Question 1. Recall that the quaternion group Q is the set

$$\{\pm 1, \pm i, \pm j, \pm k\}$$

with multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1$$

(a) Find all the subgroups of Q

(b) For each subgroup H of Q find all the left cosets of H

(c) Determine which subgroups of Q are conjugate to one another.

Question 2. Let H be a subgroup of G. Prove the following:

(a) If $h \in H$ then $hH = H$

(b) If $g \notin H$ then $gH \cap H = \varnothing$

(c) If $f \in gH$ then $fH = gH$

(d) If $f \notin gH$ then $fH \cap gH = \varnothing$

(e) That the union of all left cosets of H equals G

Question 3. Recall that the group SO_3 is the group of 3×3 orthogonal real matrices with determinant $+1$. Show that SO_3 is exactly the group of rotations of $\mathbb{R}^3$ around axes that pass through the origin.

Hint: you can break this into the following steps:

(a) Find a matrix $A \in SO_3$ that represents a rotation about the z-axis by angle θ.

(b) Find a matrix $B \in SO_3$ that sends the z-axis to the axis containing the vector (a, b, c).

(c) Find a way to combine A and B to get a rotation of angle θ about the axis containing the vector (a, b, c).

Question 4. Find some finite groups that are *not* cyclic but which only have cyclic proper subgroups.

The more groups you find the more special bonus points you get!

Special bonus points are worthless and are not exchangeable for anything.