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問題 1

求不定積分

$$\int \frac{x+5}{x^2-6x+13}\,\mathrm{d}x$$

解.

$$\begin{aligned}
\text{与式} &= \int \frac{x+5}{x^2+2\cdot 3x+3^2+4}\,\mathrm{d}x \\
&= \int \frac{x-3+8}{(x-3)^2+4}\,\mathrm{d}(x-3) \\
&= \int \frac{x-3}{(x-3)^2+4}\,\mathrm{d}(x-3) + 8\int \frac{1}{(x-3)^2+4}\,\mathrm{d}(x-3) \\
&= \frac{1}{2}\int \frac{1}{(x-3)^2+4}\,\mathrm{d}[(x-3)^2+4] + 8\int \frac{1}{(x-3)^2+2^2}\,\mathrm{d}(x-3) \\
&= \frac{1}{2}\log\left|x^2-6x+13\right| + 8\cdot\frac{1}{2}\arctan\frac{x-3}{2} + C \\
&= \frac{1}{2}\log\left(x^2-6x+13\right) + 4\arctan\frac{x-3}{2} + C
\end{aligned}$$

□

問題 2

求不定積分

$$\int \frac{\mathrm{d}x}{x(x-1)^2}$$

解. 設

$$\text{与式} = \int \frac{A}{x} + \frac{Bx+C}{x^2-2x+1}\,\mathrm{d}x$$

則

$$\begin{cases} A+B=0 \\ -2A+C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=2 \end{cases}$$

即

$$
\begin{aligned}
\text{与式} &= \int \frac{1}{x} + \frac{-x+2}{x^2-2x+1}\,\mathrm{d}x \\
&= \int \frac{1}{x}\,\mathrm{d}x - \int \frac{x-1-1}{(x-1)^2}\,\mathrm{d}(x-1) \\
&= \int \frac{1}{x}\,\mathrm{d}x - \int \frac{1}{x-1}\,\mathrm{d}(x-1) + \int \frac{1}{(x-1)^2}\,\mathrm{d}(x-1) \\
&= \log|x| - \log|x-1| - \frac{1}{x-1} + C \\
&= \log\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C
\end{aligned}
$$

□

問題 3

求不定積分

$$
\int \frac{\cos 2x - \sin 2x}{\cos x + \sin x}\,\mathrm{d}x
$$

解.

$$
\begin{aligned}
\text{与式} &= \int \frac{\cos^2 x - \sin^2 x - 2\sin x\cos x}{\cos x + \sin x}\,\mathrm{d}x \\
&= \int \frac{(\cos x + \sin x)(\cos x - \sin x) - 2\sin x\cos x}{\cos x + \sin x}\,\mathrm{d}x \\
&= \int \cos x - \sin x\,\mathrm{d}x - \int \frac{2\sin x\cos x}{\cos x + \sin x}\,\mathrm{d}x \\
&= \sin x + \cos x - \int \frac{1 + 2\sin x\cos x - 1}{\cos x + \sin x}\,\mathrm{d}x \\
&= \sin x + \cos x - \int \frac{(\cos x + \sin x)^2 - 1}{\cos x + \sin x}\,\mathrm{d}x \\
&= \sin x + \cos x - \int \cos x + \sin x - \frac{1}{\cos x + \sin x}\,\mathrm{d}x \\
&= \sin x + \cos x - \sin x + \cos x - \int \frac{1}{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)}\,\mathrm{d}x \\
&= 2\cos x - \frac{\sqrt{2}}{2}\int \csc\left(x + \frac{\pi}{4}\right)\mathrm{d}\left(x + \frac{\pi}{4}\right) \\
&= 2\cos x - \frac{\sqrt{2}}{2}\log\left|\csc(x + \frac{\pi}{4}) - \cot(x + \frac{\pi}{4})\right| + C
\end{aligned}
$$

□