Introduction to Enumerative Geometry

Jan. 11 – Jan. 22, 2021

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Lecture 6: Porteous's formula

- Porteous's formula for the vanishing of a section
- Grassmann bundle and reduction to the top Chern class
- General case
- Applications

Correction to the statement about resultants

Let $a(t)$, $b(t)$ be polynomials with coefficients in a commutative ring. Suppose $a(0) = b(0) = 1$.

Write

$$a(t) = \prod_1^e (1 + \alpha_i t) \qquad b(t) = \prod_1^f (1 + \beta_i t).$$

Then

$$\prod_{\substack{i=1,\dots,e \\ j=1,\dots,f}} (\beta_j - \alpha_i) = \Delta_f^e \left(\frac{b(t)}{a(t)} \right) = \det \begin{bmatrix} c_f & \cdots & c_{f+e-1} \\ \vdots & \ddots & \vdots \\ c_{f-e+1} & \cdots & c_f \end{bmatrix}$$

c_j are the coeffs of $b(t)/a(t)$ as an elt of the ring of power series.

Reference: Arbarello-Cornalba-Griffiths-Harris: pp.88-89

Setting

- X is a smooth algebraic variety;
- $\mathcal{E}, \mathcal{F}$ are vector bundles on X of rank e, f;
- $\varphi : \mathcal{E} \to \mathcal{F}$ is a morphism of vector bundles; we assume φ is generically injective; (if it is not, just pass to the quotient over the $\ker \varphi$)

Setting

- X is a smooth algebraic variety;
- $\mathcal{E}, \mathcal{F}$ are vector bundles on X of rank e, f;
- $\varphi : \mathcal{E} \to \mathcal{F}$ is a morphism of vector bundles; we assume φ is generically injective;
- $D_r^\varphi = \{x \in X : \mathrm{rank}(\varphi_x : \mathcal{E}_x \to \mathcal{F}_x) \leq r\}$ is the r-th determinantal variety defined by φ. We assume: (degeneracy locus)
  - D_r^φ is reduced;
  - D_r^φ has the expected codimension
$$\mathrm{codim}_X(D_r^\varphi) = \underline{(e-r)(f-r)};$$
  - D_{r-1}^φ is strictly smaller than D_r^φ. (every compt. of D^φ_{r-1} is strictly contained in a compt. of D^φ_r)

Goal:

Compute $[D_r^\varphi] \in CH^{(e-r)(f-r)}(X)$.

$$\boxed{[D_r^\varphi] = \Delta_{f-r}^{e-r}\left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})}\right).}$$

How general should sections defining Chern classes be?

Recall: if $\mathcal{E}$ has rank e and is globally generated, then

$$c_p(\mathcal{E}) = [Y(s_0, \ldots, s_{e-p})]$$

$$\shortparallel$$

for generic sections $s_j \in H^0(\mathcal{E})$.

$\{x \in X : s_0(x), \ldots, s_{e-p}(x) \text{ are lin. dependent}\}$

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Claim:

It is enough that $Y(s_0, \ldots, s_{e-p})$ is reduced and of codimension p.

Idea of pf)

Interpolate between this choice and a generic one. (As we did with a single section)

Easy case: $r = 0$

In this case

$$D_0^{\varphi} = \{x \in X : \varphi_x = 0\}$$ $\varphi : \mathcal{E} \to \mathcal{F}$

has codimension ef. $= (e-0)(f-0)$

We determine $[D_0^{\varphi}] \in CH^{ef}(X)$.

Regard $\varphi \in \underbrace{H^0(Hom(\mathcal{E}, \mathcal{F}))}$ vector bundle of rank ef

$D_0^{\varphi} = Y(s_0, \ldots, s_{ef-p})$ for $p = ef$ $s_0 = \varphi$

- D_0^{φ} is reduced
- it has codim ef

Easy case: $r = 0$

$$[D_0^?] = c_{ef}(\mathcal{H}om(\mathcal{E},\mathcal{F})) = c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}).$$

Splitting principle: $\mathcal{E} = \bigoplus_{i=1}^{e} \mathcal{L}_i \qquad c_1(\mathcal{L}_i) = \alpha_i$

$\mathcal{F} = \bigoplus_{j=1}^{f} \mathcal{M}_j \qquad c_1(\mathcal{M}_j) = \beta_j.$

$$\mathcal{E}^\vee \otimes \mathcal{F} = \bigoplus_{\substack{i=1\dots e\\ j=1\dots f}} \mathcal{L}_i^\vee \otimes \mathcal{M}_j \qquad c_1(\mathcal{L}_i^\vee \otimes \mathcal{M}_j) = -\alpha_i + \beta_j$$

$$c(\mathcal{E}^\vee \otimes \mathcal{F}) = \prod_{\substack{i=1\dots e\\ j=1\dots f}} (\overbrace{1}^{c_0} \overbrace{\underbrace{-\alpha_i + \beta_j}}^{c_1})$$

$$c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) = \prod_{\substack{i=1\dots e\\ j=1\dots f}} (\beta_j - \alpha_i)$$

Easy case: $r = 0$

$$c_{[t]}(\mathcal{E}) = \prod_{i=1\dots e} (1+\alpha_i t)$$

$$c_{[t]}(\mathcal{F}) = \prod_{1\dots f} (1+\beta_j t)$$

Deduce:

$$c_{ef}(\mathcal{E}^\vee \otimes \mathcal{F}) = \prod (\beta_j - \alpha_i) =$$

$$= \Delta_f^e \left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})} \right)$$

Which is Porteous's formula for $r=0$

Easy case: $r = 0$

Idea of general case:

Realize D_r^{φ} as push forward of some $\underbrace{D_0^{\psi}}$ in some other variety.

The case $r=0$, will tell us $[D_0^{\psi}]$.

Then we push it forward.

Grassmann bundle

Let $\mathcal{E}$ be a vector bundle of rank e on X. The Grassmann bundle of k-planes in the fibers of $\mathcal{E}$ is

$$\rho : \mathcal{G}(k, \mathcal{E}) \to X$$

the fiber bundle whose fiber at x is $\rho^{-1}(x) = G(k, \mathcal{E}_x)$.

- not a vector bundle.
- locally on U it look like

$$\rho^{-1}(U) = U \times G(k, E)$$

where E is a vector space of dim e.

Points in $\mathcal{G}(k, \mathcal{E})$ are (locally)

(x, L_x) where $L_x \in G(k, \mathcal{E}_x)$

Grassmann bundle – cont'd

$$0 \to S \to \rho^*\mathcal{E} \to Q \to 0$$

$$\begin{tikzcd}
\rho^*\mathcal{E} \arrow[d] & \mathcal{E} \arrow[d] \\
G(k,\mathcal{E}) \arrow[r, "\rho"'] & X
\end{tikzcd}$$

- $(\rho^*\mathcal{E})_{(x,L_x)} = \mathcal{E}_x$
- $S_{(x,L_x)} = L_x$
- $Q_{(x,L_x)} = \mathcal{E}_x / L_x$

$$\underbrace{S \xrightarrow{i} \rho^*\mathcal{E} \xrightarrow{\rho^*\varphi} \rho^*F}$$

$$\begin{tikzcd}
\rho^*\mathcal{E} \arrow[d] & \rho^*F \arrow[d] & & \mathcal{E} \arrow[r, "\varphi"] \arrow[d] & F \arrow[d] \\
\mathcal{G}(e-r, \mathcal{E}) \arrow[rrr, "\rho"'] & & & X & X
\end{tikzcd}$$

Define: $\tilde{\varphi} : S \longrightarrow \underbrace{\rho^*F}$ composition of i and $\rho^*\varphi$.

at $(x, K_x) \in \mathcal{G}(e-r, \mathcal{E})$

$$\tilde{\varphi}_{(x,K_x)} : K_x \longrightarrow F_x \qquad \tilde{\varphi}_{(x,K_x)} = \varphi_{|K_x}.$$
$$v \longmapsto \varphi_x(v)$$

Consider: $D_0^{\tilde{\varphi}} = \{(x, K_x) \in \mathcal{G}(e-r, \mathcal{E}) : \tilde{\varphi}_{(x,K_x)} = 0\}$

Porteous's formula: Reduction to the top Chern class

Idea:

Realize $D_r^{\varphi} \subseteq X$ as the push-forward of $D_0^{\tilde{\varphi}}$ for ~~some map bundle map $\tilde{\varphi}$.~~

Want to show $\rho^*: CH(G(e-r, \mathcal{E})) \to CH(X)$ sends $[D_0^{\tilde{\varphi}}]$ to $[D_r^{\varphi}]$.

Prop:

① $\rho^{-1}(D_r^{\varphi}) = D_0^{\tilde{\varphi}}$.

② $\rho: D_0^{\tilde{\varphi}} \to D_r^{\varphi}$ is surjective and generically 1-1

Porteous's formula: Reduction to the top Chern class – cont'd

Pf/ ① Say $(x, K_x) \in \rho^{-1}(D_r^{\varphi}) \Rightarrow x \in D_r^{\varphi}$

$x \in D_r^{\varphi}$ iff $\operatorname{rk}(\varphi_x) \leq r$ iff

iff $\exists\ K_x \subseteq \mathcal{E}_x$ such that $\dim K_x = e-r$

and $K_x \subseteq \operatorname{Ker} \varphi_x$

But then $\tilde{\varphi}_{(x, K_x)} = \varphi_x|_{K_x}$ is identically 0

So $(x, K_x) \in D_0^{\tilde{\varphi}}$.

This shows that $\rho^{-1}(D_r^{\varphi}) = D_0^{\tilde{\varphi}}$

and that $\rho : D_0^{\tilde{\varphi}} \longrightarrow D_r^{\varphi}$ is surjective.

Porteous's formula: Reduction to the top Chern class – cont'd

It remains to show that $\rho|_{\tilde{D}^{\varphi}_{\circ}}$ is generically one-to-one.

This follows from the fact that $D^{\varphi}_{r-1} \subsetneq D^{\varphi}_{r}$

because for $x \in D^{\varphi}_{r}$ generic, $rk(\varphi_x) = r$

so $\rho^{-1}(x) = (x, K_x)$ with $K_x = \ker \varphi_x$.

$$\rho^* : CH(G(e-r, \mathcal{E})) \longrightarrow CH(X)$$

sends $[Z]$ to 0 if $\rho|_Z$ is NOT finite-to-one

to $m[\rho(Z)]$ if it is $m:1$.

Porteous's formula: Reduction to the top Chern class – cont'd

So we obtain:

$$[D_r^{\varphi}] = \rho^*([D_0^{\tilde{\varphi}}]).$$

By assumption: $\operatorname{codim}_X (D_r^{\varphi}) = (e-r)(f-r)$.

$$\dim (D_r^{\varphi}) = \dim X - (e-r)(f-r).$$

$$\dim D_0^{\tilde{\varphi}} = \dim D_r^{\varphi}$$

$\dim G(e-r, \mathcal{E}) =$

$= \dim X + \dim \rho^{-1}(x) =$

$= \dim X + \dim G(e-r, E_x) =$

$= \underbrace{\dim X + r(e-r)}$

Porteous's formula: Reduction to the top Chern class – cont'd

$$\operatorname{codim}_{G(e-r,\mathcal{E})} D_0^{\tilde{\varphi}} = \dim G(e-r,\mathcal{E}) - \dim D_0^{\tilde{\varphi}} =$$

$$= \left[\underline{\dim X} + r(e-r)\right] - \left[\underline{\dim X} - (e-r)(f-\underline{r})\right] =$$

$$= \cancel{er} - \cancel{r^2} + ef - \cancel{re} - rf + \cancel{r^2} = \underline{\underline{f(e-r)}}$$

This is exactly the rank of

$$\operatorname{Hom}(\mathcal{S}_{e-r}, p^*\mathcal{F}_f) \quad \text{of which } \tilde{\varphi} \text{ is a section.}$$

We conclude: $D_0^{\tilde{\varphi}}$ has the expected codimension as a degeneracy locus of a section of $\mathcal{S}^\vee \otimes p^*\mathcal{F}$.

Porteous's formula: Reduction to the top Chern class – cont'd

From the case $r=0$ we get

$$\underline{[D_0^{\tilde\varphi}]} = \Delta_f^{e-r}\left(\frac{c_{[t]}(p^*\tilde F)}{c_{[t]}(S)}\right).$$

Therefore:

$$\underline{[D_r^{\varphi}]} = \underline{p_*}\left(\Delta_f^{e-r}\left(\frac{c_{[t]}(p^*F)}{c_{[t]}(S)}\right)\right).$$

From the Grassmann bundle to X

We obtained

$$[D_r^{\varphi}] = \rho_* \Delta_f^{e-r}\left[\frac{c_{[t]}(\rho^*\mathcal{F})}{c_{[t]}(\mathcal{S})}\right].$$

We have to resolve the pull-back and the push-forward to obtain an expression only involving elements of $CH(X)$.

By Whitney's formula:

$$0 \to \mathcal{S} \to \rho^*\mathcal{E} \to \mathcal{Q} \to 0$$

$$c_{[t]}(\mathcal{S}) = \frac{c_{[t]}(\rho^*\mathcal{E})}{c_{[t]}(\mathcal{Q})}$$

Get

$$[D_r^{\varphi}] = \rho_* \Delta_f^{e-r}\left[\rho^*\left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})}\right) c_{[t]}(\mathcal{Q})\right]$$

Consider the Sylvester matrix:

From the Grassmann bundle to X – cont'd

$$
\begin{array}{cc}
 & \\
e-r & \begin{bmatrix} S_f & \cdots & \cdots & S_{f+(e-r)-1} \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ & & & S_f \end{bmatrix}
\end{array}
$$

$$
S_p = \sum_{0}^{p} \overbrace{\rho^* \left(\left\{ \frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})} \right\}_{p-j} \right)}^{e-r} c_j(Q)
$$

where $\{-\}_j$ is the j-th coefficient (coeff. of t^j).

From the Grassmann bundle to X – cont'd

The determinant is sum of terms

$$\rho^*(Z)\cdot W$$

$\rho^*(Z)$ depends on $c_{[t]}(\mathcal{E})$, $c_{[f]}(\mathcal{F})$

and W is the product of $e-r$ Chern classes of Q.

When we push-forward, by push-pull formula we get

$$\rho_*\left(\rho^*(Z)\cdot W\right) = Z\cdot \underbrace{\rho_*(W)}$$

From the Grassmann bundle to X – cont'd

What is $\rho_*(W)$? $\rho_*([Y]) = 0$ if $\rho|_Y$ is not finite-to-one

The fibers of ρ have dim $r(e-r)$.

So if $Y \subseteq \mathcal{G}(e-r, \mathcal{E})$ has codim $\lneq r(e-r)$

then $\rho|_Y$ cannot be finite-to-one.

Therefore $\rho_* : CH(\mathcal{G}(e-r, \mathcal{E})) \to CH(X)$

maps to 0 everything of degree $< r(e-r)$.

Now: Q has rank r. so its top Chern class is

$c_r(Q)$.

From the Grassmann bundle to X – cont'd

Multiplying $e-r$ Chern classes we always get an elt of degree $< r(e-r)$ but for $\underline{c_r(Q)^{e-r}}$.

Now: $p_*\left(c_r(Q)^{e-r}\right) = m[X]$

Fact: $m=1$ Reason: Our calculation of the Chern classes of the quotient bundle of the Grassmannian.

From the Grassmann bundle to X – cont'd

Recall:

$$s_p = \sum_{j=0}^{p} \rho^* \{ \text{———} \}_{p-j} \, c_j(Q)$$

The only contribution to the final push-forward is given by the terms with $j=r$

We get:

$$[D_r^{\varphi}] = \Delta_f^{e-r} \begin{bmatrix} \tilde{s}_f & \cdots & \tilde{s}_{f+(e-r)-1} \\ & \ddots & \\ & & \tilde{s}_f \end{bmatrix} = \circledast$$

From the Grassmann bundle to X – cont'd

$$\tilde{S}_p = \left\{ \frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})} \right\}_{\underbrace{p-r}}$$

$$* \quad \Delta^{e-r}_{f-r}\left[\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})} \right]$$

and this proves the formula.

From the Grassmann bundle to X – cont'd

Let $X_r \subseteq \mathbb{P}\mathrm{Mat}_{\cancel{e\times f}}$ be the general determinantal variety of matrices. So $X_r = D_r^\varphi$ for $f \times e$

$X_r = \{M : \mathrm{rk} M \leq r\}$

- $\mathcal{E} = \underline{E}_X$;
- $\mathcal{F} = \underline{F}_X \otimes \mathcal{O}(1)$; $= \mathcal{O}(1)^{\oplus f}$
- $\varphi : \mathcal{E} \to \mathcal{F}$ defined by $\varphi_M : E \to F$.

$\mathrm{Mat}_{f \times e} = \mathrm{Hom}(E, F)$

$= E^* \otimes F$.

$c_{[t]}(\mathcal{E}) = 1$

$c_{[t]}(\mathcal{F}) = (1 + \overbrace{c_1(\mathcal{O}(1))}^{\zeta} \cdot t)^f =$

$= (1 + \zeta t)^f = \sum_{j=0}^{f} \binom{f}{j} \zeta^j t^j$

ζ hyperplane class in $\mathbb{P}\mathrm{Mat}_{f \times e}$

The general determinantal variety – cont'd

$$[X_r] = \Delta_{f-r}^{e-r}\left(\frac{c_{[t]}(\mathcal{F})}{c_{[t]}(\mathcal{E})}\right) =$$

$$= \det \begin{bmatrix} \binom{f}{f-r}\zeta^{f-r} & \cdots & \binom{f}{f-r+(e-r)-1}\zeta^{f+e-2r-1} \\ \vdots & \ddots & \\ \binom{f}{f-e+1}\zeta^{f-e+1} & & \binom{f}{f-r}\zeta^{f-r} \end{bmatrix} =$$

$$= \left[\prod_{i=0}^{e-r-1} \frac{i!\,(f+i)!}{(r+i)!\,(f-r+i)!}\right] \cdot \zeta^{(e-r)(f-r)} \qquad 0! = 1$$

$\deg X_r =$

The general determinantal variety – cont'd

Two observations:

- Compute $TC_M X_r^{(f \times r)}$ for M with $\mathrm{rk}\, M = s < r$

$$TC_M X_r^{e \times f} \cong X_{r-s}^{(e-s) \times (f-s)}$$

so the formula above gives

$$\mathrm{mult}_M (X_r^{f \times r}) = \deg TC_M (X_r^{e \times f}).$$

The general determinantal variety – cont'd

$e \leq f$

- In the case $r = e-1$

$$\deg X_r = \binom{f}{e-1}$$

The general determinantal variety – cont'd

The general determinantal variety – cont'd

Secant varieties of the rational normal curve

Let $C_d = \nu_d(\mathbb{P}^1) \subseteq \mathbb{P}S^d V$ be the rational normal curve of degree d.

For $r < d/2$, the secant variety $\sigma_r(C_d)$ is determinantal, defined by the $(r+1) \times (r+1)$ minors of

$$cat_r : S^r V^* \to S^{d-r} V. \leftarrow \quad f \in S^d V.$$

$$P \longmapsto P \cdot f$$

We interpret this as a degeneracy locus of a vector bundle map on $X = \mathbb{P}S^d V$

- $\mathcal{E} = S^r V^*$; $\quad e = r+1$
- $\mathcal{F} = S^{d-r} V \otimes \mathcal{O}(1)$; $\quad f = d-r+1$
- $\varphi : \mathcal{E} \to \mathcal{F}$ defined by

$$\varphi_f : S^r V^* \to \mathrm{Hom}(\langle f \rangle, S^{d-r} V)$$
$$P \mapsto (\varphi_f(P) : \lambda f \mapsto \lambda cat_r(P))$$

Secant varieties of the rational normal curve

$\boxed{\dim\ 2r-1 \text{ in } \mathbb{P}^d}$

$$[\sigma_r(C_d)] = \binom{d-r+1}{r} \underline{\zeta^{d-2r+1}}$$

Sanity check:

$\boxed{r=1}$ Then $\sigma_r(C_d) = C_d$

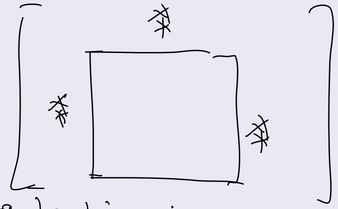
$$[\sigma_r(C_d)] = [C_d] = d\,\zeta^{d-1}$$

We conclude:

$$\deg(\sigma_r(C)) = \binom{d-r+1}{r}.$$

Secant varieties of the rational normal curve

Rank completion:



Ask membership in

$$\underbrace{J(X_r, L)}$$