Varieties of Sums of Powers, Stiefel Manifolds and their degrees

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- Hilbert scheme of points

$$\mathrm{Hilb}_r^{sm}(\mathbb{PC}^{k-1}) = \overline{\{S \subseteq \mathbb{PC}^{k-1} : S \text{ is a set of } r \text{ points}\}}$$

- Variety of sums of powers

$$\mathrm{VSP}_r(f) = \overline{\left\{ \{[\ell_1], \ldots, [\ell_r]\} : [f] \in \langle [\ell_1^d], \ldots, [\ell_r^d] \rangle \right\}} \subseteq \mathrm{Hilb}_r^{sm}(\mathbb{PC}^{k-1})$$

Alternative description of variety of decompositions

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The condition $f = \sum \ell_i^d$ determines a set of $\dim S^d\mathbb{C}^k$ polynomial equations of degree d in the $k \cdot r$ coordinates of $(\mathbb{C}^k)^{\times r}$.

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What can we say about $\mathcal{VSP}_r(f)$? Dimension? Degree?

Quadratic forms

What if $\deg(f) = 2$?

There are normal forms: $f = q_k = x_1^2 + \cdots + x_k^2 = \mathbf{x}^T \cdot \mathbb{I}_k \cdot \mathbf{x}$.

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where $C = [\mathbf{c}_1 | \cdots | \mathbf{c}_r] \in Mat_{k \times r} = (\mathbb{C}^k)^{\times r}$.

So $\{[\ell_1], \ldots, [\ell_k]\} \in \mathcal{VSP}_r(q_k)$ if and only if $CC^T = \mathbb{I}_k$.

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Therefore:

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We compute dimension and degree of $St(k, n)$.

Dimension is easy!

When $n = k$, then

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- this action is transitive on $St(k,n)$;
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So $St(k,n) = SO(n)/SO(n-k)$ is a homogeneous space and

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Degree is harder

k \ n	1	2	3	4	5	6	7	8
1	2	2	2	2	2	2	2	2
2	*	4	8	8	8	8	8	8
3	*	*	16	40	64	64	64	64
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- Green: $n \geq 2k - 1$: proof is geometric.
- Blue: $n \leq 2k - 1$: proof is representation theoretic.
- Easy: first row $St(1, n) = \{\mathbf{c} \in \mathbb{C}^n : \mathbf{c}^T\mathbf{c} = 1\}$ hypersurface of degree 2.

Main Theorem [Brysiewicz-G. 2019]

Let $n > k$ and let $s = \left\lfloor \frac{n}{2} \right\rfloor$.

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$$\deg St(k, n) = 2^{\binom{k+1}{2}}.$$

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If $n \leq 2k - 1$ then

$$\deg St(k, n) = 2^k \cdot L_{n,k}$$

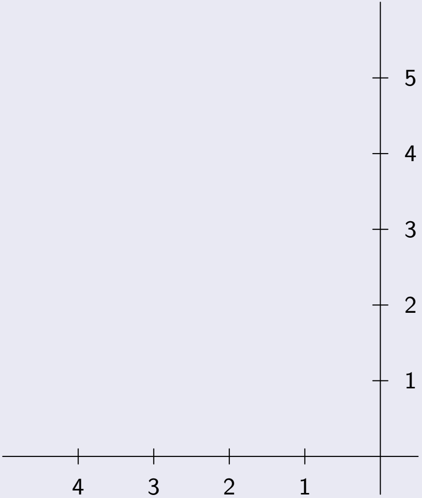
where $L_{n,k}$ is the number of non-intersecting lattice paths configuration from $A = \{(-a_i, 0) : i = 1, \ldots, s\}$ to $B = \{(0, b_i) : i = 1, \ldots, s\}$, defined by

$$(a_1, \ldots, a_s) = (\underbrace{k-1, k-2, \ldots, k-(n-k)}_{n-k}, \underbrace{2k-n-2, 2k-n-4, n-2s}_{s-(n-k)})$$

$$(b_1, \ldots, b_s) = (n-2, n-4, \ldots, n-2s).$$

Non-intersecting lattice paths

- Fix $n = 7$ so $s = \lfloor 7/2 \rfloor = 3$;
- $k = 5$, so $n - k = 2$.



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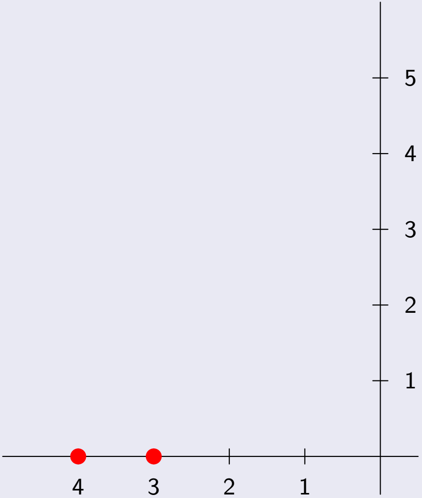

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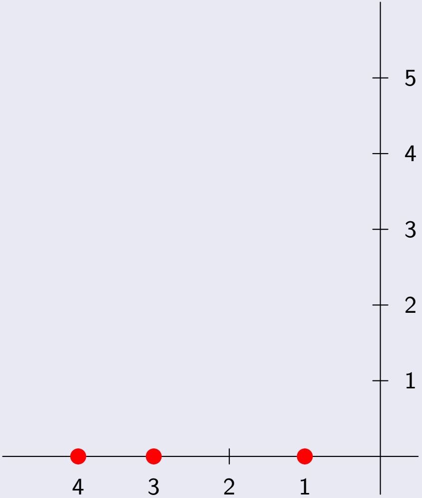

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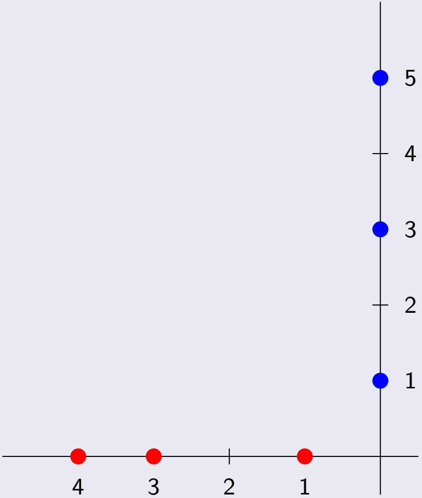

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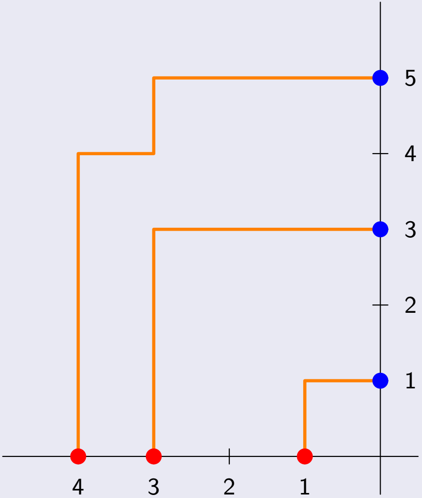

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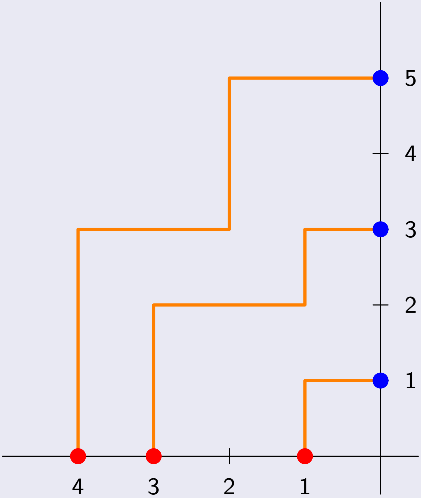

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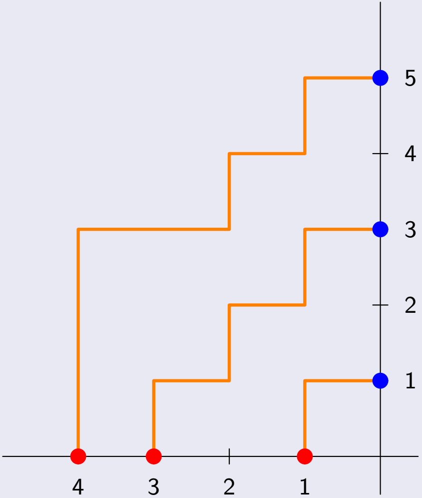

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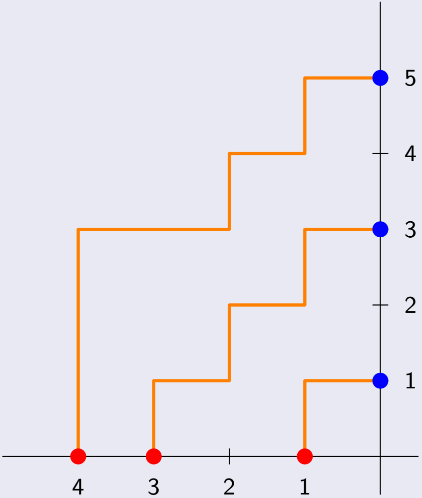

$L_{n,k} = 464$

$$\deg(St(5,7)) = 2^5 \cdot 464 \\ = 14848$$

Degree of affine and projective varieties

Consider $X \subseteq \mathbb{A}^N$ or $X \subseteq \mathbb{P}^N$ of dimension m; let $c = N - m$ be the codimension.

Let L be a generic c-dimensional linear space.

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Fix coordinates $z_0, \ldots, z_N$ on $\mathbb{P}^N$.
Identify $\mathbb{A}^N \subseteq \mathbb{P}^N$ with the affine patch $\{z_0 \neq 0\}$.

If $X \subseteq \mathbb{A}^N$ is an affine variety, let $\overline{X} \subseteq \mathbb{P}^N$ be the closure.
Then $\deg(X) = \deg(\overline{X})$.

Complete Intersections

Suppose $X \subseteq \mathbb{P}^N$ of codimension c.

Special case: the ideal has c generators: $I_X = (g_1, \ldots, g_c) \subseteq \mathbb{C}[z_0, \ldots, z_N]$. Then

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What happens in the affine setting?

Suppose $X \subseteq \mathbb{A}^N$ of codimension c, with $I_X = (g_1, \ldots, g_c) \subseteq \mathbb{C}[z_1, \ldots, z_N]$.

Homogenize the generators, to get a homogeneous ideal in $\mathbb{C}[z_0, \ldots, z_N]$ and let $Y \subseteq \mathbb{P}^N$ be the *set* (it is a projective scheme!) that this cuts out.

Certainly $\overline{X} \subseteq Y$.

If equality holds, then one gets $\deg(X) = \deg(g_1) \cdots \deg(g_c)$.

Complete Intersections: Example

Let $X \subseteq \mathbb{A}^3$ be the curve parametrized by (t, t^2, t^3). Then $I_X = (g_1, g_2)$

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Let $Y = \{\widehat{g}_1 = \widehat{g}_2 = 0\} \subseteq \mathbb{P}^3$. Then

$$Y = \overline{X} \cup L$$

where $L = \{z_0 = z_1 = 0\} \subseteq \mathbb{P}^3$ is a line.

So Y has one component supported at infinity!

These components contribute to $\deg(Y)$ but not to $\deg(\overline{X})$.

Stiefel setting

For $n \geq k$, we have

$$St(k,n) = \{C \in Mat_{k\times n} : CC^T - \mathbb{I}_k = 0\}.$$

- Homogenize: $CC^T - z_0^2\mathbb{I}_k = 0$
- Look at infinity: $CC^T = 0$.

We hope the variety

$$\mathcal{Z}_\infty = \{[C] \in \mathbb{P}Mat_{k\times n} : CC^T = 0\}$$

has dimension strictly smaller than $\dim St(k,n)$.

Stiefel manifold at infinity

Fact:

$$\mathcal{Z}_\infty = \left\{ C \in Mat_{k\times n} : \mathrm{Im}\ C^T \subseteq \{q_n = 0\} \right\}.$$

We realize $\mathcal{Z}_\infty$ as a vector bundle over the *isotropic grassmannian*.

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Recall:

- Ideal of $St(k,n)$ is defined by $\binom{k+1}{2}$ quadratic equations;
- $\operatorname{codim} St(k,n) = \binom{k+1}{2}$.

Stiefel manifold at infinity

Fact:

$$\mathcal{Z}_\infty = \left\{ C \in Mat_{k\times n} : \mathrm{Im}\ C^T \subseteq \{q_n = 0\} \right\}.$$

We realize $\mathcal{Z}_\infty$ as a vector bundle over the *isotropic grassmannian*.

Theorem: [BG'19]
If $n \geq 2k-1$, then $\dim \mathcal{Z}_\infty < \dim St(k,n)$.

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We obtain the green part of the table:

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More on degrees of algebraic varieties

Consider

- $X \subseteq \mathbb{A}^N$ an affine variety of dimension m with ideal I_X;
- $\mathbb{C}[X] = \mathbb{C}[z_1, \ldots, z_N]/I_X$ the affine coordinate ring;
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equivalently

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Algebraic Peter-Weyl Theorem

Intrinsic description for $\mathbb{C}[X]$ when $X = G/H$.

In our case $G = SO(n)$, $H = SO(n-k)$:

$$\mathbb{C}[St(k,n)] = \bigoplus_{\lambda} V_\lambda \otimes {V_\lambda^*}^{SO(n-k)}$$

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What about the grading?

Fact: V_λ occurs in $\mathbb{C}[X]_{\leq t}$ if and only if $|\lambda| := \lambda_1 + \cdots + \lambda_{s-1} + |\lambda_s| \leq t$.

Obtain

$$\deg St(k,n) = m! \lim_{t \to \infty} \frac{\sum_{|\lambda| \leq t} \dim V_\lambda \cdot \dim V_\lambda^{SO(n-k)}}{t^m}$$

Computing invariants

- V_λ representation of $SO(n)$;

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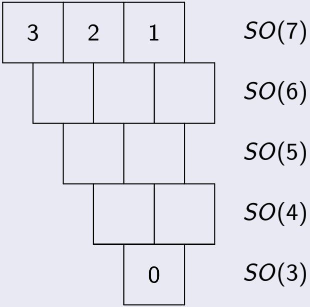
Example: Say $n = 7$ with $\lambda = (3, 2, 1)$; $k = 4$.

Draw the diagram:



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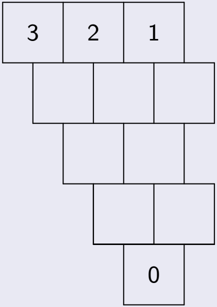

$SO(7)$

$SO(6)$

$SO(5)$

$SO(4)$

$SO(3)$

The number of ways to fill the boxes with “interlacing rule” is the number of invariants.

Computing invariants cont'd

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Gessel-Viennot: Determinants of binomial coefficients count non-intersecting lattice path configurations.

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- What about other notions of rank?