測度0集合の和集合に関するGoldsternの原理

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測度 0 集合の和集合

(Y, μ) をポーランド確率空間とする。測度の可算加法性より、測度 0 集合の可算個の和集合は再び測度 0 である。

Q. How about the union of continuum many measure 0 sets?
A. It is not necessary a measure 0 set (e.g. singletons for all points).

Q. What if we add assumptions to the given measure 0 sets?
A. If we do it in a sense, then the union of continuum many measure 0 sets is also a measure 0 set!

Goldstern の定理

(full domination order) For $x, x' \in \omega^\omega$, define a relation $x \leq x'$ by $(\forall n \in \omega)(x(n) \leq x'(n))$.

In 1993, Martin Goldstern proved the following theorem.

Goldstern's theorem $(\mathrm{ZF} + \mathrm{DC})$

Let (Y, μ) be a Polish probability space. Let $A \subseteq \omega^\omega \times Y$ be a $\mathbf{\Sigma}^1_1$ set. Assume that for each $x \in \omega^\omega$,

$$A_x := \{y \in Y : (x, y) \in A\}$$

has measure 0. Also, assume $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ has also measure 0.

原理GP(Γ)

定義

Let Γ be a pointclass. Then GP(Γ) means the following statement: Let (Y, μ) be a Polish probability space and $A \subseteq \omega^\omega \times Y$ be in Γ. Assume that for each $x \in \omega^\omega$, A_x has μ-measure 0. Also suppose that $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ has also μ-measure 0.

Goldstern's theorem says that GP($\boldsymbol{\Sigma}^1_1$) holds.
Note that if Γ is a sufficiently high pointclass (that is if $\boldsymbol{\Delta}^1_1 \subseteq \Gamma$), then we can assume that (Y, μ) is the Cantor space with the standard measure.

主結果

The symbol “all” denotes the class of all subsets of Polish spaces.

定理

GP(all) is independent from ZFC.

¬ GP(all) の無矛盾性

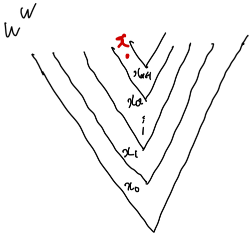
定理

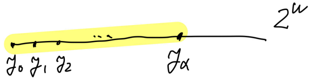
Assume CH. Then ¬ GP(all) holds.

Proof. Let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a cofinal increasing sequence in $(\omega^\omega, <^*)$. And let $\langle y_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of 2^ω. Then the set A defined by the following equation witnesses ¬ GP(all):

$$A_x = \{y_\beta : \beta < \alpha_x\},$$

where $\alpha_x = \min\{\alpha : x <^* x_\alpha\}$. □





$\neg$ GP(all) の無矛盾性

Refining the last proof, we get the following theorem.

定理

Assume that at least one of the following three conditions holds:

$$\mathrm{add}(\mathcal{N}) = \mathfrak{b},\ \mathrm{non}(\mathcal{N}) = \mathfrak{b} \text{ or } \mathrm{non}(\mathcal{N}) = \mathfrak{d}.$$

Then $\neg$ GP(all) holds.

$\mathrm{add}(\mathcal{N}) := \min\{\kappa : \text{the null ideal is not } \kappa\text{-additive}\}$

$\mathrm{non}(\mathcal{N}) := \min\{|A| : A \subseteq 2^\omega, A \text{ does not have measure } 0\}$

$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega, \neg(\exists g \in \omega^\omega)(\forall f \in F)\ f <^* g\}$

$\mathfrak{d} := \min\{|F| : F \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in F)\ g <^* f\}$

$\neg\,\mathrm{GP}(\mathrm{all})$ の無矛盾性

Assume that at least one of the following three conditions holds: $\mathrm{add}(\mathcal{N}) = \mathfrak{b}$, $\mathrm{non}(\mathcal{N}) = \mathfrak{b}$ or $\mathrm{non}(\mathcal{N}) = \mathfrak{d}$. Then $\neg\,\mathrm{GP}(\mathrm{all})$ holds.

$\mathrm{add}(\mathcal{N}) := \min\{\kappa : \text{the null ideal is not } \kappa\text{-additive}\}$

$\mathrm{non}(\mathcal{N}) := \min\{|A| : A \subseteq 2^\omega, A \text{ does not have measure } 0\}$

$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega, \neg(\exists g \in \omega^\omega)(\forall f \in F)\ f <^* g\}$

$\mathfrak{d} := \min\{|F| : F \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in F)\ g <^* f\}$

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

$V = L$ は $\neg\,\mathrm{GP}(\Delta^1_2)$ を含意する

Refining the last proof in another way again, we get the following theorem.

定理

$V = L$ implies $\neg\,\mathrm{GP}(\Delta^1_2)$.

GP(all)の無矛盾性

定理

If ZFC is consistent then so is $\mathrm{ZFC}+\mathsf{GP}(\mathsf{all})$.

In fact, “The Laver model” satisfies GP(all).

equal to $\aleph_2$ in the Laver model

$$
\begin{tikzcd}
 & \mathsf{cov}(\mathcal{N}) \arrow[r] & \mathsf{non}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathsf{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathsf{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathsf{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathsf{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

equal to $\aleph_1$ in the Laver model

決定性と Solovay モデル

定理

Assume $\mathrm{ZF} + \mathrm{AD}$. Then GP(all) holds.

定理

Solovay モデルにおいて GP(all) が成り立つ.

まとめ

- Goldstern showed $\mathsf{GP}(\boldsymbol{\Sigma}^1_1)$.
- We proved the following 3 theorems:
  1. $\mathsf{GP}(\mathrm{all})$ is independent from ZFC.
  2. $V = L$ implies $\neg\,\mathsf{GP}(\Delta^1_2)$.
  3. Under $\mathrm{ZF} + \mathrm{AD}$, $\mathsf{GP}(\mathrm{all})$ holds.

未解決問題

1. $V = L$ は $\neg\,\mathsf{GP}(\Pi^1_1)$ を導くか？
2. $\mathrm{ZFC} + (\mathfrak{c} > \aleph_2) + \mathsf{GP}(\mathsf{all})$ は無矛盾か？
3. (到達不能基数を仮定して)ZF のモデルで実数の任意の集合が Lebesgue 可測だが、$\mathsf{GP}(\mathsf{all})$ を満たさないものはあるか？
4. ある $n \geq 2$ (または全ての $n \geq 2$) について、$\mathsf{GP}(\Sigma^1_{n+1})$ と $\mathsf{GP}(\Sigma^1_n)$ を分離することは、巨大基数を使わずにできるか？

参考文献と謝辞

[Gol93] Martin Goldstern. “An Application of Shoenfield’s Absoluteness Theorem to the Theory of Uniform Distribution.”. In: *Monatshefte für Mathematik* 116.3-4 (1993), pp. 237–244.

この研究のプレプリント：arXiv:2206.08147

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