(1) Let $X = \{1, 2\}$ be the topological space with topology $\tau_X = \{\emptyset, \{1\}, X\}$, and let $Y = \{a, b\}$ have the discrete topology $\tau_Y = \mathcal{P}(Y)$. What is the product topology on $X \times Y$? List all of the open sets. Which open sets are basic open sets? (i.e. which are of the form $U \times V$, where $U \in \tau_X$ and $V \in \tau_Y$?)

Definition 1. A topological space X is said to be **Hausdorff** if for each pair of distinct points $x_1, x_2 \in X$, there is an open neighborhood U_1 of x_1 and an open neighborhood U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$.

(2) Show that $\mathbb{R}$ is a Hausdorff topological space.

(3) Show that $\mathbb{R}$ with the finite complement topology is **not** a Hausdorff topological space.

(4) Suppose that X and Y are Hausdorff topological spaces. Show that $X \times Y$ is Hausdorff too.

(5) Suppose X and Y are homeomorphic topological spaces. Prove that if X is Hausdorff, then Y is Hausdorff.

Recall:

Definition 2. A subset Z of $\mathbb{R}^n$ is said to be **closed in the Zariski topology** if it can be written as the zero locus of some set of polynomials, i.e., there are polynomials $\{f_i\}_{i\in I}$ in the variables $x_1, \ldots, x_n$ so that

$$Z = Z(\{f_i\}) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f_i(x_1, \ldots, x_n) = 0 \text{ for all } i \in I\}.$$

The open subsets in the Zariski topology are the complements of the closed subsets.

(6) Show that the Zariski topology on $\mathbb{R}$ is just the finite complement topology. (How big is $Z(\{f\})$ when f is just one polynomial in x?)

(7) Show that the Zariski topology on $\mathbb{R}^2$ is **not** the finite complement topology. (Don't think too hard about it.)

(8) Let $X = \mathbb{R}$ with the finite complement topology. (Or the Zariski topology, same thing.) Give $X \times X$ the product topology. Does $X \times X$ have the finite complement topology? Does it have the Zariski topology?