C21 Nonlinear Systems

Mark Cannon

4 lectures

Trinity Term 2020

Lecture 1

Introduction and Concepts of Stability

Organisation

- 4 lectures:

week 1	Mon	10-11 am
	Tue	10-11 am
	Thu	10-11 am
	Fri	10-11 am

- 1 class:

week 6	Thu	11-12 pm
or week 7	Thu	10-11 am

Course outline

1. Types of stability
2. Linearization
3. Lyapunov's direct method
4. Regions of attraction
5. Linear systems and passive systems

Books

- J.-J. Slotine & W. Li *Applied Nonlinear Control*, Prentice-Hall 1991.
  - Stability
  - Interconnected systems and passive systems
- H.K. Khalil *Nonlinear Systems*, Prentice-Hall 1996.
  - Stability
  - Passive systems
- M. Vidyasagar *Nonlinear Systems Analysis*, Prentice-Hall 1993.
  - Stability & passivity (more technical detail)

Why use nonlinear control?

- Real systems are nonlinear
  - friction, non-ideal components
  - actuator saturation
  - sensor nonlinearity
- Analysis via linearization
  - accuracy of approximation?
  - conservative?
- Account for nonlinearities in high performance applications
  - Robotics, Aerospace, Petrochemical industries, Process control, Power generation ...
- Account for nonlinearities if linear models inadequate
  - large operating region
  - model properties change at linearization point

Linear vs nonlinear system properties

Free response

Linear system

$$\boxed{\dot{x} = Ax}$$

- Unique equilibrium point:
  $$Ax = 0 \iff x = 0$$
- Stability independent of initial conditions

Nonlinear system

$$\boxed{\dot{x} = f(x)}$$

- Multiple equilibrium points
  $$f(x) = 0$$
- Stability dependent on initial conditions

Linear systems reminder

Linear system free response

$$\dot{x} = Ax$$

Eigen-decomposition: $Av_i = v_i\lambda_i$

let $V = \begin{bmatrix} v_1, \ldots, v_n \end{bmatrix}$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

then $A = V\Lambda V^{-1}$ (if V^{-1} exists)

$$\Rightarrow \quad \dot{z} = \Lambda z, \quad z = V^{-1}x$$

$$z(t) = e^{\Lambda t}z(0)$$

$$\Rightarrow \quad x(t) = Ve^{\Lambda t}V^{-1}x(0)$$

$$= e^{At}x(0)$$

System is stable if $\mathrm{Re}(\lambda_i) < 0 \ \forall i$

Linear vs nonlinear system properties

Forced response

Linear system

$$\boxed{\dot{x} = Ax + Bu}$$

- $\|u\|$ finite $\Rightarrow$ $\|x\|$ finite if open-loop stable
- Frequency response:
$$u = U \sin \omega t \;\;\Rightarrow\;\; x = X \sin(\omega t + \phi)$$
- Superposition:
$$u = u_1 + u_2 \;\;\Rightarrow x = x_1 + x_2$$

Nonlinear system

$$\boxed{\dot{x} = f(x, u)}$$

- $\|u\|$ finite $\not\Rightarrow$ $\|x\|$ finite
- No frequency response
$u = U \sin \omega t \;\;\not\Rightarrow\;\; x$ sinusoidal
- No linear superposition
$$u = u_1 + u_2 \;\;\not\Rightarrow x = x_1 + x_2$$

Linear systems reminder

Linear system free response

$$\dot{x} = Ax$$

Eigen-decomposition: $Av_i = v_i \lambda_i$

Let $V = \begin{bmatrix} v_1, \ldots, v_n \end{bmatrix}$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

then $A = V \Lambda V^{-1}$ (if V^{-1} exists)

$$\Rightarrow \quad \dot{z} = \Lambda z, \quad z = V^{-1} x$$

$$z(t) = e^{\Lambda t} z(0)$$

$$\Rightarrow \quad x(t) = V e^{\Lambda t} V^{-1} x(0) = e^{At} x(0)$$

System is stable if $\text{Re}(\lambda_i) < 0$

Forced response

$$\dot{x} = Ax + Bu$$

$$\Rightarrow \quad x(t) = \int_0^t e^{A(t-h)} Bu(h)\, dh + e^{At} x(0)$$

System is bounded-input bounded-output (BIBO) stable:

$$\sup_{t \geq 0} \|x(t)\| \leq \gamma \sup_{t \geq 0} \|u(t)\|$$

$\gamma = \|B\| \int_0^\infty \|e^{At}\| dt$, if $\text{Re}(\lambda_i) < 0$

Frequency response

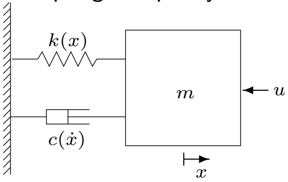
$$\dot{x} = Ax + Bu$$

$$u = U(\omega) e^{j\omega t} \;\Rightarrow\; x = X(\omega) e^{j\omega t}$$

$$\Rightarrow \quad X(\omega) = (j\omega I - A)^{-1} BU(\omega)$$

Example: step response

Mass-spring-damper system

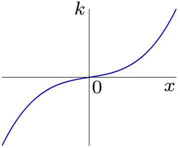


Equation of motion:

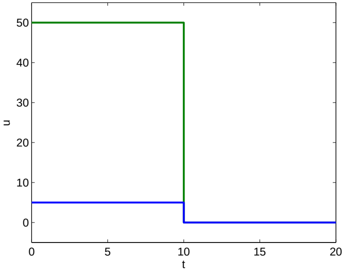
$$m\ddot{x} + c(\dot{x}) + k(x) = u$$

$c(\dot{x}) = \dot{x}$

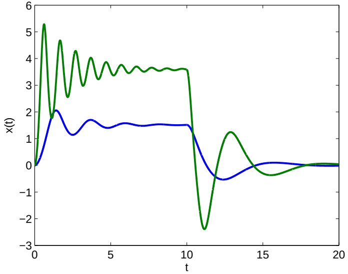
$k(x)$ nonlinear:



Input $u(t)$



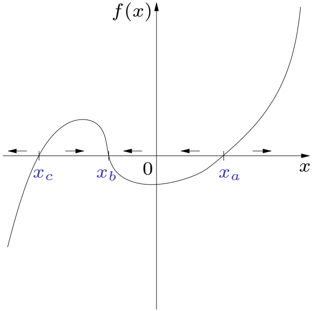
Response $x(t)$



apparent damping ratio depends on size of input step

Example: multiple equilibria

First order system: $\dot{x} = f(x)$

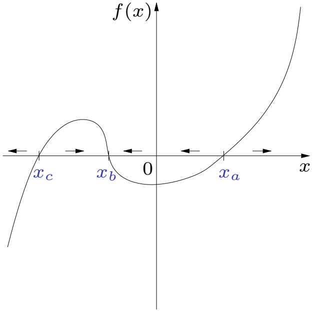


$$
\begin{array}{llllll}
x > x_a & \Longrightarrow & f(x) > 0 & \Longrightarrow & x(t) \text{ increases} \\
x_b < x < x_a & \Longrightarrow & f(x) < 0 & \Longrightarrow & x(t) \text{ decreases} \\
x_c < x < x_b & \Longrightarrow & f(x) > 0 & \Longrightarrow & x(t) \text{ increases} \\
x < x_c & \Longrightarrow & f(x) < 0 & \Longrightarrow & x(t) \text{ decreases}
\end{array}
$$

- x_a, x_c are unstable equilibrium points
- x_b is a stable equilibrium point

Example: multiple equilibria

First order system: $\dot{x} = f(x)$



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\end{array}
$$

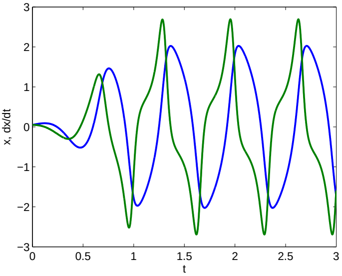
- x_a, x_c are unstable equilibrium points
- x_b is a stable equilibrium point

Example: limit cycle

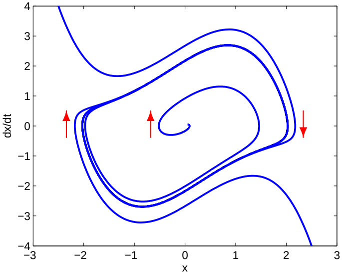
Van der Pol oscillator:

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

- Response $x(t)$ tends to a limit cycle (= trajectory forming a closed curve)
- Amplitude independent of initial conditions



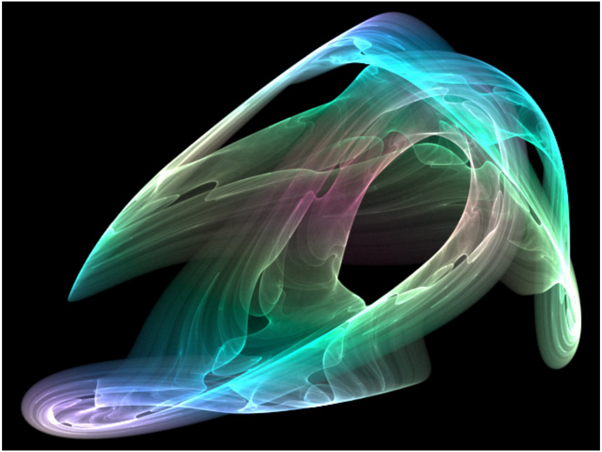
Response with $x(0) = 0.05$, $\dot{x}(0) = 0.05$



State trajectories $\big(x(t), \dot{x}(t)\big)$

Example: chaotic behaviour

Strange attractor



Example: chaotic behaviour

Lorenz attractor

- Simplified model of atmospheric convection:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

- State variables

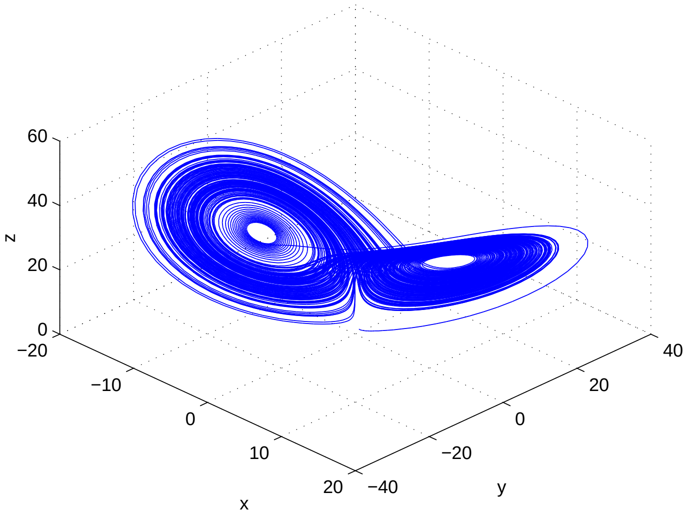
$x(t)$:	fluid velocity
$y(t)$:	difference in temperature of acsending and descending fluid
$z(t)$:	characterizes distortion of vertical temperature profile

- Parameters $\sigma = 10$, $\beta = 8/3$, $\rho =$ variable

Example: chaotic behaviour

Lorenz attractor

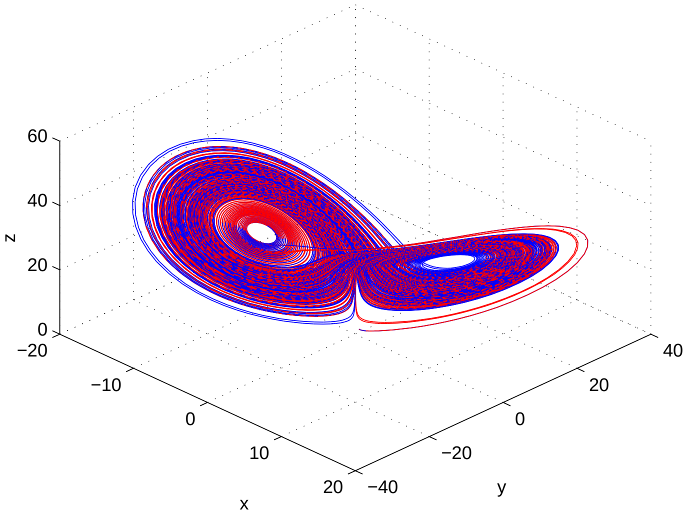
$\rho = 28 \implies$ "strange attractor":



Example: chaotic behaviour

Lorenz attractor

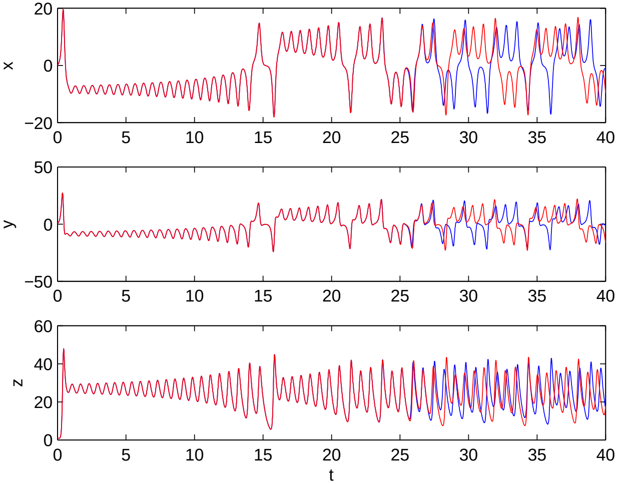
sensitivity to initial conditions



Example: chaotic behaviour

Lorenz attractor

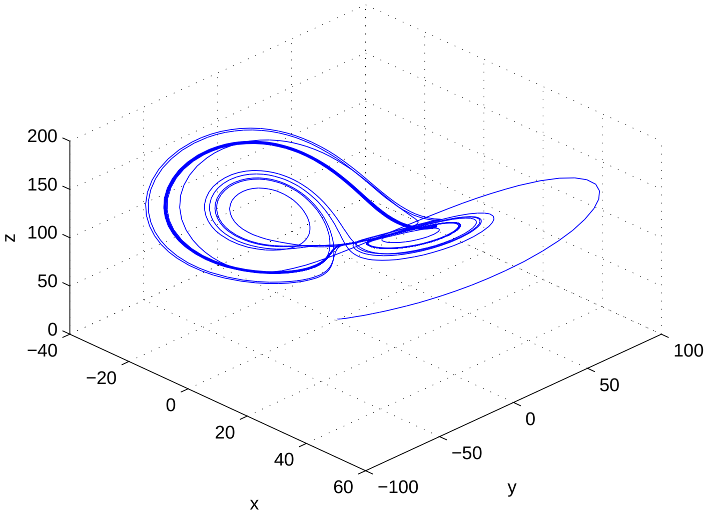
sensitivity to initial conditions blue: $(x, y, z) = (0, 1, 1.05)$
red: $(x, y, z) = (0, 1, 1.050001)$



Example: chaotic behaviour

Lorenz attractor

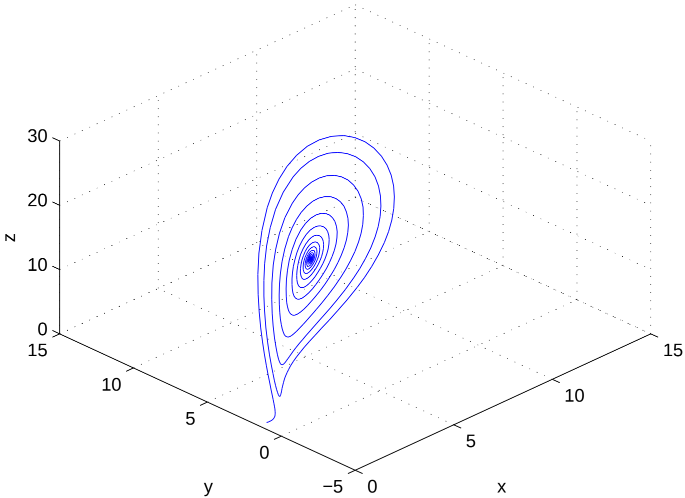
$\rho = 99.96 \implies$ limit cycle:



Example: chaotic behaviour

Lorenz attractor

$\rho = 14 \implies$ convergence to a stable equilibrium:



State space equations

A continuous-time nonlinear system

$$\dot{x} = f(x, u, t) \qquad \begin{aligned} x &: \text{ state} \\ u &: \text{ input} \end{aligned}$$

e.g. nth order differential equation:

$$\frac{d^n y}{dt^n} = h\Big(y, \ldots, \frac{d^{n-1} y}{dt^{n-1}}, u, t\Big)$$

has state vector (one possible choice)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ d^{n-1}y/dt^{n-1} \end{bmatrix}$$

and state space dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ h\big(x_1, x_2, \ldots, x_n, u, t\big) \end{bmatrix} = f(x, u, t)$$

Equilibrium points

x^* is an equilibrium point of system $\dot{x} = f(x)$ if (and only if):

$$x(0) = x^* \quad \text{implies} \quad x(t) = x^* \quad \forall t > 0$$

i.e. $f(x^*) = 0$

- ⋆ Consider local stability of individual equilibrium points
- ⋆ Convention: define f so that $x = 0$ is equilibrium point of interest
- ⋆ Autonomous system: $\dot{x} = f(x) \quad \Longrightarrow \quad x^* = \text{constant}$

Examples:

(a). $\ddot{y} + \alpha\dot{y}^2 + \beta \sin y = 0$ (damped pendulum)

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \ n = 0, \pm 1$$

(b). $\ddot{y} + (y-1)^2\dot{y} + y - \sin(\pi y/2) = 0$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Equilibrium points

EQUILIBRIUM POINT EXAMPLES

(a). $\ddot{y} + \alpha \dot{y}^2 + \beta \sin y = 0$

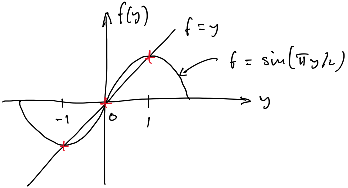
STATE $x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$: $\dot{x} = \begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{bmatrix} \dot{y} \\ -\alpha \dot{y}^2 - \beta \sin y \end{bmatrix}$

EQM: $\dot{x} = 0 \Rightarrow \left\{ \begin{array}{l} \dot{y} = 0 \\ \sin y = 0 \end{array} \right\} \Rightarrow x^* = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}, \; n = 0, \pm 1, \ldots$

(b). $\ddot{y} + (y-1)^2 \dot{y} + y - \sin(\pi y/2) = 0$

STATE $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$: $\dot{x} = \begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{bmatrix} \dot{y} \\ -(y-1)^2 \dot{y} - y + \sin(\pi y/2) \end{bmatrix}$

EQM: $\dot{x} = 0 \Rightarrow \left\{ \begin{array}{l} \dot{y} = 0 \\ y = \sin(\pi y/2) \end{array} \right\} \Rightarrow x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$



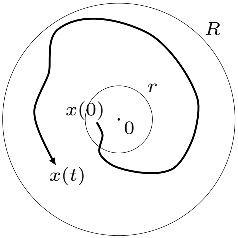
Stability definition

An equilibrium point $x = 0$ is stable iff:

$$\max_t \|x(t)\| \text{ can be made arbitrarily small}$$
$$\text{by making } \|x(0)\| \text{ small enough}$$

$$\Updownarrow$$

for any $R > 0$, there exists $r > 0$ so that
$$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > 0$$



- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
- No guarantee of convergence to the equilibrium point

Stability definition

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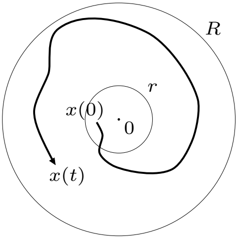
$$\boxed{\begin{array}{c}\text{for any } R > 0, \text{ there exists } r > 0 \text{ so that} \\ \|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > 0\end{array}}$$



- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
- No guarantee of convergence to the equilibrium point

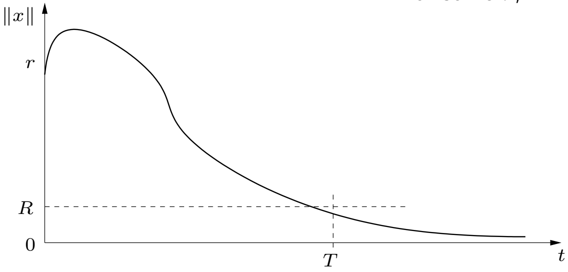
Asymptotic stability definition

An equilibrium point $x = 0$ is asymptotically stable iff:

(i). $x = 0$ is stable
(ii). $\|x(0)\| < r \implies \|x(t)\| \to 0$ as $t \to \infty$

(ii) is equivalent to:

for any $R > 0$,
$$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > T$$
for some r, T

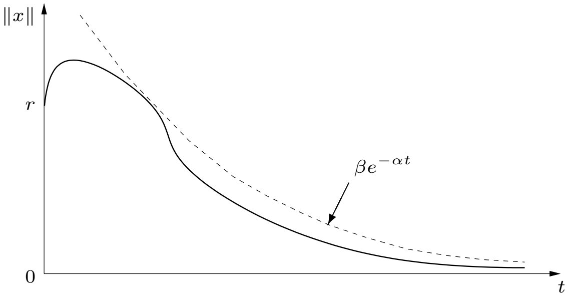


Exponential stability definition

An equilibrium point $x = 0$ is exponentially stable iff:

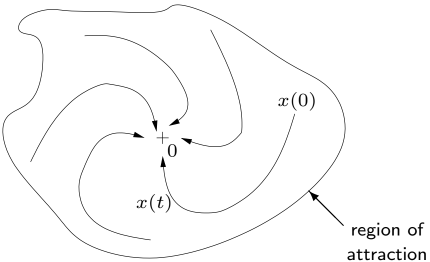
$$\boxed{\|x(0)\| < r \implies \|x(t)\| \leq \beta e^{-\alpha t} \quad \forall t > 0}$$

exponential stability is a special case of asymptotic stability



Region of attraction

The region of attraction of $x = 0$ is the set of all initial conditions $x(0)$ for which $x(t) \to 0$ as $t \to \infty$



- Every asymptotically stable equilibrium point has a region of attraction
- $r = \infty \implies$ entire state space is a region of attraction
  $\implies x = 0$ is globally asymptotically stable
- Are stable linear systems asymptotically stable?

Summary

- Nonlinear state space equations: $\dot{x} = f(x, u)$
  $x =$ state vector, $u =$ control input

- Equilibrium points: x^* is an equilibrium point
  of $\dot{x} = f(x)$ if $f(x^*) = 0$

- Stable equilibrium point: x^* is stable if state trajectories starting
  close to x^* remain near x^* at all times

- Asymptotically stable equilibrium point: x^* must be stable and state trajectories starting near x^* must tend to x^* asympotically

- Region of attraction: the set of initial conditions from which state trajectories converge asymptotically to equilibrium x^*

Lecture 2

Linearization and Lyapunov's direct method

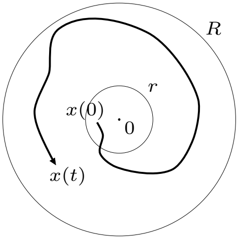
Linearization and Lyapunov's direct method

- Review of stability definitions
- Linearization method
- Direct method for stability
- Direct method for asymptotic stability
- Linearization method revisited

Review of stability definitions

System: $\dot{x} = f(x)$

- $\star$ unforced system (i.e. closed-loop)
- $\star$ consider stability of individual equilibrium points



0 is a stable equilibrium if:

$$\|x(0)\| \le r \implies \|x(t)\| \le R$$
for any $R > 0$

0 is asymptotically stable if:

$$\|x(0)\| \le r \implies \|x(t)\| \to 0$$
as $t \to \infty$

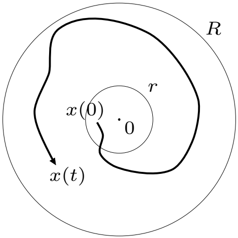
Stability $\to$ local property

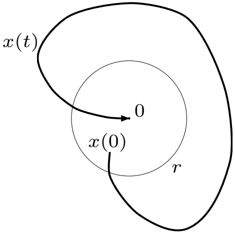
Asymptotic stability $\to$ global if $r = \infty$ allowed

Review of stability definitions

System: $\dot{x} = f(x)$

- $\star$ unforced system (i.e. closed-loop)
- $\star$ consider stability of individual equilibrium points





0 is a stable equilibrium if:

$$\|x(0)\| \le r \implies \|x(t)\| \le R$$

for any $R > 0$

0 is asymptotically stable if:

$$\|x(0)\| \le r \implies \|x(t)\| \to 0$$

as $t \to \infty$

Stability	$\rightarrow$	local property
Asymptotic stability	$\rightarrow$	global if $r = \infty$ allowed

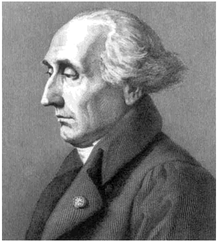
Historical development of Stability Theory

- Potential energy in conservative mechanics (Lagrange 1788):

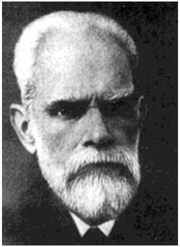
  An equilibrium point of a conservative system is stable if it corresponds to a minimum of the potential energy stored in the system

- Energy storage analogy for general ODEs (Lyapunov 1892)

- Invariant sets (Lefschetz, La Salle 1960s)



J-L. Lagrange 1736-1813



A. M. Lyapunov 1857-1918

S. Lefschetz 1884-1972

Lyapunov's linearization method

- Determine stability of equilibrium at $x = 0$ by analyzing the stability of the linearized system at $x = 0$.

- Jacobian linearization:

$$\begin{aligned} \dot{x} &= f(x) && \text{original nonlinear dynamics} \\ &= f(0) + \frac{\partial f}{\partial x}\Big|_{x=0} x + R_1(x) && \text{Taylor's series expansion} \\ &\approx Ax && \text{since } f(0) = 0 \end{aligned}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \text{Jacobian matrix}$$

$$R_1(x) \to 0 \ \text{ as } x \to 0 \qquad \text{remainder}$$

Taylor's Theorem reminder

– IF $f(x)$ IS DIFFERENTIABLE AT $x=0$, THEN

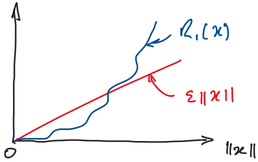
$$f(x) = f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + R_1(x)$$

WHERE $R_1(x) \to 0$ AS $x \to 0$

– MORE PRECISELY : $\lim_{x \to 0} \frac{R_1(x)}{\|x\|} = 0$

WHICH IMPLIES THAT, FOR ANY $\varepsilon > 0$, THERE EXISTS AN $r > 0$ SUCH THAT

$$\|R_1(x)\| \leq \varepsilon \|x\| \quad \text{if} \quad \|x\| \leq r$$



Lyapunov's linearization method

Conditions on A for stability of original nonlinear system at $x = 0$:

stability of linearization	stability of nonlinear system at $x = 0$
$\text{Re}\big(\lambda(A)\big) < 0$	asymptotically stable (locally)
$\max \text{Re}\big(\lambda(A)\big) = 0$	stable or unstable
$\max \text{Re}\big(\lambda(A)\big) > 0$	unstable

Lyapunov's linearization method: examples

* $\dot{x} = x\cos x \Rightarrow \dot{x} \simeq \left[\cos x - x\sin x\right]_{x=0} \cdot x = x$

  or $\dot{x} = x\left(1 - \frac{x^2}{2} + \dots\right) \approx x$

  LINEARISATION: $\dot{x} = x \Rightarrow \lambda = 1 \Rightarrow x = 0$ IS AN <u>UNSTABLE</u> EQM

* $\ddot{x} + \dot{x}e^{x} = 0 \Rightarrow \ddot{x} + \dot{x}\left(1 + x + \frac{x^2}{2} + \dots\right) = 0$

  $\Rightarrow \ddot{x} + \dot{x} + \text{h.o.t.} = 0$

  $\Rightarrow \ddot{x} + \dot{x} \approx 0$

  LINEARISATION: $\ddot{x} + \dot{x} = 0 \Rightarrow \lambda = -1, 0 \Rightarrow$ <u>INCONCLUSIVE</u>

Lyapunov's linearization method

- Linearization may not provide enough information:

(stable)	$\dot{x} = -x^3$	$\xrightarrow{\text{linearize}}$	$\dot{x} = 0$	(indeterminate)
(unstable)	$\dot{x} = x^3$	$\xrightarrow{\text{linearize}}$	$\dot{x} = 0$	(indeterminate)

$\uparrow$

higher order terms determine stability

- Why does linear control work?
  1. Linearize the model:

$$\begin{aligned} \dot{x} &= f(x, u) \\ &\approx Ax + Bu, \qquad A = \frac{\partial f}{\partial x}(0,0), \;\; B = \frac{\partial f}{\partial u}(0,0) \end{aligned}$$

  2. Design a linear feedback controller using the linearized model:

$$u = -Kx, \quad \max \mathrm{Re}\big(\lambda(A - BK)\big) < 0$$

closed-loop linear model strictly stable

nonlinear system $\dot{x} = f(x, -Kx)$ is locally asymptotically stable at $x = 0$

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$\uparrow$
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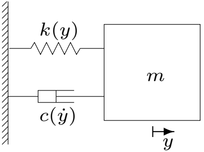
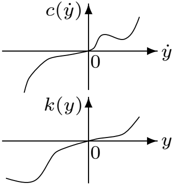
Lyapunov's direct method: mass-spring-damper example





Equation of motion: $\quad m\ddot y + c(\dot y) + k(y) = 0$

Stored energy: $\quad V = \text{K.E.} + \text{P.E.} \quad \begin{cases} \text{K.E.} = \frac{1}{2}m\dot y^2 \\ \text{P.E.} = \int_0^y k(y)\,dy \end{cases}$

Rate of energy dissipation
$$\dot V = \tfrac{1}{2}m\ddot y \frac{d}{d\dot y}\dot y^2 + \dot y \frac{d}{dy}\left[\int_0^y k(y)\,dy\right]$$
$$= m\ddot y\dot y + \dot y k(y)$$

but $m\ddot y + k(y) = -c(\dot y)$, so
$$\dot V = -c(\dot y)\dot y$$
$$\le 0 \qquad \leftarrow \text{since } \operatorname{sign}\big(c(\dot y)\big) = \operatorname{sign}(\dot y)$$

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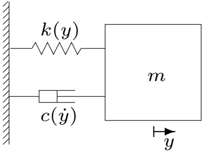

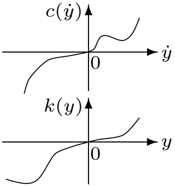

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but $m\ddot{y} + k(y) = -c(\dot{y})$, so

$$\begin{aligned} \dot{V} &= -c(\dot{y})\dot{y} \\ &\le 0 \qquad \leftarrow \text{since } \operatorname{sign}\big(c(\dot{y})\big) = \operatorname{sign}(\dot{y}) \end{aligned}$$

Mass-spring-damper example contd.

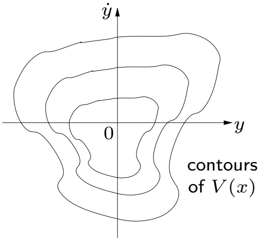
- System state: e.g. $x = [y \;\; \dot{y}]^T$
- $\dot{V}(x) \le 0$ implies that $x = 0$ is stable

$$\uparrow$$

$V(x(t))$ must decrease over time

but

$V(x)$ increases with increasing $\|x\|$



- Formal argument:

for any given $R > 0$:

$$\|x\| < R \quad \text{whenever} \quad V(x) < \overline{V} \quad \text{for some } \overline{V}$$

$$\text{and } V(x) < \overline{V} \quad \text{whenever} \quad \|x\| < r \quad \text{for some } r$$

$$\therefore \|x(0)\| < r \implies V\big(x(0)\big) < \overline{V}$$

$$\implies V\big(x(t)\big) < \overline{V} \quad \text{for all } t > 0$$

$$\implies \|x(t)\| < R \quad \text{for all } t > 0$$

Mass-spring-damper example contd.

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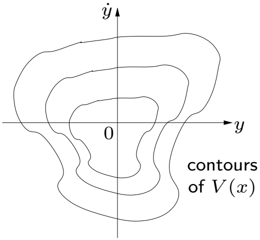

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Positive definite functions

- What if $V(x)$ is not monotonically increasing in $\|x\|$?
- Same arguments apply if $V(x)$ is continuous and positive definite, i.e.

$$\boxed{\begin{array}{rll} \text{(i).} & V(0) = 0 & \\ \text{(ii).} & V(x) > 0 & \text{for all } x \neq 0 \end{array}}$$

for any given $\overline{V} > 0$,
can always find r so that

$$V(x) < \overline{V} \quad \text{whenever} \quad \|x\| < r$$

$V(x) \geq \alpha\|x\|^n$
for some constants α, n, so

$$\|x\| < (\overline{V}/\alpha)^{1/n} \quad \text{whenever} \quad V(x) < \overline{V}$$

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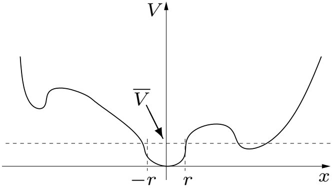

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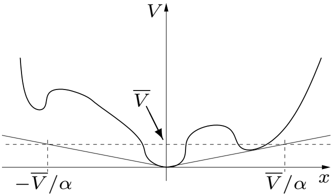

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Lyapunov stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{l} V(x) \text{ is positive definite} \\ \dot{V}(x) \le 0 \end{array}}$$

then $x = 0$ is stable.

To show that this implies $\|x(t)\| < R$ for all $t > 0$ whenever $\|x(0)\| < r$ for any R and some r:

1. choose $\overline{V}$ as the minimum of $V(x)$ for $\|x\| = R$
2. find r so that $V(x) < \overline{V}$ whenever $\|x\| < r$
3. then $\dot{V}(x) \le 0$ ensures that

$$V(x(t)) < \overline{V} \quad \forall t > 0 \quad \text{if } \|x(0)\| < r$$
$$\therefore \|x(t)\| < R \quad \forall t > 0$$

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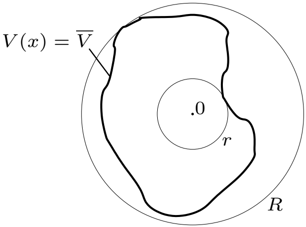

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$$\begin{array}{ll} \text{(i).} & V(0) = 0 \\ \text{(ii).} & V(x) > 0 \quad \text{for } x \neq 0 \text{ and } \|x\| < R_0 \end{array}$$

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- Apply the theorem without determining R, r

  – only need to find p.d. $V(x)$ satisfying $\dot{V}(x) \leq 0$.

- Examples

(i). $\dot{x} = -a(t)x, \quad a(t) > 0$

$$V = \tfrac{1}{2}x^2 \quad \Longrightarrow \quad \begin{aligned} \dot{V} &= x\dot{x} \\ &= -a(t)x^2 \leq 0 \end{aligned}$$

(ii). $\dot{x} = -a(x), \quad \mathrm{sign}\big(a(x)\big) = \mathrm{sign}(x)$

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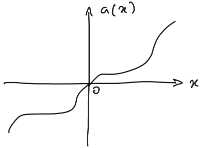
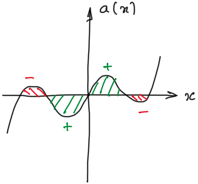


Lyapunov stability theorem

- More examples

(iii). $\dot{x} = -a(x), \quad \int_0^x a(x)\,dx > 0$



$$V = \int_0^x a(x)\,dx \quad \Longrightarrow \quad \dot{V} = a(x)\dot{x}$$
$$= -a^2(x) \leq 0$$

(iv). $\ddot{\theta} + \sin\theta = 0$

$$V = \tfrac{1}{2}\dot{\theta}^2 + \int_0^\theta \sin\theta\,d\theta \quad \Longrightarrow \quad \dot{V} = \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta$$
$$= 0$$

Asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}}$$

then $x = 0$ is locally asymptotically stable.

($\dot{V}$ negative definite $\iff$ $-\dot{V}$ positive definite)

Asymptotic convergence $x(t) \to 0$ as $t \to \infty$ can be shown by contradiction:

if $\|x(t)\| > R'$ for all $t \geq 0$, then

$$\left.\begin{array}{l} \dot{V}(x) < -W \\ \\ V(x) \geq \underline{V} \end{array}\right\} \quad \text{for all } t \geq 0$$

↑

contradiction

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$\uparrow$

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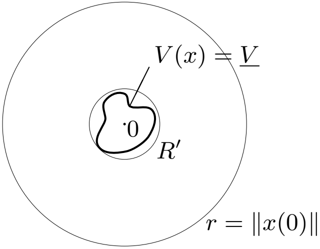

Linearization method and asymptotic stability

- Asymptotic stability result also applies if $\dot{V}(x)$ is only locally negative definite.

- Why does the linearization method work?

  ⋆ consider 1st order system: $\dot{x} = f(x)$
  linearize about $x = 0$: $\quad = -ax + R(x)$

  ⋆ assume $a > 0$ and try Lyapunov function V:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^2 \\ \dot{V}(x) &= x\dot{x} = -ax^2 + xR(x) = -x^2(a - R(x)/x) \\ &\le -x^2(a - |R(x)/x|) \end{aligned}$$

  ⋆ but we can choose ϵ so that $|R(x)/x| < \epsilon$ whenever $|x| \le r$, so

$$\begin{aligned} \dot{V} &\le -x^2(a - \epsilon) \\ &\le -\gamma x^2 \qquad \text{with } a - \epsilon = \gamma > 0 \text{ if } |x| \le r \end{aligned}$$

$\implies$ $\dot{V}$ negative definite for $|x|$ small enough
$\implies$ $x = 0$ locally asymptotically stable

Generalization to nth order systems is straightforward

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Generalization to nth order systems is straightforward

Global asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{l} \left.\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}\right\} \text{for all } x \\ V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}}$$

then $x = 0$ is globally asymptotically stable

- If $V(x) \to \infty$ as $\|x\| \to \infty$, then $V(x)$ is radially unbounded
- Test whether $V(x)$ is radially unbounded by checking if $V(x) \to \infty$ as each individual element of x tends to infinity (necessary).

Global asymptotic stability theorem

- Global asymptotic stability requires:

$$\|x(t)\| \text{ finite } \begin{cases} \text{for all } t > 0 \\ \text{for all } x(0) \end{cases}$$

$$\uparrow$$

not guaranteed by $\dot{V}$ negative definite

in addition to asymptotic stability of $x = 0$

- Hence add extra condition: $V(x) \to \infty$ as $\|x\| \to \infty$

$\updownarrow$ equiv. to

level sets $\{x \ : \ V(x) = \overline{V}\}$ are bounded

$\updownarrow$ equiv. to

$\|x\|$ is finite whenever $V(x)$ is finite

$\uparrow$

prevents $x(t)$ drifting away from 0 despite $\dot{V} < 0$

Asymptotic stability example

System: $$\begin{aligned}\dot{x}_1 &= (x_2 - 1)x_1^3 \\ \dot{x}_2 &= -\frac{x_1^4}{(1+x_1^2)^2} - \frac{x_2}{1+x_2^2}\end{aligned}$$

- Trial Lyapunov function $V(x) = x_1^2 + x_2^2$:

$$\begin{aligned}\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= -2x_1^4 + \underbrace{2x_2x_1^4 - 2\frac{x_2x_1^4}{(1+x_1^2)^2}}_{\substack{\uparrow \\ \text{change } V \text{ to make} \\ \text{these terms cancel}}} - 2\frac{x_2^2}{1+x_2^2} \quad \color{red}{\not\leq 0}\end{aligned}$$

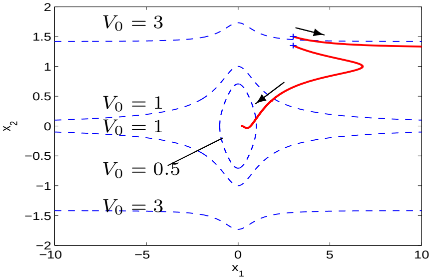
Asymptotic stability example

- New trial Lyapunov function $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$:

$$\dot{V}(x) = 2\Big[\frac{x_1}{1+x_1^2} - \frac{x_1^3}{(1+x_1^2)^2}\Big]\dot{x_1} + 2x_2\dot{x_2}$$
$$= -2\frac{x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2} \leq 0$$

$V(x)$ positive definite, $\dot{V}(x)$ negative definite $\implies$ $x = 0$ a.s.
But $V(x)$ not radially unbounded, so cannot conclude global asymptotic stability

State trajectories:



Summary

- Positive definite functions
- Derivative of $V(x)$ along trajectories of $\dot{x} = f(x)$
- Lyapunov's direct method for: stability
asymptotic stability
global stability
- Lyapunov's linearization method

Lecture 3

Convergence and invariant sets

Convergence and invariant sets

- Review of Lyapunov's direct method
- Convergence analysis using Barbalat's Lemma
- Invariant sets
- Global and local invariant set theorems

Review of Lyapunov's direct method

Positive definite functions

- If
$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0 \quad \text{for all } x \neq 0 \end{aligned}$$
then $V(x)$ is positive definite

- If $\mathcal{S}$ is a set containing $x = 0$ and
$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0 \quad \text{for all } x \neq 0,\ x \in \mathcal{S} \end{aligned}$$
then $V(x)$ is locally positive definite (within S)

- e.g.

$$V(x) = x^\top x \qquad \leftarrow \qquad \text{positive definite}$$

$$V(x) = x^\top x(1 - x^\top x) \qquad \leftarrow \qquad \text{locally positive definite within } \mathcal{S} = \{x \ : \ x^\top x < 1\}$$

Review of Lyapunov's direct method

System: $\dot{x} = f(x), \quad f(0) = 0$

Storage function: $V(x)$

Time-derivative of V: $\dot{V}(x) = \dfrac{\partial V}{\partial x}\dfrac{dx}{dt} = \nabla V(x)^\top \dot{x} = \nabla V(x)^\top f(x)$

- If

$$\left.\begin{array}{r l} \text{(i). } & V(x) \text{ is positive definite} \\ \text{(ii). } & \dot{V}(x) \le 0 \end{array}\right\} \text{ for all } x \in \mathcal{S}$$

  then the equilibrium $x = 0$ is stable

- If

$$\text{(iii). } \dot{V}(x) \text{ is negative definite} \qquad \text{for all } x \in \mathcal{S}$$

  then the equilibrium $x = 0$ is asymptotically stable

- If

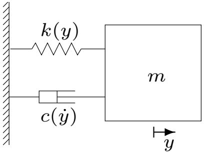
$$\begin{array}{r l} \text{(iv). } & S = \text{entire state space} \\ \text{(v). } & V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}$$

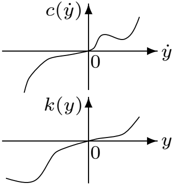
  then the equilibrium $x = 0$ is globally asymptotically stable

Convergence analysis

- What can be said about convergence of $x(t)$ to 0
  if $\dot{V}(x) \leq 0$ but $\dot{V}(x)$ is not negative definite?

- Revisit m-s-d example:





Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$

Storage function:
$$V = \text{K.E.} + \text{P.E.} = \tfrac{1}{2}m\dot{y}^2 + \int_0^y k(y)\,dy$$
$$\dot{V} = -c(\dot{y})\dot{y}$$

Convergence analysis

- V is p.d. and $\dot{V} \le 0$ so: $(y, \dot{y}) = (0, 0)$ is stable
  and $V(y, \dot{y})$ tends to a finite limit as $t \to \infty$

- but does $(y, \dot{y})$ converge to $(0, 0)$?

$\updownarrow$ equivalent to

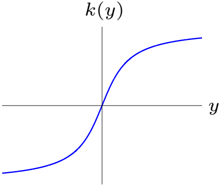
can $V(y, \dot{y})$ "get stuck" at $V = V_0 \neq 0$ as $t \to \infty$?

$\downarrow$

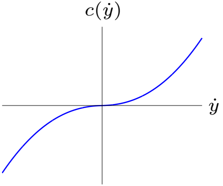
need to consider motion at points $(y, \dot{y})$ for which $\dot{V} = 0$

Example

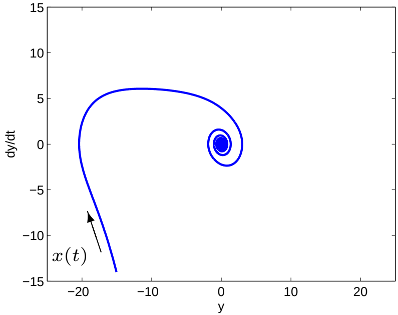
Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



$k(y) = 5\tan^{-1}(y/5)$



$c(\dot{y}) = 0.1\dot{y}(0.5 + |\dot{y}|)(2 - e^{-0.1|\dot{y}|})$



Storage function:

$V = \frac{1}{2}\dot{y}^2 + \int_0^y 5\tan^{-1}(y/5)\,dy$

$\dot{V} = -c(\dot{y})\dot{y} \leq 0$

$\Downarrow$

$\dot{V} = 0$ when $\dot{y} = 0$

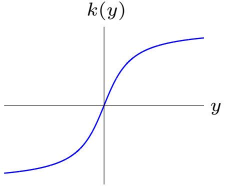
but $k(y) \neq 0 \implies \ddot{y} \neq 0 \implies \ddot{V} \neq 0$

$\Downarrow$

V continues to decrease until $y = \dot{y} = 0$

Example

Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



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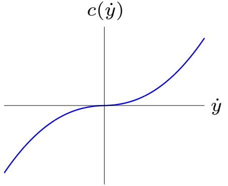

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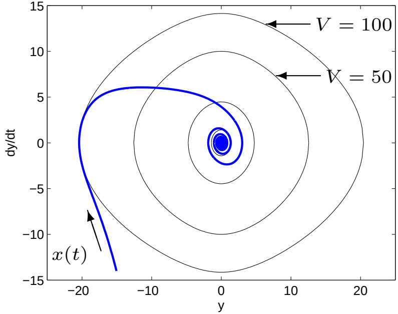

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$$\downarrow$$

$$\dot{V} = 0 \text{ when } \dot{y} = 0$$

but $k(y) \neq 0 \implies \ddot{y} \neq 0 \implies \ddot{V} \neq 0$

$$\Downarrow$$

V continues to decrease until $y = \dot{y} = 0$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times

$$\boxed{\text{then } x(t) \text{ has to converge to } \mathcal{M} \text{ as } t \to \infty}$$

This approach is the basis of the invariant set theorems

Barbalat's Lemma

Barbalat's lemma: For any function $\phi(t)$, if

$$\boxed{\begin{array}{l} \text{(i). } \displaystyle\int_0^t \phi(\tau)\,d\tau \text{ converges to a finite limit as } t \to \infty \\ \text{(ii). } \dot{\phi}(t) \text{ exists and remains finite for all } t \end{array}}$$

then $\lim_{t\to\infty} \phi(t) = 0$

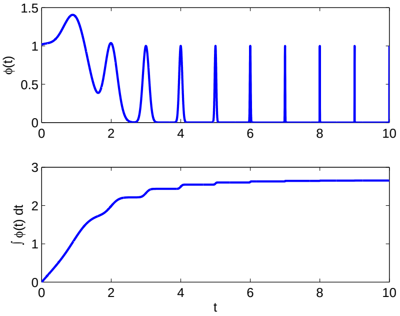
- ⋆ If ϕ is uniformly continuous, then
$$\int_0^t \phi(\tau)\,d\tau \to \text{constant} \quad \Longrightarrow \quad \phi(t) \to 0 \text{ as } t \to \infty$$
- ⋆ Condition (ii) ensures that $\phi(t)$ is continuous for all t
- ⋆ Without (ii) we could have
$$\left.\begin{array}{r} \displaystyle\int_0^t \phi(\tau)\,d\tau \to \text{constant} \\ \text{and} \quad \phi(t) \not\to 0 \end{array}\right\} \text{as } t \to \infty$$

Barbalat's Lemma

Example: pulse train $\phi(t) = \sum_{k=0}^{\infty} e^{-4^k (t-k)^2}$:

$\phi(t)$:

$\int_0^t \phi(\tau)\, d\tau$:



From the plots it is clear that

$$\int_0^t \phi(s)\, ds \text{ tends to a finite limit}$$

but $\quad \phi(t) \not\to 0$ as $t \to \infty \qquad$ because $\;\dot{\phi}(t) \to \infty$ as $t \to \infty$

Barbalat's Lemma

Apply Barbalat's Lemma to $\dot{V}\big(x(t)\big) = \phi(t) \leq 0$:

(a) Integrate:

$$\int_0^t \phi(s)\, ds = V\big(x(t)\big) - V\big(x(0)\big) \qquad \leftarrow \text{finite limit as } t \to \infty$$

(b) Differentiate:

$$\begin{aligned}\dot{\phi}(t) = \ddot{V}\big(x(t)\big) &= f(x)^\top \frac{\partial^2 V}{\partial x^2}(x) f(x) + \nabla V(x)^\top \frac{\partial f}{\partial x}(x) f(x) \\ &= \text{finite for all } t \text{ if } f(x) \text{ continuous and } V(x) \text{ continuously differentiable}\end{aligned}$$

$$\Downarrow$$

$$\boxed{\dot{V}(x) \to 0 \text{ as } t \to \infty}$$

(a) and (b) rely on $\|x(t)\|$ remaining finite for all t,
which is implied by:

$$\begin{aligned}&V(x) \text{ positive definite} \\ &\dot{V}(x) \leq 0 \\ &V(x) \to \infty \text{ as } \|x\| \to \infty\end{aligned}$$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
   $\to$ true whenever $\dot{V} \leq 0$ & V, f are smooth & $\|x(t)\|$ is bounded
   [by Barbalat's Lemma]

2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
   $\to$ algebra!

3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times
   $\to$ $\mathcal{M}$ must be invariant

$$\boxed{\text{then } x(t) \text{ has to converge to } \mathcal{M} \text{ as } t \to \infty}$$

This approach is the basis of the invariant set theorems

Convergence analysis

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1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
   $\to$ true whenever $\dot{V} \leq 0$ & V, f are smooth & $\|x(t)\|$ is bounded
   [by Barbalat's Lemma]
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This approach is the basis of the invariant set theorems

Invariant sets

- A set of points $\mathcal{M}$ in state space is invariant if

$$\boxed{x(t_0) \in \mathcal{M} \quad \Longrightarrow \quad x(t) \in \mathcal{M} \quad \text{for all } t > t_0}$$

Examples:

  - ⋆ Equilibrium points
  - ⋆ Limit cycles
  - ⋆ If $\dot{V}(x) \leq 0$, then sublevel sets of $V(x)$ are invariant

$\uparrow$

sublevel sets of $V(x)$ are $\{x : V(x) \leq \alpha\}$ for constant α

- If $\dot{V}(x) \to 0$ as $t \to \infty$, then

$$\boxed{\begin{array}{l}x(t) \text{ must converge to an invariant set } \mathcal{M} \text{ contained} \\ \text{within the set of points on which } \dot{V}(x) = 0\end{array}}$$

as $t \to \infty$

Global invariant set theorem

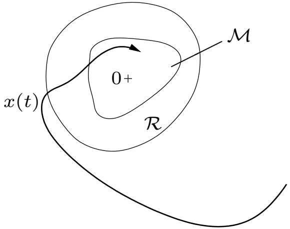
If there exists a continuously differentiable function $V(x)$ such that

$$\boxed{\begin{array}{l} V(x) \text{ is positive definite} \\ \dot{V}(x) \le 0 \\ V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}}$$

then: (i). $\dot{V}(x) \to 0$ as $t \to \infty$

(ii). $x(t) \to \mathcal{M} =$ the largest invariant set contained in $\mathcal{R}$

where $\mathcal{R} = \{x : \dot{V}(x) = 0\}$



- $\dot{V}(x)$ negative definite $\implies \mathcal{M} = 0$ (c.f. Lyapunov's direct method)
- Determine $\mathcal{M}$ by considering system dynamics within $\mathcal{R}$

Global invariant set theorem

Revisit m-s-d example

- $V(x)$ is positive definite, $V(x) \to \infty$ as $\|x\| \to \infty$, and
$$\dot{V}(y, \dot{y}) = -c(\dot{y})\dot{y} \leq 0$$

- therefore $\dot{V} \to 0$, implying $\dot{y} \to 0$ as $t \to \infty$
i.e. $\mathcal{R} = \{(y, \dot{y}) : \dot{y} = 0\}$

- but $\dot{y} = 0$ implies $\ddot{y} = -k(y)/m$

- therefore $\ddot{y} \neq 0$ unless $y = 0$, so $\dot{y}(t) = 0$ for all t only if $y(t) = 0$
i.e. $\mathcal{M} = \{(y, \dot{y}) : (y, \dot{y}) = (0, 0)\}$

$$\Downarrow$$

$(y, \dot{y}) = (0, 0)$ is a globally asymptotically stable equilibrium!

Local invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

> the sublevel set $\Omega = \{x : V(x) \leq \alpha\}$ is bounded for some α
> and $\dot{V}(x) \leq 0$ whenever $x \in \Omega$

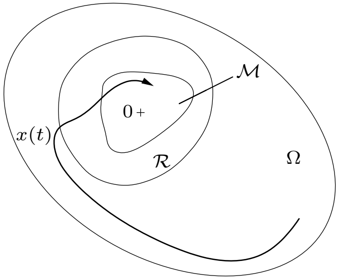
then:

(i). Ω is an invariant set

(ii). $x(0) \in \Omega \quad \Longrightarrow \quad \dot{V}(x) \to 0$ as $t \to \infty$

(iii). $x(t) \to \mathcal{M} =$ largest invariant set contained in $\mathcal{R} \cap \Omega$

where $\mathcal{R} = \{x : \dot{V}(x) = 0\}$



Local invariant set theorem

- $V(x)$ doesn't have to be positive definite or radially unbounded

- Result is based on Barbalat's Lemma applied to $\dot{V}$

  $\uparrow$

  applies here because boundedness of Ω implies $\|x(t)\|$ finite for all t since $x(0) \in \Omega$ and $\dot{V} \leq 0$

- Ω is a region of attraction for $\mathcal{M}$

Example: local invariant set theorem

- Second order system: $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 - 1)^2 x_2^3 - x_1 + \sin(\pi x_1/2) \end{aligned}$

- Equilibrium points: $(x_1, x_2) = (0,0), (1,0), (-1,0)$

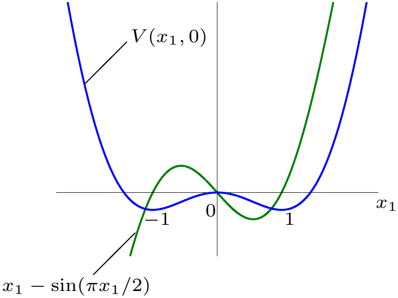
- Trial storage function:

$$V(x) = \tfrac{1}{2}x_2^2 + \int_0^{x_1} \big(y - \sin(\pi y/2)\big)\, dy$$

V is not positive definite
but $V(x) \to \infty$ if $x_1 \to \infty$ or $x_2 \to \infty$

$\Downarrow$

sublevel sets of V are bounded



Example: local invariant set theorem

- Differentiate: $\dot{V}(x) = -(x_1 - 1)^2 x_2^4 \leq 0$

$$\dot{V}(x) = 0 \quad \iff \quad x \in \mathcal{R} = \{x : x_1 = 1 \ \text{ or } \ x_2 = 0\}$$

- From the system model, $x \in \mathcal{R}$ implies:

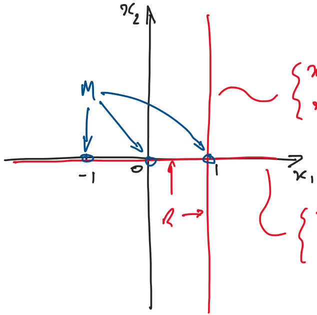
$$\begin{aligned} x_1 = 1 \quad &\implies \quad (\dot{x}_1, \dot{x}_2) = (x_2, 0) \\ \text{and} \quad & \\ x_2 = 0 \quad &\implies \quad (\dot{x}_1, \dot{x}_2) = (0, \sin(\pi x_1/2) - x_1) \end{aligned}$$

$$\text{therefore} \begin{cases} x(t) \text{ remains on line } x_1 = 1 \text{ only if } x_2 = 0 \\ x(t) \text{ remains on line } x_2 = 0 \text{ only if } x_1 = 0,\ 1 \text{ or } -1 \end{cases}$$

$$\implies \mathcal{M} = \{(0,0), (1,0), (-1,0)\}$$

Example: local invariant set theorem

SYSTEM:
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(x_1-1)^2 x_2^3 - x_1 + \sin(\pi x_1/2)$$



$$\begin{cases} x_1 = 1 \\ \dot{x}_1 = x_2 = 0 \text{ iff } x_2 = 0 \end{cases}$$

$$\begin{cases} x_2 = 0 \\ \dot{x}_2 = -x_1 + \sin(\pi x_1/2) = 0 \text{ iff } x_1 = 0, \pm 1 \end{cases}$$

Example: local invariant set theorem

- Apply the local invariant set theorem to any sublevel set $\Omega = \{x : V(x) \le \alpha\}$ containing $x(0)$:

$$\left.\begin{array}{l} \Omega \text{ is bounded} \\ \dot{V} \le 0 \end{array}\right\} \implies x(t) \to \mathcal{M} = \{(0,0), (1,0), (-1,0)\} \text{ as } t \to \infty$$

- For any given $x(0)$, we can choose sufficiently large α so that $\Omega = \{x : V(x) \le \alpha\}$ contains $x(0)$

  so $x(t) \to \mathcal{M} = \{(0,0), (1,0), (-1,0)\}$ as $t \to \infty$ for all $x(0)$

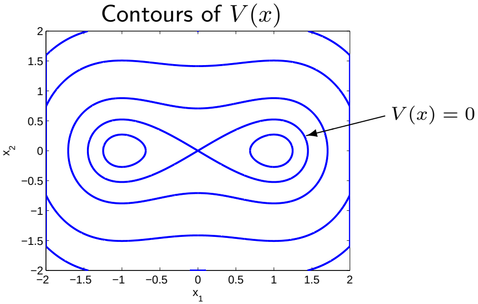
Can we find more precise limits for $x(t)$?

Example: local invariant set theorem

We have shown $x(t)$ converges asymptotically to $(0,0)$, $(1,0)$ or $(-1,0)$ but:

(a). $x = (0,0)$ is unstable since the linearization at $(0,0)$ has poles $\pm\sqrt{\frac{\pi}{2} - 1}$

(b). $V(x)$ has sublevel sets that contain only $(1,0)$ or $(-1,0)$



apply the local invariant set theorem to $\Omega = \{x : V(x) \leq \alpha\}$ for $\alpha < 0$

$\downarrow$

$x = (1,0)$, $x = (-1,0)$ are stable equilibrium points

Summary

- Convergence analysis using Barbalat's lemma
- Invariant sets
- Invariant set methods for convergence analysis:

  local invariant set theorem

  global invariant set theorem

Lecture 4

Linear systems, passivity, and the circle criterion

Linear systems, passivity, and the circle criterion

- Summary of stability methods
- Lyapunov functions for linear systems
- Passive linear systems
- The circle criterion

Summary of stability methods

- Linearization method

$$\dot{x} = Ax \text{ is strictly stable, } A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$
$$\Downarrow$$
$$x = 0 \text{ locally asymptotically stable}$$

- Lyapunov's direct method

$V(x)$ locally p.d. $\dot{V}(x) \le 0$ locally	$V(x)$ locally p.d. $\dot{V}(x)$ locally n.d.	$V(x)$ p.d. $\dot{V}(x)$ n.d. $V(x) \to \infty$ as $\Vert x\Vert \to \infty$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$x = 0$ stable	$x = 0$ locally asymptotically stable	$x = 0$ globally asymptotically stable

- Invariant set theorems

$V(x)$ p.d. $\dot{V}(x) \le 0$ $V(x) \to \infty$ as $\Vert x\Vert \to \infty$	$\Omega = \{x \;:\; V(x) \le V_0\}$ bounded $\dot{V}(x) \le 0$ for all $x \in \Omega$

$$\Downarrow$$

$x(t)$ converges to the union of invariant sets contained in $\{x \;:\; \dot{V}(x) = 0\}$

Summary of stability methods

- Instability theorems analogous to Lyapunov's direct method, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot V(x) \text{ p.d.} \end{array}\right\} \quad \Longrightarrow \quad x = 0 \text{ unstable}$$

- Lyapunov stability criteria are only sufficient, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot V(x) \not\leq 0 \end{array}\right\} \quad \not\Longrightarrow \quad x = 0 \text{ unstable}$$

(some other $V(x)$ demonstrating stability may exist)

- Converse theorems

$$x = 0 \text{ stable} \quad \Longrightarrow \quad V(x) \text{ demonstrating stability exists}$$

(can swap premises and conclusions in Lyapunov's direct method)

↑

But no general method for constructing $V(x)$

Linear systems

- Systematic method for constructing storage function $V(x) = x^\top P x$

  $\dot{x} = Ax$ strictly stable $\implies$ can always find constant matrix P so that $\dot{V}(x)$ is negative definite

- Only need consider symmetric P

$$x^\top P x = \tfrac{1}{2}x^\top P x + \tfrac{1}{2}x^\top P^\top x = \tfrac{1}{2}x^\top \underbrace{(P + P^\top)}_{\text{SYMMETRIC}} x$$

- Need $\lambda(P) > 0$ for positive definite $V(x) = x^\top P x$

  $P = U\Lambda U^\top$ eigenvector/value decomposition

  $\Downarrow$

  $x^\top P x = z^\top \Lambda z$ $\quad z = U^\top x$

  $\Downarrow$

  $x^\top P x$ positive definite iff Λ strictly positive $\quad$ notation: $P \succ 0$ or "P is positive definite"

Linear systems

- Systematic method for constructing storage function $V(x) = x^\top P x$

$$\dot{x} = Ax \text{ strictly stable} \quad \Longrightarrow \quad \begin{array}{l}\text{can always find constant matrix } P \\ \text{so that } \dot{V}(x) \text{ is negative definite}\end{array}$$

- Only need consider symmetric P

$$x^\top P x = \tfrac{1}{2} x^\top P x + \tfrac{1}{2} x^\top P^\top x = \tfrac{1}{2} x^\top \underbrace{(P + P^\top)}_{\text{SYMMETRIC}} x$$

- Need $\lambda(P) > 0$ for positive definite $V(x) = x^\top P x$

$$\begin{array}{ll} P = U \Lambda U^\top & \text{eigenvector/value decomposition} \\ \Downarrow & \\ x^\top P x = z^\top \Lambda z & z = U^\top x \\ \Downarrow & \\ x^\top P x \text{ positive definite} & \left\{ \begin{array}{l} \text{notation: } P \succ 0 \\ \text{or ``} P \text{ is positive definite''} \end{array} \right. \\ \text{iff } \Lambda \text{ strictly positive} & \end{array}$$

Linear systems

- How is P computed?

$$\left.\begin{aligned} \dot{x} &= Ax \\ V(x) &= x^\top P x \end{aligned}\right\} \implies \begin{aligned} \dot{V}(x) &= x^\top P\dot{x} + \dot{x}^\top P x \\ &= x^\top (PA + A^\top P)x \end{aligned}$$

$\therefore x = 0$ is globally asymptotically stable if, for some Q:

$$\boxed{PA + A^\top P = -Q \qquad Q = Q^\top \succ 0}$$

Lyapunov matrix equation

- Pick $Q \succ 0$ and solve $PA + A^\top P = -Q$ for P, then

$$\boxed{\mathrm{Re}[\lambda(A)] < 0 \qquad \Longleftrightarrow \qquad \text{unique solution for } P \text{ and } P = P^\top \succ 0}$$

Linear systems

CLAIM : $PA + A^T P = -Q$ HAS A UNIQUE SOLUTION $P > 0$ FOR EVERY $Q > 0$

IF AND ONLY IF $\mathrm{Re}[\lambda(A)] < 0$

PROOF: LET $\dot{x} = Ax$ AND $V = \frac{1}{2} x^T P x$

① IF $PA + A^T P = -Q$ WITH $P, Q > 0$ THEN:

V IS POSITIVE DEFINITE

AND $\dot{V} = \frac{1}{2} x^T (A^T P + PA) x = -\frac{1}{2} x^T Q x$ IS NEGATIVE DEFINITE

SO $\mathrm{Re}[\lambda(A)] < 0$ BY LYAPUNOV'S DIRECT METHOD

② IF $\mathrm{Re}[\lambda(A)] < 0$ THEN $x(t) = e^{At} x(0)$ AND $\dot{V} = -\frac{1}{2} x^T Q x$ IMPLIES

$$\int_0^\infty \dot{V}(t)\, dt = -\frac{1}{2} x^T(0) \int_0^\infty e^{A^T t} Q e^{At}\, dt \; x(0)$$

$$\therefore\; V(0) - \underbrace{\lim_{t \to \infty} V(t)}_{=0} = x^T(0) \cdot \underbrace{\frac{1}{2} \int_0^\infty e^{A^T t} Q e^{At}\, dt}_{=P} \cdot x(0)$$

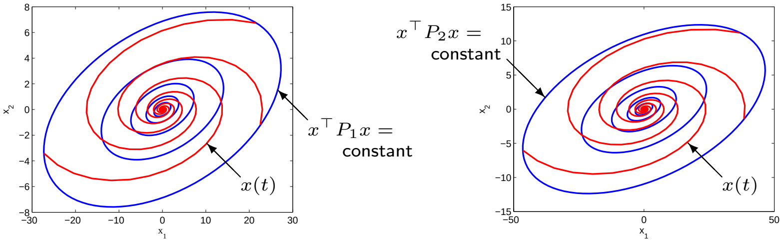
SO $V = x^T P x$ AND $P > 0$ IF $Q > 0$

Example: Lyapunov matrix equation

Stable linear system $\dot{x} = Ax$: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -16 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\qquad \lambda(A) = -1 \pm i\sqrt{15}$

Solve $PA + A^\top P = -Q$ for P:

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\Rightarrow\; P_1 = \begin{bmatrix} 0.33 & -0.5 \\ -0.5 & 4.25 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.41 & -0.19 \\ -0.19 & 0.11 \end{bmatrix} \;\Rightarrow\; P_2 = \begin{bmatrix} 0.12 & -0.21 \\ -0.21 & 1.67 \end{bmatrix}$$

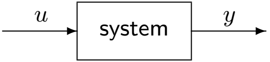


- ⋆ any choice of $Q \succ 0$ gives $P \succ 0$ (since A is strictly stable)
- ⋆ but not every $P \succ 0$ gives $Q \succ 0$

Passive systems

Systematic method for constructing storage functions

based on the input-output representation of a system:



input: u
output: y

The system mapping u to y is:

– passive if

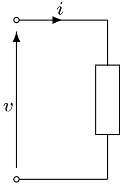
$$\boxed{\dot{V} = yu - g \quad \text{with} \quad V(t) \geq 0, \quad g(t) \geq 0}$$

– dissipative if it is passive and

$$\int_0^\infty g\,dt > 0 \quad \text{whenever} \quad \int_0^\infty yu\,dt \neq 0$$

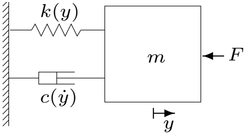
Passive systems

- Passivity is motivated by electrical networks with no internal power generation



$$\left.\begin{array}{r}\text{input: } i\\ \text{output: } v\end{array}\right\}\quad \text{stored energy:}\quad \begin{array}{l} V = \displaystyle\int_0^t vi\,dt \geq 0 \\ \dot{V} = iv \end{array}$$

- Passive mechanical systems (robotics, automotive, aerospace ...)
  e.g. passive m-s-d system mapping input F to output $\dot{y}$:



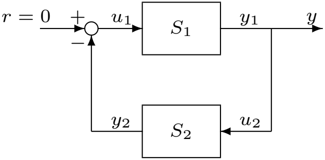
$$m\ddot{x} + c(\dot{x}) + k(x) = F$$
$$\text{sign}(k(y)) = \text{sign}(y)$$
$$\text{sign}(c(\dot{y})) = \text{sign}(\dot{y})$$

$$V = \frac{1}{2}m\dot{y}^2 + \int_0^y k(x)\,dx \quad \Longrightarrow \quad \dot{V} = F\dot{y} - \dot{y}c(\dot{y})$$

Passive systems

Passivity is useful for determining storage functions for feedback systems

- Closed-loop system with passive subsystems S_1, S_2:



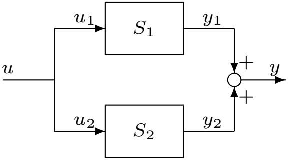
$$S_1: \quad V_1 \geq 0 \quad \dot{V}_1 = y_1 u_1 - g_1$$
$$S_2: \quad V_2 \geq 0 \quad \dot{V}_2 = y_2 u_2 - g_2$$

$$
\begin{aligned}
V_1 + V_2 &\geq 0 \\
\dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\
&= y_1(-y_2) + y_2 y_1 - g_1 - g_2 \\
&= -g_1 - g_2 \\
&\leq 0
\end{aligned}
$$

$\implies$ $V = V_1 + V_2$ is a Lyapunov function for the closed-loop system
if V is a p.d. function of the system state

Interconnected passive systems

- Parallel connection:

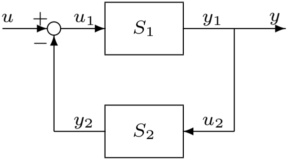


$$
\begin{aligned}
V_1 + V_2 &\geq 0 \\
\dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\
&= (y_1 + y_2) u - g_1 - g_2 \\
&= y u - g_1 - g_2
\end{aligned}
$$

$\Downarrow$

Overall system from u to y is passive

- Feedback connection:



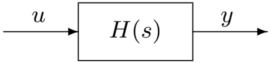
$$
\begin{aligned}
V_1 + V_2 &\geq 0 \\
\dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\
&= y(u - y_2) + y_2 y - g_1 - g_2 \\
&= y u - g_1 - g_2
\end{aligned}
$$

$\Downarrow$

Overall system from u to y is passive

Passive linear systems

Transfer function : $\dfrac{Y(s)}{U(s)} = H(s)$



- H is passive if and only if

(i).	$\mathrm{Re}(p_i) \le 0$, where $\{p_i\}$ are the poles of $H(s)$
(ii).	$\mathrm{Re}\big[H(j\omega)\big] \ge 0$ for all $0 \le \omega \le \infty$

⋆ H must be stable, otherwise $V(t) = \displaystyle\int_0^t yu\,dt$ is not defined for all u

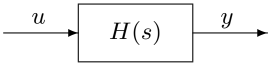
⋆ From Parseval's theorem:

$$\mathrm{Re}\big[H(j\omega)\big] \ge 0 \quad \Longleftrightarrow \quad \int_0^t yu\,dt \ge 0 \text{ for all } u(t) \text{ and } t$$

frequency domain criterion for passivity

Passive linear systems

Transfer function : $\dfrac{Y(s)}{U(s)} = H(s)$



- H is dissipative if and only if $\mathrm{Re}(p_i) \le 0$ and

  $\mathrm{Re}\big[H(j\omega)\big] > 0$ for all $0 \le \omega < \infty$

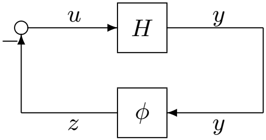
- Kalman-Yakubovich-Popov (KYP) Lemma:

> If H is dissipative, then there exists $P \succ 0$ such that
>
> $V = x^\top P x$ and $\dot{V} = yu - x^\top Q x$, $Q \succ 0$

⋆ x is the state (of any controllable state space realization) of H

⋆ $x = 0$ is globally asymptotically stable with passive output feedback

Linear system + static nonlinearity



H linear: $\quad \dfrac{Y(s)}{U(s)} = H(s)$

ϕ static nonlinearity: $\quad z = \phi(y)$

What are the conditions on H and ϕ for closed-loop stability?

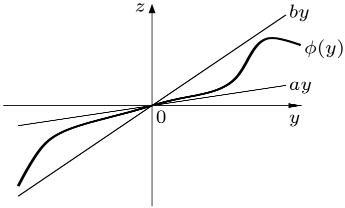
- A common problem in practice, due to e.g.
  - ⋆ actuator saturation (valves, dc motors, etc.)
  - ⋆ sensor nonlinearity
- Determine closed-loop stability given:

ϕ belongs to sector $[a, b]$

$$\Updownarrow$$

$$\boxed{a \leq \frac{\phi(y)}{y} \leq b}$$

"Absolute stability problem"



Linear system + static nonlinearity

- Aizerman's conjecture (1949):

  Closed-loop system is stable if stable for $\phi(y) = ky$, $a \le k \le b$

  false (necessary but not sufficient)

- Sufficient conditions for closed-loop stability:

$$\left.\begin{array}{l}\text{Popov criterion (1960)}\\ \text{Circle criterion}\end{array}\right\}\ \text{based on passivity}$$

- The passivity approach:

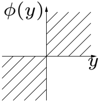
  (1). If H is dissipative (i.e. if $\mathrm{Re}\big[H(j\omega)\big] > 0$ and H is stable), then:

$$\left.\begin{aligned} V &= x^\top P x \\ \dot V &= yu - x^\top Q x \end{aligned}\right\}\ \text{for some } P > 0,\ \ Q > 0$$

$$= -y\phi(y) - x^\top Q x \qquad \leftarrow x = \text{state of } H$$

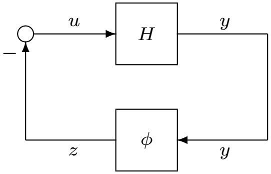
  (2). If ϕ belongs to sector $[0, \infty)$, then:

$$y\phi(y) \ge 0 \qquad \leftarrow$$

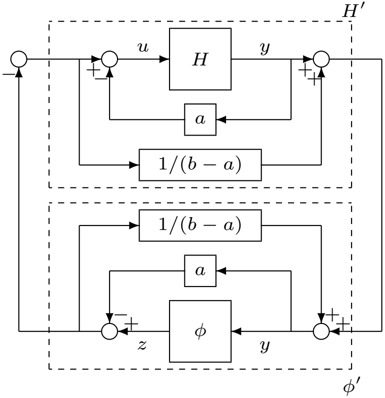


$$\begin{aligned} \text{(1) \& (2)} \quad &\Longrightarrow \quad \dot V \le -x^\top Q x \\ &\Longrightarrow \quad x = 0 \text{ is globally asymptotically stable} \end{aligned}$$

Use loop transformations to generalize the approach for $\begin{cases} H \text{ not passive} \\ \phi \notin [0, \infty) \end{cases}$



$\longleftrightarrow$ equiv. to



$\phi \in [a, b] \quad a, b$ arbitrary

$$\phi \in [a, b] \implies \phi' \in [0, \infty]$$

$$H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b - a}$$

Circle criterion

To make $H'(j\omega) = \dfrac{H(j\omega)}{1 + aH(j\omega)} + \dfrac{1}{b-a}$ dissipative, need:

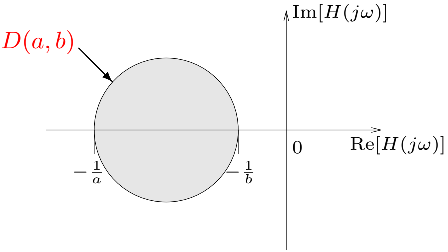
(i). H' stable $\iff$ $\dfrac{H(j\omega)}{1 + aH(j\omega)}$ stable

$\Updownarrow$

Nyquist plot of $H(j\omega)$ goes through ν anti-clockwise encirclements of $-1/a$ as ω goes from $-\infty$ to ∞

($\nu =$ no. poles of $H(j\omega)$ in RHP)

(ii). $\mathrm{Re}\big[H'(j\omega)\big] > 0 \quad \iff \quad \begin{cases} H(j\omega) \text{ lies outside } D(a,b) & \text{if } ab > 0 \\ H(j\omega) \text{ lies inside } D(a,b) & \text{if } ab < 0 \end{cases}$

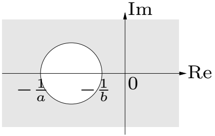


Graphical interpretation of circle criterion

$x = 0$ is globally asymptotically stable if:

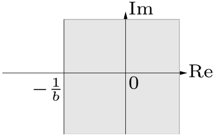
- $\star$ $0 < a < b$

  $H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(a, b)$



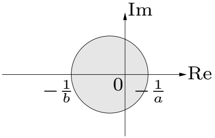
- $\star$ $b > a = 0$

  $H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



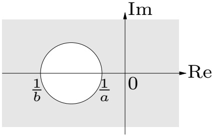
- $\star$ $a < 0 < b$

  $H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



- $\star$ $a < b < 0$

  $-H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(-b, -a)$



Circle criterion

- Circle criterion is equivalent to Nyquist criterion for $a = b > 0$

$$\uparrow$$

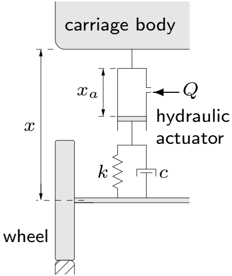
$$D(a,b) = -\frac{1}{a} \quad \text{(single point)}$$

- Circle criterion is only sufficient for closed-loop stability for general a, b

- Results apply to time-varying static nonlinearity: $\phi(y,t)$

Example: Active suspension system

- Active suspension system for high-speed train:



$$
\begin{aligned}
Q &= \phi(u) \\
\dot{x}_a &= Q/A
\end{aligned}
$$

u : valve input signal
Q : flow rate
ϕ : valve characteristics, $\phi \in [0.005, 0.1]$
A : actuator working area

- Force exerted by suspension system on carriage body: F_{susp}

$$
\begin{aligned}
F_{\text{susp}} &= k(x_a - x) + c(\dot{x}_a - \dot{x}) \\
&= \Big(k \int^t Q \, dt + cQ\Big)/A - kx - c\dot{x}, \qquad Q = \phi(u)
\end{aligned}
$$

- Design controller to compensate for the effects of (constant) unknown load on displacement x despite uncertain valve characteristics $\phi(u)$.

Active suspension system contd.

- Dynamics:

$$
\begin{aligned}
& F_{\text{susp}} - F = m\ddot{x} \\
\Longrightarrow \qquad & m\ddot{x} + c\dot{x} + kx = \Big(k\int^t Q\,dt + c\,Q\Big)/A - F, \qquad Q = \phi(u)
\end{aligned}
$$

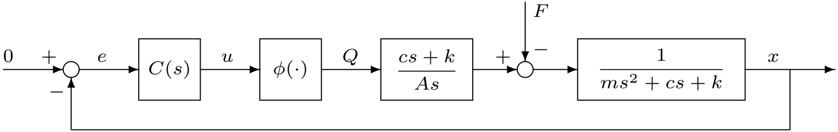
F : unknown load on suspension unit
m : effective carriage mass

- Transfer function model:

$$
X(s) = \frac{cs+k}{ms^2+cs+k}\cdot\frac{Q(s)}{As} - \frac{F}{ms^2+cs+k} \qquad Q = \phi(u)
$$

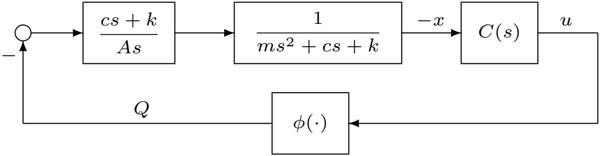
- Try linear compensator $C(s)$:

$$
U(s) = C(s)E(s) \qquad e = -x, \quad \text{setpoint: } x = 0
$$



Active suspension system contd.

- For constant F, we need to stabilize the closed-loop system:



linear system: $H(s) = \dfrac{cs+k}{As(ms^2+cs+k)} \cdot C(s)$

static nonlinearity: $\phi \in [0.005, 0.1]$

- P+D compensator (no integral term needed):

$$C(s) = K(1+\alpha s) \quad \Longrightarrow \quad H(s) = \frac{K(1+\alpha s)(cs+k)}{As(ms^2+cs+k)}$$

H open-loop stable ($\nu = 0$)

- From the circle criterion, closed-loop (global asymptotic) stability is ensured if:

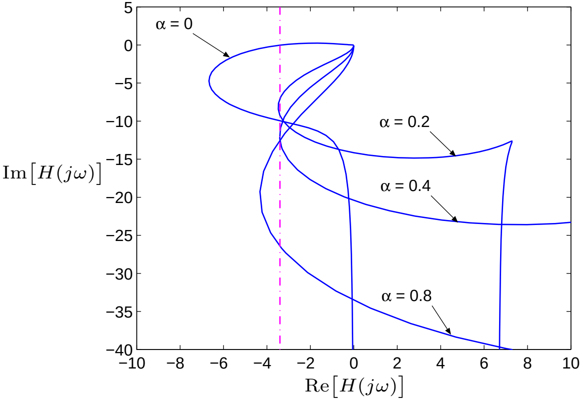
$H(j\omega)$ lies outside $D(0.005, 0.1)$

$\uparrow$

sufficient condition: $\mathrm{Re}\big[H(j\omega)\big] > -10$

Active suspension system contd.

- Nyquist plot of $H(j\omega)$ for $K = 1$ and $\alpha = 0, 0.2, 0.4, 0.8$:



- To maximize gain margin:

$$\text{choose} \quad \alpha = 0.2 \qquad \leftarrow \text{allows for largest } K$$
$$K \leq 10/3.4 = 2.94$$

Summary

At the end of the course you should be able to do the following:

- Understand the basic Lyapunov stability definitions (lecture 1)
- Analyse stability using the linearization method (lecture 2)
- Analyse stability by Lyapunov's direct method (lecture 2)
- Determine convergence using Barbalat's Lemma (lecture 3)
- Understand how invariant sets can determine regions of attraction (lecture 3)
- Construct Lyapunov functions for linear systems and passive systems (lecture 4)
- Use the circle criterion to design controllers for systems with static nonlinearities (lecture 4)