Tufts University
Department of Mathematics

Math 135 **Homework 5**[1] **Fall 2022**

Readings for the week of October 3, 2022

§**3.5:** Continuity implies ε-δ
§**3.3:** The intermediate-value theorem (statement), existence of roots
§**3.4:** Uniform continuity

Problem Set 5
(Due Wednesday, October 12, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (15 points) **ε-δ criterion for continuity**. §3.5, p. 73, #1 (only at $x = 2$).
   Define $f(x) = x^2$ for all $x \in \mathbb{R}$. Verify the ε-δ criterion for continuity at $x = 2$. (To verify the ε-δ criterion at a point a means given $\varepsilon > 0$, you need to find a $\delta > 0$ that satisfies the condition at a.)

2. (15 points) **ε-δ criterion for continuity**. §3.5, p. 73, #5.
   Define $h(x) = 1/(1+x^2)$ for all $x \in \mathbb{R}$. Prove that the function $h\colon \mathbb{R} \to \mathbb{R}$ satisfies the ε-δ criterion on $\mathbb{R}$. (To prove the ε-δ condition on $\mathbb{R}$ means to verify the condition at every point a of $\mathbb{R}$.)

3. (10 points) **Existence of a root**. §3.3, p. 65, #3.
   Prove that there is a solution of the equation
   $$\frac{1}{\sqrt{x+x^2}} + x^2 - 2x = 0, \qquad x > 0.$$

4. (10 points) **Fixed point theorem**.
   For a function $f\colon D \to \mathbb{R}$, a solution of the equation
   $$f(x) = x, \quad x \text{ in } D,$$
   is called a *fixed point* of f. A fixed point corresponds to a point at which the graph of the function f intersects the line $y = x$. Prove that every continuous function $f\colon [a,b] \to [a,b]$ has a fixed point. (*Hint*: Consider the function $g(x) = f(x) - x$.)

5. (6 points) **Limits**.
   For any sequence a_n, prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$. (*Hint*: Write down the ε-N definition of a limit.)

6. (10 points) **Uniform continuity**. §3.4, p. 69, #5.
   Define $f(x) = x^3$ for all x. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.

7. (10 points) **Lipschitz $\Rightarrow$ uniform continuity**. §3.4, p. 69, #11.
   A function $f\colon D \to \mathbb{R}$ is called **Lipschitz** if there is a nonnegative number C such that

   $$|f(u) - f(v)| \leq C|u - v| \qquad \text{for all points } u, v \in D.$$

   Prove that if $f\colon D \to R$ is Lipschitz, then it is uniformly continuous.

8. (14 points) **Uniformly continuous but not Lipschitz**. §3.5, p. 74, #7.
   Define $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$.
   (a) Prove that the function $f\colon [0,1] \to \mathbb{R}$ is continuous.
   (b) Use part (a) to show that $f\colon [0,1] \to \mathbb{R}$ is uniformly continuous.
   (c) Show that $f\colon [0,1] \to \mathbb{R}$ is not Lipschitz.
   (*Hint*:

   $$\sqrt{x} - \sqrt{x_0} = (\sqrt{x} - \sqrt{x_0})\frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}}.\Big)$$

9. (10 points) **ε-δ criterion for uniform continuity**. §3.5, p. 74, #10.
   Prove Theorem 3.22 (ε-δ criterion for uniform continuity) following the strategy of the proof of Theorem 3.20 (ε-δ criterion for continuity).

(End of Problem Set 5)