INTRODUCTION TO ENUMERATIVE GEOMETRY

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ABSTRACT. Lecture notes for the course *Introduction to Enumerative Geometry* which will be held January 11 - January 22, 2021.

`https://sites.google.com/view/intro-enumerative-geometry/`.

The course covers an introduction to intersection theory, and applies the acquired techniques to some classical problems. We will introduce the basics of intersection theory: Chow ring, Chern classes, and basics of Schubert calculus. The theoretical tools which are developed will be applied to the enumerative geometry of some Grassmannian problem and to the Thom-Porteous formula for the calculation of the degree of determinantal varieties. If time permits, we will draw connections to the representation theory of the general linear group.

Lecture notes are in preliminary and incomplete form.

The main reference is [EH16]. Other references that we follow are [Man98, ACGH85].

LECTURE 1: THE CHOW RING

1.1. The Chow ring.

Definition 1.1 (Cycles)**.** Let X be a scheme[1]. The *group of cycles* on X, denoted $Z(X)$ is the free abelian group of formal integral linear combinations of irreducible subvarieties of X. The group $Z(X)$ decomposes according to the dimension of the subvarieties: $Z(X) = \bigoplus_k Z_k(X)$ where $Z_k(X)$ is the group of formal linear combinations of irreducible subvarieties of dimension k. We say that a k-cycle Z is effective if $Z = \sum n_i Y_i$ with $n_i \geq 0$. Elements of $Z_{\dim(X)-1}(X)$ are called divisors. Clearly $Z(X) = Z(X_{red})$ where X_{red} denotes the reduced structure of the scheme X.

If $Y \subseteq X$ is a subscheme, we associate an effective cycle to Y. If Y is reduced and its irreducible components are $Y_1, \dots, Y_s$, the associated effective cycle is $Y = \sum Y_i$. If Y is not reduced, let $Y_1, \dots, Y_s$ be the associated components of Y_{red}.

Write $\mathcal{O}_{Y,Y_i}$ for the quotient $\mathcal{O}_Y / \mathcal{I}_{Y_i}$ where $\mathcal{I}_{Y_i}$ is the ideal sheaf of Y_i in $\mathcal{O}_Y$. Then $\mathcal{O}_{Y,Y_i}$ has finite length as a $\mathcal{O}_Y$-module: write $\mathrm{mult}_{Y_i}(Y)$ for the length, called the *multiplicity* Y along Y_i. Define the effective cycle associated to Y to be $Y = \sum \mathrm{mult}_Y(Y_i) \cdot Y_i$.

1.2. Rational equivalence.

Let X be a scheme. Let W be an irreducible subvariety of $X \times \mathbb{P}^1$ which is not contained in a "fiber", that is there is no $t \in \mathbb{P}^1$ such that $W \subseteq X \times \{t\}$. By irreducibility, we have that the image of the projection of W on the second factor is dense in $\mathbb{P}^1$.

[1] Almost all schemes in these notes can be assumed to be varieties. But the theory is exactly the same, so the notes are written in the slightly more general setting.

We say that two irreducible subvarieties $Y_0, Y_\infty \in Z(X)$ are *rationally equivalent* if there exists an irreducible variety $W \subseteq X \times \mathbb{P}^1$ not contained in a fiber such that $W \cap (X \times \{0\}) = Y_0$ and $W \cap (X \times \{\infty\}) = Y_\infty$. We say that W interpolates between Y_0 and Y_∞

Rational equivalence is an equivalence relation. Let $\mathrm{Rat}(X) \subseteq Z(X)$ be the subgroup generated by differences of rationally equivalent varieties:

$$\mathrm{Rat}(X) = \langle Y_0 - Y_\infty : Y_0, Y_\infty \text{ rationally equivalent}\rangle.$$

Example 1.2 (Two hypersurfaces of the same degree)**.** Let $X := V(f)$ and $Y := V(g)$ be hypersurfaces in $\mathbb{P}^n$ defined by two polynomials f, g of the same degree. Then they are rationally equivalent: define $W = V(t_0 f + t_1 g) \subseteq \mathbb{P}^1 \times \mathbb{P}^n$; then W interpolates between X at $(t_0, t_1) = (1, 0)$ and Y at $(t_0, t_1) = (0, 1)$. ♠

Definition 1.3 (Chow group)**.** Let X be a scheme. The Chow group of X is

$$\mathrm{CH}(X) = Z(X)/\mathrm{Rat}(X).$$

For a subscheme $Y \subseteq X$, write $[Y]$ for the class in $\mathrm{CH}(X)$ of its associated effective divisor.

Lemma 1.4. *If $Y_0, Y_\infty \subseteq X$ are rationally equivalent and non-empty, then* $\dim Y_0 = \dim Y_\infty$. *In particular,* $\mathrm{Rat}(X)$ *is generated by* homogeneous *elements.*

Proof. Let $W \subseteq X \times \mathbb{P}^1$ be the irreducible variety which interpolates between Y_0 and Y_∞. Let (t_0, t_1) be coordinates on $\mathbb{P}^1$. Then $Y_0 = W \cap \{t_1 = 0\}$ and $Y_\infty = W \cap \{t_0 = 0\}$. So Y_0, Y_∞ are cut out by a single equation $t_1 = 0$ and $t_0 = 0$ in $W \times \mathbb{P}^1$. By irreducibility t_0, t_1 are nonzero divisors, hence Y_0, Y_∞ are either empty or of codimension 1 in W. □

By Lemma 1.4, the decomposition of $Z(X)$ by dimension descends to the Chow group: $\mathrm{CH}(X) = \bigoplus \mathrm{CH}_k(X)$, where $\mathrm{CH}_k(X) = Z_k(X)/(\mathrm{Rat}_k(X))$. If X is equidimensional, we write $\mathrm{CH}^k(X) = \mathrm{CH}_{\dim X - k}$.

Rationality defines a natural exact sequence

$$Z(\mathbb{P}^1 \times X) \xrightarrow{\rho} Z(X) \to \mathrm{CH}(X) \to 0$$

where $\rho(W) = 0$ if W is contained in a fiber of $\mathbb{P}^1 \times X$ and $\rho(W) = (W \cap (\{\infty\} \times X)) - (W \cap (\{0\} \times X))$ otherwise.

Definition 1.5 (Transversality)**.** Let X be an irreducible variety and let Y_1, Y_2 be subvarieties. We say that Y_1 and Y_2 intersect transversely at $p \in Y_1 \cap Y_2$ if Y_1, Y_2 and X are smooth at p and

$$T_p Y_1 + T_p Y_2 = T_p X.$$

We say that Y_1 and Y_2 are generically transverse if they intersect transversely at the general point of every irreducible component of $Y_1 \cap Y_2$; this terminology extends naturally to cycles.

Theorem 1.6 (Moving Lemma)**.** *Let X be a smooth variety. Then*

- *For every $\alpha, \beta \in \mathrm{CH}(X)$ there are generically transverse cycles $A, B \in Z(X)$ such that $\alpha = [A]$ and $\beta = [B]$;*
- *If A and B are transverse, then the class $[A \cap B]$ is independent from the choice of the cycles A, B.*

Theorem 1.7. *Let X be a smooth variety. Then there is a unique product structure on* $\mathrm{CH}(X)$ *such that whenever A, B are generically transverse subvarieties of X, then $[A][B] = [A \cap B]$. This product makes* $\mathrm{CH}(X)$ *into a graded ring, where the grading is given by codimension.*

Proposition 1.8. *Let X be a scheme. Then* $\mathrm{CH}(X) = \mathrm{CH}(X_{red})$*. If X is equidimensional and $X_1, \dots, X_s$ are its irreducible components, then* $\mathrm{CH}^0(X) = \bigoplus_i \mathbb{Z} \cdot [X_i]$*, the free abelian group generated by the classes of the irreducible components.*

Proof. Cycles and rational equivalence are defined via reduced varieties, so $Z(X) = Z(X_{red})$ and $\mathrm{Rat}(X) = \mathrm{Rat}(X_{red})$. Hence $\mathrm{CH}(X) = \mathrm{CH}(X_{red})$.

As for the second assertion, it suffices to show that $\mathrm{CH}(X)$ is generated by $[X_1], \dots, [X_s]$ and that there are no relations among them. Both assertions follow from the irreducibility of the interpolating variety:

$$W \subseteq X \times \mathbb{P}^1 = \bigcup (X_i \times \mathbb{P}^1).$$

Since W is irreducible, $W \subseteq X_j \times \mathbb{P}^1$ for some j. □

For every scheme X of dimension n, the class $[X] \in \mathrm{CH}^0(X)$ is called *the fundamental class* of X.

Example 1.9 (Affine space)**.** We prove that $\mathrm{CH}(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n]$ is the free abelian group generated by the fundamental class.

To see this, we show that every proper subvariety of $\mathbb{A}^n$ is rationally equivalent to the empty set. Let Y be a proper subvariety and suppose that $0 \notin Y$. Define

$$W^\circ = \{(tz, t) : z \in Y, t \in \mathbb{A}^1 \setminus \{0\}\} \subseteq \mathbb{A}^n \times \mathbb{A}^1.$$

Let $W = \overline{W^\circ} \subseteq \mathbb{A}^n \times \mathbb{P}^1$. The fiber of W at $t = 1$ is Y. Let $g \in I(Y)$ with $g(0) = c \neq 0$ (which exists because $0 \notin Y$). The function $G(z, t) = g(z/t)$ is an equation for W. Its value at $t = \infty$ is c, so the fiber of W at $t = \infty$ is empty.

This shows that Y is rationally equivalent to the empty set, hence $[Y] = 0$. ♠

Proposition 1.10 (Mayer-Vietoris and Excision)**.**

- *Let X_1, X_2 be closed subschemes of X. Then there is a right exact sequence*

$$\mathrm{CH}(X_1 \cap X_2) \to \mathrm{CH}(X_1) \oplus \mathrm{CH}(X_2) \to \mathrm{CH}(X_1 \cup X_2) \to 0.$$

- *Let $Y \subseteq X$ be a closed subscheme and let $U = X \smallsetminus Y$. Then there is a right exact sequence*

$$\mathrm{CH}(Y) \to \mathrm{CH}(X) \to \mathrm{CH}(U) \to 0.$$

  Moreover, if X is smooth, then $\mathrm{CH}(X) \to \mathrm{CH}(U)$ *is a ring homomorphism.*

Definition 1.11 (Pushforward)**.** Let $f : Y \to X$ be a proper morphism of schemes. We define a *pushforward map* $f_* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$ as follows; for a subscheme $A \subseteq Y$, we define extending it linearly from

- $f_*([A]) = 0$ if $f|_A$ is not generically finite on A;
- $f_*([A]) = d[f(A)]$ if $f|_A$ is generically finite and the generic fiber has d points.

The dual notion of the pushforward map is a pullback map; we can give a good definition exploiting the following theorem:

Theorem 1.12 (Good definition of pullback)**.** *Let $f : Y \to X$ be a map of smooth quasi-projective varieties. There is a unique map $f^* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ such that, $A \subseteq X$ is generically transverse to f, then $f^*[A] = [f^{-1}(A)]$.*

Moreover, the map f^* satisfies the following push-pull formula: if $\alpha \in \mathrm{CH}^k(X)$ and $\beta \in \mathrm{CH}^{n-\ell}(Y)$, then

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in \mathrm{CH}(X).$$

The map f^* is called pullback map of f.

Definition 1.13 (Dimensional Transversality)**.** Let X be a scheme and let A, B be two irreducible subschemes of X. We say that A, B are dimensionally transverse if every irreducible component C of $A \cap B$ satisfies $\operatorname{codim}_X C = \operatorname{codim}_X A + \operatorname{codim}_X B$. The definition extends naturally to cycles.

Theorem 1.14 (Product and dimensionally transverse cycles)**.** *Let X be a smooth scheme and let $A, B \subseteq X$ be irreducible dimensionally transverse subvarieties. Then*

$$[A][B] = \sum_{C \text{ component}} m_C(A,B)[C] \in \mathrm{CH}(X)$$

where the sum runs over the irreducible components of $A \cap B$ and $m_C(A,B)$ are integers called the intersection multiplicities of A and B at C. If A, B intersect transversely at C, then $m_C(A,B) = 1$.

Definition 1.15 (Stratification)**.** Let X be a scheme and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a collection of locally closed subschemes of X. We say that $\mathcal{U}$ is a *stratification* of X if X is disjoint union of the U_i and for every i $\overline{U_i} \smallsetminus U_i$ is disjoint union of some of the U_j's. Each U_i is called a *stratum* of the stratification; the closure $Y_i = \overline{U_i}$ is called a *closed stratum*.

A stratification $\mathcal{U}$ is called a *affine stratification* if the strata are isomorphic to affine spaces. It is called *quasi-affine stratification* if the strata are isomorphic to open subset of affine spaces.

For instance, the projective space $\mathbb{P}^n$ has a stratification given by $\mathbb{P}^n = \bigcup_{i=0}^n \mathbb{A}^i$.

Theorem 1.16 (Chow group of affinely stratifiable schemes)**.** *Let X be a scheme that admits a quasi affine stratification. Then $\mathrm{CH}(X)$ is generated by the classes of the closed strata. Moreover, if the stratification is affine, the closed strata form a basis of $\mathrm{CH}(X)$ as free $\mathbb{Z}$-module.*

Example 1.17 (Projective spaces)**.** Let $\mathbb{P}^n$ be the projective space. We prove that, as a ring,

$$\mathrm{CH}(\mathbb{P}^n) \simeq \mathbb{Z}[\zeta]/(\zeta^{n+1})$$

where $\zeta = [H]$ is the hyperplane class of $\mathbb{P}^n$. More generally if X is an irreducible variety of codimension k and degree d, then $[X] = d\zeta^k$.

The result about the additive group follows from Thm. 1.16, using the stratification given by the complement of a flag $\mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \cdots \subseteq \mathbb{P}^n$; this shows that $\mathrm{CH}^k(\mathbb{P}^n) = \mathbb{Z}$ for every $k = 0, \ldots, n$. The intersection product follows from the fact that a generic plane L of codimension k is transverse intersection of k generic hyperplanes, so $[L] = \zeta^k$.

If X is an irreducible variety of codimension k and degree d, and L is a transverse plane of dimension L then $[X]\zeta^{n-k} = [X \cap L] = [d \text{ points}] = d\zeta^n$, so $[X] = d\zeta^k$. ♠

Theorem 1.18 (Bezout's Theorem)**.** *Let $X_1, \dots, X_k \subseteq \mathbb{P}^n$ be subvarieties of codimension $c_1, \dots, c_k$, with $\sum c_i \le n$ and suppose the X_i intersect generically transversely.*

Then

$$\deg(X_1 \cap \dots \cap X_k) = \prod \deg(X_i).$$

Example 1.19 (Veronese varieties)**.** Let $\nu_d = \nu_{d,n} : \mathbb{P}V \to \mathbb{P}S^dV$ be the d-th Veronese embedding, where V is a vector space of dimension $n+1$. Identify V with the space of linear forms on V^* and S^dV with the space of homogeneous polynomials of degree d on V^*. Then $\nu_d(\ell) = \ell^d$ sends a linear form to its d-th power.

The degree of the Veronese variety $\nu_{d,n}(\mathbb{P}^n)$ is the number of points in the intersection of the Veronese variety $\nu_d(\mathbb{P}^n)$ with n generic hyperplanes $H_1, \dots, H_n$. Since ν_d is injective, we have

$$\#(\nu_d(\mathbb{P}^n) \cap H_1 \cap \dots \cap H_n) = \#(\nu_d^{-1}(H_1) \cap \dots \cap \nu_d^{-1}(H_n)).$$

If H is a hyperplane, then $\nu_d^{-1}(H)$ is a hypersurface of degree d in $\mathbb{P}^n$. Hence

$$\#(\nu_d^{-1}(H_1) \cap \dots \cap \nu_d^{-1}(H_n))$$

equals the degree of the intersection of n generic hypersurfaces in $\mathbb{P}^n$. We conclude

$$\#\nu_d^{-1}(H_1) \cap \dots \cap \nu_d^{-1}(H_n) = (d\zeta)^n = d^n\zeta^n,$$

therefore $\deg(\nu_d(\mathbb{P}^n)) = d^n$. ♠

Example 1.20 (Dual varieties)**.** Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface and let $X^\vee \subseteq \mathbb{P}^{n*}$ be its dual variety, which is the image of X under the Gauss map:

$$\begin{aligned} \mathcal{G}_X : X &\to \mathbb{P}^{n*} \\ p &\mapsto \mathbb{P}T_pX \end{aligned}$$

where $\mathbb{P}T_pX$ is the projective tangent space to X at p. In coordinates, if $X = V(f) \subseteq \mathbb{P}^n$, where f is homogeneous of degree d in $x_0, \dots, x_n$, then

$$\begin{aligned} \mathcal{G}_X : X &\to \mathbb{P}^{n*} \\ p &\mapsto \ker[\partial_0 f(p), \dots, \partial_n f(p)]; \end{aligned}$$

this expression defines a map $\mathcal{P}_X : \mathbb{P}^n \to \mathbb{P}^{n*}$ called polar map.

We compute the degree of $X^\vee$ under the assumption that $\mathcal{G}_X$ is birational, which is true if X is smooth of degree at least 2.

The degree of $X^\vee$ is the cardinality of the intersection of $X^\vee$ with $n-1$ generic hyperplanes in $\mathbb{P}^{n*}$.

Let $H_1, \dots, H_{n-1}$ be generic hyperplanes in $\mathbb{P}^{n*}$. We have

$$\deg(X^\vee) = X^\vee \cap H_1 \cap \dots \cap X_{n-1}.$$

Equivalently, since $\mathcal{G}_X$ is birational,

$$\deg(X^\vee) = \mathcal{G}_X^{-1}(H_1) \cap \dots \cap \mathcal{G}_X^{-1}(H_{n-1}) = X \cap \mathcal{P}_X^{-1}(H_1) \cap \dots \cap \mathcal{P}_X^{-1}(H_{n-1})$$

If H is a hyperplane in $\mathbb{P}^{n*}$, say $H = \{L = 0\}$ then

$$\mathcal{P}_X^{-1}(H) = \{p \in X : L(\partial_0(f), \dots, \partial_n(f))(p) = 0\}$$

which is an equation of degree $d-1$.

Since $\deg(X) = d$, we conclude

$$\deg(X^\vee)\zeta^n = (d\zeta)((d-1)\zeta)^{n-1} = d(d-1)^{n-1}\zeta^n$$

from which we have $\deg(X^\vee) = d(d-1)^{n-1}$. ♠

Example 1.21. Let $S \subseteq \mathbb{P}^3$ be a smooth cubic surface and let $L \subseteq \mathbb{P}^3$ be a general line. How many planes in $\mathbb{P}^3$ containing L are tangent to S?

The set of planes in $\mathbb{P}^3$ containing L is a generic line $\widetilde{L} \subseteq \mathbb{P}^{3*}$. The set of planes tangent to X is $X^\vee$: from Example 1.20, $\deg X^\vee = 3 \cdot (3-1)^{3-1} = 12$; so by Bezout's Theorem, $X^\vee \cap \widetilde{L}$ consists of 12 points, corresponding to 12 planes containing L and tangent to X. ♠

Example 1.22 (Two factors Segre products)**.** Let U, V be vector spaces of dimension $r+1, s+1$ respectively. Then

$$\mathrm{CH}(\mathbb{P}U \times \mathbb{P}V) \simeq \mathrm{CH}(\mathbb{P}U) \otimes_{\mathbb{Z}} \mathrm{CH}(\mathbb{P}V) = \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1})$$

where α, β are the pullbacks of the hyperplane classes of $\mathbb{P}U, \mathbb{P}V$ via the projection maps, respectively. If $X \subseteq \mathbb{P}U \times \mathbb{P}V$ is a hypersurface defined by bihomogeneous forms of bidegree (d, e) then $[X] = d\alpha + e\beta$. The proof of this fact uses Theorem 1.16, as in the case of the projective space.

Now consider the Segre embedding $Seg : \mathbb{P}U \times \mathbb{P}V \to \mathbb{P}(U \otimes V)$; we will often drop Seg from the notation. We compute the degree of the Segre variety $\mathbb{P}U \times \mathbb{P}V$. Notice that $\dim(\mathbb{P}U \times \mathbb{P}V) = r+s$, so the degree of the Segre variety is the number of points of intersection of $\mathbb{P}U \times \mathbb{P}V$ with $r+s$ hyperplanes in $\mathbb{P}(U \otimes V)$. A generic hyperplane H is rationally equivalent to one of the form $H_U \otimes V + U \otimes H_V$ for hyperplanes H_U, H_V in U, V respectively. Such a hyperplane has generically transverse intersection with $\mathbb{P}A \times \mathbb{P}B$ and pulls back to the class $\alpha + \beta$; therefore

$$\deg(\mathbb{P}U \times \mathbb{P}V) = \deg(\alpha+\beta)^{r+s} = \deg(\textstyle\sum_0^{r+s} \binom{r+s}{j} \alpha^j \beta^{r+s-j}) = \deg(\binom{r+s}{s} \alpha^r \beta^s)$$

therefore $\deg(\mathbb{P}U \times \mathbb{P}V) = \binom{r+s}{s}$. ♠

Lecture 2: Grassmannians

Definition 2.1. The Grassmannian of k-planes in a vector space V of dimension $n+1$, denoted $G(k, V)$, is the variety of k-dimensional subspaces of V. It can be realized as a projective variety in its Plücker embedding.

$$\begin{aligned} G(k, V) &\to \mathbb{P} \textstyle\bigwedge^k V \\ \langle v_1, \dots, v_k \rangle &\mapsto [v_1 \wedge \cdots \wedge v_k]. \end{aligned}$$

After fixing a basis $e_0, \dots, e_n$ of V, for every $I \subseteq \{0, \dots, n\}$ with $\# I = k$, we write p_I for the Plucker coordinates of a plane $E \in G(k, V)$.

The map $G(k, V) \to G(n+1-k, V^*)$ defined by $E \mapsto E^\perp$ defines an isomorphism of projective varieties.

The Grassmannian has two natural *universal* bundles. Fix V and let $\underline{V} = G(k, V) \times V$ be the trivial bundle with constant fiber V. The *tautological bundle* of $G(k, V)$ is the bundle whose fiber at the point $E \in G(k, V)$ is the plane E itself. The tautological bundle is a vector bundle of rank k. The *quotient bundle* on $G(k,V)$ is the quotient $\mathcal{Q} = \underline{V}/\mathcal{S}$, whose fibers are $\mathcal{Q}_E = V/E$; the quotient bundle is a vector bundle of rank $n+1-k$.

Proposition 2.2 (Universal property of the Grassmannian)**.** *Let X be a scheme and let $\mathcal{F}$ be a vector bundle of rank k contained in a trivial bundle $\underline{V} = V \times X$. Then there exists a unique map $f : X \to G(k,V)$ such that $\mathcal{F} = f^*\mathcal{S}$, the pull back of the tautological bundle via f. Moreover, the tautological inclusion $\mathcal{S} \to G(k,V) \times V$ pulls back to the inclusion of $\mathcal{F}$ into $X \times V$.*

Sketch of proof. Define the map f as $f : X \to G(k,V)$, $f(x) = \mathcal{F}_x \in G(k,V)$. One can check that this assignment works. □

Proposition 2.3 (Tangent bundle to Grassmannian)**.** *The tangent bundle $TG(k,V)$ to the Grassmannian of k-planes in V is isomorphic to $\mathcal{S}^\vee \otimes \mathcal{Q}$.*

Proof. Let $E = \langle v_1, \dots, v_k \rangle \in G(k,V)$ be a k-plane. We prove $T_E G(k,V) = E^* \otimes V/E$. Let $\Lambda(t)$ be a curve on $G(k,V) \subseteq \mathbb{P} \bigwedge^k V$ such that $\Lambda(0) = E$. In particular $\Lambda(t) = v_1(t) \wedge \cdots \wedge v_k(t)$ with $v_j(0) = v_j$. By Leibniz rule $\frac{d}{dt}|_0 \Lambda(t) = \sum_j v_1 \wedge \cdots \wedge v_j' \wedge \cdots \wedge v_k$ where $v_j' = v_j'(0)$. Since the tangent vectors v_j' are arbitrary, we deduce that

$$T_\Lambda G(k,V) = \left\{ \sum_j v_1 \wedge \cdots \wedge w_j \wedge \cdots \wedge v_k : w_1, \dots, w_k \in V \right\}.$$

Now, given a map $\varphi : E \to V$, define $v_j(t) = v_j + t\varphi(v_j)$ and let ω be the corresponding tangent vector. Two maps φ, ψ generate the same ω if and only if $\varphi = \psi \mod E$, $T_\Lambda G(k,V)$ is isomorphic to the space of linear maps $\{\varphi : E \to V/E\} = E^* \otimes V/E$. These are the fibers of $\mathcal{S}^* \otimes \mathcal{Q}$. □

We start our first explicit study of the Chow ring of a Grassmannian. Let V be a vector space with $\dim V = 4$ and let $k = 2$. Chow rings of Grassmannians are generated by Schubert cycles. They depend on the choice of a complete flag variety $F_\bullet$ on V, that is a nested sequence of vector spaces $0 = F_0 \subseteq \cdots \subseteq F_{\dim V} = V$ with $\dim F_j = j$. Let

$$F_\bullet = (0 = F_0 \subseteq \cdots \subseteq F_4 = V)$$

be a complete flag on V. Given (a,b) with $2 \geq a \geq b \geq 0$, define the Schubert varieties of $G(2,V)$:

$$\Sigma_{a,b} = \{\Lambda : \dim(\Lambda \cap F_{3-a}) \geq 1, \dim(\Lambda \cap F_{4-b}) \geq 2\},$$

where F_j is the j-dimensional plane in the flag $F_\bullet$. Explicitly

$$\begin{aligned}
\Sigma_{0,0} &= G(1,3);\\
\Sigma_{1,0} &= \{\Lambda : \Lambda \cap F_2 \neq 0\};\\
\Sigma_{2,0} &= \{\Lambda : F_1 \subseteq \Lambda\};\\
\Sigma_{1,1} &= \{\Lambda : \Lambda \subseteq F_3\};\\
\Sigma_{2,1} &= \{\Lambda : F_1 \subseteq \Lambda \subseteq F_3\};\\
\Sigma_{2,2} &= \{\Lambda : \Lambda = F_2\}.
\end{aligned}$$

Schubert varieties are closed, irreducible and $\operatorname{codim}\Sigma_{a,b} = a+b$. Moreover, $\Sigma_{a,b} \supseteq \Sigma_{a',b'}$ if $(a,b) \leq (a',b')$ componentwise. For every (a,b) define $\Sigma^\circ_{a,b} = \Sigma_{a,b} \smallsetminus \bigcup_{(a',b')\geq(a,b)} \Sigma_{a',b'}$. These are called Schubert cells.

The Schubert cells form an affine stratification of $G(2,V)$. We only have to show that $\Sigma^\circ_{a,b}$ are affine spaces.

We show this explicitly for the case of Σ_1. Let

$$\Sigma^\circ_1 = \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{(1,1)}) = \{\Lambda : \Lambda \cap F_2 \neq 0, F_1 \not\subseteq \Lambda, \Lambda \not\subseteq F_3\}.$$

Lemma 2.4. $\Sigma^\circ_1 \simeq \mathbb{A}^3$

Proof. Fix a hyperplane H such that $F_1 \subseteq H$ and $F_2 \not\subseteq H$. If $\Lambda \in \Sigma^\circ_1$, then $\Lambda \cap F_2$ is a line L with $L \neq F_1$. Therefore L determines a point in $\mathbb{P}F_2 \setminus \mathbb{P}F_1 = \mathbb{A}^1$. Now consider $\mathbb{P}(V/L) \ni \mathbb{P}(F_2/L)$: since $F_2 \not\subseteq F_3$, $\mathbb{P}(F_2/L)$ uniquely defines a point of $\mathbb{P}(V/L) \setminus \mathbb{P}(F_3/L) \simeq \mathbb{A}^2$. This gives a map $\Sigma^\circ_1 \to \mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3$ which is a bijection. □

By Theorem 1.16, the Chow ring $\mathrm{CH}(G(2,V))$ is generated by the classes $\sigma_{a,b} = [\Sigma_{a,b}] \in \mathrm{CH}^{a+b}(G(2,V))$.

The multiplicative structure is given by

$$\begin{aligned}
\sigma_1^2 &= \sigma_{1,1} + \sigma_2 \\
\sigma_1\sigma_{1,1} &= \sigma_1\sigma_2 = \sigma_{2,1} \\
\sigma_1\sigma_{2,1} &= \sigma_{2,2} \\
\sigma_{1,1}^2 &= \sigma_2^2 = \sigma_{2,2} \\
\sigma_2\sigma_{1,1} &= 0.
\end{aligned}$$

We compute few of these products explicitly. In order to prove these relations, we assume that Schubert cycles corresponding to distinct generic flags are transverse. This will be shown more precisely later.

Example 2.5. We show $\sigma_2^2 = \sigma_{2,2}$. Let $\Sigma_2(F_\bullet^{(1)})$ and $\Sigma_2(F_\bullet^{(2)})$ be the corresponding Schubert varieties given by two generic flags $F_\bullet^{(1)}, F_\bullet^{(2)}$. Then

$$\Sigma_2(F_\bullet^{(1)}) \cap \Sigma_2(F_\bullet^{(2)}) = \{\Lambda : F_1^{(1)}, F_1^{(2)} \subseteq \Lambda\} = [\langle F_1^{(1)}, F_1^{(2)}\rangle]$$

which is a single element. So $\sigma_2^2 = \sigma_{2,2}$.

Similarly $\sigma_{1,1}^2 = \sigma_{2,2}$, resulting from

$$\Sigma_{1,1}(F_\bullet^{(1)}) \cap \Sigma_{1,1}(F_\bullet^{(2)}) = [F_3^{(1)} \cap F_3^{(2)}].$$

Moreover $\Sigma_2(F_\bullet^{(1)}) \cap \Sigma_{1,1}(F_\bullet^{(2)}) = \{\Lambda : F_1^{(1)} \subseteq \Lambda \subseteq F_3^{(2)}\} = \emptyset$ since by genericity assumption $F_1^{(1)} \not\subseteq F_3^2$. This shows $\sigma_2\sigma_{1,1} = 0$. ♠

From the multiplicative relations, one obtains

$$\mathrm{CH}(G(2,V)) = \frac{\mathbb{Z}[\sigma_1,\sigma_2]}{\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2}.$$

Example 2.6 (Lines meeting four given lines in $\mathbb{P}^3$)**.** How many lines meet four generic lines in $\mathbb{P}^3$?

Given a flag $F_\bullet = (F_1, F_2, F_3)$ in V, consider its projectivization (p, L, H) in $\mathbb{P}V = \mathbb{P}^3$. The Schubert variety $\Sigma_1 \subseteq G(2,V)$ is the set of planes meeting F_2, which projectively is the set of lines in $\mathbb{P}^3$ meeting L. Therefore, the intersection of four varieties Σ_1 corresponding to four distinct flags gives the locus of lines meeting four given (generic) lines.

We have $\sigma_1^4 = \sigma_1^2 \cdot (\sigma_2 + \sigma_{1,1}) = \sigma_1 \cdot (2\sigma_{2,1}) = 2\sigma_{2,2}$. We conclude that the number of lines meeting four generic lines is $\deg(\sigma_1^4) = 2$. ♠

Example 2.7 (Lines meeting four curves in $\mathbb{P}^3$)**.** How many lines meet four generic curves of degrees $d_1, \dots, d_4$ in $\mathbb{P}^3$?

First we study the locus of lines meeting a single curve. Let $C \subseteq \mathbb{P}^3$ be a curve of degree d. Define $\Gamma_C = \{L \in G(2,V) : \mathbb{P}L \cap C \neq \emptyset\}$; Γ_C is a closed subvariety of codimension 1 in $G(2,V)$ (it is called the Chow form of C). Let $\gamma_C = [\Gamma_C] \in \mathrm{CH}(G(2,V))$. We show $\gamma_C = d\sigma_1$. To prove this, we observe that $\gamma_C \cdot \sigma_{2,1} = d$: indeed let $\Sigma_{2,1}\{\Lambda : F_1 \subseteq \Lambda \subseteq F_3\}$ for a fixed generic flag $F_\bullet$. Then

$$\#(\Gamma_C \cap \Sigma_{2,1}) = \#\{\Lambda : F_1 \subseteq \Lambda \subseteq F_3, \mathbb{P}\Lambda \cap C \neq \emptyset\}.$$

Projectively these are through $p = \mathbb{P}F_1$, contained in $H = \mathbb{P}F_3$ which intersect C. Now, $C \cap \mathbb{P}F_3$ consists of d distinct points because $\deg(C) = d$. For each of these points, consider the line Λ joining it with p. These are d distinct lines. So $\Gamma_C \cap \Sigma_{2,1}$ consists of d distinct lines, showing $\gamma_C \cdot \sigma_{2,1} = d$.

Now, if $C_1, \dots, C_4$ are four distinct curves, we have

$$\deg(\Gamma_{C_1} \cap \cdots \cap \Gamma_{C_4}) = \deg(\gamma_{C_1} \cdots \gamma_{C_4}) = (d_1\sigma_1) \cdots (d_4\sigma_1) = d_1 \cdots d_4(\sigma_1^4) = 2d_1 \cdots d_4.$$

♠

Example 2.8 (Variety of secant lines)**.** Let $C \subseteq \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g. Define a rational map

$$\begin{aligned} \Psi_2 : C \times C &\dashrightarrow G(2,V) \\ (p,q) &\mapsto \langle p, q \rangle. \end{aligned}$$

Let $\mathfrak{s}(C) = \overline{\mathrm{Im}\,(\Psi_2)} \subseteq G(2,V)$; one can show that $\dim \mathfrak{s}(C) = 2$.

We determine $[\mathfrak{s}(C)] \in \mathrm{CH}^2(G(2,V))$. Since σ_2 and $\sigma_{1,1}$ generate $\mathrm{CH}^2(G(2,V))$, one has $[\mathfrak{s}(C)] = a\sigma_2 + b\sigma_{1,1}$ for some integers a, b characterized by

$$\begin{aligned} a &= \deg(\sigma_2 \cdot [\mathfrak{s}(C)]) \\ b &= \deg(\sigma_{1,1} \cdot [\mathfrak{s}(C)]), \end{aligned}$$

because $\sigma_2 \cdot \sigma_{1,1} = 0$.

Let $H = \mathbb{P}F_3$ be a generic hyperplane and consider $\Sigma_{1,1} = \{\Lambda : \Lambda \subseteq H\}$. Then

$$b = \#(\Sigma_{1,1} \cap \mathfrak{s}(C)) = \#\{\Lambda : \Lambda \subseteq H, \Lambda \in \mathfrak{s}(C)\}.$$

The intersection $H \cap C$ consists of d points. By genericity, the lines joining pairs of such points are all distinct. This gives $b = \binom{d}{2}$.

Now let $p = \mathbb{P}F_1$ be a point and let $\Sigma_2 = \{\Lambda : p \in \Lambda\}$ be the corresponding Schubert variety. Then

$$a = \#(\Sigma_2 \cap \mathfrak{s}(C)) = \#\{\Lambda : p \in \Lambda \text{ and } \Lambda \in \mathfrak{s}(C)\}.$$

Let $\pi_p : C \to \mathbb{P}^2$ be the projection from p, mapping every point $q \in C$ to the line $\langle q, p\rangle$. The number of lines which are secant to C and pass through p correspond to double points of $\pi_p(C)$. Now $\pi_p(C)$ is a plane curve of degree d and genus g, therefore it has $\binom{d-1}{2} - g$ double points. This shows $a = \binom{d-1}{2} - g$. ♠

Example 2.9 (Common secant lines to twisted cubics)**.** Let $C_1, C_2 \subseteq \mathbb{P}^3$ be two generic twisted cubic curves. Then, how many secant lines do they have in common?

This number is given by the cardinality of the intersection $\mathfrak{s}(C_1) \cap \mathfrak{s}(C_2)$. We have $d = 3, g = 0$, therefore

$$\begin{aligned}\#(\mathfrak{s}(C_1) \cap \mathfrak{s}(C_2)) &= \deg([\mathfrak{s}(C_1)] \cdot [\mathfrak{s}(C_2)]) = \\ &= (3\sigma_{1,1} + \sigma_2)^2 = 9 + 1 = 10.\end{aligned}$$

♠

Example 2.10 (Tangent lines to a surface)**.** Let $S \subseteq \mathbb{P}^3$ be a smooth surface of degree d. Define $\mathfrak{t}(S) = \{\Lambda : \mathbb{P}\Lambda \text{ is tangent to } S\}$. We want to compute $\tau = [\mathfrak{t}(S)] \in \mathrm{CH}(G(2, V))$. Consider the incidence correspondence

$$\mathcal{T} = \{(q, \Lambda) \in S \times G(2, V) : \mathbb{P}\Lambda \subseteq T_q S\}.$$

This is a bundle over S such that the fiber at $q \in S$ is $\mathbb{P}T_q S$. In particular $\dim \mathcal{T} = 3$; the projection to $G(2, V)$ surjects onto $\mathfrak{t}(S)$, showing that $\mathfrak{t}(S)$ is irreducible and $\dim \mathfrak{t}(S) = 3$. Therefore $\tau = a\sigma_1$ for some $a \in \mathbb{Z}$.

To compute a, we consider the product $a = \deg(\tau \cdot \sigma_{2,1})$. Fix generic $F_1 \subseteq F_3$ and let $\Sigma_{2,1} = \{\Lambda : F_1 \subseteq \Lambda \subseteq F_3\}$. Set $p = \mathbb{P}F_1$ and $H = \mathbb{P}F_3$. Therefore $\Sigma_{2,1} \cap \mathfrak{t}(S)$ contains lines $\mathbb{P}\Lambda$ such that

- $p \in \mathbb{P}\Lambda$;
- $\mathbb{P}\Lambda \subseteq H$;
- $\mathbb{P}\Lambda$ is tangent to S.

By genericity $C = S \cap H$ is a smooth curve of degree d. Therefore $\mathbb{P}\Lambda$ is a tangent line to a plane curve of degree d passing through a fixed point p.

Dually, $\mathbb{P}\Lambda$ is an element of $C^\vee$ contained in a line $p^\vee \subseteq \mathbb{P}^{2*}$. The number of such elements equals $\deg(C^\vee) = d(d-1)$.

We conclude $\tau = d(d-1)\sigma_1$. ♠

Example 2.11 (Common tangent lines)**.** Let $S_1, \ldots, S_4$ be four generic surfaces of degree $d_1, \ldots, d_4$ respectively. How many lines are tangent to all of them?

This is the number of points in the intersection $\mathfrak{t}(S_1) \cap \cdots \cap \mathfrak{t}(S_4)$. Therefore, this is

$$\begin{aligned}\deg(\tau(S_1) \cdots \tau(S_4)) &= (d_1(d_1 - 1))\sigma_1 \cdots (d_4(d_4 - 1))\sigma_1 = \\ &= \textstyle\prod(d_i(d_i - 1))\sigma_1^4 = 2\prod(d_i(d_i - 1)).\end{aligned}$$

♠

Lecture 3: More Grassmannians

We generalize the construction of Schubert varieties to any Grassmannian:

Let n, k be integers and let $F_\bullet = (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V)$ be a complete flag in the n-dimensional vector space V, with $\dim V_i = i$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a non-increasing sequence of integers with $\lambda_1 \leq n-k$. In this case, λ is called a *partition* and it is often represented by a Young diagram, contained in the $k \times (n-k)$ box.

The Schubert variety associated to λ with flat $F_\bullet$ is

$$\Sigma_\lambda(F_\bullet) = \left\{ \Lambda \in G(k,n) : \forall i = 0, \ldots, k \quad \dim(V_{n-k+i-\lambda_i} \cap \Lambda) \geq i \right\}.$$

The class $\sigma_\lambda = [\Sigma_\lambda(F_\bullet)] \in \mathrm{CH}(G(k,n))$ is called the *Schubert class* of λ and it does not depend on the choice of $F_\bullet$.

If μ is a partition not contained in the rectangle $k \times (n-k)$, then we set $\sigma_\mu = 0$.

Remark 3.1. We provide some intuition on the condition definiting Σ_λ.

Given $\Lambda \in G(k,n)$, consider the induced flag $0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_n = \Lambda$, where $\Lambda_i = \Lambda \cap F_i$. For dimension reasons, this flag has repetitions. If Λ is generic, then $\Lambda_1 \subseteq \cdots \subseteq \Lambda_{k-1}$ is a complete flag in Λ and $\Lambda = \Lambda_k = \cdots = \Lambda_n$. In particular, all *dimension jumps* in $\Lambda_1 \subseteq \cdots \subseteq \Lambda_n$ occur as early as possible and all repetitions occur as late as possible.

If $\Lambda \in \Sigma_\lambda$ then the i-th dimension jump occurs at least λ_i steps early.

Example 3.2. We record three easy examples of Σ_λ.

- $\lambda = (\lambda_1)$. If λ has only one part, then

  $$\Sigma_\lambda = \{\Lambda \in G(k,V) : \Lambda \cap V_{n-k+1-\lambda_1} \neq 0\}.$$

  Since $\lambda_1 \leq n-k$, Σ_λ is non-empty. The larger λ_1 is, the more restrictive is the condition $V_{n-k+1-\lambda_1} \cap \Lambda \neq 0$.

  In the particular case $\lambda_1 = 1$, Σ_λ is the variety of subspaces intersecting V_{n-k} non-trivially. This is a hyperplane section of the Grassmannian in its Plücker embedding: condition in the Plucker embedding: if $V_{n-k} = v_1 \wedge \cdots \wedge v_{n-k}$ then

  $$\Sigma_1 = \{\Lambda = w_1 \wedge \cdots \wedge w_k : v_1 \wedge \cdots \wedge v_{n-k} \wedge w_1 \wedge \cdots \wedge w_k = 0\}.$$

  In particular $\dim \Sigma_1 = \dim G(k,n) - 1$.

- $\lambda = (n-p)^k = (\underbrace{n-p, \ldots, n-p}_{k})$. In this case

  $$\Sigma_\lambda = \{\Lambda \in G(k,V) : \Lambda \subseteq V_p\}.$$

  This is the sub-Grassmannian of planes contained in V_p.

- $\lambda = (n-k)^\ell$. In this case

  $$\Sigma_\lambda = \{\Lambda \in G(k,V) : V_\ell \subseteq \Lambda\}.$$

  This is the sub-Grassmannian of planes containing V_ℓ.

- $\lambda = (n-k)^k$. In this case $\Sigma_\lambda = \{V_k\}$ is a point, corresponding to the k-th plane of the flag.

♠

Lemma 3.3. *If λ, μ are two partitions such that $\lambda \geq \mu$ componentwise, then $\Sigma_\lambda \subseteq \Sigma_\mu$.*

Proof. From the definition, $\Lambda \in \Sigma_\lambda$ if and only if $\dim(\Lambda \cap V_{n-k+1-\lambda_i}) \geq i$. Since $\mu_i \leq \lambda_i$, $V_{n-k+1-\lambda_i} \subseteq V_{n-k+1-\mu_i}$, therefore $\dim(\Lambda \cap V_{n-k+1-\mu_i}) \geq i$. □

Lemma 3.4. *Let W be a subspace disjoint from V_1. Consider the inclusion maps*

$$\begin{aligned} i_{F_\bullet} &: G(k-1, W) \to G(k, V) \\ j_{F_\bullet} &: G(k, V_{n-1}) \to G(k, V) \end{aligned}$$

where $i_{F_\bullet}(E) = E + V_1$ and $j_{F_\bullet}(\Lambda) = \Lambda$. Then, for every λ

$$\begin{aligned} i_{F_\bullet}^{-1}(\Sigma_\lambda) &= \Sigma_\lambda, \\ j_{F_\bullet}^{-1}(\Sigma_\lambda) &= \Sigma_\lambda. \end{aligned}$$

3.1. **The affine stratification of Grassmannians.** We will prove that the Schubert varieties form an affine stratification of the Grassmannian.

Define $\Sigma_\lambda^\circ = \Sigma_\lambda \setminus \bigcup_{\mu > \lambda} \Sigma_\mu$. These are the *Schubert cells* in $G(k, V)$.

The following result shows that the Schubert varieties are an affine stratification of the Grassmannian. The proof is a more advanced version of the one of Lemma 2.4.

Theorem 3.5. *Fix a partition λ. Then Σ_λ° is isomorphic to the affine space $\mathbb{A}^{k(n-k)-|\lambda|}$; in particular Σ_λ is irreducible of codimension $|\lambda|$ in $G(k, V)$. If $\Lambda \in \Sigma_\lambda^\circ$, then the tangent space $T_\Lambda \Sigma_\lambda \subseteq T_\Lambda G(k, n) = \mathrm{Hom}(\Lambda, V/\Lambda)$ is the subspace of linear maps respecting the flag, namely it consists of those linear maps sending $V_{n-k+i-a_i} \cap \Lambda \subseteq \Lambda$ into $(V_{n-k+i-a_i} + \Lambda)/\Lambda$.*

In particular, from Theorem 3.5 and Theorem 1.16, we have that the classes σ_λ of the Schubert classes generate the Chow ring $\mathrm{CH}(G(k, V))$ of the Grassmannian.

Notice that the number of partitions contained in the $(n-k) \times k$ box is $\binom{n}{k}$. Therefore, $\mathrm{CH}(G(k, V))$ has rank $\binom{n}{k}$ as an abelian group.

Moreover, Remark 3.4, together with the fact that $\operatorname{codim} \Sigma_\lambda$ only depends on λ (and not on the Grassmannian in which it is contained) guarantees that the Schubert classes behave well with respect to pullback.

Lemma 3.6. *In $\mathrm{CH}(G(k, V))$ with $\dim V = n$, we have*

$$\sigma_{1^k}^{n-k} = \sigma_{n-k}^k = \sigma_{(n-k)^k}.$$

Proof. The component $\mathrm{CH}^{k(n-k)}(G(k, V))$ is generated by $\sigma_{(n-k)^k}$, so it suffices to show that $\deg(\sigma_{1^k}^{n-k}) = \deg(\sigma_{n-k}^k) = 1$.

We prove the statement for $\lambda = (1^k)$. The Schubert variety Σ_{1^k} depends on the choice of a hyperplane $H \subseteq V$ and it is defined as

$$\Sigma_{1^k}(H) = \{\Lambda : \Lambda \subseteq H\}.$$

The tangent space at Λ is $T_\Lambda \Sigma_{1^k} = \{\varphi : \Lambda \to V/\Lambda : \mathrm{Im}\, \varphi \subseteq H/\Lambda\}$.

Now,

$$\deg(\sigma_{n-k}^k) = \#\left(\bigcap_1^k \Sigma_{1^k}(H_j)\right)$$

for generic hyperplanes $H_1, \dots, H_k$. The intersection is transverse because $\bigcap_1^k H_j/\Lambda = 0$. Therefore $\deg(\sigma_{n-k}^k)$ is the cardinality of the intersection, which consists of only the element $\Lambda = \bigcap H_j$.

The proof for the case $\lambda = \sigma_{n-k}$ is similar. □

It is a fact that Schubert varieties associated to generic flags meet transversely. The genericity condition can be made very precise. Two flags $E_\bullet$ and $F_\bullet$ are transverse if $E_i \cap F_{n-i} = \emptyset$ for every i. Schubert varieties associated to transverse flags meet transversely.

3.2. **Ring structure in** $\mathrm{CH}(G(k, V))$**.** The ring structure in $\mathrm{CH}(G(k, V))$ is not very easy to understand. In general

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{\pi \subseteq (n-k)\times k \\ |\pi| = |\lambda| + |\mu|}} c_{\lambda\mu}^\pi \sigma_\pi$$

where $c_{\lambda\mu}^\pi$ are the *Littlewood-Richardson coefficients.*

Theorem 3.7 (Schubert cycles of complementary dimension)**.** *Let λ, μ be two partitions with $|\lambda| + |\mu| = k(n-k)$. Then*

$$c_{\lambda,\mu}^{(n-k)\times k} = \begin{cases} 1 & \text{if } \lambda, \mu \text{ are complementary in } (n-k)\times k \\ 0 & \text{otherwise} \end{cases}$$

Proof. We are going to compute the degree of the intersection

$$\Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(E_\bullet)$$

for two transverse flags $F_\bullet$, $E_\bullet$.

We have

$$\Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(E_\bullet) = \left\{ \Lambda : \begin{array}{l} \dim(\Lambda \cap F_{n-k+i-\lambda_i}) \geq i, \\ \dim(\Lambda \cap E_{n-k+i-\lambda_i}) \geq i \end{array} \right\}.$$

The i-th condition for $F_\bullet$ and the $(k-i+1)$-th condition for $E_\bullet$ provide

$$\Lambda \cap F_{n-k+i-\lambda_i} \geq i, \quad \Lambda \cap E_{n-i+1-\mu_{k-i+1}} \geq k-i+1.$$

Therefore the two subspaces $\Lambda \cap F_{n-k+i-\lambda_i}, \Lambda \cap E_{n-i+1-\mu_{k-i+1}}$ have non trivial intersection. In particular $F_{n-k+i-\lambda_i} \cap E_{n-i+1-\mu_{k-i+1}}$ have non-trivial intersection. By the transversality of the flags, we have

$$n + 1 \leq (n-k+i-\lambda_i) + (n-i+1-\mu_{k-i+1}) = 2n - k - \lambda_i + 1 - \mu_{k-i+1}$$

which implies $\lambda_i + \mu_{k-i+1} \leq n-k$. Adding over $i = 1, \dots, k$, since $|\lambda| + |\mu| = k(n-k)$, we obtain $\lambda_i + \mu_{k-i+1} = n-k$ for every i. This shows that if $\Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(E_\bullet) \neq \emptyset$ then λ and μ are complementary in the $(n-k)\times k$ rectangle.

If indeed they are complementary, then $\lambda_i + \mu_{k-i+1} = n-k$; in this case, the intersection $F_{n-k+i-\lambda_i} \cap E_{n-i+1-\mu_{k-i+1}}$ is a one-dimensional space P_i and since $F_{n-k+i-\lambda_i} \cap E_{n-i+1-\mu_{k-i+1}} \cap \Lambda$ is non-trivial, we have $P_i \subseteq \Lambda$. By genericity, the P_i's are linearly independent, therefore they span Λ.

This shows that $\Lambda \in \Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(E_\bullet)$ is uniquely determined by the choices of the flags, therefore $\deg(\sigma_\lambda\sigma_\mu) = 1$. □

Theorem 3.7 naturally defines a pairing between $\mathrm{CH}^p(G(k,V))$ and $CH^{k(n-k)-p}(G(k,V))$. For a partition λ, let λ^* be its complementary in the $k \times (n-k)$ rectangle, namely

$$\lambda^*_i = n - k + 1 - \lambda_{k+1-i}.$$

This defines an isomorphism

$$\begin{aligned}(\mathrm{CH}^p(G(k,V)))^\vee &\to \mathrm{CH}^{k(n-k)-p}(G(k,V))\\ \langle \sigma_\lambda, - \rangle &\mapsto \sigma_{\lambda^*}\end{aligned}$$

where $\langle \sigma_\lambda, - \rangle$ is the element dual to σ_λ in the basis of $\mathrm{CH}^p(G(k,V))$ dual to the Schubert basis.

This gives a convenient way to determine the coefficients of a class $\alpha \in \mathrm{CH}^p(G(k,V))$. We have

$$\alpha = \sum_{|\lambda|=p} \deg(\alpha\sigma_{\lambda^*})\sigma_\lambda.$$

In particular, the Littlewood-Richardson coefficient $c^\pi_{\lambda\mu}$, which is the coefficient of σ_π in $\sigma_\lambda\sigma_\mu$, coincides with $\deg(\sigma_\lambda\sigma_\mu\sigma_{\pi^*})$.

Proposition 3.8 (Pieri's formula)**.** *Let λ be a partition in the $k \times (n-k)$ box and let $p \geq 1$. Then*

$$\sigma_\lambda\sigma_p = \sum_{\substack{|\mu|=|\lambda|+p\\ \lambda_i \leq \mu_i \leq \lambda_{i-1}}} \sigma_\mu$$

Proof. We want to prove that if $|\mu| = |\lambda| + p$ then the Littlewood-Richardson coefficient $c^\mu_{\lambda,p}$ is 1 if μ interlaces λ and 0 otherwise. From the discussion above, $c^\mu_{\lambda,p} = \deg(\sigma_\lambda\sigma_p\sigma_{\mu^*})$.

Consider three generic flags $F_\bullet$, $G_\bullet$ and $E_\bullet$ and the three corresponding Schubert varieties $\Sigma_\lambda(F_\bullet)$, $\Sigma_p(G_\bullet)$ and $\Sigma_{\mu^*}(E_\bullet)$. The only relevant element for $G_\bullet$ is the $(n-k+1-p)$-th plane.

First, we show that $\Sigma_\lambda(F_\bullet) \cap \Sigma_p(G_\bullet) \cap \Sigma_{\mu^*}(E_\bullet)$ is empty if μ does not interlace λ.

By definition

$$\begin{aligned}\Sigma_\lambda(F_\bullet) &= \{\Lambda \in G(k,V) : \dim(\Lambda \cap F_{n-k+i-\lambda_i}) \geq i\}\\ \Sigma_p(G_\bullet) &= \{\Lambda \in G(k,V) : \dim(\Lambda \cap G_{n-k+1-p}) \geq 1\}\\ \Sigma^*_\mu(E_\bullet) &= \{\Lambda \in G(k,V) : \dim(\Lambda \cap E_{i+\mu_{k+1-i}}) \geq i\},\end{aligned}$$

Define $A_i = F_{n-k+i-\lambda_i} \cap E_{k+1-i+\mu_i}$. Since the flags are transverse,

$$\dim A_i = (n-k+i-\lambda_i) + (k+1-i+\mu_i) - n = \mu_i - \lambda_i + 1 \text{ (or 0 is this is negative).}$$

Let $\Lambda \in \Sigma_\lambda(F_\bullet) \cap \Sigma_{\mu^*}(E_\bullet)$.

The i-th condition for $\Sigma_\lambda(F_\bullet)$ and the $(k+1-i)$-th condition of $\Sigma_{\mu^*}(E_\bullet)$ guarantee $\Lambda \cap A_i \neq 0$ because $i + k + 1 - i = 1$. In particular $\dim A_i = \mu_i - \lambda_i + 1 \geq 1$ so $\mu_i \geq \lambda_i$. Moreover, Λ is spanned by its intersections with the A_i because it is spanned by the induced flags.

One can show that the A_i are linearly independent if and only if $\mu_i \leq \lambda_{i-1}$.

Let $A = A_1 + \cdots + A_k$. We have

$$\dim(A_1 + \cdots + A_k) = \sum \dim A_i \leq p + k$$

and equality holds if and only if $\mu_i \leq \lambda_{i-1}$.

Now, $G_{n-k+1-p}$ has generic intersection with A, so if $\dim A < p-k$ we have $G_{n-k+1-p} \cap A = 0$ and so $G_{n-k+1-p} \cap \Lambda = 0$. This shows that

$$\Sigma_\lambda(F_\bullet) \cap \Sigma_p(G_\bullet) \cap \Sigma_{\mu^*}(E_\bullet) = \emptyset$$

if μ does not interlace λ.

If μ does interlace λ, then $\dim A = p+k$ and by genericity $A \cap G_{n-k+1-p} = \langle v \rangle$ is 1-dimensional. Since $v \in A_1 \oplus \cdots \oplus A_k$, we have $v = v_1 + \cdots + v_k$ with $v_j \in A_j$ and by the genericity condition on $G_{n-k+1-p}$, all v_j's are nonzero.

Define $\Lambda = \langle v_1, \ldots, v_k \rangle$. By construction $\Lambda \in \Sigma_\lambda(F_\bullet) \cap \Sigma_p(G_\bullet) \cap \Sigma_{\mu^*}(E_\bullet)$ so this is not empty. Moreover, for any element Λ of the intersection, the subspaces $\Lambda \cap A_i$ uniquely determine Λ, so $\langle v_1, \ldots, v_k \rangle$ is the only element in $\Sigma_\lambda(F_\bullet) \cap \Sigma_p(G_\bullet) \cap \Sigma_{\mu^*}(E_\bullet)$.

This shows

$$\deg(\Sigma_\lambda(F_\bullet) \cap \Sigma_p(G_\bullet) \cap \Sigma_{\mu^*}(E_\bullet)) = 1$$

if μ interlaces λ. □

The correspondence $G(k, V) \leftrightarrow G(n-k, V^*)$ provides a column-wise Pieri's formula as well

Corollary 3.9. *Let λ be a partition in the $k \times (n-k)$ box and let $p \geq 1$. Then*

$$\sigma_\lambda \sigma_{1^p} = \sum_{\substack{|\mu| = |\lambda| + p \\ \lambda_i \leq \mu_i \leq \lambda_i + 1}} \sigma_\mu$$

Giambelli's formula allows us to write a Schubert class in terms of special Schubert classes.

Corollary 3.10 (Giambelli's formula). *Let λ be a partition in the $k \times (n-k)$ rectangle. Then*

$$\sigma_\lambda = \det \begin{pmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \cdots & \sigma_{\lambda_1+k-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \cdots & \sigma_{\lambda_2+k-2} \\ \vdots & & \ddots & \\ \sigma_{\lambda_k-k+1} & & & \sigma_{\lambda_k} \end{pmatrix}.$$

Lecture 4: Chern classes

Let X be a smooth variety and let $\mathcal{E}$ be a vector bundle of rank k over X with bundle map $\pi : \mathcal{E} \to X$. Suppose that $\mathcal{E}$ is globally generated, that is there exist sections $s_1, \ldots, s_N$ such that, for every $x \in X$, $s_1(x), \ldots, s_N(x)$ span $\mathcal{E}_x$.

Chern classes are elements of the Chow ring $\mathrm{CH}^\bullet(X)$ associated to $\mathcal{E}$.

4.1. **Line bundles.** Suppose $k = 1$. Let $s \in \Gamma(\mathcal{E})$ be a generic linear section. Let

$$Y = \{x \in X : s_x(x) = 0\}.$$

Locally, Y is given by a single equation because if $U \ni x$ is a trivializing open set then $s|_U : U \to \mathbb{C}$. Therefore $\operatorname{codim}(Y) = 1$ or $Y = \emptyset$.

Lemma 4.1. *If $Y = \emptyset$ then $\mathcal{E}$ is a trivial bundle.*

Proof. If $Y = \emptyset$ then there exists a no-where vanishing global section, say $s \in \Gamma(\mathcal{E})$. Define the bundle map

$$\begin{aligned} X \times \mathbb{C} &\to \mathcal{E} \\ (x, \lambda) &\mapsto (x, \lambda s(x)). \end{aligned}$$

This is a bundle isomorphism. □

Definition 4.2. Let $\mathcal{E}$ be a line bundle on X. Assume $\mathcal{E}$ is generated by global sections. The first Chern class of $\mathcal{E}$ is $[Y] \in \mathrm{CH}^1(X)$ where Y is the vanishing locus of a generic section of $\mathcal{E}$.

In particular, $c_1(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a trivial line bundle.

Definition 4.3. The Picard group of a variety X is a

$$\mathrm{Pic}(X) = \{\mathcal{L} : \ \mathcal{L} \text{ line bundle on } X\}/\simeq$$

the set of isomorphism classes of line bundles on X. It is a fact that $\mathrm{Pic}(X)$ is an abelian group under the operation of tensor product, and the trivial bundle is the identity element.

Lemma 4.4. *The first Chern class $c_1 : \mathrm{Pic}(\mathcal{L}) \to \mathrm{CH}^1(X)$ is a group homomorphism. If X is smooth, the it is an isomorphism. In particular:*

- *If $\mathcal{L}_1, \mathcal{L}_2$ are line bundles, then $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$;*
- *$c_1(\mathcal{L}) = -c_1(\mathcal{L}^\vee)$ where $\mathcal{L}^\vee$ denotes the dual bundle.*

Example 4.5 (Projective space)**.** Let $\mathcal{O}_{\mathbb{P}V}(d)$ be the d-th twist of the hyperplane bundle on $\mathbb{P}V$, that is the vector bundle whose fiber at the element $[v] \in \mathbb{P}V$ is $\langle v^* \rangle^{\otimes d}$. Then $H^0(\mathcal{O}_{\mathbb{P}V}(d)) = S^d V^*$. If F is a homogeneous polynomial of degree d on V, then it naturally defines a section of $\mathcal{O}_{\mathbb{P}V}(d)$ via $F : [v] \mapsto F|_{\langle v \rangle^{\otimes d}}$.

The vanishing locus of F is a hypersurface of degree d. Therefore $c_1(\mathcal{O}_{\mathbb{P}V}(d)) = d\zeta \in \mathrm{CH}^1(\mathbb{P}V)$, where ζ is the hyperplane class of $\mathbb{P}V$.

As a consequence $c_1(\mathcal{O}_{\mathbb{P}V}(-1)) = -\zeta$. Consider the tautological sequence of

$$0 \to \mathcal{S} \simeq \mathcal{O}_{\mathbb{P}V}(-1) \to \underline{V} \to \mathcal{Q} \to 0$$

of vector bundles on $\mathbb{P}V$. By Whitney's formula $c(\mathcal{Q}) = \frac{1}{c(\mathcal{S})} = \frac{1}{1-\zeta} = 1 + \zeta + \cdots + \zeta^{n-1}$. ♠

4.2. **Higher rank bundles.** The definition of Chern classes for higher rank vector bundles uses the same idea as in the line bundle case but the "vanishing" condition is replaced by the "degeneracy" of a subspace of sections.

Let $\mathcal{E}$ be a vector bundle of rank k over X. Then $\Lambda^k\mathcal{E}$ is a line bundle. We define $c_k(\mathcal{E}) := c_1(\Lambda^k\mathcal{E})$, the class of the locus where k generic sections are linearly dependent.

Lemma 4.6. *Let $\mathcal{E}$ be a globally generated vector bundle of rank k. Let $p \leq k$ and let $s_0, \dots, s_{k-p}$ be $k-p+1$ generic sections of $\mathcal{E}$. Let $D_p = \{x \in X : s_0(x) \wedge \cdots \wedge s_{k-p}(x) = 0\} \subseteq X$. Then*

- *For every component D' of D, we have $\operatorname{codim}_X(D') \leq p$;*
- *D' is generically reduced and it has codimension exactly p.*

Proof. Refer to [EH16, Lemma 5.1]. □

We have $D_p = \{x : \ k-p+1$ generic sections of $\mathcal{E}$ are linearly dependent$\}$ is a variety of codimension p. Define

$$c_p(\mathcal{E}) = [D_p] \in \mathrm{CH}^p(X).$$

Two objects which "know" all Chern classes:

- full Chern class of $\mathcal{E}$: $c(\mathcal{E}) = \sum_{p \geq 0} c_p(\mathcal{E})$. It is an element of $\mathrm{CH}^\bullet(X)$.
- Chern polynomial of $\mathcal{E}$: $c_{[t]}(\mathcal{E}) = \sum_{p \geq 0} c_p(\mathcal{E}) t^p$. It is an element of $\mathrm{CH}^\bullet(X)[t]$.

The full Chern class of $\mathcal{E}$ is characterized by the following Theorem

Theorem 4.7 ([EH16], Theorem 5.3)**.** *Let $\mathcal{E}$ be a globally generated vector bundle of rank k on a smooth variety X. Then there exists a unique element $c(\mathcal{E}) = \sum_{p>0} c_p(\mathcal{E}) \in \mathrm{CH}^\bullet(X)$ such that*

- *(Line bundles) If $\mathcal{E}$ is a line bundle on X, then $c(\mathcal{E}) = 1 + c_1(\mathcal{E})$ where c_1 is the class of the vanishing locus of a generic section of $\mathcal{E}$.*
- *(Bundles with enough sections) If $\tau_0, \dots, \tau_{r-p}$ are global sections of $\mathcal{E}$ and $D := V(\tau_0 \wedge \cdots \wedge \tau_{k-p})$ has codimension exactly p, then $c_p(\mathcal{E}) = [D] \in \mathrm{CH}^p(X)$.*
- *(Whitney's formula) If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is a short exact sequence of vector bundles, then $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}) \in \mathrm{CH}(X)$.*
- *(Functoriality) If $\varphi : Y \to X$ is a morphism of smooth schemes, then $\varphi_*(c(\mathcal{E})) = c(\varphi^*\mathcal{E}) \in \mathrm{CH}(Y)$.*

Two important consequences:

- If $\mathcal{E} = \bigoplus_1^r \mathcal{L}_j$ is direct sum of line bundles, then

  $$c(\mathcal{E}) = \prod_{j=1}^{k} (1 + c_1(\mathcal{L}_j)).$$

  And the Chern classes of $\mathcal{E}$ can be interpreted as symmetric functions of the first Chern classes of the line bundles.
- When we do calculations with Chern classes, we can always pretend that bundles split as direct sums of line bundles. The Chern classes of these line bundles are sometimes called *virtual Chern classes* of $\mathcal{E}$; their opposites are the roots of the Chern polynomial of $\mathcal{E}$.

Example 4.8 (Grassmannian)**.** Let V be a vector space and let $G(k,V)$ be the Grassmannian of k-planes in V. We compute the Chern classes of the tautological and the quotient bundle over $G(k,V)$.

We start with the universal quotient bundle $\mathcal{Q}$, whose fiber at $E \in G(k,V)$ is the quotient V/E. An element $v \in V$ induces a section $s : E \mapsto (E, v \mod E)$. Therefore, $v_1,\dots,v_p \in V$ induce sections $s_1,\dots,s_p$ of $\mathcal{Q}$. The degeneracy locus of these is the locus of the Λ such that

$$\langle v_1,\dots,v_p\rangle \mod \Lambda \subseteq V/\Lambda$$

has dimension at most $p-1$, that is $\dim(\Lambda \cap \langle v_1,\dots,v_p\rangle) \geq 1$. Regarding $\langle v_1,\dots,v_p\rangle$ as the p-th element of a generic flag, we see that this is the Schubert condition for $\Sigma_{n-k-p+1}$. We deduce $c_p(\mathcal{Q}) = \sigma_p$.

From the exact sequence $0 \to \mathcal{S} \to \underline{V} \to \mathcal{Q} \to 0$, we deduce

$$c(\mathcal{S})c(\mathcal{Q}) = 1$$

and we obtain

$$c(\mathcal{S}) = 1 - \sigma_1 + \sigma_{1^2} - \sigma_{1^3} + \cdots \pm \sigma_{1^k}.$$

which can be verified via Pieri's rule. ♠

Lecture 5: Determinantal varieties

5.1. **Definition, desingularization and dimension.** Fix $r,m,n > 0$. Let $\mathrm{Mat}_{m\times n}$ be the space of $m \times n$ matrices. The r-th generic determinantal variety is

$$D_r(m,n) = \{[A] \in \mathbb{P}\mathrm{Mat}_{m\times n} : \mathrm{rk}(A) \leq r\} \subseteq \mathbb{P}\mathrm{Mat}_{m,n}.$$

Remark 5.1. The set $D_r(m,n)$ is an algebraic variety because it is the zero set of $(r+1)\times(r+1)$.

Fix the notation $E = \mathbb{C}^m$, $F = \mathbb{C}^n$ and identify $\mathrm{Mat}_{m\times n} \simeq E^* \otimes F$.

Proposition 5.2. *The variety $D_r(m,n)$ is irreducible of codimension $(m-r)(n-r)$.*

Proof. Define the incidence correspondence

$$\mathcal{I} = \{([A],L) \subseteq \mathbb{P}\mathrm{Mat}_{m\times n} \times G(r,F) : \mathrm{Im}\,(A) \subseteq L\}$$

There are natural projections

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\pi_{\mathrm{Mat}}"'] \arrow[dr, "\pi_G"] & \\ \mathbb{P}\mathrm{Mat}_{m\times n} & & G(r,F). \end{tikzcd}$$

First, we show that $\mathcal{I}$ is irreducible. To see this, observe that $\pi_G : \mathcal{I} \to G(r,F)$ is surjective and all its fibers are linear spaces of the same dimension: indeed, for $L \subseteq F$, we have

$$\pi_G^{-1}(L) = \{A : \mathrm{Im}\,A \subseteq L\} \simeq E^* \otimes L.$$

This shows that $\mathcal{I}$ is irreducible of dimension

$$\dim \mathcal{I} = \dim G(r,F) + \dim \pi_G^{-1}(L) = r(n-r) + mr = (m+n)r - r^2.$$

The projection π_{Mat} surjects onto $D_r(m,n)$ and it is generically one-to-one. Therefore $D_r(m,n)$ is irreducible of dimension $r(n-r)+mr$. Its codimension is $mn-(m+n)r+r^2 = (m-r)(n-r)$. □

In fact, $\mathcal{I}$ is a vector bundle over $G(r,F)$, therefore it is smooth. This shows that $\mathcal{I}$ is a *desingularization* of $D_r(m,n)$.

Equivalently, we can define "another" desingularization using kernels instead of images

$$\widetilde{D}_r(m,n) = \{([A],K) \subseteq \mathbb{P}\mathrm{Mat}_{m\times n} \times G(m-r,E) : K \subseteq \ker(A)\}.$$

5.2. **Degeneracy loci.** Let X be a (smooth) algebraic variety. Let $\mathcal{E},\mathcal{F}$ be vector bundles on X of rank e,f respectively; let $\varphi : \mathcal{E} \to \mathcal{F}$ be a bundle map.

The r-the degeneracy locus of φ is

$$D_r^{\varphi}(\mathcal{E},\mathcal{F}) = \{x \in X : \mathrm{rk}(\varphi_x) \leq r\}.$$

Remark 5.3. We have $\operatorname{codim} D_r^{\varphi}(\mathcal{E},\mathcal{F}) \leq (m-r)(n-r)$. We say that $D_r^{\varphi}(\mathcal{E},\mathcal{F})$ has the expected codimension if equality holds.

The general determinantal variety corresponds to the case:

- $X = \mathbb{P}\mathrm{Mat}_{e\times f}$;
- $\mathcal{E} = {\mathcal{O}_{\mathbb{P}\mathrm{Mat}_{e\times f}}}^{\oplus e}$: the trivial bundle of rank e;
- $\mathcal{F} = \mathcal{O}_{\mathbb{P}\mathrm{Mat}_{e\times f}}(1)^{\oplus f}$;
- $\varphi : \mathcal{E} \to \mathcal{F}$ defined by $\varphi_A : v \mapsto Av$.

5.3. **Introduction to Porteous's formula.** Porteous's formula expresses $[D_r^{\varphi}(\mathcal{E},\mathcal{F})] \in \mathrm{CH}^{(m-r)(n-r)}(X)$ in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{F}$.

Let $\mathbf{c} = (c_0, c_1, \ldots)$ be a sequence of elements in a commutative ring. For integers e,f, define the element

$$\Delta_f^e(\mathbf{c}) = \det \begin{bmatrix} c_f & c_{f+1} & \cdots & c_{f+e-1} \\ c_{f-1} & c_f & & c_{f+e-2} \\ \vdots & & \ddots & \\ c_{f-e+1} & & & c_f \end{bmatrix};$$

this is the *Sylvester determinant* of $\mathbf{c}$ of order f and degree e. Write $S_f^e(\mathbf{c})$ for the matrix above.

Let $a(t) = \sum_0^e a_i t^i$ and $b(t) = \sum_0^f b_j t^j$ be two polynomials of degree e and f respectively. Then we have the following formula for the Sylvester resultant:

$$\mathrm{Res}_t(a(t),b(t)) = \Delta_f^e \left[\frac{b(t)}{a(t)}\right] = (-1)^{ef} \Delta_e^f \left[\frac{a(t)}{b(t)}\right],$$

where $a(t)/b(t)$ is identified with the sequence of the coefficients of its power series.

Theorem 5.4 (Porteous's Formula)**.** *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles of ranks e and f over X whose r-th degeneracy locus has the expected codimension. Then*

$$[D_r^{\varphi}(\mathcal{E},\mathcal{F})] = \Delta_{f-r}^{e-r}\left(c(\mathcal{E})/c(\mathcal{F})\right).$$

Lecture 6: Porteous's Formula

In this lecture, we prove Porteous's formula and we use it to compute the degree of general determinantal varieties.

6.1. **Easiest case: $r = 0$ and expected codimension.**

If $r = 0$, then

$$D_0^{\varphi}(\mathcal{E},\mathcal{F}) = \{x \in X : \varphi_x : \mathcal{E}_x \to \mathcal{F}_x \text{ is identically } 0\}.$$

Therefore $D_0^{\varphi}(\mathcal{E},\mathcal{F})$ is the 0-locus of $\varphi \in H^0(\mathcal{E}^* \otimes \mathcal{F})$. Suppose $\operatorname{codim} D_0^{\varphi}(\mathcal{E},\mathcal{F}) = ef$ coincides with the expected dimension. Then, by definition of Chern class

$$[D_0^{\varphi}(\mathcal{E},\mathcal{F})] = c_{ef}(\mathcal{E}^* \otimes \mathcal{F}).$$

Theorem 6.1. *Let $\mathcal{E},\mathcal{F}$ be vector bundle on X. Then*

$$c_{ef}(\mathcal{E}^*,\mathcal{F}) = \operatorname{Res}_t(c_t(\mathcal{E}), c_t(\mathcal{F})) = \Delta_f^e(c(\mathcal{E})/c(\mathcal{F})),$$

where $c_t(-)$ is the Chern polynomial and $\operatorname{Res}_t(-,-)$ is the Sylvester resultant.

Proof. Apply the splitting principle. Suppose $\mathcal{E} = \bigoplus_{i=1}^e \mathcal{L}_i$ and $\mathcal{F} = \bigoplus_{j=1}^f \mathcal{M}_j$. Then

$$\mathcal{E}^* \oplus \mathcal{F} = \bigoplus_{ij} \mathcal{L}_i^* \otimes \mathcal{M}_j.$$

Write $\alpha_i := c_1(\mathcal{L}_i)$ and $\beta_j := c_1(\mathcal{M}_j)$, so that $c_1(\mathcal{L}_i^* \otimes \mathcal{M}_j) = \beta_j - \alpha_i$. Then

$$c(\mathcal{E}^* \otimes \mathcal{F}) = c(\bigoplus_{ij} \mathcal{L}_i^* \otimes \mathcal{M}_j) = \prod_{ij}(1 - \alpha_i + \beta_j)$$

and the top Chern class $c_{ef}(\mathcal{E}^* \otimes \mathcal{F})$ equals the term of highest degree at the right hand side, which is

$$c_{ef}(\mathcal{E}^* \otimes \mathcal{F}) = \prod_{ij}(\beta_j - \alpha_i).$$

Now, the α_i are the roots of the Chern polynomial of $\mathcal{E}$ and the β_j are the roots of the Chern polynomial of $\mathcal{F}$. Therefore $c_{ef}(\mathcal{E}^* \otimes \mathcal{F}) = \operatorname{Res}_t(c_t(\mathcal{E}), c_t(\mathcal{F}))$. □

6.2. **Reduction to the case of correct codimension.**

Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a vector bundle map on X. It is possible that the r-th degeneracy locus of φ does not have the *expected* codimension $(e-r)(f-r)$. In this section, we apply a Grassmann bundle construction which reduces the general case to the one of expected codimension.

First, we discuss the construction and the universal properties of the Grassmann bundle. Suppose $\mathcal{V}$ is a vector bundle of rank n on X. The Grassmann bundle of k-planes in $\mathcal{E}$ on X is

$$\begin{array}{c} \mathcal{G}(k,\mathcal{V}) = \{(x, L_x) : L_x \in G(k, \mathcal{V}_x)\} \\ \big\downarrow \\ X \end{array}$$

Then $\mathcal{G}(k,\mathcal{V})$ is a fiber bundle on X whose fiber at $x \in X$ is $G(k,\mathcal{V}_x)$. Write $\rho : \mathcal{G}(k,\mathcal{E}) \to X$. This fiber bundle $\mathcal{G}(k,\mathcal{V})$ has a tautological bundle whose fiber at (x, L_x) is the plane L_x itself; the bundle $\mathcal{S}$ has a natural inclusion into $\rho^*\mathcal{E}$, the vector bundle over $\mathcal{G}(k,\mathcal{V})$ whose fiber at (x, E_x) is the plane E_x.

Recall the universal property of the Grassmannian: if $\mathcal{F}$ is a vector sub-bundle of $\underline{V} = X \times V$ of rank k over X, then there exists a unique morphism $f : X \to G(k, V)$ such that $\mathcal{F} = f^*\mathcal{S}$. Moreover, the inclusion $\mathcal{S} \to G(k, V) \times V$ lifts to the inclusion inclusion $\mathcal{F} \to X \times V$, in the sense of the following diagram

$$\begin{tikzcd}
 & X \times V \arrow[r, "f\times\mathrm{id}"] \arrow[ddl] & G(k,V)\times V \arrow[ddl] \\
\mathcal{F} \simeq f^*S \arrow[ur] \arrow[r, "f^*\pi"'] \arrow[d] & \mathcal{S} \arrow[ur] \arrow[d, "\pi"'] & \\
X \arrow[r, "f"'] & G(k,V) &
\end{tikzcd}$$

More generally, this construction generalizes to the case of arbitrary inclusions of vector bundles $\psi : \mathcal{E} \to \mathcal{V}$. If $\mathcal{E}$ has rank k, such an inclusion uniquely defines a map $f : X \to \mathcal{G}(k, \mathcal{V})$ such that $\mathcal{E} = f^*\mathcal{S}$ and the inclusion map $\mathcal{S} \to \rho^*\mathcal{V}$ lifts to ψ.

$$\begin{tikzcd}
 & \mathcal{V} \arrow[r] \arrow[ddl] & \rho^*\mathcal{V} \arrow[ddl] & \\
\mathcal{E} \simeq f^*\mathcal{S} \arrow[ur] \arrow[r] \arrow[d] & \mathcal{S} \arrow[ur] \arrow[d] & & \\
X \arrow[r, "f"'] & \mathcal{G}(k,\mathcal{V}) \arrow[r, "\rho"'] & X &
\end{tikzcd}$$

Now, starting from a bundle map $\varphi : \mathcal{E} \to \mathcal{F}$ of vector bundles over X, consider the augmented map

$$\psi \oplus \mathrm{id}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \oplus \mathcal{F},$$

which defines an inclusion of vector bundles. From the discussion above, we have the diagram

$$\begin{tikzcd}
 & \mathcal{E}\oplus\mathcal{F} \arrow[r] \arrow[ddl] & \rho^*(\mathcal{E}\oplus\mathcal{F}) & \\
\mathcal{E} \simeq f^*\mathcal{S} \arrow[ur, "\psi"] \arrow[r] \arrow[d] & \mathcal{S} \arrow[ur] \arrow[d] & & \\
X \arrow[r, "f"'] & \mathcal{G}(e,\mathcal{E}\oplus\mathcal{F}) \arrow[r, "\rho"'] & X &
\end{tikzcd}$$

The inclusion map $\mathcal{S} \to \rho^*(\mathcal{E} \to \mathcal{F})$ pulls back to ψ and therefore the composition φ' given by $\mathcal{S} \to \rho^*(\mathcal{E} \oplus \mathcal{F}) \to \rho^*\mathcal{F}$ (where the last map is the projection induced by the direct summand) pulls back to φ. We obtain that $D_r^{\varphi}(\mathcal{E}, \mathcal{F}) = f^{-1}(D_r^{\varphi'}(\mathcal{S}, \rho^*\mathcal{F}))$.

Now, since $\mathcal{E}$ is the kernel of the projection $\mathcal{E} \to \mathcal{E} \oplus \mathcal{F}$, an element $\xi = (x, H_x) \in \mathcal{G}(e, \mathcal{E} \oplus \mathcal{F})$ belongs to $D_r^{\varphi'}(\mathcal{S}, \rho^*\mathcal{F})$ if and only if $\dim(\mathcal{S}_\xi \cap (\rho^*\mathcal{E})_\xi) \geq e - r$. This is the "vector bundle analog" of the Schubert condition for $\Sigma_{(e-r)^{f-r}}$. In fact, this shows that on a trivializing open set for $\mathcal{G}(e, \mathcal{E} \oplus \mathcal{F})$, the variety $D_r^{\varphi'}(\mathcal{S}, \rho^*\mathcal{F})$ is rationally equivalent to $X \times \Sigma_{(e-r)^{f-r}} \subseteq X \times G(e, e + f)$. This shows that $D_r^{\varphi'}(\mathcal{S}, \rho^*\mathcal{F})$ is irreducible, reduced and of codimension

$(e-r)(f-r)$. Moreover $D^{\varphi'}_{r-1}(\mathcal{S},\rho^*\mathcal{F})$ has dimension strictly smaller than $D^{\varphi'}_{r}(\mathcal{S},\rho^*\mathcal{F})$ because it corresponds to the Schubert condition for $\Sigma_{(e-r+1)^{f-r+1}}$.

It turns out, via the Cohen-Macaulay property, that the pull-back map $f^* : \mathrm{CH}(\mathcal{G}(e,\mathcal{E}\oplus\mathcal{F}) \to \mathrm{CH}(X)$ satisfies $f^*([D^{\varphi'}_r(\mathcal{S},\rho^*\mathcal{F})]) = [D^{\varphi}_r(\mathcal{E},\mathcal{F})]$.

Moreover, the Chern classes of $\mathcal{S}$ and of $\rho^*\mathcal{F}$ pull back to the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. This guarantees that a Chern class expression for $[D^{\varphi'}_r(\mathcal{S},\rho^*\mathcal{F})]$ pulls back to a Chern class expression for $[D^{\varphi}_r(\mathcal{E},\mathcal{F})]$.

6.3. **Reduction to the top Chern class case.** Following the discussion in the previous section, we may assume $\varphi : \mathcal{E} \to \mathcal{F}$ is a vector bundle map on X such that

- $D^{\varphi}_r(\mathcal{E},\mathcal{F})$ has codimension $(e-r)(f-r)$;
- $D^{\varphi}_r(\mathcal{E},\mathcal{F})$ is reduced;
- $D^{\varphi}_{r-1}(\mathcal{E},\mathcal{F})$ is strictly smaller than $D^{\varphi}_r(\mathcal{E},\mathcal{F})$.

Next, we reduce to the top Chern class case, extending the desingularization construction to the bundle setting.

Define $\mathcal{G}(e-r,\mathcal{E}) = \{(x,K_x) : K_x \subseteq \mathcal{E}_x\}$, the Grassmann bundle of $e-r$-planes in the fibers of $\mathcal{E}$. Let $\rho_{\mathcal{E}} : \mathcal{G}(e-r,\mathcal{E}) \to X$ be the bundle map. The variety $\mathcal{G}(e-r,\mathcal{E})$ has a tautological and a quotient bundle; there are bundles $\mathcal{S},\mathcal{Q}$ over $\mathcal{G}(e-r,\mathcal{E})$ such that

$$0 \to \mathcal{S} \to \rho_{\mathcal{E}}^*\mathcal{E} \to \mathcal{Q} \to 0$$

is exact, where

- $\mathcal{S}_{(x,K_x)} = K_x$
- $(\pi_{\mathcal{E}}^*\mathcal{E})_{(x,K_x)} = \mathcal{E}_x$
- $\mathcal{Q}_{(x,K_x)} = \mathcal{E}_x/K_x$.

The map φ pulls back on the bundles, defining $\rho_{\mathcal{E}}^*\varphi : \rho_{\mathcal{E}}^*\mathcal{E} \to \rho_{\mathcal{E}}^*\mathcal{F}$. Now $\rho_{\mathcal{E}}^*\varphi$ restricts to $\mathcal{S}$, giving a map $\widetilde{\varphi} : \mathcal{S} \to \rho_{\mathcal{E}}^*\mathcal{F}$. Punctually, on the fibers:

$$\begin{aligned} \widetilde{\varphi}_{(x,K_x)} : K_x &\to \mathcal{F}_x \\ v &\mapsto \varphi_x(v). \end{aligned}$$

Define

$$\widetilde{D}^{\varphi}_r(\mathcal{E},\mathcal{F}) = \{(x,K_x) \in \mathcal{G}(e-r,\mathcal{E}) : \widetilde{\varphi}_x = 0\}.$$

We deduce $\rho^{-1}(D^{\varphi}_r(\mathcal{E},\mathcal{F})) = \widetilde{D}^{\varphi}_0(\mathcal{E},\mathcal{F})$. Moreover, the restriction of ρ to $\widetilde{D}^{\varphi}_0(\mathcal{E},\mathcal{F})$ is surjective and generically one-to-one. Therefore, by definition of push-forward,

$$[D^{\varphi}_r(\mathcal{E},\mathcal{F})] = \rho_*([\widetilde{D}^{\varphi}_r(\mathcal{E},\mathcal{F})]).$$

Moreover $\dim D^{\varphi}_r(\mathcal{E},\mathcal{F}) = \dim X - (e-r)(f-r)$ by assumption. Since the restriction of ρ is surjective and generically one-to-one, we deduce $\dim \widetilde{D}^{\varphi}_r(\mathcal{E},\mathcal{F}) = \dim D^{\varphi}_r(\mathcal{E},\mathcal{F})$, hence

$$\mathrm{codim}_{\mathcal{G}(e-r,\mathcal{E})}(\widetilde{D}^{\varphi}_r(\mathcal{E},\mathcal{F})) = (e-r)f.$$

This shows that $\widetilde{D}^{\varphi}_r(\mathcal{E},\mathcal{F}) = D^{\rho^*\varphi}_0(\mathcal{S},\rho^*\mathcal{F})$ has the expected codimension. We conclude

$$[D^{\varphi}_r(\mathcal{E},\mathcal{F})] = \rho_*(\mathrm{Res}_t(c_t(\mathcal{S}),c_t(\rho^*\mathcal{F}))).$$

6.4. **From the Grassmann bundle to X.** The discussion of the previous paragraph showed:

$$[D_r^\varphi(\mathcal{E},\mathcal{F})] = \rho_* \Delta_f^{e-r}\left(\frac{c(\rho^*\mathcal{F})}{c(\mathcal{S})}\right).$$

In this section, we conclude the proof of Porteous's formula, resolving the pull-back and push-forward in the formula above.

By Whitney's formula

$$c(\mathcal{S}) = \frac{c(\rho^*\mathcal{E})}{c(\mathcal{Q})}$$

where $\mathcal{Q}$ is the quotient bundle. Therefore

$$[D_r^\varphi(\mathcal{E},\mathcal{F})] = \rho_* \Delta_f^{e-r}\left[\rho^*\left(\frac{c(\mathcal{F})}{c(\mathcal{E})}\right)c(\mathcal{Q})\right].$$

Consider the Sylvester matrix definiting the above determinant. Let $s_{f-(e-r)+1},\dots,s_{f+(e-r)-1}$ be its entries. Here, s_p is the coefficient of t^p in the expression of $\rho^*\left(\frac{c(\mathcal{F})}{c(\mathcal{E})}\right)c(\mathcal{Q})$, that is

$$s_p = \sum_0^p \rho^*\left\{\frac{c(\mathcal{F})}{c(\mathcal{E})}\right\}_{p-j} c_j(\mathcal{Q}).$$

where $\{-\}_j$ indicates the coefficient of t^j.

Therefore, the determinant of the Sylvester matrix can be written as a sum of terms of the form $\rho^*(\alpha)\beta$ where β is the product of $e-r$ Chern classes of $\mathcal{Q}$. Then the push-pull formula will guarantee that applying ρ_* to these terms we obtain $\alpha\rho_*(\beta)$.

Consider push-forward map $\rho_* : \mathrm{CH}(\mathcal{G}(e-r,\mathcal{E})) \to \mathrm{CH}(X)$. Let Y be a subvariety of $\mathcal{G}(e-r,\mathcal{E})$. Then $\rho_*([Y]) = [\rho(Y)]$ if ρ is finite to 1 on Y and 0 otherwise. The fibers of $\mathcal{G}(e-r,\mathcal{E})$ have dimension $r(e-r)$, therefore if $Y \subseteq \mathcal{G}(e-r,\mathcal{E})$ has codimension smaller than $r(e-r)$, then the restriction of ρ to Y is not finite; this implies that any class of degree smaller than $r(e-r)$ pushes forward to 0.

Now, $\mathcal{Q}$ has rank r, therefore its Chern classes have degree at most r. The only product of $e-r$ of them which has degree at least $r(e-r)$ is $c_r(\mathcal{Q})^{e-r}$ and this pushes forward to $m[X] \in CH^0(X)$ for some integer m. The value of m is the degree of the intersection of $c_r(\mathcal{Q})^{e-r}$ with the general fiber of $\mathcal{G}(e-r,\mathcal{E})$, say $\rho^{-1}(x)$; this is exactly the restriction of $\mathcal{Q}$ to $\rho^{-1}(x) = G(e-r,\mathcal{E}_x)$, that is the universal quotient bundle $\overline{\mathcal{Q}}$ of $G(e-r,e)$. From example Example 4.8, we deduce $m = \deg(c_r(\mathcal{Q})^{e-r}) = 1$.

This shows that the only terms of the Sylvester matrix that contribute to the final result are the ones where $c_r(\mathcal{Q})$ appears and their pushes forward to 1 via the push-forward map. Therefore, we can drop the $c_r(\mathcal{Q})$ coefficient in the Sylvester matrix, obtaining the matrix whose entries $\overline{s}_{f-(e-r)+1},\dots,\overline{s}_{f+(e-r)-1}$ are

$$\overline{s}_p = \rho^*\left\{\frac{c(\mathcal{F})}{c(\mathcal{E})}\right\}_{p-r}.$$

These are exactly the coefficients of $\frac{c(\mathcal{F})}{c(\mathcal{E})}$ (shifted back by r).

We conclude

$$[D_r^\varphi(\mathcal{E},\mathcal{F})] = \rho_* \Delta_f^{e-r}\left[\rho^*\left(\frac{c(\mathcal{F})}{c(\mathcal{E})}\right)c(\mathcal{Q})\right] = \Delta_{f-r}^{e-r}\left(\frac{c(\mathcal{F})}{c(\mathcal{E})}\right).$$

This proves Porteous's formula.

6.5. **Degree of determinantal varieties.** We will prove

Theorem 6.2. *Let r, e, f be nonnegative integers with $r \leq e \leq f$. Let*
$$D_r(e,f) = \{A \in \mathbb{P}\mathrm{Mat}_{e\times f} : \mathrm{rk}(A) \leq r\} \subseteq \mathbb{P}\mathrm{Mat}_{e\times f}.$$
Then
$$\deg D_r(e,f) = \prod_{i=0}^{f-r-1} \frac{(e+i)!i!}{(r+i)!(e-r+i)!}$$

Consider the class
$$[D_r(e,f)] \in \mathrm{CH}^{(e-r)(f-r)}(\mathbb{P}\mathrm{Mat}_{e\times f}).$$
in the Chow ring of $\mathbb{P}\mathrm{Mat}_{m\times n}$. Then $[D_r(e,f)] = \deg(D_r(e,f))h^{(e-r)(f-r)}$ where $h = [H]$ is the class hyperplane class of $\mathbb{P}\mathrm{Mat}_{e\times f}$.

Therefore the Sylvester determinant in Porteous's formula will give us the value of $\deg(D_r(e,f))$.

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