Algebraic Geometry in Complexity Theory

Fulvio Gesmundo

Institut de Mathématiques de Toulouse

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- Evaluating polynomials;
- Computing products in algebraic structures (e.g., matrix multiplication);
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General framework:

- Points in a vector space represent problems.
- The complexity is measured by some *geometric* invariant, for instance membership in an algebraic variety.

Complexity measures for Polynomials

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We consider two important complexity measures (but there are many others):

- Waring rank.
- Algebraic Branching Program (ABP) Width.

Waring Rank

The **Waring rank** of f, denoted $\mathrm{wr}(f)$, is the smallest r such that

$$f = \ell_1^d + \cdots + \ell_r^d$$

is a sum of powers of linear forms $\ell_i = a_{i1}x_1 + \cdots + a_{iN}x_N$.

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shows $\mathrm{wr}(h) \leq 8$ (and equality holds).

ABP Width

The **algebraic branching program width** of f, denoted $\mathrm{abpw}(f)$, is the smallest w such that

$$f = \begin{bmatrix} \ell_{\mathbf{1},\mathbf{1}} & \cdots & \ell_{\mathbf{1},w} \end{bmatrix} \begin{bmatrix} \ell_{\mathbf{2},\mathbf{1},\mathbf{1}} & \cdots & \\ \vdots & \ddots & \\ & & \ell_{\mathbf{2},w,w} \end{bmatrix} \cdots \begin{bmatrix} \ell_{d-\mathbf{1},\mathbf{1},\mathbf{1}} & \cdots & \\ \vdots & \ddots & \\ & & \ell_{d-\mathbf{1},w,w} \end{bmatrix} \begin{bmatrix} \ell_{d,\mathbf{1}} \\ \vdots \\ \ell_{d,w} \end{bmatrix}$$

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shows $\mathrm{abpw}(h) \leq 2$ (and equality holds).

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Valiant's Flagship Conjecture
$\mathrm{perm}_n \notin \mathsf{VBP}$, or equivalently $\mathrm{abpw}(\mathrm{perm}_n)$ is superpolynomial.

[If this conjecture is false, then P = NP]

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- Membership:
  if $h \in C(r)$ then $h = \lim_{\varepsilon \to 0} h_\varepsilon$ with $c(h_\varepsilon) \leq r$.

  We can use geometry for upper bounds.

Strategy for lower bounds

To prove $c(h) > r$, we look for polynomial equations on $S^d\mathbb{C}^N$ such that:

- they vanishes on $C(r)$ (e.g. on $\{f \in S^4\mathbb{C}^N : \mathrm{abpw}(f) \leq 3\}$);
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Often, instead of explicit polynomial equations we simply study geometric conditions which can be "translated" into equations
(even if we rarely do this translation).

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Lower bounds for Waring rank

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Easy theorem for lower bounds.
If $f \in C(r)$ then the space of k-th order partial derivatives of f has dimension at most r.

Consequence.
If the k-th order partials of f span a space of dimension r, then $\mathrm{wr}(f) \geq r$.

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More complicated rank conditions:

- Young flattenings $\to$ representation theory;
- homological methods $\to$ commutative algebra;
- apolarity $\to$ deformation theory.

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Corollary.
VWaring $\neq$ VBP.

Proof.
m_n belongs to VBP but not to VWaring.

Lower bounds for ABP width

We study geometric properties of the *hypersurface*

$$\mathcal{Z}(f) = \{(x_1, \ldots, x_N) \in \mathbb{C}^N : f(x_1, \ldots, x_N) = 0\}.$$

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If $\mathrm{abpw}(f) \leq w$ then $\{f = 0\} \subseteq \mathbb{C}^N$ contains a linear space of dimension at least $N - w$.

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Idea of the Proof.
From an ABP, get an expression $f = \ell_1 g_1 + \cdots + \ell_w g_w$ with $\deg(\ell_j) = 1$. Then

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Open problem.
Determine a geometric method which can potentially give super-polynomial lower bounds on abpw.

Matrix multiplication complexity

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$$\mathrm{R}(T) = \min\{r : T = u_1 \otimes v_1 \otimes w_1 + \cdots + u_r \otimes v_r \otimes w_r\}.$$

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Conjecture. The exponent ω of matrix multiplication is 2:

$$\mathrm{R}(\mu_n) = O(n^{2+\varepsilon}) \text{ for every } \varepsilon.$$

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- [Strassen, 1987] Laser method: Find *intermediate tensors* to simulate matrix multiplication: $\omega \leq 2.48$.

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- [Coppersmith-Winograd, 1990] $\omega \leq 2.39$.

Overview of upper bounds on ω

- Classical: $\mathrm{R}(\mu_n) \leq n^3$ so $\omega \leq 3$.
- [Strassen 1969] $\mathrm{R}(\mu_2) \leq 7$;

  Consequence: $\omega \leq \log_2(7) \approx 2.81$.
- [Bini-Capovani-Lotti-Romani, 1979]
  ω is controlled by membership in a variety

  $$C(r) = \overline{\{T \in U \otimes V \otimes W : \mathrm{R}(T) \leq r\}}.$$

  Membership is sufficient for upper bounds! $\omega \leq 2.78$.
- [Schönhage, 1981]
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- [Coppersmith-Winograd, 1990] $\omega \leq 2.39$.
- More improvements by Stothers, Williams, LeGall, Alman-Williams, ...
  [Williams-Xu-Xu-Zhou, 2023] $\omega \leq 2.371552$.

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  Flattening methods give non-membership for much bigger varieties than the $C(r)$'s.

  We cannot hope to prove lower bounds when these varieties fill the ambient space.

  [Buczyński, Ballico-Bernardi-G.]
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- [Alman-Williams, Ambainis-Filmus-LeGall, Christandl-Vrana-Zuiddam]
  Strassen's laser method, applied to the original Coppersmith-Winograd tensor, cannot prove $\omega < 2.3$.

What can we do?

Lower bounds based on new geometric invariants:

- [Landsberg-Manivel-Ressayre] Degeneracy of dual variety
- [G.-Ghosal-Ikenmeyer-Ghosal] Noether-Lefschetz type conditions
- [Iarrobino, Buczyńska-Buczyński, Jelisiejew-Landsberg-Pal] Classical apolarity, Hilbert scheme of points, deformation theory.

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Upper bounds via more refined version of laser method:

- [Conner-G.-Landsberg-Ventura]
  Use geometry to find better intermediate tensors.
- [Homs-Michałek-Jelisiejew-Seynnaeve]
  Refine the laser method using ideas from commutative algebra.

Conclusions and more open directions

- Complexity is (almost) controlled by membership in certain sets $C(r)$
  - Lower bounds via geometric obstructions to membership.
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- Barriers: Geometric methods have structural limitations.
- Boundary: To what extent taking closure matters? Debordering in complexity theory
- Asymptotic geometry: Strassen developed an asymptotic spectrum of tensors to study matrix multiplication via the laser method. Can we use similar ideas in other settings?