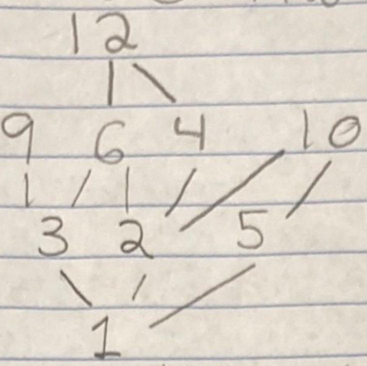
# Math 65 HW 8

1a)




b) The minimal element is 1, as for all elements in A, there is no element on a chain lower than 1.

c) 1 is a minimum element, as for a $1 \in A$, and for $\forall a \in A$, $1 \preceq a$.

d) The maximal elements are 9, 10, and 12, as they are the highest respective values on each chain.

e) There is no maximum, as we showed in class a maximum must be unique and the only maximal, which doesn't exist.

2a) Conditions for a poset are reflexive, antisymmetric and transitive

reflexive: Every subset of $P(A)$ is contained within itself, meaning for $A_1 \in P(A)$, $A_1 \subseteq A_1$

antisymmetric: If $A_1 \preceq A_2$ and $A_2 \preceq A_1$ then $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$, and by definition of equality, $A_1 = A_2$

Transitive: $A_1, A_2, A_3 \in P(A)$, If $A_1 \preceq A_2$, $A_2 \preceq A_3$, $A_1 \subseteq A_2$, $A_2 \subseteq A_3$, meaning $A_1 \subseteq A_3$ so $A_1 \preceq A_3$ showing transitivity

b) $A$ has $n$ elements. A chain of maximal length will start from $a_1 \in A$ without loss of generality, then add an element each path up to the chain until we reach $A$. This chain will be of length $n+1$. as goes $\emptyset \to \{a_1\} \to \{a_1, a_2\} \ldots \}$

c) $A = \{a_1, a_2, a_3 \ldots a_n\}$
$P(A)$ = all subsets of $A$.
Starting from $a_1$, by inclusion can add any element as the next step up on the chain, so $\{a_1\} \to \{a_1, a_j\}$ $1 < j \leq n$, this goes on for each iteration, so there are $n-j$ paths to reach $A = \{a_1 \ldots a_4\}$

ex) $\{a_1\} \to \{a_1, a_2\} \to \{a_1, a_2, a_3\} \to \{a_1, a_2, a_3, a_4\}$
$\{a_1, a_2\} \to \{a_1, a_2, a_4\}$
$\{a_1\} \to \{a_1, a_3\}$
$\{a_1\} \to \{a_1, a_4\}$ All originate from $\emptyset$

Based off this pattern, there are $n!$ maximal length chains, as $n-1$ less patterns for each successive element

d) Consider all $A_1, A_2 \in P(A)$ with cardinality $K$. If $A_1 \leq A_2$, then $A_1 \subseteq A_2$ and since $|A_1| = |A_2|$, $A_1 = A_2$. However for each subset $\in P(A)$ they are unique, meaning sets of $P(A)$ with cardinality $K$ aren't related and form an antichain.

e) $\{\{1\}, \{2\}, \{3\}\}$
$\{\{1,2\}, \{3\}\}$
$\{\{1,3\}, \{2\}\}$
$\{\{2,3\}, \{1\}\}$

3 a) 40 cards that aren't J, Q, K and are $\binom{52}{13}$ total ways to be dealt a hand. There are $\binom{40}{13}$ ways to pick no J, Q, K.

So $P(\text{no } J, Q, K) = \frac{\binom{40}{13}}{\binom{52}{13}} = \boxed{0.0189}$

b) $\binom{4}{4}$ ways to get 4 Kings, and $\binom{4}{3}$ ways to get 3 queens. $\binom{44}{6}$ to pick 6 remaining cards from no Q, K

$P = \frac{\binom{4}{4}\binom{4}{3}\binom{44}{6}}{\binom{52}{13}} = \boxed{0.00004}$

c) $\binom{4}{2}$ ways to choose a pair. Also have to pick 5 categories to have pairs which selects $\binom{13}{5}$, then so $\binom{4}{2}^5 \binom{13}{5}$, there are

For the 3 unique cards, there are $\binom{8}{3}$ ways to pick the categories and 4 possibilities for each card or $\binom{4}{1}$

Multiply this and:

$P(5 \text{ pair}, 3 \text{ unique}) = \boxed{\frac{\binom{4}{2}^5 \binom{13}{5} \binom{8}{3} \cdot 4^3}{\binom{52}{13}}}$

4 a) Even # 1's w/ 0, 2, 4 ones. total cases is

$\binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 8$

$P(E) = \frac{8}{16} = \frac{1}{2} \qquad P(F) = \frac{1}{2}$

$P(E \cap F) = \frac{4}{16}$, as 4 combinations w/ even number of 1's and starting with a 0.

b) $P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{1/2} = \boxed{\frac{1}{2}}$

c) $P(F \mid E) = \frac{P(F \cap E)}{P(E)} = \frac{1/4}{1/2} = \boxed{\frac{1}{2}}$

d) yes yas $P(E \cap F) = P(E)P(F)$

5 $P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{\sum_{i=1}^{r} P(A \mid B_i)\, P(B_i)}$

$$\frac{P(B_i \cap A)}{P(A)} = \frac{\frac{P(A \cap B_i)}{P(B_i)} P(B_i)}{\frac{P(A \cap B_1)}{P(B_1)} P(B_1) + \frac{P(A \cap B_2)}{P(B_2)} P(B_2) + \dots + \frac{P(A \cap B_r)}{P(B_r)} P(B_r)}$$

$$\frac{P(B_i \cap A)}{P(A)} = \frac{P(A \cap B_i)}{P(A \cap B_1) + P(A \cap B_2) + \dots P(A \cap B_r)}$$

Since $B_1 \dots B_r$ forms a partition of S, so

$\sum_{i=1}^{r} P(A \cap B_i) = P(A)$ so our equation is

$$\frac{P(B_i \cap A)}{P(A)} = \frac{P(A \cap B_i)}{P(A)}$$ and $P(B_i \cap A) = P(A \cap B_i)$

So this proves Bayes' theorem

6 $P(F \mid HHHTH) = \frac{P(HHHTH \mid F)\, P(F)}{P(HHHTH)}$

$$= \frac{P(HHHTH \mid F)\, P(F)}{P(HHHTH \mid F)\, P(F) + P(HHHTH \mid L)\, P(L)}$$

$\frac{1}{32}$ $\qquad \frac{9^4}{10^5}$

$$= \frac{\frac{1}{32} \cdot \frac{9}{10}}{\frac{1}{32} \cdot \frac{9}{10} + \frac{9^4}{10^5} \cdot \frac{1}{10}}$$

$$= \boxed{0.81084}$$

7a) $X =$ money get paid

$$E(x) = \sum_{k=1}^{n} x P(x) = \sum_{k=1}^{n} \frac{1}{n} \cdot 2^k = \frac{1}{n} \sum_{k=1}^{n} 2^k$$

$$E(x) = \frac{1}{n} \sum_{k=1}^{n} 2^k = \frac{1}{n}\left(\frac{2(1-2^n)}{1-2}\right)$$ Geometric series sum

$$= \frac{1}{n}(2(2^n - 1))$$

$$\boxed{E(x) = \frac{2(2^n-1)}{n}}$$

7b) On regions $2 \to n-1$, $E(x)$ if you flip is

$$E(k) = \frac{1}{2}(2^{k-1}) + \frac{1}{2}(2^{k+1})$$

$$= 2^{k-2} + 2^k = 2^k\left(1 + \frac{1}{4}\right) = \frac{5}{4} \cdot 2^k$$

$\frac{5}{4} \cdot 2^k > 2^k$, so should flip on regions $2 \to n-1$

On region $n$, $\frac{1}{2}$ chance win $2^{n-1}$, so no reason to flip. On region 1, can go to 4 or $2^n$ so you should flip on region 1.

$$\boxed{\text{Flip on regions } 1 \to n-1}$$