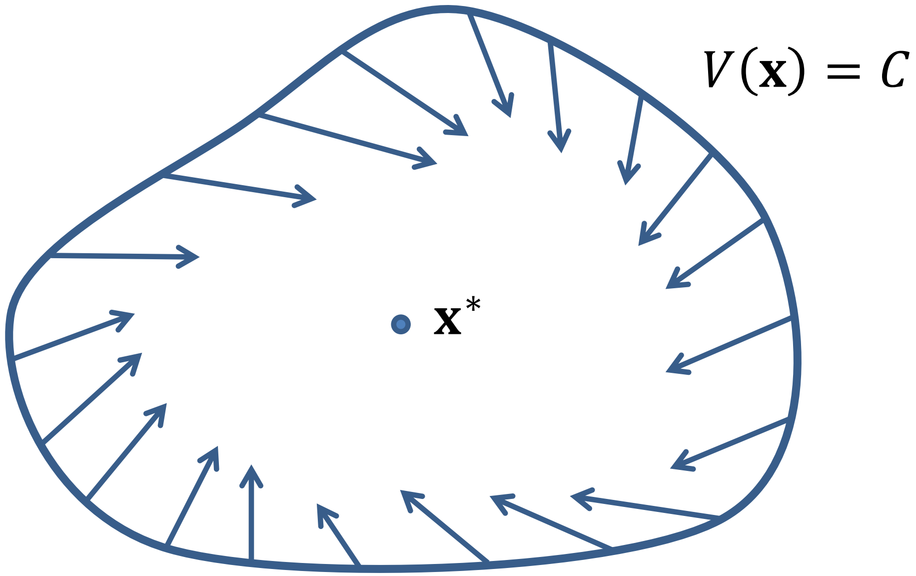
Lecture 4: Lyapunov Functions

- Suppose there exists a connected orientable region (meaning there is an inside and outside) defined by $\{\mathbf{x}\colon V(\mathbf{x}) \leq C\}$ surrounding an equilibrium point $\mathbf{x}^*$ so that all flows crossing the boundary point remain inside the region.

- Once inside the region,
  the flow cannot escape



Nested boundaries

Consider a nested sequence of surfaces defined by a reducing set of constants. The increasing normal to each surface is given by $\underline{\nabla} V$

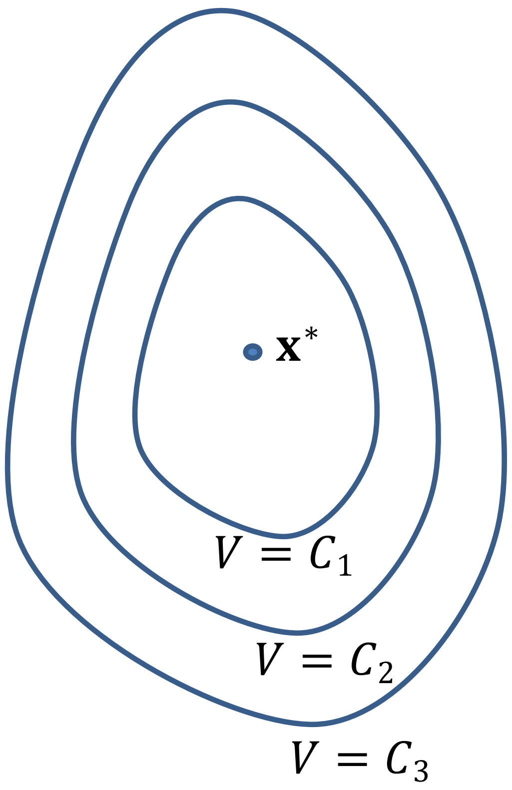
We require the flow to point inwards, i.e. $\underline{\boldsymbol{\nabla}} V . \dot{\mathbf{x}} \leq 0$ But $\dot{\mathbf{x}} = f(\mathbf{x})$. We thus require

$$\underline{\nabla} V . f(\mathbf{x}) \leq 0$$

But

$$\frac{dV}{dt} = \sum \frac{\delta V}{\delta x_i} \frac{dx_i}{dt} = \underline{\nabla} V . \dot{\mathbf{x}} \leq 0$$

so flows go downhill and end up at the bottom



Lyapunov's Theorem

- Let $\mathbf{x}^*$ be an equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, i.e. $\mathbf{f}(\mathbf{x}^*) = 0.$ Let D be an open set surrounding $\mathbf{x}^*$ and let $V(\mathbf{x}): D \to \mathbb{R}$ be a continuously differentiable function on D such that

  1. $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$
  2. $\dot{V}(\mathbf{x}) = \underline{\nabla} V.\mathbf{f}(\mathbf{x}) \leq 0$

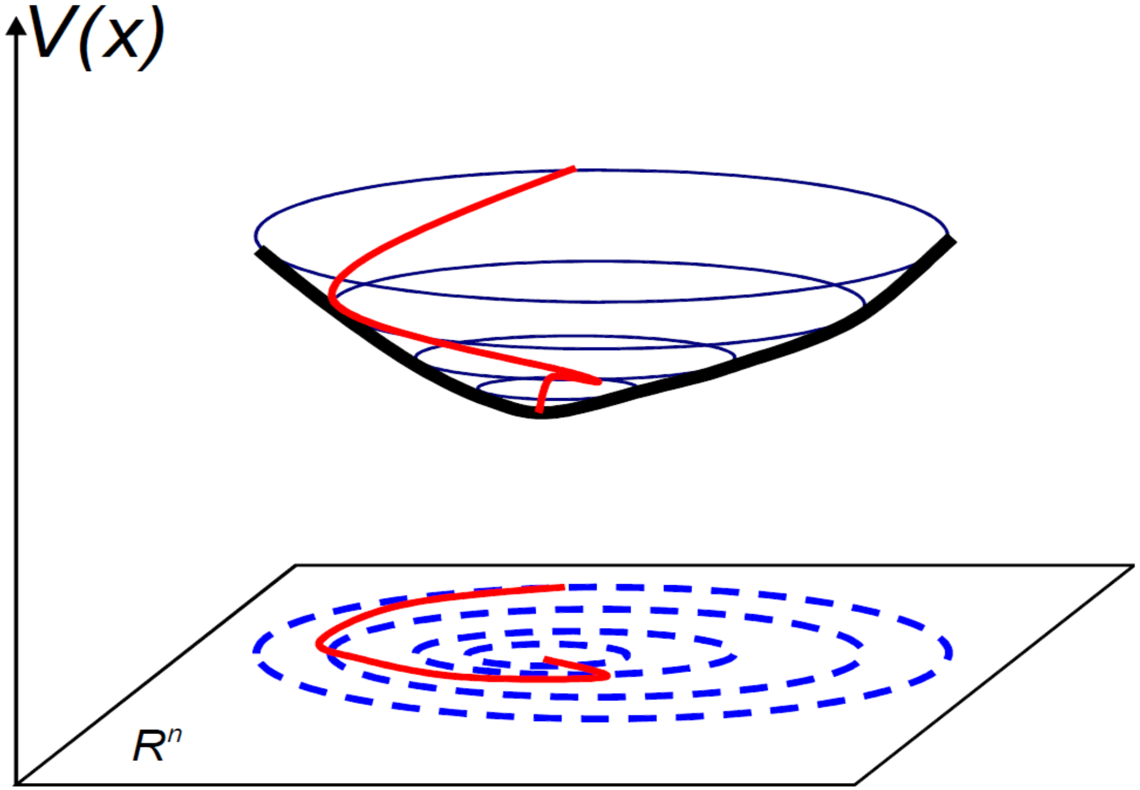
  then $\mathbf{x}^*$ is **stable**. If, in addition

  3. $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$

  then $\mathbf{x}^*$ is **asymptotically stable**

- $V(\mathbf{x})$ is called a Lyapunov function
- If $\lim_{\|\mathbf{x}\| \to \infty} V(\mathbf{x}) = \infty$ and $D = \mathbb{R}^n$ then $\mathbf{x}^*$ is **globally asymptotically stable**

Illustration



$V(\mathbf{x})$ decreases along solution trajectories

Example 1

Consider the dynamical system

$$\dot{x} = y$$
$$\dot{y} = -x + \epsilon x^2 y$$

Equilibrium: $(0,0)$ has Jacobian: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with eigenvalues: $\lambda = \pm j$

so Hartman-Grobman doesn't apply

Let $V(x,y) = \frac{1}{2}(x^2 + y^2)$,

$$\frac{dV}{dt} = \underline{\nabla} V . \dot{\mathbf{x}} = x\dot{x} + y\dot{y} = xy - xy + \epsilon x^2 y^2 = \epsilon x^2 y^2$$

If $\epsilon < 0$ then (0,0) is stable

Example 2

$$\dot{x}_1 = -2x_2 + x_2 x_3$$
$$\dot{x}_2 = x_1 - x_1 x_3$$
$$\dot{x}_3 = x_1 x_2$$

- Equilibrium point: $(0, 0, 0)$ is a linear centre. Let

$$V(\mathbf{x}) = \tfrac{1}{2}(c_1 {x_1}^2 + c_2 {x_2}^2 + c_3 {x_3}^2)$$

  Then

$$\dot{V} = \underline{\nabla} V . \dot{\mathbf{x}} = c_1 x_1(-2x_2 + x_2 x_3) + c_2 x_2 (x_1 - x_1 x_3) + c_3 x_3 x_1 x_2$$
$$= (c_1 - c_2 + c_3) x_1 x_2 x_3 + (-2c_1 + c_2) x_1 x_2$$

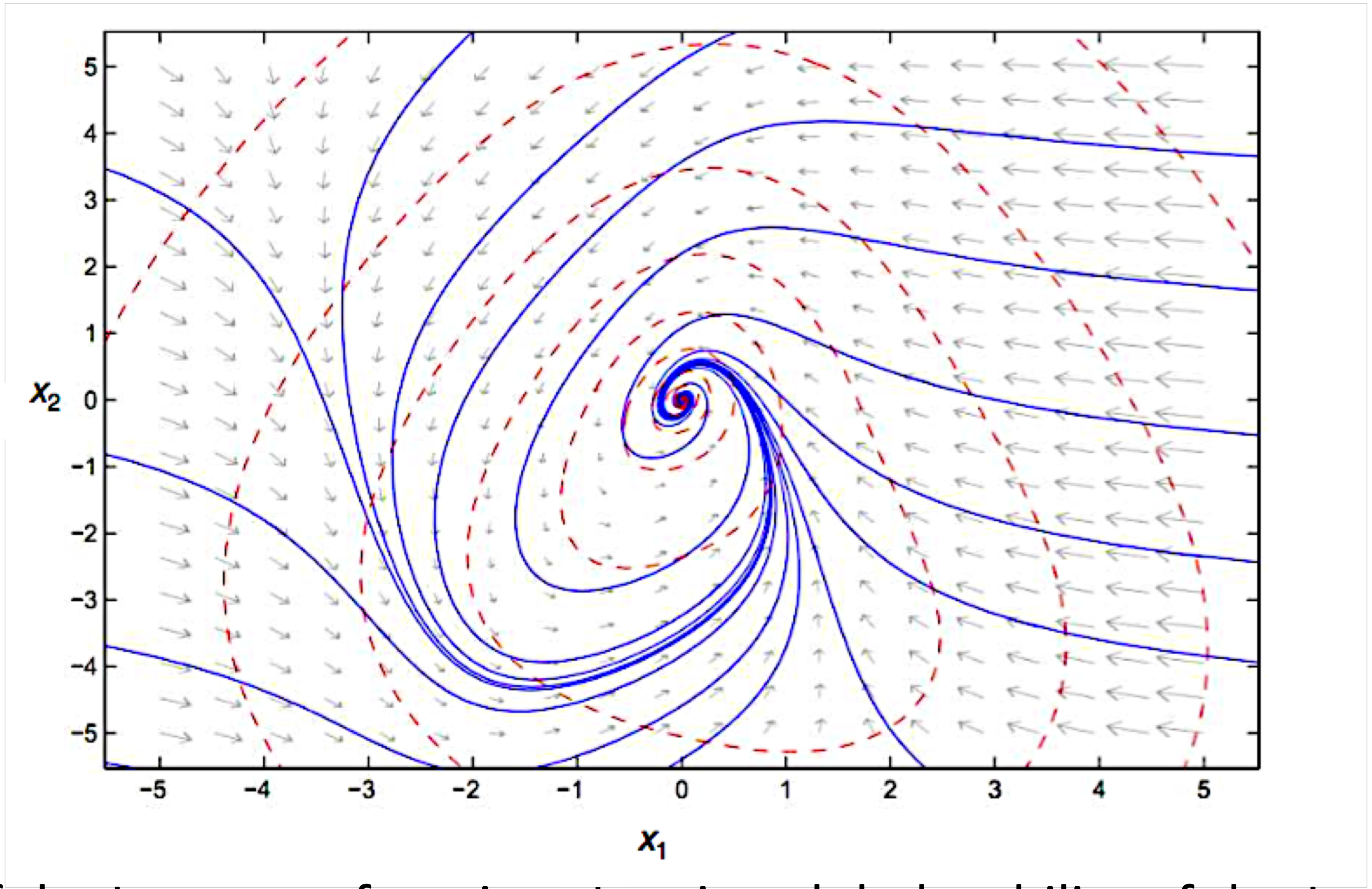
- Choose $c_2 = 2c_1 > 0$ and $c_3 = c_1$, then $\dot{V} = 0$, so equilibrium is stable
- $\dot{V} = 0$ on $V(\mathbf{x}) = \frac{1}{2}({x_1}^2 + 2{x_2}^2 + {x_3}^2)$ so $\mathbf{x}(t)$ lies on $V(\mathbf{x}) = \mathrm{const}$

Jet Engine Example

$$\dot{x}_1 = -x_2 + 1.5{x_1}^2 - 0.5{x_1}^3$$
$$\dot{x}_2 = 3x_1 - x_2$$

- Equilibrium point: $(0,0)$ has Jacobian $\begin{bmatrix} 0 & -1 \\ 3 & -1 \end{bmatrix}$ with $\lambda = \frac{-1 \pm j\sqrt{11}}{2}$ so is a linear stable focus.
- Hartman-Grobman theorem states that the non-linear system is stable (but only close to the origin)
- Lyapunov functions can extend this result globally using specially constructed functions – see the lecture notes

Jet Engine Example



Level curves of the Lyapunov function showing global stability of the Jet engine model

Vector fields possessing an integral

- Consider the flow associated with the solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ as a vector field

- This is said to have an integral $I(\mathbf{x})$ (a scalar function) if
$$\frac{dI(\mathbf{x})}{dt} = \frac{\partial I(\mathbf{x})}{\partial \mathbf{x}}\dot{\mathbf{x}} = \frac{\partial I(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = 0$$

- $\frac{\partial I(\mathbf{x})}{\partial \mathbf{x}}$ is the gradient vector of $I(\mathbf{x})$

- $I(\mathbf{x})$ defines level sets which contain the flow

Pendulum example

$$\dot{q} = p$$
$$\dot{p} = -\frac{g}{l}\sin q$$

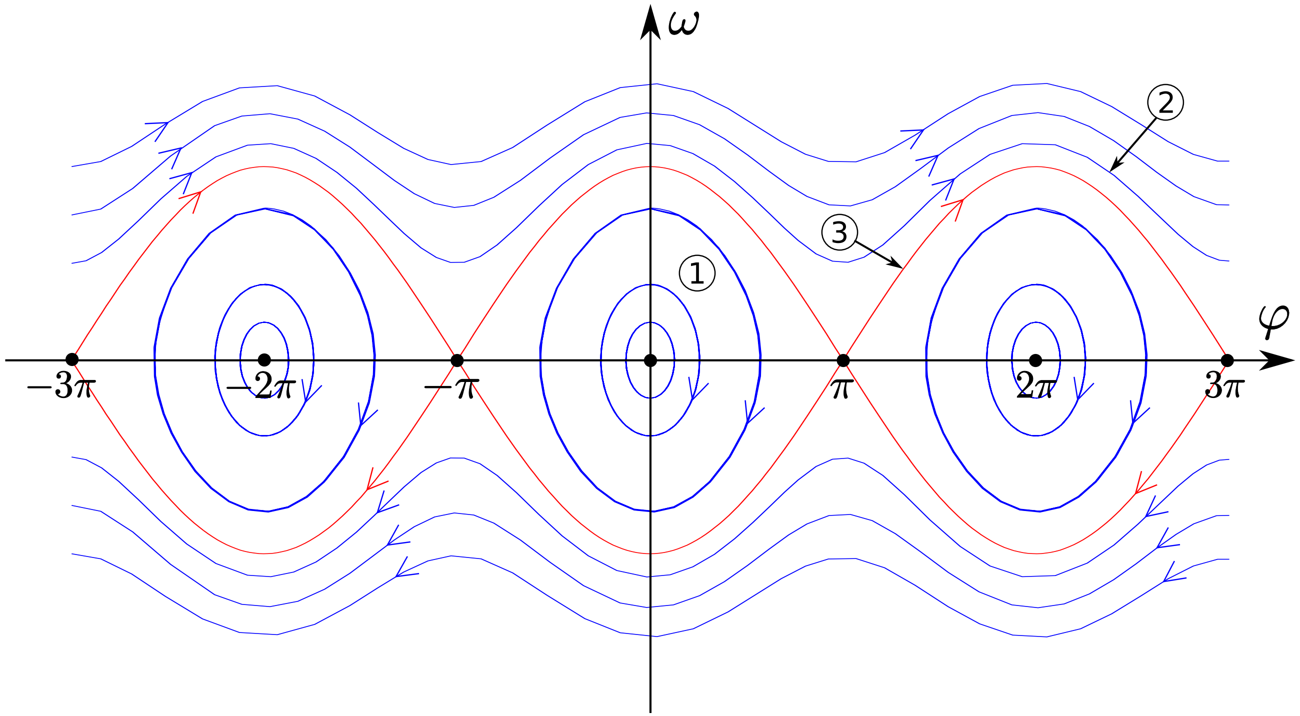
The total stored energy E is conserved

$$E = \frac{1}{2}p^2 - \frac{g}{l}\cos q$$

i.e.

$$\frac{dE}{dt} = p\dot{p} + \dot{q}\frac{g}{l}\sin q = 0$$

Pendulum example



Phase plane of pendulum and level sets of constant energy

Duffing Oscillator for δ=0

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

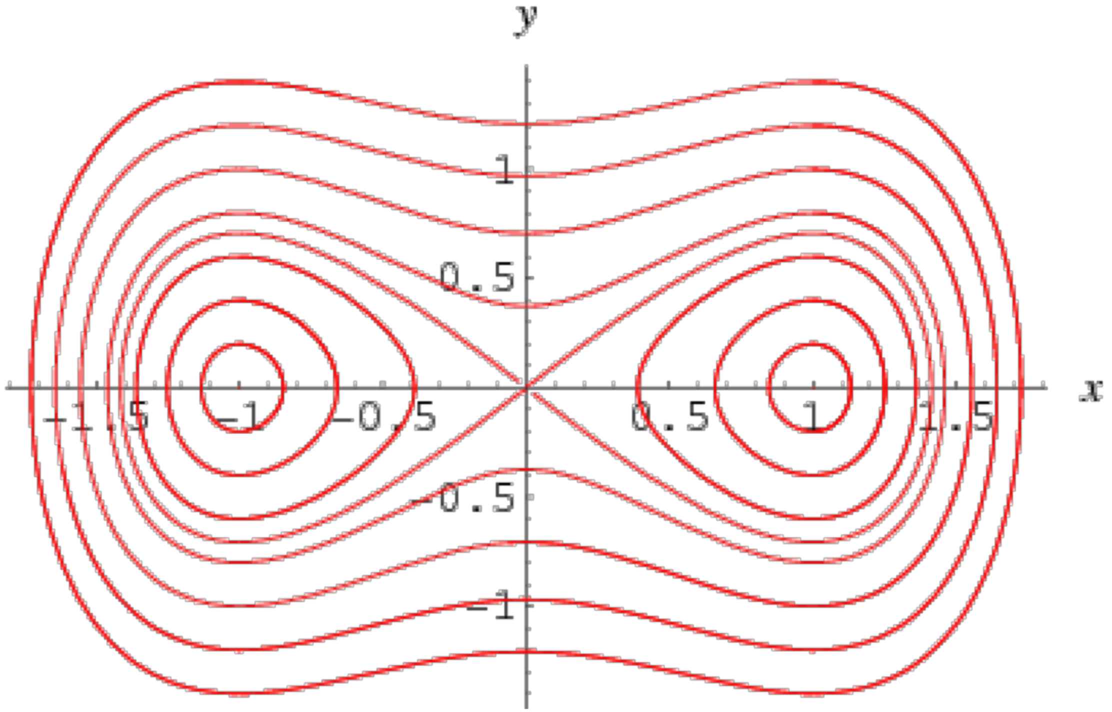
We require

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} = 0$$
$$\frac{\partial I}{\partial x}y + \frac{\partial I}{\partial y}(x - x^3) = 0$$

So, for example

$$I = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

Duffing Oscillator for δ=0



Level sets of $I(\mathbf{x})$ in the phase plane of the duffing oscillator

Hamiltonian systems

Hamiltonian systems have vector fields that possess an integral

Definition: Systems of the form

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}, \mathbf{q})$$
$$\dot{\mathbf{q}} = \mathbf{g}(\mathbf{p}, \mathbf{q})$$

such that

$$\mathbf{f}(\mathbf{p}, \mathbf{q}) = \partial H(\mathbf{p}, \mathbf{q})/\partial \mathbf{q}, \qquad \mathbf{g}(\mathbf{p}, \mathbf{q}) = -\partial H(\mathbf{p}, \mathbf{q})/\partial \mathbf{p}$$

are called **Hamiltonian Systems**.

- **p** and **q** are real vectors with n elements
- H is a twice differentiable function called the Hamiltonian
- **q** is the vector of generalised positions, **p** the vector of generalised momenta
- All Hamiltonian systems are conservative by construction

More on Hamiltonian systems

- If $(\mathbf{p}^*, \mathbf{q}^*)$ is an equilibrium and $H(\mathbf{p}, \mathbf{q}) > 0$ in a region surrounding the equilibrium, then the equilibrium is stable
- A Newtonian system of the form $\ddot{x} = f(x)$ can be written as a Hamiltonian system by summing the potential energy and kinetic energy

$$\dot{x} = v$$
$$\dot{v} = f(x)$$
$$H(x, y) = \frac{v^2}{2} - \int_{x_0}^{x} f(s)ds$$

Gradient Systems

Definition: Let $V(\mathbf{x})$ be a twice differentiable function in a region $D \subseteq \mathbb{R}^n$. The system

$$\dot{x}_i = -\frac{\partial V}{\partial x_i}$$

is called a **gradient** system.

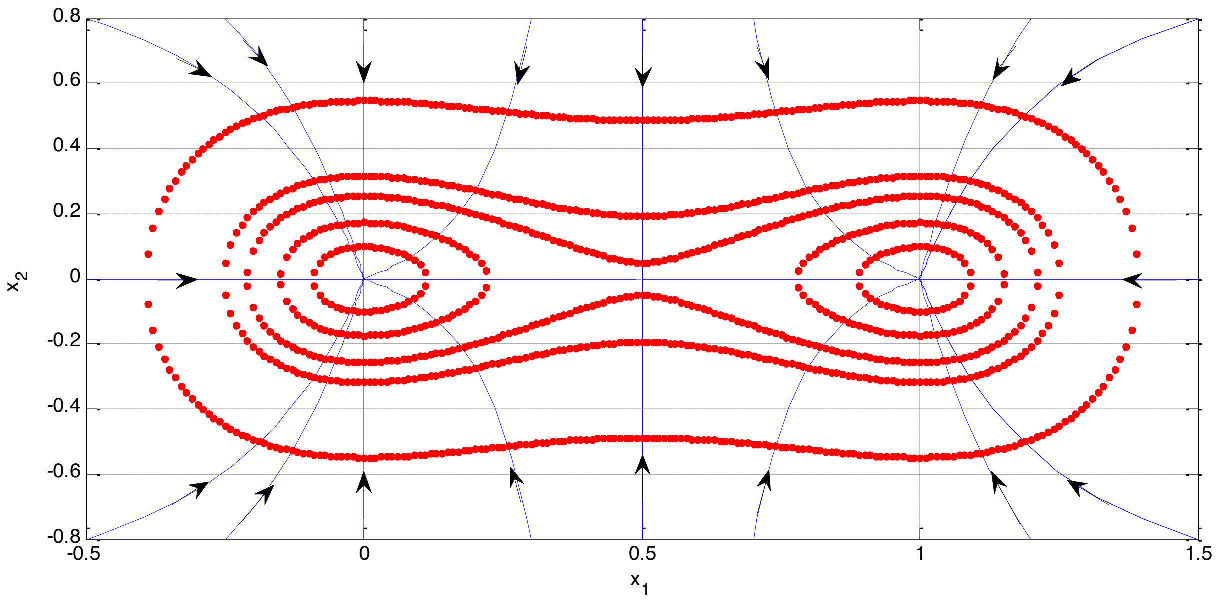
- Equilibrium points are the critical points of V. Away from critical points the trajectories are orthogonal to the level sets of V.
- If $\mathbf{x}^*$ is a strict local minimum of V then $V(\mathbf{x}) - V(\mathbf{x}^*)$ is a Lyapunov function for $\mathbf{x}^*$, showing that $\mathbf{x}^*$ is asymptotically stable. If $\mathbf{x}^*$ is a strict local maximum, then the equilibrium is unstable.

Example Gradient System

$$\dot{x} = -4x(x-1)(x-0.5)$$
$$\dot{y} = -2y$$

Has

$$V(x, y) = x^2(x-1)^2 + y^2$$



Relationship between Gradient and Hamiltonian Systems

- The system

$$\dot{x} = f(x, y) = \frac{\partial H}{\partial y}$$
$$\dot{y} = g(x, y) = -\frac{\partial H}{\partial x}$$

  is orthogonal to

$$\dot{x} = g(x, y)$$
$$\dot{y} = -f(x, y)$$

- They have the same equilibria, centres map to nodes, saddles to saddles and foci to foci.