問題 1

求極限

$$\lim_{x\to 0}\frac{\sqrt{1-x^2}-\cos 3x}{\mathrm{e}^x-x-1}$$

解.

$$\begin{aligned}
\text{与式} &= \lim_{x\to 0}\frac{\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)+3\sin 3x}{\mathrm{e}^x-1}\\
&= \lim_{x\to 0}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-\frac{3}{2}}(-2x)^2+(-2)\left[\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\right]+9\cos 3x}{\mathrm{e}^x}\\
&= 8
\end{aligned}$$

□

問題 2

求極限

$$\lim_{x\to 0}\frac{\mathrm{e}^{\tan x}-\mathrm{e}^x}{x-\sin x}$$

解. Maclaurin Expansion: $\mathrm{e}^{\tan x}=1+\tan x+o(\tan x)$, $\mathrm{e}^x=1+x+o(x)$.

$$\begin{aligned}
\text{与式} &= \lim_{x\to 0}\frac{1+\tan x+o(\tan x)-1-x-o(x)}{x-\sin x}\\
&= \lim_{x\to 0}\frac{\tan x-x+o(x)}{x-\sin x}\\
&= \lim_{x\to 0}\frac{\tan x-x}{x-\sin x}\\
&= \lim_{x\to 0}\frac{1+\tan^2 x-1}{1-\cos x}\\
&= \lim_{x\to 0}\frac{2\tan x\sec^2 x}{\sin x}\\
&= \lim_{x\to 0}\frac{2}{\cos x}\\
&= 2
\end{aligned}$$

□

問題 3

求不定積分

$$\int\frac{x+5}{x^2-6x+13}\,\mathrm{d}x$$

解.

$$
\begin{aligned}
\text{与式} &= \int \frac{x+5}{x^2+2\cdot 3x+3^2+4}\,\mathrm{d}x \\
&= \int \frac{x-3+8}{(x-3)^2+4}\,\mathrm{d}(x-3) \\
&= \int \frac{x-3}{(x-3)^2+4}\,\mathrm{d}(x-3) + 8\int \frac{1}{(x-3)^2+4}\,\mathrm{d}(x-3) \\
&= \frac{1}{2}\int \frac{1}{(x-3)^2+4}\,\mathrm{d}[(x-3)^2+4] + 8\int \frac{1}{(x-3)^2+2^2}\,\mathrm{d}(x-3) \\
&= \frac{1}{2}\log\left|x^2-6x+13\right| + 8\cdot\frac{1}{2}\arctan\frac{x-3}{2} + C \\
&= \frac{1}{2}\log\left(x^2-6x+13\right) + 4\arctan\frac{x-3}{2} + C
\end{aligned}
$$

□

問題 4

求不定積分

$$\int \frac{\cos 2x - \sin 2x}{\cos x + \sin x}\,\mathrm{d}x$$

解.

$$\text{与式} = unimplemented$$

□

問題 5

求不定積分

$$\int \frac{\mathrm{d}x}{x(x-1)^2}$$

解. 設

$$\text{与式} = \int \frac{A}{x} + \frac{Bx+C}{x^2-2x+1}\,\mathrm{d}x$$

則

$$\begin{cases} A+B=0 \\ -2A+C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=2 \end{cases}$$

即

$$
\begin{aligned}
与式 &= \int \frac{1}{x} + \frac{-x+2}{x^2-2x+1}\,\mathrm{d}x \\
&= \int \frac{1}{x}\,\mathrm{d}x - \int \frac{x-1-1}{(x-1)^2}\,\mathrm{d}(x-1) \\
&= \int \frac{1}{x}\,\mathrm{d}x - \int \frac{1}{x-1}\,\mathrm{d}(x-1) + \int \frac{1}{(x-1)^2}\,\mathrm{d}(x-1) \\
&= \log|x| - \log|x-1| - \frac{1}{x-1} + C \\
&= \log\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C
\end{aligned}
$$

$\square$