Approximation guarantees of local search algorithms via localizability of set functions

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Set functions

A function that assigns a value to every subset

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$f(\{$🐬, 🐍, 🐘$\}) = 50$

Set functions

A function that assigns a value to every subset

$$f(\{\text{🐄}, \text{🦒}\}) = 20$$

Set function maximization

Finding a subset maximizing the objective value

Maximize $f(X)$

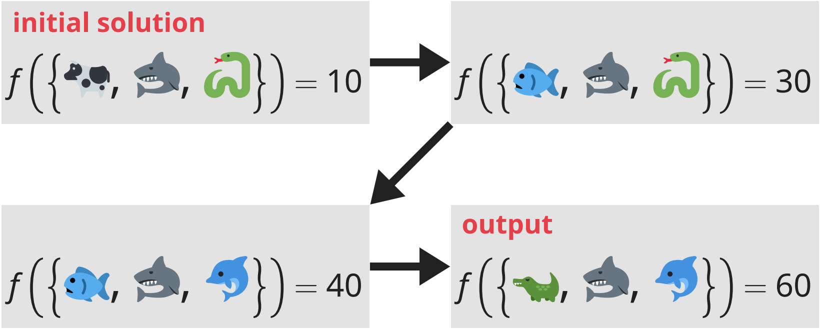
subject to $X \in \mathcal{I}$

$f : 2^N \rightarrow \mathbb{R}$ set function

N is the ground set

Local search

Algorithmic framework for set function max.



Q **When do local search algorithms work well?**

Our contribution

We propose a property called *localizability*

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2. The objective function of sparse optimization satisfies localizability
3. Local search for sparse optimization can be accelerated

Warmup: Cardinality constraints

Finding a subset of size at most s

$$\begin{aligned} &\text{Maximize} \quad f(X) \\ &\text{subject to} \quad |X| \leq s \end{aligned}$$

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Assume f is monotone

i.e. $A \subseteq B \Rightarrow f(A) \leq f(B)$

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An algorithm is α-approximation ($\alpha \in [0, 1]$)

$\overset{\triangle}{\Longleftrightarrow} \quad f(X) \geq \alpha f(X^*),$

where X is the output and X^* is an optimal

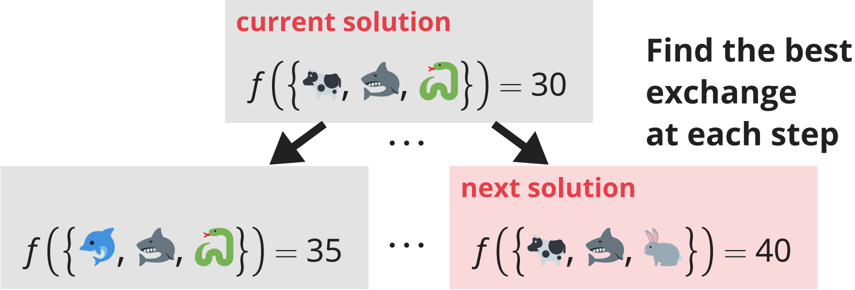
Local search: Cardinality constraints

Let X be any maximal feasible solution

For $i = 1, \cdots, T$:

  For each $a \in N \setminus X,\ b \in X$, compute $f(X \setminus \{b\} \cup \{a\})$

  Update X with the best exchange



Localizability (simple version)

current solution X

$f(\{$🐄, 🦈, 🐍$\}) = 10$

optimal solution X^*

$f(\{$🐊, 🐖, 🐬$\}) = 70$

decompose the difference

remove 🐄 add 🐊	remove 🦈 add 🐖	remove 🐍 add 🐬

f is (α, β)-localizable

$\overset{\triangle}{\Longleftrightarrow}$ **sum of local improvement** $\geq \alpha f(X^*) - \beta f(X)$

Approx. guarantee: Cardinality constr.

Theorem

If the objective function is (α, β)-localizable,

our local search algo. is $\frac{\alpha}{\beta}\left(1 - \exp\left(-\frac{\beta T}{s}\right)\right)$-approx.

T: #iterations, s: maximum solution size

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Guarantees for well-known classes

	(α, β)	approx. guarantee
linear	$(1, 1)$	$(1 - \exp(T/s))$
submodular	$(1, 2)$	$\frac{1}{2}(1 - \exp(T/s))$

General settings: Structured constraints

Maximize $f(X)$

subject to $X \in \mathcal{I}$

$\mathcal{I} \subseteq 2^N$ represents a family of feasible subsets

Various classes of constraints

p-exchange systems $\not\subseteq$ $\not\supseteq$ *p*-matroid intersection

$\cup\!\!\!\mid$

matroid

$\cup\!\!\!\mid$

cardinality $\mathcal{I} = \{X \subseteq N \mid |X| \leq s\}$

Matroids

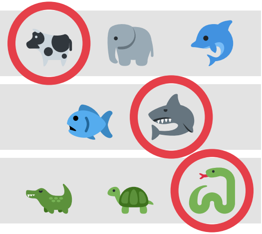
N finite set, $\mathcal{I} \subseteq 2^N$ s.t. $\emptyset \in \mathcal{I}$ and $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$

$\mathcal{M} = (N, \mathcal{I})$ is a **matroid**

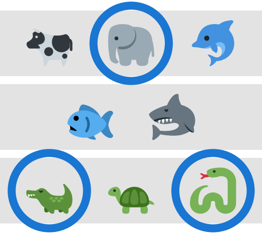
$\Updownarrow$ **def**

$\forall A, B \in \mathcal{I},\ |A| < |B| \Rightarrow \exists i \in B \setminus A,\ A \cup \{i\} \in \mathcal{I}$

e.g.) partition matroid



independent



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$(N, \mathcal{I})$ is a ***p*-matroid intersection**

$\Updownarrow$ **def**

$\exists (N, \mathcal{I}_1), \cdots, (N, \mathcal{I}_p)$ matroids s.t. $\mathcal{I} = \bigcap_{i=1}^{p} \mathcal{I}_p$

p-Exchange systems [Feldman–Naor–Schwartz–Ward'11]

N finite set, $\mathcal{I} \subseteq 2^N$ s.t. $\emptyset \in \mathcal{I}$ and $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$

$(N, \mathcal{I})$ is a **p-exchange system**

$\Updownarrow$ **def**

$\exists \phi : A \setminus B \to 2^{B \setminus A}$ s.t.

- $|\phi(v)| \leq p \ (\forall v \in A \setminus B)$
- Each $v' \in B \setminus A$ appears at most p sets of $(\phi(v))_{v \in A \setminus B}$
- For all $B' \subseteq B \setminus A$, $(A \setminus (\bigcup_{v \in B'} \phi(v))) \cup B' \in \mathcal{I}$

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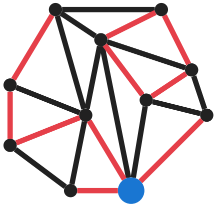
N finite set, $\mathcal{I} \subseteq 2^N$ s.t. $\emptyset \in \mathcal{I}$ and $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$

$(N, \mathcal{I})$ is a ***p*-exchange system**

e.g.) b-matching

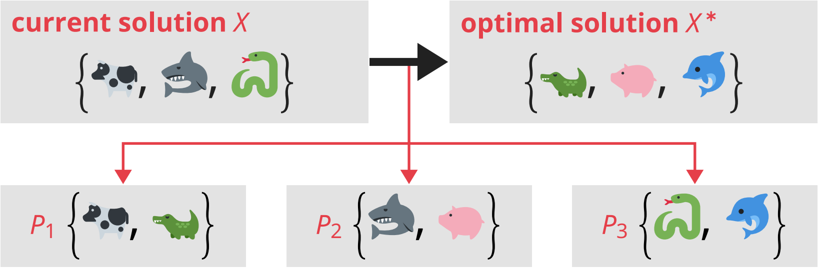
2-matching



not 2-matching

$\mathcal{I} = \{X \subseteq E \mid \forall v \in V,\ \deg_v(X) \leq b\}$ is a 2-exchange sys.

Localizability (general version)



$\mathcal{P}$ a multiset of 2^N s.t. $\begin{cases} \text{each of } X^* \setminus X \text{ appears } k \text{ times} \\ \text{each of } X \setminus X^* \text{ appears } \ell \text{ times} \end{cases}$

f is $(\alpha, \beta_1, \beta_2)$-localizable

$$\overset{\Delta}{\Longleftrightarrow} \quad \sum_{P \in \mathcal{P}} \{f(X \triangle P) - f(X)\} \;\geq\; \alpha k\, f(X^*) - (\beta_1 \ell + \beta_2 k)\, f(X)$$

Approximation guarantees

Theorem

If the objective function is $(\alpha, \beta_1, \beta_2)$-localizable

Matroids the local search algorithm is

$$\frac{\alpha}{\beta_1 + \beta_2}\left(1 - \exp\left(-\frac{(\beta_1 + \beta_2)T}{s}\right)\right)\text{-approximation}$$

p-MI/p-ES the local search algo. with paramter

$q \in \mathbb{Z}_{>0}$ is $\dfrac{\alpha\left(1 - \exp\left(\frac{(\beta_1(p-1+1/q)+\beta_2)T}{s}\right)\right)}{\beta_1(p - 1 + 1/q) + \beta_2}$-approx.

q is a parameter of the algorithms you can choose

(you must check $n^{O(q)}$ solutions at each step)

Sparse optimization

Finding a sparse solution for continuous opt.

$$\begin{aligned} \text{Maximize}_{\mathbf{w}\in\mathbb{R}^n} \quad & u(\mathbf{w}) \\ \text{subject to} \quad & \text{supp}(\mathbf{w}) \in \mathcal{I} \end{aligned}$$

Notation

- $u : \mathbb{R}^n \to \mathbb{R}$ continuous function with $u(\mathbf{0}) \geq 0$
- $N \triangleq \{1, \cdots, n\}$
- $\text{supp}(\mathbf{w}) \triangleq \{i \in N \mid \mathbf{w}_i \neq 0\}$ indices of non-zeros

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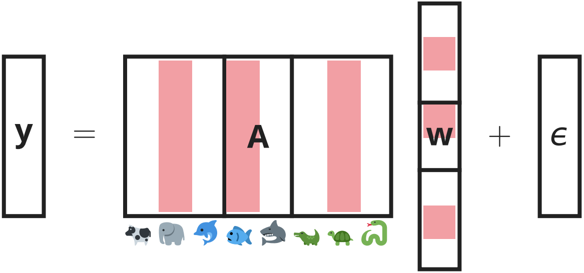
Feature selection with matroid constraints

Selecting one feature from each category



$$\text{Maximize} \quad u_{R^2}(\mathbf{w}) \triangleq 1 - \frac{\|\mathbf{y} - \mathbf{A}\mathbf{w}\|_2^2}{\|\mathbf{y}\|_2^2}$$

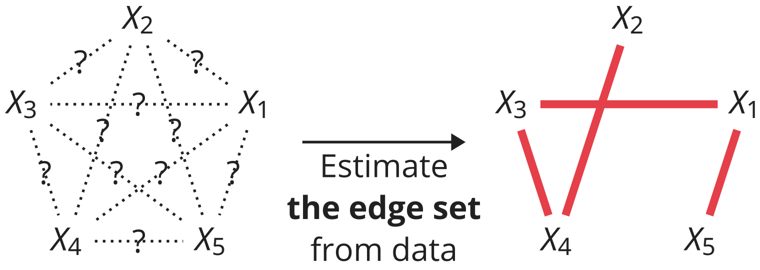
$$\text{subject to} \quad \mathrm{supp}(\mathbf{w}) \in \mathcal{I}$$

Structure learning of graphical models

Estimate supp($\mathbf{w}$) **of a sparse Ising model**

$$p(\mathbf{x}|\mathbf{w}) \propto \exp\left(\sum\nolimits_{(u,v)\in E} w_{uv}x_u x_v + \sum\nolimits_{u\in V} w_u x_u\right)$$

from data $\{\mathbf{x}^1, \cdots, \mathbf{x}^m\} \sim p(\mathbf{x}|\mathbf{w})$



A degree constraint is a special case of a 2-ES constr.

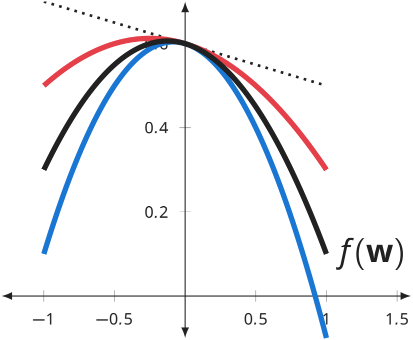
Assumptions [Negahban+'12, Elenberg+'18]

u is restricted strongly concave

$$\Leftrightarrow \forall \mathbf{x}, \mathbf{y} \in \Omega,\ u(\mathbf{y}) - u(\mathbf{x}) - \langle \nabla u(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \leq -\frac{m_\Omega}{2} \|\mathbf{y} - \mathbf{x}\|_2^2$$

u is restricted smooth

$$\Leftrightarrow \forall \mathbf{x}, \mathbf{y} \in \Omega,\ u(\mathbf{y}) - u(\mathbf{x}) - \langle \nabla u(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq -\frac{M_\Omega}{2} \|\mathbf{y} - \mathbf{x}\|_2^2$$



Restricted strong concavity constant m_s on $\Omega_s = \{\mathbf{x}, \mathbf{y} \mid \|\mathbf{x} - \mathbf{y}\|_0 \leq s\}$

Restricted smoothness constant $M_{s,t}$ on $\Omega_{s,t} = \{\mathbf{x}, \mathbf{y} \mid \|\mathbf{x}\|_0 \leq s,\ \|\mathbf{y}\|_0 \leq s, \|\mathbf{x} - \mathbf{y}\|_0 \leq t\}$

Sparse optimization $\rightarrow$ set function opt.

We can reduce sparse opt. to set function opt.

$$\begin{aligned} \text{Maximize}_{\mathbf{w}\in\mathbb{R}^n} \quad & u(\mathbf{w}) \\ \text{subject to} \quad & \mathrm{supp}(\mathbf{w}) \in \mathcal{I} \end{aligned}$$

$\downarrow$ Define $f_u(X) \overset{\triangle}{=} \max_{\mathbf{w}\colon \mathrm{supp}(\mathbf{w})\subseteq X} u(\mathbf{w})$

$$\begin{aligned} \text{Maximize}_{X\subseteq N} \quad & f_u(X) \\ \text{subject to} \quad & X \in \mathcal{I} \end{aligned}$$

f_u is $\left(\frac{m_{2s}}{M_{s,t}}, \frac{M_{s,t}}{m_{2s}}, 0\right)$-localizable with size s and exchange size t

Approximation guarantees for sparse opt.

Constraint	Local search	Greedy
Cardinality	$\frac{m_{2s}^2}{M_{s,2}^2}(1-\epsilon_1(T))$	$1-\exp\left(-\frac{m_{2s}}{M_{s,1}}\right)$ †
Matroids	$\frac{m_{2s}^2}{M_{s,2}^2}(1-\epsilon_1(T))$	$\frac{1}{(1+\frac{M_{s,1}}{m_s})^2}$ ‡
p-MI/p-ES	$\frac{1}{p-1+1/q}\frac{m_{2s}^2}{M_{s,2}^2}(1-\epsilon_2(T))$	N/A

$\epsilon_1(T)$ and $\epsilon_2(T)$ are terms converging to 0 as $T \to \infty$

† [Elenberg–Khanna–Dimakis–Negahban'18]

‡ [Chen–Feldman–Karbasi'18]

Acceleration

It takes much time to decide $\mathop{\mathrm{argmax}}_{a \in N \setminus X, b \in X} f_u(X \setminus \{a\} \cup \{b\})$

→ **Approximately evaluate the value of f_u**

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$\mathbf{w}^{(X)} \in \operatorname{argmax}_{\mathbf{w}:\, \mathrm{supp}(\mathbf{w}) \subseteq X} u(\mathbf{w})$

semi-oblivious

$\operatorname{argmax}_{a \in N \setminus X} f_u(X \setminus \{b\} \cup \{a\})$, where $b \in \operatorname{argmin}_{b \in X} (\mathbf{w}^{(X)})_b^2$

Quickly decides the element to be removed

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$\operatorname*{argmax}_{a \in N \setminus X} f_u(X \setminus \{b\} \cup \{a\})$, where $b \in \operatorname*{argmin}_{b \in X} (\mathbf{w}^{(X)})_b^2$

Quickly decides the element to be removed

non-oblivious

$$\operatorname*{argmax}_{a \in N \setminus X, b \in X} \left\{ \frac{1}{2M_{s,2}} (\nabla u(\mathbf{w}^{(X)}))_a^2 - \frac{M_{s,2}}{2} (\mathbf{w}^{(X)})_b^2 \right\}$$

Our contribution

We propose a property called *localizability*

1. If the objective function satisfies localizability, local search is guaranteed to work well
2. The objective function of sparse optimization satisfies localizability
3. Local search for sparse optimization can be accelerated