# Math 135 HW 3

1) Let $\varepsilon > 0$, want some $N \in \mathbb{N}$ s.t. $\forall n \geq N$
$|a_nb_n - 10| < \varepsilon$.

Scratch work)

$|a_nb_n - 10| = |a_nb_n - 5a_n + 5a_n - 10|$
$= |a_n(b_n - 5) + 5(a_n - 2)|$
$\leq |a_n(b_n-5)| + |5(a_n-2)|$
$\leq |a_n||b_n - 5| + 5|a_n - 2|$ Since $a_n$ converges,
$\leq M \cdot \frac{\varepsilon}{2M} + \frac{5\varepsilon}{12} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ $|a_n| \leq M$, $M \in \mathbb{R}$

Since $a_n \to 2$, $\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $|a_n| \leq M$ and $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $|a_n - 2| < \frac{5\varepsilon}{2 \cdot 12}$

As $b_n \to b$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$, $|b_n - 5| < \frac{\varepsilon}{2M}$

Let $N = \mathrm{Max}(N_1, N_2)$ Then:
$|a_nb_n - 10| \leq |a_n||b_n - 5| + 5|a_n - 2| < M(\frac{\varepsilon}{2M}) + \frac{5\varepsilon}{12} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

$\boxed{WM}$

2a) $a_n = \frac{1}{n}$ and $a_n \to 0$

Let $\varepsilon < 0$ so $|\frac{1}{n}| < \varepsilon$, but $|\frac{1}{n}| > 0$ and $\varepsilon < 0$ which isn't true.

b) $a_n = \frac{1}{n}$ and $a_n \to 0$

Let $\varepsilon = \frac{1}{10}$, so $|\frac{1}{n}| < \frac{1}{10}$. If $N = \{1, 2, 3, \ldots 9\}$ then this inequality doesn't hold as $1 \not< \frac{1}{10}$, $\frac{1}{2} \not< \frac{1}{10}$, and so on.

c) Let $a_n = \frac{1}{n}$ and though $a_n \to 0$, assume $a_n \to -1$
So $|\frac{1}{n} + 1| < \varepsilon' < \varepsilon + 1$ by archimedian property.
~~this~~ shows the sequence converges to $-1$, which isn't true.

3) $S_n = \frac{1}{2\cdot1} + \frac{1}{3\cdot2} + \cdots + \frac{1}{(n+1)n}$

$\frac{1}{(n+1)n} = \left(\frac{1}{n} - \frac{1}{n+1}\right) \rightarrow \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$

So $S_n = \left(1-\cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

-This means every term except for the first and last cancel out.

$\lim_{n\to\infty} S_n = 1 + \lim_{n\to\infty} \frac{-1}{n+1} = 1 - \lim_{n\to\infty}\frac{1}{n+1} = 1$

$\lim S_n = 1$

4a) $\left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$

$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$\sqrt{n+1}+\sqrt{n} > 2\sqrt{n}$ as $\sqrt{n+1} > \sqrt{n}$

So $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$

Note $\frac{1}{\sqrt{n+1}+\sqrt{n}} > 0$ for all $n$.

$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \lim_{n\to\infty} \frac{1}{2\sqrt{n}}$

$\lim_{n\to\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2}\lim_{n\to\infty}\frac{1}{\sqrt{n}} = \frac{1}{2}\lim_{n\to\infty}\sqrt{\frac{1}{n}}$, by square root lemma,

as $\lim_{n\to\infty}\frac{1}{n} = 0$, $\lim_{n\to\infty}\sqrt{\frac{1}{n}} = 0$, so $\lim_{n\to\infty}\frac{1}{2\sqrt{n}} = 0$.

$0 \leq \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} \leq 0$ ; by Sandwich theorem:

So $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$ which is equivalent to $\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}$ Converges to 0.

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4 b) $(\sqrt{n+1}-\sqrt{n})\sqrt{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{(n+1-n)\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \approx \frac{1}{2}$, as $2\sqrt{n}$,

$\lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\lim\limits_{n\to\infty}\sqrt{n}}{\lim\limits_{n\to\infty}\sqrt{n+1}+\sqrt{n}} = \frac{\lim\limits_{n\to\infty}\sqrt{n}}{\lim\limits_{n\to\infty}2\sqrt{n}}$ as for large $n$. $\sqrt{n+1} \approx \sqrt{n}$

$\rightarrow \frac{\lim\limits_{n\to\infty}\sqrt{n}}{\lim\limits_{n\to\infty}\sqrt{n+1}+\sqrt{n}} = \cancel{\frac{\lim\limits_{n\to\infty}\sqrt{n}}{2\lim\limits_{n\to\infty}\sqrt{n}}} = \frac{1}{2}$

$$\boxed{\text{So } \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n} = \frac{1}{2}}$$

c) $(\sqrt{n+1}-\sqrt{n})n \cdot \left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$

$= \frac{(n+1-n)n}{\sqrt{n+1}+\sqrt{n}} = \frac{n}{\sqrt{n+1}+\sqrt{n}}$

$\lim\limits_{n\to\infty} \frac{n}{\sqrt{n+1}+\sqrt{n}}$ note $\sqrt{n+1}+\sqrt{n} < 3\sqrt{n}$

$(\sqrt{n+1} < 2\sqrt{n})^2$

$n+1 < 4n$, $1 < 3n$ for $n > \frac{1}{3}$
$-n \quad -n$, so $\forall n \in \mathbb{N}$.

Since $\sqrt{n+1}+\sqrt{n} < 3\sqrt{n}$ $\forall n \in \mathbb{N}$,

$\frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{3\sqrt{n}}$

$\lim\limits_{n\to\infty} \frac{n}{\sqrt{n+1}+\sqrt{n}} > \lim\limits_{n\to\infty} \frac{n}{3\sqrt{n}} = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{3} = \frac{1}{3}\lim\limits_{n\to\infty}\sqrt{n}$ so this diverges. to $\infty$

Since $\frac{n}{\sqrt{n+1}+\sqrt{n}} > \frac{n}{3\sqrt{n}}$, then $\frac{n}{\sqrt{n+1}+\sqrt{n}}$ must also diverge to $\infty$.

5) Let $\mathbb{Q}^c$ denote the irrationals.

$\mathbb{Q}^c$ is closed if $s_n \in S$ converges to $s \in S$

Let $s_n = \left\{\frac{\sqrt{2}}{n}\right\}_{n=1}^{\infty}$ $\sqrt{2}$ is irrational from the book, and $\frac{1}{n}$ is rational. From textbook, product of a rational and irrational number is irrational, so $s_n \subset \mathbb{Q}^c$.

$$\lim_{n\to\infty} \frac{\sqrt{2}}{n} = \lim_{n\to\infty} \sqrt{2} \cdot \lim_{n\to\infty} \frac{1}{n} = \sqrt{2} \cdot \lim_{n\to\infty} \frac{1}{n} = \sqrt{2} \cdot 0 = 0$$

$0 \notin \mathbb{Q}^c$, so the set of irrational numbers are not closed

6) Let $S = (-\infty, 0]$ and $a_n \in S$ and $\lim a_n = a$

So $a_n \leq 0$, $\lim a_n \leq \lim 0$

$a \leq 0$ so $a \in S$ and $S$ is closed.

7) From textbook, $S \subset \mathbb{R}$ iff every $x \in \mathbb{R}$ is the limit of $s_n \in S$.

Let $S = [a, b]$, a closed interval where $a < b$ and $a, b \in \mathbb{R}$.

By definition of a closed set, then $a_n \in [a, b]$ converges to some $x \in [a, b]$.

Note, by Archimedian property, $b+1 \in \mathbb{R}$, but is not in $[a, b]$.

However as $a_n \to x \in [a, b]$ there is no $a_n \to b+1$, meaning any closed subset cannot be dense in $\mathbb{R}$.

8 a) Let $a_n = 0, 0, -1, -1, -2, -2, \ldots$

$b_n = 0, 1, 1, 2, 2, 3 \ldots$

$a_n$ is monotonically decreasing and $b_n$ is monotonically increasing

$a_n + b_n = \{0, 1, 0, 1, 0 \ldots\}$ which is not a monotone.

So the sum of monotone sequences isn't monotone.

8 b) Let $a_n = \left\{\left(\frac{3}{2}\right)^n\right\}_{n=1}^{\infty}$ and $b_n = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

$a_n$ is monotonically increasing, as $\frac{\left(\frac{3}{2}\right)^{n+1}}{\left(\frac{3}{2}\right)^n} \geq \left(\frac{3}{2}\right)$, $\left(\frac{3}{2}\right)^1 \geq 1$ which is true $\forall n \in \mathbb{N}$.

$b_n$ is monotonically decreasing as $\frac{1}{n} > \frac{1}{n+1}$, $n+1 > n$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \frac{\left(\frac{3}{2}\right)^n}{n} = \left\{\frac{3}{2}, \frac{9}{8}, \frac{9}{8}, \frac{81}{64} \ldots\right\}$$

$\frac{3}{2} > \frac{9}{8}$, but $\frac{9}{8} < \frac{81}{64}$ so this sequence is not monotone, and the product of monotone sequences isn't neccesarily monotone

c) Let $a_n = (-1)^n$ $|a_n| \leq 1$ $\forall n \in \mathbb{N}$ as $a_n = \{1, -1, 1, -1, 1 \ldots\}$

So $a_n$ is bounded, but doesn't converge so a bounded sequence doesn't need to converge

d) Let $a_n = n$, $a_n$ is monotonically increasing as $a_n \leq a_{n+1} \to n \leq n+1$. However $a_n$ doesn't converge, so not all monotone sequences converge.

9 a) $\Rightarrow$ So $l = \inf A$

- $l$ is a lower bound by definition. Next, so
- $\forall x \in A$, $l \leq x$. Let $\varepsilon = x - l$, so $x - l = \varepsilon \geq 0$, so there is $x = \varepsilon + l$, and by completeness, some $x < l + \varepsilon$ is in $A$, and $l = \inf A$.

$\Leftarrow$ $l = \inf A$ and $\varepsilon > 0$, note $l < l + \varepsilon$, so $l + \varepsilon$ isn't a lower bound of $A$. Therefore, by definition of $l = \inf A$, $\exists x \in A$ s.t. $x < l + \varepsilon$.

9 b) By definition, if $l = \inf A$, then
$\exists x \in A$ s.t. $x < l + \varepsilon$
So, let $\{a_n\} \in A$ and, then $a_n \leq l + \varepsilon$, and
as $\{a_n\} \in A$, then by definition of infinum,
$\lim (l \leq a_n \leq l + \varepsilon)$
$\lim l \leq \lim a_n \leq \lim l + \varepsilon$
$l \leq \lim a_n \leq l + \varepsilon$, so $a_n \to l$ as it's $< l + \varepsilon$ for any $\varepsilon > 0$
Since $a_n \to l$ then $a_n \to \inf A$ $\square$
$\square$

c) Suppose $A \subset \mathbb{R}$ is a nonempty set bounded above and $s \in \mathbb{R}$.
$s = \sup A$ if and only if $s$ is an upper bound of $A$ and $\forall \varepsilon > 0$, there is some $x \in A$ such that $x > s - \varepsilon$