C24: Dynamical Systems

Examples Sheet C24/1

Michaelmas 2018

You are expected to refer to textbooks as well as the lecture notes when attempting the questions on this sheet. Feel free to use MATLAB.

1. **Equilibria of time-varying systems.** Consider the system

$$\dot{x} = -x + t$$

(a). Show that if time is frozen, $x = t$ is an 'instantaneous' equilibrium of this system.

(b). Show that the solution to this differential equation is

$$x(t) = t - 1 + e^{-t}(x_0 + 1)$$

where $x(0) = x_0$.

(c). What is the equilibrium of the nonautonomous system? Draw the trajectories of the system.

(This question shows that looking at 'instantaneous' equilibria can result in incorrect conclusions.)

2. **Nature of Equilibria of Maps** Find the stable, unstable and centre subspaces of the system

$$\begin{aligned} x_{k+1} &= \lambda x_k + y_k \\ y_{k+1} &= \mu y_k \end{aligned}$$

where $\lambda \neq 0$ and $\mu \neq 0$, for the following cases:

(a). $|\lambda|, |\mu| > 1$.

(b). $|\lambda|, |\mu| < 1$.

(c). $|\lambda| > 1, |\mu| < 1$.

(d). $|\lambda| = 1, |\mu| > 1$.

3. **Linear Flows**

(a). For the following systems, solve the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, determine the stable, unstable and centre subspaces and sketch the phase portraits:

(i). $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$

(ii). $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}$

(iii). $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x}$.

(iv). $\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}$.

(b). Show that

$$e^{\theta \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(c). Consider a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ has eigenvalues λ_i and corresponding eigenvectors $\mathbf{v}_i$. Representing the initial condition by

$$\mathbf{x}(0) = c_1\mathbf{v}_1 + ... + c_n\mathbf{v}_n$$

show that

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + ... + c_n e^{\lambda_n t}\mathbf{v}_n.$$

(d). For a 2-D linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, show that the eigenvalues of $\mathbf{A}$ satisfy

$$\lambda^2 - \tau\lambda + D = 0$$

where τ is the Trace of $\mathbf{A}$ and D is its determinant. Then, show on a graph with axes D and τ, the regions where one expects to have spirals (stable or unstable), nodes (stable or unstable), saddles and centres.

4. **Nonlinear Flows**

(a). Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1(3 - x_1 - x_2)\\ \dot{x}_2 &= x_2(x_1 - 1)\end{aligned}$$

Draw the phase plane of this system, showing clearly the position of the equilibria, the behaviour of the system close to them and the far-from-equilibrium behaviour.

(b). Draw the phase plane of the system

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_1 x_2\\ \dot{x}_2 &= 0.5x_2^2 + x_1 x_2\end{aligned}$$

HINT: You may have to use MATLAB.

(c). Draw the phase plane of the system

$$\begin{aligned}\dot{x}_1 &= x_1^2\\ \dot{x}_2 &= x_2\end{aligned}$$

HINT: You may have to use MATLAB. Think about the trajectories of the system.

(d). Draw the phase plane of the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2\end{aligned}$$

HINT: You may have to use MATLAB. Think about the trajectories of the system.

5. **Polar transformations**

(a). Show that a two-dimensional nonlinear system with states x_1 and x_2 can be written in polar coordinates using

$$\begin{aligned}\dot{r} &= \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} \\ \dot{\theta} &= \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}\end{aligned}$$

(b). Consider the following system

$$\begin{aligned}\dot{x}_1 &= -x_2 + ax_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + ax_2(x_1^2 + x_2^2)\end{aligned}$$

What is the nature of the zero equilibrium as a function of a?

6. **Nonlinear Centres**

(a). Show that a system of the form $\ddot{x} = f(x)$ can be written as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1)\end{aligned}$$

by appropriate choice of x_1, x_2.

(b). By considering the expression

$$x_1\dot{x}_1 + x_2\dot{x}_2$$

show that if the system is conservative, one can obtain a non-constant function $V(x_1, x_2)$ such that $\frac{d}{dt}V(x_1, x_2) = 0$ by integrating

$$-f(x_1)\dot{x}_1 + x_2\dot{x}_2 = 0.$$

(c). Apply your results to the system

$$\begin{aligned}\dot{z}_1 &= -z_2 - z_2^3 \\ \dot{z}_2 &= z_1\end{aligned}$$

to obtain the function $V(z_1, z_2)$.

7. **Lyapunov functions**

(a). Show that the origin of the system

$$\begin{aligned}\dot{x} &= -y - x(x^2 + y^2) \\ \dot{y} &= x - y(x^2 + y^2)\end{aligned}$$

is globally asymptotically stable using Lyapunov's direct method.

(b). Using the function $V(x, y) = 2y^2 - 2x^2 + x^4$ show that the $(\pm 1, 0)$ equilibria of the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^3 - \gamma y, \quad \gamma > 0\end{aligned}$$

are stable. [In later lectures, we will show that these equilibria are actually asymptotically stable.]

(c). Show that the zero equilibrium of the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2^3 - 2x_2^4 \\ \dot{x}_2 &= -x_1 - x_2 + x_1 x_2\end{aligned}$$

is globally asymptotically stable. (Hint: try a function of the form $V(x_1, x_2) = x_1^{\alpha_1} + kx_2^{\alpha_2}$)

8. **Hamiltonian and Gradient Systems**

(a). Show that the system

$$\begin{aligned}\dot{x} &= 2\cos x + \cos y \\ \dot{y} &= 2\cos y + \cos x\end{aligned}$$

has a symmetry around the origin but that it is not conservative.

(b). Show that the system

$$\begin{aligned}\dot{x}_1 &= \sin x_2 \\ \dot{x}_2 &= x_1 \cos x_2\end{aligned}$$

is a gradient system and find the system potential. Hence construct a system which is Hamiltonian, that is related to this gradient system.

Some Answers/Hints

2(a). $E^U = \mathbb{R}^2$, $E^S = \emptyset$, $E^C = \emptyset$.

3(b). Hint: Expand the exponential in series.

4(a). $(0, 0)$ is a saddle point; $(1, 2)$ is a counterclockwise stable spiral; $(3, 0)$ is a saddle point.

5. Hint: Differentiate $x_1^2 + x_2^2 = r^2$. $a = 0$: nonlinear centre.

6(c). $V(z_1, z_2) = \frac{1}{2}z_2^2 + \frac{1}{2}z_1^2 + \frac{1}{4}z_2^4$.

7(c). $V(x_1, x_2) = x_1^2 + x_2^4$.

8(a). The $(\pi/2, \pi/2)$ equilibrium is attracting.