

Math 145 Practicum

8 December 2023

Question 1. Let G be a finite group. Show that the order of any conjugacy class divides the size of the group.

(Hint: let G act on itself by conjugation—first show this is an action.)

Question 2. Let G be a group acting on a set X . Show that the orbits of any two conjugate elements in G have the same number of elements.

Question 3. Prove (carefully, with all details) that every group of order 15 is cyclic. Is it true that every group of order pq where p and q are distinct primes is cyclic?

Question 4. Let $m \geq 2$ be an integer. Begin with $4m$ elements

$$e, x, x^2, \dots, x^{2m-1}, y, xy, x^2y, \dots, x^{2m-1}y$$

and multiply them via

$$\begin{aligned}x^a x^b &= x^{a+b} \\x^a (x^b y) &= x^{a+b} y \\(x^a y) x^b &= x^{a-b} y \\(x^a y) (x^b y) &= x^{a-b+m}\end{aligned}$$

where $0 \leq a, b \leq 2m - 1$ and the powers of x are read modulo $2m$.

- Check that this defines a group G . We call G the **dicyclic group** of order $4m$.
- Check that for $m = 2$, G is isomorphic to the quaternion group.
- Check that for $m = 3$, G is **not** isomorphic to $\mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, D_6$, or A_4 . (Hint: the Sylow subgroups have different properties.)