5) Let $\{u_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in $[0, \delta)$, and this is countable

Define $E_{\epsilon, n} = \{x \in [0,1] \,\|f(x, y_n) - f(x,0)| \leq \epsilon\} \forall y < n\}$

Now, as $f(x, \cdot)$ is continuous, then each E_n is measurable.

Define $E = \bigcap_{n=1}^{\infty} E_{\epsilon, n}$. E is measurable as $E_{\epsilon, n}$ measurable and countable intersection of measurable sets is measurable so E is measurable.

Now, to show $E = E_{\epsilon, \delta}$. Now, clearly $E \subset E_{\epsilon, \delta}$ as enumerating over rationals so now w.t.s $E_{\epsilon, \delta} \supset E$

To show this, let $x \in E$ this means that $\forall u_n \in u_n$, $|f(x, u_n) - f(x, 0)| \leq \epsilon$.

Now, for $y < \delta$, then can pick some subsequence of u_n, u_{n_k} s.t $u_{n_k} \to y$

So from there we see that it follows $|f(x, y) - f(x, 0)| \leq \epsilon$ meaning $x \in E_{\epsilon, \delta}$

Therefore, $E_{\epsilon, \delta} = E$, and as E is measurable, $E_{\epsilon, \delta}$ is measurable

$\square$

2.1) $\forall r \in \mathbb{R}$, $\exists \{a_n\}_{n=1}^{\infty}$ where $a_n \in \mathbb{Q}$ $\forall n \in \mathbb{N}$ and $a_n \to r$. We can also pick a_n s.t. $a_n \to r$ as a monotonically decreasing sequence from above.

f is measurable if $\forall r \in \mathbb{R}$, $f^{-1}(r,\infty]$ is measurable.

$$f^{-1}(r,\infty] = f^{-1}\left(\bigcup_{n=1}^{\infty}(r_n,\infty]\right) = \bigcup_{n=1}^{\infty} f^{-1}(r_n,\infty]$$

Now, by assumption, $f^{-1}(r_n,\infty]$ is Lebesgue measurable as $r_n \in \mathbb{Q}$, and union of measurable sets is measurable, so $f^{-1}(r,\infty]$ is measurable $\forall r \in \mathbb{R}$, meaning f is a measurable function on E

2.2)

$$\text{let } a \in \mathbb{R},\ g^{-1}(a,\infty] = \begin{cases} \emptyset & a > 1 \\ \mathbb{R} & a \leq -1 \\ (0,\infty) & 0 < a \leq 1 \\ [0,\infty) & -1 < a \leq 0 \end{cases}$$

Now $\emptyset$, $\mathbb{R}$ are measurable. $(0,\infty)$ is measurable as it is open.

Now, as $[0,\infty) = \{0\} \cup (0,\infty)$ and $\{0\}$ is measurable as singleton, union of measurable sets is measurable so $[0,\infty)$ is measurable.

Therefore, as $g^{-1}(a,\infty]$ is measurable $\forall a \in \mathbb{R}$, then g is a measurable function.