Math 70 Worksheet 6

Instructions: This worksheet is due on Gradescope at 11:59 p.m. Eastern Time on Monday, November 2. You are encouraged to work with others, but the final results must be your own.[1]

1. (6 points) Let $T : M_{3\times 3} \longrightarrow \mathbb{R}^3$ be defined by $T(A) = A\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$. Note that T is a transformation on matrices $A \in M_{3\times 3}$ (not $\mathbf{x}$, which is the fixed vector given above).

   (a) Show that T is a linear transformation.

   (b) Find a basis for the kernel (null space) of this transformation.

   (c) Find a basis for the range of this transformation.

Solution:

Part a) let $U, V \in M_{3\times 3}$, $a, b \in R$. Thus,

$$\begin{aligned} T(aU + bV) &= (aU + bV)\mathbf{x} && \text{by the definition of } T(A) \\ &= aU\mathbf{x} + bV\mathbf{x} && \text{by the distributive property of matrix multiplication} \\ &= a(T(U)) + b(T(V)) \end{aligned}$$

In addition,

$$T(\mathbf{0}) = \mathbf{0x} = \vec{0}$$

Thus T is a linear transformation by definition.

Part b) By definition, we need to find $A \in M_{3\times 3}$ such that $A\mathbf{x} = \mathbf{0}$. Thus,

$$A\mathbf{x} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This yields the system

$$\begin{aligned} a_1 + 3b_1 &= 0 \quad => a_1 = -3b_1 \\ a_2 + 3b_2 &= 0 \quad => a_2 = -3b_2 \\ a_3 + 3b_3 &= 0 \quad => a_3 = -3b_3 \end{aligned}$$

Thus, the kernel of T is all matrices in $M_{3\times 3}$ of the form

$$A = \begin{bmatrix} -3b_1 & b_1 & c_1 \\ -3b_2 & b_2 & c_2 \\ -3b_3 & b_3 & c_3 \end{bmatrix}.$$

We can put this in parametric vector form separating each free variable and get the basis

$$\left\{ \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Part c) If we write $A \in M_{3\times 3}$ as

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$$

where $\vec{a}_i$ is the i^{th} column of A, then $T(A) = \vec{a}_1 + 3\vec{a}_2$. Since $\vec{a}_1$ and $\vec{a}_2$ can be any vectors in $\mathbb{R}^3$, the range of T is $\mathbb{R}^3$.
Thus a basis for the range of T would be any basis of $\mathbb{R}^3$, e.g. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

2. (4 points) Assume the matrices A and B are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer the following:

(a) Give a basis for the Nul(A)

(b) Give a basis for the Col(A)

(c) If C is a matrix and Nul$(C) = \{\mathbf{0}\}$ is the matrix transformation $T(\mathbf{x}) = C\mathbf{x}$ guaranteed to be onto? In this case, is it guaranteed to be one-to-one? Explain.

Solution:

(a) We start by reducing B to reduced echelon form.

$$B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -5 & 6 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can then write x_1 and x_2 in terms of the free variable x_3, x_4, x_5, write the null space in parametric vector form, and get a basis

$$\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) From the matrix B, we say that the two first columns are pivot columns. Thus the column space is spanned by the two corresponding vectors in A. The column space, therefore, has basis

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}.$$

(c) If Nul$(C) = \{\mathbf{0}\}$, then the kernel of T is the zero subspace, and so T is one-to-one. If Nul$(C) = \{\mathbf{0}\}$, then C has no free variables, and hence there is a pivot in every column of C. This condition says nothing about whether T is onto.

3. (6 points)

(a) Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a *one-to-one* linear transformation and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$. Show $\mathcal{B}' = \{T(\mathbf{b}_1), T(\mathbf{b}_2), \cdots, T(\mathbf{b}_n)\}$ is also a basis for $\mathbb{R}^n$.

(b) Suppose for $m > n$, $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation (not necessarily one-to-one) and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$. Is $\mathcal{B}' = \{T(\mathbf{b}_1), T(\mathbf{b}_2), \cdots, T(\mathbf{b}_n)\}$ necessarily a basis for $\mathbb{R}^m$? Explain.

(c) Suppose for $m < n$, $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation (not necessarily one-to-one) and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$. Is $\mathcal{B}' = \{T(\mathbf{b}_1), T(\mathbf{b}_2), \cdots, T(\mathbf{b}_n)\}$ a basis for $\mathbb{R}^m$? Explain.

Solution:

(a) Since $\mathcal{B}'$ is a collection of n vectors in $\mathbb{R}^n$, to show that it is a basis, it sufficient to show that it is linearly independent.
Let $c_1, \ldots, c_n$ be weights such that

$$c_1 T(\mathbf{b}_1) + \cdots + c_n T(\mathbf{b}_n) = \mathbf{0}.$$

Our goal is to show that $c_1 = \cdots = c_n = 0$. By linearity of T, we have that

$$T(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) = \mathbf{0}.$$

Since T is one-to-one, $T(\mathbf{v}) = 0$ if and only if $\mathbf{v} = 0$. Thus

$$c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \mathbf{0}.$$

Since $\mathcal{B}$ is a basis, it is a linearly independent set, and so $c_1 = \cdots = c_n = 0$, proving that $\mathcal{B}'$ is linearly independent and hence a basis.

(b) (And (c)) In both cases, $\mathcal{B}'$ is not a basis. When $m > n$ there are too few vectors to span $\mathbb{R}^n$, and when $m < n$, there are too many vectors for the set to be linearly independent. For example, no matter whether $m > n$ or $m < n$, we can consider the case where T is the zero map, sending everything in $\mathbb{R}^n$ to $\mathbf{0} \in \mathbb{R}^m$. Then $\mathcal{B}'$ is a collection of zero vectors, and so clearly not a basis.

4. (4 points) Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \in M_{2\times 2}$. You may assume that

$$W = \{Y \in M_{2\times 2} | YA = AY\}$$

is a subspace of $M_{2\times 2}$. Find a basis for W.

Solution:

Let $Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $YA = \begin{bmatrix} a & 5b \\ c & 5d \end{bmatrix}$ and $AY = \begin{bmatrix} a & b \\ 5c & 5d \end{bmatrix}$. If $YA = AY$, then $a = a$, $5b = b$, $c = 5c$, and $5d = 5d$. For this to be true, a and d can be anything, while b and c must be 0. Thus a basis for W is given by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$