Geometry of tensor network varieties

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Tensor networks

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- Machine Learning (Linear Networks)
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Algebraic varieties of tensors with rich representation theoretic structure

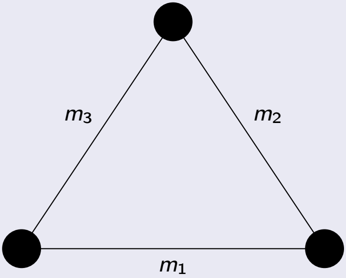
Plan for today

- Define tensor network varieties
- Basic geometric properties
- Dimension
- Equations

Source:

- [Bernardi, De Lazzari, G. - 2021]
- [Christandl, Stilck-França, G., Werner - 2020]

How to build tensor network varieties



Fix a weighted graph $\Gamma = (\mathbf{v}(\Gamma), \mathbf{e}(\Gamma))$ with weights $\mathbf{m} = (m_e : e \in \mathbf{e}(\Gamma))$ called *bond dimensions*. Let $d = \#\mathbf{v}(\Gamma)$.

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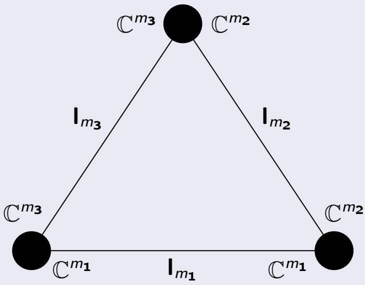

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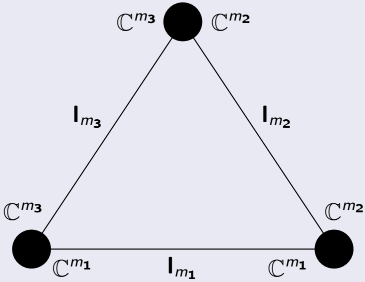

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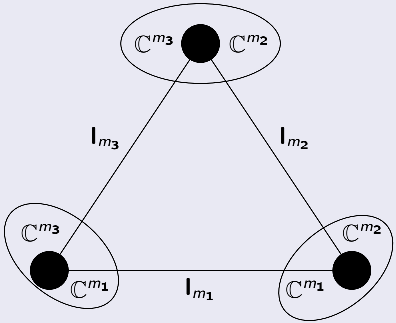

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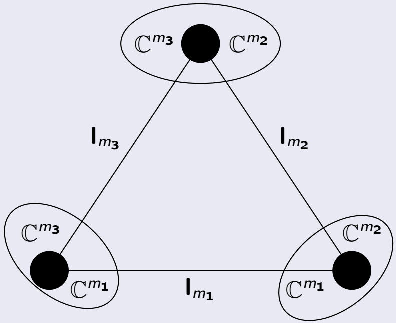

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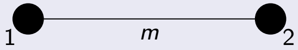

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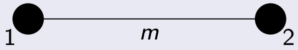

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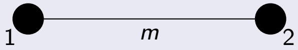

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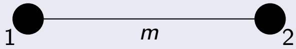

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So

$$\mathcal{TNS}_\Gamma^\circ(m, \mathbf{n}) = \{T \in V_1 \otimes V_2 : \mathrm{rank}(T) \leq m\}$$

In this case $\mathcal{TNS}_\Gamma(m, \mathbf{n}) = \mathcal{TNS}_\Gamma^\circ(m, \mathbf{n})$.

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- Irreducibility and invariancy under rescaling:

  We can think of (the projectivization of) $\mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n})$ as a projective irreducible variety in $\mathbb{P}(V_1 \otimes \cdots \otimes V_d)$.

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- hope to find linear maps $X_1, \ldots, X_d$ such that

$$(X_1 \otimes \cdots \otimes X_d)(T(\Gamma, \mathbf{m})) = T;$$

- use evaluation of $T(\Gamma, m)$ to evaluate T.

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Questions

- What if $T \in \mathcal{TNS}_{\Gamma}(\mathbf{m}, \mathbf{n}) \setminus \mathcal{TNS}^{\circ}_{\Gamma}(\mathbf{m}, \mathbf{n})$?
- How large is $\mathcal{TNS}^{\circ}_{\Gamma}(\mathbf{m}, \mathbf{n})$?
- How to test if $T \in \mathcal{TNS}_{\Gamma}(\mathbf{m}, \mathbf{n})$?

The boundary of $\mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n})$

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Theorem. [Landsberg-Qi-Ye'12]

- If Γ does not have cycles, then $\mathcal{TNS}^\circ_\Gamma(\mathbf{m},\mathbf{n})$ is Zariski closed.
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In [CGSW'20], we determine a new ansatz class with two properties:

- evaluation is *easy*
- the class includes tensors at the boundary

We use the geometry of osculating spaces to $\mathcal{TNS}_\Gamma(\mathbf{m},\mathbf{n})$.

Dimension of $\mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n})$

If $f : X \to Y$ map between varieties, then

$$\dim(\overline{\mathrm{Im}(f)}) = \dim X - \dim f^{-1}(y)$$

for y generic in $\mathrm{Im}(f)$.

Strategy: Study the fibers of

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In [BDG'21]:

- We provide an upper bound for $\dim \mathcal{TNS}_\Gamma(\mathbf{m},\mathbf{n})$.
- The upper bound is *easy to compute* in most cases.
- It is sharp in a certain range.
- It is not sharp in few small cases and we explain why.

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Back to matrices of bounded rank:

$$\Phi : \mathrm{Hom}(\mathbb{C}^m, V_1) \times \mathrm{Hom}(\mathbb{C}^m, V_2) \to V_1 \otimes V_2$$

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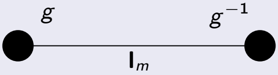

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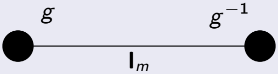

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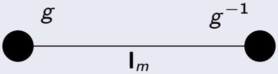

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In general, the fiber containing $\mathbf{X} = (X_v : v \in \mathbf{v}(\Gamma))$ contains its entire $\mathcal{G}_{\Gamma,\mathbf{m}}$-orbit, where

$$\mathcal{G}_{\Gamma,\mathbf{m}} = \bigtimes_{e \in \mathbf{e}(\Gamma)} GL_{m_e} \qquad \textit{gauge subgroup of } \Gamma.$$

Dimension of $\mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n})$ – cont'd

Two issues:

- What is the dimension of the $\mathcal{G}_{\Gamma,\mathbf{m}}$-orbit of a generic element $\mathbf{X}$?
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Theorem [BDG'21]
The bound is sharp if $n_i \geq \prod_{e\ni i} m_e$ for every $i = 1,\dots,d$.

Equations for $\mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n})$

$\mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n})$ is NOT linearly degenerate in $\mathbb{P}(V_1 \otimes \cdots \otimes V_d)$.

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Proof

For matrices: $\Gamma = \bullet\!\overset{m}{\longrightarrow}\!\bullet$. In this case

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In general:

$$\mathcal{TNS}_\Gamma(k \cdot \mathbf{m},\mathbf{n}) \subseteq \sigma_R\left(\mathcal{TNS}_\Gamma(\mathbf{m},\mathbf{n})\right)$$

for some R large enough.

Equations for $\mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n})$ – cont'd

Flattening equations

Fix $I \subseteq \{1, \ldots, d\}$.

If $T \in V_1 \otimes \cdots \otimes V_d$, I defines a *flattening map*

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Easy Fact.
If $T \in \mathcal{TNS}_\Gamma(\mathbf{m},\mathbf{n})$ then $\mathrm{rank}(T_I) \leq w(I)$.

Proof.
True for the graph tensors, and rank of linear maps is semicontinuous.

One small example when Γ is a triangle

- bond dimensions $\mathbf{m} = (m_1, m_2, m_3) = (2, 2, 2)$
- local dimensions $\mathbf{n} = (n_1, n_2, n_3) = (2, 3, 4)$

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In particular $\dim \mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n}) \leq 24 - 2 = 22$ (and equality holds).

In this case the upper bound gives $\dim \mathcal{TNS}_\Gamma(\mathbf{m}, \mathbf{n}) \leq 24$.

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