Maximising the guaranteed feasible set for uncertain MPC schemes with chance constraints

Problem Setup

We consider the linear time invariant system

$$x^+ = Ax + Bu + w$$

where the state x, control input u and the disturbance w are subject to constraints defined in terms of compact, convex polytopic sets X, U and W:

$$\begin{aligned} X &= \{x \in \mathbb{R}^d : E_i x \leq 1, i \leq M_X\} \\ U &= \{u \in \mathbb{R}^{q_U} : F_i u \leq 1, i \leq M_U\} \\ W &= \{w \in \mathbb{R}^{q_W} : G_i w \leq 1, i \leq M_W\} \end{aligned}$$

The output variable y = Cx + Du + v is subject to the random variable v with

$$v \in V = \{v \in \mathbb{R}^{q_Y} : \Gamma_i v \leq 1, i \in M_V\}$$

which is uniformly distributed over V. The output variable y is constrained to satisfy the chance constraint

$$\mathbb{P}\{v \in V : y = Cx + Du + v \in Y\} \geq p$$

for $0 < p \leq 1$ and $Y = \{y \in \mathbb{R}^{q_Y} : H_i y \leq 1, i \leq M_Y\}$.

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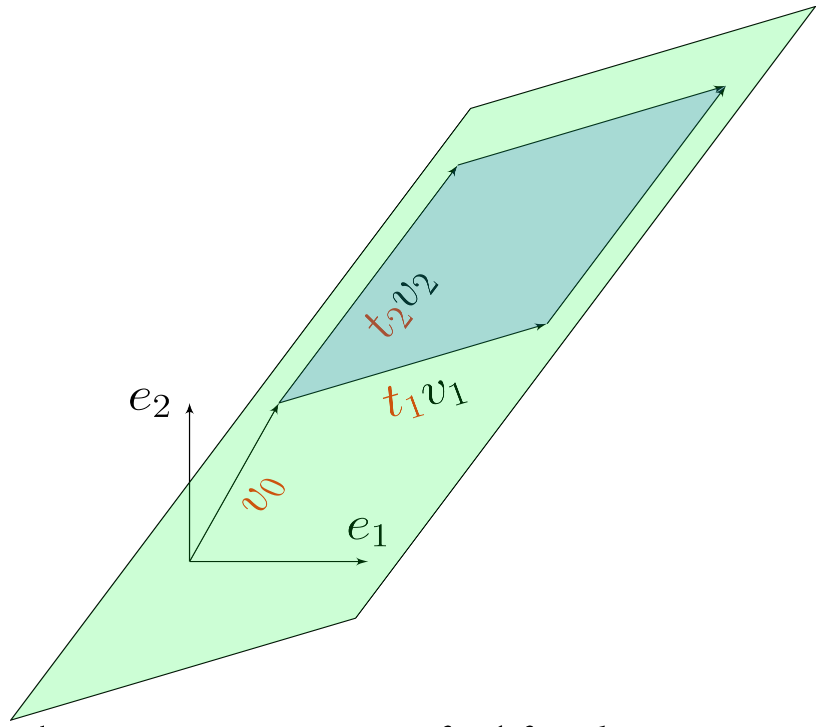
Proposition: Use established methods from Robust Model Predictive Control framework with an admissibly chosen auxiliary set V'.

Key Question: How to choose V' and can it be chosen optimally?

Key Idea: Restrict the combinatorial structure of the auxiliary set V' and optimise to obtain the best choice with respect to the size of the maximal invariant set.

$$\mathscr{X}^\infty / X^\infty = \Big\{ x_0 \in \mathbb{R}^d : \\ x_{k+1} = (A+BK)^{k+1} x_0 + \textstyle\sum_{i=0}^{k} (A+BK)^{k-i} w_i \\ x_{k+1} \in X \\ K x_k \in U \\ \mathbb{P}\{v \in V : (C+DK)x_k + v \in Y\} \geq p \\ (C+DK)x_k + v_k \in Y \\ \forall w_k \in W, \forall v_k \in V', k \geq 0 \Big\}.$$



The parametrisation of V' for the proposed optimisation.

Restricting the Combinatorial Structure of V'

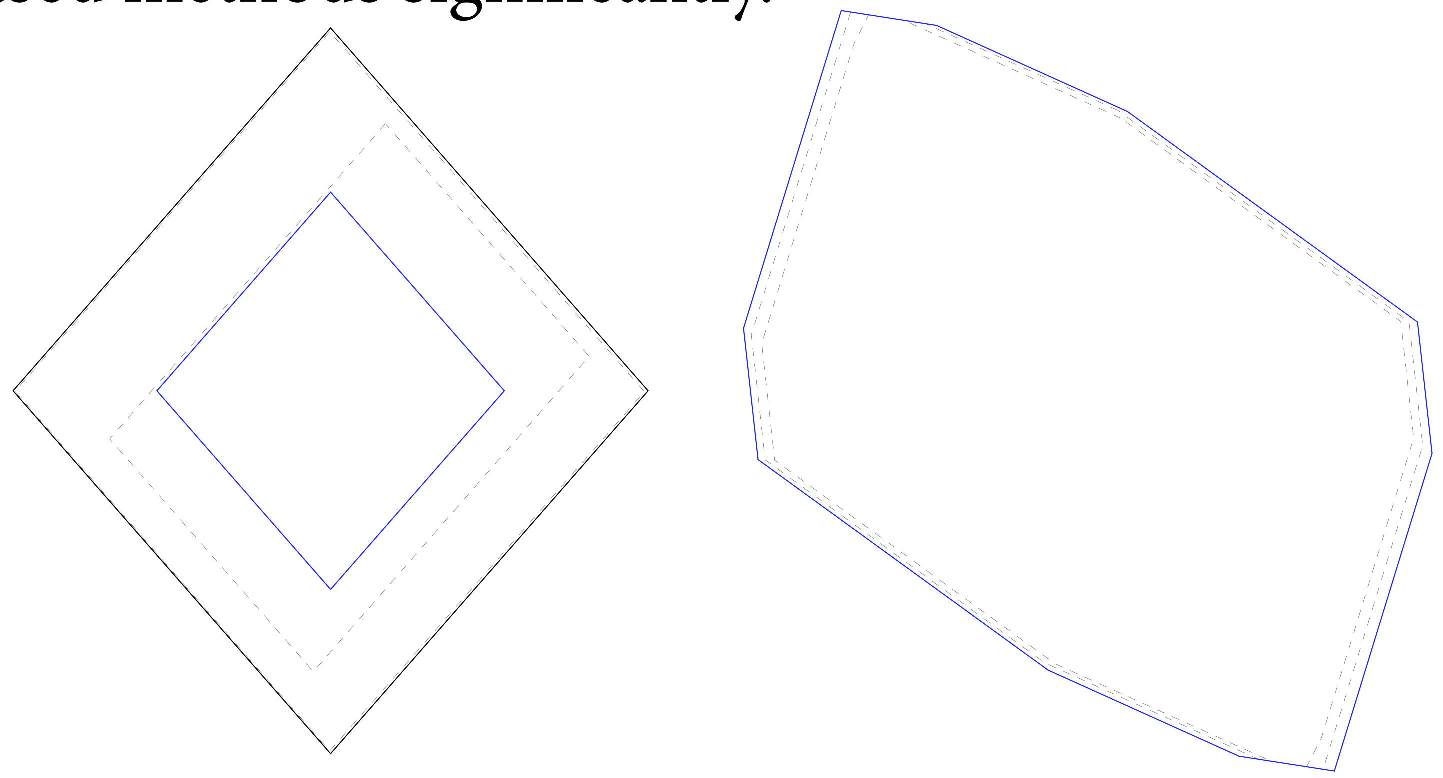
Enforcing V' to be a parallelotope allows several simplifications.

We introduce a minimal number of variables.
The probabilistic measure of V' is proportional to the produce of positive decision variables (convex).

Computational Example

For a comparison with scenario based alternatives we use the method proposed in Zhang et al. with 100 scenarios and compare the best, average and worst performing ones with our proposed method.

The optimised parallelotope outperforms scenario based methods significantly.



Comparison between proposed method blue, and best and worst performing auxiliary sets for scenario approach with 26 samples and 100 scenarios. The best and worst performing scenario correspond to a probability of 0.5573 and 0.9827 respectively - the proposed method achieves the desired probability 0.5000.

Problem formulation

Linear system $\begin{cases} x^+ = Ax + Bu + w \\ y = Cx + Du + v \end{cases}$

with linear state and control constraints $x \in \mathcal{X} = \{x : Ex \leq \underline{1}\}$
$u \in \mathcal{U} = \{u : Fu \leq \underline{1}\}$

unknown state disturbance $w \in \mathcal{W} = \{w : Gw \leq \underline{1}\}$
stochastic output disturbance $v \in \mathcal{V} = \{v : \Gamma v \leq \underline{1}\}$

Chance constraint $\Pr[y \in \mathcal{Y} \geq p]$

with $\mathcal{Y} = \{y : Hy \leq \underline{1}\}$
v uniformly distributed on $\mathcal{V}$
$p \in [0, 1]$

Problem formulation

Proposition

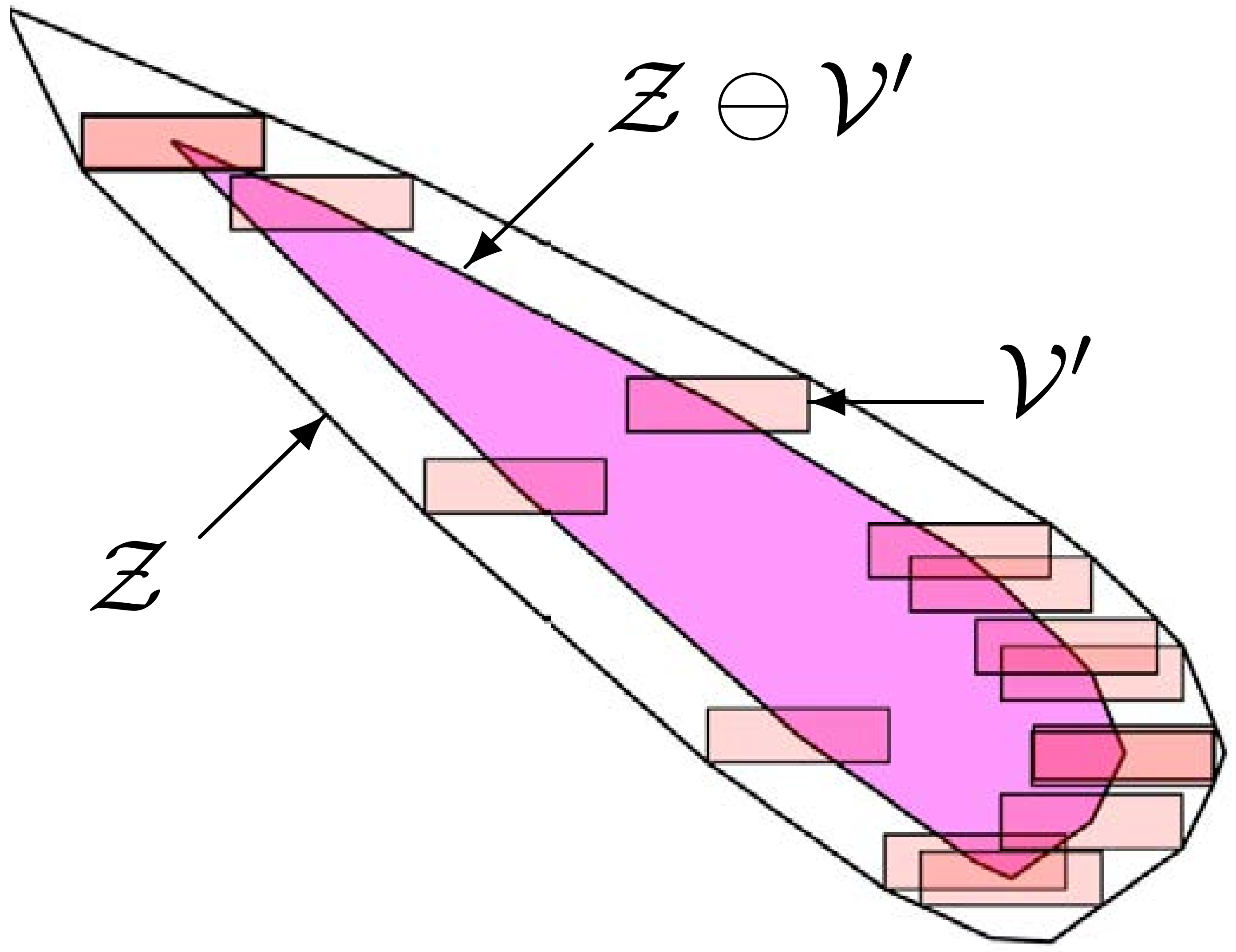
Let v be uniformly distributed on $\mathcal{V}$ and let $\mathcal{V}' \subseteq \mathcal{V}$ be such that $\mathrm{vol}(\mathcal{V}') \geq p$, then every $q \in \mathcal{Z} \ominus \mathcal{V}'$ satisfies

$$\Pr[q + v \in \mathcal{Z}] \geq p$$

Proof:

by definition $\mathcal{Z} \ominus \mathcal{V}' = \{q : q + v \in \mathcal{Z} \ \text{ for all } \ v \in \mathcal{V}'\}$

hence $\Pr[q + v \in \mathcal{Z}] \geq \Pr[v \in V'] = \mathrm{vol}(\mathcal{V}') \geq p$



Problem formulation

Choose $\mathcal{V}'$ to obtain

	the largest n-step controllable set to a given target set	
or	the largest positively invariant set	
subject to	robust constraints	$x \in \mathcal{X},\ u \in \mathcal{U}$
	& chance constraints	$\Pr[y \in \mathcal{Y} \geq p]$

Invariant sets

Sets of initial conditions from which all possible trajectories satisfy robust constraints & chance constraints:

$$\mathscr{X}_\infty = \Big\{ x_0 \in \mathcal{X} : \; x_k = (A+BK)^k x_0 + \sum_{i=0}^{k-1} (A+BK)^i w_{k-1-i}$$

$$x_k \in \mathcal{X}, \quad Kx_k \in \mathcal{U}$$

$$\Pr\big[v \in \mathcal{V} : (C+DK)x_k + v \in \mathcal{Y}\big] \geq p \quad \forall w_k \in \mathcal{W} \;\; \forall k \geq 0 \Big\}$$

$$\boxed{\begin{array}{rcl} \Pr[v \in \mathcal{V}'] \geq p & \Longrightarrow & X_\infty \subseteq \mathscr{X}_\infty \\ \mathrm{vol}(X_\infty) \text{ maximized} & \Longrightarrow & X_\infty \text{ approximates } \mathscr{X}_\infty \end{array}}$$

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$$(C+DK)x_k + v_k \in \mathcal{Y} \quad \forall v_k \in \mathcal{V}' \;\; \forall w_k \in \mathcal{W} \;\; \forall k \geq 0 \Big\}$$

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Approximating the maximal invariant set

Define $\mathcal{R}$ as the set $(C + DK)^{-1}(\mathcal{Y} \ominus \mathcal{V}')$

$\mathscr{D}_k(\mathcal{E})$ as the k-step successor of $\mathcal{E}$, $\mathscr{D}_k(\mathcal{E}) = (A + BK)^k \mathcal{E} \oplus \bigoplus_{i=0}^{k-1} (A + BK)^i \mathcal{W}$

$\mathcal{E}_k$ as the largest set such that $\mathscr{D}_k(\mathcal{E}_k) \subseteq \mathcal{X} \cap K^{-1}\mathcal{U} \cap \mathcal{R}$

Then

$$\mathcal{E}_k = (A + BK)^{-k} \left[(\mathcal{X} \cap K^{-1}\mathcal{U} \cap \mathcal{R}) \ominus \bigoplus_{i=0}^{k-1} (A + BK)^i \mathcal{W} \right]$$

$$X_\infty = \bigcap_{k=0}^{\infty} \mathcal{E}_k$$

Approximating the maximal invariant set

Iteration: $$X_0 = \mathcal{X} \cap K^{-1}\mathcal{U} \cap \mathcal{R}$$

$$X_{k+1} = X_k \cap (A + BK)^{-1}(X_k \ominus \mathcal{W}), \quad k > 0$$

Properties

(i). $X_k = \bigcap_{i=0}^{k} \mathcal{E}_i$

(ii). $X_k = X_{k+1}$ for all $k \geq N$, for some finite N

(iii). $X_\infty = X_N$

Approximating the maximal invariant set

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Properties

(i). $X_k = \bigcap_{i=0}^{k} \mathcal{E}_i$

Proof: $X_0 = \mathcal{E}_0$ by definition, and if $X_k = \bigcap_{i=0}^{k} \mathcal{E}_i$, then

$$\begin{aligned} X_{k+1} &= \bigcap_{i=0}^{k} \mathcal{E}_i \cap (A+BK)^{-1}\Big(\bigcap_{i=0}^{k} \mathcal{E}_i \ominus \mathcal{W}\Big) \\ &= \bigcap_{i=0}^{k} \mathcal{E}_i \cap \bigcap_{i=0}^{k} \Big((A+BK)^{-1}\mathcal{E}_i \ominus \mathcal{W}\Big) \\ &= \bigcap_{i=0}^{k} \mathcal{E}_i \cap \bigcap_{i=1}^{k+1} \mathcal{E}_i = \bigcap_{i=0}^{k+1} \mathcal{E}_i \end{aligned}$$

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Proof: If $\big((A+BK), \Gamma\big)$ is observable, then X_k is compact for $k \geq \dim(x) - 1$

hence if $\rho(A+BK) = \varrho < 1$, then $X_{k+1} = X_k \cap \mathcal{E}_{k+1}$ where

$$\mathcal{E}_{k+1} \supset \mathcal{B}_P\left(\varrho^{-(k+1)}\left(r_1 - \frac{r_2}{1-\varrho}\right)\right)$$

for some $r_1, r_2 > 0$ and $P \succ 0$, where $\mathcal{B}_P = \{x : x^T P x \leq 1\}$

Approximating the maximal invariant set

Iteration:
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(iii). $X_\infty = X_N$

Proof: Follows from (i) and (ii)

Approximating the maximal invariant set

Optimization to approximate $\mathscr{X}_\infty$:

$$\begin{array}{ll} \underset{\mathcal{V}'}{\text{maximize}} & \operatorname{vol}(X_\infty) \\ \text{subject to} & \mathcal{V}' \subseteq \mathcal{V} \\ & \operatorname{vol}(\mathcal{V}') \geq p \operatorname{vol}(\mathcal{V}) \end{array}$$

Solve using e.g. a parallelotope parameterization of $\mathcal{V}'$

n-step controllable sets

Sets of initial conditions from which all trajectories can be driven into a target set subject to robust constraints & chance constraints:

$$
\begin{aligned}
\mathscr{C}_n = \Big\{ x_0 \in \mathcal{X} :\ & \exists\{u_0(\cdot),\ldots,u_{n-1}(\cdot)\} : \\
& x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} B u_i(x_i) + \sum_{i=0}^{k-1} A^{k-1-i} w_i \\
& x_k \in \mathcal{X}, \quad x_n \in \mathcal{T}, \quad u_k(x_k) \in \mathcal{U} \\
& \Pr\big[C x_k + D u_k(x_k) + v \in \mathcal{Y}\big] \geq p \quad \forall w_k \in \mathcal{W} \ \ \forall k = 0,\ldots,n-1 \Big\}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\Pr[v \in \mathcal{V}'] \geq p \quad & \Longrightarrow \quad \mathcal{C}_n \subseteq \mathscr{C}_n \\
\mathrm{vol}(\mathcal{C}_n) \text{ maximized} \quad & \Longrightarrow \quad \mathcal{C}_n \text{ approximates } \mathscr{C}_n
\end{aligned}
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& x_k \in \mathcal{X}, \quad x_n \in \mathcal{T}, \quad u_k(x_k) \in \mathcal{U} \\
& C x_k + D u_k(x_k) + v_k \in \mathcal{Y} \quad \forall v_k \in \mathcal{V}' \ \ \forall w_k \in \mathcal{W} \ \ \forall k = 0, \dots, n-1 \Big\}
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\end{array}
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Approximating the maximal n-step controllable set

Optimization to approximate $\mathscr{C}_n$:

$$\begin{aligned} \underset{\mathcal{V}'}{\text{maximize}} \quad & \text{vol}(\mathcal{C}_n) \\ \text{subject to} \quad & \mathcal{V}' \subseteq \mathcal{V} \\ & \text{vol}(\mathcal{V}') \geq p\, \text{vol}(\mathcal{V}) \end{aligned}$$

Solve using e.g. a parallelotope parameterization of $\mathcal{V}'$

Improve approximation by introducing optimization variables $\mathcal{V}'_0, \ldots, \mathcal{V}'_{n-1}$

Parameterisation using parallelotopes

Define a parallelotope, for given $\{v_1, \dots, v_d\}$ spanning $\mathbb{R}^d$, as

$$\mathcal{Z}(v_1, \dots, v_d) = \left\{ \sum_{i=1}^{d} t_i v_i \ : \ t_i \in [0,1] \right\}$$

then $\mathcal{Z}(t_1 v_1, \dots, t_d v_d) \oplus \{v_0\}$ has volume

$$\mathrm{vol}\big(\mathcal{Z}(t_1 v_1, \dots, t_d v_d) \oplus \{v_0\}\big) = |\det[v_1 \ \cdots \ v_d]| \prod_{i=1}^{d} t_i$$

hence optimize $\mathcal{V}'$ over $2d$ decision variables $(v_0, t_1, \dots, t_d)$ with

$$\begin{aligned} \mathcal{V}' &= \mathcal{Z}(t_1 v_1, \dots, t_d v_d) \oplus \{v_0\} \\ &= \operatorname*{conv}_{i=1,\dots,2^d} \big\{ [t_1 v_1 \ \cdots \ t_d v_d] \lambda_i + v_0 \big\} \end{aligned}$$

where $\{\lambda_1, \dots, \lambda_{2^d}\} = \mathrm{vert}\big([0,1]^d\big)$

Parameterisation using parallelotopes

Optimization to approximate $\mathscr{X}_\infty$:

$$
\begin{aligned}
\underset{v_0, t_1, \ldots, t_d}{\text{maximize}} \quad & \operatorname{vol}(X_\infty) \\
\text{subject to} \quad & \Gamma\Big([t_1 v_1 \ \cdots \ t_d v_d] \lambda_i + v_0 \Big) \leq \underline{1} \quad && i = 1, \ldots, 2d \\
& \prod_{i=1}^{d} t_i \geq p \, \frac{\operatorname{vol}(\mathcal{V})}{|\det[v_1 \ \cdots \ v_d]|} \\
& t_j \in [0, 1] && j = 1, \ldots, d
\end{aligned}
$$

Constraints are convex and ensure $\mathcal{V}' \subseteq \mathcal{V}$ and $\operatorname{vol}(\mathcal{V}')/\operatorname{vol}(\mathcal{V}) \leq p$

but maximizer is non-unique since objective is non-convex

Parameterisation using parallelotopes

Optimization to approximate $\mathscr{C}_n$:

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\text{subject to} \quad & \Gamma\Big([t_1 v_1 \ \cdots \ t_d v_d] \lambda_i + v_0 \Big) \leq \underline{1} \qquad i = 1, \ldots, 2d \\
& \prod_{i=1}^{d} t_i \geq p \, \frac{\mathrm{vol}(\mathcal{V})}{|\det[v_1 \ \cdots \ v_d]|} \\
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Alternative parameterisations

Computation of $\mathrm{vol}(\mathcal{V}')$ is simplified if $\mathcal{V}'$ has a fixed combinatorial structure

since if $\mathcal{V}'$ is combinatorially equivalent to $\mathcal{V}^0$ (i.e. induced graphs $\mathcal{G}(\mathcal{V}') = \mathcal{G}(\mathcal{V}^0)$) then

$$\begin{array}{l} \mathcal{V}', \mathcal{V}^0 \text{ have the same simplex decomposition} \\ \text{and } \mathcal{V}' = \bigcup_i \mathcal{S}_i \text{ implies } \mathrm{vol}(\mathcal{V}') = \sum_i \mathrm{vol}(\mathcal{S}_i) \end{array}$$

Fix the combinatorial structure of $\mathcal{V}'$, for example:

(a). by defining $\mathcal{V}'$ as a projective transformation of some fixed $\mathcal{V}^0$, and optimizing the (rotation & projection) parameters defining this transformation

or

(b). by imposing explicit constraints on $\mathrm{vert}(\mathcal{V}')$

Computational tools for general polytopic sets

Conversion between half-space and vertex representations performed using the LRS Library http://cgm.cs.mcgill.ca/∼avis/C/lrs.html

lrs: A Revised Implementation of the Reverse Search Vertex Enumeration Algorithm

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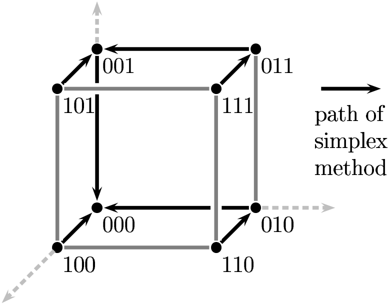
January 26, 1999

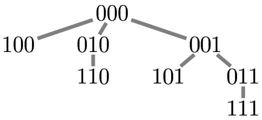
[DMV Seminar]

Computational tools for general polytopic sets

Reverse search vertex enumeration algorithm



(a) The “simplex tree” induced by the objective $(-\sum x_i)$.



(b) The corresponding reverse search tree.

Matlab interface by R.M. Schaich http://worc4021.github.io