Math 125 **Project 2** **Due: Monday 12/12 at 11:59 p.m. EST**

Instruction: You only have to do two out of the three problems. In addition to the solution of the problems, include with your submission a brief report that summarizes in non-technical terms the problem, the method, merits of the method e.g. simplicity, scalability, convergence, stability, limitations of the method and limitations of the model if one was to use the proposed numerical methods in a practical setting. For both problems, include a printout of your code with your project submission. You should submit the project on Gradescope.

Remark: For all questions that ask you to do a numerical implementation, the implementation must be your own and not based on calling routines/functions from existing libraries in MATLAB/Python.

1 Numerical integration

In the following two problems, we consider the approximation of integrals via numerical quadrature.

1.1 Integral of curves

We consider a curve whose parametric equations are given by

$$x(t) = a\cos(t)\,, y(t) = b\sin(t) \quad t \in (-\pi, \pi)$$

The arc-length of the parametric curve is given by

$$L = \int_{-\pi}^{\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

(a) Let $a = b$. Find the arc-length of the curve. What is the curve?

(b) For a general a and b, prove that the arc-length of the curve is

$$L = 4a \int_{0}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2(t)}\, dt,$$

where $k^2 = 1 - \frac{b^2}{a^2}$.

(c) Let $a = 3$ and $b = 2$. The resulting curve is shown below. Using a numerical integration method of your choice, compute L with an error that is at most 10^{-6}.

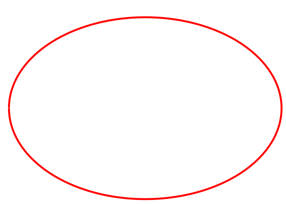

Figure 1: Curve with parametric equation $x = 3\cos(t)$ and $y = 2\sin(t)$ with $-\pi \leq t \leq \pi$.

(d) **Extra credit**: Unlike the case of a circle, finding the arc-length of an ellipse is not simple. Do research to find out what other approximations exist to estimate the perimeter of an ellipse. Compare the estimations to your result in (c).

1.2 Integral of $\exp\left(-\frac{x^2}{2}\right)$

In this problem, we consider the integral of $\exp\left(-\frac{x^2}{2}\right)$ using Gaussian quadrature.

(a) In brief terms, describe how Gaussian quadrature can be applied to compute the integral of a function $f(x)$ on the interval $[-1, 1]$ i.e. $\int_{-1}^{1} f(x)\, dx$.

(b) Implement Gaussian quadrature for the following integral

$$\int_{-1}^{1} \exp\left(-\frac{x^2}{2}\right) dx$$

Compute the integral using number of points $n = 2$, $n = 3$ and $n = 4$. The roots of the n-degree Legendre polynomial are noted in the table below.

n	x_i
2	$-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{3}}$
3	$-\sqrt{\frac{3}{5}}$ 0 $\sqrt{\frac{3}{5}}$
4	-0.86113631159405 -0.33998104358486 0.33998104358486 0.86113631159405

Table 1: The roots x_i of the n-degree Legendre polynomial.

(c) **Extra credit**: Relate the integral in (b) to the probability density function of the normal distribution. Using numerical values from exisiting solvers, check the error for your answer in b for all the different values of n.

2 Iterative methods for linear systems

We study the problem of solving the following linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ where $\mathbf{A} \in \mathcal{R}^{n\times n}$ is the matrix defined as:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 2 \end{pmatrix}$$

and $\boldsymbol{b} \in \mathcal{R}^n$ is the right hand size vector defined as:

$$\boldsymbol{b} = h^2 \begin{pmatrix} \sin(\pi h) \\ \sin(2\pi h) \\ \vdots \\ \vdots \\ \sin(n\pi h) \end{pmatrix}$$

(a) Show that the Jacobi iteration converges for this problem.

(b) Prove that Gauss Seidel iteration converges for this problem.

(c) Implement the Jacobi method to solve the linear system with $n = 1000$ and $h = 0.1$. Plot the number of iterations versus the residual error in a semilog y plot.

(d) Implement the Gauss Seidel method to solve the linear system with $n = 1000$ and $h = 0.1$. Plot the number of iterations versus the residual error in a semilog y plot.

(e) Using your results in (c) and (d), what can you say about the convergence rate of these methods to solve this problem?

3 Steady state of a dynamical system

A matrix $\mathbf{A} \in \mathcal{R}^{n\times n}$ is column-stochastic if all of its entries are non-negative and the entries of each column sum to 1. Here on, assume that $\mathbf{A}$ is a column-stochastic matrix.

(a) Prove that $\lambda = 1$ is an eigenvalue of $\mathbf{A}$.

(b) Prove that the largest eigenvalue of $\mathbf{A}$ is 1.

(c) Let $\mathbf{x}^{(0)} \in \mathcal{R}^n$ be a non-negative vector whose entries sum to 1. Consider the following iterative procedure

$$\mathbf{x}^{(k)} = A\mathbf{x}^{(k-1)} \quad \text{for } k = 1, 2, 3, ...$$

If $\mathbf{x}^{(k)}$ converges to $\mathbf{x}^*$, what limiting equation do we obtain?

(d) Perron–Frobenius Theorem states that the dominant eigenvector of a positive column-stochastic matrix has positive entries and its entries sum to 1. Load the matrix `stochastic_matrix.mat` or `stochastic_matrix.csv` from the `Project2` folder. Implement the power method and find the dominant eigenvector and eigenvalue of the matrix.

(e) **Extra credit**: Comment how your work in (a)-(d) connects to the page rank algorithm we discussed in class.