Tufts University
Department of Mathematics

Math 135 | **Homework 9 v. 2**[1] | **Fall 2022**

Readings for Problem Set 9

§11.1: Continuity of functions with domain $\mathbb{R}^n$
§11.2: Sequential compactness, extreme values, uniform continuity

Problem Set 9
(Due Wednesday, November 16, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (10 points) (**Product of sequentially compact sets**) §11.2, p. 304, # 8.
   Let A and B be sequentially compact subsets of $\mathbb{R}$. Prove that $A \times B$ is sequentially compact in $\mathbb{R}^2$. (The Cartesian product $A \times B$ is the set $\{(a,b) \in \mathbb{R}^2 \mid a \in A \text{ and } b \in B\}$.)

2. (15 points) (**Continuity in terms of closed sets**)
   (a) Prove the lemma: Let X and Y be two sets. For any map $F: X \to Y$ and $C \subset Y$, $F^{-1}(Y \setminus C) = X \setminus F^{-1}(C)$. ("The inverse image preserves the complement.") (*Hint*: The easiest way to proceed is to write down a sequence of equivalent statements starting with $x \in F^{-1}(Y \setminus C)$ iff $\cdots$.)
   (b) Let $F : \mathbb{R}^n \to \mathbb{R}^m$. Prove that F is continuous on domain $\mathbb{R}^n$ if and only if for every closed subset C in $\mathbb{R}^m$, the inverse image $F^{-1}(C)$ is closed in $\mathbb{R}^n$.

3. (20 points) (**Distance from a point to a set**)
   (a) (5 points) Let $\vec{v} \in \mathbb{R}^n$. Define $D : \mathbb{R}^n \to \mathbb{R}$ by $D(\vec{u}) = \text{dist}(\vec{u}, \vec{v})$ for $\vec{u} \in \mathbb{R}^n$. Prove D is continuous.
   (b) (10 points) §11.2, p. 304, # 9.
   Let A be a sequentially compact subset of $\mathbb{R}^n$ and let $\vec{v}$ be a point in $\mathbb{R}^n \setminus A$. Prove that there is a point $\vec{u}_0$ in the set A such that
   $$\text{dist}(\vec{u}_0, \vec{v}) \leq \text{dist}(\vec{u}, \vec{v}) \quad \text{for all points } \vec{u} \in A.$$
   Is this point unique? (Note that the proof works even if $\vec{v} \in A$.)
   (c) (5 points) Is the conclusion of (b) true if A is closed (and not necessarily sequentially compact)? Why or why not? (*Hint*: Consider the closed ball $\bar{B}_r(\vec{v})$ of radius r centered at $\vec{v}$. Let the radius be large enough so that $\bar{B}_r(\vec{v}) \cap A$ is nonempty. Explain why the point of A at minimum distance from $\vec{v}$ must be in $\bar{B}_r(\vec{v}) \cap A$. What kind of set if $\bar{B}_r(\vec{v}) \cap A$?)

4. (10 points) (**Boundary of a set in** $\mathbb{R}$)
   Let A be a bounded and nonempty subset of $\mathbb{R}$.
   (a) Prove that $\inf(A) \in \mathrm{bd}(A)$.
   (b) Now assume that A is closed. Prove that $\inf(A) \in A$.

5. (20 points) **True or False.**
   If a statement is true, prove it, and if it is false, come up with a counterexample.
   (a) $\left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x^4 + y^4 = 1\right\}$ is sequentially compact.
   (b) Let $A \subset \mathbb{R}^n$ and assume $F : A \to \mathbb{R}^m$ is continuous. If A is closed, then the image $F(A)$ is closed.
   (c) Let $A \subset \mathbb{R}^n$ and assume $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on domain $\mathbb{R}^n$. If A is bounded, then its image $F(A)$ is bounded.
   (d) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and C is sequentially compact in $\mathbb{R}^m$, then the inverse image $f^{-1}(C)$ is sequentially compact in $\mathbb{R}^n$.

6. (10 points) (**Sequentially compact**) §11.2, p. 304, # 7.
   Suppose that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(\vec{u}) \geq \|\vec{u}\|$ for every point $\vec{u}$ in $\mathbb{R}^n$. Prove that $f^{-1}\big([0,1]\big)$ is sequentially compact.

7. (10 points) (**Cubes versus balls**)
   Define $K(u,r)$ to be the cube with center u and radius r in $\mathbb{R}^n$:
   $$K(u,r) = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n \mid |x_i - u_i| < r \text{ for all } i\}.$$
   Prove that
   $$B(0,r) \subset K(0,r) \subseteq B(0,\sqrt{n}\,r).$$
   (*Hint*: Draw a picture of the 2-dimensional case when the cube is a square. Do the problem in $\mathbb{R}^2$ first.)

(End of Problem Set 9)