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問題 1

求極限

$$\lim_{x \to 0} \frac{\sqrt{1-x^2} - \cos 3x}{e^x - x - 1}$$

解.

$$\begin{aligned}
\text{与式} &= \lim_{x \to 0} \frac{\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x) + 3\sin 3x}{e^x - 1} \\
&= \lim_{x \to 0} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-\frac{3}{2}}(-2x)^2 + (-2)\left[\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\right] + 9\cos 3x}{e^x} \\
&= 8 \qquad\qquad \square
\end{aligned}$$

問題 2

求極限

$$\lim_{x \to 0} \frac{e^{\tan x} - e^x}{x - \sin x}$$

解. Maclaurin Expansion: $e^{\tan x} = 1 + \tan x + o(\tan x)$, $e^x = 1 + x + o(x)$.

$$\begin{aligned}
\text{与式} &= \lim_{x \to 0} \frac{1 + \tan x + o(\tan x) - 1 - x - o(x)}{x - \sin x} \\
&= \lim_{x \to 0} \frac{\tan x - x + o(x)}{x - \sin x} \\
&= \lim_{x \to 0} \frac{\tan x - x}{x - \sin x} \\
&= \lim_{x \to 0} \frac{1 + \tan^2 x - 1}{1 - \cos x} \\
&= \lim_{x \to 0} \frac{2\tan x \sec^2 x}{\sin x} \\
&= \lim_{x \to 0} \frac{2}{\cos x} \\
&= 2 \qquad\qquad \square
\end{aligned}$$

問題 3

求不定積分

$$\int \frac{x+5}{x^2 - 6x + 13}\,\mathrm{d}x$$

解.

$$\begin{aligned}
\text{与式} &= \int \frac{x+5}{x^2 + 2 \cdot 3x + 3^2 + 4}\,\mathrm{d}x \\
&= \int \frac{x - 3 + 8}{(x-3)^2 + 4}\,\mathrm{d}(x-3) \\
&= \int \frac{x-3}{(x-3)^2 + 4}\,\mathrm{d}(x-3) + 8\int \frac{1}{(x-3)^2 + 4}\,\mathrm{d}(x-3) \\
&= \frac{1}{2}\int \frac{1}{(x-3)^2 + 4}\,\mathrm{d}[(x-3)^2 + 4] + 8\int \frac{1}{(x-3)^2 + 2^2}\,\mathrm{d}(x-3) \\
&= \frac{1}{2}\log\left|x^2 - 6x + 13\right| + 8 \cdot \frac{1}{2}\arctan\frac{x-3}{2} + C \\
&= \frac{1}{2}\log\left(x^2 - 6x + 13\right) + 4\arctan\frac{x-3}{2} + C \qquad\qquad \square
\end{aligned}$$

問題 4

求不定積分

$$\int \frac{\cos 2x - \sin 2x}{\cos x + \sin x}\, \mathrm{d}x$$

解.

$$与式 = unimplemented$$

$\square$

問題 5

求不定積分

$$\int \frac{\mathrm{d}x}{x(x-1)^2}$$

解. 設

$$与式 = \int \frac{A}{x} + \frac{Bx + C}{x^2 - 2x + 1}\, \mathrm{d}x$$

則

$$\begin{cases} A + B = 0 \\ -2A + C = 0 \\ A = 1 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -1 \\ C = 2 \end{cases}$$

即

$$\begin{aligned}
与式 &= \int \frac{1}{x} + \frac{-x + 2}{x^2 - 2x + 1}\, \mathrm{d}x \\
&= \int \frac{1}{x}\, \mathrm{d}x - \int \frac{x - 1 - 1}{(x-1)^2}\, \mathrm{d}(x-1) \\
&= \int \frac{1}{x}\, \mathrm{d}x - \int \frac{1}{x-1}\, \mathrm{d}(x-1) + \int \frac{1}{(x-1)^2}\, \mathrm{d}(x-1) \\
&= \log|x| - \log|x-1| - \frac{1}{x-1} + C \\
&= \log\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C
\end{aligned}$$

$\square$