

PROBLEM SETS FOR INTRODUCTION TO ENUMERATIVE GEOMETRY

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ABSTRACT. This document collects a few problems which should be useful to practice on the material that is covered during the lectures.

LECTURE 1

Problem 1. Let V be a vector space of dimension $n + 1$ and let $\nu_{d,n} : \mathbb{P}V \rightarrow \mathbb{P}S^dV$ be the Veronese embedding. Show that if $X \subseteq \mathbb{P}V$ is a variety of dimension k and degree e , then $\nu_{d,n}(X)$ has degree $d^k e$.

In particular, the degree of a k -dimensional subvariety of $\nu_{d,n}(\mathbb{P}V)$ is a multiple of d^k .

Problem 2. Let

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : p, r, s \text{ are collinear}\}.$$

Show that Ψ is a subvariety of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ of codimension $n - 1$.

Determine the class $[\Psi]$ in the Chow ring $\text{CH}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$.

Note: The Chow ring of a product of several projective spaces is what one expects. If $\dim V_j = n_j + 1$, then

$$\text{CH}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_s) = \mathbb{Z}[\alpha_1, \dots, \alpha_s] / (\alpha_1^{n_1+1}, \dots, \alpha_s^{n_s+1})$$

where α_j is identified with the class of the pull back of the hyperplane section in $\mathbb{P}V_j$ via the projection map. In other words

$$\alpha_j = [\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{j-1} \times H_j \times \mathbb{P}V_{j+1} \times \cdots \times \mathbb{P}V_s].$$

Problem 3. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables. Let

$$\mathcal{T} = \{f : f = \ell_1 \ell_2 \ell_3 \text{ for some linear forms } \ell_j\},$$

that is the space of *triangles* (i.e., cubic curves which are union of three lines).

Determine the dimension and the degree of \mathcal{T} .

Hint: Write \mathcal{T} is the image of a map defined on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Problem 4. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{A} = \{f : f = \ell_1 \ell_2 \ell_3 \text{ for some linear forms } \ell_j \text{ with a common zero}\},$$

that is the space of *asterisks* (i.e., cubic curves which are union of three lines passing through the same point).

Determine the dimension and the degree of \mathcal{A} .

Hint: It is similar to the previous problem, but one needs something more complicated than $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Problem 5. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{C} = \{f = \ell_1^2 \ell_2 : \ell_j \text{ is a linear form}\},$$

that is the space of cubic curves which are union of a double line and a line.

Determine the dimension and the degree of \mathcal{C} .

LECTURE 2

Problem 6. Let $C_1, C_2 \subseteq \mathbb{P}^3$ be two curves of degree d_1, d_2 respectively and genera g_1, g_2 respectively. Suppose C_1, C_2 are in general position with respect to each other. How many lines are secant both two C_1 and C_2 ?

Problem 7. Let C be a smooth non-degenerate curve in \mathbb{P}^3 of degree d and genus g . Let

$$TC = \{\Lambda \in G(2, 4) : \Lambda = T_p C \text{ for some } p \in C\}.$$

What is the class of TC in $G(2, 4)$?

Problem 8. Let $S_1, \dots, S_4 \subseteq \mathbb{P}^3$ be four surfaces with $\deg(S_i) = d_i$ in general position. How many lines are tangent to all of them?

Problem 9. Let C be a smooth non-degenerate curve of degree d and genus g in \mathbb{P}^3 . Let S be a smooth surface of degree e . Suppose C and S are in general position with respect to each other. How many lines are tangent to both S and C ?

Problem 10. Let $X \subseteq G(2, 4)$ be an irreducible variety of codimension 2. Then $[X] = \alpha\sigma_2 + \beta\sigma_{1,1}$. Show that if $\alpha = 0$ then $\beta = 1$. What if $\beta = 0$?

LECTURE 3

Problem 11. Let C be a smooth non-degenerate curve of degree d and genus g in $\mathbb{P}V$. Let

$$s_2(C) = \overline{\{\Lambda \in G(2, V) : \mathbb{P}\Lambda \text{ is a secant line to } C\}}.$$

Determine the class of $s_2(X)$ in $G(2, V)$.

Problem 12. Let C be a smooth non-degenerate curve of degree d and genus g in $\mathbb{P}V$. Let

$$T(C) = \overline{\{\Lambda \in G(2, V) : \mathbb{P}\Lambda \text{ is a tangent line to } C\}}.$$

Determine the class of $T(C)$ in $G(2, V)$.

Problem 13. In $G(3, 6)$, compute the product

$$\sigma_{2,1} \cdot \sigma_{2,1}.$$

Problem 14. Let λ be a partition contained in the $k \times (n - k)$ box and let Σ_λ be the corresponding Schubert variety in $G(k, n)$. Consider the identification

$$i : G(k, n) \rightarrow G(n - k, n)$$

Show that i maps Σ_λ to Σ_{λ^T} , where λ^T is the partition in the $(n - k) \times k$ whose Young diagram is the transpose of the Young diagram of λ .

Problem 15. Let X be an irreducible smooth variety of codimension c and degree d in $\mathbb{P}V$. Let

$$H(X) = \{\Lambda \in G(c, V) : \mathbb{P}\Lambda \cap X \neq \emptyset\}.$$

This is the Chow form of X .

- Prove that $\text{codim}_{G(c, V)}(H(X)) = 1$;
- Determine the class $[H(X)]$ in $CH^1(G(c, V))$.

LECTURE 4

Problem 16. Prove the statements about global sections mentioned during the lectures. In particular prove:

- $H^0(\mathcal{S}) = 0$ where \mathcal{S} is the tautological bundle on $G(k, V)$;
- $H^0(\mathcal{S}^\vee) = V^*$ where \mathcal{S}^\vee is the dual of the tautological bundle on $G(k, V)$;
- $H^0(\mathcal{Q}) = V$ where \mathcal{Q} is the universal quotient bundle on $G(k, V)$.

Problem 17. Compute the Chern classes of the tangent bundle of $G(2, 4)$.

Problem 18. Let X be a generic hypersurface of degree $2n - 3$ in \mathbb{P}^n . Prove that X contains a finite number of lines and determine this number.

Problem 19. Let X be a generic hypersurface of degree 4 in \mathbb{P}^7 . Prove that X contains a finite number of 2-planes and determine this number.

Problem 20. Let X_1, X_2 be two generic cubic hypersurfaces in general position in \mathbb{P}^5 . How many lines are contained in both of them?

LECTURE 5

Problem 21. Let Sym_n and Skew_n be the spaces of symmetric and skew-symmetric matrices respectively. Let $A_r = \{S \in \text{Sym}_n : \text{rank}(S) \leq r\}$ and $K_r = \{\Lambda \in \text{Skew}_n : \text{rank}(\Lambda) \leq r\}$. Compute the dimensions of A_r and K_r .

Problem 22. Let $X_r = \{A \in \mathbb{P}\text{Mat}_{e \times f} : \text{rank}(A) \leq r\}$ be the r -th general determinantal variety. Let $M \in X_r$ be a matrix of rank $s \leq r$. Compute

- $T_M X_r$, the tangent space to X_r at M ;
- $TC_M X_r$, the tangent cone to X_r at M .

Deduce that M is a smooth point of X_r if and only if $s = r$.

Problem 23. Let C_d be the rational normal curve of degree d in \mathbb{P}^d . Let $r < d/2$ and let $\sigma_r(C_d)$ be the r -th secant variety of C_d . Prove that $\dim \sigma_r(C_d) = 2r - 1$. Is this true for every curve?

Problem 24. Let C_d be the rational normal curve of degree d in $\mathbb{P}^d = \mathbb{P}S^d \mathbb{C}^2$. Let $e < d/2$ and let $\text{cat}_e : S^e \mathbb{C}^{2*} \rightarrow S^{d-e} \mathbb{C}^2$ be the e -th catalecticant map. Show that the 2×2 minors of cat_e define C_d set-theoretically (in fact, they generate the ideal). In particular, C_d has the expected dimension as a determinantal variety with respect to cat_1 but it does not have the expected dimension as a determinantal variety with respect to cat_e for $e \geq 2$.

Problem 25. Let $\nu_{2,2}(\mathbb{P}^2)$ be the Veronese surface in $\mathbb{P}^5 = \mathbb{P}S^2\mathbb{C}^3$. Prove that $\sigma_2(\nu_{2,2}(\mathbb{P}^2))$ does not have the expected dimension (as a secant variety).