Computational Intelligence Laboratory

Lecture 1: Course Overview & Linear Algebra Primer

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CIL Team

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Machine Learning Laboratory

Credit

- Laboratory has been developed by Prof. Joachim Buhmann and various research associates and assistants
- Some changes and amendments have been made (by TH) compared to 2014, yet the overall structure and the majority of material has been kept.

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Linear Algebra Primer

- Vectors and Matrices
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Learning Goals

- Acquire fundamental **theoretical concepts** of machine learning related to **matrix factorization**.
- Implement and compare **alternative techniques** to solve the same application problem:

Collaborative Filtering

Inpainting

Compression

- Combine and extend techniques $\Rightarrow$ **novel solution**
  - compare to baselines developed during the course
- Produce a **write up** of your findings = scientific short paper
  - we will emphasize an ethically sound way to conduct research.

Course Structure

Method/application matrix structure:

- basic methods are useful for several applications
- moderate adaption to use case (e.g. in class projects)

Applications

Theory	Collaborative Filtering	Inpainting	Image Compression
Dimension Reduction	✔	✔	✔
Clustering	✔	✔	✔
Sparse Coding		✔	✔
Robust PCA			✔

Organizational Form

Weekly Schedule: 2 + 2 + 1

- Lecture: Fri 10-12, CAB G61
- Exercises: Thu 15-17, CAB G51 **or** Thu 16-18, CAB G61 **or** Fri 15-17, CAB G61
  - all three exercise sessions are "identical"
  - first hour: pen-and-paper **exercise**, immediate discussions
  - second hour: group work on programming **assignment**
  - we reserve the right to "load balance", if the distribution is highly skewed
- Voluntary presence time: Mo 11-12, CAB H53
  - TAs help completing programming assignments
- **Webpage:** `http://cil.inf.ethz.ch/`
  - only accessible from the ETH network

Organizational Form

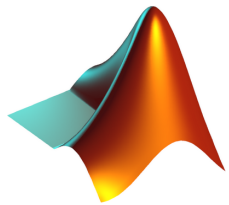
Programming Assignments

- Joint work in groups of **two or three** students
- Solving problems by applying techniques learned in class
- Submitting solution to the CIL online evaluation and ranking system

Been here before? :-/

- You can re-submit previous year's project. The competitive part will be re-graded!
- You have to let us know that you are a "one person group"
- You have the choice to redo the project and join a group (as anyone else)

First Week



Coming week: exercise session on efficient **Matlab** programming

- Bring laptops with Matlab installed
- "Hands on" exercises in groups of 2-3 students

Reading material

- Linear algebra background – on the course website

Grading Criteria

Semester Work

- Develop a **novel solution** for one of the application problems
- **Compare** with two baseline techniques already implemented in the weekly programming assignments
- Write up in the form of a short **paper**
- **Competitive criteria:** run-time, accuracy, ..
- **Non-competitive criteria:** paper review, creativity of solution, quality of implementation

Written Exam

- 120 minutes written exam
- up to 5 problems in the spirit of the pen-and-paper exercises

Grading Formula

You need to satisfy two **requirements** to pass this course:

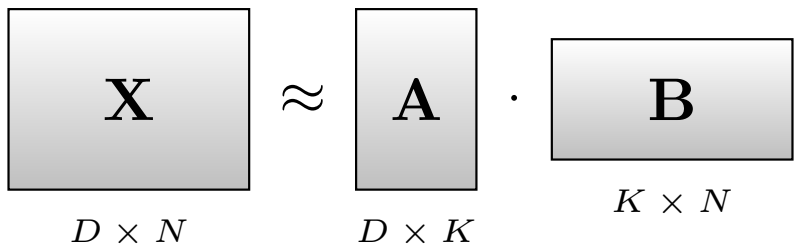
1. Your **average grade** (2/3 written exam, 1/3 group project) is greater or equal to 4.
2. Your **written exam grade** is greater or equal to 3.5.

Therefore, your **final grade** will be:

1. Your average grade, if your written exam grade is greater or equal to 3.5.
2. Your written exam grade if it is below 3.5.

Matrix Factorization

Core problem: **matrix factorization** (MF) of a given data matrix into two or three unknown matrices, e.g.



Many important data analysis techniques can be written as MF problems by specifying:

- **Type of data:** e.g. Boolean: $x_{d,n} \in \{0, 1\}$ or non-negative real: $x_{d,n} \in \mathbb{R}_+$, i.e., $x_{d,n} \geq 0$
- **Approximation:** e.g. exact: $\mathbf{X} = \mathbf{A} \cdot \mathbf{B}$ or minimal Frobenius norm: $\min_{\mathbf{A},\mathbf{B}} \|\mathbf{X} - \mathbf{A} \cdot \mathbf{B}\|_F$
- **Constraints on unknown matrices:** e.g. $\mathbf{A}$ has to be a basis or $\mathbf{B}$ must be sparse.

Matrix Factorization

For many variants, MF is computationally NP-hard, therefore we need **efficient** and **low-error** approximation algorithms.

Techniques studied in this course:

1. Dimension reduction
2. Clustering
3. Sparse coding
4. Robust PCA

Theory Part I: Dimension Reduction

Given N vectors in D dimensions, find the K most important axes to project them.

Why dimensionality reduction?

- some features may be irrelevant, projection may reduce noise
- visualize high dimensional data, e.g., in 2 or 3 dimensions.
- "intrinsic" dimensionality may be smaller than D

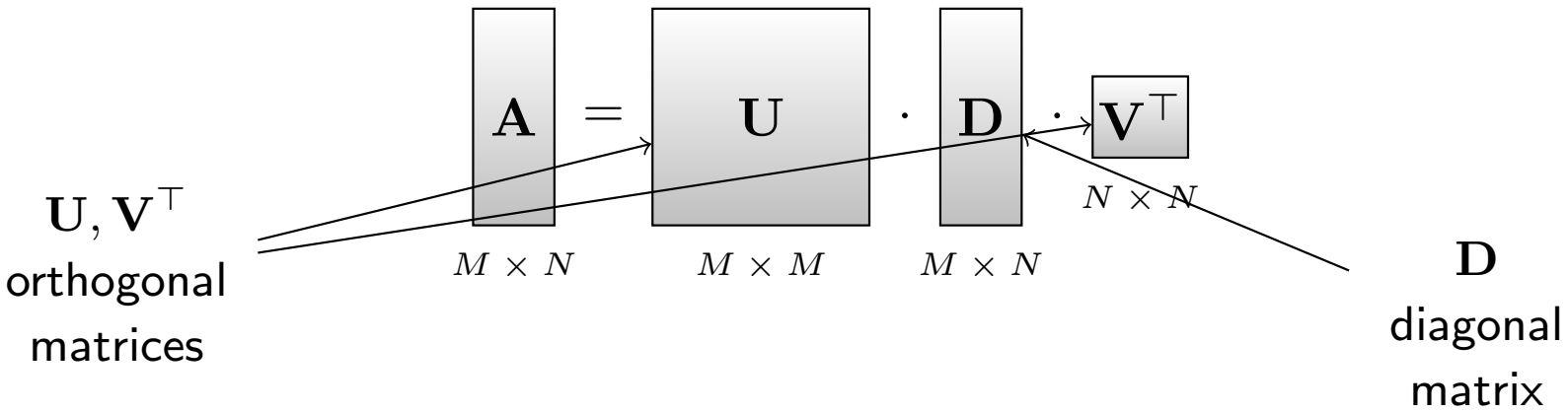
Many important dimensionality reduction techniques are based on the **eigen-decomposition** of the data matrix.

In this course we review the singular value decomposition (SVD) and principal component analysis (PCA).

Theory Part I: Dimension Reduction

SVD

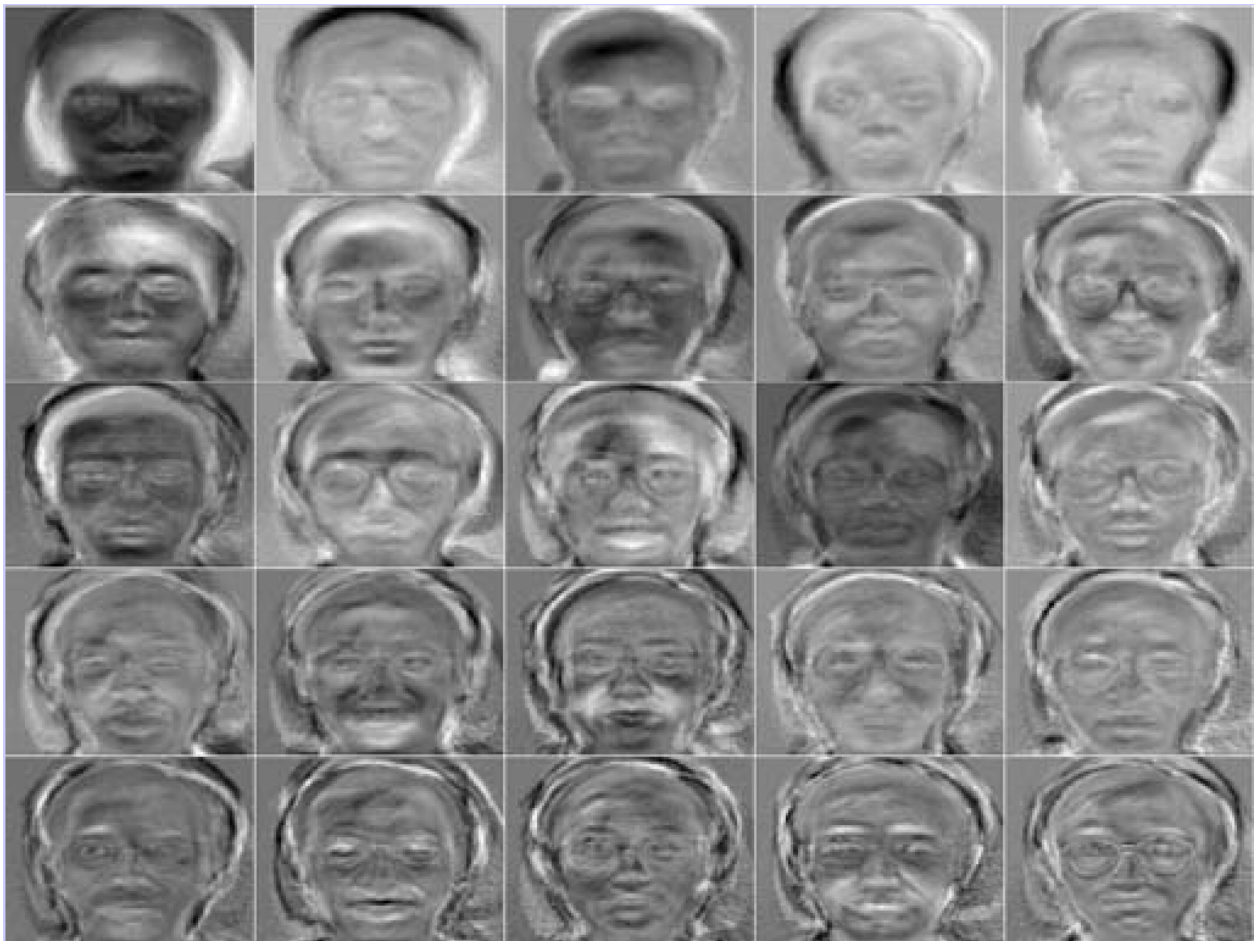
Any matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ can be decomposed **uniquely** as $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ (up to permutations and signs):



Eckart-Young Theorem

Let $\mathbf{A}$ be a matrix of rank R, if we wish to approximate $\mathbf{A}$ using a matrix of a lower rank K then, $\tilde{\mathbf{A}} = \sum_{k=1}^{K} d_k \mathbf{u}_k \mathbf{v}_k^\top$ is the closest matrix in the Frobenius norm. (Assumes ordering of singluar values $d_k \geq d_{k+1}$)

Theory Part I: Dimension Reduction

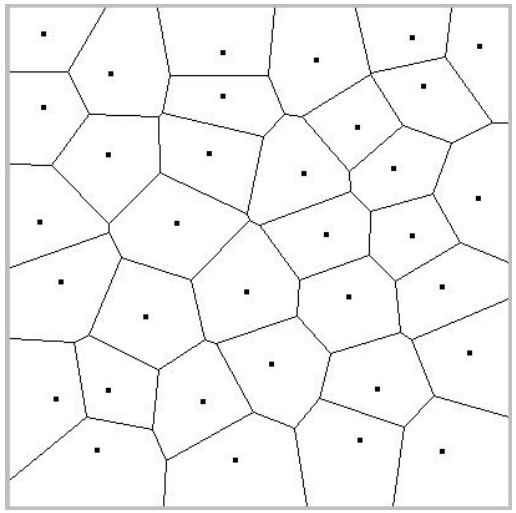


Christopher DeCoro `http://www.cs.princeton.edu/~cdecoro/eigenfaces`

"Eigenfaces": basis vectors produced by SVD on a set of face images

Theory Part II: Clustering

Quantization of the space $\mathbb{R}^D$



- data points $\mathbf{x} \in \mathbb{R}^D$ are represented by K **centroids** $\mathbf{u}_k$
- assign $\mathbf{x}$ to the centroid $\mathbf{u}_{k^*}$ that has minimal distance to $\mathbf{x}$.
- induces Voronoi tesselation of $\mathbb{R}^D$

Theory Part II: Clustering for Color Quantization

Figure : Top: original image, Bottom: same image where each pixel is represented by one of 10 colors.

Theory Part II: Clustering

K-means algorithm

Alternate between two steps until convergence:

1. Update assignments $z_{k,n}$ of data points to centroids:

$$z_{k,n} = \begin{cases} 1 & \text{if } k = \operatorname{argmin}_j \|\mathbf{x}_n - \mathbf{u}_j\|_2 \\ 0 & \text{otherwise.} \end{cases}$$

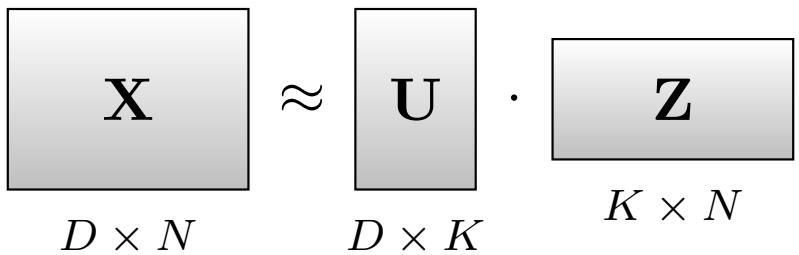
2. Update centroid positions:

$$\mathbf{u}_k = \frac{\sum_n z_{k,n} \mathbf{x}_n}{\sum_n z_{k,n}}.$$

Theory Part II: Clustering

K-means clustering formulated as matrix factorization

Assign each datapoint (column of $\mathbf{X}$) to one of K clusters:

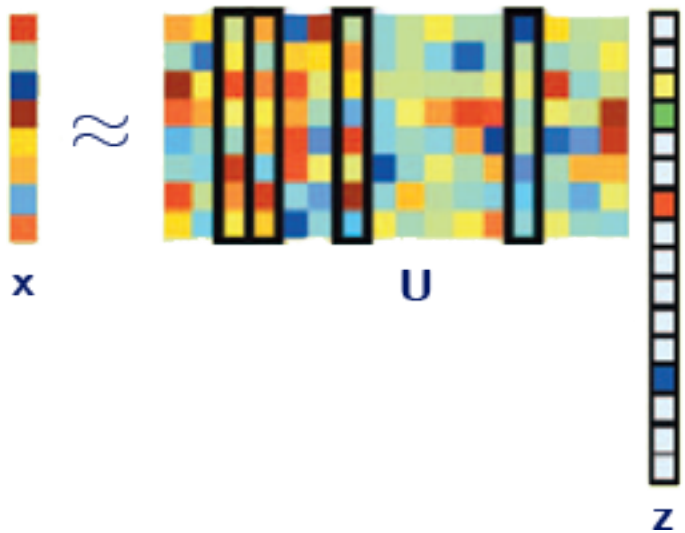


with $\mathbf{X} \in \mathbb{R}^{D \times N}$, $\mathbf{U} \in \mathbb{R}^{D \times K}$ and $\mathbf{Z} \in \{0, 1\}^{K \times N}$. In addition we enforce the unique assignment constraint $\sum_k z_{k,n} = 1 \ (\forall n)$.

The centroids define the columns of $\mathbf{U}$.

Theory Part III: Sparse Coding

Approximate representation of a signal $\mathbf{x} \in \mathbb{R}^D$ by a **sparse** linear combination $\mathbf{z} \in \mathbb{R}^L$ of K atoms of a **dictionary** $\mathbf{U} \in \mathbb{R}^{D \times L}$:



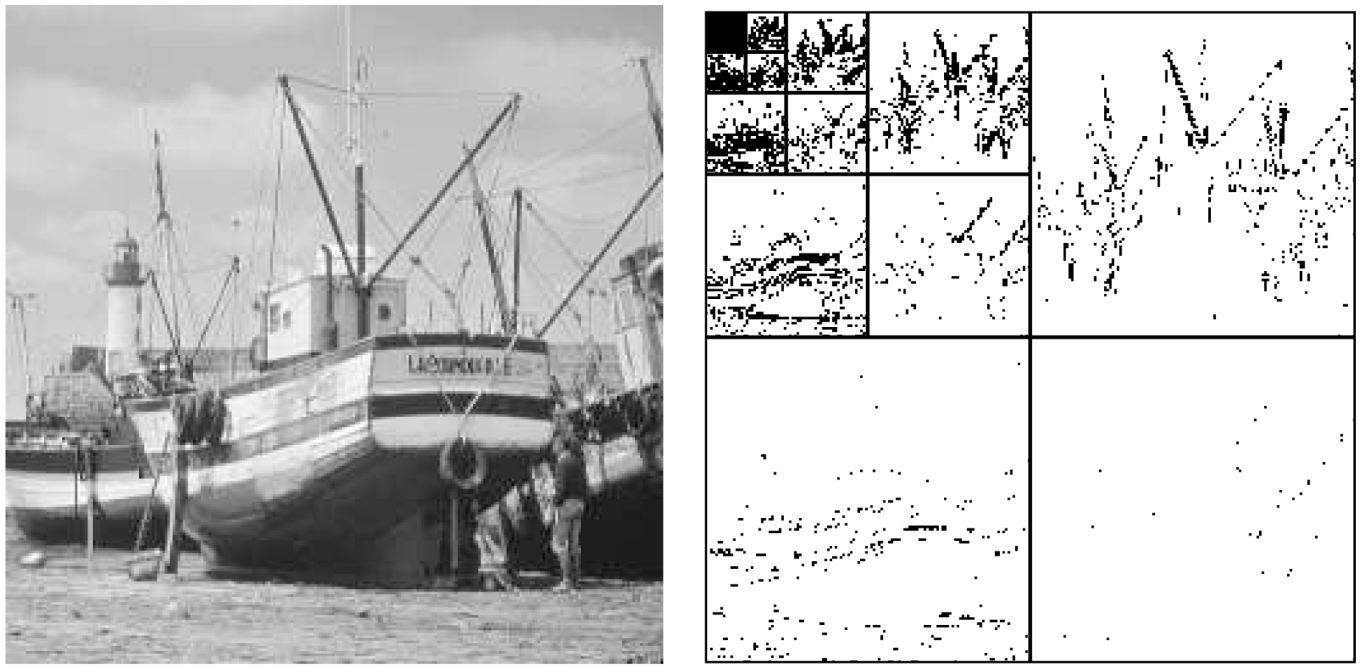
Formally, minimize the number of non-zero coefficients

$$\min_{\mathbf{z}} \|\mathbf{z}\|_0 , \quad \text{s.t.} \quad \|\mathbf{U}\mathbf{z} - \mathbf{x}\|_2 < \sigma,$$

while keeping the approximation error below σ.

Theory Part III: Sparse Coding

Representation of grayscale image in a **wavelet** dictionary:



Only few coefficients are above a certain magnitude threshold (black dots in right figure). Efficient **compression** approach: keep large coefficients, disregard small ones.

Theory Part III: Dictionary Learning as Matrix Factorization

Can we find even sparser representations? Yes, by adapting $\mathbf{U}$ to the image.

Dictionary learning: Find dictionary $\mathbf{U}$ and coding $\mathbf{Z}$ by solving

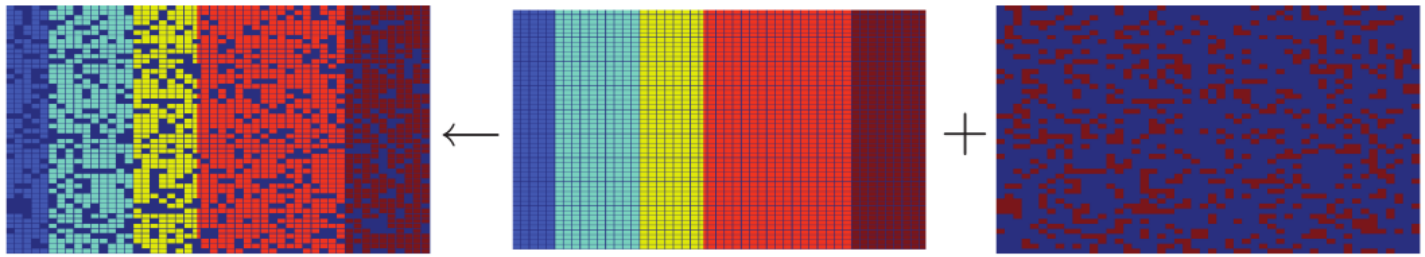
$$\min_{\mathbf{U},\mathbf{Z}} \|\mathbf{X} - \mathbf{U} \cdot \mathbf{Z}\|_F^2 ,$$

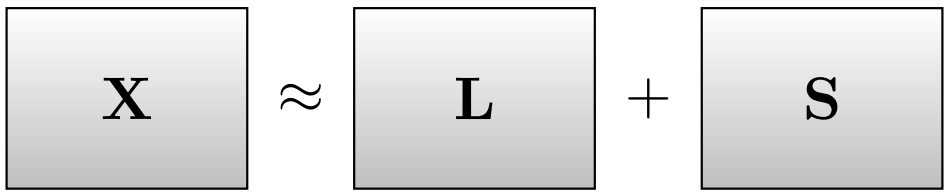
where $\mathbf{Z}$ must be sparse.

- Problem can be made convex in $\mathbf{U}$ or $\mathbf{Z}$, but not in both arguments.
- Simple algorithm: iterate between optimizing $\mathbf{U}$ and $\mathbf{Z}$.

Theory Part IV: Robust PCA

Separate a matrix $\mathbf{X}$ into *low-rank* $\mathbf{L}$ and *sparse* $\mathbf{S}$?





- Highly corrupted measurements ($\|\mathbf{S}\|_2$ can be big).
- Number of corrupted measurements ($\|\mathbf{S}\|_0$) should be small.

Theory Part IV: Robust PCA

Corrupted data in computer vision

Figure : Sunglasses or shadows occluding the face; old films

Figure : Decompose a video into static background and activity in the foreground

Theory Part IV: Robust PCA

Principal Component Pursuit

$$\begin{aligned} &\text{minimize} && \|\mathbf{L}\|_* + \lambda\|\mathbf{S}\|_1 \\ &\text{subject to} && \mathbf{L} + \mathbf{S} = \mathbf{X} \end{aligned}$$

Unobserved values

	Ben	Tom	John	Fred	Jack
Star Wars	?	?	1	?	4
WallE	5	?	3	4	?
Avatar	3	4	?	4	4
Trainspotting	?	1	5	?	?
Shrek	5	?	?	5	?
Ice Age	5	?	4	?	1

- Robust PCA: can also be applied to matrix $\mathbf{X}$ with unknown values
- $\Rightarrow$ collaborative filtering in recommender systems

Theory and Application

Group Project Example

	Collaborative Filtering	Inpainting	Compression
Dim. Reduction	**o**		✓
Clustering			✓
Robust PCA			
Sparse Coding			

Programming assignments: You implemented several solutions for each application, e.g. image compression by dimension reduction and by clustering (✓ in the matrix).

Group project: Develop a novel solution for one application, e.g. by transferring a technique to this new application domain (**o** in the matrix).

Lab Application I: Collaborative Filtering

Recommender system: present items that are of likely interest to the user.

- Products: Amazon
- Movies: MovieLens, IMDB
- Music: LastFM

In collaborative filtering (CF), we base our recommendations:

- On the (known) preference of the user towards other items,
- Take into account the preferences of other users.

Lab Application I: Collaborative Filtering

Viewers were asked to rate some movies (items):

	Ben	Tom	John	Fred	Jack
Star Wars	?	?	1	?	4
WallE	5	?	3	4	?
Avatar	3	4	?	4	4
Trainspotting	?	1	5	?	?
Shrek	5	?	?	5	?
Ice Age	5	?	4	?	1

- Not all viewers rated all movies.
- Unrated user-movie pairs are **missing values**: we want to predict them.
- Should we recommend Fred to watch "Ice Age"?

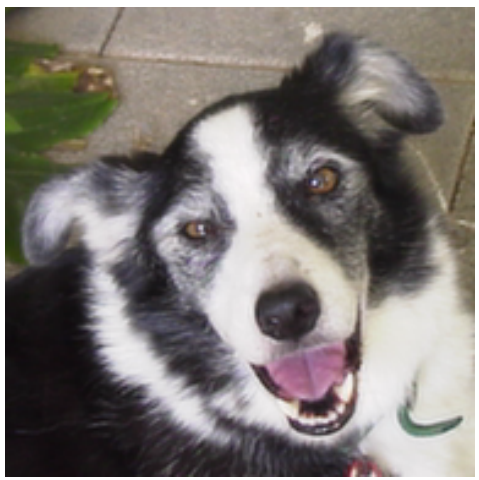
Lab Application II: Inpainting

Inpainting aims at the **restoration of missing pixels**, based on the intact surroundings:

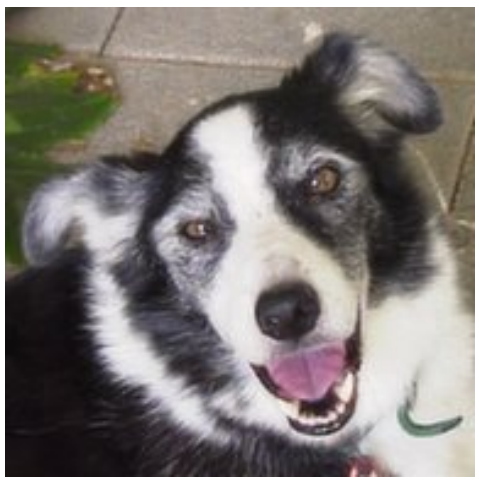
As with collaborative filtering, the missing values are predicted based on the statistics of known pixels of the target image, and possibly other images, too.

Lab Application III: Compression

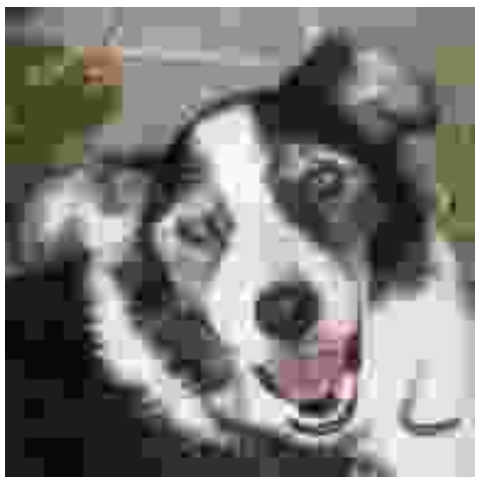
- Encode information using fewer bits than original representation
- **Lossy compression**: loss between original and compressed version is acceptable according to some loss function
- Achieve this by neglecting irrelevant parts of the data
  - noise vs. signal
- Here: Compression of images



Original (61 KB)



Low Compression (10 KB)



High Compression (2 KB)

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Notation

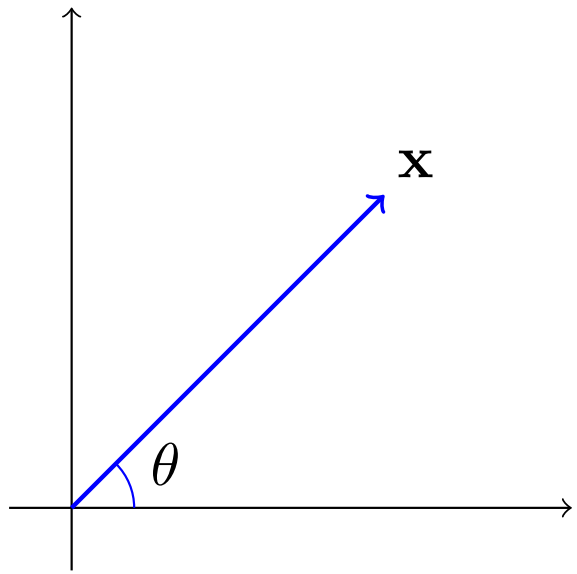
- $\mathbf{x}$ is a column vector
- $^\top$ denotes the **transpose** operator, so $\mathbf{x}^\top$ is a row vector
- The elements of a vector are denoted as $\mathbf{x} = (x_1, x_2, \ldots, x_D)^\top$
- $\mathbf{U}$ is a matrix:
  - $\mathbf{u}_k$ or $\mathbf{u}_{\cdot,k}$ is the k-th column of $\mathbf{U}$
  - $\mathbf{u}_d^\top$ or $\mathbf{u}_{d,\cdot}$ is the d-th row of $\mathbf{U}$
  - $u_{d,k}$ is the element in the d-th row and k-th column of $\mathbf{U}$

Vectors

- n-th data sample $\mathbf{x}_n$ is a D-dimensional vector containing D features:

$$\mathbf{x}_n \in \mathbb{R}^{D \times 1}$$

- Vectors are basic **geometric** entities that allow you to talk about **magnitude** (or length) and **direction**.



Matrices

- **Data matrix**: represent a dataset of N datapoints $\mathbf{x}_1, ..., \mathbf{x}_N$:

$$\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) = \begin{pmatrix} x_{1,1} & \dots & x_{1,N} \\ \vdots & & \vdots \\ x_{D,1} & \dots & x_{D,N} \end{pmatrix}$$

- Matrices can also be used to apply geometric transformations (rotations, scaling) to vectors.
- In particular: express dataset in a different coordinate system (using projections).

Example

Viewers were asked to rate the some movies (items):

	Ben	Tom	John	Fred	Jack
Star Wars	0	0	1	0	4
WallE	5	0	3	4	0
Avatar	3	4	0	4	4
Trainspotting	0	1	5	0	0
Shrek	5	0	0	5	0
Ice Age	5	0	4	0	1

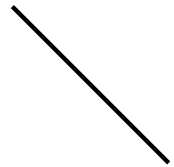
$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 & 0 & 4 \\ 5 & 0 & 3 & 4 & 0 \\ 3 & 4 & 0 & 4 & 4 \\ 0 & 1 & 5 & 0 & 0 \\ 5 & 0 & 0 & 5 & 0 \\ 5 & 0 & 4 & 0 & 1 \end{pmatrix}$$

$\mathbf{X}$: Users are the samples, ratings on movies are the features.

$\mathbf{X}^\top$: Movies are the samples, ratings by users are the features.

Matrix Transpose

Transpose: reflect vector/matrix on a diagonal line



Two examples:

$$\begin{pmatrix} a \\ b \end{pmatrix}^\top = (a\ b) \qquad \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}^\top = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$$

Note that $(\mathbf{A}\mathbf{x})^\top = \mathbf{x}^\top \mathbf{A}^\top$.

Special Matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $\mathbf{I}$ - identity matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$ diagonal matrix

$$\begin{pmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix}$$ upper triangular

$$\begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{pmatrix}$$ lower triangular

$$\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}$$ symmetric: $\mathbf{A}^\top = \mathbf{A}$

Scalar Product

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{D \times 1}$, then define:

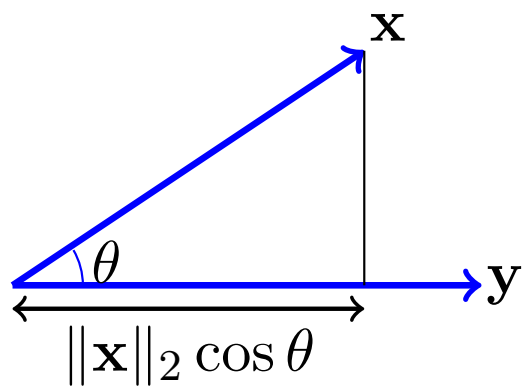
$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^\top \mathbf{y} = (x_1, x_2, \ldots, x_D) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_D \end{pmatrix}$$

$$= \sum_{d=1}^{D} x_d y_d$$

- $\mathbf{x}^\top \mathbf{x} = \langle \mathbf{x}, \mathbf{x} \rangle$ gives the squared Euclidean length of $\mathbf{x}$.
- L_2 Norm: $\|\mathbf{x}\|_2 := \sqrt{\mathbf{x}^\top \mathbf{x}}$
- Generally: $\mathbf{x}^\top \mathbf{y} = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos\theta$, where $\theta := \angle(\mathbf{x}, \mathbf{y})$

Scalar Product

- If $\|\mathbf{y}\|_2 = 1$ (unit vector), then $\mathbf{x}^\top \mathbf{y} = \|\mathbf{x}\|_2 \cos\theta$ gives the magnitude of the projection of $\mathbf{x}$ onto $\mathbf{y}$.
  - the scalar component of $\mathbf{x}$ in the direction of $\mathbf{y}$:



Question: When is $\mathbf{x}^\top \mathbf{y} = 0$?

Vector Matrix Multiplications

Let $\mathbf{x}$ be a D-dimensional column vector and $\mathbf{A}$ be a $M \times D$ matrix. Then $\mathbf{b} = \mathbf{A}\mathbf{x}$ is an M vector defined as

$$b_m = \sum_{d=1}^{D} a_{m,d} x_d, \quad m = 1, \ldots, M.$$

We interpret this by saying that "$\mathbf{A}$ acts on $\mathbf{x}$ to produce $\mathbf{b}$".

This product can be rewritten as $\mathbf{b} = \mathbf{A}\mathbf{x} = \sum_{d=1}^{D} x_d \mathbf{a}_d$ and visualized in the following manner

$$\begin{bmatrix} \\ b \\ \\ \end{bmatrix} = \left[\begin{array}{c|c|c|c} & & & \\ a_1 & a_2 & \cdots & a_n \\ & & & \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \\ a_1 \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \\ a_2 \\ \\ \end{bmatrix} + \cdots + x_n \begin{bmatrix} \\ a_n \\ \\ \end{bmatrix}.$$

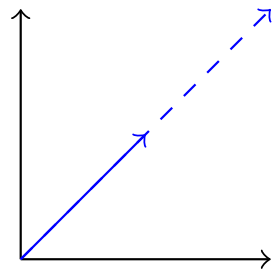
Matrices as Linear Transformations

We can transform $\mathbf{x}$ into $\mathbf{y} \in \mathbb{R}^{M \times 1}$ using a transformation matrix $\mathbf{A} \in \mathbb{R}^{M \times D}$

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

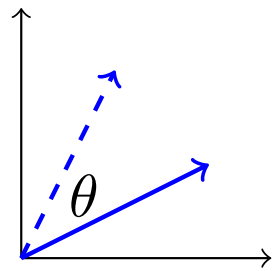
Some special cases (2-D):

$$\mathbf{A} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$



scaling in all directions by α

$$\mathbf{A} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$



counterclockwise rotation by θ

Vector Spaces

- Formally, a vector space is a set of vectors which is closed under vector addition and (scalar) multiplication
  - typically: scalars = real numbers
- A subspace is a subset of a vector space which is itself a vector space
  - example: the plane $z = 0$ is a subspace of $\mathbb{R}^3$ (essentially $\mathbb{R}^2$).
- Note: subspaces must include the origin (zero vector).

Column Space

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- Linear systems define certain subspaces
- $\mathbf{Ax} = \mathbf{b}$ is solvable iff $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.
- The set of possible vectors $\mathbf{b}$ forms a subspace called the **column space** of $\mathbf{A}$.

Range and Nullspace

Every linear mapping from $\mathbb{R}^D$ to $\mathbb{R}^M$ can be represented by an appropriate $\mathbf{A} \in \mathbb{R}^{M \times D}$.

- range$(\mathbf{A})$ is the set of vectors $\{\mathbf{b} \,|\, \exists \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$.
  - this is just the column space of $\mathbf{A}$
- null$(\mathbf{A})$ is the set of vectors $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$.
  - the entries of each $\mathbf{x} \in$ null$(\mathbf{A})$ form an expansion of zero as a linear combination of the columns of $\mathbf{A}$:

$$\mathbf{0} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_D\mathbf{a}_D.$$

Nullspace

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \text{ null space: } \{(0,0)\}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \text{ null space: } \{(c, \frac{2}{3}c, -c)\}$$

Linear Independence

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called **linearly independent** if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = 0 \quad \text{implies}$$
$$c_1 = c_2 = \ldots = c_n = 0$$

Note that in this case the nullspace of the matrix is the origin.

$$\begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In the previous example, the nullspace contained only $(0, 0)$ therefore the columns of the matrix are linearly independent

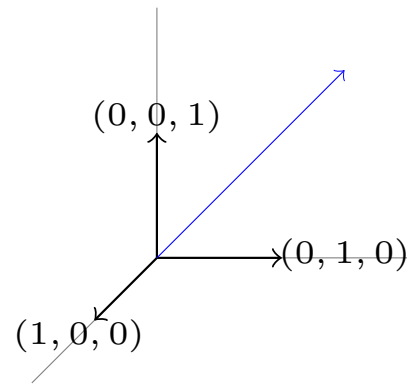
$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

null space: $\{(0, 0)\}$

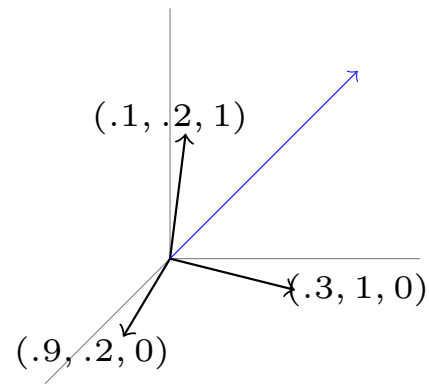
Spanning the Space

If all vectors in a vector space can be expressed as linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ **span** the space.

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 1.57 \begin{pmatrix} .9 \\ .2 \\ 0 \end{pmatrix} + 1.29 \begin{pmatrix} .3 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} .1 \\ .2 \\ 1 \end{pmatrix}$$



Basis

A **basis** B of a vector space V is a **linearly independent** subset of V that **spans** V.

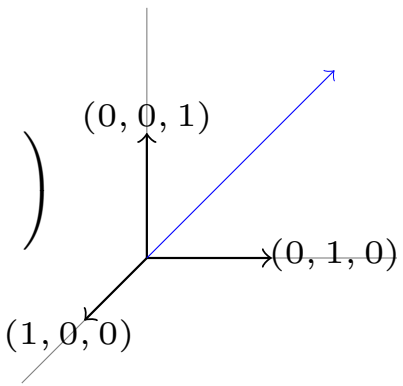
- There can be multiple different bases for a space.
- It can be shown: all bases have the same number of vectors
- Define: **dimension** of a space $=$ number of vectors in a/any basis for the space.
- A basis is a maximal set of linearly independent vectors and a minimal set of spanning vectors.

Basis Transformation

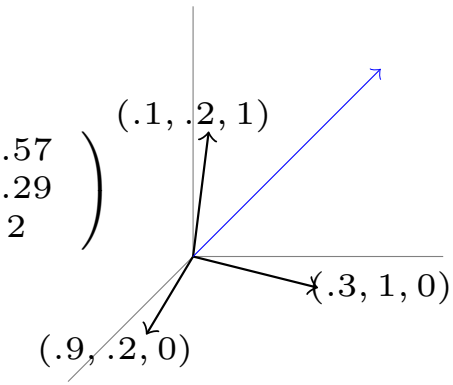
We may write $\mathbf{x} = (2, 2, 2)$ in terms of an alternate basis:

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$



$$\begin{pmatrix} .9 & .3 & .1 \\ .2 & 1 & .2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1.57 \\ 1.29 \\ 2 \end{pmatrix}$$



* The components of $(1.57, 1.29, 2)$ are projections of $\mathbf{x}$ onto the new vectors basis, normalized such that the new $\mathbf{x}$ still has the same length.

Rank

The column **rank** of $\mathbf{A}$ is the dimension of range($\mathbf{A}$).

- equivalently: maximal number of linearly independent column vectors
- one can also define the row rank of a matrix $\mathbf{A}$, however it can be shown that row and column rank are always equal (see below)

Exercise

Consider:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1. What is the range?
2. What is the rank?
3. What is the null space?

Theorem

Fundamental Theorem of Linear Algebra:

If $\mathbf{A}$ is $M \times N$ with rank R, then

- range($\mathbf{A}$) has dimension R
- null($\mathbf{A}$) has dimension $N - R$ (= nullity of $\mathbf{A}$)
- row space of $\mathbf{A}$ = range($\mathbf{A}^\top$), has dimension R

Rank-Nullity Theorem: rank + nullity = N

Inverse

A **nonsingular** or **invertible** matrix $\mathbf{A}$ is a square $M \times M$ matrix of full rank.

- its columns form a basis for the whole space $\mathbb{R}^M$.
- any vector can be uniquely expressed as a linear combination of the basis vectors
- in particular: canonical unit vectors $\mathbf{e}_j$ can be expanded as

$$\mathbf{e}_j = \sum_{m=1}^{M} z_{m,j} \mathbf{a}_m \quad (\forall j = 1, \ldots, M)$$

- this fact can be compactly written as

$$\left[\begin{array}{c|c|c} & & \\ \mathbf{e}_1 & \cdots & \mathbf{e}_M \\ & & \end{array}\right] = \mathbf{I} = \left[\begin{array}{c|c|c} & & \\ \mathbf{a}_1 & \cdots & \mathbf{a}_M \\ & & \end{array}\right] \mathbf{Z},$$

  where $\mathbf{Z}$ is the matrix with entries $z_{m,j}$.
- $\mathbf{Z}$ is the **inverse** $\mathbf{A}^{-1}$ of $\mathbf{A}$, satisfying $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Eigenvalues and Eigenvectors

Given a square matrix $\mathbf{A}$,

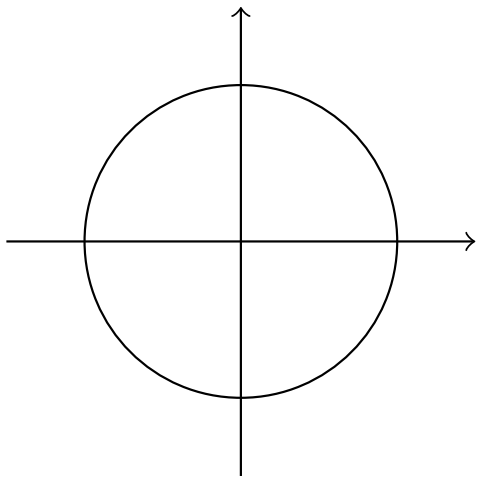
- The set of solutions to $\mathbf{Au} = \lambda\mathbf{u}$ is in the form of eigen-pairs $(\lambda, \mathbf{u})$ = (eigenvalue,eigenvector) where $\mathbf{u}$ is non-zero.

$$\mathbf{Au} = \overbrace{\lambda}^{\text{eigenvalue}} \underbrace{\mathbf{u}}_{\text{eigenvector}}$$

Eigenvector $\mathbf{u}$: Direction of $\mathbf{u}$ is not changed by transformation $\mathbf{A}$. It is only scaled by a factor λ (eigenvalue)

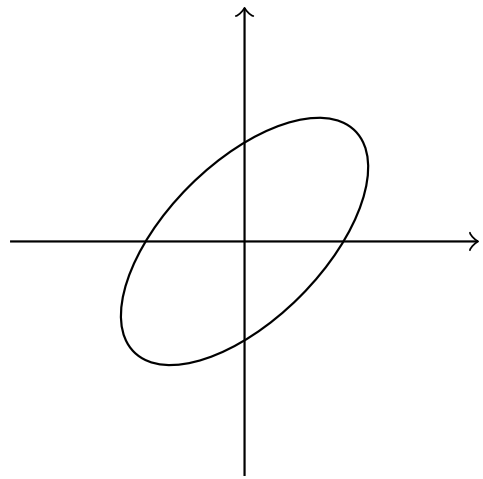
Eigendecomposition: $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$, where $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues on the diagonal.

Eigenvalues and Eigenvectors



$$\Longrightarrow$$

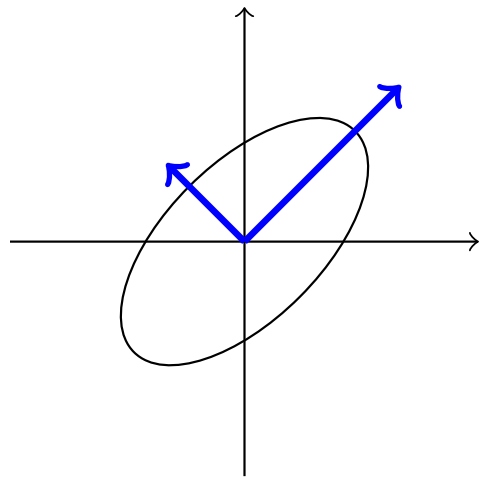
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$



What are the (geometric) eigenvectors of this projection?

Solution

The axes of the ellipse do not change their directions:



Equivalent Conditions

For $\mathbf{A} \in \mathbb{R}^{M \times M}$, the following conditions are equivalent:

- $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$,
- $\text{rank}(\mathbf{A}) = M$,
- $\text{range}(\mathbf{A}) = \mathbb{R}^M$,
- $\text{null}(\mathbf{A}) = \{\mathbf{0}\}$,
- 0 is not an eigenvalue of $\mathbf{A}$,
- 0 is not a singular value of $\mathbf{A}$,