Introduction to Enumerative Geometry

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Max-Planck-Institut für Mathematik in den Naturwissenschaften

Lecture 5: Determinantal varieties

- The variety of matrices of bounded rank
- Determinantal varieties as degeneracy loci
- Other examples of determinantal varieties
- Introduction to Porteous's formula

Determinantal varieties

Let X be an algebraic variety in $\mathbb{P}V$. Coords $x_0, \dots, x_n$ on $\mathbb{P}V$

We say that X is *determinantal* if there exists a matrix M whose entries are linear forms on V and an integer r such that

$$I_X \text{ is the ideal of } r \times r \text{ minors of } M.$$

$$M = \begin{pmatrix} l_{11}(\underline{x}) & & l_{1e}(\underline{x}) \\ \vdots & & \vdots \\ & \cdots & \end{pmatrix} e$$

$$f$$

Determinantal varieties

Let X be an algebraic variety in $\mathbb{P}V$.

We say that X is *determinantal* if there exists a matrix M whose entries are linear forms on V and an integer r such that

$$I_X \text{ is the ideal of } r \times r \text{ minors of } M.$$

Other definition in the literature:

$$X \text{ is cut out scheme-theoretically by the } r \times r \text{ minors of } M.$$

If M_r is the ideal generated by $r \times r$ minors

then $(I_X)_D = (M_r)_D$ for $D \gg 0$

The general determinantal variety

Let $V = \underbrace{\mathrm{Mat}_{e\times f}}$. And define

$$X_r = \{M \in \mathbb{P}V : \mathrm{rank}(M) \le r\}.$$

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$$X_r = \{M \in \mathbb{P}V : \mathrm{rank}(M) \leq r\}.$$

Claim:

X_r is an algebraic variety

X_r is the zero set of $(r+1) \times (r+1)$ minors

so it is Zariski closed.

linear space

Every determinantal variety is $X_r \cap L$ for some r, e, f.

The general determinantal variety

Let $V = \mathrm{Mat}_{e \times f}$. And define

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Claim:

X_r is an algebraic variety

Fact:
The $(r+1) \times (r+1)$ minors generate the ideal I_{X_r}.

Second fundamental theorem of invariant theory

The general determinantal variety – cont'd

Recall the Segre embedding:

$$\begin{aligned} Seg : \mathbb{P}E \times \mathbb{P}F^* &\to \mathbb{P}(E \otimes F^*) \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

E column vectors

F^* row vectors

col x row multiplication giving a rank one matrix.

Regard $E \otimes F^* = \mathrm{Hom}(F, E)$ as $\mathrm{Mat}_{e \times f}$.

The general determinantal variety – cont'd

Then $Seg(\mathbb{P}E \times \mathbb{P}F^*) = X_1$.

Exercise: A matrix M has rank at most r if and only if it can be written as a sum of r matrices of rank one.

[SVD decomposition]

The general determinantal variety – cont'd

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Exercise: A matrix M has rank at most r if and only if it can be written as a sum of r matrices of rank one.

is Zariski closed.

Therefore:

$$X_r = \left\{ M : \begin{array}{l} \text{there exist } Z_1, \ldots, Z_r \in X_1 \\ \text{such that } M \in \langle Z_1, \ldots, Z_r \rangle \end{array} \right\}$$

A short detour on secant varieties

Let $X \subseteq \mathbb{P}V$. The r-th secant variety of X is

closure is needed

$$\sigma_r(X) = \overline{\left\{ p \in \mathbb{P}V : \begin{array}{l} \text{there exist } z_1, \ldots, z_r \in X \\ \text{such that } p \in \underbrace{\langle z_1, \ldots, z_r \rangle}_{\to\ r\text{-secant plane to } X} \end{array} . \right\}}$$

What is ~~dim X?~~ $\dim \sigma_r(X)$?

$$\dim \sigma_r(X) \leq r \cdot \dim X + r - 1 = r(\dim X + 1) - 1$$

We say that $\sigma_r(X)$ has the expected dimension (as a secant variety) if equality holds.

A short detour on secant varieties – cont'd

If $X \subseteq \mathbb{P}V$ is determinental, given by the $(r+1)\times(r+1)$ minors of a matrix M of size $e \times f$

we say that X has the expected dim as a determinantal variety is

$$\operatorname{codim}_{\mathbb{P}V}(X) = (e-r)(f-r)$$

Dimension of X_r

Easy cases:

$e \leq f$

- $r = 1$.

$$X_1 = \mathbb{P}E \times \mathbb{P}F^*$$

$$\dim X_1 = e-1+f-1 = e+f-2$$

$$\dim \sigma_r(X_1) \leq \underbrace{r(e+f-1)-1}$$

If $\underline{r \sim e}$ then this number is larger than $ef-1 = \dim \mathrm{Mat}_{e \times f}$

So X_r does not have the expected dimension as a secant variety

Dimension of X_r – cont'd

Easy cases:

- $e = f$ and $r = e - 1$:

$$X_r = \{ M : M \text{ is singular} \} =$$

$$= \{ M : \det(M) = 0 \}$$

$$\operatorname{codim} X_r = 1$$

Dimension of X_r – cont'd

What happens in general?

Thm: $\operatorname{codim}_{\mathbb{P}(E\otimes F^*)} X_r = \underbrace{(e-r)(f-r)}_{\text{if } \operatorname{rk}(M)=r}$ (*)

$\pi_2^{-1}(M) = \{(M, \operatorname{Im} M)\}$

Pf/ Incidence correspondence:

$$\tilde{I} = \{ (M, L) \in \mathbb{P}(E\otimes F^*) \times G(r, E) : \operatorname{Im}(M) \subseteq L \}$$

$M : F \to E$

$$\begin{tikzcd} & \tilde{I} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \underline{\mathbb{P}(E\otimes F^*)} & & G(r, E) \end{tikzcd}$$

$\dim \tilde{I} = \dim X_r$
$= r(e+f-r)-1$

$\operatorname{codim} X_r = (ef-1) - \dim X_r$
$= (*)$

Claim 1: $\operatorname{Im}(\pi_1) = X_r$

$\operatorname{rk}(M) \leq r$ iff $\dim(\operatorname{Im}(M)) \leq r$ iff $\exists\, L \in G(r, E)$ such that $\operatorname{Im}(M) \subseteq L$

Claim 2

All fibers of π_2 are linear spaces of dim $= f\cdot r - 1$ (projective)

If $L \leq G(r,E)$ what is $\pi_2^{-1}(L)$?

$\pi_2^{-1}(L) = \{M : \operatorname{Im}(M) \subseteq L\}$

Suppose: $L = \langle \underline{e}_1, \dots, \underline{e}_r \rangle$

$$M = \begin{array}{c} \left[\begin{array}{c} * \\ \hline 0 \end{array}\right] \begin{array}{l} r \\ e-r \end{array} \\ f \end{array}$$

$\pi_2^{-1}(L) \simeq \mathbb{P}\mathbb{C}^{f\cdot r} \Longrightarrow$ dim $= fr - 1$.

$\dim I = \dim \underbrace{G(r,E)} + \dim \pi_2^{-1}(L) = r(e-r) + fr - 1$

And I irreducible $\Longrightarrow X_r$ is irreducible

Claim 3: π_1 generically 1-1 because the generic elt of X_r has rk exactly r

Determinantal varieties as degeneracy loci

Let X be a (smooth) algebraic variety. Let $\mathcal{E}, \mathcal{F}$ be vector bundles on X of rank e, f respectively; let $\varphi : \mathcal{F} \to \mathcal{E}$ be a bundle map.

$\varphi \in H^0(Hom(\mathcal{F}, \mathcal{E}))$

$= \mathcal{F}^* \otimes \mathcal{E}$.

The r-th degeneracy locus of φ is

$$\underbrace{D_r^{\varphi}(\mathcal{F}, \mathcal{E}) = \{x \in X : \mathrm{rank}(\varphi_x) \le r\}}_{\varphi_x : \mathcal{F}_x \to \mathcal{E}_x}.$$

$\psi : F \longrightarrow E$

For every k, ψ induces a map

$$\psi^{\wedge k} : \Lambda^k F \longrightarrow \Lambda^k E$$

$$v_1 \wedge \dots \wedge v_k \longmapsto \psi(v_1) \wedge \dots \wedge \psi(v_k)$$

and extended linearly

if $\mathrm{rk}\,\psi < k$ this is 0

Determinantal varieties as degeneracy loci – cont'd

Fact: $\operatorname{rk}(\psi) = \max\{k : \psi^{\wedge k} \text{ is not identically zero}\}$

$\varphi^{\wedge k} \in H^0(\underbrace{\mathcal{H}om(\Lambda^k \mathcal{F}, \Lambda^k \mathcal{E})}_{\mathcal{H}})$.

and

$$D_r^{\varphi} = \{x : \varphi_x^{\wedge r} = 0\}$$

This is the vanishing locus of a section of $\mathcal{H}$.

The general determinantal variety as a degeneracy locus

What are $\mathcal{E}, \mathcal{F}$ and φ in the case of the general determinantal variety?

- $X = \mathbb{P}\mathrm{Mat}_{\underbrace{e \times f}} = \mathbb{P}(\underbrace{E \otimes F^*}_{= V})$

- $\mathcal{F} = \underline{F}_X$

- $\mathcal{E} = \underline{E}_X \otimes \mathcal{O}(1)$ $\qquad H^0(\mathcal{O}(1)) = V^*$

- $\varphi : \mathcal{F} \longrightarrow \mathcal{E}$

if $M \in \mathbb{P}(E \otimes F^*)$ $\quad \underbrace{\varphi_M} : F \longrightarrow \overbrace{E \otimes \langle M \rangle^*}^{\cong \mathrm{Hom}(\langle M \rangle, E)}$

$\underline{v} \longmapsto \left[\varphi(v) : \langle M \rangle \longrightarrow E, \; \lambda M \longmapsto \lambda M v \right]$

$\in E$

$$D_r^{\varphi} = X_r \quad \text{because}$$

$$\{M : \mathrm{rk}(\varphi_M) \le r\} = \{M : \mathrm{rk}(M) \le r\}$$

The rational normal curve as a determinantal variety

Let C_d be the rational normal curve in $\mathbb{P}^d$. $V = \langle x_0, x_1 \rangle$

Regard $C_d = \underline{\nu_d(\mathbb{P}V)} \subseteq \mathbb{P}S^d V$ for a vector space V with $\underline{\dim V = 2}$.

$$\begin{array}{rcll} \nu_d : \mathbb{P}V & \longrightarrow & \mathbb{P}S^d V & \longrightarrow \text{ hom. polys of deg } d \\ \ell & \longmapsto & \ell^d & \qquad \text{in } x_0, x_1 \end{array}$$

Use coords $z_0, \dots, z_d$ on $\mathbb{P}S^d V$

z_j is the coord of $\binom{d}{j} x_0^j x_1^{d-j}$ in the expression of $f \in S^d V$.

Claim:

$C_d = \nu_d(\mathbb{P}V)$ is the zero set of the 2×2 minors of:

$$(*)\quad \begin{bmatrix} z_0 & \cdots & z_{d-1} \\ z_1 & \cdots & z_d \end{bmatrix}$$

$$\alpha_i(x_j) = \delta_{ij}$$

Given $f = \sum_{0}^{d} z_j \boxed{\binom{d}{j} x_0^j x_1^{d-j}}$ define: basis elts

cat_1 : catalecticant

$$\mathrm{cat}_1 : V^* \longrightarrow S^{d-1} V$$

$$\alpha_0 \longmapsto \frac{\partial}{\partial x_0} f$$

$$\alpha_1 \longmapsto \frac{\partial}{\partial x_1} f$$

- The representing matrix of cat_1 is $(*)$
- cat_1 has rank 1 iff $f = l^d$ for some l.

The rational normal curve as a determinantal variety – cont'd

- $\sigma_r(C_d)$ the r-th secant variety of C_d.

Fact: $\dim \sigma_r(C) = 2r-1 = r\cdot(\dim C + 1) - 1$ which is the expected dimension as a secant variety.

Fact: $\sigma_r(C)$ is determinantal:

$$(**)\quad \begin{array}{cc} & d-r+1 \\ r+1 & \begin{bmatrix} z_0 & z_1 & \cdots & z_{d-r} \\ \vdots & & & \vdots \\ z_r & \cdots & \cdots & z_d \end{bmatrix} \end{array}$$

$$I_{\sigma_r(C)} = \left((r+1)\times(r+1) \text{ minors of } (**) \right)$$

~~and this is the matrix representing~~

The rational normal curve as a determinantal variety – cont'd

And this is the matrix representing

$\rightarrow \mathrm{cat}_r : S^r V^* \longrightarrow S^{d-r} V$

which applies order r partials.

exp codim (as a det. variety) of $\sigma_r(C) = \underbrace{(r+1-r)}_{1}(d-r+1-r) =$

$= d-2r+1 = d-(2r-1)$

which is codim $\sigma_r(C)$.

The class of a determinantal variety

How can we determine the class of a determinantal variety in $CH(X)$?

Wrong idea:

$$\varphi : \mathcal{F} \longrightarrow \mathcal{E}$$

$$D_r^{\varphi} = \{ x : \varphi_x^{\wedge(r+1)} \equiv 0 \}$$

We would like to use Chern classes to compute $[D_r^{\varphi}]$

The class of a determinantal variety – cont'd

Chern classes are defined via generic sections.

$\varphi^{\wedge(r+1)}$ is <u>not</u> generic.

<u>So this does not work</u>

A generic section of $\mathcal{H}om\left(\Lambda^{r+1}\mathcal{F}, \Lambda^{r+1}\mathcal{E}\right)$ will not vanish in "the right codimension"

Toward Porteous's formula

Porteous's formula expresses $[D_r^\varphi]$ in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{F}$, under the assumption that codim $D_r^\varphi = (e-r)(f-r)$.

→ general determinantal variety

→ secants of RNC

Toward Porteous's formula

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Historical Deteour

→ 1904 : Giambelli proved when $\mathcal{E}, \mathcal{F}$ split as sums of line bundles

→ 50–60 Chern introduces Chern classes

→ 60s Thom: $[D_r^\varphi]$ only depend on the Chern classes of $\mathcal{E}, \mathcal{F}$.

1971 : Porteous gives the formula we study

Lascov-Kempf Pragatz : generalizations to more general fiber bundles

Resultants

Let $\mathbf{c} = (c_0, c_1, \cdots)$ be a sequence of elements in a commutative ring. For integers e, f, define the element

$e \leq f$

$$\Delta_f^e(\mathbf{c}) = \det \begin{array}{cc} & e \\ \begin{bmatrix} c_f & c_{f+1} & \cdots & c_{f+e-1} \\ c_{f-1} & c_f & & \vdots \\ \vdots & & \ddots & c_{f+1} \\ c_{f-e+1} & \cdots & c_{f-1} & c_f \end{bmatrix} & e \end{array}$$

this is the *Sylvester determinant* of $\mathbf{c}$ of order f and degree e.

Resultants

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this is the *Sylvester determinant* of $\mathbf{c}$ of order f and degree e.

Let $a(t) = \sum_0^e a_i t^i$ and $b(t) = \sum_0^f b_j t^j$ be two polynomials of degree e and f respectively. Let $\alpha_1, \ldots, \alpha_e$ be the roots of a and $\beta_1, \ldots, \beta_f$ the roots of b.

The resultant of a and b is

$$\mathrm{Res}_t(a, b) = \prod_{\substack{i=1,\ldots,e \\ j=1,\ldots,f}} (\alpha_i - \beta_j).$$

vanishes if a, b have common roots

$a(t) = a_e \prod (t - \alpha_i)$

$b(t) = b_f \prod (t - \beta_j)$

Res is a polynomial in the $a_0 \ldots a_e, b_0, \ldots, b_f$

Toward Porteous's formula – cont'd

Suppose $a(0) \neq 0$, so that $c(t) = \dfrac{b(t)}{a(t)}$ is well defined in 0.

$\hookrightarrow$ $a(0)$ is invertible in the ring

Toward Porteous's formula – cont'd

Suppose $a(0) \neq 0$, so that $c(t) = \dfrac{b(t)}{a(t)}$ is well defined in 0.

Claim:

$$\mathrm{Res}_t(a, b) = \underbrace{\Delta_f^e(\mathbf{c})}_{}$$

where $\mathbf{c}$ is the sequence of the coefficients of $c(t)$.

Toward Porteous's formula – cont'd

Theorem

Let $\varphi : \mathcal{F} \to \mathcal{E}$ be a morphism of vector bundles on a smooth variety X.

Suppose codim $D_r^\varphi = (e-r)(f-r)$. Then

$$[D_r^\varphi] = \Delta^{e-r}_{f-r}\left(\frac{c_{[t]}(\mathcal{E})}{c_{[t]}(\mathcal{F})}\right)$$

↳ ratio of the Chern polynomials

What is $\dfrac{1}{a(t)} = \sum_0 p_j t^j$

The condition $1 = \boxed{a(t) \cdot \dfrac{1}{a(t)}}$ tell us: $\cdot\ p_0 = a_0^{-1}$

$\cdot\ a_0 p_1 + a_1 p_0 \to$ get p_1

$\vdots$