PROBLEM SETS FOR
INTRODUCTION TO ENUMERATIVE COMBINATORICS

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ABSTRACT. This document collects a few problems which should be useful to practice on the material that is covered during the lectures.

LECTURE 1

Problem 1. Let V be a vector space of dimension $n+1$ and let $\nu_{d,n} : \mathbb{P}V \to \mathbb{P}S^d V$ be the Veronese embedding. Show that if $X \subseteq \mathbb{P}V$ is a variety of dimension k and degree e, then $\nu_{d,n}(X)$ has degree $d^k e$.

In particular, the degree of a k-dimensional subvariety of $\nu_{d,n}(\mathbb{P}V)$ is a multiple of d^k.

Problem 2. Let

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : p, r, s \text{ are collinear}\}.$$

Show that Ψ is a subvariety of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ of codimension $n-1$.

Determine the class $[\Psi]$ in the Chow ring $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$.

Note: The Chow ring of a product of several projective spaces is what one expects. If $\dim V_j = n_j + 1$, then

$$\mathrm{CH}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_s) = \mathbb{Z}[\alpha_1, \ldots, \alpha_s]/(\alpha_1^{n_1+1}, \ldots, \alpha_s^{n_s+1})$$

where α_j is identified with the class of the pull back of the hyperplane section in $\mathbb{P}V_j$ via the projection map. In other words

$$\alpha_j = [\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{j-1} \times H_j \times \mathbb{P}V_{j+1} \times \cdots \times \mathbb{P}V_s].$$

Problem 3. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables. Let

$$\mathcal{T} = \{f : f = \ell_1\ell_2\ell_3 \text{ for some linear forms } \ell_j\},$$

that is the space of *triangles* (i.e., cubic curves which are union of three lines).

Determine the dimension and the degree of $\mathcal{T}$.

Hint: Write $\mathcal{T}$ is the image of a map defined on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Problem 4. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{A} = \{f : f = \ell_1\ell_2\ell_3 \text{ for some linear forms } \ell_j \text{ with a common zero}\}\},$$

that is the space of *asterisks* (i.e., cubic curves which are union of three lines passing through the same point).

Determine the dimension and the degree of $\mathcal{A}$.

Hint: It is similar to the previous problem, but one needs something more complicated than $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Problem 5. Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{C} = \{f = \ell_1^2 \ell_2 : \ell_j \text{ is a linear form}\},$$

that is the space of cubic curves which are union of a double line and a line.

Determine the dimension and the degree of $\mathcal{C}$.