Math HW 9

1 For $\vec{x} \in V^\perp$, $\vec{v} \in V$, $\vec{v} \cdot \vec{x} = \vec{0}$ since $\vec{v} \cdot \vec{x} = v^T \vec{x} = 0$, we have $V^T \vec{x} = 0$, and this ~~is of form~~ $A\vec{x} = 0$, meaning to find $V^\perp$, we need null(V^T)

$$V^T \vec{x} = [v_1 \ldots\ldots v_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

$$[v_1 x_1 + \ldots\ldots v_n x_n] = 0$$

dim(Kernel) = 1, as $V^T \vec{x}$ has 1 column. By rank nullity theorem, dim(Kernel) + dim(image) = n. 1 + dim(image) = n, dim(image) = n-1. V^T spans n-1 dimensions. Since $V^T \vec{x} = 0$, dim(image) = dim(V^T), Meaning dim $V^\perp$ = n-1 $\boxtimes$

2 $\left\| \sum_{i=1}^{p} a_i \vec{v}_i \right\|^2 = \|a_1 v_1 + \ldots + a_p v_p\|^2$ $\vec{v}_1 \ldots \vec{v}_p$ forms an orthogonal set, this means that by pythagorean theorem, for orthogonal vectors, $\|v_1 \ldots v_p\|^2 = \sum_{n=1} \|v_i\|^2$, since $a_1 \ldots a_p$ are scalars, then we can use pythagorean theorem to write the sum as $\|a_1 \vec{v}_1\|^2 + \|a_2 \vec{v}_2\|^2 + \ldots \|a_p \vec{v}_p\|^2$, by properties of vector norms, $\|c\vec{v}\| = |c|\|\vec{v}\|$, $c \in \mathbb{R}$, and $\|c\vec{v}\|^2 = \|c\vec{v}\|\|c\vec{v}\| = c^2\|\vec{v}\|^2$, this makes our sum into $a_1^2\|\vec{v}_1\|^2 + \ldots\ldots a_p^2\|v_p\|^2$, which can be rewritten as the infinite series $\sum_{i=1}^{p} |a_i|^2 \|v_i\|^2$ $\boxtimes$

$\|av\|^2 = |a|^2|v|^2$

3 a) If $\vec{v} \perp \vec{w}$, $\vec{v}\cdot\vec{w} = 0$, $\vec{w} = \vec{u} - \frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}$

$$\vec{v}\cdot\left(\vec{u} - \frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}\right) = 0$$

$$\vec{u}\cdot\vec{v} - \vec{v}\cdot\left(\frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}\right) = 0$$

$$\vec{u}\cdot\vec{v} - (\vec{v}\cdot\vec{v})\left(\frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\right) = 0$$, as $\vec{v}\cdot(c\vec{w}) = c(\vec{v}\cdot\vec{w})$

$$\vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} = 0$$

$$0 = 0 \quad \boxtimes$$

b) $proj_v\vec{u}\cdot\vec{w} = 0$ $\quad proj_v\vec{u} = \frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}$, $w = u - proj_v\vec{u}$

$$proj_v\vec{u}\cdot(\vec{u} - proj_v\vec{u}) = 0$$

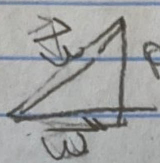
$$\vec{u}\,proj_v\vec{u} - proj_v\vec{u}\cdot proj_v\vec{u} = 0$$

$$\vec{u}\cdot\left(\frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}\right) - \left(\frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}\right)\cdot\left(\frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\vec{v}\right) = 0$$

$$\vec{u}\cdot\vec{v}\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right) - \left(\frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\right)^2\vec{v}\cdot\vec{v} = 0$$

$$\frac{(\vec{u}\cdot\vec{v})^2}{(\vec{v}\cdot\vec{v})} - \frac{(\vec{u}\cdot\vec{v})^2}{(\vec{v}\cdot\vec{v})} = 0$$

$$0 = 0 \quad \boxtimes$$

c) 

By pythagorean theorem

$$||\vec{u}||^2 = ||proj_v\vec{u}||^2 + ||w||^2$$

$$||\vec{u}||^2 = \left|\left|\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right|\right|^2 + ||u - proj_v\vec{u}||^2$$

$$||\vec{u}||^2 - ||\vec{w}||^2 = \left|\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right|^2 ||\vec{v}||^2$$

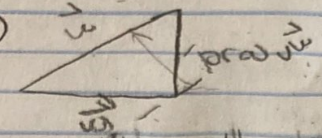
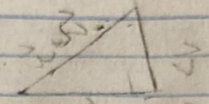
$$||\vec{u}||^2 - ||\vec{w}||^2 = \frac{|\vec{u}\cdot\vec{v}|^2}{(||\vec{v}||^2)^2}||\vec{v}||^2$$

$$||\vec{u}||^2 - ||\vec{w}||^2 = \frac{|\vec{u}\cdot\vec{v}|^2}{||\vec{v}||^2}$$

$$||\vec{v}||^2(||\vec{u}||^2 - ||w||^2) = (\vec{u}\cdot\vec{v})^2$$

$$||\vec{v}||^2||\vec{u}||^2 - ||\vec{v}||^2||\vec{w}||^2 = (\vec{u}\cdot\vec{v})^2$$

We know that $||\vec{v}||^2||\vec{w}||^2 \ge 0$, and $(\vec{u}\cdot\vec{v})^2 \ge 0$,
meaning that $\sqrt{||\vec{v}||^2||\vec{u}||^2 \ge |\vec{u}\cdot\vec{v}|^2}$, $||\vec{v}||\,||\vec{u}|| \ge |\vec{u}\cdot\vec{v}|$,
meaning $|\vec{u}\cdot\vec{v}| \le ||\vec{u}||\,||\vec{v}||$, $\boxtimes$

3 d)  

$$\left(\|\vec{u}+\vec{v}\| \le \sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}\right)$$

$$\|\vec{u}+\vec{v}\| = \sqrt{\vec{u}\cdot\vec{u}+2\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v}} = \sqrt{\|\vec{u}\|+\|\vec{v}\|+2\vec{u}\cdot\vec{v}}$$

$$\left(\sqrt{\|\vec{u}\|+\|\vec{v}\|+2\vec{u}\cdot\vec{v}} \le \|\vec{u}\|+\|\vec{v}\|\right)^2$$

$$\|\vec{u}\|+\|\vec{v}\|+2\vec{u}\cdot\vec{v} \le \|\vec{u}\|^2+2\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|^2$$

$$\sqrt{\vec{u}\cdot\vec{u}}+\sqrt{\vec{v}\cdot\vec{v}}+2\vec{u}\cdot\vec{v} \le \vec{u}\cdot\vec{u}+2\sqrt{\vec{u}\cdot\vec{u}}\sqrt{\vec{v}\cdot\vec{v}}+\vec{v}\cdot\vec{v}$$

We know that $\sqrt{\vec{u}\cdot\vec{u}}+\sqrt{\vec{v}\cdot\vec{v}} \le \vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{v}$, so to prove

$$2\vec{u}\cdot\vec{v} \le 2\sqrt{\vec{u}\cdot\vec{u}}\sqrt{\vec{v}\cdot\vec{v}}$$

$$\vec{u}\cdot\vec{v} \le \sqrt{\vec{u}\cdot\vec{u}}\sqrt{\vec{v}\cdot\vec{v}}$$

$\vec{u}\cdot\vec{v} \le \|\vec{u}\|\|\vec{v}\|$, which was proven true in part c.

We don't have to worry about absolute value, as $\|\vec{u}\|\|\vec{v}\| \ge 0$, meaning it doesn't matter if $\vec{u}\cdot\vec{v}$ is positive/negative.

This proves that $\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$