Introduction to Enumerative Geometry

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Max-Planck-Institut für Mathematik in den Naturwissenschaften

Recap:

- Chow Ring (integer lin. comb. of irred. subvarieties of X modulo rational equivalence.

- $CH(\mathbb{P}^n) = \mathbb{Z}[z]/z^{n+1}$

Lecture 2: Grassiannians: Introduction and $G(2,4)$

- Definitions
- Tautological and quotient bundle
- $\boxed{G(2,4)}$
- Schubert calculus in $G(2,4)$ $\leftarrow$
- Examples

The k-th Grassmannian in V is the variety of k-dimensional subspaces in V:

$$G(k, V) = \{E \subseteq V : E \text{ lin. subspace, } \dim E = k\}.$$

The Grassmannian has a natural *Plücker* embedding

$$G(k, V) \to \mathbb{P}\Lambda^k V$$

$$E = \langle v_1, \ldots, v_k\rangle \mapsto v_1 \wedge \cdots \wedge v_k = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

$$\Lambda^k V \subseteq V^{\otimes k}$$

$$\| $$

sign of the permutation

$$\left\langle \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \ : \ v_j \in V \right\rangle$$

$\mathfrak{S}_k$ is the symmetric acting by permutation on the factors

Fact: The class of $v_1 \wedge \cdots \wedge v_k$ in $\mathbb{P}\Lambda^k V$ only depends on $E = \langle v_1, \ldots, v_k\rangle$.

First properties

If dim $V = n$ there is a natural isomorphism

$$G(k, V) \to G(n-k, V^*)$$
$$E \mapsto E^{\perp}$$

$$E^{\perp} = \{\alpha \in V^* : \alpha|_E = 0\}$$ annihilator of E in V^*.

Vector bundle on X of rank k: An algebraic variety $\mathcal{F}$ with a map $\pi : \mathcal{F} \to X$ such that all fibers of π are isomorphic to linear spaces all of the same dimension k

Fiber at $p \in X$: $\mathcal{F}_p = \pi^{-1}(p)$

Tautological and quotient bundle

$$\begin{tikzcd}
 & \{(E,v) : v \in E\} \arrow[d, phantom, "\shortparallel"] & \{(E,v) : E \in G,\ v \in V\} \arrow[d, phantom, "\shortparallel"] & & \\
0 \arrow[r] & S \arrow[r] \arrow[dr, "\pi_S"'] & \underbrace{G(k,V) \times V}_{} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[dl, "\pi_Q"] \arrow[d, phantom, "\shortparallel"] & 0 \\
 & & G(k,V) & \{(E,w) : w \in V/E\} \ (\text{locally}) &
\end{tikzcd}$$

$$\begin{tikzcd}
S^\vee \otimes Q \arrow[d, "\pi_{S^\vee} \otimes \pi_Q"] \\
G
\end{tikzcd}$$

- S tautological bundle on $G(k,V)$: vector bundle whose fiber at $E \in G(k,V)$ is the space E itself

- Q quotient bundle is the vector bundle with fiber at E equal to $Q_E = V/E$

Universal property of Grassmannian

"Every vector bundle is the pullback of the some tautological bundle".

$$\begin{tikzcd} g^*F \arrow[d] & F \arrow[d, "\pi"] \\ g : X \arrow[r] & Y \end{tikzcd}$$

$$(g^*F)_x = F_{g(x)}.$$

If F is a vector subbundle of rank k in the trivial bundle $\begin{tikzcd} X \times V \arrow[d] \\ X \end{tikzcd}$ for some space V.

Then $F = g^*S$ where

$$\begin{tikzcd} F = g^*S \arrow[d] & S \arrow[d] \\ g : X \arrow[r] & G(k, V) \end{tikzcd}$$

$g(x) = F_x$

Tangent bundle of $G(k, V)$

The tangent bundle $TG(k, V)$ of $G(k, V)$ is isomorphic to $\underline{\mathcal{S}}^\vee \otimes \underline{\mathcal{Q}}$.

$TG(k,V)$ = vector bundle whose fiber at E is $T_E G(k,V)$

Sketch of pf:

The fibers of $S^\vee \otimes Q$ are $(S^\vee \otimes Q)_E = \overline{E^* \otimes V/E} = \mathrm{Hom}(E, V/E)$.

Fix $E \in G(k,V) \subseteq \mathbb{P}\Lambda^k V$. Tangent vectors at E are

$\Lambda'(0)$ where $\Lambda(t)$ is a curve in $G(k,V)$ with $\Lambda(0) = E$.

If $E = \langle v_1, \dots, v_k \rangle$ then $\Lambda(t) = v_1(t) \wedge \dots \wedge v_k(t)$ where $v_j(0) = v_j$

$v_j(t)$ curve in V

Tangent bundle – cont'd

- $\dot{\Lambda}(0) = \dot{v}_1(0) \wedge v_2 \wedge \dots \wedge v_k + v_1 \wedge \dot{v}_2(0) \wedge \dots \wedge v_k + \dots + v_1 \wedge \dots \wedge v_{k-1} \wedge \dot{v}_k(0).$

Tangent vectors have the form:

$$w_1 \wedge v_2 \wedge \dots \wedge v_k + \dots + v_1 \wedge \dots \wedge v_{k-1} \wedge w_k$$

Now if $\varphi : E \to V/E$ define

$v_j(t) = v_j + t\varphi(v_j)$ and pick $\Lambda_\varphi(t) = v_1(t) \wedge \dots \wedge v_k(t)$

This defines a map $E^* \otimes V/E \longrightarrow T_E G(k, V)$

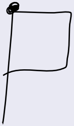
Two maps φ, ψ give same tangent vector if

$$\varphi - \psi \in E$$

Flags and Schubert classes

Let V be a vector space of dimension n. A *complete flag* in V is a nested sequence of vector spaces:

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = V$$



with $\dim F_k = k$.

Schubert varieties are subvarieties of $G(k, V)$ consisting of planes with "non-generic intersection" with the planes of a fixed flag.

A generic E intersects F_p in dimension

$$\operatorname{codim}(E \cap F_p) = \operatorname{codim} E + \operatorname{codim} F_p \quad \left(\text{or } E \cap F_p = 0\right)$$

Schubert varieties in $G(2,4)$

Fix $\dim V = 4$ and $\underline{k = 2}$.

If $k = 1 \to$ project. space

$k = n-1 \to$ dual proj. space

Schubert varieties in $G(2, 4)$

Fix $\dim V = 4$ and $k = 2$.

Consider a complete flag $F_\bullet = (0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 = V)$ in V.

Schubert varieties in $G(2,4)$

Fix $\dim V = 4$ and $k = 2$.

Consider a complete flag $F_\bullet = (0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 = V)$ in V.

For $0 \leq a, b \leq 2$, we define a Schubert variety in $G(2,4)$

$$\Sigma_{a,b} = \{\Lambda : \dim(\Lambda \cap F_{3-a}) \geq 1, \dim(\Lambda \cap F_{4-b}) \geq 2\}.$$

Schubert varieties in $G(2,4)$

Fix $\dim V = 4$ and $k = 2$.

Consider a complete flag $F_\bullet = (0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 = V)$ in V.

For $0 \leq a, b \leq 2$, we define a Schubert variety in $G(2,4)$ $\quad a \geq b$

$$\Sigma_{a,b} = \{\Lambda : \dim(\Lambda \cap F_{3-a}) \geq 1, \dim(\Lambda \cap F_{4-b}) \geq 2\}.$$

Explicitly

$$\begin{aligned}
\Sigma_{0,0} &= G(1,3); \\
\Sigma_{1,0} &= \{\Lambda : \Lambda \cap F_2 \neq 0\}; \quad \dim \Lambda \cap F_2 \geq 1 \\
\Sigma_{2,0} &= \{\Lambda : F_1 \subseteq \Lambda\}; \\
\Sigma_{1,1} &= \{\Lambda : \Lambda \subseteq F_3\}; \\
\Sigma_{2,1} &= \{\Lambda : \underline{F_1 \subseteq \Lambda} \subseteq F_3\}; \quad \dim \Lambda \cap F_1 \geq 1 \\
\Sigma_{2,2} &= \{\Lambda : \Lambda = F_2\}. \quad \dim \underset{\Lambda}{\underbrace{\Lambda \cap F_3}} \geq 2
\end{aligned}$$

Schubert varieties in $G(2,4)$ – cont'd

$$\Sigma_{0,0} = G(1,3);$$
$$\Sigma_{1,0} = \{\Lambda : \Lambda \cap F_2 \neq 0\};$$
$$\Sigma_{2,0} = \{\Lambda : F_1 \subseteq \Lambda\};$$
$$\Sigma_{1,1} = \{\Lambda : \Lambda \subseteq F_3\};$$
$$\Sigma_{2,1} = \{\Lambda : F_1 \subseteq \Lambda \subseteq F_3\};$$
$$\Sigma_{2,2} = \{\Lambda : \Lambda = F_2\}.$$

If $a \leq a'$, $b \leq b'$, then
$$\Sigma_{a,b} \supseteq \Sigma_{a',b'}.$$

$\Sigma_{(1,0)} \supseteq \Sigma_{(2,0)}$: $\Lambda \in \Sigma_2 \Rightarrow F_1 \subseteq \Lambda \Rightarrow \Lambda \cap F_2 \supseteq F_1 \Rightarrow$

$\Rightarrow \Lambda \cap F_2 \neq 0 \Rightarrow$

$\Rightarrow \Lambda \in \Sigma_1$

Schubert varieties in $G(2,4)$ – cont'd

$$\Sigma_{0,0} = G(1,3);$$
$$\Sigma_{1,0} = \{\Lambda : \Lambda \cap F_2 \neq 0\};$$
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If $a \leq a'$, $b \leq b'$, then

$$\Sigma_{a,b} \supseteq \Sigma_{a',b'}.$$

The Schubert cell associated to (a,b) is

$$\Sigma^\circ_{a,b} = \Sigma_{a,b} \setminus \bigcup_{(a',b') \geq (a,b)} \Sigma_{a',b'}$$

$a' \geq a \quad b' \geq b$

(at least one strict)

Schubert varieties in $G(2,4)$ – cont'd

$$\begin{aligned}
\Sigma_{0,0} &= G(1,3); \\
\Sigma_{1,0} &= \{\Lambda : \Lambda \cap F_2 \neq 0\}; \\
\Sigma_{2,0} &= \{\Lambda : F_1 \subseteq \Lambda\}; \\
\Sigma_{1,1} &= \{\Lambda : \Lambda \subseteq F_3\}; \\
\Sigma_{2,1} &= \{\Lambda : F_1 \subseteq \Lambda \subseteq F_3\}; \\
\Sigma_{2,2} &= \{\Lambda : \Lambda = F_2\}.
\end{aligned}$$

If $a \leq a'$, $b \leq b'$, then

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The Schubert cell associated to (a,b) is

$$\Sigma^{\circ}_{a,b} = \Sigma_{a,b} \setminus \bigcup_{(a',b') \geq (a,b)} \Sigma_{a',b'}$$

By definition, the Schubert cells define a stratification of $G(2,4)$.

Schubert varieties in $G(2,4)$ – cont'd

The Schubert cell Σ_1° is isomorphic to $\mathbb{A}^3$.

Pf/ $\Sigma_1^\circ = \{ \Lambda \in G(2,4) : \Lambda \cap F_2 \neq 0, \underline{\Lambda \not\subseteq F_3 \quad F_1 \not\subseteq \Lambda} \}$

$\quad\;\; \shortparallel$

$\Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{1,1})$

We show $\Sigma_1^\circ \simeq \mathbb{A}^3$:

Fix H hyperplane in V such that $F_1 \subseteq H$, $F_2 \not\subseteq H$.

Note: $H \cap F_3$ has dim 2: $\dim H \cap F_3 \geq 3+3-4 = 2$

but $H \neq F_3$ because $F_2 \not\subseteq H$.

Schubert varieties in $G(2,4)$ – cont'd

$\mathbb{A}^1 = \mathbb{P}F_2 \setminus \mathbb{P}F_1 = \mathbb{P}^1 \setminus p_0$

$\mathbb{A}^2 = \mathbb{P}H \setminus \mathbb{P}(H \cap F_3) = \mathbb{P}^2 \setminus \mathbb{P}^1$

Given $\Lambda \in \Sigma_1^\circ$ define $L'_\Lambda = \Lambda \cap F_2$

So L'_Λ is a point in $\mathbb{P}F_2 \setminus \mathbb{P}F_1$

Define: $L''_\Lambda = \Lambda \cap H$.

Claim: $\dim L''_\Lambda = 1$. $\dim L''_\Lambda \geq 2+3-4 = 1$

Suppose $\dim L''_\Lambda > 1 \implies L''_\Lambda = \Lambda \implies \Lambda \subseteq H \implies L'_\Lambda \subseteq H$

$F_2 = F_1 + L'_\Lambda \subseteq H$ contradiction because $F_2 \not\subseteq H$.

Moreover:

$L''_\Lambda \not\subseteq F_3$.

$\Downarrow$

$L''_\Lambda \in \mathbb{P}H \setminus \mathbb{P}(H \cap F_3)$

Define:

$\Sigma_1^\circ \longrightarrow \mathbb{A}^1 \times \mathbb{A}^2$

$\Lambda \longmapsto (L'_\Lambda, L''_\Lambda)$

$L' + L'' = \Lambda \longleftarrow (L', L'')$

Schubert varieties in $G(2,4)$ – cont'd

In general

$$\underline{\operatorname{codim}_{G(2,4)} \Sigma_{a,b} = a + b}$$

and $\Sigma^{\circ}_{a,b}$ is isomorphic to an affine space.

$\dim G(2,4) =$
$= 2 \cdot (4-2) = 4$

$\underset{3}{\overset{\|}{\dim \Sigma_1}} = \dim G(2,4) - 1$

Schubert varieties in $G(2,4)$ – cont'd

In general

$$\operatorname{codim}_{G(2,4)} \Sigma_{a,b} = a + b$$

and $\Sigma^{\circ}_{a,b}$ is isomorphic to an affine space.

Therefore the Schubert cells form an affine stratification of $G(2,4)$.

The Chow ring $CH(G(2,4))$ is generated by the classes of the Schubert varities $\sigma_{a,b} = [\Sigma_{a,b}]$.

Transversality of Schubert varieties

Let $F_\bullet, E_\bullet$ be two flags. They are *transverse* if

$$\cancel{\operatorname{codim}(F_i \cap E_j) = \max\{n-(i+j), 0\}.}$$

$$\text{dim } E_i \cap F_j = i+j-n \quad \text{or} \quad E_i \cap F_j = 0$$

Transversality of Schubert varieties

Let $F_\bullet, E_\bullet$ be two flags. They are *transverse* if

$$\mathrm{codim}(F_i \cap E_j) = \max\{n - (i + j), 0\}.$$

Fact: Schubert varieties defined by transverse flags intersect transversely.

Transversality of Schubert varieties

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Fact: Schubert varieties defined by transverse flags intersect transversely.

We can use transverse flags to determine the product structure.

Product structure in $CH(G(2,4))$

We have $\sigma_{a,b} \in CH^{a+b}(G(2,4))$ and they satisfies these relations:

$$\begin{aligned}
\sigma_1^2 &= \sigma_{1,1} + \sigma_2 \\
\sigma_1\sigma_{1,1} &= \sigma_1\sigma_2 = \sigma_{2,1} \\
\sigma_1\sigma_{2,1} &= \sigma_{2,2} \\
\sigma_{1,1}^2 &= \sigma_2^2 = \sigma_{2,2} \\
\sigma_2\sigma_{1,1} &= 0
\end{aligned}$$

$\sigma_2 \cdot \sigma_2 = \sigma_{2,2}$

$$\Sigma_2(F_\bullet) = \{\Lambda : F_1 \subseteq \Lambda\}$$

$$\Sigma_2(E_\bullet) = \{\Lambda : E_1 \subseteq \Lambda\}$$

$$\sigma_2 \cdot \sigma_2 = [\Sigma_2(F_\bullet)] \cdot [\Sigma_2(E_\bullet)] =$$

$$= [\Sigma_2(F_\bullet) \cap \Sigma_2(E_\bullet)] = [\{\Lambda : F_1, E_1 \subseteq \Lambda\}]$$

$$\|$$

$$[\ \{\Lambda_\bullet\}\ ] = [\Sigma_{22}] =$$

$$= \sigma_{22}$$

$\sigma_1 \cdot \sigma_2 = \sigma_{2,1}$

$$\Sigma_1(F_\cdot) = \{\Lambda : \Lambda \cap F_2 \neq 0\}$$

$$\Sigma_2(E_\cdot) = \{\Lambda : E_1 \subseteq \Lambda\}$$

Want to determine the intersection $X = \Sigma_1(F_\cdot) \cap \Sigma_2(E_\cdot)$

Define a flag: $P_\cdot = (E_1 \subseteq P_2 \subseteq E_1 + F_2)$

Claim: $X = \Sigma_{2,1}(P_\cdot) = \{\Lambda : P_2 \subseteq \Lambda \subseteq P_3\}$

$\dim X = 1$ and $\dim \Sigma_{2,1}(P_\cdot) = 1$, $\Sigma_{2,1}(P_\cdot)$ irreducible.

We show: $X \subseteq \Sigma_{2,1}(P)$

$\Lambda \in X \Rightarrow \underline{\Lambda \cap F_2 \neq 0}$ and $E_1 \subseteq \Lambda \Rightarrow$ E_1, F_2 are indep. $\Downarrow$ $\Lambda = (\Lambda \cap F_2) + E_1 \Rightarrow \Lambda \subseteq E_1 + F_2$

$\boxed{\sigma_1 \cdot \sigma_1 = \sigma_2 + \sigma_{1,1}}$

easier to show

We saw $\sigma_2^2 = \sigma_{22}$. Similarly $\sigma_{11}^2 = \sigma_{22}$.

and $\sigma_2 \cdot \sigma_{11} = 0$. $\sigma_1 \sigma_{21} = \sigma_{22}$

We want:

$$\sigma_1 \cdot \sigma_1 = a\sigma_2 + b\sigma_{11}$$

$$\sigma_1^2 \cdot \sigma_2 = (a\sigma_2 + b\sigma_{11}) \cdot \sigma_2 = a\sigma_{22}$$

$$\|$$

$$\sigma_1 \cdot (\sigma_1 \sigma_2) = \sigma_1 \cdot \sigma_{21} = \sigma_{22} \qquad \Rightarrow a = 1$$

Same argument shows $b = 1$.

Example 1

How many lines in $\mathbb{P}^3$ meet four given lines in general position?

Note: Projectively $G(2,4) = \{\Lambda : \mathbb{P}\Lambda \text{ is a line in } \mathbb{P}^3\}$.

Let L_1, L_2, L_3, L_4 $\quad L_j = \mathbb{P}F_2^{(j)}$ $\quad L_j$ is the 2-dimensional space of a flag.

$X_j = \{\mathbb{P}\Lambda : \mathbb{P}\Lambda \cap L_j \neq \emptyset\} =$

$= \{\Lambda : \Lambda \cap F_2^{(j)} \neq 0\} = \Sigma_1(F_\bullet^{(j)})$

$\deg(X_1 \cap X_2 \cap X_3 \cap X_4) = \deg(\sigma_1^4) = \deg((\sigma_{11} + \sigma_2)(\sigma_{11} + \sigma_2)) =$

$\deg(\sigma_{11}^2 + \sigma_2^2) = \deg(\sigma_{22} + \sigma_{22}) = 2$

$CH^4(G(2,4)) = [\sigma_{22}]$

Example 2

Let $C_1, \ldots, C_4$ be four curves in $\mathbb{P}^3$, $\deg(C_i) = d_i$, in general position. How many lines in $\mathbb{P}^3$ meet all of them?

$$\Gamma(C) = \{ \Lambda : \mathbb{P}\Lambda \cap C \neq \emptyset \}$$

Fact: $\Gamma(C)$ is a subvar. of codim 1 in $G(2,4)$ (Chow form)

Claim: $[\Gamma(C)] = d\sigma_1$ if $\deg(C) = d$.

Pf/ $[\Gamma(C)] = m \cdot \sigma_1$ for some m because $CH^1(G(2,4)) = \langle \sigma_1 \rangle$.

$$m = \deg([\Gamma(C)] \cdot \sigma_{21}) = \deg(\Gamma(C) \cap \Sigma_{21}(F_\bullet)) =$$

because $\sigma_1 \sigma_{21} = \sigma_{22}$

$$= \deg\left(\left\{ \Lambda : \begin{array}{c} \mathbb{P}\Lambda \cap C \neq \emptyset, \\ F_1 \subseteq \Lambda \subseteq F_3 \end{array} \right\}\right).$$

$$p_0 = \mathbb{P}F_1 \subseteq \mathbb{P}\Lambda \subseteq \mathbb{P}F_3$$

Example 2 – cont'd

This number is the number of lines in $\mathbb{P}F_3$ such that:

- $\mathbb{P}\Lambda \subseteq \mathbb{P}F_3$
- $p_0 \in \mathbb{P}\Lambda$
- $\mathbb{P}\Lambda \cap C \neq \phi$

$\Rightarrow$ $\mathbb{P}\Lambda$ is spanned by p_0 and an elt of $C \cap \mathbb{P}F_3$

How many of these lines? $\#(C \cap \mathbb{P}F_3) = \deg(C) = d$

How many lines meet $C_1, \dots, C_4$:

$$\deg\left([\Gamma(C_1)] \cdots [\Gamma(C_4)]\right) = d_1\sigma_1 \cdots d_4\sigma_1 = d_1 \cdots d_4 (\sigma_1^4) =$$
$$= 2 d_1 \cdots d_4 .$$

Variety of secant lines

not contained in hyperplane

Let $C \subseteq \mathbb{P}^3$ be a smooth $\boxed{\text{non-degenerate}}$ curve of degree d and genus g. Let

$$\mathfrak{s}(C) = \overline{\{\Lambda \in G(2,4) : \mathbb{P}\Lambda = \langle p, q\rangle \text{ for some } p, q \in C\}}.$$

What is the class of $\mathfrak{s}(C)$?

Claim: $\dim s(C) = 2$

$$\Psi : C \times C \dashrightarrow G(2,4)$$
$$(p, q) \longmapsto \langle p, q \rangle$$

is rational and generically finite.

Moreover $s(C) = \overline{\operatorname{Im} \Psi}$ so $\operatorname{codim} s(C) = 2$

$$[s(C)] = \alpha \sigma_2 + \beta \sigma_{11}$$

Variety of secant lines – cont'd

Variety of secant lines – cont'd

Common secant lines

Let C_1, C_2 be two twisted cubics in $\mathbb{P}^3$ in general position. How many lines are secant to both of them?

Variety of tangent lines

Let $S \subseteq \mathbb{P}^3$ be a surface of degree $d \geq 2$. Let

$$\mathfrak{t}(S) = \{\Lambda \in G(2,4) : \mathbb{P}L \text{ is tangent to } S\}.$$

What is the class of $\mathfrak{t}(S)$?

Variety of tangent lines

Variety of tangent lines

Common tangent lines

Let $S_1, , \ldots, S_4$ be four surfaces of degrees $d_1, \ldots, d_4$, in general position. How many lines are tangent to all of them?