Math 235 HW Kop)

4.4.22) If $\int_a^x f(t)\,dt = 0$, well suppose $\exists E \subseteq [a,b]$ s.t $|E| > 0$ and $f(x) > 0$ on E.

There exists a closed $F \subseteq E$ s.t $0 < |F| \leq |E|$ and as $f > 0$ on F, then if $\int_F f = 0$ well $f = 0$ a.e on F which is contradiction.
We repeat this argument for open $U = (a,b) \setminus F$, where $U = \bigcup (a_k, b_k)$ as disjoint intervals.
$0 = \int_a^b f = \int_U f + \int_F f$, so $\int_U f = -\int_F f < 0$ so there is an (a_k, b_k) s.t $\int_{a_k}^{b_k} f \neq 0$, but then
$\int_a^b f = \int_a^{a_k} f + \int_{a_k}^{b_k} f + \int_{b_k}^{b} f = 0 + \int_{a_k}^{b_k} f \neq 0$ So contradiction.

This implies $f \leq 0$ a.e, and we perform a symmetric argument w/ $f(x) < 0$ on E to get $f \geq 0$ a.e so $f = 0$ a.e.

4.5.17)
$F(x)$ being uniform continuous means $\forall \epsilon > 0, \exists \delta > 0$ s.t if $|x - y| < \delta$ then $|F(x) - F(y)| \leq \epsilon$
So let $x = \delta$ in this case.
$|F(x+\delta) - F(x)| = \left|\int_0^{x+\delta} f(x)\,dx - \int_0^x f(x)\,dx\right|$

$$= \left|\int_x^{x+\delta} f(x)\,dx\right| \leq \int_x |f(x)|\,dx$$

As f is integrable, we can pick a $\delta > 0$ s.t if $|E| < \delta$ then $\int_E |g| < \epsilon$ where g measurable fn.

Therefore, as $|(x+\delta) - x| = \delta$, then $\int_x^{x+h} |f(x)|\,dx \leq \epsilon$ meaning $F(x)$ is uniformly continuous.

4.5.22)

So $|f_n| \le g$ a.e and $f_n \xrightarrow{m} f$. Let $\{g_k\}$ bc subsequence of f_n, then $g_k \xrightarrow{m} f$ and g_k has subsequence h_k s.t. $h_k \to f$ a.c

Now, $|h_k| \le g$ a.e, so dominated convergence thm $\Rightarrow \int h_k \to \int f$

As h_k is subsequence of subsequence of f_n, and $\int h_k \to \int f$, then $\int f_n \to \int f$

Now, as $|f - f_n| \xrightarrow{m} 0$ so f_n has subsequence $f_{n_k} \to f$ a.e

As $|f_{n_k}| \le g$ a.e then $|f| \le g$ a.e,

So $|f - f_n| = |(f - f_{n_k}) + (f_{n_k} - f_n)| \le |g| + |g| = 2g$

So $\|f - f_n\|_1 = \int_E |f - f_n| \to \int_E 0 = 0$

as $\int f_n \to \int f$ which concludes the pf

4.5.26)

Now $2|E| = |E| + |E+h|$

$= |E \cup (E+h)| + |E \cap (E+h)|$

$= |E \cap (E+h)| + |E \setminus (E+h)| + |(E+h) \setminus E| + |E \cap (E+h)|$

$2|E| = 2|E \cap (E+h)| + |E \setminus (E+h)| + |(E+h) \setminus E|$

$|(E \setminus (E+h)| + |(E+h) \setminus E| = \int_{\mathbb{R}^d} \chi_{E \setminus (E+h)} + \int_{\mathbb{R}^d} \chi_{(E+h) \setminus E}$

$= \int_{\mathbb{R}^d} |\chi_E - \chi_{E+h}| = \|\chi_E - \chi_{E+h}\|$ and

as $h \to 0$ see clearly $\|\chi_E - \chi_{E+h}\| \to 0$

So $2|E \cap (E+h)| = 2|E| - |E \setminus (E+h)| + |(E+h) \setminus E|$

and as $h \to 0$ then follows

$2|E \cap (E+h)| \to 2|E|$ so $|E \cap (E+h)| \to |E|$

Which proves the statement

4.5.27)

So first $\forall x$ a.e, $|f(x)| = \lim |f_n(x)| \leq \lim g_n(x) = g(x)$

Now, that means $f(x) \in L^1(E)$ as $g \in L^1(E)$

From above, $g + g_n - |f - f_n| \geq 0$ a.e

will use Fatou's lemma, but first note

$$\int_E 2g = \int_E \liminf (g + g_n - |f - f_n|) \text{ by above}$$

(Fatou's) $\leq \liminf\limits_{n \to \infty} \int_E (g + g_n - |f - f_n|)$

$= \liminf\limits_{n \to \infty} \left(\int_E g + \int_E g_n - \int_E |f - f_n| \right)$

$= \int_E g + \liminf\limits_{n \to \infty} \left(\int_E g_n + \int_E |f - f_n| \right)$

$\leq \int_E g + \liminf\limits_{n \to \infty} \int_E g_n + \liminf\limits_{n \to \infty} - \int_E |f - f_n|$

As $g_n \to g$: $\leq \int_E g + \int_E g + \liminf\limits_{n \to \infty} - \int_E |f - f_n|$

$= 2 \int_E g + \liminf\limits_{n \to \infty} \int_E -|f - f_n|$

$= 2 \int_E g - \limsup\limits_{n \to \infty} \int_E |f - f_n|$ (inf sup swap)

So $\int_E 2g \leq 2 \int_E g - \limsup \int_E |f - f_n|$

So: $0 \leq \limsup\limits_{n \to \infty} \int_E |f - f_n| \leq 0$ so

$\lim\limits_{n \to \infty} \int_E |f - f_n| = 0$ meaning the statement has been proven $\square$

4.5.32)

Let's define $f_k(x,y) = \frac{f(x+\frac{1}{k},y) - f(x,y)}{1/k}$ $k \in \mathbb{N}$

f_k is measurable as f measurable in y, and as $\frac{\partial f}{\partial x}(x,y)$ exists

$$\frac{\partial f}{\partial x}(x,y) = \lim_{k\to\infty} f_k(x,y) \quad \text{so}$$

also $\frac{\partial f}{\partial x}(x,y)$ is measurable. $\frac{\partial f}{\partial x}(x,y)$ is bdd, so $\int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy$ exists.

$f(x,y)$ is integrable of y so define

$F(x) = \int_0^1 f(x,y)\,dy$, which exists $\forall x \in [0,1]$

$\lim_{h\to 0} \frac{F(x+h) - F(x)}{F(h)}$ but want to use bded convergence thm.

So can define sequence $h_k \to 0$ as $k \to \infty$ and have:

$\lim_{k\to\infty} \frac{F(x+h_k) - F(x)}{F(h_k)}$ and w.t.s this $= \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy$ $\forall y \in [0,1]$

So now we know by def of F that

$$\frac{F(x+h_k) - F(x)}{F(h_k)} = \int_0^1 \frac{f(x+h_k,y) - f(x,y)}{h_k}\,dy$$

using integral properties.

We know $\frac{\partial f}{\partial x}$ exists so integrand will converge

Furthermore, $\lim_{k\to\infty} \frac{f(x+h_k,y)-f(x,y)}{h_k} = \frac{\partial f}{\partial x}(x,y)$

using defs.

Now, need to show in $\mathbb{C}$, so $f = f_r + i f_i$
It follows $\partial f = \partial f_r + i \partial f_i$

Note as $\frac{\partial f}{\partial x}(x,y)$ bded, denote $\sup |\frac{\partial f}{\partial x}(x,y)| = M$

Therefore, the real and imaginary components of ∂f are bded.
Now, pick $y \in [0,1]$. As $f_r \in \mathbb{R}$, by MVT $\exists c \in (x, x+h)$ s.t.

$$\frac{f_r(x+h_k) - f_r(x,y)}{h_k} = \left|\frac{\partial f_r}{\partial x}(c,y)\right| \leq M$$

and we can repeat this logic for the imaginary part.
Now, as $|f| \leq |f_r| + |f_i|$:

$$\left|\frac{f(x+h_k,y) - f(x,y)}{h_k}\right| \leq \frac{|f_r(x+h_k) - f_r(x,y)|}{h_k} + \text{Same for imaginary}$$

$$\leq M + M = 2M$$

So, this means we actually satisfy the criteria for the bded convergence thm as $|f_n| \leq 2M \; \forall n$ and $f_n \to f$ a.e. but our sequence is different so

$$\lim_{k\to\infty} \frac{F(x+h_k) - F(x)}{h_k} = \lim_{k\to\infty} \int_0^1 \frac{f(x+h_k) - f(x,y)}{h_k} dy$$

$$= \int_0^1 \frac{\partial f}{\partial x}(x,y)$$

This concludes the proof $\square$