Question 1

Part a)

(i) At (0,0) the derivatives are zero – thus the origin is an equilibrium point. Also this is the only equilibrium point. The Jacobian is

$$J = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 & 1 - {x_1}^2 - 3{x_2}^2 \end{bmatrix}$$

At (0,0)

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Eigenvalues

$$det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda - 1 \end{bmatrix} = 0 \rightarrow \lambda^2 - \lambda + 1 = 0 \rightarrow \lambda = 0.5 \pm i\sqrt{0.75}$$

Real part of the eigenvalue is positive, the equilibrium point is thus an unstable spiral.

(ii) The only possible limit cycle is around the origin as the index for a curve encircling the origin is +1 and the index is 0 for any curve that does not encircle the origin.

Try converting to Polar Co-ords

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} = \frac{x_1x_2 + x_2(-x_1 + (1 - r^2)x_2)}{r} = \frac{(1 - r^2){x_2}^2}{r}$$

$$\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2} = \frac{x_1(-x_1 + (1 - r^2)x_2) - {x_2}^2}{r^2} = \frac{-r^2 + x_1x_2(1 - r^2)}{r^2}$$

If $r = 1$ then $\dot{r} = 0, \dot{\theta} = -1$, indicating a limit cycle. Also, if $r < 1$ then $\dot{r} \geq 0$ and if $r > 1$ then $\dot{r} \leq 0$, so the limit cycle is attractive.

(iii) A circle with radius $r > 1$ centred at the origin is positively invariant because $\dot{r} \leq 0$ if $r > 1$. Also, since $\dot{r} \geq 0$ if $r < 1$, the region of phase plane lying between any r_0 and r_1, with $0 < r_0 < 1$ and $r_1 > 1$, is positively invariant. Hence, by the Poincaré-Bendixson Theorem, a trajectory passing through any point in this region must have as its ω-limit set a either an equilibrium, a closed orbit or a finite number of equilibria making up heteroclinic and homoclinic orbits within this region. The limit cycle at lying on the unit circle is the only attractor in this region (there are no equilibrium points), so all trajectories that do not start at the origin must end up at this limit cycle.

Part b)

The derivatives are zero at (0,0) thus it is an equilibrium point. $V(x, y)$ is positive definite (and radially unbounded) and is thus a candidate Lyapunov function. Now check its derivative:

$$\dot{V} = x\dot{x} + y\dot{y}$$

Along a trajectory we have

$$\dot{V} = x(-x^3 + 2y^3) - 2xy^3 = -x^4 \leq 0$$

The origin is therefore stable. However, we cannot show asymptotic stability using Lyapunov's method as $\dot{V}$ is not a negative definite function of the state (x, y). We therefore look to LaSalle's Invariant Set Theorem.

The set $E = \{x \in D \text{ such that } dV/dt = 0\}$ is the solution to $x^4 = 0$ which is the set $(0, y)$, i.e. the y axis.

We now consider the set $M =$ {the union of all trajectories in E that are positively invariant}. Since $x = 0$ within E, this set corresponds to the positively invariant solutions of

$$\dot{x} = 2y^3$$
$$\dot{y} = 0$$

The trajectories thus travel to the right for all points on the y axis except for the origin. Therefore the only trajectory in E that is positively invariant is the equilibrium point $(0,0)$. Thus $M = \{(0,0)\}$ and $(0,0)$ is asymptotically stable.

Question 2

Part a)

For equilibrium $\dot{x}_1 = \dot{x}_2 = 0$. Add the equations: we get $0 = b - x_1$, thus $x_1 = b$. Substitute back into the first equation $0 = -b + ax_2 + b^2 x_2$ which is the negative of the second equation, thus $x_2 = \frac{b}{a+b^2}$. This is the only equilibrium point.

Find the Jacobian

$$J = \begin{bmatrix} -1 + 2x_1 x_2 & a + {x_1}^2 \\ -2x_1 x_2 & -a - {x_1}^2 \end{bmatrix}$$

At the equilibrium point

$$J = \begin{bmatrix} -1 + 2\dfrac{b^2}{a+b^2} & a + b^2 \\ -2\dfrac{b^2}{a+b^2} & -a - b^2 \end{bmatrix} = \begin{bmatrix} \dfrac{b^2 - a}{a+b^2} & a + b^2 \\ -2\dfrac{b^2}{a+b^2} & -a - b^2 \end{bmatrix}$$

$$det(\lambda I - J) = \lambda^2 - \left\{\frac{b^2 - a}{a+b^2} - (a + b^2)\right\}\lambda + a + b^2$$

The equation is of the form $\lambda^2 - B\lambda + C$ roots are $\frac{B \pm \sqrt{B^2 - 4C}}{2}$. $C = b^2 + a$ and a, b are both positive. Thus C is never negative or zero. As C is positive, stability is determined solely by B. If B is positive the system is unstable.

Instability is when $\frac{b^2-a}{a+b^2} - (a + b^2) \geq 0$ or $b^2 - a - (a + b^2)^2 \geq 0$. We thus need

$$(b^2)^2 + (b^2)\{2a - 1\} + a + a^2 \leq 0$$

Thus

$$b^2 = \frac{1 - 2a \pm \sqrt{\{2a - 1\}^2 - 4\{a + a^2\}}}{2}$$

$$b^2 = \frac{1 - 2a \pm \sqrt{1 - 8a}}{2}$$

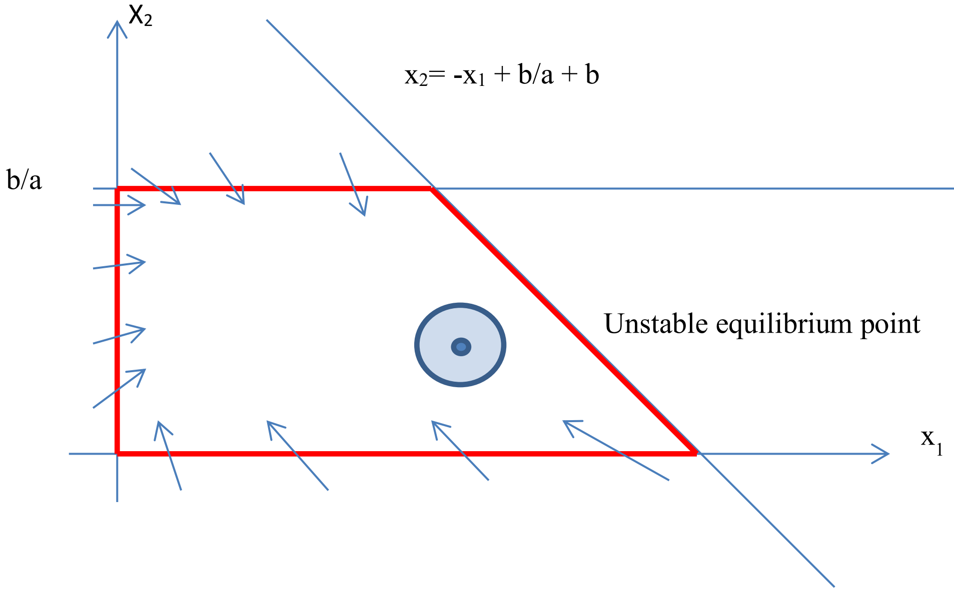
No real solution if a>1/√8. We thus require a<1/√8 . B is negative for large 'b', for instability we thus require

$$a < \frac{1}{\sqrt{8}}$$

$$\frac{1 - 2a - \sqrt{1 - 8a}}{2} < b^2 < \frac{1 - 2a + \sqrt{1 - 8a}}{2}$$

Part b)

Region in question is shown below



Consider trajectories as they cross the boundaries.
For $x_1 = 0$ $0<x_2<b/a$ (left boundary)

$$\dot{x}_1 = ax_2 > 0$$
$$\dot{x}_2 = b - ax_2 > 0$$

Dot product with outward facing normal is negative.

For $x_2 = 0$ $0<x_1<b/a+b$ (bottom boundary)

$$\dot{x}_1 = -x_1 < 0$$
$$\dot{x}_2 = b > 0$$

Dot product with outward facing normal is negative.

For $x_2 = b/a$ $0<x_1<b$ (top boundary)

$$\dot{x}_1 = -x_1 + b + {x_1}^2\frac{b}{a} > 0$$
$$\dot{x}_2 = b - b - {x_1}^2\frac{b}{a} < 0$$

Dot product with outward facing normal is negative.

Right boundary is more difficult – outward normal is $n = \begin{bmatrix}1\\1\end{bmatrix}$. Compute $\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}$.

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix} = \{-x_1 + ax_2 + {x_1}^2x_2\} + \{b - ax_2 - {x_1}^2x_2\} = b - x_1$$

On this boundary $x_1 > b$, thus the dot product is negative and the trajectories cannot leave via this boundary.

We have shown all trajectories crossing the boundaries must enter the region and none leave, thus the region is a trapping region.

Part c)

We have a positively invariant region that contains an unstable equilibrium point (which is the only equilibrium point within the region). We can thus encircle this equilibrium point with a small circle and all trajectories must leave this small circle (as shown in the diagram). We have thus defined a positively invariant region outside this small circle that contains no other equilibrium points – thus is must contain a closed orbit (limit cycle), by Poincaré-Bendixson.

Question 3

Part a)

Dulac's theorem tests $\frac{\partial Bf}{\partial x}+\frac{\partial Bg}{\partial y}$ for changes of sign in a region. Using the given function, we have

$$\frac{\partial\left\{\frac{x(2-x-y)}{x^a y^b}\right\}}{\partial x}+\frac{\partial\left\{\frac{y(4x-x^2-3)}{x^a y^b}\right\}}{\partial y}=\frac{1}{y^b}\frac{\partial\{(2-y)x^{1-a}-x^{2-a}\}}{\partial x}+\frac{1}{x^a}\frac{\partial\{(4x-x^2-3)y^{1-b}\}}{\partial y}$$

$$=\frac{1}{y^b}\{(1-a)(2-y)x^{-a}-(2-a)x^{1-a}\}+\frac{1}{x^a}(1-b)(4x-x^2-3)y^{-b}$$

$$=\frac{1}{x^a y^b}\{(1-a)(2-y)-(2-a)x+(1-b)(4x-x^2-3)\}$$

If a=b, b=1 then we get -1/y <0 in first quadrant. There is thus no change of sign and thus there are no closed orbits.

To find the equilibria

$$\dot{x}=x(2-x-y)=0\rightarrow x=0,\qquad x+y=2$$

$$\dot{y}=y(4x-x^2-3)=0\rightarrow y=0,\qquad x^2-4x+3=0$$

Equilibrium points are thus (0,0) (2,0) (1,1) (3,-1)

The Jacobian is given by

$$J=\begin{bmatrix}2-2x-y & -x\\ 4y-2xy & 4x-x^2-3\end{bmatrix}$$

At (0,0)

$$J=\begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}\ saddle$$

At (2,0)

$$J=\begin{bmatrix}-2 & -2\\ 0 & 1\end{bmatrix}\ saddle$$

Eigenvectors $\left\{-2\rightarrow\begin{bmatrix}1\\0\end{bmatrix}\right\}\left\{1\rightarrow\begin{bmatrix}-2/3\\1\end{bmatrix}\right\}$

At (1,1)

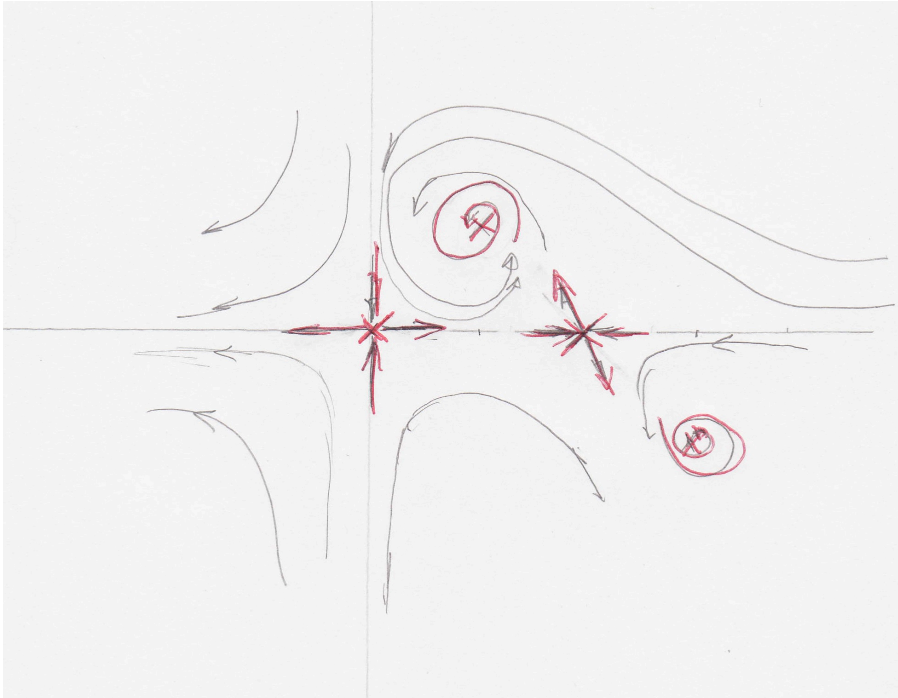
$$J=\begin{bmatrix}-1 & -1\\ 2 & 0\end{bmatrix}$$

$$\lambda^2+\lambda+2=0\rightarrow\lambda=\frac{-1\pm i\sqrt{7}}{2}\ stable\ spiral$$

At (3,-1)

$$J=\begin{bmatrix}-3 & -3\\ 2 & 0\end{bmatrix}$$

$$\lambda^2+3\lambda+6=0\rightarrow\lambda=\frac{-3\pm i\sqrt{15}}{2}\ stable\ spiral$$



Part b)

Using Dulac test the following for changes in sign

$$\frac{\partial\{be^{-2\beta x}y\}}{\partial x}+\frac{\partial\{be^{-2\beta x}(-ax-by+\alpha x^2+\beta y^2)\}}{\partial y}$$
$$=-2\beta b e^{-2\beta x}y+be^{-2\beta x}(-b+2\beta y)$$
$$=-b^2e^{-2\beta x}$$

Not identically zero and no change in sign in the plane. Thus there is no limit cycle.

Part c)

Using Bendixson, test $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}$ for changes in sign:

$$\frac{\partial\left(-\varepsilon\left\{\frac{x^3}{3}-x+y\right\}\right)}{\partial x}+\frac{\partial(-x)}{\partial y}=-\varepsilon\{x^2-1\}$$

If |x|<1 then there is no change in sign and thus no limit cycles within this strip (but there may be one passing through the strip!)

Question 4

Part a)

We are looking for the possibility of an index of '1' for a closed orbit

Equilibria:

Equation 1 gives x=0 or $4-y-x^2=0$, equation 2 gives y=0 or x = 1.

Letting y= 0 from equation 2 we get x=0 or x=$\pm$2; letting x=1 in equation 2 gives y=3. The equilibria are thus (0,0) (2,0) (-2,0) (1,3).

Jacobian:

$$J = \begin{bmatrix} 4-y-3x^2 & -x \\ y & x-1 \end{bmatrix}$$

At (0,0)

$$J = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \; saddle, index -1$$

At (2,0)

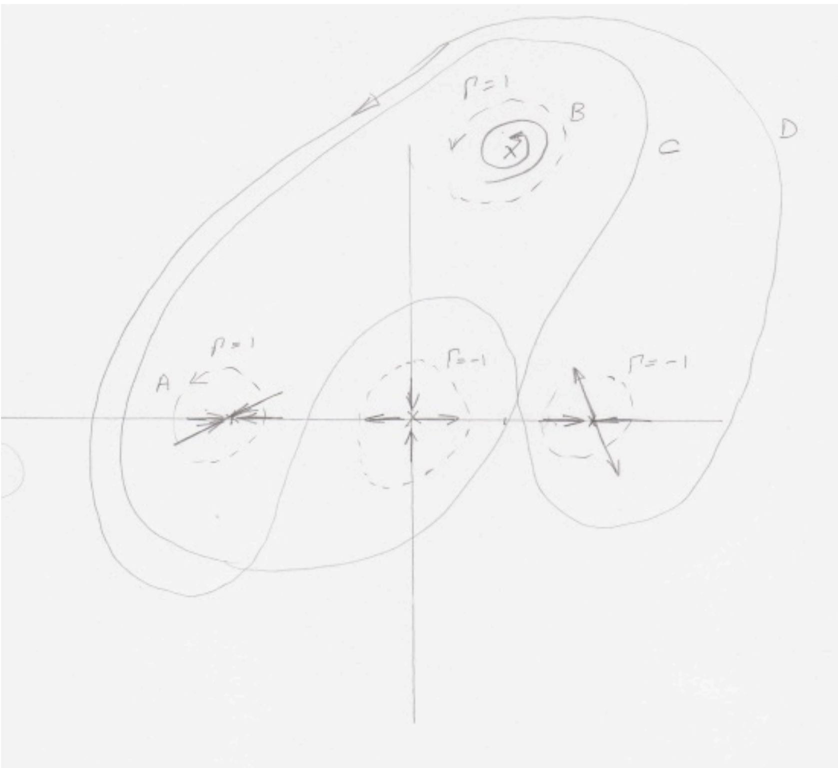
$$J = \begin{bmatrix} -8 & -2 \\ 0 & 1 \end{bmatrix} \; saddle, index -1$$

At (-2,0)

$$J = \begin{bmatrix} -8 & 2 \\ 0 & -3 \end{bmatrix} \; stable\ node, index\ 1$$

At (1,3)

$$J = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix} \rightarrow \lambda^2 + 2\lambda + 3 \rightarrow \lambda = \frac{-2 \pm i\sqrt{8}}{2} \; stable\ spiral, index\ 1$$



There are 4 possible limit cycles that have indices of 1. 'A' is around (-2,0), 'B' is around (1,3), 'C' is around {(-2,0) (0,0) (1,3)} and 'D' is around {(-2,0) (2,0) (1,3)}. Index theory can take us no further.

Equation are very similar to question 3a - so try the same function in Dulac B = 1/xy. Look for changes in sign of

$$\frac{\partial\left\{\frac{x(4-y-x^2)}{xy}\right\}}{\partial x}+\frac{\partial\left\{\frac{y(x-1)}{xy}\right\}}{\partial y}=\frac{1}{y}\frac{\partial(4-y-x^2)}{\partial x}+\frac{1}{x}\frac{\partial(x-1)}{\partial y}=-2\frac{x}{y}$$

There is no change in sign for this function in a single quadrant – thus cycle 'B' is not possible.

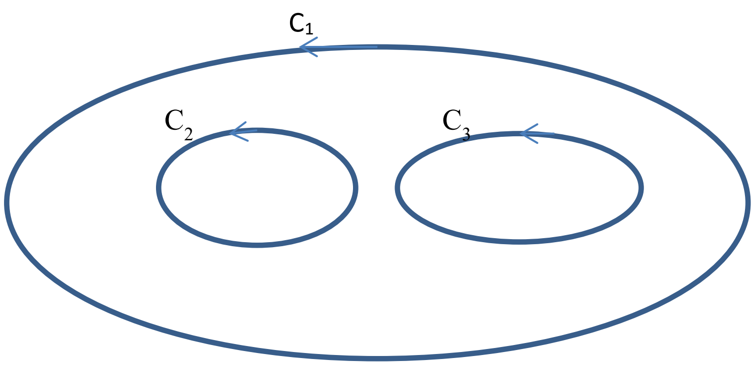
Further reasoning is required beyond Dulac.

If y = 0 then $\dot{y}=0$. Thus no trajectory may cross from +y to –y or visa versa. Similarly for x. Thus every trajectory must remain within its starting quadrant. This argument removes the possibility of trajectories 'A', 'C' and 'D'. 'B' was taken out by Dulac. As Index Theory ensures these are the only possible orbits, there can be no orbits.

Part b)

Equation 1 indicates that at any equilibrium $x_1=0$. There is then no real solution to equation 2. There are thus no equilibrium points and there can be no orbits as the index is zero for the whole plane.

Part c)



As each closed curve is trajectory, the index of each one is '1'. Thus C2 and C3 must both contain equilibria whose indices add to '1'. As C1 encloses both C2 and C3, then it also encloses the equilibrium points within C2 and C3 and the sum of their indices is '2'. There must be a set of equilibrium points outside C2 and C3 that sums to '-1', thus there must be at least one saddle node within C1 and not within either C2 or C3.

Part d)

The Jacobian at the equilibrium point is $\begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}$ and is a linear centre – but note there is a repeated root x=0. Modify the equations to read

$$\dot{x}=x^2-\varepsilon x$$
$$\dot{y}=-y$$

Choose ε to be very small. The equilibrium points are now (0,0) and (ε,0). One is a saddle and the other a stable node. Any contour enclosing these two equilibrium points will have an index of zero. Now let ε go to zero.

Part e)

The equilibrium points are $x=\pm y$ from equation 1 and $x=0$ or $y=0$ from equation 2. A repeated root again. Modify the equations to be

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = 2xy + \varepsilon$$

Choose ε to be small and positive. Then the equations have a solution for $x=-y$, with equilibrium points $\left(-\sqrt{\frac{\varepsilon}{2}}, \sqrt{\frac{\varepsilon}{2}}\right)$ $\left(\sqrt{\frac{\varepsilon}{2}}, -\sqrt{\frac{\varepsilon}{2}}\right)$

The Jacobian is

$$J = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

At $\left(-\sqrt{\frac{\varepsilon}{2}}, \sqrt{\frac{\varepsilon}{2}}\right)$

$$J = 2\sqrt{\frac{\varepsilon}{2}}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \textit{ stable spiral index } 1$$

At $\left(\sqrt{\frac{\varepsilon}{2}}, -\sqrt{\frac{\varepsilon}{2}}\right)$

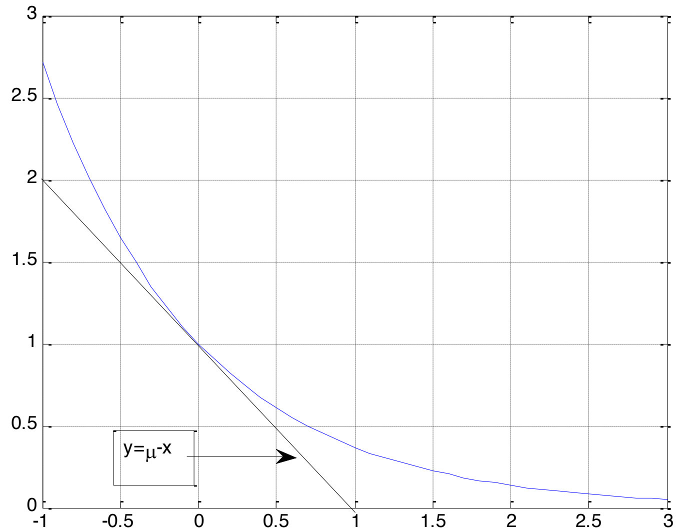
$$J = 2\sqrt{\frac{\varepsilon}{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \textit{ unstable spiral index } 1$$

The total index is thus 2. Now let ε go to zero.

Question 5

Part a)

$$f(x_0, \mu_0) = 0 \rightarrow \mu_0 - x_0 = e^{-x_0}$$



Plot of $y = \mu_0 - x_0$ and $y = e^{-x_0}$

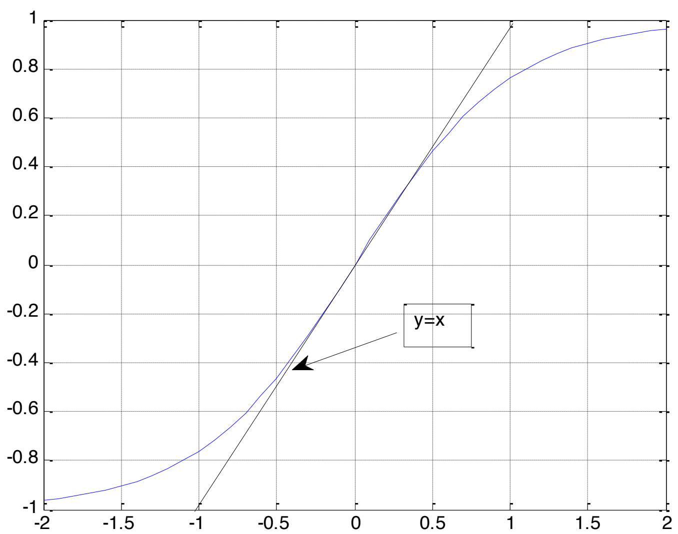
There is no solution for $\mu < 1$. At $\mu = 1$ there is a point of tangency at x=0. This means there will be a change at this point. What is the nature of this change?

$$f_x(x_0, \mu_0) = -1 + e^{-x} = 0 \; at \; x = 0$$
$$f_\mu(x_0, \mu_0) = 1 \neq 0$$
$$f_{xx}(x_0, \mu_0) = -e^{-x} \neq 0$$

We thus have a saddle-node bifurcation.

Part b)

$$f(x_0, \mu_0) = 0 \rightarrow \mu_0 \tanh x_0 = x_0$$



Plot of y=x and y=tanh x

There is only one solution at $x = 0$ provided the slope of the line is greater than the slope of $\mu \tanh(x)$ at the origin. When μ=1 the line and curve are tangential at the origin and then further solutions come into existence, thus $x_0 = 0$ and $\mu_0 = 1$.

$$x_0 = 0, \mu_0 = 1 \rightarrow f(x_0, \mu_0) = 0$$
$$f_x(x_0, \mu_0) = -1 + \mu_0 \mathrm{sech}^2 x_0 = 0$$
$$f_\mu(x_0, \mu_0) = \tanh x_0 = 0$$
$$f_{xx}(x_0, \mu_0) = -2\mu_0 \,\mathrm{sech}\, x_0 \,(\tanh x_0 \,\mathrm{sech}\, x_0) = -2\mu_0 \,\mathrm{sech}^2 x_0 \tanh x_0 = 0$$
$$f_{x\mu}(x_0, \mu_0) = \mathrm{sech}^2 x_0 \neq 0$$
$$f_{xxx}(x_0, \mu_0) = -2\mu_0\{-2\,\mathrm{sech}^2 x_0 \tanh^2 x_0 + \mathrm{sech}^4 x_0\} \neq 0$$

This is a pitchfork bifurcation.

Part c)

$$f(x, \mu) = x(\mu + x^3) = 0$$

There is always an equilibrium at x=0, but when μ goes from negative to positive the number of other equilibrium points at the origin is instantaneously 4

$$x_0 = 0, \mu_0 = 0 \rightarrow f(x_0, \mu_0) = 0$$
$$f_x(x_0, \mu_0) = 4x^3 = 0$$
$$f_\mu(x_0, \mu_0) = x_0 = 0$$
$$f_{xx}(x_0, \mu_0) = 12x^2 = 0$$
$$f_{x\mu}(x_0, \mu_0) = 1 \neq 0$$
$$f_{xxx}(x_0, \mu_0) = 24x_0 = 0$$

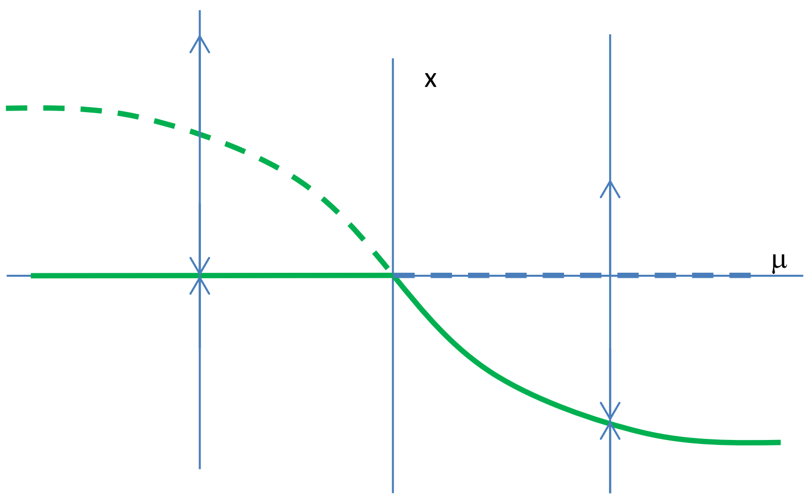
It is not a pitchfork bifurcation.

Reason further:

For μ<0 equilibrium for $x = \sqrt[3]{-\mu}$ (positive) is unstable and for $x = 0$ is stable.

For μ>0 equilibrium for $x = \sqrt[3]{-\mu}$ (negative) is stable and for $x = 0$ is unstable.

We thus have a form of transcritical bifurcation but on a curve rather than a straight line.



Question 6

For a Hopf Bifurcation we need complex equilibrium points that pass through a centre

The Jacobian is

$$J = \begin{bmatrix} \mu + \sigma(3x^2 + y^2) & -1 + 2\sigma xy \\ 1 + 2\sigma xy & \mu + \sigma(3y^2 + x^2) \end{bmatrix}$$

AT (0,0)

$$J = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

For μ<0 we have a stable spiral for μ>0 an unstable spiral passing through a linear centre. We thus have a Hopf bifurcation.

To test for criticality convert the equations to polar form

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{x\{\mu x - y + \sigma x r^2\} + y\{x + \mu y + \sigma y r^2\}}{r} = \frac{\mu r^2 + \sigma r^4}{r} = \mu r + \sigma r^3$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x\{x + \mu y + \sigma y r^2\} - y\{\mu x - y + \sigma x r^2\}}{r^2} = 1$$

If σ=-1 and μ>0 then $\mu r - r^3 > 0$ for small r and $\mu r - r^3 < 0$ for large r. There is thus an unstable spiral inside a stable limit cycle.

If μ=0 $-r^3 < 0$ and we have a stable spiral.

If μ<0 then then $\mu r - r^3 < 0$ and we always have a stable spiral. We thus have a supercritical bifurcation.

If σ=1 and μ>0 then $\mu r + r^3 > 0$ and r grows – an unstable spiral.

If μ=0 r still grows – an unstable spiral.

If μ<0 then $\mu r + r^3 < 0$ for small r, but $\mu r + r^3 > 0$ for r large. Thus the limit cycle is unstable, either falling into the spiral centre or growing without bound. We thus have a subcritical bifurcation.

If σ=0 $\dot{r} = \mu r$ thus for μ>0 we have an unstable spiral and for μ<0 a stable spiral.

For μ=0 we have $\dot{r} = 0$ for all r, thus we have a non-linear centre (actually a linear centre as all the non-linear bits disappear). This is therefore a degenerate bifurcation.

Question 7

Part a)

For an equilibrium

$$x^* = {x^*}^2 + c$$

Thus

$$x^* = \frac{1 \pm \sqrt{1-4c}}{2}$$

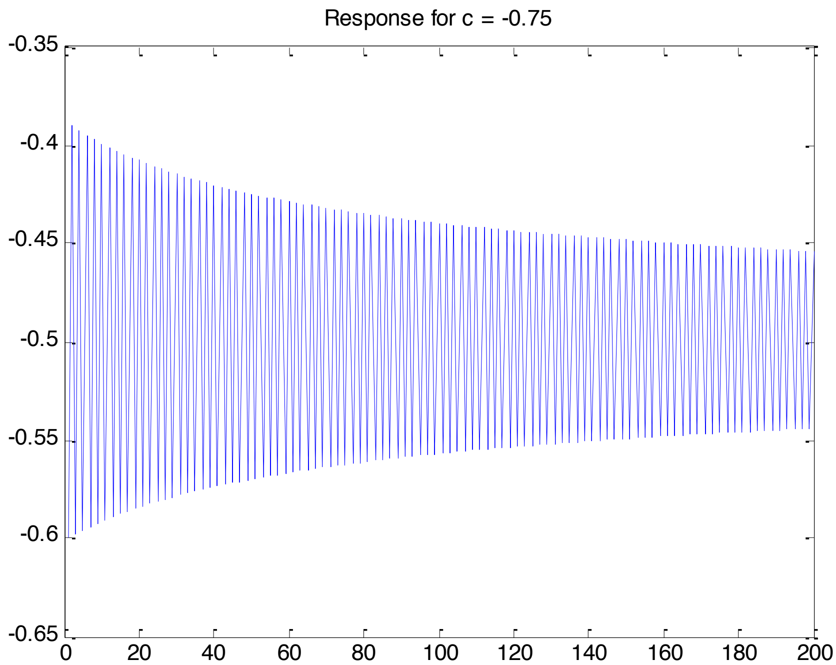
Require $c \leq 1/4$ for an equilibrium to exist.

Part b)

For stability consider the linearization: $w_{k+1} = 2x^* w_k$. We require $|2x^*| < 1$ for this to be stable. If $c \leq 1/4$, then $x^* = \frac{1+\sqrt{1-4c}}{2}$ is unstable. Also $x^* = \frac{1-\sqrt{1-4c}}{2}$ is stable if -3/4 ≤ c ≤ ¼

We know that for $c > 1/4$ there is no equilibrium. What happens when $c = -3/4$? Then we have $|2x^*| = 1$. This is the discrete equivalent of a zero eigenvalue in a continuous time system. It is a centre. We expect a bifurcation at $c = -3/4$.

Part c)



We appear to have a two-cycle about the linear equilibrium point -0.5. Check this out.

For a two-cycle

$$x_{k+2} = f\big(f(x_k)\big) = ({x_k}^2 + c)^2 + c = {x_k}^4 + 2c{x_k}^2 + c^2 + c$$

The equilibrium condition is

$$x^* = {x^*}^4 + 2c{x^*}^2 + c^2 + c$$

For stability we consider the linearization

$$w_{k+2} = \left(4x^{*3} + 4cx^*\right)w_k$$

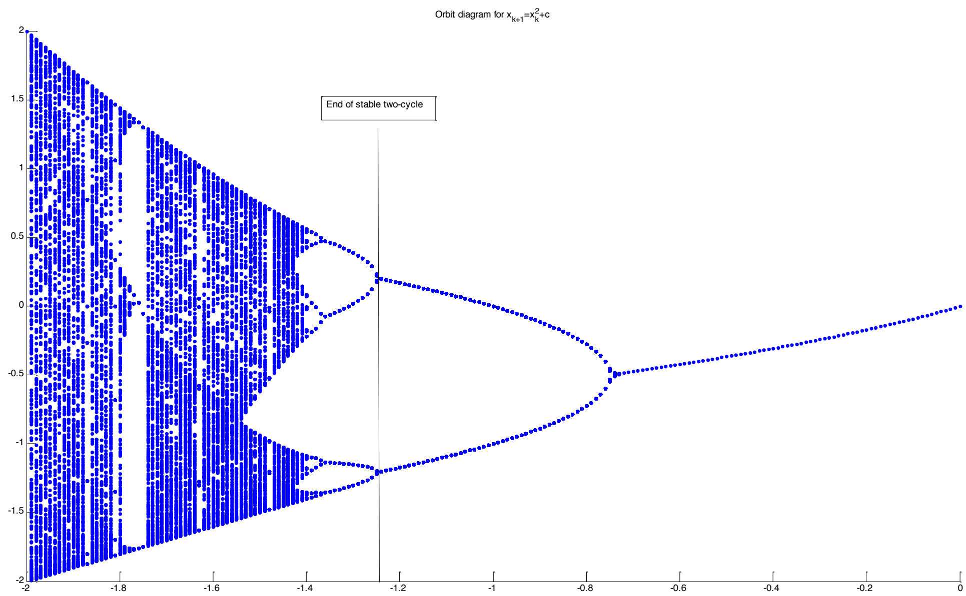
We require $\left|4x^{*3} + 4cx^*\right| < 1.$

Getting tricky now. We need to first solve for equilibrium and then plug into the stability test. Best to do this using the results from part d!

Part d)

Process: For a range of c iterate the diagram and plot the set of points against c.

The diagram shows that the two-cycle ends when c drops below -1.24. The system admits a stable 2-cycle for -1.24 < c < -0.75.




```
% Plot of orbit diagram for Q7 Class 2

%Initial burn in iterations to ensure a 'random' start point
nStart = 200;

% Number of iterations to run.  this is the maximum number of vertical
% spots
nIterations = 300;

% Where we will store the result for plotting
xMap = zeros(1,nIterations);

% Used for plotting
cOnes = ones(1,nIterations);

% Some starting point
xStart = 0;

figure(1)
clf
hold on

for c = 0:-0.01:-2

    % Initialise the system
    x = xStart;
    for i = 1:nStart
```

```
        x = x^2+c;
    end

    % Generate line of the orbit
    xMap(1) = x;
    for i = 2:nIterations
        xMap(i) = xMap(i-1)^2+c;
    end

    % Generate a coloumn of the orbit diagram
    plot(c*cOnes,xMap,'.')

end

title('Orbit diagram for x_{k+1}=x_{k}^{2}+c')
```

Question 8
Part a)

Does the volume in state-space contract with time, i.e. is it dissipative? Test div **f.**

$$\nabla\cdot\mathbf{f}=\frac{\partial(-\mu x+zy)}{\partial x}+\frac{\partial(-\mu y+(z-1)x)}{\partial y}+\frac{\partial(1-xy)}{\partial z}=-2\mu$$

It is dissipative if μ is positive, i.e. a volume in state space will contract.

Part b)
For fixed points

$$\mu x = zy$$
$$\mu y = (z-a)x$$
$$xy = 1$$

Multiply equation 1 and 2

$$\mu^2 xy = zy(z-a)x$$

Substitute equation 3

$$\mu^2 = z(z-a)$$

Solve for z and equate to the suggested substitution

$$z=\frac{a\pm\sqrt{a^2+4\mu^2}}{2}=\mu k^2$$

Using the above substitution for 'k' for equation 1

$$\mu x = zy \Rightarrow x = k^2 y$$

Then for equation 2

$$\mu y=(z-a)x\Rightarrow \mu y=(\mu k^2-a)k^2 y\Rightarrow \mu(k^2-k^{-2})=a$$

Also substituting into equation 3

$$xy=1\Rightarrow k^2y^2=1\Rightarrow y=\pm\frac{1}{k}\Rightarrow x=\pm k$$

Given this result we must have $z=\frac{a+\sqrt{a^2+4\mu^2}}{2}=\mu k^2$ for real solutions.

Part c)
Jacobian

$$J=\begin{bmatrix}-\mu & z & y\\ z-a & -\mu & x\\ -y & -x & 0\end{bmatrix}=\begin{bmatrix}-\mu & \mu k^2 & \frac{1}{k}\\ \mu k^2-a & -\mu & k\\ -\frac{1}{k} & -k & 0\end{bmatrix},\quad \begin{bmatrix}-\mu & \mu k^2 & -\frac{1}{k}\\ \mu k^2-a & -\mu & -k\\ \frac{1}{k} & k & 0\end{bmatrix}$$

Eigenvalues?!! First, note that the 3rd column and 3rd rows change signs between +k and –k, we thus expect the same determinant as we have two changes in sign.

Try using the symbolic package in Matlab to compute det(J-λI).

The input using the Matlab symbolic package:

```
syms mu lambda k a
A = [-mu-lambda mu*k^2 1/k;mu*k^2-a -mu-lambda k;-1/k -k -lambda]
r=collect(expand(det(A)),lambda)
A2 = [-mu-lambda mu*k^2 -1/k;mu*k^2-a -mu-lambda -k;1/k k -lambda]
r2=collect(expand(det(A2)),lambda)
```

The Output:

```
A =
[ - lambda - mu,       k^2*mu,     1/k]
[    mu*k^2 - a, - lambda - mu,       k]
[          -1/k,           -k, -lambda]

r =
(-2*mu)*lambda^2 - lambda^3 + (k^4*mu^2 - k^2 - mu^2 - 1/k^2 - a*k^2*mu)*lambda + a - mu/k^2
- 3*k^2*mu

A2 =
[ - lambda - mu,       k^2*mu,    -1/k]
[    mu*k^2 - a, - lambda - mu,      -k]
[           1/k,            k, -lambda]

r2 =
(-2*mu)*lambda^2 - lambda^3 + (k^4*mu^2 - k^2 - mu^2 - 1/k^2 - a*k^2*mu)*lambda + a - mu/k^2
- 3*k^2*mu
```

Both +k and –k give the same characteristic equation as expected – not much to be said about the roots at present!

$$-\lambda^3 - 2\mu\lambda^2 + \left(\mu^2 k^4 - k^2 - \mu^2 - \frac{1}{k^2} - \mu a k^2\right)\lambda + a - \frac{\mu}{k^2} - 3\mu k^2 = 0$$

OR

$$\lambda^3 + 2\mu\lambda^2 + \left(-\mu^2 k^4 + k^2 + \mu^2 + \frac{1}{k^2} + \mu a k^2\right)\lambda - a + \frac{\mu}{k^2} + 3\mu k^2 = 0$$

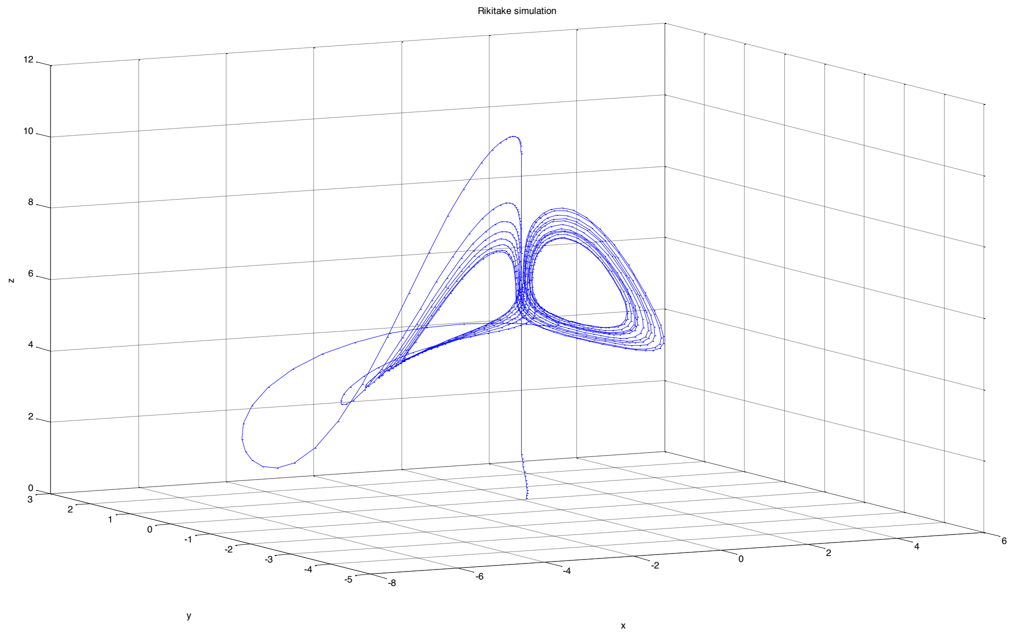
We have shown above $\mu(k^2 - k^{-2}) = a$ thus $-\mu^2 k^4 + \mu^2 = -\mu^2 k^2(k^2 - k^{-2}) = -\mu a k^2$. We can thus simplify the characteristic equation to

$$\lambda^3 + 2\mu\lambda^2 + \left(k^2 + \frac{1}{k^2}\right)\lambda + 2\mu\left(k^2 + \frac{1}{k^2}\right) = 0$$

One root is $\lambda = -2\mu$ (by substitution). Thus the characteristic equation factorises as

$$(\lambda + 2\mu)\left(\lambda^2 + k^2 + \frac{1}{k^2}\right) = 0$$

We thus have a negative real root (a stable subspace) and a pair of imaginary roots (centre subspace).




```
% simulation of Rikitake system for question 8 in class 2

% Initial condition
x0 = [0.1;0;0];

tFinal = 100;

% Solve the differential equation using Runge Kuta - update equations are
% in the function Rikitake
[t,x] = ode23(@Rikitake,[0,tFinal],x0);

figure(1)
clf
plot3(x(:,1),x(:,2),x(:,3))
xlabel('x')
ylabel('y')
zlabel('z')
title('Rikitake simulation')
grid
```

The function

```
% State update equations for Question 8 Class 2
% Note that x is x(1), y is x(2) and z is x(3)
function dx=Rikitake(t,x)
dx = zeros(3,1);
mu = 2;
a = 5;
dx(1) = -mu*x(1)+x(3)*x(2);
dx(2) = -mu*x(2)+(x(3)-a)*x(1);
dx(3) = 1 - x(1)*x(2);
```

Looks chaotic to me!