PROBLEM SETS FOR INTRODUCTION TO ENUMERATIVE COMBINATORICS

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Abstract. This document collects a few problems which should be useful to practice on the material that is covered during the lectures.

Lecture 1

Problem 1. Let V be a vector space of dimension $n+1$ and let $\nu_{d,n} : \mathbb{P}V \to \mathbb{P}S^dV$ be the Veronese embedding. Show that if $X \subseteq \mathbb{P}V$ is a variety of dimension k and degree e, then $\nu_{d,n}(X)$ has degree $d^k e$.

In particular, the degree of a k-dimensional subvariety of $\nu_{d,n}(\mathbb{P}V)$ is a multiple of d^k.

Problem 2. Let

$$\Psi = \{(p,q,r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : p, r, s \text{ are collinear}\}.$$

Show that Ψ is a subvariety of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ of codimension $n-1$.

Determine the class $[\Psi]$ in the Chow ring $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$.

Note: The Chow ring of a product of several projective spaces is what one expects. If $\dim V_j = n_j + 1$, then

$$\mathrm{CH}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_s) = \mathbb{Z}[\alpha_1, \dots, \alpha_s]/(\alpha_1^{n_1+1}, \dots, \alpha_s^{n_s+1})$$

where α_j is identified with the class of the pull back of the hyperplane section in $\mathbb{P}V_j$ via the projection map. In other words

$$\alpha_j = [\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{j-1} \times H_j \times \mathbb{P}V_{j+1} \times \cdots \times \mathbb{P}V_s].$$

Problem 3. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables. Let

$$\mathcal{T} = \{f : f = \ell_1\ell_2\ell_3 \text{ for some linear forms } \ell_j\},$$

that is the space of *triangles* (i.e., cubic curves which are union of three lines).

Determine the dimension and the degree of $\mathcal{T}$.

Hint: Write $\mathcal{T}$ is the image of a map defined on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Problem 4. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{A} = \{f : f = \ell_1\ell_2\ell_3 \text{ for some linear forms } \ell_j \text{ with a common zero}\}\},$$

that is the space of *asterisks* (i.e., cubic curves which are union of three lines passing through the same point).

Determine the dimension and the degree of $\mathcal{A}$.

Hint: It is similar to the previous problem, but one needs something more complicated than $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Problem 5. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{C} = \{f = \ell_1^2\ell_2 : \ell_j \text{ is a linear form}\},$$

that is the space of cubic curves which are union of a double line and a line.

Determine the dimension and the degree of $\mathcal{C}$.

Lecture 2

Problem 6. Let $C_1, C_2 \subseteq \mathbb{P}^3$ be two curves of degree d_1, d_2 respectively and genera g_1, g_2 respectively. Suppose C_1, C_2 are in general position with respect to each other. How many lines are secant both two C_1 and C_2?

Problem 7. Let C be a smooth non-degenerate curve in $\mathbb{P}^3$ of degree d and genus g. Let

$$TC = \{\Lambda \in G(2,4) : \Lambda = T_pC \text{ for some } p \in C\}.$$

What is the class of TC in $G(2,4)$?

Problem 8. Let $S_1, \dots, S_4 \subseteq \mathbb{P}^3$ be four surfaces with $\deg(S_i) = d_i$ in general position. How many lines are tangent to all of them?

Problem 9. Let C be a smooth non-degenerate curve of degree d and genus g in $\mathbb{P}^3$. Let S be a smooth surface of degree e. Suppose C and S are in general position with respect to each other. How many lines are tangent to both S and C?

Problem 10. Let $X \subseteq G(2,4)$ be an irreducible variety of codimension 2. Then $[X] = \alpha\sigma_2 + \beta\sigma_{1,1}$. Show that if $\alpha = 0$ then $\beta = 1$. What if $\beta = 0$?

Lecture 3

Problem 11. Let C be a smooth non-degenerate curve of degree d and genus g in $\mathbb{P}V$. Let

$$s_2(C) = \overline{\{\Lambda \in G(2,V) : \mathbb{P}\Lambda \text{ is a secant line to } C\}}.$$

Determine the class of $s_2(X)$ in $G(2,V)$.

Problem 12. Let C be a smooth non-degenerate curve of degree d and genus g in $\mathbb{P}V$. Let

$$T(C) = \overline{\{\Lambda \in G(2,V) : \mathbb{P}\Lambda \text{ is a tangent line to } C\}}.$$

Determine the class of $T(C)$ in $G(2,V)$.

Problem 13. In $G(3,6)$, compute the product

$$\sigma_{2,1} \cdot \sigma_{2,1}.$$

Problem 14. Let λ be a partition contained in the $k \times (n-k)$ box and let Σ_λ be the corresponding Schubert variety in $G(k,n)$. Consider the identification

$$i : G(k,n) \to G(n-k,n)$$

Show that i maps Σ_λ to Σ_{λ^T}, where λ^T is the partition in the $(n-k)\times k$ whose Young diagram is the transpose of the Young diagram of λ.

Problem 15. Let X be an irreducible smooth variety of codimension c and degree d in $\mathbb{P}V$. Let

$$H(X) = \{\Lambda \in G(c, V) : \mathbb{P}\Lambda \cap X \neq \emptyset\}.$$

This is the Chow form of X.

- Prove that $\operatorname{codim}_{G(c,V)}(H(X)) = 1$;
- Determine the class $[H(X)]$ in $CH^1(G(c, V))$.