Introduction to Enumerative Geometry

Jan. 11 – Jan. 22, 2021

Max-Planck-Institut für
Mathematik
in den Naturwissenschaften

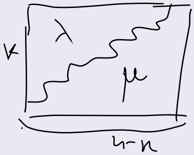
Lecture 4: Chern classes

- Final remarks from last lecture
- Sections of vector bundles $\longrightarrow$ lines on cubic surface
- Chern classes of lines bundles
- Chern classes in general
- 27 lines on a cubic surface

Remarks on Littlewood-Richardson coefficients

Let V be a vector space, $\dim V = n$. We saw that $CH(G(k, V))$ is generated by Schubert classes

$$\sigma_\lambda \text{ for } \lambda \text{ partition in the } k \times (n-k) \text{ box.}$$



If $|\lambda| + |\mu| = k(n-k)$, then

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} \sigma_{(n-k)^k} & \text{if } \lambda, \mu \text{ are complementary in } \cancel{(n-k)\times k} \\ 0 & \text{otherwise} \end{cases}$$

$k \times (n-k)$

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We get a pairing:

$$CH^p(G(k, V)) \times CH^p(G(k, V)) \to \mathbb{Z}$$
$$(\sigma_\lambda, \sigma_\mu) \mapsto \deg(\sigma_\lambda \sigma_{\mu^*}) = \deg(\sigma_{\lambda^*} \cdot \sigma_\mu)$$

where μ^* is the complement of μ in the $k \times (n-k)$ box.

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where μ^* is the complement of μ in the $k \times (n-k)$ box.

This gives us the coefficients of a class $\alpha \in CH^p(G(k,V))$ as $\qquad \alpha = [Y]$

$$\alpha = \sum_{|\lambda| = p} \deg(\alpha \cdot \sigma_{\lambda^*}) \sigma_\lambda$$

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This gives us the coefficients of a class $\alpha \in CH^p(G(k,V))$ as

$$\alpha = \sum_{|\lambda|=p} \deg(\alpha \cdot \sigma_{\lambda^*}) \sigma_\lambda$$

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{|\pi|=|\lambda|+|\mu|} c^\pi_{\lambda\mu} \sigma_\pi$$

and in particular $\boxed{c^\pi_{\lambda\mu} = \deg(\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\pi^*}).}$

Sections of vector bundles

Let $\mathcal{E}$ be a vector bundle on X of rank e with projection $\pi : \mathcal{E} \to X$.

$\mathcal{E}$ locally looks like $X \times \mathbb{C}^e$

in the sense that there are open sets such that

$U \subseteq X$ with $\underbrace{\pi^{-1}(U) \cong U \times \mathbb{C}^e}$

and "they glue together nicely".

Sections of vector bundles

Let $\mathcal{E}$ be a vector bundle on X of rank e with projection $\pi : \mathcal{E} \to X$.

Let $U \subseteq X$ be an open set of X. An algebraic section on U is an algebraic function $s : U \to \mathcal{E}$ such that $\pi \circ s = \mathrm{id}_U$.

Denote by $H^0(U, \mathcal{E})$ the space of sections on U. If $U = X$, we say that s is a global section and we write $\boxed{H^0(\mathcal{E})} = H^0(X, \mathcal{E})$.

$$\begin{array}{ccc} \mathcal{E} & \supseteq & \pi^{-1}(U) \\ \downarrow & & \uparrow{\scriptstyle s} \\ X & \supseteq & U \end{array}$$

$s(x) \in \mathcal{E}_x = \pi^{-1}(x)$

Section of $\mathcal{E}$ form a "sheaf" on X.

Exercise:

Zariski closed subset of $\mathbb{P}^N$

Let X be a smooth projective variety and let V be a vector space. Write $\underline{V}_X$ for the trivial bundle on X with fiber V, that is $\underline{V}_X = X \times V$. Then the only global sections of $\underline{V}_X$ are constant functions, that is $H^0(\underline{V}_X) = V$.

V-valued

$$\begin{array}{c} X \times V \\ \big\downarrow \pi \\ X \end{array}$$

π : proj. on first factor.

If $s \in H^0(\underline{V}_X)$ then

$$s : X \longrightarrow X \times V$$
$$x \longmapsto (x, \underline{\tilde{s}(x)})$$

$\tilde{s}$ is a V-valued function.

The only allowed $\tilde{s}$ are constants.

Reason: Liouville's Thm in Compl. An: Holomorphic function on compact manifolds are constant.

Two exercises: Bundles with very few global sections

Exercise:

Let X be a smooth projective variety and let V be a vector space. Write $\underline{V}_X$ for the trivial bundle on X with fiber V, that is $\underline{V}_X = X \times V$. Then the only global sections of $\underline{V}_X$ are constant functions, that is $H^0(\underline{V}_X) = V$.

Exercise:

Let $\mathcal{S}$ be the tautological bundle on $G(k, V)$. Show that $\mathcal{S}$ has "no global sections", that is $H^0(\mathcal{S}) = 0$.

Recall: $\begin{array}{c} \mathcal{S} \\ \downarrow \\ G(k,V) \end{array}$ $\qquad \mathcal{S}_\Lambda = \Lambda$ for $\Lambda \in G(k,V)$.

Global sections of $\mathcal{S}^\vee$

Let $\mathcal{S}^\vee$ be the dual of the tautological bundle on $G(k,V)$.

$$S_\Lambda = \Lambda^*$$

Claim: Every element $\ell \in V^*$ defines a global section of $\mathcal{S}^\vee$.

$$s : G(k,V) \longrightarrow S^\vee$$

$$\Lambda \longmapsto \text{some elt of } \Lambda^*$$

$$\left(\ell|_\Lambda : \Lambda \longrightarrow \mathbb{C}\right) \in \Lambda^*$$

This works. So $V^* \subseteq H^0(S^\vee)$

Global sections of $\mathcal{S}^\vee$

Let $\mathcal{S}^\vee$ be the dual of the tautological bundle on $G(k, V)$.

Claim: Every element $\ell \in V^*$ defines a global section of $\mathcal{S}^\vee$.

Fact: $H^0(\mathcal{S}^\vee) = V^*$.

Global sections of $\mathcal{Q}$

Let $\mathcal{Q}$ be the universal quotient bundle on $G(k, V)$.

$\mathcal{Q}_\Lambda = V/\Lambda$

Claim: Every element $v \in V$ defines a global section of $\mathcal{Q}$.

$s : G(k, V) \longrightarrow \mathcal{Q}$

$\Lambda \longmapsto$ some elt of V/Λ

$v \bmod \Lambda :=$ image of the projection $V \to V/\Lambda$.

Global sections of $\mathcal{Q}$

Let $\mathcal{Q}$ be the universal quotient bundle on $G(k, V)$.

Claim: Every element $v \in V$ defines a global section of $\mathcal{Q}$.

Fact: $H^0(\mathcal{Q}) = V$.

All global sections of $\mathcal{Q}$ arise as "$v \bmod \Lambda$" for some $v \in V$.

Globally generated bundles

Let $\mathcal{E} \to X$ be a vector bundle of rank e. We say that $\mathcal{E}$ is *globally generated* if there exists global sections $s_1, \ldots, s_N$ such that

$$\underline{\langle s_1(x), \ldots, s_N(x) \rangle} = \mathcal{E}_x \text{ for every } x.$$

$s_j(x) \in \mathcal{E}_x$ They span $\mathcal{E}_x$ at every point

Globally generated bundles

Let $\mathcal{E} \to X$ be a vector bundle of rank e. We say that $\mathcal{E}$ is *globally generated* if there exists global sections $s_1, \dots, s_N$ such that

$$\langle s_1(x), \dots, s_N(x) \rangle = \mathcal{E}_x \text{ for every } x.$$

Examples

$\mathcal{Q}$ and $\mathcal{S}^\vee$ are globally generated.

For Q: Fix a basis $v_1, \dots, v_n$ of V $\rightsquigarrow$ associated sections $s_1, \dots, s_n$

Of course $v_1 \bmod \Lambda, \dots, v_n \bmod \Lambda$ span V/Λ for every Λ

Toward Chern classes: Lines on Cubic surfaces

Let $X \subseteq \mathbb{P}^3$ be a (generic) cubic surface. How many lines are contained in X?

$\mathbb{P}^3 = \mathbb{P}V$, $\dim V = 4$

Idea:

Answer: If X smooth then 27.

We will answer for generic X.

Say $X = \{g = 0\}$ g hom. eq. of deg 3 on V

$$g \in S^3 V^*$$

Want to compute $\#\{\Lambda \in G(2,4) : \mathbb{P}\Lambda \subseteq X\} =$

$$= \#\{\Lambda \in G(2,4) : g|_\Lambda \equiv 0\}$$

We saw: $H^0(S^\vee) = V^*$. Similarly: $H^0(\mathrm{Sym}^3 S^\vee) = S^3 V^*$

Toward Chern classes: Lines on Cubic surfaces – cont'd

$\mathrm{Sym}^3 S^\vee$ is the vector bundle with

$$(\mathrm{Sym}^3 S^\vee)_\Lambda = S^3\Lambda^* \quad \text{hom. pol. of deg } 3 \text{ on } \Lambda$$

If $f \in S^3 V^*$ then

$$\underbrace{\begin{aligned} s_f : G(2,4) &\longrightarrow \mathrm{Sym}^3 S^\vee \\ \Lambda &\longmapsto f|_\Lambda \end{aligned}}$$

defines a global section and they all look like this.

The set we care about is the vanishing locus of the section g defining X.

Question: What is the degree of the vanishing locus of a generic section?

The first Chern class of a line bundle

Let $\mathcal{L}$ be a line bundle on X. Suppose $\mathcal{L}$ is globally generated. Let $s \in H^0(\mathcal{L})$ be a generic section and consider

$$Y = \{x \in X : s(x) = 0\}.$$

$0 \in L_x$.

The first Chern class of a line bundle

Let $\mathcal{L}$ be a line bundle on X. Suppose $\mathcal{L}$ is globally generated. Let $s \in H^0(\mathcal{L})$ be a generic section and consider

$$Y = \{x \in X : s(x) = 0\}.$$

Y is an algebraic subvariety of X.

Locally Y is given by a single equation, so $\boxed{\operatorname{codim}_X(Y) = 1 \text{ or } Y = \emptyset.}$

Locally L looks like $X \times \mathbb{C}$

Vanishing of s gives locally a single equation.

Theorem

If $Y = \emptyset$ then $\mathcal{L}$ is the trivial bundle, i.e. $\mathcal{L} \overset{\sim}{=} X \times \mathbb{C}$.

Pf/ If $Y = \phi$ then there is $\underset{H^0(\mathcal{L})}{\underset{\cap}{s}}$ which never vanishes

Define:

$$\begin{aligned} X \times \mathbb{C} &\longrightarrow \mathcal{L} \\ (x, \lambda) &\longmapsto \lambda s(x) \in \mathcal{L}_x \end{aligned}$$

Claim: This is an isomorphism of vector bundles $\square$

The first Chern class of a line bundle – cont'd

Let $\mathcal{L}$ be a line bundle on X. Suppose $\mathcal{L}$ is globally generated. Let $s \in H^0(\mathcal{L})$ be a generic section and consider

$$Y = \{x \in X : s(x) = 0\}.$$

The class of Y in $CH(X)$ does not depend on the choice of s.

Same proof that shows two hypersurfaces of the same degree are rationally equivalent.

The first Chern class of a line bundle – cont'd

Let $\mathcal{L}$ be a line bundle on X. Suppose $\mathcal{L}$ is globally generated. Let $s \in H^0(\mathcal{L})$ be a generic section and consider

$$Y = \{x \in X : s(x) = 0\}.$$

The class of Y in $CH(X)$ does not depend on the choice of s.

It only depends on $\mathcal{L}$.

The *first Chern class* of $\mathcal{L}$ is

$$c_1(\mathcal{L}) = [Y] \in CH^1(X).$$

$Y = \{x : s(x) = 0\}$ s generic section of $\mathcal{L}$

Properties of the first Chern class of a line bundle

Fact: Let $\mathcal{L}$ be a line bundle on X. Then there exists a globally generated line bundle $\mathcal{P}$ such that $\mathcal{L} \otimes \mathcal{P}$ is globally generated.

If $z < 0$ then $\exists N$ with $N \geq 0$ and $z + N > 0$

Properties of the first Chern class of a line bundle

Fact: Let $\mathcal{L}$ be a line bundle on X. Then there exists a globally generated line bundle $\mathcal{P}$ such that $\mathcal{L} \otimes \mathcal{P}$ is globally generated.

$\mathrm{Pic}(X) = \{L : \text{line bundle on } X\}/\sim$

with the operation of $\otimes$.

Using this fact, we define the Chern class of any line bundle:

$$c_1(\mathcal{L}) = \underbrace{c_1(\mathcal{L} \otimes \mathcal{P})} - \underbrace{c_1(\mathcal{P})}.$$

$$c_1 : \mathrm{Pic}(X) \longrightarrow CH^1(X)$$
$$L \longmapsto c_1(L)$$

is a gp homomorphism

Properties:

- $c_1(\underline{\mathcal{L} \otimes \mathcal{M}}) = c_1(\mathcal{L}) + c_1(\mathcal{M})$
- $c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L})$

Line bundles on projective space

Let $\mathcal{O}(d)$ be the line bundle on $\mathbb{P}V$ whose fiber at $[v]$ is $\langle v^{\otimes d}\rangle^*$.

Fact: $H^0(\mathcal{O}(d)) = \underline{S^d V^*}$

$\mathcal{O}(1) = S^\vee$ regarding $\mathbb{P}V = G(1, V)$

A generic section is $f \in S^d V^*$, in the sense that

$$s : \mathbb{P}V \longrightarrow \mathcal{O}(d)$$
$$[v] \longmapsto f|_{\langle v^{\otimes d}\rangle} \in S^d V^* \simeq (S^d V)^*$$

$Y = \{[v] : f|_{\langle v^{\otimes d}\rangle} \equiv 0\} = \{[v] : f(v) = 0\}$ hypersurface defined by f

So $c_1(\mathcal{O}(d)) = [Y] = d \cdot \zeta$ $\quad$ ζ is the hyperplane class.

Line bundles on projective space – cont'd

What if we take $\mathcal{O}(-m)$ for some $m \geq 0$.

$$\mathcal{O}(-m) = \mathcal{O}(m)^{\vee}$$

So $c_1(\mathcal{O}(-m)) = -m\zeta$.

Alternatively, consider $P = \mathcal{O}(N)$ for $N > m$

so that

$$c_1(\mathcal{O}(-m)) = c_1(\mathcal{O}(-m) \otimes P) - c_1(P) =$$

$$= c_1(\mathcal{O}(-m+N)) - c_1(\mathcal{O}(N)) =$$

$$= (-m+N-N)\zeta = -m\zeta.$$

Line bundles on projective space – cont'd

What are the fibers of $\mathcal{O}(-m) \otimes \mathcal{O}(N)$.

$$(\mathcal{O}(-m))_{[v]} = \langle v^{\otimes m} \rangle \qquad (\mathcal{O}(N))_{[v]} = \langle v^{\otimes N} \rangle^*$$

Claim:

$$\langle v^{\otimes m} \rangle \otimes \langle v^{\otimes N} \rangle^* \simeq \langle v^{\otimes (N-m)} \rangle^*$$

$$\underbrace{\langle v^{\otimes m} \rangle \otimes \langle v^{\otimes m} \rangle^*}_{\mathbb{C}} \otimes \langle v^{\otimes (N-m)} \rangle^* = \langle v^{\otimes (N-m)} \rangle^*$$

$\xi : v \longmapsto 1$

$$\langle v \rangle \otimes \langle v \rangle^* \longrightarrow \mathbb{C}$$

$$\lambda v \otimes \mu \xi \longmapsto \lambda \mu$$

Degeneracy loci

Let $\mathcal{E}$ be a vector bundle of rank e on X and suppose it is globally generated. For $p \leq e$, fix $\underline{s_0, \ldots, s_{e-p}} \in H^0(\mathcal{E})$. Define

$$Y(s_0, \ldots, s_{e-p\cancel{+1}}) = \{\underline{x \in X} : \underline{s_0(x), \ldots, s_{e-p}(x)} \text{ are linearly dependent}\}$$

the degeneracy locus of $s_0, \ldots, s_{e-p\cancel{+1}}$.

Locally $Y(s_0, \ldots, s_{e-p+1})$ is defined by a rank condition on some matrix whose entries are functions on X.

Y is a possibly reducible algebraic variety

The class of a degeneracy locus

Theorem

Every component of $Y(s_0, \ldots, s_{e-p})$ has codimension at most p in X (or Y is empty).

Moreover, if the s_j's are chosen generically, then $\operatorname{codim}_X Y(s_0, \ldots, s_{e-p}) = p$ (or Y is empty).

Idea of proof:

Call $h_0 = \dim H^0(\mathcal{E})$.

Define
$$\Phi : X \longrightarrow G(h_0 - e, H^0(\mathcal{E}))$$
$$x \longmapsto \operatorname{Ker}\left(ev_x : H^0(\mathcal{E}) \longrightarrow \mathcal{E}_x\right)$$
$$s \longmapsto s(x)$$

Since $\mathcal{E}$ globally generated,

ev_x is surjective at every point.

$\dim(\operatorname{Ker} ev_x) = h_0 - e$

The class of a degeneracy locus – cont'd

Φ defines a morphism of varieties.

Let $s_0, \dots, s_{e-p}$ be generic sections of $\mathcal{E}$.

Define $F_{e-p+1} = \langle s_0, \dots, s_{e-p} \rangle \subseteq H^0(\mathcal{E})$

Consider: $\Sigma_{(p)}(F_{e-p+1}) = \{ \Lambda \in G(h_0 - e, H^0(\mathcal{E})) : \dim \big(\Lambda \cap F_{\underbrace{h_0 - (h_0 - e) + 1 - p}_{e-p+1}}\big) \geq 1 \}$

$\Phi^{-1}(\Sigma_{(p)}(F_{e-p+1})) = \{ x \; \dim(\mathrm{Ker}(ev_x) \cap \underset{\langle s_0, \dots, s_{e-p+1} \rangle}{\underset{\shortparallel}{F_{e-p+1}}}) \geq 1 \}$

$= \{ x : s_0(x), \dots, s_{e-p+1}(x) \text{ are lin. dep} \}$.

$\operatorname{codim} \Sigma_p = p \quad \underbrace{}$

$\to$ More work to get $\operatorname{codim} = p$ in X.

The class of a degeneracy locus – cont'd

Fact: If the s_j's are chosen generically, then

- $Y(s_0, \ldots, s_{e-p})$ is generically reduced;
- $[Y(s_0, \ldots, s_{e-p})] \in \underbrace{CH^p(X)}_{\text{Codim}=p}$ does not depend on the chosen sections.

So $Y(s_0, \ldots, s_{e-p})$ only depends on $\mathcal{E}$.

The class of a degeneracy locus – cont'd

Fact: If the s_j's are chosen generically, then

- $Y(s_0, \dots, s_{e-p})$ is generically reduced;
- $[Y(s_0, \dots, s_{e-p})] \in CH^p(X)$ does not depend on the chosen sections.

Define the p-th Chern class of $\mathcal{E}$ to be

$$c_p(\mathcal{E}) = \underline{[Y(s_0, \dots, s_{e-p})]} \in \underline{CH^p(X)}$$

for generic $s_0, \dots, s_{e-p}$.

In particular $c_0(\mathcal{E}) = [X] \in CH^0(X)$.

$\to$ $e+1$ sections are always depend.

$\to$ We write $1_{CH(X)}$

Properties of Chern classes

Setting: $\mathcal{E}$ vector bundle of rank e on X. $\mathcal{E}$ globally generated

Notation:

total Chern class

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots + c_e(\mathcal{E}) \in CH(X)$$

$$c_{[t]}(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + \cdots + c_e(\mathcal{E})t^e \in CH(X)[t]$$

Chern polynomial

Properties of Chern classes

Setting: $\mathcal{E}$ vector bundle of rank e on X.

Notation:

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots + c_e(\mathcal{E}) \in CH(X)$$
$$c_{[t]}(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + \cdots + c_e(\mathcal{E})t^e \in CH(X)[t]$$

Properties:

- Whitney's formula: If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is a short exact sequence of vector bundles, then
$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}).$$
- Functoriality: If $\varphi : X \to Y$ is a morphism of varieties and $\mathcal{E}$ is a bundle on Y
$$\varphi^*(c(\mathcal{E})) = c(\varphi^*\mathcal{E})$$

pull-back of $\mathcal{E}$

$\varphi^* : CH(Y) \to CH(X)$

Properties of Chern classes – cont'd

Splitting principle

Whitney's formula implies that if $\mathcal{E} = \bigoplus_1^e \mathcal{L}_i$ for line bundles $\mathcal{L}_i$ then

$$\underbrace{c(\mathcal{E}) = c(\mathcal{L}_1)\cdots c(\mathcal{L}_e)} = \prod_1^e (1 + c_1(\mathcal{L}_i)).$$

$c_1(\mathcal{L}_i)$

If $\mathcal{E}$ splits as sum of line bundles then $c_p(\mathcal{E})$ is the p-th elementary symmetric polynomial in $c_1(\mathcal{L}_1), \dots, c_1(\mathcal{L}_e)$.

$\mathcal{E}^\vee = \bigoplus \mathcal{L}_i^\vee$ $\qquad c_p(\mathcal{E}^\vee) = (-1)^p c_p(\mathcal{E})$

$$c_{[t]}(\mathcal{E}) = c_{[-t]}(\mathcal{E}^\vee).$$

Splitting principle

When we do calculations with Chern classes, we can pretend that bundles split as direct sums of line bundles. The Chern classes of these line bundles are called *virtual Chern classes* of $\mathcal{E}$; their opposites are the "roots" of the Chern polynomial of $\mathcal{E}$.

Thm:

X with a v.bun. $\mathcal{E}$

Then there is $\varphi : Y \longrightarrow X$ such that

- $\varphi^* : CH(X) \longrightarrow CH(Y)$ is injective
- $\varphi^*\mathcal{E}$ has a filtration with "line" factors:

$$\begin{array}{c} \varphi^*\mathcal{E} \\ \downarrow \\ Y \end{array} \qquad 0 \subseteq \mathcal{E}_1 \subseteq \dots \subseteq \mathcal{E}_e = \varphi^*\mathcal{E}$$

such that $\mathcal{E}_{j+1}/\mathcal{E}_j$ is a line bundle

Grassmannian

We compute the Chern classes of $\mathcal{S}$ and $\mathcal{Q}$ on $G(k, V)$. $\dim V = n$

Start with Q.

$H^0(Q) = V$ Given $v \in V$

$$s_v : G(k,V) \longrightarrow Q$$
$$\Lambda \longmapsto v \bmod \Lambda$$

To compute $c_p(Q)$ consider $v_0, \dots, v_{n-k-p}$ generic

Define $F_{n-k-p+1} = \langle v_0, \dots, v_{n-k-p} \rangle$.

$$Y(v_0, \dots, v_{n-k-p}) = \{ \Lambda : v_0 \bmod \Lambda, \dots, v_{n-k+p} \bmod \Lambda \text{ are dep.} \} =$$
$$= \{ \Lambda : \dim(\Lambda \cap F_{n-k-p+1}) \geq 1 \} = \Sigma_p(F_{n-k+1-p})$$

Grassmannian

So $c_p(Q) = \sigma_p$.

What about S:

$$0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0$$

$$\underset{\underset{1}{\shortparallel}}{c(\underline{V})} = c(S)\,c(Q)$$

$\to$ Claim $c(S) = 1 - \sigma_1 + \sigma_{11} - \sigma_{111} + \dots \pm \sigma_{1^k}$.

Pf/ Via Pieri's rule. $\square$

Compute directly $c(S^\vee) = 1 + \sigma_1 + \sigma_{1^2} + \dots + \sigma_{1^k}$

Lines on cubic surface

Let $g \in S^3 V^*$ with $\dim V = 4$. Let $X = \{g = 0\} \subseteq \mathbb{P}V$. How many lines are contained in X? When g is generic.

The number of lines is the degree of

$$\{\Lambda : g|_\Lambda \equiv 0\} = Y(s_0) \quad \text{for } s_0 = s_g$$

as a section of $\mathrm{Sym}^3 S^\vee$.

$$\operatorname{rank} \mathrm{Sym}^3 S^\vee = \dim S^3 S_\Lambda^\vee = 4$$

The vanishing of a single section gives $c_4(\mathrm{Sym}^3 S^\vee)$

because $0 + 4 - p = 0$ for $p = 4$.

Lines on cubic surface

Pretend $S^\vee$ splits $\quad S^\vee = \mathcal{A} \oplus \mathcal{B}$

$\alpha = c_1(\mathcal{A})$
$\beta = c_1(\mathcal{B})$
virtual Char classes

$$c(S^\vee) = (1+\alpha)(1+\beta)$$

$$c_1(S^\vee) = \sigma_1 = \alpha + \beta$$
$$c_2(S^\vee) = \sigma_{1,1} = \alpha\beta$$

$$\mathrm{Sym}^3(\mathcal{A}\oplus\mathcal{B}) = \underline{\mathcal{A}^{\otimes 3}} \oplus \underbrace{\mathcal{A}^{\otimes 2} \otimes \mathcal{B}} \oplus \mathcal{A} \otimes \mathcal{B}^{\otimes 2} \oplus \mathcal{B}^{\otimes 3}$$

$$c(\mathrm{Sym}^3(\mathcal{A}\oplus\mathcal{B})) = (1 + \underline{3\alpha})(1 + \underbrace{2\alpha+\beta})(1+\alpha+2\quad)(1+3\beta)$$

$$c_4(\text{——})$$

Lines on cubic surface

$$c_4(\mathrm{Sym}^3(S^\vee)) = (3\alpha)(2\alpha+\beta)(\alpha+2\beta)(3\beta) =$$

$$= 9\alpha\beta\left(2(\alpha+\beta)^2+\alpha\beta\right) =$$

$$= 9\sigma_{11}\cdot\left(2\sigma_1^2+\sigma_{11}\right) =$$

$$= 9\sigma_{11}\left(\underbrace{2(\sigma_2+\sigma_{11})+\sigma_{11}}\right) = 27\sigma_{11}^2 = 27\sigma_{22}$$

$$\underline{\deg(\{\Lambda : \Lambda \subseteq X\}) = 27}$$

Lines on cubic surface