Sample Problems in Discrete Mathematics

This handout lists some sample problems that you should be able to solve as a pre-requisite to Computer Algorithms. Try to solve all of them. You should also read Chapters 2 and 3 of the textbook, and look at the Exercises at the end of these chapters. If you are unfamiliar with some of these topics, or cannot solve many of these problems, then you should take a Discrete Math course before taking Computer Algorithms.

1 Using Mathematical Induction

The task:	Given property $P = P(n)$, prove that it holds for all integers $n \geq 0$.

Base Case:	show that $P(0)$ is correct;
Induction hypothesis:	assume that for some fixed, but arbitrary integer $n \geq 0$, P holds for all integers $0, 1, \ldots, n$
Induction step:	prove that the induction hypothesis, $P(n)$, implies that P is true of $n + 1$: $P(n) \Longrightarrow P(n + 1)$
Conclusion:	using the principle of Mathematical Induction conclude that $P(n)$ is true for arbitrary $n \geq 0$.

Variants of induction:	(although they are really all the same thing)
Strong Induction:	The induction step is instead: $P(0) \wedge P(1) \wedge \ldots \wedge P(n) \Longrightarrow P(n + 1)$
Structural Induction:	We are given a set S with a well-ordering $\prec$ on the elements of this set. For example, the set S could be all the nodes in a tree, and the ordering is that $v \prec w$ if v is below w in the tree. The Base Case is now to show $P(w)$ for all $w \in S$ which have no element preceding it (i.e., such that there is no $v \prec w$). The Inductive Step is now to show that for any $w \in S$, we have that $(P(v)$ for all $v \prec w) \Longrightarrow P(w)$.

Problem 1 Prove that for all $n > 10$

$$n - 2 < \frac{n^2 - n}{12}.$$

Proof: Define property $P(n)$ by

$$P(n): \quad \forall k \le n \; (k > 10;\; n > 10), \quad k - 2 < \frac{k^2 - k}{12}.$$

Then,

Base case: for $n = 11$,

$$11 - 2 < \frac{11^2 - 11}{12} \Longrightarrow 9 < \frac{110}{12} = \frac{55}{6} = 9 + \frac{1}{6}.$$

Notice that the base of the induction proof start with $n = 11$, rather than with $n = 0$. Such a shift happens often, and it does not change the principle, since this is nothing more than the matter of notations. One can define a property $Q(m)$ by $Q(m) = P(n - 11)$, and consider Q for $m \ge 0$.

Induction step. Suppose, given $n \ge 11$, P holds true for all integers up n. Then

$$\begin{array}{rcl}
P(n) & \Longrightarrow & n - 2 < \frac{n^2 - n}{12} \\
 & \Longrightarrow & (n+1) - 2 - 1 < \frac{(n+1)^2 - (n+1) - 2n - 1 + 1}{12} \\
 & \Longrightarrow & (n+1) - 2 < \frac{(n+1)^2 - (n+1) - 2n - 1 + 1}{12} + 1 \\
 & \Longrightarrow & (n+1) - 2 < \frac{(n+1)^2 - (n+1)}{12} - \frac{2n}{12} + 1 \\
 & \Longrightarrow & (n+1) - 2 < \frac{(n+1)^2 - (n+1)}{12} \;\; (\text{since } n > 10)
\end{array}$$

The last inequality is $P(n+1)$. □

Problem 2 *For every integer $n \ge 1$,*

$$\sum_{i=1}^{n} \sqrt{i} > 2n\sqrt{n}/3.$$

Proof: We prove it by induction on n.

Base. For $n = 1$, the left part is 1 and the right part is $2/3$: $1 > 2/3$.

Inductive step. Suppose the statement is correct for some $n \ge 1$; we prove that it is correct for $n + 1$. Thus,

Given: $\sum_{i=1}^{n} \sqrt{i} > 2n\sqrt{n}/3$; **Prove:** $\sum_{i=1}^{n+1} \sqrt{i} > 2(n+1)\sqrt{n+1}/3$.

$$\sum_{i=1}^{n+1} \sqrt{i} = (\sum_{i=1}^{n} \sqrt{i}) + \sqrt{n+1} \tag{1}$$

Using the given inequality, we evaluate the right part of (1)

$$(\sum_{i=1}^{n} \sqrt{i}) + \sqrt{n+1} \ge 2n\sqrt{n}/3 + \sqrt{n+1}. \tag{2}$$

The following sequence of equivalent inequalities completes the proof:

$$\begin{array}{rcll}
2n\sqrt{n}/3 + \sqrt{n+1} & \ge & 2(n+1)\sqrt{n+1}/3 & (3) \\
2n\sqrt{n} + 3\sqrt{n+1} & \ge & 2(n+1)\sqrt{n+1} & (4) \\
2n\sqrt{n} & \ge & (2n-1)\sqrt{n+1} & (5) \\
4n^2 n & \ge & (2n-1)^2(n+1) & (6) \\
4n^3 & \ge & (4n^2 - 4n + 1)(n+1) = 4n^3 - 3n + 1 & (7) \\
0 & \ge & -3n + 1. & (8)
\end{array}$$

The last inequality holds true for any $n > 0$. □

Problem 3 *For every integer $n \geq 0$,*

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Prove.

Problem 4 *For any integers a and b with $0 \leq a \leq b$, and every integer $n \geq 1$, $b^n - a^n$ is divisible by $b - a$.*

Problem 5 *Prove that every integer greater than 1 is a product of prime numbers.*

Proof: We will use strong induction. Base Case: 2 is a product of prime numbers, since 2 itself is a prime. Inductive Step: Let n be an arbitrary integer greater than 1. The Inductive Hypothesis is that for all integers k such that $1 < k < n$, we have that k can be written as a product of primes. Then, either n is a prime itself (in which case it is a product of prime numbers), or n can be written as $n = n_1 n_2$ with both n_1 and n_2 being integers greater than 1 and less than n. Thus, the Inductive Hypothesis applies to both n_1 and n_2, and so each can be written as a product of prime numbers. Thus, n can be written as a product of prime numbers as well. □

Problem 6 *What is wrong with the following classic "proof" that all horses have the same color?*

Let $P(n)$ be the proposition that for any set of n horses, all horses in this set have the same color.

Base Case: $P(1)$ is clearly true.

Inductive Step: The Inductive Hypothesis is that $P(n)$ holds; we will show that this implies $P(n+1)$. Let A be an arbitrary set of $n+1$ horses. Consider the first n horses in A, and the last n horses in A. Both of these are sets of n horses, and so the inductive hypothesis holds for them, telling us that both the first n and the last n horses of A have the same color. Since the set of the first n and the last n horses overlap, then all $n+1$ horses of A must have the same color, thus proving $P(n+1)$.

Problem 7 *Prove by induction that*

$$F_1 + F_3 + \ldots + F_{2n-1} = F_{2n},$$

where F_i denotes the i^{th} Fibonacci number.

2 Order of Functions

See Chapter 2 of the textbook, and the exercises therein.

Problem 8 *Compare the following pairs of functions in terms of order of growth. In each case, determine if $f(n) = O(g(n)), f(n) = \Omega(g(n)), f(n) = \Theta(g(n))$. Prove your answers.*

	$f(n)$	$g(n)$
(1)	$(\lg n)^a$	n^b *(here $a, b > 0$)*
(2)	$2^{n \log_2 n}$	$10n!$
34)	$\sqrt{n}$	$(\log_2 n)^5$
(3)	$\frac{n^2}{\log_2 n}$	$(n\log_2 n)^4$
(4)	$\log_2 n$	$\log_2(66n)$
(5)	$1000(\log_2 n)^{0.999}$	$(\log_2 n)^{1.001}$
(6)	n^2	$n(\log_2 n)^{15}$

3 Graph Theory

See also Chapter 3 of the textbook and the exercises therein.

Problem 9 *Here is an example of Structural Induction in trees. Consider a rooted tree $T = (V, E)$, where nodes are labeled with positive integers: each node $v \in V$ is labeled with an integer a_v. Assume that T obeys the* heap property, *i.e., if w is a child of v, then $a_w < a_v$. Prove that the root node r has the largest label of all nodes in the tree.*

Proof: Let T_v be the subtree of T rooted at v. We will prove the proposition $P(v)$ which states that v has the largest label in T_v. Base Case: If v is a leaf, then T_v consists of a single node v, and so $P(v)$ is trivially true. Inductive Step: Assume that $P(w)$ is true for all children w of v. By the heap property, $a_w < a_v$, and by the inductive hypothesis, a_w is the largest label in T_w. Therefore, a_v is the largest label in T_v, as desired.

We have now shown that $P(v)$ holds for all nodes $v \in V$. Therefore, $P(r)$ holds, and so r has the largest label of all nodes in the tree. $\square$

Problem 10

By analyzing the algorithm for Breadth-First-Search (BFS), show that in the case of a connected, undirected graph G, its every edge is either a tree edge (belongs to the BFS tree), or a cross edge (connects two vertices, neither is a descendant of the other in the BFS tree.)

Problem 11 *Find an example of a directed graph and a DFS-forest such that vertex v is not a descendant of u, but graph G has a path from u to v and $dis[u] < dis[v]$.*

Problem 12

Describe an algorithm (in words, not a code), which for a given connected undirected graph G, directs its edges in such a way that the following two conditions are satisfied:

1. the resulting directed graph contains a rooted tree all of whose edges are directed away from the root;
2. every edge not in the tree above forms a directed cycle with some edges of the tree.

What is the complexity of your algorithm? Explain.

Problem 13 *Show how to tell if graph is bipartite (in linear time).*

4 Additional Problems in Discrete Math and Logic

Problem 14 *How many eight digit numbers are there that contain a 5 and a 6? Explain.*

Problem 15 *How many nine digit numbers are there that contain exactly two 5's?*

Problem 16 *What are the coefficients of the terms $\frac{1}{x}$, $\frac{1}{x^2}$ in the expansion of $(x + \frac{1}{x})^n$, where $(n > 2)$? Explain.*

Problem 17 *Prove that for every $n \geq 1$ and every $m \geq 1$, the number of functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$ is m^n.*

Problem 18 Prove the following identities.

Idempotency	$A \cup A = A$; $A \cap A = A$
Commutativity	$A \cup B = B \cup A$; $A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
	$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
DeMorgan's Laws	$U - (B \cup C) = (U - B) \cap (U - C)$
	$U - (B \cap C) = (U - B) \cup (U - C)$

Problem 19 *(Converse) Consider the following two statements: "100 percent of convicted felons consumed bread during their childhood." and "People who consume bread as a child are likely to become felons." Does the first statement logically imply the second?*

Problem 20 *(Contrapositive) Consider the following two statements: "100 percent of convicted felons consumed bread during their childhood." and "People who do not consume bread as a child do not become felons." Does the first statement logically imply the second?*

Problem 21 Fill in the truth tables:

$\mathbf{P} \wedge \mathbf{Q}$	**P** is true	**P** is false
Q is true		
Q is false		

$\mathbf{P} \vee \mathbf{Q}$	**P** is true	**P** is false
Q is true		
Q is false		

If Q then P	**P** is true	**P** is false
Q is true		
Q is false		

	true	false
P		
$\neg$ **P**		