M 125

1 a) When $K=1$

$$f(\theta) = a_0 + a_1 \cos\left(\frac{2\pi\theta}{360}\right) + b_1 \sin\left(\frac{2\pi\theta}{360}\right)$$

This can be applied to the least squares model discussed in class:

So $A^T A x = A^T y \rightarrow$ where $x = \begin{bmatrix} a_0 \\ a_1 \\ b_1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & \cos\left(\frac{2\pi\theta_1}{360}\right) & \sin\left(\frac{2\pi\theta_1}{360}\right) \\ \vdots & \vdots & \vdots \\ 1 & \cos\left(\frac{2\pi\theta_n}{360}\right) & \sin\left(\frac{2\pi\theta_n}{360}\right) \end{bmatrix}$$

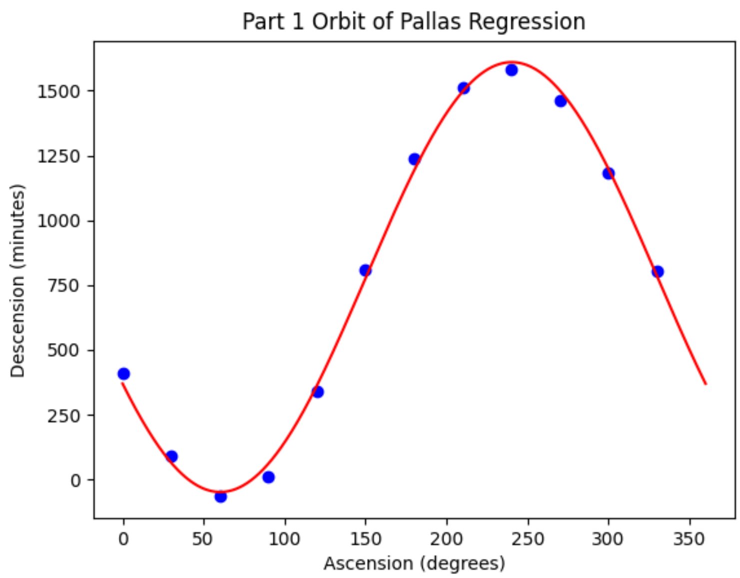
and y = declinations so $= \begin{bmatrix} 408 \\ 89 \\ \vdots \\ 804 \end{bmatrix}$

Solving this system under $K=1$ gives the following graph w/ an equation

Part 1a)

Equation: $f(\theta) = 780.53 - 411.014cos(\frac{2\pi\theta}{360}) - 720.227sin(\frac{2\pi\theta}{360})$

Graph: (Code at end)



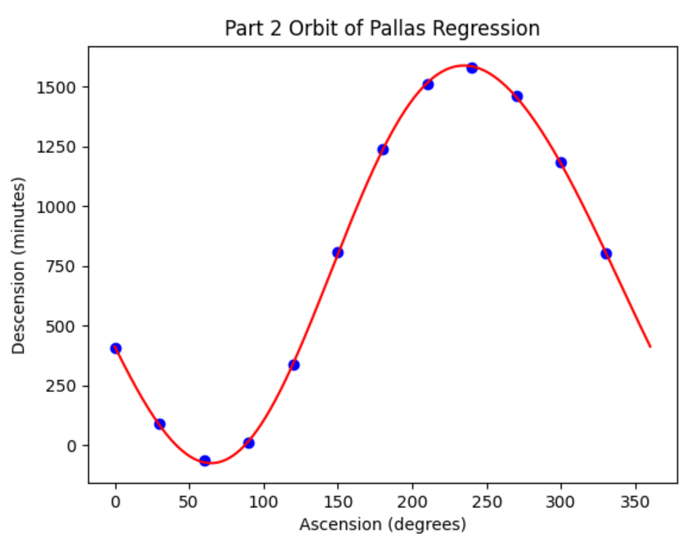
Part 1b)

Repeating the approach in part a, but adding two additional columns gives the following least squares equation:

Equation:

$f(\theta) = 780.53 - 411.014cos(\frac{2\pi\theta}{360}) - 720.227sin(\frac{2\pi\theta}{360}) + 43.416cos(\frac{4\pi\theta}{360}) - 2.165sin(\frac{4\pi\theta}{360})$

Graph:



2 a) $\lim_{\sigma\to\infty} C(\sigma) = \lim_{\sigma\to\infty} S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_1 - \sigma\sqrt{T-t})$

$= \lim S\Phi\left(\frac{\ln\frac{S}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1 - \sigma\sqrt{T-t})$

$= \lim S\Phi\left(\frac{\ln\frac{S}{K}}{\sigma\sqrt{T-t}} + \frac{(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\ln\frac{S}{K}}{\sigma\sqrt{T-t}} + \frac{(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}\right)$

$= \lim S\Phi(\infty) - Ke^{-r(T-t)}\Phi\left(\frac{\ln\frac{S}{K} + (r+\frac{1}{2}\sigma^2)(T-t) - \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$

$= \lim S - Ke^{-r(T-t)}\Phi\left(\frac{\ln\frac{S}{K} - \frac{\sigma^2}{2}(T-t) + r(T-t)}{\sigma\sqrt{T-t}}\right)$

$= S - Ke^{-r(T-t)}\Phi(-\infty)$

$\boxed{\lim_{\sigma\to\infty} = S}$

b) $S - Ke^{-r(T-t)} > 0 \rightarrow \frac{S}{K} > e^{-r(T-t)}$ $\ln\frac{S}{K} + r(T-t) > 0$

$\lim_{\sigma\to 0^+} S\Phi(d_1) - Ke^{-r(T-t)}(d_1 - \sigma\sqrt{T-t})$

$= \lim_{\sigma\to 0^+} S\Phi\left(\frac{\ln\frac{S}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1)$

$= \lim_{\sigma\to 0^+} S\Phi\left(\frac{\ln\frac{S}{K} + r(T-t) + \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1)$

$= \lim_{\sigma\to 0^+} S\Phi\left(\frac{\ln\frac{S}{K} + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1)$

$= \lim_{\sigma\to 0^+} S\Phi(\infty) - Ke^{-r(T-t)}\Phi(\infty)$

$= \boxed{S - Ke^{-r(T-t)}}$

c) $S - Ke^{-r(T-t)} < 0 \rightarrow \ln\frac{S}{K} + r(T-t) < 0$

$\lim_{\sigma\to 0^+} S\Phi\left(\frac{\ln\frac{S}{K} + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1)$

$= \lim_{\sigma\to 0^+} S\Phi(-\infty) - Ke^{-r(T-t)}\Phi(-\infty) = \boxed{0}$

2d) $S - Ke^{-r(T-t)} = 0 \to \ln\frac{S}{K} + r(T-t) = 0$

$$\lim_{\sigma\to 0^+} S\Phi\left(\frac{\ln\frac{S}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1)$$

$$= \lim_{\sigma\to 0^+} S\Phi\left(\frac{\ln\frac{S}{K} + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1)$$

$$= \lim_{\sigma\to 0^+} S\Phi\left(\frac{\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi(d_1)$$

$$= \lim_{\sigma\to 0^+} S\Phi(0) - Ke^{-r(T-t)}\Phi(0) = \frac{1}{2}(S - Ke^{-r(T-t)})$$

But from above $S - Ke^{-r(T-t)} = 0$, so

$$\boxed{\lim_{\sigma\to 0^+} C(\sigma) = 0}$$

2e) $\lim_{\sigma\to 0^+} C(\sigma) = \max(0, S - Ke^{-r(T-t)})$

2f) $\max(S - Ke^{-r(T-t)}, 0) < S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) < S$

As $d_2 = d_1 - \sigma\sqrt{T-t}$, $d_2 \le d_1$

$C'(\sigma) = S\sqrt{T-t}\, N'(d_1)$

$\lim_{\sigma\to\infty} C(\sigma) = S$ but this is a limit and not a maximum as $\nexists\, \sigma$ s.t. $C(\sigma) = S$

The limit is bounded by it's supremum S, so

$S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) < S$

To prove the second half, that

$\max(S - Ke^{-r(T-t)}, 0) < S\Phi(d_1) - Ke^{-r(T-t)}$

$\lim_{\sigma\to 0^+} C(\sigma) = \max(S - Ke^{-r(T-t)}, 0)$

$C'(\sigma) = \sigma\sqrt{T-t}\, N'(d_1) \ge 0$. So this

$C(\sigma)$ is always increasing, so it will always be greater than $\max(S - Ke^{-r(T-t)}, 0)$ thus proving the inequality.

2g) Using $C''(\sigma) = \frac{d_1 d_2}{\sigma} C'(\sigma) = 0$

Either $d_1 = 0$, $d_1 (d_1 - \sigma\sqrt{T-t}) = 0$

If $C'(\sigma) = 0$ then $S\sqrt{T-t}\, N'(d_1) = 0$ which is never true as $N'(d_1) \neq 0$ always.

If $d_1 = 0$, then $\frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = 0$

$$\ln(\tfrac{S}{K}) + (r + \tfrac{1}{2}\sigma^2)(T-t) = 0$$

$$\frac{\ln(\frac{S}{K}) + r(T-t) = -\frac{1}{2}\sigma^2(T-t)}{-2(T-t)}$$

$$= \sqrt{-2\left|\frac{\ln(\frac{S}{K}) + r(T-t)}{T-t}\right|} = \sigma^2 \quad \sigma^* = \sqrt{-2\left(\frac{\ln(\frac{S}{K}) + r(T-t)}{T-t}\right)}$$

But not neccesarily defined

If $d_1 = \sigma\sqrt{T-t}$ then

$$\frac{\ln\frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \sigma\sqrt{T-t}$$

$$\ln\frac{S}{K} + r(T-t) + \tfrac{1}{2}\sigma^2(T-t) - \sigma^2(T-t) = 0$$

$$\ln\frac{S}{K} + r(T-t) - \frac{\sigma^2}{2}(T-t) = 0$$

$$\ln\frac{S}{K} + r(T-t) = \frac{\sigma^2}{2}(T-t), \quad \sigma^* = \sqrt{2\left|\left(\frac{\ln\frac{S}{K} + r(T-t)}{T-t}\right)\right|}$$

So as ~~that~~ $\pm$ is critical point

σ^* has unique max at $\sigma^* = \sqrt{2\left|\frac{\ln(\frac{S}{K}) + r(T-t)}{T-t}\right|}$

2h) $C^* = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$. For a specific value of C^*, we can say $0 = S\Phi(d_1) - Ke^{-r(T-t)} - C^*$. As C^* = constant, we know it's derivative, as it's the vega and then as all other values are constant, we can use Newton's method to get the σ that satisfies the equation.

2 J) With initial value $\sigma^0 = 0.64$ here are the volatilities for each option. Code is attached.

Exercise price	Volatility (σ)
5125	0.2099
5225	0.2061
5325	0.2021
5425	0.1979
5525	0.1929
5625	0.1892
5725	0.1852
5825	0.1813

K) The volatility calculations are all pretty similar, which makes sense as these options come from the same market. The variance among these volatility calculations could be based on real world factors that impacted how the option was priced, such as a different model

Part 1 Orbit of Pallas Code:

```
import numpy as np
import math
import matplotlib.pyplot as plt

#Data
asc = np.array([0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300,
330]).astype('float64')
dec = np.array([408, 89, -66, 10, 338, 807, 1238, 1511, 1583, 1462,
1183, 804])

#Data setup for part a
x1 = np.zeros(12)
x2 = np.zeros(12)
for i in range(0, len(x1)):
    k = (2*math.pi*asc[i])/360
    x1[i] = np.cos(k)
    x2[i] = np.sin(k)

#Does initial setup for part b
x3 = np.zeros(12)
x4 = np.zeros(12)
for i in range(0, len(x3)):
    k = (4*math.pi*asc[i])/360
    x3[i] = np.cos(k)
    x4[i] = np.sin(k)

def part1(x1, x2, dec): #Least squares function for part a
    cons = np.ones(12)
    At = np.vstack((cons, x1, x2))
    A = np.transpose(At)
    LHS = np.matmul(At, A)
    RHS = np.matmul(At, dec)
    return np.linalg.solve(LHS, RHS)

def part2(x1, x2, x3, x4, dec): #Least squares function for part 2
    cons = np.ones(12)
    At = np.vstack((cons, x1, x2, x3, x4))
    A = np.transpose(At)
    LHS = np.matmul(At, A)
    RHS = np.matmul(At, dec)
    return np.linalg.solve(LHS, RHS)

res = part1(x1, x2, dec)
print(res)
```

```
res_2 = part2(x1, x2, x3, x4, dec)
print(res_2)
x = np.linspace(0, 360, 1000)

#Just change which plot is showing via comments
plt.title('Part 1 Orbit of Pallas Regression')
plt.xlabel('Ascension (degrees)')
plt.ylabel('Descension (minutes)')
plt.scatter(asc, dec, color = 'blue')
plt.plot(x, res[0]+res[1]*np.cos(k)+res[2]*np.sin(k), color ='red')
#plt.plot(x, res_2[0] + res_2[1]*np.cos(k)+ res_2[2]*np.sin(k)+
#res_2[3]*np.cos(k_2)+ res_2[4]*np.sin(k_2), color='red')
plt.show()
```

Part 2 Black Scholes Code:

```
import numpy as np
import math
import scipy.stats as stats

eprice = np.array([5125, 5225, 5325, 5425, 5525, 5625, 5725, 5825])
oprice = np.array([475, 405, 340, 280.5, 226, 179.5, 139, 105])
init = 5420.3

#Handles all Black Scholes math, same as Black-Scholes, except T-t=T
#Finds d1, d2, then evaluates Black-Scholes, and Vega in that order
d1 = lambda S, K, r, sig, T: (np.log(S/K) + (r+0.5*sig**2)
*(T))/(sig*math.sqrt(T))
d2 = lambda S, K, r, sig, T: d1(S, K, r, sig, T) - sig*np.sqrt(T)
BS_1 = lambda S, K, r, sig, T: S*stats.norm.cdf(d1(S, K, r, sig, T))
BS_2 = lambda S, K, r, sig, T:
K*np.exp(-1*r*(T))*stats.norm.cdf(d2(S, K, r, sig, T))
BS = lambda S, K, r, sig, T: BS_1(S, K, r, sig, T) - BS_2(S, K, r,
sig, T)
vega = lambda S, K, r, sig, T: S*np.sqrt(T)*stats.norm.pdf(d1(S, K,
r, sig, T))

#Newton’s Method Black-Scholes
def newt(init, eprice, oprice, r, dur,tol):
    vol_old = 0.64
    vol_new = 0
    for i in range(0, 100):
        temp = BS(init, eprice, r, vol_old, dur)-oprice
        vol_new = vol_old - temp/vega(init, eprice, r, vol_old, dur)
        if (abs(vol_new-vol_old) < tol): #Tolerance check
```

```
            break
        vol_old = vol_new
    return vol_old

vols = np.zeros(8)
for i in range(0, len(vols)): #Does it for each price
    vols[i] = newt(init, eprice[i], oprice[i], 0.04, 4/12,tol=10**-6)
print(vols)
```