Time-average constraints in stochastic Model Predictive Control

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Outline

1. Motivation
2. Problem statement
3. MPC for parametric uncertainty
4. Handling time-average constraints
5. Numerical example

Motivation

For linear dynamics $+$ additive disturbances
parameter uncertainty
state/control constraints:

- ▷ Enforcing constraints for all uncertainty realisations can be conservative
- ▷ Some applications allow non-zero constraint violation rate

e.g. engine knock control
wind turbine control
building climate control

hence consider constraint softening methods:

- ⋆ Chance constraints — Schwarm 99, Oldewurtel 08, Cannon 11
- ⋆ Time-average constraints — Oldewurtel 13, Korda 14

Motivation

▷ Chance constraints:

- ⋆ computationally hard & possibly non-convex if invoked explicitly

  Prékopa 95

- ⋆ hold with confidence < 1 if imposed approximately

  Campi 08, Calafiore 10

▷ Time-average constraints

- ⋆ constraints tightened according to observed violation frequency
- ⋆ asymptotic limit on average violation rate

  Oldewurtel 13, Korda 14

This paper's approach:

- ⋆ limit on number of violations on any time-interval of a given length

Problem formulation

Control objective:

regulate state x_t to 0 subject to $Fx_t \leq \mathbb{1}$ "sufficiently frequently"

System model:

$$x_{t+1} = A(\omega_t)x_t + B(\omega_t)u_t + w(\omega_t)$$

ω_t : unknown parameter with polytopic bound, hence

$$x_{t+1} \in \mathrm{Conv}\big\{A^{(i)}x_t + B^{(i)}u_t + w^{(i)}\big\}, \quad i = 1, \ldots, p$$

Stabilizability assumption:

K, V, g exist such that $\{x \in \mathbb{R}^n : Vx \leq g\}$ is:

- ⋆ invariant for $x_{t+1} \in \mathrm{Conv}\big\{(A^{(i)} + B^{(i)}K)x_t + w^{(i)}\big\}$
- ⋆ a subset of $\{x : Fx \leq \mathbb{1}\}$

Problem formulation

Consider two types of time-average constraint

Type 1: average number of violations in interval T must not exceed ε

for all t and given T, require $$\frac{1}{T}\sum_{k=t}^{t+T-1} M_k \le \varepsilon$$

where $$M_t = \begin{cases} 0, & Fx_t \le \mathbb{1} \\ 1, & Fx_t \not\le \mathbb{1} \end{cases}$$

- ⋆ implies $\dfrac{1}{t}\displaystyle\sum_{k=0}^{t-1} M_k \le \varepsilon$ for all $t \ge T$
- ⋆ analogous to chance constraint: $\mathbb{P}\big\{Fx_t \not\le \mathbb{1}\big\} \le \epsilon$

Problem formulation

Consider two types of time-average constraint

Type 2: average loss over interval T must not exceed $\bar{l}$

for all t and given T, require $$\frac{1}{T}\sum_{k=t}^{t+T-1} l(Fx_k - \mathbb{1}) \leq \bar{l}$$

where $l : \mathbb{R}^q \to \mathbb{R}$ convex loss function

- ⋆ allows higher penalties on larger violations
- ⋆ analogous to expected value constraint: $\mathbb{E}\big\{l(Fx_t - \mathbb{1})\big\} \leq \bar{l}$

Outline

Dual Mode Predictions

Predicted control input at time t: $u_{k|t} = Kx_{k|t} + c_{k|t}$, $k = 0, 1, \ldots,$

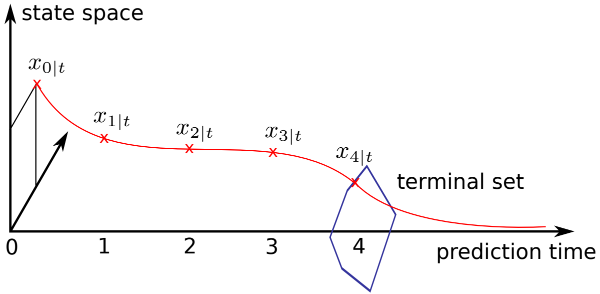
$$c_{k|t} = \begin{cases} \text{decision variable} & k < N \quad \text{(mode 1)} \\ 0 & k \geq N \quad \text{(mode 2)} \end{cases}$$

Predicted state trajectories

$$x_{k+1|t} \in \mathrm{Conv}\{\Phi^{(i)} x_{k|t} + B^{(i)} c_{k|t} + w^{(i)}, \quad i = 1 \ldots p\}$$

$$\Phi^{(i)} = A^{(i)} + B^{(i)} K$$

e.g. $N = 4$
(1 realisation)



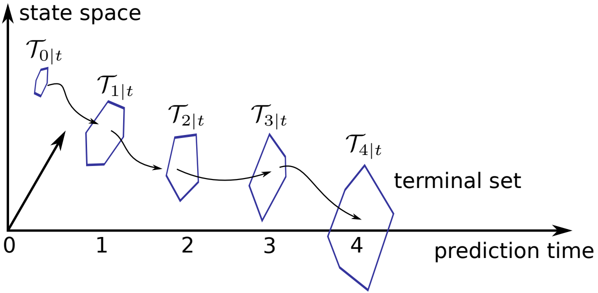
Tube MPC

p uncertainty vertices $\implies$ $x_{k|t}$ belongs to a set with p^k vertices

Bound $x_{k|t}$ using fixed-complexity polytopes $\mathcal{T}_{k|t}$ to avoid exponential growth:

$$x_{0|t} \in \mathcal{T}_{0|t},$$
$$x_{k|t} \in \mathcal{T}_{k|t} \implies x_{k+1|t} \in \mathcal{T}_{k+1|T}$$
$$\mathcal{T}_{N|t} : \text{robustly invariant under } u = Kx$$

e.g. $N = 4$



Tube MPC

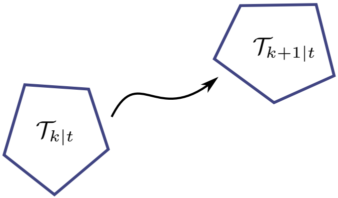
Define $\mathcal{T}_{k|t} = \{x : Vx \leq \alpha_{k|t}\}, \;\; k = 0, 1, \ldots$

- $\star$ fixed face normals, flexibility due to variable $\alpha_{k|t}$
- $\star$ set $\alpha_{k|t} = g$ for all $k \geq N \quad \Longrightarrow \quad \mathcal{T}_{N|t}$ robustly invariant
- $\star$ optimize $\underline{c}_t = (c_{0|t}, \ldots, c_{N-1|t})$ **and** $\underline{\alpha}_t = (\alpha_{0|t}, \ldots, \alpha_{N-1|t})$ at each time t

Tube constraints

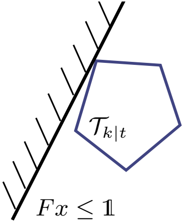
Inclusion constraints:

$$\mathcal{T}_{k|t} \subseteq \Big\{x : \Phi^{(i)}x + B^{(i)}c_{k|t} + w^{(i)} \in \mathcal{T}_{k+1|t},\quad i = 1, \ldots, p\Big\}$$



Feasibility constraints:

$$\mathcal{T}_{k|t} \subseteq \big\{x : Fx \le \mathbb{1}\big\}$$



Impose these conditions using

$\{x : V_1 x \le g_1\} \subseteq \{x : V_2 x \le g_2\}$ iff $\exists\, H \ge 0$ s.t. $HV_1 = V_2,\ Hg_1 \le g_2$

Tube constraints

Proposition

- $\mathcal{T}_{k|t} \subseteq \left\{x : \Phi^{(i)}x + B^{(i)}c_{k|t} + w^{(i)} \in \mathcal{T}_{k+1|t},\ i = 1, \ldots, p\right\}$ if
$$\Gamma^{(i)}\alpha_{k|t} + VB^{(i)}c_{k|t} + Vw^{(i)} \leq \alpha_{k+1|t}$$
for some $\Gamma^{(i)} \geq 0$ satisfying $\Gamma^{(i)}V = V\Phi^{(i)}$

- $\mathcal{T}_{k|t} \subseteq \left\{x : Fx \leq \mathbb{1}\right\}$ if
$$\Lambda\alpha_{k|t} \leq \mathbb{1}$$
for some $\Lambda \geq 0$ satisfying $\Lambda V = F$

Determine $\Gamma^{(i)}$, Λ offline by minimizing the row-sums:

$$\Gamma_j^{(i)} = \arg\min_y \{\mathbb{1}^\top y : y^\top V = V_j\Phi^{(i)},\ y \geq 0\}$$
$$\Lambda_j = \arg\min_y \{\mathbb{1}^\top y : y^\top V = F_j,\ y \geq 0\}$$

then $\Gamma^{(i)}$, Λ are sparse with at most n non-zero elements in each row

Cost function

Penalise deviation from a target set $\Omega = \{x : Wx \leq h\}$,

Ω robustly invariant under $u = Kx$

Quadratic cost:

$$J_t = \min_{s_{0|t} \in \Omega} \sum_{k=0}^{\infty} \mathbb{E}\Big(\|x_{k|t} - s_{k|t}\|_Q^2 + \|u_{k|t} - K s_{k|t}\|_R^2 \Big)$$

where $s_{k+1|t} = \Phi(\omega_{k|t}) s_{k|t} + w(\omega_{k|t})$, $k = 0, 1, \dots$

Proposition

Let $x_{0|t} = x_t$, then

$$J_t = \min_{s_t \in \Omega} \left\| \begin{bmatrix} x_t - s_t \\ \underline{c}_t \end{bmatrix} \right\|_P^2$$

where $P - \mathbb{E}\big\{\Psi^\top(\omega) P \Psi(\omega)\big\} = \bar{Q}$, with

$$\Psi(\omega) = \begin{bmatrix} A(\omega) + B(\omega)K & B(\omega)E \\ 0 & S \end{bmatrix} \qquad \bar{Q} = \begin{bmatrix} Q + K^\top RK & K^\top RE \\ E^\top RK & E^\top RE \end{bmatrix}$$

and $E\underline{c}_t = c_{0|t}$, $S\underline{c}_t = [c_{1|t} \ \cdots \ c_{N-1|t} \ 0]^\top$

Outline

Time-average constraints

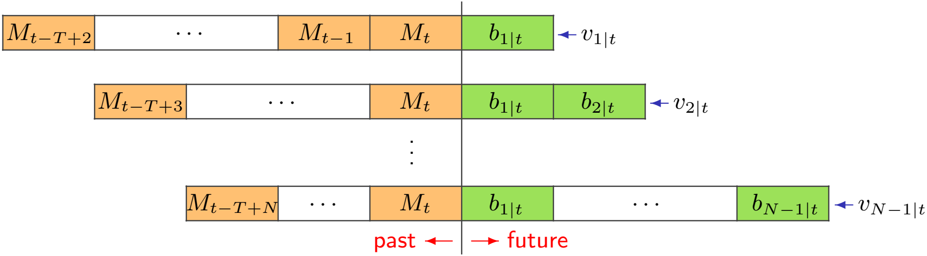
Relaxed constraints: $Fx_{k|t} \le \mathbb{1} + r\, b_{k|t}$, where

r : determines maximum constraint violation

$b_{k|t}$: 0 or 1 depending on $M_{t-T+1}, \dots, M_t$

Enforce the time-average violation limit via constraints, for $k = 1, \dots, N-1$:

$$v_{k|t} \le \epsilon \quad \text{where} \quad v_{k|t} = \frac{1}{T} \sum_{j=t-(T-1)+k}^{t} M_j + \frac{1}{T} \sum_{j=1}^{k} b_{k|t}$$



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hence

$$\begin{aligned} v_{0|t} &= v_{1|t-1} - \tfrac{1}{T} b_{1|t-1} + \tfrac{1}{T} M_t \\ v_{k|t} &= v_{k-1|t} - \tfrac{1}{T} M_{t-T+k} + \tfrac{1}{T} b_{k|t}, \quad \text{for all } k \geq 1 \end{aligned}$$

so if $v_{k-1|t} - \frac{1}{T} M_{t-T+k} + \frac{1}{T} \leq \epsilon$, then $b_{k|t} = 1$ is allowed

otherwise, set $b_{k|t} = 0$

MPC Algorithm

Initialize: $M_{-T} = \cdots = M_{-1} = 0$. At each time $t = 0, 1, \ldots$:

1. determine M_t and $v_{0|t}$ given x_t
2. for each $k = 1, \ldots, N-1$:

   (a). $b_{k|t} := \begin{cases} 1 & \text{if } v_{k-1|t} - \frac{1}{T} M_{t-T+k} + \frac{1}{T} \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}$

   (b). $v_{k|t} := v_{k-1|t} - \frac{1}{T} M_{t-T+k} + \frac{1}{T} b_{k|t}$

3. solve the QP:

$$
\begin{aligned}
(\underline{c}_t^*, \underline{\alpha}_t^*, s_t^*) := \operatorname*{argmin}_{\underline{c}_t, \underline{\alpha}_t, s_t} \quad & \left\| \begin{bmatrix} x_t - s_t \\ \underline{c}_t \end{bmatrix} \right\|_P^2 \\
\text{s.t.} \quad & V x_t \leq \alpha_{0|t}, \;\; W s_t \leq h \\
& \Lambda \alpha_{k|t} \leq \mathbb{1} + b_{k|t}\, r \\
& \Gamma^{(i)} \alpha_{k|t} + V B^{(i)} c_{k|t} + V w^{(i)} \leq \alpha_{k+1|t} \\
& \qquad \text{for } k = 0, \ldots, N-1, \; \alpha_{t+N|t} = g
\end{aligned}
$$

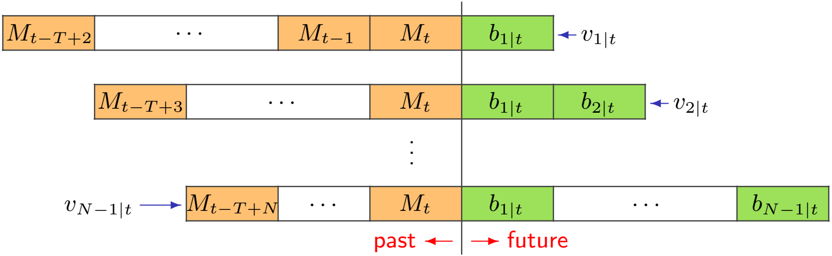
4. $u_t := K x_t + c_{0|t}^*$

Feasibility and constraint satisfaction

Lemma

If step 3 is feasible at time t, then for $k < N-1$ we have

$$v_{k|t+1} \leq v_{k+1|t} \quad \text{and} \quad b_{k+1|t} = 1 \implies b_{k|t+1} = 1$$



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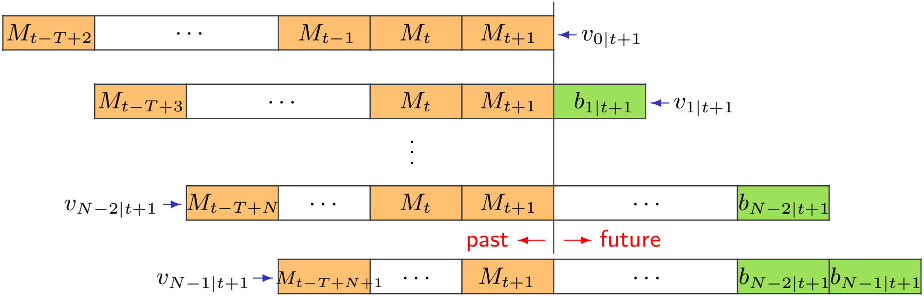

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- ⋆ for all $k < N-1$ we have

$$v_{k|t+1} - v_{k+1|t} = \frac{1}{T}(M_{t+1} - b_{1|t}) + \frac{1}{T}\sum_{j=1}^{k}(b_{j|t+1} - b_{j+1|t})$$

and $M_{t+1} = 1 \implies b_{1|t} = 1$ since step 3 is feasible at t

$\implies v_{k|t+1} = v_{k+1|t}$ in step 2

$M_{t+1} = 0 \implies b_{1|t} = 1$ or 0

$\implies v_{k|t+1} = v_{k+1|t} - \frac{1}{T}$ or $v_{k+1|t}$ in step 2

- ⋆ therefore step 2 cannot tighten constraints at $t+1$ for $k < N-1$

hence step 3 feasible at $t \implies$ step 3 feasible at $t+1$

Feasibility and constraint satisfaction

Theorem

If step 3 is feasible at $t = 0$ then for all $t \geq 0$ we have

$$\frac{1}{T} \sum_{k=t}^{t+T-1} M_k \leq \epsilon$$

- ⋆ Step 3 enforces the constraint $Fx_{1|t} \leq \mathbb{1} + rb_{1|t}$, so that $M_{t+1} \leq b_{1|t}$
- ⋆ Step 2 chooses $b_{1|t}$ so that $v_{1|t} \leq \epsilon$

Hence

$$\frac{1}{T} \sum_{k=t-T-2}^{t+1} M_k \leq \frac{1}{T} \sum_{k=t-T-2}^{t} M_k + \frac{1}{T} b_{1|t} = v_{1|t} \leq \epsilon$$

Closed loop stability

Theorem

Let $\phi_t = (x_t - s_t, \underline{c}_t)$, then there exists $\delta \in (0,1)$ such that, for all $t \geq 0$,

$$\mathbb{E}\big\{\|\phi_t\|_P^2\big\} \leq \delta^{(t/n)-1}\|\phi_0\|_P^2$$

- $\star$ Hence convergence: $x_t \to s_t$ and $c_t \to 0$ in mean square
- $\star$ Result follows from feasibility at time $t+1$ of

$$(c^*_{1|t}, \ldots, c^*_{N-1|t}, 0), \quad (\alpha^*_{1|t}, \ldots, \alpha^*_{N-1|t}, g), \quad \Phi(\omega_t)s^*_t + w(\omega_t)$$

which implies

$$\mathbb{E}\big\{\|\phi_{t+1}\|_P^2\big\} - \|\phi_t\|_P^2 \leq -\|\phi_t\|_{\bar{Q}}^2 \qquad \forall t \geq 0$$

Example

- Model parameters: $$\begin{aligned} A(\omega) &= A_0 + \Delta_A^{(1)}\omega_1 + \Delta_A^{(2)}\omega_2 + \Delta_A^{(3)}\omega_3 \\ B(\omega) &= B_0 + \Delta_B^{(1)}\omega_4 + \Delta_B^{(2)}\omega_5 \\ w(\omega) &= w^{(1)}\omega_6 + w^{(2)}\omega_7 \end{aligned}$$

  where $\omega_1, \ldots, \omega_7$ are independent, uniformly distributed on $[0, 1]$ and

$$A_0 = \begin{bmatrix} -1.9 & -1.4 \\ 0.7 & 0.5 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ -0.25 \end{bmatrix}$$

$$\Delta_A^{(1)} = \begin{bmatrix} 0.01 & 0.05 \\ -0.05 & -0.01 \end{bmatrix}, \quad \Delta_A^{(2)} = \begin{bmatrix} -0.01 & -0.05 \\ 0 & -0.01 \end{bmatrix}, \quad \Delta_A^{(3)} = \begin{bmatrix} 0 & 0 \\ 0.05 & 0.02 \end{bmatrix}$$

$$\Delta_B^{(1)} = \begin{bmatrix} 0.03 \\ -0.02 \end{bmatrix}, \quad \Delta_B^{(2)} = \begin{bmatrix} -0.03 \\ 0.02 \end{bmatrix}, \quad w^{(1)} = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}, \quad w^{(2)} = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

Example

- Cost parameters: $Q = I$, $R = 1$
  $\Omega =$ min. RPI set under $u = Kx$

- Constraint: $[-0.5 \;\; 1]x_t \leq 1$
  relaxed version: $[-0.5 \;\; 1]x_{k|t} \leq 1 + 0.8\, b_{k|t}, \;\; r = 0.8$

- Prediction and constraint horizons: $N = T = 3$

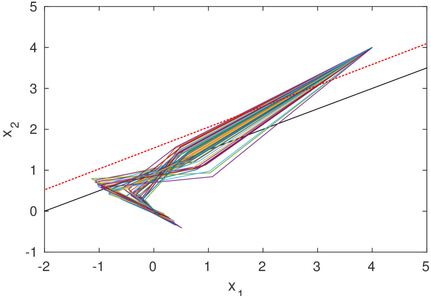
- Time-average constraint: $\displaystyle\sum_{k=t}^{t+2} M_t \leq 1, \;\; \epsilon = 1/3$

- Compare sample-based MPC with chance constraint

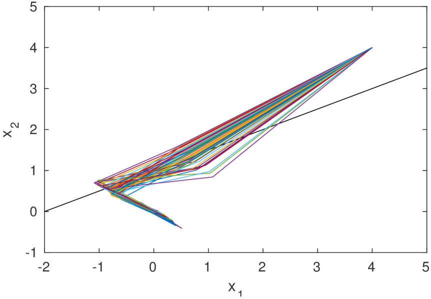
$$\Pr\big\{[-0.5 \; 1]x_{1|t} > 1\big\} \leq 1/3$$

Example

Time-average constraints



Chance constraints



Example

	MPC with chance constraints	MPC with time-average constraints
% constraints satisfied, at $t = 2$	68	66
average solver time (ms)	132	69
closed-loop cost	198.3	195.2

- ⋆ Chance constraints imposed via sampling with (greedy) discarding, with # samples limited by requirement for similar computation
- ⋆ Fair comparison since both methods satisfy ensemble-average constraints

Conclusions

- Time-average constraints:
  - enforced over an arbitrary time-interval
  - satisfied robustly in transients as well as asymptotically
- MPC algorithm:
  - recursively feasible and robustly stable
  - computational requirement similar to robust MPC
- Impose limits on time-average of loss function using similar approach
  (details in proceedings)
- Sequential computation of tightening parameters is suboptimal
  but same approach allows for simultaneous optimization (requires MIQP)

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Thank you for your attention!