Introduction to Enumerative Geometry

Jan. 11 – Jan. 22, 2021

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Lecture 3: Grassmannians in general

- Finish examples from last time
- Schubert varieties in general
- Ring structure of $CH(G(k, V))$
- Pieri's and Giambelli's formulas
- Start with Chern classes

Variety of secant lines

Let $C \subseteq \mathbb{P}^3$ be a smooth non-degenerate curve of degree d and genus g. Let

$$\mathfrak{s}(C) = \overline{\{\Lambda \in G(2,4) : \mathbb{P}L = \langle p, q\rangle \text{ for some } p, q \in C\}}.$$

What is the class of $\mathfrak{s}(C)$?

last time $\dim s(C) = 2$ $\quad CH^2(G(2,4)) = \langle \sigma_2, \sigma_{1,1} \rangle.$

$[s(C)] = \alpha \sigma_2 + \beta \sigma_{11}$

$\beta = \deg([s(C)] \cdot \sigma_{11}) =$

$= \deg(s(C) \cap \Sigma_{1,1}(F_3)) =$

$= \deg(s(C) \cap \{\Lambda : \Lambda \subseteq F_3\})$

$\Sigma_{11}(F_3) = \{\Lambda : \Lambda \subseteq F_3\}$

Variety of secant lines – cont'd

β is the number of secant lines to C contained in F_3

Note: $C \cap F_3$ is a set of $\deg(C)$ many points

- $\beta = \binom{d}{2}$ because every two points determine a unique line

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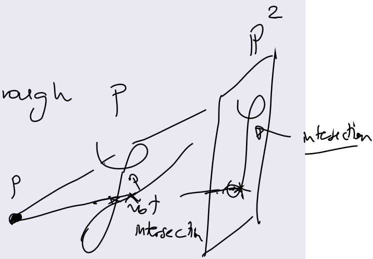
$$\alpha = \deg\left([s(C)] \cdot \sigma_2 \right) =$$

$$= \#\{ s(C) \cap \underbrace{\{ \Lambda : F_1 \subseteq \Lambda \}}_{\Sigma_2(F_1)} \}$$

Variety of secant lines – cont'd

Let $p = \mathbb{P}F_2 \in \mathbb{P}^3$



$\alpha = \#$ secant lines to C through p

$$\pi_p : C \longrightarrow \mathbb{P}^2$$

π_p is generically 1-1

$$\Downarrow$$

$\deg(\pi_p(C)) = \deg(C) \Longrightarrow$ $\pi(C)$ is a plane curve of deg d genus g

$$\alpha = \#\text{ nodes of } \pi(C) = \binom{d-1}{2} - g$$

$$[s(C)] = \left(\binom{d-1}{2} - g\right)\sigma_2 + \binom{d}{2}\sigma_{11}$$

Variety of tangent lines

Let $S \subseteq \mathbb{P}^3$ be a $\overset{\text{smooth}}{}$ surface of degree $d \geq 2$. Let

$$\mathfrak{t}(S) = \{\Lambda \in G(2,4) : \mathbb{P}\Lambda \text{ is tangent to } S\}.$$

What is the class of $\mathfrak{t}(S)$?

$$T = \{(q, \Lambda) \in S \times G(2,4) : \mathbb{P}\Lambda \text{ is tangent at } q\}$$

$$T = \{(q, \Lambda) : \mathbb{P}\Lambda \subseteq T_q S\}$$

$$\begin{tikzcd} & T \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_S"] & \\ \Lambda \quad G(2,4) \arrow[ur, bend left] & & S \subseteq \mathbb{P}^3 \end{tikzcd}$$

$$\pi_S^{-1}(q) = \{(q, \Lambda) : \underbrace{\mathbb{P}\Lambda}_{\text{dim } 1} \subseteq \underbrace{T_q S}_{\text{2 dim}}\} = \mathbb{P}^1 \qquad \dim T = 3$$

Variety of tangent lines – cont'd

$$\pi_G(T) = t(S).$$

Claim: π_G is generically finite

Pf/ Suppose it is not. The generic line tangent to S is tangent along a curve $\to$ this curve has to be the line itself.

This shows that every tangent line to S is contained in S $\Longrightarrow$ S is a plane. but $\deg(S) \geq 2$ $\square$

$\dim(t(S)) = 3$ so $[t(S)] \in CH^1(G(2,4)) = \langle \sigma_1 \rangle$

$[t(S)] = \alpha\, \sigma_1$

$C, S_1, S_2 \subseteq \mathbb{P}^3$ in general position.

How many lines satisfy: $\cdot\, \Lambda$ secant to C
$\cdot\, \Lambda$ tangent to S_1, S_2

$\Lambda \in s(C) \cap t(S_1) \cap t(S_2) \to \deg([s(C)] \cdot [t(S_1)] \cdot$

$(t(S_2 L))$

Variety of tangent lines – cont'd

$$\alpha = \deg\left([t(S)] \cdot \sigma_{21}\right) = \qquad \Sigma_{21} = \{\Lambda : F_1 \subseteq \Lambda \subseteq F_3\}$$

$$= \#\{\Lambda : \mathbb{P}\Lambda \text{ is tangent to } S, \text{ and } \underline{F_1 \subseteq \Lambda \subseteq F_3}\}$$

$\Lambda \subseteq F_3$ and tangent to $S \Longrightarrow \Lambda$ tangent to $C = S \cap F_3$.

smooth plane curve of deg d.

$\mathbb{P}\Lambda$ is a tangent line to C through $\underset{p}{\overset{\shortparallel}{\mathbb{P}F_1}} \in \mathbb{P}F_3 = \mathbb{P}^2$

Dualizing: $\mathbb{P}\Lambda$ is a point in $C^\vee \cap H_p$

$\#$ possible Λ's $= \deg(C^\vee) = d(d-1)$

$[t(S)] = d(d-1) \cdot \sigma_1$.

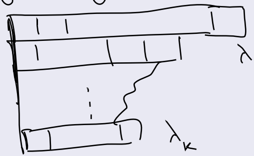
Partitions

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a *partition*. ← non-increasing sequence of non-negative integers

$\lambda_1 \geq \cdots \geq \lambda_k$

$\lambda_j \geq 0 \quad \lambda_j \in \mathbb{N}$

→ Young diagram:



$|\lambda| = \lambda_1 + \ldots + \lambda_k$

Schubert varieties

Let V be a vector space of dimension n and let $k \geq 1$. $G(k,n)$

Let λ be a partition contained in the $k \times (n-k)$ box. $= \underbrace{(n-k, \ldots, n-k)}_{k \text{ times}}$

Fix a complete flag $F_\bullet = (F_1 \subseteq \cdots \subseteq F_n)$ in V.

The Schubert variety associated λ in $G(k, V)$ is

$$\Sigma_\lambda = \{\Lambda \in G(k,V) : \underline{\dim(F_{n-k+i-\lambda_i} \cap \Lambda)} \geq i \text{ for all } i\}$$

$F_1 \subseteq \quad - - - - \quad \subseteq F_{n-1}$

Pick $\Lambda \in G(k,n)$

$(F_1 \cap \Lambda) \subseteq \ldots \subseteq (F_{n-k} \cap \Lambda) \subseteq (F_{n-k+1} \cap \Lambda) \subseteq \ldots (F_{n-1} \cap \Lambda) \subseteq (F_n \cap \Lambda)$

If Λ generic: $\longleftarrow \overset{\shortparallel}{0} \qquad \overset{\shortparallel}{1} \quad \ldots \underset{\substack{i \\ n-k+i}}{\uparrow} \quad \overset{\shortparallel}{k-1} \qquad \overset{\shortparallel}{\Lambda}$

If $\Lambda \in \Sigma_\lambda$

Then intersection of dimension i occurs λ_i steps earlier than generically.

Schubert varieties – cont'd

If $\lambda_i = \lambda_{i+1}$

$$i: \dim\Big(\overbrace{\Lambda \cap F_{n-k+i-\lambda_i}}^{= P_1}\Big) \geq i$$

$$i+1 \quad \dim\Big(\underbrace{\Lambda \cap F_{n-k+(i+1)-\lambda_i}}_{P_2}\Big) \geq i+1 \; . \qquad \Uparrow$$

P_1 is a hyperplane in P_2

Examples

What is Σ_λ if $\lambda = (\lambda_1)$? $\lambda = \underbrace{\boxed{1 \cdots 1}}_{\lambda_1}$

$$\Sigma_{\lambda_1} = \{ \Lambda \in G(k,V) : \underline{\Lambda \cap F_{n-k+1-\lambda_1} \neq 0} \}$$

In particular: if $\lambda_1 = 1$

$$\Sigma_1 = \{ \Lambda \in G(k,V) : \Lambda \cap F_{n-k} \neq 0 \}$$

Chow form of F_{n-k}.

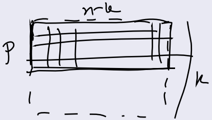
codim 1 condition which is linear in the Plücker embedding

$$F_{n-k} = \langle w_1, \dots, w_{n-k} \rangle \Rightarrow \Sigma_1 = \{ \langle v_1, \dots, v_k \rangle : v_1 \wedge \dots \wedge v_k \wedge w_1 \wedge \dots \wedge w_{n-k} = 0 \}.$$

Examples

What is Σ_λ if $\lambda = (n-k)^p$?

$$(n-k)^p = \underbrace{(n-k, \dots, n-k)}_{p \text{ times}}$$



$$\Sigma_\lambda = \{\Lambda : \Lambda \cap F_{\cancel{n-k}+p-\cancel{(n-k)}} \geq p\} =$$

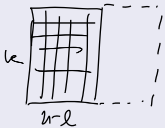
$$= \{\Lambda : \Lambda \cap F_p \geq p\} = \{\Lambda : F_p \subseteq \Lambda\}$$

$$= G(k-p, V/F_p).$$

$$\operatorname{codim}_{G(k,V)} \Sigma_\lambda = k(n-k) - (k-p)\cdot((n-p)-(k-p)) = p\cdot(n-k)$$

Examples

What is Σ_λ if $\lambda = (n-\ell)^k$?



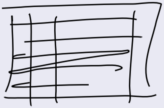
$$\Sigma_\lambda = \{ \Lambda : \Lambda \cap \underbrace{F_{\cancel{n}-\cancel{k}+\cancel{k}-(\cancel{n}-\ell)}}_{= F_\ell} \geq k \} =$$

$$= \{ \Lambda : \Lambda \subseteq F_\ell \} = G(k, F_\ell)$$

$$\operatorname{codim} \Sigma_\lambda = k(n-k) - k\cdot(\ell-k) = k(n-\ell)$$

Examples

What is Σ_λ if $\lambda = (n-k)^k$?



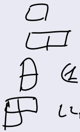
$$\Sigma_\lambda = \{ \Lambda : F_k \le \Lambda \le F_k \} = \{ F_k \}$$ a point

$$\operatorname{codim}_{G(k,V)} \Sigma_\lambda = |\lambda| \leftarrow$$

$\Sigma_{(a,b)}$

The two indices of $G(2,4)$ are Young diagrams in

$(2,2)$:



1

2

$(1,2)$

$(2,1)$.

Containment of Schubert varities

Write $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$.

Write $\lambda > \mu$ if $\lambda \geq \mu$ and $\lambda_i > \mu_i$ for at least one i.

If $\mu \geq \lambda$ then

$$\Sigma_\mu \subseteq \Sigma_\lambda.$$

"The larger the partition, the earlier the "non generic intersections occur""

Affine stratification

Define the Schubert cells

$$\Sigma_\lambda^\circ = \Sigma_\lambda \setminus \bigcup_{\mu > \lambda} \Sigma_\mu .$$

all $\Sigma_\mu \subsetneq \Sigma_\lambda$.

Claim: The Schubert cells form a stratification of the Grassmannian.

Clear: $\Sigma_\lambda \setminus \Sigma_\lambda^\circ$ is union of Schubert cells

To prove: the cells are disjoint.

- $\Sigma_\lambda \cap \Sigma_\mu = \Sigma_\pi$ $\qquad \pi = (\pi_1, \dots, \pi_k)$, $\quad \pi_j = \max(\lambda_j, \mu_j)$

Affine stratification – cont'd

Theorem

Σ_λ° is isomorphic to the affine space $\mathbb{A}^{k(n-k)-|\lambda|}$.

Idea of proof:

Choose $F_1 \subseteq \ldots \subseteq F_n$

$F_1 = \langle e_1 \rangle$, $F_2 = \langle e_1, e_2 \rangle$ ...

and write down the conditions "explicitly"

- Collect the basis vectors of Λ in a matrix
- column-reduce and check that $k(n-k)-|\lambda|$ parameters give Λ uniquely. $\square$

Affine stratification – cont'd

Theorem

Σ_λ° is isomorphic to the affine space $\mathbb{A}^{k(n-k)-|\lambda|}$.

In particular:

- Σ_λ is irreducible, (closure of $\mathbb{A}^{k(n-k)-|\lambda|}$).
- $\underline{\operatorname{codim}_{G(k,V)} \Sigma_\lambda = |\lambda|}$,
- The stratification by Schubert cells is affine.

Affine stratification – cont'd

Theorem

Σ_λ° is isomorphic to the affine space $\mathbb{A}^{k(n-k)-|\lambda|}$.

In particular:

- Σ_λ is irreducible,
- $\operatorname{codim}_{G(k,V)} \Sigma_\lambda = |\lambda|$,
- The stratification by Schubert cells is affine.

So the classes of the Schubert varieties in $CH(G(k,V))$ generate $CH(G(k,V))$ as an abelian group.

$$CH(G(k,V)) = \bigoplus_{d\geq 0} CH^d(G(k,V))$$

$$CH^d(G(k,V)) = \langle \sigma_\lambda : |\lambda| = d \rangle \qquad \sigma_\lambda := [\Sigma_\lambda]$$

Tangent space to Schubert varieties

Recall: If $\Lambda \in G(k, V)$ then

$$T_\Lambda G(k, V) \simeq \mathrm{Hom}(\Lambda, V/\Lambda). \simeq \Lambda^* \otimes V/\Lambda$$

What is the tangent space to the Schubert variety Σ_λ?

$$T_\Lambda \Sigma_\lambda \subseteq \mathrm{Hom}(\Lambda, V/\Lambda).$$

Tangent space to Schubert varieties

Recall: If $\Lambda \in G(k, V)$ then

$$T_\Lambda G(k, V) \simeq \mathrm{Hom}(\Lambda, V/\Lambda).$$

What is the tangent space to the Schubert variety Σ_λ?

It is the space of linear maps $\varphi : \Lambda \to V/\Lambda$ which "respect the flag structure":

$$\underline{F_{n-k+i-\lambda_i} \cap \Lambda} \text{ is mapped to } \underline{(F_{n-k+i-\lambda_i} + \Lambda)/\Lambda}.$$

Using this one can show that
Schubert varieties defined by transverse flags
intersect generically transversely.

Compatibility of Schubert varieties

Let $W \subseteq V$ be a subspace with $W \cap F_1 = 0$. $\operatorname{codim}_V W = 1$

Let $F_\bullet^W$ be flag in W induced by $F_\bullet$: $F_i^W = F_{i+1} \cap W$.

There are natural maps

$$i : G(k, F_{n-1}) \to G(k, V)$$

$$\overline{\Lambda} \longmapsto \Lambda$$

$$j : G(k-1, W) \to G(k, V)$$

$$\Lambda \longmapsto \Lambda + F_1$$

satisfying

$$i^{-1}(\Sigma_\lambda(F_\bullet)) = \Sigma_\lambda(F_\bullet)$$

$$j^{-1}(\Sigma_\lambda(F_\bullet)) = \Sigma_\lambda(F_\bullet^W)$$

in particular codimensions are the same.

Schubert varieties work well with pull-back

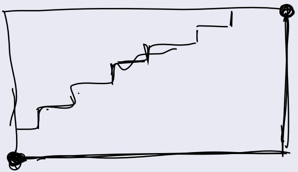
The Chow ring of $G(k, V)$

Let $\sigma_\lambda = [\Sigma_\lambda] \in CH(G(k, V))$.

How many Schubert classes are there?

What is the rank of $CH(G(k,V))$ as an abelian gp.

As many as Young diagrams in the $k \times (n-k)$ box



$\Updownarrow$

paths from South-West corner to the N-E corner of the box

$\Updownarrow$

paths of length $k+n-k=n$ with $n-k$ horizontal step, k vertical

$\Updownarrow$

$\binom{n}{k}$ many such paths

Ring structure of $CH(G(k, V))$

Let λ, μ be partitions in the $k \times (n-k)$ box.

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\pi c^\pi_{\lambda,\mu} \sigma_\pi .$$

range: π : $|\pi| = |\lambda| + |\mu|$ by the grading.

$c^\pi_{\lambda,\mu}$ = Littlewood-Richardson coefficient (good algorithms to compute them).

Special cases 1

We have

$$\sigma_{1^k}^{n-k} = \sigma_{n-k}^{k} = \cancel{\sigma_{(n-k)^{n-k}}} \ \sigma_{(n-k)^k}$$

Let's do $\sigma_{(n-k)}^{k} = m \cdot \sigma_{(n-k)^k}$.

$$\sigma_{n-k} = \left[\Sigma_{n-k}(F_1) \right] = \left[\{ \Lambda : F_1 \le \Lambda \} \right].$$

Pick generic $F_1^{(1)}, \dots, F_1^{(k)}$ then:

$$\sigma_{n-k}^{k} = \left[\Sigma_{n-k}(F_1^{(1)}) \cap \dots \cap \Sigma_{n-k}(F_1^{(k)}) \right] =$$

$$= \left[\{ \Lambda : F_1^{(1)} + \dots + F_1^{(k)} \le \Lambda \} \right] = \left[\{ \Lambda = F_1^{(1)} + \dots + F_1^{(k)} \} \right] =$$

$$= \sigma_{(n-k)^k}.$$

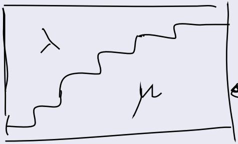
Special cases 1 – cont'd

$\sigma^{n-k}_{(1^k)}$: similar argument with the condition of being contained in a hyperplane.

Let λ, μ such that $|\lambda| + |\mu| = k(n-k)$. Then

$$c_{\lambda,\mu}^{(n-k)\times k} = \begin{cases} 1 & \text{if } \lambda, \mu \text{ are complementary in } (n-k)\times k \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma_\lambda \cdot \sigma_\mu = m \cdot \sigma_{(n-k)^k}.$$



← complementary

← i

$$\boxed{\lambda, \mu \text{ complementary iff } \quad \mu_i = n-k-\lambda_{k-i+1}}$$

Pf/ Consider transverse flags $F_\bullet$, $E_\bullet$

And compute $\#\left(\Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(E_\bullet)\right)$

Special cases 2 – cont'd

Consider condition i for Σ_λ

$k-i+1$ for Σ_μ

$$\dim(\underbrace{\Lambda \cap F_{n-k+i-\lambda_i}}) \geq i$$

$$\dim(\overbrace{\Lambda \cap \underbrace{E_{n-k+(k-i+1)-\mu_{k-i+1}}}_{= \, E_{n-i+1-\mu_{k-i+1}}}}) \geq k-i+1$$

these are two subspaces of Λ of dim i, $k-i+1$

So they intersect;

So $\underbrace{F_{n-k+i-\lambda_i} \cap E_{n-i+1-\mu_{k-i+1}}}\neq 0$

Special cases 2 – cont'd

$$n+1 \le n-k+i-\lambda_i + n-i+1-\mu_{k-i+1} =$$

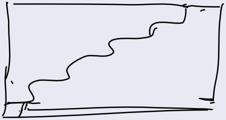
$$= 2n-k-\lambda_i-\mu_{k-i+1}+1$$

$$\lambda_i+\mu_{k-i+1} \le n-k$$

Together λ, μ reach $k\cdot(n-k)$

So for every i $\lambda_i+\mu_{k-i+1} = n-k$

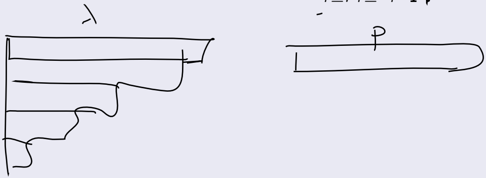
So if $\Lambda \in \Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(E_\bullet)$ exist, λ, μ are complementary



Pieri's formula

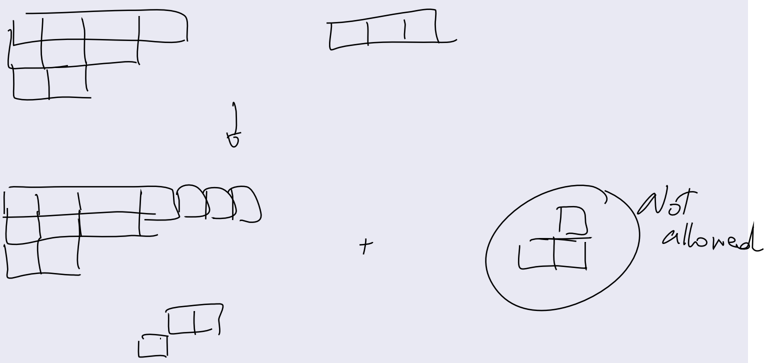
Let λ be a partition and let $p \leq n-k$.

$$\sigma_\lambda \sigma_p = \sum_{\substack{|\mu|=|\lambda|+p \\ \lambda_i \leq \mu_i \leq \lambda_{i-1}}} \sigma_\mu$$



The summation ranges over all μ that can be obtained from λ adding p boxes with the property that no two of them are in the same column

Pieri's formula – cont'd



Pieri's formula – cont'd

Giambelli's formula

Let λ be a partition in the $k \times (n-k)$ rectangle. Then

$\sigma_{\lambda_1} = [\Sigma_{\lambda_1}]$

$$\sigma_\lambda = \det \begin{pmatrix} \sigma_{\lambda_1} & \underbrace{\sigma_{\lambda_1+1}} & \cdots & \sigma_{\lambda_1+k-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \cdots & \sigma_{\lambda_2+k-2} \\ \vdots & & \ddots & \\ \sigma_{\lambda_k-k+1} & & & \sigma_{\lambda_k} \end{pmatrix}.$$

$\lambda_1 = \boxed{|||}$

$\sigma_\lambda \cdot \sigma_\mu = \det[\quad] \cdot \sigma_\mu =$

$=$ determinant expansion $\cdot\, \sigma_\mu$

but this can be done with Pieri.

Sections of vector bundles

Let $\mathcal{F}$ be a vector bundle on X.

Sections of vector bundles – cont'd