MATH235 HOMEWORK 2 SOLUTION

- 2.3.17. Assume that $E \subseteq \mathbb{R}^d$ is measurable, $0 < |E| < \infty$, and $A_n \subseteq E$ are measurable sets such that $|A_n| \to |E|$ as $n \to \infty$. Prove that there exists a subsequence $\{A_{n_k}\}_{k\in\mathbb{N}}$ such that $|\cap A_{n_k}| > 0$. Show by example that this can fail if $|E| = \infty$.

  Proof. $\forall k \geq 1$, $\exists n_1 < n_2 < \cdots$ such that

  $$||E| - |A_{n_k}|| = |E| - |A_{n_k}| < 2^{-k}|E|$$

  Consider $A = \cap_{k=1}^{\infty} A_{n_k}$, then we have

  $$|E| - |A| = |E \setminus A| = |\cup_{k=1}^{\infty} (E \setminus A_{n_k})| < |E| \sum 2^{-k} = |E|$$

  Therefore $|A| = |\cap_{k=1}^{\infty} A_{n_k}| > 0$.

  Consider $E \in \mathbb{R}$, set $A_n = [2^n, 2^{n+1}]$, then $\lim_{n\to\infty} |A_n| = \infty = |\mathbb{R}|$. Notice that for all n_k as a subsequence we have $\cap_{k=1}^{\infty} A_{n_k} = \emptyset$. □

- 2.3.19. Let E be a measurable subset of $\mathbb{R}^d$, and set $f(t) = |E \cap B_t(0)|$ for $t > 0$. Prove the following statements. (a). f is monotonic increasing and continuous on $(0, \infty)$. (b). $\lim_{t\to 0^+} f(t) = 0$. (c). $\lim_{t\to\infty} f(t) = |E|$. (d). If $|E| < \infty$, then f is uniformly continuous on $(0, \infty)$.

  Proof. (a). Set $t_1 > t_2 > 0$, then we have $B_{t_1}(0) \subset B_{t_2}(0)$. Therefore $E \cap B_{t_1}(0) = (E \cap B_{t_2}(0)) \cup (E \cap (B_{t_2}(0) \setminus B_{t_1}(0)))$. Hence we have

  $$|E \cap B_{t_2}(0)| = |E \cap B_{t_1}(0)| + |E \cap (B_{t_2}(0) \setminus B_{t_1}(0))|$$

  Notice that

  $$f(t_2) = f(t_1) + |E \cap (B_{t_2}(0) \setminus B_{t_1}(0))|$$

  Hence we have

  $$f(t_2) \geq f(t_1)$$

  Therefore f is monotonically increasing. To prove continuity, consider

  $$|f(t_2) - f(t_1)| \leq |B_{t_2}(0) - B_{t_1}(0)| \leq |t_2^d - t_1^d|$$

  which goes to 0 as $t_2 \to t_1$.

  (b). Since f is monotone and bounded in the neighborhood of 0^+, $\lim_{t\to 0^+} f(t)$ exists. Consider

  $$\lim_{t\to 0^+} f(t) = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} |E \cap (B_{\frac{1}{n}}(0)| = |E \cap_{n=1}^{\infty} B_{\frac{1}{n}}(0)| = 0$$

  The second last equality is from the continuity property of Lebesgue measure.

  (c). As f monotone, $\lim_{t\to\infty} f(t)$ exists. Consider

  $$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} |E \cap (B_t(0)| = |E \cap \mathbb{R}^d| = |E|$$

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The second last equality is from the continuity property of Lebesgue measure.

(d). Let $\epsilon > 0$, then $\exists R > 0$ such that $f(R) = |E_R| > |E| - \epsilon$, hence $E \setminus E_R < \epsilon$. Consider $s \leq t \leq R+1$, we have

$$|f(t) - f(s)| \leq C|t^d - s^d| = C|t-1||t^{d-1} + t^{d-2}s + \cdots + s^{d-1}| \leq C(R+1)^{d-1}d|t-s|$$

Choose $\delta = \min\left(\epsilon, \frac{\epsilon}{c(R+1)^{d-1}d}\right)$, then $|t-s| < \delta$ implies $|f(t) - f(s)| < \epsilon$. If $R < s < t$ then we have $E_t \setminus E_s \subseteq E \setminus E_s$ and therefore $|f(t) - f(s)| = |E_t \setminus E_s| \leq |E \setminus E_s| < \epsilon$. Hence f is uniformly continuous on $(0, \infty)$. □

- 2.3.25

Proof. (a). Notice that σ algebras are closed under countable intersections. Then by definition of $\Sigma(\mathcal{U})$, it contains all open sets. $\mathcal{B}$ also contains all closed sets as they are complements of open sets. F_σ and G_δ sets as countable union and intersection of open and closed sets, can also exist in $\mathcal{B}$. The rest of the claim follows from this pattern.

(b). Since every open set is Lebesgue measurable, we have $\mathcal{B} \subset \mathcal{L}$.

(c). By approximation of any measurable set by G_δ set, we have the desired claim. □

- 2.4.8. (a). Prove that continuity from below holds for exterior Lebesgue measure. (b). Show that there exist sets $E_1 \supseteq E_2 \cdots$ in $\mathbb{R}^d$ such that $|E_k|_e < \infty$ for every k and

$$|\cap_{k=1}^{\infty} E_k|_e < \lim_{k\to\infty} |E_k|_e$$

Hence the continuity from above doesn't hold for exterior Lebesgue measure.

Proof. (a). Consider $E = \cup_k E_k$, then there exist G_δ sets $H_k \supset E_k$ such that $|E_k|_e = |H_k|$ and $|E|_e = |H|$. Let $F_k = H \cap (\cap_{n=k}^{\infty} H_n)$ and $F = \cup_{k=1}^{\infty} F_k$ which are both measurable, then we have

$$E_k = \cap_{n=k}^{\infty} E_n \subset H \cap (\cap_{n=k}^{\infty} H_n) = F_k \subset H_k$$

Notice that $|E_k|_e \leq |F_k| \leq H_k = |E_k|_e$, hence $|F_k| = |E_k|_e$. Also,

$$E = \cup_k E_k \subset \cup_k F_k = F \subset H$$

Notice that $|E|_e \leq |F| \leq |H| = |E|_e$, hence $|F| = |E|_e$. Also, $F_1 \supseteq F_2 \cdots$ therefore

$$\lim_{n\to\infty} |F_n| = \lim_{n\to\infty} |E_n|_e = |\cup_{n=1}^{\infty} F_k| = |F| = |E|_e$$

where the second last equality is from continuity from below property of Lebesgue measure.

(b). Let $N \subseteq \mathbb{R}$ be the nonmeasurable set such satisfying $(N + r) \cap (N + r') = \emptyset$ if $r \neq r' \in \mathbb{Q}$. Consider $N = \cup_{k=-\infty}^{\infty}(N \cap [k, k+1])$, then without loss of generality, there exists some $k_0 \in [0, 1]$ such that $N_0 = N \cap [k_0, k_0 + 1]$ is nonmeasurable. Enumerate all rational numbers between $[0, 1]$ by $\{q_k\}$ and consider $N_k = N_0 + q_k$.

Consider a sequence $\{E_k\}$ defined by $E_k = \cup_{j=k} N_j$, then clearly $E_1 \supseteq E_2 \cdots$ and $E_1 \subseteq [0,2]$, while

$$|E_k|_e \geq |N_k|_e > 0$$

and

$$|\cap_{k=1} E_k|_e = |\emptyset|_e = 0$$

which is a contradiction. Hence continuity from above doesn't hold for exterior Lebesgue measure. $\square$

- 2.4.10. Given any integer $d > 0$, show that there exists a set $N \subseteq \mathbb{R}^d$ that is not Lebesgue measurable.

Proof. Generalize the Steinhaus Theorem to higher dimensions.

Step 1. Let $Q = [0,s]^d$. We claim that if $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, then

$$|Q \cap (Q+t)| \leq \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) s^k \|t\|^{d-k}.$$

First assume that $0 \leq t_k$ for every k. In this case we have $0 \leq t_k \leq \|t\|$ for every k, so

$$Q \cup (Q+t) \subseteq [0, s + \|t\|]^d,$$

and therefore, by the Binomial Theorem,

$$|Q \cup (Q+t)| \leq (s + \|t\|)^d = \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) s^k \|t\|^{d-k}.$$

A similar argument applies if any t_k is negative, so the claim follows.

Step 2. Now we generalize the Steinhaus Theorem. We claim that if $E \subseteq \mathbb{R}^d$ is Lebesgue measurable and $|E| > 0$, then the set of differences

$$E - E = \{x - y : x, y \in E\}$$

contains an open ball $B_r(0)$ for some $r > 0$.

To see this, we apply Problem 2.2.39 and conclude that there exists a cube Q such that the measure of the set $F = E \cap Q$ satisfies

$$|F| = |E \cap Q| > \frac{3}{4}|Q|. \qquad \text{(A)}$$

The statement of Steinhaus' Theorem is invariant under translations, so by translating E, F, and Q we can assume that $Q = [0,s]^d$ where $s > 0$. Choose any $t \in \mathbb{R}^d$. If F and $F + t$ are disjoint, then we must have

$$\begin{aligned}
2s^d = 2|Q| &< 2 \cdot \frac{4}{3}|F| && \text{(by equation (A))} \\
&= \frac{4}{3}|F \cup (F+t)| && \text{(since } F \text{ and } F+t \text{ are disjoint)} \\
&\leq \frac{4}{3}|Q \cup (Q+t)| && \text{(by monotonicity)} \\
&\leq \frac{4}{3} \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) s^k \|t\|^{d-k} && \text{(by the Lemma). (B)}
\end{aligned}$$

However,

$$\lim_{\|t\|\to 0} \frac{4}{3} \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) s^k \|t\|^{d-k} = \frac{4}{3} s^d < 2s^d$$

Therefore if $\|t\|$ is small enough then equation (B) cannot hold. Hence there is some $r > 0$ such that

$$\|t\| < r \implies F \text{ and } F + t \text{ are not disjoint.}$$

Therefore, if $\|t\| < r$ then there is some point $x \in F \cap (F + t)$. So, $x = y + t$ for some $y \in F$, which implies that $t = x - y \in F - F$. This shows that $F - F$ contains the open ball $B_r(0)$, and therefore $E - E$ must contain this ball as well.

Step 3. Define a relation on $\mathbb{R}^d$ by declaring that $x \sim y$ if and only if every component of $x - y$ is rational. This is an equivalence relation, so by the Axiom of Choice there exists a set N that contains exactly one element of each distinct equivalence class of this relation.

The distinct equivalence classes partition $\mathbb{R}^d$, so their union is $\mathbb{R}^d$. Therefore

$$\mathbb{R}^d = \bigcup_{x \in N} \left(\mathbb{Q}^d + x \right) = \bigcup_{x \in N} \bigcup_{r \in \mathbb{Q}^d} \{r + x\} = \bigcup_{r \in \mathbb{Q}^d} (N + r)$$

Since exterior Lebesgue measure is translation-invariant, the exterior measure of $N + r$ is exactly the same as the exterior measure of N. Combining this fact with countable subadditivity, we see that

$$\infty = |\mathbb{R}|_e = \left| \bigcup_{r \in \mathbb{Q}^d} (N + r) \right|_e \leq \sum_{r \in \mathbb{Q}^d} |N + r|_e = \sum_{r \in \mathbb{Q}^d} |N|_e.$$

Consequently, we must have $|N|_e > 0$. However, any two distinct points $x \neq y$ in N belong to distinct equivalence classes of the relation $\sim$, so some component of x and y must differ by an irrational amount. Therefore $N - N$ contains no open balls, so the Steinhaus Theorem implies that N cannot be Lebesgue measurable.

$\square$