Introduction to Enumerative Geometry

Jan. 11 – Jan. 22, 2021

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Course plan:

- Lecture 1: Chow ring and first examples
- Lecture 2: Grassmannians: Examples on $G(2,4)$
- Lecture 3: Grassmannians in general
- Lecture 4: Chern classes
- Lecture 5: Determinantal varieties
- Lecture 6: Porteous's formula

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- Lecture 1: Chow ring and first examples
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- Lecture 6: Porteous's formula

Varieties are quasi-projective varieties over complex numbers.

Schemes are quasi-projective schemes.

Lecture 1: Chow ring and first examples

- Cycles
- Rational equivalence
- Chow group
- Chow ring
- Examples

Group of cycles

Let X be a variety or a scheme.

The group of k-cycles of X, denoted $Z_k(X)$, is the group of formal integer linear combinations of k-dimensional irreducible subvarieties of X.

If X irred. curve $\dim X = 1$.

$$Z_0(X) = \{ w_1 p_1 + \dots + w_s p_s \ : \ p_1, \dots, p_s \in X,\ s \in \mathbb{N},\ w_j \in \mathbb{Z} \}$$

$$Z_1(X) = \mathbb{Z} \cdot X$$ free group on one generator.

$$Z_k(X) = 0 \quad \text{if } k \geq 2.$$

If X has dim n:

$$Z_k(X) = \bigoplus_{\substack{Y : \dim Y = k \\ Y \text{ irred.}}} \mathbb{Z} \cdot Y.$$

Group of cycles – cont'd

The group of all cycles is $Z(X) = \bigoplus_{k \geq 0} Z_k(X)$.

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If $Y \subseteq X$ is a reducible subvariety, with components $Y_1, \ldots, Y_s$, then we define the *effective cycle associated to* Y to be

$$Y = Y_1 + \cdots + Y_s.$$

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If $Y \subseteq X$ is a scheme with components $Y_1, \ldots, Y_s$ and multiplicities $p_1, \ldots, p_s$, then we define the effective associated cycle to be

$$Y = p_1 Y_1 + \cdots + p_s Y_s.$$

Rational equivalence

Let X be a variety and let W be an irreducible subvariety of $X \times \mathbb{P}^1$. Let $\pi : X \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection.

Then either

- there is $t \in \mathbb{P}^1$ such that $W \subseteq X \times \{t\}$. (1)
- $\pi(W)$ is dense in $\mathbb{P}^1$. (2)

$\pi(W)$ is irreducible in $\mathbb{P}^1$

Either it is a point or it is dense

- If a point: case (1)
- otherwise case (2)

Rational equivalence – cont'd

We say that two subvarieties Y_0, Y_1 of X are *rationally equivalent* if there exists a $W \subseteq X \times \mathbb{P}^1$ irreducible such that

$$W \cap (X \times \{0\}) = Y_0 \times \{0\}$$
$$W \cap (X \times \{1\}) = Y_1 \times \{1\}$$

$\pi : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

$\pi^{-1}(0) = Y_0 \qquad \pi^{-1}(1) = Y_1$

Rational equivalence – cont'd

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Exercise:
Rational equivalence is an equivalence relation on $Z(X)$.

Rational equivalence – cont'd

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Exercise:
Rational equivalence is an equivalence relation on $Z(X)$.

Let

$$Rat(X) = \langle Y_0 - Y_1 : Y_0, Y_1 \text{ are rationally equivalent} \rangle \subseteq Z(X)$$

Rational equivalence – cont'd

Example: Two hypersurfaces of the same degree are always rationally equivalent.

$Y_0, Y_1 \subseteq \mathbb{P}^n$

$Y_0 = \{g_0 = 0\}$ $\quad g_j \in \mathbb{C}[x_0, \ldots, x_n]_d$

$Y_1 = \{g_1 = 0\}$

Define:

$W = \{g_0 t_0 + g_1 t_1 = 0\}$ $\quad (t_0, t_1)$ coords on $\mathbb{P}^1$.

- at $(0,1)$: $W_1 = Y_1$
- at $(1,0)$: $W_0 = Y_0$

Ex. If Y_0 is irred. then W irred.

$\Downarrow$

any Y_1 is equivalent to Y_0 if Y_0 irreducible.

Rational equivalence – cont'd

Lemma If $Y_0, Y_1 \subseteq X$ are rationally equivalent and non-empty, then

$$\dim Y_0 = \dim Y_1.$$

$\mathbb{P}_1$ coords $(1,t)$

Pf/ There is W such that $W_0 = Y_0$, $W_1 = Y_1$

$W_0 = W \cap (X \times \{0\})$: the equation $\{t=0\}$ cuts W_0 in $W \Rightarrow \operatorname{codim}_W (W_0) = 1$

Same holds for W_1 so equality holds.

Chow group

The Chow group of X is

$$CH(X) = Z(X)/Rat(X)$$

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$$Z(X) = \bigoplus Z_k(X)$$

By the Lemma

$$CH(X) = \bigoplus_{k \geq 0} CH_k(X)$$

where $CH_k(X) = Z_k(X)/Rat_k(X)$. Write $CH^c(X) = CH_{\dim X - c}(X)$.

Transversality

Let $Y_1, Y_2 \subseteq X$ be two subvarieties. Let $p \in Y_1 \cap Y_2$. We say that they intersect transversely at p if

- X, Y_1, Y_2 are smooth at p
- $\underline{T_p Y_1 + T_p Y_2 = T_p X}$ $\longrightarrow$ $\operatorname{codim}_X (Y_1 \cap Y_2) =$

$$= \operatorname{codim}_X Y_1 + \operatorname{codim}_X Y_2$$

We say that Y_1, Y_2 are generically transverse if they are transverse at a generic point of every irreducible component of $\underline{Y_1 \cap Y_2}$.

We say that two cycles are generically transverse if every component of one is transverse to every component of the other.

Moving Lemma

Theorem Let X be a smooth variety. Then

- For every $\alpha, \beta \in CH(X)$ there exist generically transverse cycles A, B such that $\alpha = [A]$ and $\beta = [B]$.
- If A and B are generically transverse subvarieties, then the class $[A \cap B]$ is independent of the choice of representative for $[A]$ and $[B]$.

Change A with A' $[A] = [A']$

B with B' $[B] = [B']$

if A, B are gen. transverse A', B' are gen. tr.

then $[A \cap B] = [A' \cap B']$

Chow ring

Theorem

Let X be a smooth variety. Then there is a unique product structure on $CH(X)$ such that whenenver A, B are generically transverse subvarieties of X, then $[A][B] = [A \cap B]$.

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The class $[X] \in CH^0(X)$ is the identity element of the ring and is called the fundamental class of X. (If X irreducible).

Example: The affine space

Let $\mathbb{A}^n$ be the affine space of dimension n. Its Chow ring is the free abelian group generated by the fundamental class:

$$CH(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n] = \{ m[\mathbb{A}^n] : m \in \mathbb{Z} \}.$$

$$\overset{\shortparallel}{CH^0(\mathbb{A}^n)}.$$

Pf/ If $Y \subseteq \mathbb{A}^n$ is an $\overset{\text{irreducible}}{\text{proper}}$ subvariety then $Y \sim \phi$.

Assume $\underline{0 \notin Y}$. and define:

$$\underline{W^0} = \{ (tz, t) : z \in Y,\ t \in \mathbb{A}^1 \setminus \{0\} \} \subseteq \mathbb{A}^n \times \underline{\mathbb{A}^1 \setminus \{0\}}.$$

$$W = \overline{W^0} \subseteq \underline{\mathbb{A}^n \times (\mathbb{P}^1 \setminus \{0\})}.$$

Since $0 \notin Y$, there is $g \in I(Y)$ with $g(0) \neq 0$.

Define: $G(z,t) = g(z/t)$.

Claim: G is an eqn for W.

If $t = \infty$: $G(z, \infty) = g(0) \neq 0$

Cohomological properties of the Chow ring

Theorem (Mayer-Vietoris sequence and Excision)

- Let X_1, X_2 be closed subschemes of X. Then there exists an exact sequence

  $(\alpha, \beta) \longmapsto \alpha - \beta$

  $$CH(X_1 \cap X_2) \to CH(X_1) \oplus CH(X_2) \to CH(X_1 \cup X_2) \to 0.$$

  $\alpha \qquad (\alpha, \alpha)$

- Let $Y \subseteq X$ be a closed subscheme and let $U = X \setminus Y$. Then there is an exact sequence

  $\beta \longrightarrow \beta$

  $$CH(Y) \to CH(X) \to CH(U) \to 0.$$

  $\alpha \longmapsto \alpha$

  if $P \subseteq X$ subvar then $P \cap U$ is a subvar. of U.

Cohomological properties of the Chow ring – pushforward and pullback

Let $f : Y \to X$ be a morphism of schemes. The *pushforward* map of f is

$$f_* : CH(Y) \to CH(X)$$

defined by

- $f_*([A]) = 0$ if $f|_A$ has infinite fiber;
- $f_*([A]) = d\underbrace{[f(A)]}$ if the generic fiber of $f|_A$ has cardinality d.

this map is not graded because $\dim A = \dim f(A)$

so codim might be different.

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The *pullback* of f is

$$f^* : CH(X) \to CH(Y)$$

defined by

- $f^*([A]) = [f^{-1}(A)]$ if $f^{-1}(A)$ is generically reduced and $\text{codim}_X(A) = \text{codim}_Y(f^{-1}(A))$ the map is of degree 0.
- Fact: This uniquely determines f^*. (via the moving lemma)

Cohomological properties of the Chow ring – pushforward and pullback

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$\alpha \in CH(X)$

Push-pull formula

$$f_*(\underline{f^*(\alpha) \cdot \beta}) = \alpha \cdot \underline{f_*(\beta)}$$

$\beta \in CH(Y)$

Stratifications

Let X be a scheme and let $\mathcal{U} = \{U_i : i \in I\}$ be a collection of locally closed subschemes of X. We say that $\mathcal{U}$ is a *stratification* of X if

- X is disjoint union of the U_i's;
- for every i, $\overline{U}_i \setminus U_i$ is disjoint union of some of the U_j's.

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Example

$\mathbb{P}^n$

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We say that the stratification is affine if the U_i's are isomorphic to affine spaces.

Example

Coords $(x_0, \dots, x_n)$

$$\mathbb{P}^n = \underset{x_0 \neq 0}{\mathbb{A}^n} \cup \underset{x_0 = 0}{\mathbb{P}^{n-1}} = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \mathbb{P}^{n-2} \dots$$

$$\overline{\mathbb{A}^{n-1}} = \mathbb{P}^{n-1} = \mathbb{A}^{n-1} \cup \mathbb{A}^{n-2} \cup \dots \cup \mathbb{A}^0.$$

Stratifications – cont'd

Theorem The Chow group of affinely stratifiable schemes is generated by the classes of the closed strata.

Pf/ Via Excision.

Projective space

The Chow ring of the projective space $\mathbb{P}^n$ is

$$CH(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$$

where $\zeta = [H]$ is the class of a hyperplane. If X is a variety of degree d and codimension k, then $[X] = d\zeta^k$.

Pf/ $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \dots \cup \mathbb{A}^1 \cup \underset{p_0}{\underset{\parallel}{\mathbb{A}^0}}$

$[\mathbb{P}^n], \underset{[H] = \zeta}{\underset{\parallel}{[\mathbb{P}^{n-1}]}}, \dots, [\mathbb{P}^1], [p_0]$ generate the Chow group

Claim: If L_k is plane of codim. k then $[L_k] = \zeta^k$.

Pf/ $L_k = H_1 \cap \dots \cap H_k$ for k hyperplanes.

$[L_k] = [H_1] \cdots [H_k] = \zeta^k$.

Projective space - cont'd

If X is irred. of codim k deg d then

$$[X] = d\,\zeta^k.$$

Pf/ $[X] = m\,\zeta^k$. Let P generic k-dimensional plane. so $[P] = \zeta^{n-k}$

$$[X]\cdot[P] = m\,\zeta^k \cdot \zeta^{n-k} = m\cdot\zeta^n$$

$$\overset{\shortparallel}{[X\cap P]} = [\deg X \text{ many points } P_1, \dots, P_{\deg X}] = \sum_{1}^{\deg X} [P_j] = d\,\zeta^n$$

so $m = d$. $\square$

Bezout's Theorem

Let $X_1, \dots, X_k \subseteq \mathbb{P}^n$ be subvarieties of codimension $c_1, \dots, c_k$, with $\sum c_i \leq n$ and suppose the X_i intersect generically transversely.

Then

$$\deg(X_1 \cap \cdots \cap X_k) = \prod \deg(X_i).$$

Pf/ $[X_j] = d_j \zeta^{c_j}$

$$\underbrace{[X_1 \cap \cdots \cap X_k]}_{\text{has codim } c_1 + \cdots + c_k} = d_1 \zeta^{c_1} \cdots d_k \zeta^{c_k} = d_1 \cdots d_k \cdot \zeta^{c_1 + \cdots + c_k}$$

So $\deg(X_1 \cap \cdots \cap X_k) = d_1 \cdots d_k$.

If $\alpha \in CH^a(\mathbb{P}^n)$, $\beta \in CH^{n-a}(\mathbb{P}^n)$ then $\boxed{\alpha \cdot \beta} = m\,\zeta^n$

$\deg(\alpha\beta)$

Veronese variety

Let $\nu_d =: \mathbb{P}V \to \mathbb{P}S^d V$ be the d-th Veronese embedding, where V is a vector space of dimension $n+1$. Then

$$\deg(\nu_d(\mathbb{P}V)) = d^n.$$

—

$V = \langle x_0, \dots, x_n \rangle = \mathbb{C}[x_0, \dots, x_n]_1$

$\nu_d : \mathbb{P}V \longrightarrow \underline{\mathbb{P}S^d V}$

$\ell \longmapsto \ell^d$

$S^d V = \mathbb{C}[x_0, \dots, x_n]_d$.

$\dim \nu_d(\mathbb{P}V) = n$ and ν_d is an embedding.

$\deg \nu_d(\mathbb{P}V) = \#\big(\nu_d(\mathbb{P}V) \cap \underbrace{H_1 \cap \dots \cap H_n}_{\text{hyperplanes}}\big) =$ since ν_d is injective

Veronese variety – cont'd

$$= \#\left(\mathbb{P}^n \cap \underline{\nu_d^{-1}(H_1)} \cap \ldots \cap \nu_d^{-1}(H_n)\right) = (*)$$

If H is a hyperplane in $\underline{\mathbb{P}S^d\mathbb{C}^{n+1}}$ then

$\underline{\nu_d^{-1}(H)}$ is a hypersurface ~~of~~ of deg d in $\mathbb{P}^n$.

$$(*) = \deg(\underbrace{dZ \cdot dZ \cdot \ldots \cdot dZ}_{n \text{ times}}) = d^n$$

$$\deg \nu_d(\mathbb{P}^n) = d^n$$

Degree of dual varieties

Let $X \subseteq \mathbb{P}V$ be a smooth hypersurface and let $X^\vee \subseteq \mathbb{P}V^*$ be its dual variety, which is the image of X under the Gauss map:

$$\mathcal{G}_X : X \to \mathbb{P}V^*$$
$$p \mapsto \ell_p$$

$(x_0, \dots, x_n)$ coords on $\mathbb{P}V$.

where ℓ_p is the equation of the tangent space $T_pX \subseteq V$.

If $X = \{f = 0\}$ then $\mathcal{G}_X(p) = \frac{\partial f}{\partial x_0}(p) \cdot x_0 + \dots + \frac{\partial f}{\partial x_n}(p) \cdot x_n$

Extend $\mathcal{G}_X$ to $\mathbb{P}V$: $P_X : \mathbb{P}V \longrightarrow \mathbb{P}V^*$

$$p \longmapsto \frac{\partial f}{\partial x_0}(p) x_0 + \dots + \frac{\partial f}{\partial x_n}(p) x_n$$

(polar map).

Exercise: If X smooth deg ≥ 2 then P_X is well defined.

Degree of dual varieties – cont'd

Fact: X smooth of $\deg \geq 2$ then $\mathcal{G}_X$ is birational (generically injective).

$X^\vee$ is a hypersurface:

$$\deg X^\vee = \deg\left([X^\vee]\cdot \zeta^{n-1}\right) =$$

$$= \#\left(X^\vee \cap H_1 \cap \ldots \cap H_{n-1}\right) =$$

$$= \#\left(X \cap P_X^{-1}(H_1) \cap \ldots \cap P_X^{-1}(H_{n-1})\right) =$$

$$= \circledast \deg\left(d\zeta\left((d-1)\zeta\right)^{n-1}\right) =$$

$$= d\cdot(d-1)^{n-1}$$

$H = \{l = 0\} \Rightarrow P_X^{-1}(H) = \{p \in \mathbb{P}V : l\left(\frac{\partial f}{\partial x_0}(p), \ldots, \frac{\partial f}{\partial x_n}(p)\right) = 0\}$

is a hypers. of deg $d-1$

Example:

Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree d and let $L \subseteq \mathbb{P}^3$ be a generic line. How many planes in $\mathbb{P}^3$ containing L are tangent to S?

• The set of hyperplanes in $\mathbb{P}^3$ containing L is a line in $\mathbb{P}^{3*}$; call it $\tilde{L}$.

Among the elts of $\tilde{L}$, how many are tangent to S.

This number is

$$\deg(\underbrace{S^\vee \cap \tilde{L}}_{\text{codim } 1 \,+\, \text{codim } 2 \,=\, \text{codim } 3}) = \deg S^\vee = \underline{d(d-1)^2}.$$

Two factors Segre

Let U, V be vector spaces of dimension $r+1, s+1$ respectively. Let $X = \mathbb{P}U \times \mathbb{P}V \subseteq \mathbb{P}(U \otimes V)$ be the image of the Segre embedding. Then

$$\deg(\mathbb{P}U \times \mathbb{P}V) = \binom{r+s}{s}.$$

$$\begin{aligned} \mathrm{Seg}: \mathbb{P}U \times \mathbb{P}V &\longrightarrow \mathbb{P}(U \otimes V) \\ (u, v) &\longmapsto u \otimes v \quad \leftarrow \text{rank one matrix} \end{aligned}$$

First:

$$CH(\mathbb{P}U \times \mathbb{P}V) = \mathbb{Z}[\alpha, \beta] \big/ (\alpha^{r+1}, \beta^{s+1}).$$

where α, β are defined as follows:

$$\begin{tikzcd} & \mathbb{P}U \times \mathbb{P}V \arrow[dl, "\pi_U"'] \arrow[dr] & \\ \mathbb{P}U & & \mathbb{P}V \end{tikzcd}$$

$H_U \subseteq \mathbb{P}U$ gives $\pi_U^{-1}(H_U) \subseteq \mathbb{P}U \times \mathbb{P}V$

$\alpha = [\pi_U^{-1}(H_U)]$ similar for β

Two factors Segre – cont'd

To compute $\deg X$ where $X = \mathrm{Seg}(\mathbb{P}U \times \mathbb{P}V)$

$\dim X = r+s$

we compute $\#\left(X \cap H_1 \cap \dots \cap H_{r+s}\right)$ H_j generic.

Claim:

If H is a hyperplane in $\mathbb{P}(U \otimes V)$ then

H is rationally equivalent to

$$H \sim \underline{\underline{H_U \otimes V + U \otimes H_V}}$$

where $H_U \subseteq \mathbb{P}U$ hyperp.

$H_V \subseteq \mathbb{P}V$ hyper.

and those of this for are transverse to X.

Two factors Segre – cont'd

$$[Seg^{-1}(H)] = \alpha + \beta$$

So $\# (X \cap H_1 \cap \dots \cap H_{r+s}) =$

$$= \# \left(Seg^{-1}(H_1) \cap \dots \cap Seg^{-1}(H_{r+s}) \right) =$$

$$= \deg \left((\alpha+\beta)^{r+s} \right) = \deg \left(\sum_{j=0}^{r+s} \binom{r+s}{j} \alpha^j \beta^{r+s-j} \right) =$$

$$= \deg \left(\binom{r+s}{r} \alpha^r \beta^s \right) = \binom{r+s}{r}.$$