

問題 1

求極限

$$\lim_{x \rightarrow 0} \frac{\sqrt{1-x^2} - \cos 3x}{e^x - x - 1}$$

解.

$$\begin{aligned} \text{与式} &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) + 3\sin 3x}{e^x - 1} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(1-x^2)^{-\frac{3}{2}}(-2x)^2 + (-2)\left[\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\right] + 9\cos 3x}{e^x} \\ &= 8 \end{aligned} \quad \square$$

問題 2

求極限

$$\lim_{x \rightarrow 0} \frac{e^{\tan x} - e^x}{x - \sin x}$$

解. Maclaurin Expansion: $e^{\tan x} = 1 + \tan x + o(\tan x)$, $e^x = 1 + x + o(x)$.

$$\begin{aligned} \text{与式} &= \lim_{x \rightarrow 0} \frac{1 + \tan x + o(\tan x) - 1 - x - o(x)}{x - \sin x} \\ &= \lim_{x \rightarrow 0} \frac{\tan x - x + o(x)}{x - \sin x} \\ &= \lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x} \\ &= \lim_{x \rightarrow 0} \frac{1 + \tan^2 x - 1}{1 - \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2 \tan x \sec^2 x}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{2}{\cos x} \\ &= 2 \end{aligned} \quad \square$$

問題 3

求不定積分

$$\int \frac{x+5}{x^2-6x+13} dx$$

解.

$$\begin{aligned} \text{与式} &= \int \frac{x+5}{x^2+2 \cdot 3x+3^2+4} dx \\ &= \int \frac{x-3+8}{(x-3)^2+4} d(x-3) \\ &= \int \frac{x-3}{(x-3)^2+4} d(x-3) + 8 \int \frac{1}{(x-3)^2+4} d(x-3) \\ &= \frac{1}{2} \int \frac{1}{(x-3)^2+4} d[(x-3)^2+4] + 8 \int \frac{1}{(x-3)^2+2^2} d(x-3) \\ &= \frac{1}{2} \log |x^2-6x+13| + 8 \cdot \frac{1}{2} \arctan \frac{x-3}{2} + C \\ &= \frac{1}{2} \log (x^2-6x+13) + 4 \arctan \frac{x-3}{2} + C \end{aligned} \quad \square$$

問題 4

求不定積分

$$\int \frac{\cos 2x - \sin 2x}{\cos x + \sin x} dx$$

解.

与式 = *unimplemented*

□

問題 5

求不定積分

$$\int \frac{dx}{x(x-1)^2}$$

解. 設

$$\text{与式} = \int \frac{A}{x} + \frac{Bx+C}{x^2-2x+1} dx$$

則

$$\begin{cases} A+B=0 \\ -2A+C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=2 \end{cases}$$

即

$$\begin{aligned} \text{与式} &= \int \frac{1}{x} + \frac{-x+2}{x^2-2x+1} dx \\ &= \int \frac{1}{x} dx - \int \frac{x-1-1}{(x-1)^2} d(x-1) \\ &= \int \frac{1}{x} dx - \int \frac{1}{x-1} d(x-1) + \int \frac{1}{(x-1)^2} d(x-1) \\ &= \log|x| - \log|x-1| - \frac{1}{x-1} + C \\ &= \log \left| \frac{x}{x-1} \right| - \frac{1}{x-1} + C \end{aligned}$$

□