x_k)\big)$$

- The fixed points of this map are called 2-cycles (they have period 2) and are solutions of

  $$r^2 x^*(1 - x^*)\{1 - r x^*(1 - x^*)\} = x^*.$$

  The roots (see lecture notes) are stable for $3 < r < 1 + \sqrt{6} = 3.4495.$

- The next bifurcation is to a 4-cycle, then 8, etc. The changes in period occur as doublings with increasingly complex expressions for r and narrower stability ranges.

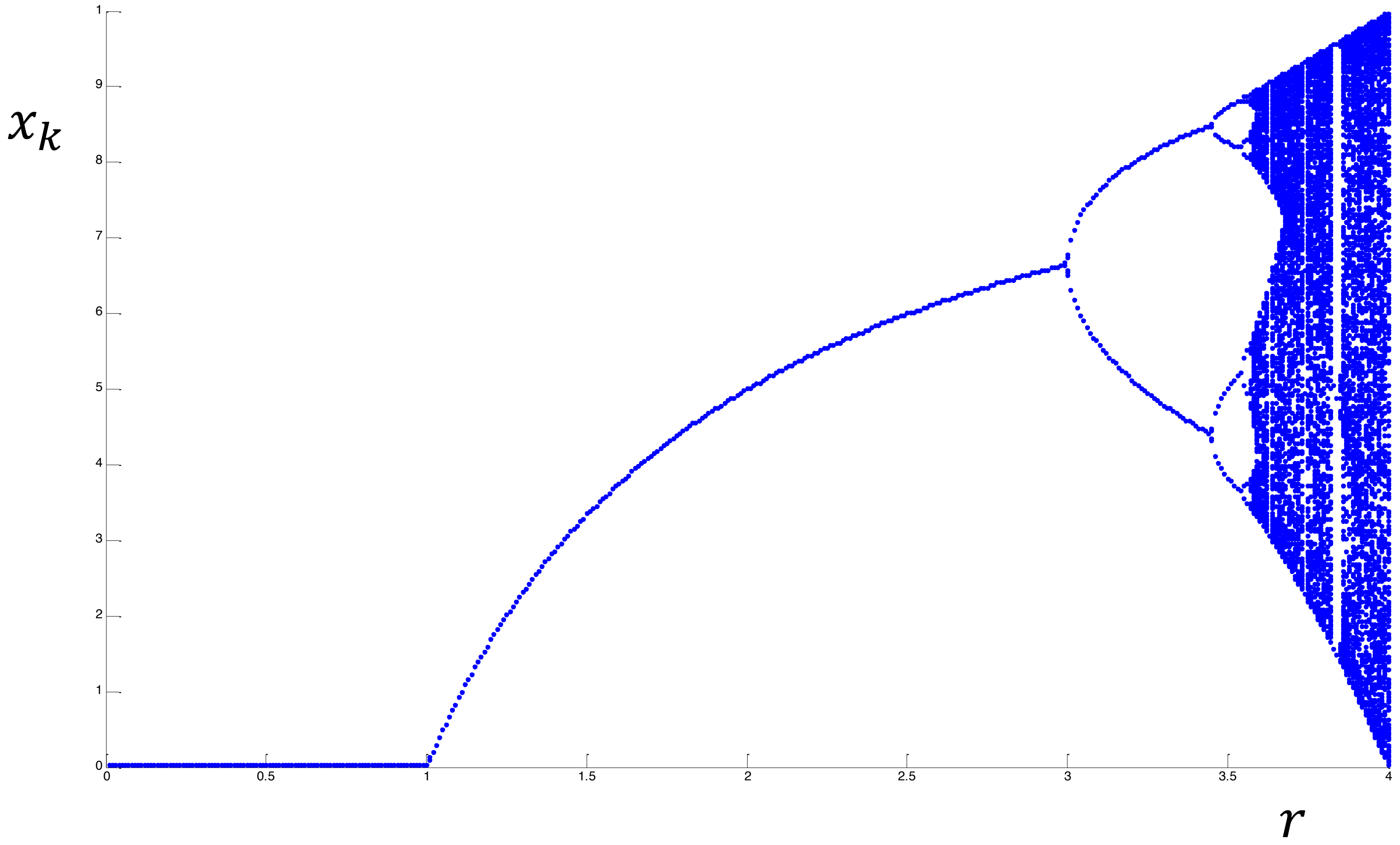
Orbit diagrams

These are a way of displaying how a map changes with a parameter.

- Choose a value of the parameter r and a starting value for x.
- Run the iteration for, say, 300 iterations.
- Record the values of x for a further 300 or so iterations and plot them on the map.
- Change r and repeat.

Various plots are given in the lecture notes.

The logistic map orbit



Chaos

- The map illustrates dynamical chaos.
- Chaos is **aperiodic long-term** behaviour in a **deterministic** system that exhibits **sensitive dependence on initial conditions**.
  - Aperiodic long-term: These are trajectories that never settle down to fixed points or periodic orbits.
  - Deterministic: The trajectory is the solution of an equation with no noise – everything is certain and precise.
  - Sensitive to initial conditions: Points on trajectories that are near to each other diverge with time.

Lyapunov exponent for maps

- Measures how fast solutions that are initially close diverge (sensitivity of initial conditions).

- Consider the effect on long-term behaviour of changing initial condition from x_0 to $x_0 + w_0$:

$$\begin{array}{ccc} x_1 + w_1 = f(x_0 + w_0) & \Longrightarrow & w_1 = f(x_0 + w_0) - x_1 \\ x_2 + w_2 = f(f(x_0 + w_0)) & \Longrightarrow & w_2 = f(f(x_0 + w_0)) - x_2 \\ \vdots & \vdots & \vdots \\ x_k + w_k = f(\cdots f(x_0 + w_0)) & \Longrightarrow & w_k = f(\cdots f(x_0 + w_0)) - x_k \end{array}$$

The **Lyapunov exponent**, λ, measures the rate of growth of w_k:

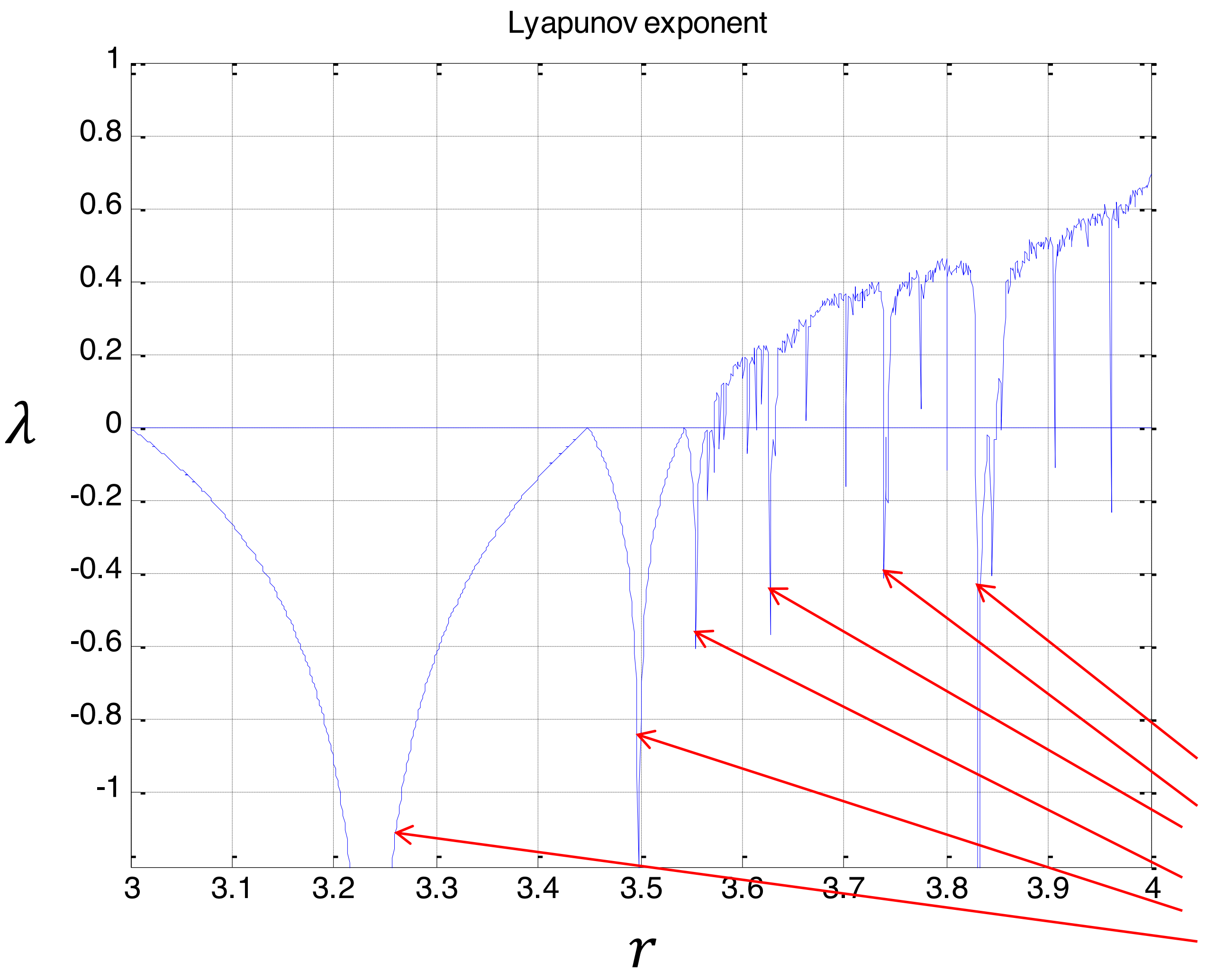
$$|w_k| \approx |w_0| e^{\lambda k}$$

Lyapunov exponent for maps

Therefore
$$
\begin{aligned}
w_1 &= f(x_0 + w_0) - x_1 \\
&\approx f(x_0) - x_1 + Df(x_0)w_0 = Df(x_0)w_0 \\
w_2 &\approx Df(x_1)w_1 \\
&= Df(x_1)Df(x_0)w_0 \\
&\ \vdots \\
w_k &= \prod_{i=1}^{k-1} Df(x_i)\, w_0
\end{aligned}
$$

$$\Rightarrow \qquad \lambda = \lim_{k\to\infty} \lim_{|w_0|\to 0} \frac{1}{k} \ln\left|\frac{w_k}{w_0}\right| \approx \frac{1}{k} \sum_{i=0}^{k-1} |Df(x_i)| \qquad \text{(accurate for large } k\text{)}$$

Lyapunov exponents of the logistic map



Periodic solutions – dips in the exponent

Chaos in Flows

- The **Lorenz Equations**: model of convection in the atmosphere

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

  $r =$ Rayleigh number, $\sigma =$ Prandtl number, b is a positive constant.

- If $r < 1$ the only equilibrium is the origin $(0,0,0)$

- For $r > 1$ two more equilibria appear via a pitchfork bifurcation:

$$(x^*, y^*, z^*) = \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$$
$$(x^*, y^*, z^*) = \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$$

- The equations are symmetric in x and y.

Lorenz equations: volume contraction

- Compute div $\mathbf{f}$:

$$\nabla.\mathbf{f} = \frac{\partial\sigma(y-x)}{\partial x} + \frac{\partial(rx-y-xz)}{\partial y} + \frac{\partial(xy-bz\,)}{\partial z}$$
$$= -(1+\sigma+b) < 0$$

  Hence the integral of $\nabla.\mathbf{f}$ over any control volume is negative. So trajectories converge to a zero volume region of phase space.

- Is this zero volume solution a point or limit cycle? Neither!

Lorenz equations: stability

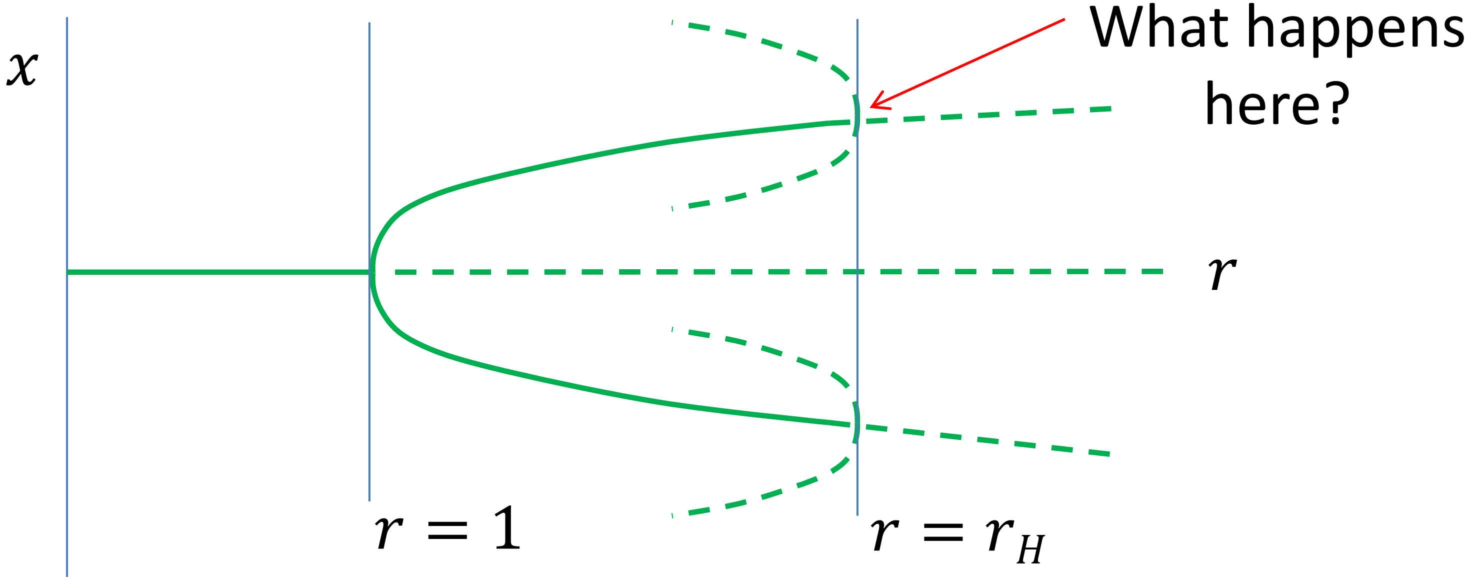
- The solution cannot have unstable equilibrium points or unstable periodic orbits – such solutions imply expansion of the state space, not contraction.

  Thus any fixed points must be stable or saddles, or if there are limit cycles they must be stable.
- Linearisation about the origin reveals a stable node for $r < 1$ and a saddle $r > 1$.
- For $r < 1$ the system is globally asymptotically stable (via Lyapunov) – there are no limit cycles and all trajectories fall into the origin.

Lorenz equations: stability

- For
$$1 < r < \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} = r_H$$
the other two equilibrium points are stable and are surrounded by a saddle cycle (a type of unstable limit cycle). At $r = r_H$ they undergo a Hopf bifurcation (the critical linearised eigenvalues are complex).

- For $r > r_H$ we have a saddle point, and there are no other attractors nearby

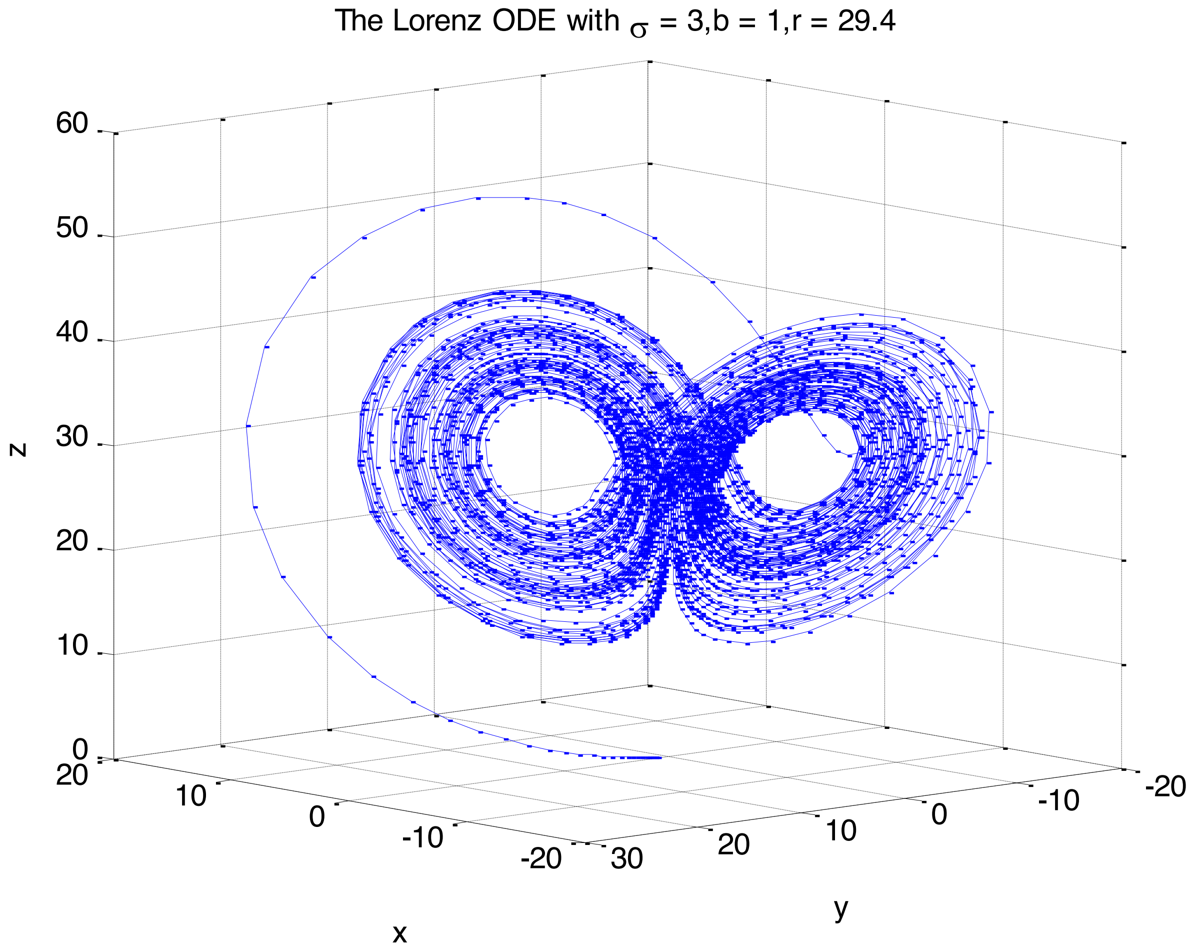


Clues to a Strange Attractor

- The volume is contracting.
- There are no stable classical attractors.
- There are no stable limit cycles for $r > r_H$ (proved by Lorenz).
- Trajectories cannot go to infinity.

- There must be a zero volume object that attracts the trajectories – a **Strange Attractor**

The Lorenz Butterfly



Properties

- There is sensitivity to initial conditions.

- We can compute the rates of contraction along the principal axes of the Jacobian – these numbers are called the Lyapunov exponents (remember the Lyapunov exponent for maps). If the largest exponent is positive, phase space is stretching in that direction.

- The volume is contracting because the sum of the exponents is negative – but one or two may be positive and so the volume is growing in some direction(s) and shrinking in others.

The Mandelbrot Set

- The map

$$z_{k+1} \longmapsto z_k^2 + c$$

  z and c are complex. The point c is in the Mandelbrot set if this iteration remains bounded for all k. We can use colours to indicate the rate of divergence at c

- The graph is a strange attractor for the map. The colours indicate how close the points shown are to the attractor.

- The set is a very complex object.

The Mandelbrot Set