Sunday help session 2/26

Sunday, February 26, 2023 19:30

$\lim_{x\to x_0} f(\bar x) = 0$ iff $\lim_{\bar x\to \bar x_0} |f(x)| = 0$

First order approx thm.

$O \subset \mathbb{R}^n$ open $f\in C^1(O,\mathbb{R})$

$\bar x_0 \in O$ then

affine 1st order approx to f near $\bar x_0$

$$\lim_{\bar x\to\bar x_0} \frac{|f(\bar x) - (f(\bar x_0) + \langle \nabla f(\bar x_0), \bar x - \bar x_0\rangle)|}{\|\bar x-\bar x_0\|} = 0$$

(version for $f: O\to\mathbb{R}^m$.

Thm $O\subset\mathbb{R}^2$ open $f\in C^1(O,\mathbb{R})$

$(x_0,y_0)\in O$ $f(x_0,y_0)=c$

$\frac{\partial f}{\partial y}(x_0,y_0)\neq 0$

then $\exists$ open interval I, $x_0\in I$ and C^1 fn

$g: I\to\mathbb{R}$ $g(x_0)=y_0$

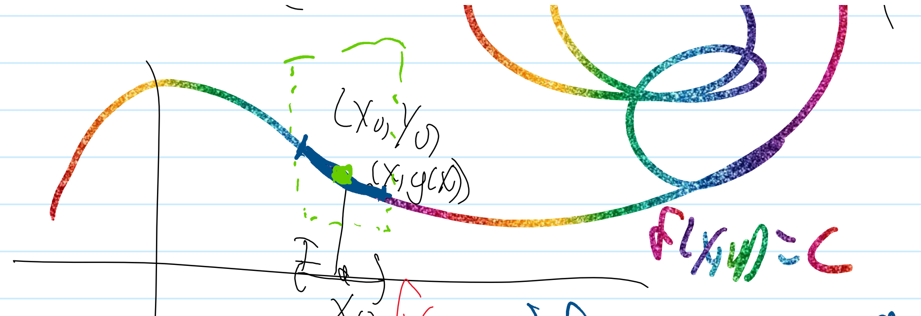
st $\forall x\in I$ $f(x,g(x)) = c$

ie the graph of $y=g(x)$ for $x\in I$ is the curve

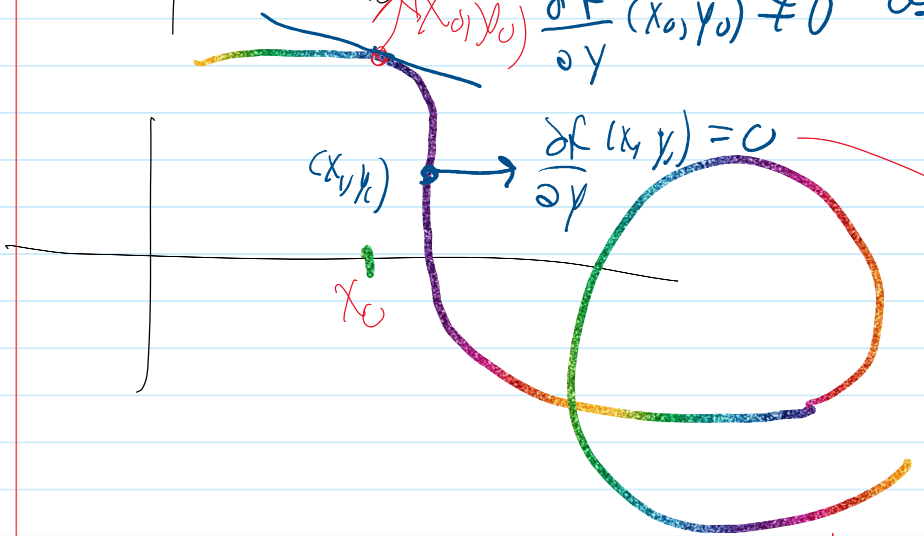
$f(x,y)=c$, at least near (x_0,y_0)

$\frac{\partial f}{\partial x}(x_0,y_0)\neq 0$

then $\exists$ open interval J $y_0\in J$



$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ as gradient not horiz (vertical tangent on graph $F(x,y)=c$



The graph has vertical tangent ∴ can't write y as a C' fn of x near x_1

~~#6 review~~

Meeting ID 963 8378 0643

pass code 924828

6 review. $F(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \\ F_3(x,y) \end{pmatrix}$ is cont. diff

$DF(1,1) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$ show ∃ nbhd U of $(1,1)$

st $F: U \to \mathbb{R}^3$ is 1-1

hint ... $F(x \ldots) = (F(x \ldots))$ $\tilde{F}: \mathbb{R}^2 \to \mathbb{R}^2$

hint let $\tilde{F}(x,y) = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$ $\quad \tilde{F}: \mathbb{R}^2 \to \mathbb{R}^2$

use IFT on $\tilde{F}$.

as $F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ and F is C^1

so F_1 and F_2 are C^1

$\therefore \tilde{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ is a C^1 fn from $\mathbb{R}^2$ to $\mathbb{R}^2$

$DF(1,1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$ so $D\tilde{F}(1,1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$J\tilde{F}(1,1) = \det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -1 \neq 0$

so can apply IVT

To say $\exists$ open nbd U of $(1,1)$

and open nbd V of $\tilde{F}(1,1)$

st $\tilde{F}: U \to V$ is bijective

F- tilde

now show F is 1-1

$(x_0, y_0) \in U$

$(x_1, y_1) \in U$

$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$

$F(x_0, y_0) = F(x_1, y_1)$

$$\begin{pmatrix} F_1(x_0, y_0) \\ F_2(x_0, y_0) \\ F_3(x_0, y_0) \end{pmatrix} = \begin{pmatrix} F_1(x_1, y_1) \\ F_2(x_1, y_1) \\ F_3(x_1, y_1) \end{pmatrix}$$

So $\tilde{F}(x_0, y_0) = \tilde{F}(x_1, y_1)$

a $\tilde{F}$ is 1-1 on domain U, $(x_0, y_0) = (x_1, y_1)$

3 $g(x,y) = \frac{x^2 y}{x^2+y^4}$ for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

Find $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^4}$ or show it d.n.e.

guess $\bar{x}_k = \left(\frac{1}{k^2}, \frac{1}{k}\right)$

$$g(\bar{x}_k) = \frac{\frac{1}{k^4}\frac{1}{k}}{\frac{1}{k^4}+\frac{1}{k^4}} = \frac{k^4 \frac{1}{k^4}\frac{1}{k}}{2} = \frac{1}{2k} \to 0$$

guess limit is 0

$$\frac{x^2 y}{x^2+y^4} \qquad \left(\frac{1}{k}, \frac{1}{k}\right) \qquad \frac{\frac{1}{k^3}\, k^4}{\left(\frac{1}{k^2}+\frac{1}{k^4}\right) k^4} = \frac{k}{k^2+1}$$

$$\frac{x^2}{x^2+y^4} \leq \frac{x^2+y^4}{x^2+y^4} = 1$$

$$0 \leq \left|\frac{x^2 y}{x^2+y^4}\right| \leq 1 \cdot |y| = |y|$$

as $(x,y)\to 0$

$0 \qquad 0 \qquad 0$

sandw. thm.

Let $O \subset \mathbb{R}^n$ be open $f: O \to \mathbb{R}$

Let $\mathcal{O} \subset \mathbb{R}^n$ be open $f: \mathcal{O} \to \mathbb{R}$

$\bar{x}_0 \in \mathcal{O}$ & f is diff at $\bar{x}_0$

$\therefore \exists\, \bar{b} \in \mathbb{R}^n$ s.t.

$$\lim_{\bar{x} \to \bar{x}_0} \frac{|f(\bar{x}) - (f(\bar{x}_0) + \langle \bar{b}, \bar{x} - \bar{x}_0 \rangle|}{\|x - x_0\|} = 0$$

To show f is cont at $\bar{x}_0$

$$\lim_{\bar{x} \to \bar{x}_0} \|\bar{x} - x_0\| = 0$$

so $$\lim_{\bar{x} \to \bar{x}_0} \frac{|f(\bar{x}) - f(\bar{x}_0) - \langle \bar{b}, \bar{x} - \bar{x}_0 \rangle|}{\|x - x_0\|} \|\bar{x} - \bar{x}_0\|$$

$$= \lim_{\bar{x} \to \bar{x}_0} \frac{|f(\bar{x}) - f(\bar{x}_0) - \langle \bar{b}, \bar{x} - \bar{x}_0 \rangle|}{\|x - x_0\|} \lim_{x \to x_0} \|\bar{x} - \bar{x}_0\|$$

$$= 0 \cdot 0 = 0$$

So $$|f(x) - f(x_0) - \langle b, \bar{x} - x_0 \rangle| \underset{\bar{x} \to \bar{x}_0}{\to} 0$$

$$|f(\bar{x}) - f(\bar{x}_0)| \le |f(\bar{x}) - f(\bar{x}_0) - \langle \bar{b}, \bar{x} - \bar{x}_0 \rangle| + |\langle \bar{b}, \bar{x} - \bar{x}_0 \rangle|$$

show $|\langle \bar{b}, \bar{x} - x_0 \rangle| \to 0$ as $\bar{x} \to \bar{x}_0$

$$0 \le |\langle \bar{b}, x - x_0 \rangle| \le \|\bar{b}\| \|x - \bar{x}_0\| \quad \text{CSB}$$

$\downarrow$ $\quad$ so $\uparrow$ $\quad\quad$ $\downarrow$ as $\bar{x} \to \bar{x}_0$

$0 \quad\quad 0$

$$\therefore 0 \le |f(\bar{x}) - f(x_0)| \le |f(\bar{x}) - f(x_0) - \langle \bar{b}, \bar{x} - \bar{x}_0 \rangle| + |\langle \bar{b}, \bar{x} - \bar{x}_0 \rangle|$$

$\downarrow \quad\quad\quad \downarrow \quad\quad\quad \downarrow$

$0 \quad\quad\quad 0$

$\downarrow$ 0 $\qquad \downarrow$ 0 $\qquad \downarrow$ 0

$\therefore |f(\bar{x}) - f(\bar{x}_0)| \to 0$

f is cont at $\bar{x}_0$

5 2018 dis $\quad f(x,y) = x^2 + y^2$

$g(x,y) = x^2 - y^2$

$F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{matrix} x^2 + y^2 \\ x^2 - y^2 \end{matrix}$

Explain why F is C^1 and find DF

$$DF(x,y) = \begin{array}{c|cc} & \partial/\partial x & \partial/\partial y \\ \hline f & 2x & 2y \\ g & 2x & -2y \end{array}$$

$F(1,1) = (2,0)$

Check IFT at $(x_0,y_0) = (1,1)$ $\quad F$ is C^1

$DF(1,1) = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$

$\det DF(1,1) = -4 - 4 = -8 \neq 0$

$\therefore \exists$ nbd U of $(1,1)$ & V of $F(1,1) = (2,0)$

st $F: U \to V$ is bij

$F^{-1}: V \to U$ (is bij)

$F(x,y) = \begin{pmatrix} u \\ v \end{pmatrix} \qquad F^{-1}(u,v) = \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in U$

$F(x,y) = \begin{pmatrix} u \\ v \end{pmatrix} \qquad DF(1,1) = \begin{array}{c|cc} & \partial/\partial x & \partial/\partial y \\ \hline u & 2 & 2 \\ v & 2 & -2 \end{array}$

$F(x,y) = \begin{pmatrix} u \\ v \end{pmatrix}$ $DF(1,1) = \begin{array}{c|cc} & \partial x & \partial y \\ u & 2 & 2 \\ v & 2 & -2 \end{array}$

what is $\frac{\partial v}{\partial x} = 2$.

Find $\frac{\partial x}{\partial u}$ by Finds $D(F^{-1})(2,0)$ $F(1,1)$

$((DF)(1,1))^{-1} = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}^{-1}$

$F^{-1}(u,v) = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{array}{c|cc} & \partial u & \partial v \\ x & & \\ y & & \end{array} = \frac{1}{-8}\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$$

$\frac{\partial x}{\partial u} = \frac{-2}{-8} = \frac{1}{4}$.