Tufts University
Department of Mathematics

Math 235 | **Final Exam**[1] | **Fall, 2022**

This is an open book and open notes exam but you may not consult anyone other than Shuang Guan or myself, and you may not ask for help online (e.g., message boards, forums). You are allowed to consult your class notes and Heil's textbook.

The test will be available on Gradescope starting 8:30 a.m. Eastern Time on Thursday December 22nd. You will have 2 hours from the time you download the test to finish it. You will have an extra 15 minutes to upload it to Gradescope. Since it can take time to upload, do not wait until this 15 minutes is almost up. You must finish uploading your answers by 11:59 p.m. on Thursday December 22nd. When you upload your answers to Gradescope, please scan this signed signature page first and then scan all your answers into one PDF file. Please number the problems clearly and in order.

Please sign the following pledge and submit the signed pledge with your answers:

The Tufts University statement on academic integrity holds that: "Academic integrity is the joint responsibility of faculty, students, and staff. Each member of the community is responsible for integrity in their own behavior and for contributing to an overall environment of integrity at the university." I accept this responsibility, affirm that I am an honest person who can be trusted with to do the right thing, and certify that the work I will do on this exam is mine alone.

I pledge that I have used only the reference sources I have cited in my answers.

Signature____________________

The test problems start on the next page.

Problem 1. Let $E = \{(x, x^3), x \in \mathbb{R}\} \subset \mathbb{R}^2$. Using only the definition, prove that E is Lebesgue measurable and find its measure $|E|$.

Let $n \geq 1$ and Consider $E_n = \{(x, x^3) \; x \in [-n, n]\}$.

$$E \subseteq \bigcup_{k=1}^{\infty} E_n \Rightarrow |E|_e \leq \sum_{k=1}^{\infty} |E_k|_e$$

let $\varepsilon > 0$, $f(x) = x^3$ is uniformly Continuous on $[-k, k]$

$\Rightarrow \exists \delta > 0: \; \forall x, y \in [-k, k] \; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2k}$

Next we can subdivide $[-k, k]$ into L intervals each of length of δ. So $[-k, k] \subseteq \bigcup_{j=1}^{L} I_j$ $f(I_j) = J_j$ is an interval and $|J_j| < \frac{\varepsilon}{2k}$

Then $E_k \subset \bigcup_{j=1}^{L} I_j \times J_j$

$$|E_k|_e \leq \sum_{j=1}^{L} |I_j \times J_j|_e \leq \sum_{j=1}^{L} |I_j| \, |J_j| \leq \sum_{j=1}^{L} |I_j| \frac{\varepsilon}{2k} < \varepsilon$$

$\Rightarrow \forall k \; \forall \varepsilon > 0 \quad |E_k|_e < \varepsilon \Rightarrow |E_k|_e = 0 \Rightarrow |E|_e = 0$

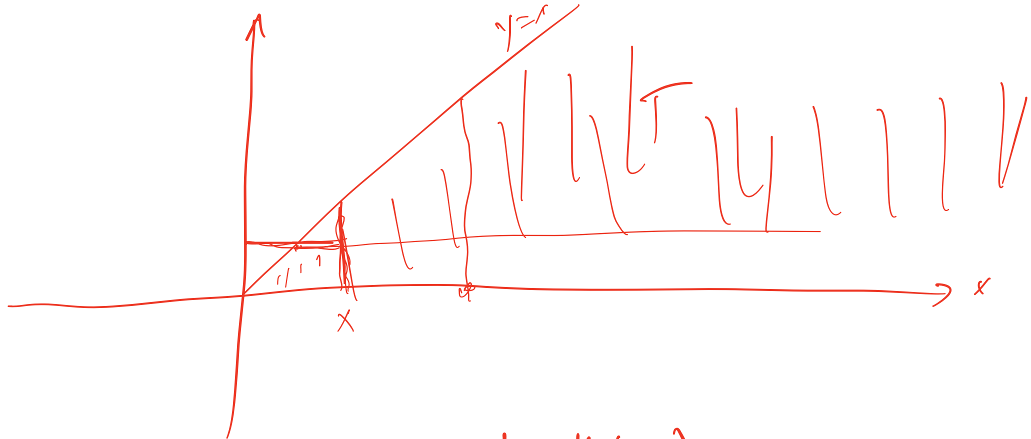
$\Rightarrow E$ is measurable and $|E| = 0$

$(0,1)$

$(1,0)$

Problem 2. Let $T = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le x < \infty\}$.

Evaluate $\lim_{n\to\infty} \iint_T e^{-(x+\frac{1}{n}+\frac{n+1}{n}y)}(dxdy)$ justifying your answer.



Let $f_n(x,y) = e^{-(x+\frac{1}{n}+\frac{n+1}{n}y)} = e^{-(x+y+\frac{1}{n}+\frac{y}{n})}$ $(x,y)\in T$

f_n is measurable. and nonnegative

$\lim_{n\to\infty} f_n(x,y) = e^{-(x+y)}$

$f_n(x,y) \le e^{-(x+y)} e^{-(\frac{1}{n}+\frac{y}{n})} \le e^{-(x+y)}$

Next, because $g(x,y) = e^{-(x+y)}$ is nonnegative and measurable we can apply Tonelli's theorem on the measurable set T to write

$$\iint_T g(x,y)\,dxdy = \int_0^\infty \left(\int_0^x g(x,y)\,dy \right) dx$$

$$= \int_0^\infty \left(\int_0^x e^{-(x+y)}\,dy \right) dx$$

$$= \int_0^\infty e^{-x} \left(\int_0^x e^{-y}\,dy \right) dx = \int_0^\infty e^{-x}(1-e^{-x})\,dx$$

$$= \int_0^\infty (e^{-x} - e^{-2x})\,dx = -e^{-x} + \frac{1}{2}e^{-2x} \Big|_0^\infty = +1-\frac{1}{2} = \frac{1}{2} < \infty$$

$g \in L^1(T)$

we can apply the LDCT and see that

$$\lim_{n\to\infty} \iint_T f_n(x,y)\,dx\,dy = \iint_T \lim_n f_n(x,y) = \iint_T g(x,y) = \frac{1}{2}.$$

Problem 3. Assume that $f \in AC[0,1]$ and there is a function g continuous on $[0,1]$ such that $f' = g$ a.e. Show that f is differentiable everywhere on $[0,1]$, and that $f'(x) = g(x)$ for all $x \in [0,1]$. Show by an example that the hypothesis of absolute continuity is necessary.

$f \in AC[0,1] \Rightarrow$ $f(x) = \int_0^x f'(t)\,dt + f(0)$ $\forall x \in [0,1]$

f is diff a.e. on $[0,1]$ and $f' \in L^1[0,1]$

Let $h(x) = \int_0^x g(t)\,dt$ Since g is contin.

$\Rightarrow$ h is diff everywhere and $h \in AC[0,1]$

and $h'(x) = g(x)$ $\forall x \in [0,1]$

but $g(x) = f'(x)$ a.e $\Rightarrow$ $h'(x) = g(x) = f'(x)$ a.e

$h' = f'$ a.e and h, f are $AC[0,1]$

$\Rightarrow$ $h(x) = f(x) - f(0)$ $\forall x \in [0,1]$. Since h is differentiable everywhere $\Rightarrow$ f is differentiable everywhere

and $h'(x) = f'(x) = g(x)$ $\forall x \in [0,1]$.

Let $\varphi : [0,1] \to [0,1]$ be the Lebesgue-Cantor function

$\varphi'(x) = 0$ a.e. and $g(x) = 0$ $\forall x \in [0,1]$ is continuous

but φ is not differentiable everywhere!

Problem 4. Let $f \in L^1(\mathbb{R})$ and define $F(x) = \int_{\mathbb{R}} f(t)\frac{\sin tx}{t}dt$.

(a) Prove that F is differentiable on $\mathbb{R}$ and find F'

(b) Is F absolutely continuous on each compact subinterval of $\mathbb{R}$? Justify your answer.

Let $x_0 \in \mathbb{R}$ and $h \neq 0$

$$\frac{F(x_0+h)-F(x_0)}{h} = \int_{\mathbb{R}} \frac{f(t)}{t}\,\frac{\sin t(x_0+h)-\sin tx_0}{h}\,dt$$

$$= \int_{\mathbb{R}} f(t)\,\frac{\sin t(x_0+h)-\sin tx_0}{th}\,dt$$

$$\lim_{h\to 0} \frac{\sin t(x_0+h)-\sin tx_0}{th} = \lim_{h\to 0} \frac{t\cos(tx_0+th)}{t} = \cos tx_0$$

$$\Rightarrow \lim_{h\to 0} f(t)\cdot\frac{\sin(tx_0+th)-\sin tx_0}{th} = f(t)\cos(tx_0)$$

$$\left| f(t)\cdot\frac{\sin(tx_0+th)-\sin(tx_0)}{th}\right| = \left|f(t)\cdot\cos(tx_0+t\tilde{h})\right|$$ where

$\tilde{h}$ is between 0 and h and we use the MVT for the function $h \to \sin(tx_0+th)$ (t is fixed and x_0 is fixed)

$$\Rightarrow \left| f(t)\cdot\frac{\sin(tx_0+th)-\sin(tx_0)}{th}\right| \le |f(t)| \in L^1$$

By the Generalized LDCT $\displaystyle\lim_{h\to 0}\frac{F(x_0+h)-F(x_0)}{h} = \lim_{h\to 0}\int_{\mathbb{R}} f(t)\,\frac{\sin(tx_0+th)-\sin tx_0}{th}$

$$= \int_{\mathbb{R}} f(t)\cos(tx_0)\,dt \quad\Rightarrow\quad F'(x_0) = \int f(t)\cos tx_0$$

(b) Take a compact interval $[a,b]$ in $\mathbb{R}$

$\forall x \in [a,b]$ $F'(x) = \int f(t)\,dx$

$\Rightarrow |F'(x)| \leq \int |f(t)\,dx| = \int |f(t)| = \|f\|_{L^1}$

$\Rightarrow F$ is Lipschitz on $[a,b]$

$\Rightarrow F \in AC[a,b]$

Problem 5. Let $f \in L^p(\mathbb{R})$, where $1 \le p < \infty$. For $\alpha > 0$, define

$$E_\alpha(f) = \{x \in \mathbb{R} : |f(x)| > \alpha\}.$$

(a) Show that E_α has finite Lebesgue measure.
(b) Use (a) to show that every $f \in L^p(\mathbb{R})$, $1 \le p \le 2$, can be decomposed as $f_1 + f_2$ where $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$.

$$|E_\alpha(f)| = |\{|f| > \alpha\}| = |\{|f|^p > \alpha^p\}| \le \frac{1}{\alpha^p}\int |f|^p < \infty$$

b) $f_1 = f\chi_{E_\alpha}$ and $f_2 = f\chi_{E_\alpha^c}$ take $\alpha = 1$

Note $f_1 + f_2 = f(\chi_{E_\alpha} + \chi_{E_\alpha^c}) = f \cdot 1 = f$.

f_1, f_2 are measurable.

$$\int_{\mathbb{R}} |f_1| = \int |f| \cdot \chi_{E_\alpha} \le \left(\int |f|^p\right)^{\frac{1}{p}} \cdot \left(\int \chi_{E_\alpha}^{p'}\right)^{\frac{1}{p'}} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

$$\le |E_\alpha|^{\frac{1}{p'}} \|f\|_p < \infty.$$

$$\int |f_2|^2 = \int |f|^2 \chi_{E_\alpha^c}^2 = \int_{[|f| \le 1]} |f|^2 \quad \text{Sul } p < 2$$

and $|f| < 1 \Rightarrow |f|^2 < |f|^p$

$$\Rightarrow \int |f_2|^2 \le \int_{[|f| \le 1]} |f|^p \le \int |f|^p \checkmark$$