Math 125 **Project 1** **Due: Tuesday 11/29 at 11:59 p.m. EST**

Instruction: In addition to the solution of the problems, include with your submission a brief report that summarizes in non-technical terms the problem, the method, merits of the method e.g. simplicity, scalability, convergence, stability, limitations of the method and limitations of the model if one was to use the proposed numerical methods in a practical setting. For both problems, include a printout of your code with your project submission. You should submit the project on Gradescope.

Orbit of Pallas

For it is the duty of an astronomer to record celestial motions through careful observation. Then, turning to the causes of these motions, he must conceive and devise hypotheses about them, since he cannot in any way attain the true cause. But, from whatever assumptions he adopts, the motions can be correctly calculated from the principles of geometry for the past as well as the future.
- Andreas Osiander, author of the introduction to the 1543 book De Revolutionibus orbium coelestium by Copernicus. Source: [Cunningham, 2016].

In March 28, 1802, the astronomer Heinrich Wilhelm Matthäus Olbers discovered the asteroid Pallas. It was initially conjectured that Pallas was a small planet between Mars and Jupiter. Pallas is the second asteroid to be discovered after the asteroid Ceres. Before discussing further the observational data of Pallas, we will note the definition of the word "celestial sphere". In astronomy, it is usually the case that we have very little information on exact distances to celestial objects. In this setting, one defines a reference coordinate system and specifies information about celestial objects using direction. The celestial sphere is a sphere of arbitrarily large radius which is concentric with Earth. This sphere is fictitious and its role is that it allows us to project celestial bodies. Using the Earth's coordinate system of longitude and latitude and projecting it onto the celestial sphere, we get the celestial coordinates right ascension and declination.

The declination of Pallas as a function of the right ascension is noted below.

Ascension (θ)	0	30	60	90	120	150	180	210	240	270	300	330
Declination	408	89	-66	10	338	807	1238	1511	1583	1462	1183	804

Table 1: Ascension and declination of Pallas. Ascension is in units of degrees and declination is in minutes.

Given this data, consider using the following 360°-trigonometric polynomial to approximate the observations.

$$f(\theta) = a_0 + \sum_{i=1}^{k} \left[a_i \cos\left(\frac{2i\pi\theta}{360}\right) + b_i \sin\left(\frac{2i\pi\theta}{360}\right) \right].$$

(a) Propose a least squares approach to fit the data to $f(\theta)$ for $k = 1$. Plot your fit against the data.

(b) Use your approach in (a) and apply it to $k = 2$. Plot your fit against the data.

(c) **Extra credit**: Define $g(\theta)$ as follows:

$$g(\theta) = a_0 + \sum_{i=1}^{5} \left[a_i \cos\left(\frac{2i\pi\theta}{360}\right) + b_i \sin\left(\frac{2i\pi\theta}{360}\right) \right] + a_6 \cos\left(\frac{12\pi\theta}{360}\right)$$

Fit the observational data exactly to $g(\theta)$.

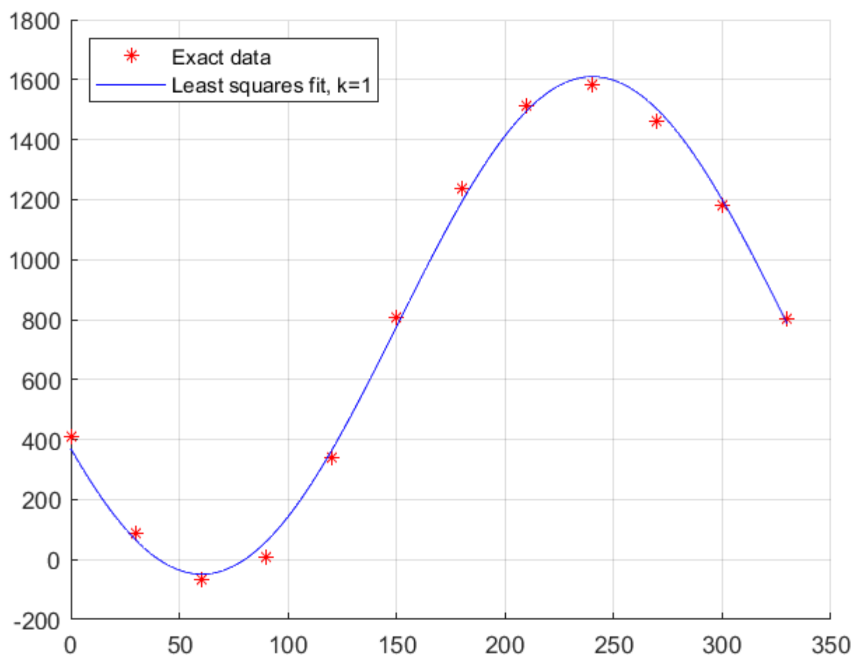
Solution:

For details of implementation, refer to `orbit_pallas.m` in the `project1_soln` folder.

(a) Let $f_1 = \mathbf{1}$, $f_2 = \{\cos\left(\frac{2i\pi\theta}{360}\right)\}_{i=1}^{12}$ and $f_3 = \{\sin\left(\frac{2i\pi\theta}{360}\right)\}_{i=1}^{12}$. Our goal is to find a_0, a_1 and b_1 such that $\{f(\theta_i)\}_{i=1}^{12} \approx a_0 f_1 + a_1 f_2 + b_1 f_3$. The least squares formulation of this problem is $\min_{\mathbf{x}} \; ||\mathbf{A}\mathbf{x} - \mathbf{y}||$ where

$$\mathbf{A} = \begin{pmatrix} 1 & \cos\left(\frac{2\pi\theta_0}{360}\right) & \sin\left(\frac{2\pi\theta_0}{360}\right) \\ 1 & \cos\left(\frac{2\pi\theta_1}{360}\right) & \sin\left(\frac{2\pi\theta_1}{360}\right) \\ \vdots & \vdots & \vdots \\ 1 & \cos\left(\frac{2\pi\theta_{12}}{360}\right) & \sin\left(\frac{2\pi\theta_{12}}{360}\right) \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 408 \\ 89 \\ -66 \\ \vdots \\ 804 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} a_0 \\ a_1 \\ b_1 \end{pmatrix}$$

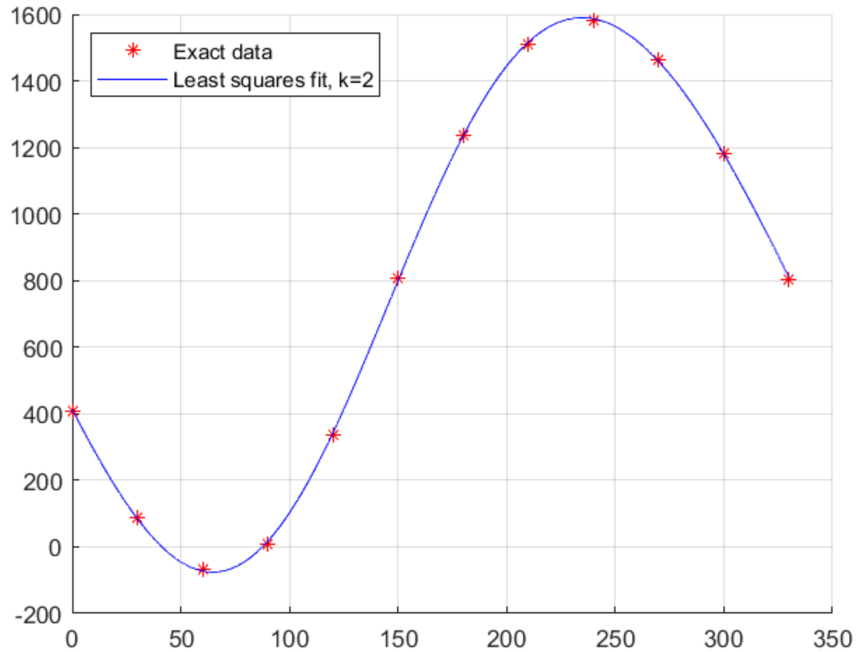
The normal equations to solve the least square problem are $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{y}$. Below, we show a plot of the least square fit with $k = 1$ along with the data points.



(b) Let $f_1 = \mathbf{1}$, $f_2 = \{\cos\left(\frac{2i\pi\theta}{360}\right)\}_{i=1}^{12}$ and $f_3 = \{\sin\left(\frac{2i\pi\theta}{360}\right)\}_{i=1}^{12}$, $f_4 = \{\cos\left(\frac{4i\pi\theta}{360}\right)\}_{i=1}^{12}$ and $f_5 = \{\sin\left(\frac{4i\pi\theta}{360}\right)\}_{i=1}^{12}$. Our goal is to find a_0, a_1, a_2, b_1 and b_2 such that $\{f(\theta_i)\}_{i=1}^{12} \approx a_0 f_1 + a_1 f_2 + a_2 f_4 + b_1 f_3 + b_2 f_5$. The least squares formulation of this problem is $\min_{\mathbf{x}} \; ||\mathbf{A}\mathbf{x} - \mathbf{y}||$ where

$$\mathbf{A} = \begin{pmatrix} 1 & \cos\left(\frac{2\pi\theta_0}{360}\right) & \sin\left(\frac{2\pi\theta_0}{360}\right) & \cos\left(\frac{4\pi\theta_0}{360}\right) & \sin\left(\frac{4\pi\theta_0}{360}\right) \\ 1 & \cos\left(\frac{2\pi\theta_1}{360}\right) & \sin\left(\frac{2\pi\theta_1}{360}\right) & \cos\left(\frac{4\pi\theta_1}{360}\right) & \sin\left(\frac{4\pi\theta_1}{360}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cos\left(\frac{2\pi\theta_{12}}{360}\right) & \sin\left(\frac{2\pi\theta_{12}}{360}\right) & \cos\left(\frac{4\pi\theta_{12}}{360}\right) & \sin\left(\frac{4\pi\theta_{12}}{360}\right) \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 408 \\ 89 \\ -66 \\ \vdots \\ 804 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}$$

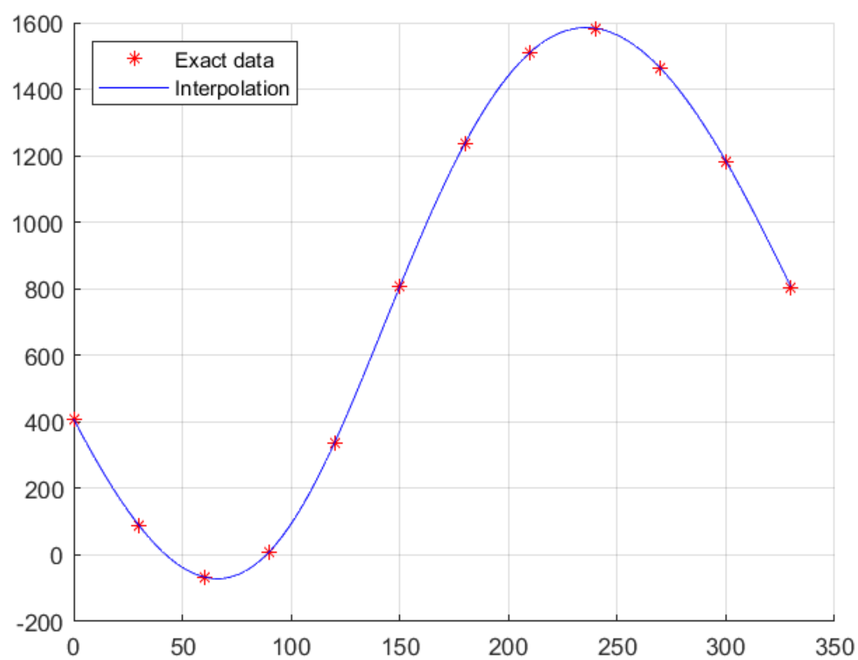
The normal equations to solve the least square problem are $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{y}$. Below, we show a plot of the least square fit with $k = 2$ along with the data points.



(c) For exact interpolation, we solve $\mathbf{Ax} = \mathbf{y}$ where

$$\mathbf{A}_j = \begin{cases} \mathbf{1} & j = 1 \\ \{\cos\left(\frac{2i\pi\theta_i}{360}\right)\}_{i=1}^{12} & \text{j is even and } 2 \leq j \leq 12 \\ \{\sin\left(\frac{2i\pi\theta_i}{360}\right)\}_{i=1}^{12} & \text{j is odd and } 3 \leq j \leq 11 \end{cases}$$

where $\mathbf{A}_j$ indicates the j-th column of $\mathbf{A}$ and the vector $\mathbf{y}$ is a vector of the declination values. Below, we will show that the obtained solution exactly interpolates the data points as desired.



Implied volatility

Volatility is a measure of the rate at which the price of a stock or index of stocks changes over a fixed period of time. Volatility is one metric that is considered to assess the risk in an investment. Generally, low volatility is associated with positive returns while high volatility is associated with negative returns. One way to measure volatility is by considering the past prices of a stock or index of stock. This is called *historical volatility*. In contrast, implied volatility is based on the dynamics of the market itself and is a measure of what the markets think volatility will be over a given period of time. In short, historical volatility is based on recent and past data while implied volatility is forward looking. Before defining implied volatility more precisely, we review some basic terminologies in finance.

Options: There are two common types of option contracts, *calls* and *puts*. A call option gives you the option to buy while a put option gives you an option to sell. In a call option, the holder/buyer has the right but not the obligation to buy a stock at a specified price. In contrast, in a put option, the holder/seller has the right but not the obligation to sell a stock at a specified price. In both cases, the right lasts on a particular date which is known as the *maturity* or *expiration* date. The specified or agreed price for the call or put is known as the *exercise* or *strike price*. There are two main types of options depending when the rights can be exercised. In *American options*, the options can be exercised any time before or on the expiration date. For *European options*, the options can only be exercised at the expiration date. Few notations are in order:

- $S(t)$: the stock price at time t
- K: the strike price
- T: the expiration time
- r: interest rate
- σ: volatility

One of the most popular approach to price an option is based on the Black-Scholes formula. For an European call option, the Black-Scholes formula for the price of the option is

$$C = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where d_1 and d_2 are given by

$$\begin{aligned} d_1 &= \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

and Φ is the cumulative distribution function for the standard normal distribution. Note that, with the exception of the volatility parameter, all the variables in the Black-Scholes formula can be inferred directly. One approach to estimate the volatility is to use market data. Assume that we have a quote for a call option denote by C^*. Then, we can apply the Black-Scholes formula and find a σ that leads to C^*. This estimate of volatility which is implied by the market price of an option is known as *implied volatility*. Formally, the goal is to find σ such that

$$S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = C^*.$$

Here on, consider C to be a function of σ where we assume that all other parameters are known.

(a) What value does $C(\sigma)$ approach as $\sigma \to \infty$?

(b) Assume $S - Ke^{-r(T-t)} > 0$. What value does $C(\sigma)$ approach as $\sigma \to 0^+$?

(c) Assume $S - Ke^{-r(T-t)} < 0$. What value does $C(\sigma)$ approach as $\sigma \to 0^+$?

(d) Assume $S - Ke^{-r(T-t)} = 0$. What value does $C(\sigma)$ approach as $\sigma \to 0^+$?

(e) Using (b), (c) and (d), evaluate $\lim_{\sigma \to 0^+} C(\sigma)$.

(f) The derivative of C with respect to σ is known as *vega* and is given by

$$C'(\sigma) = S\sqrt{T-t}N'(d_1),$$

where $N'(d_1)$ is the standard normal probability density function. Prove that

$$\max(S - Ke^{-r(T-t)}, 0) < C^* < S.$$

(g) **Extra credit**: Prove that $C'(\sigma)$ has a unique maximum value at σ^* where

$$\sigma^* = \sqrt{2\left|\frac{\ln(\frac{S}{K}) + r(T-t)}{T-t}\right|}.$$

Hint: You can use the fact that $C''(\sigma) = \frac{d_1 d_2}{\sigma} C'(\sigma)$.

(h) Argue that estimating the implied volatility can be formulated as a root finding problem. What methods can be employed?

(i) **Extra credit**: Prove that using the initial guess $\sigma^0 = \sigma^*$, Newton's method converges quadratically to the implied volatility.

(j) We now compute the implied volatility for call options using data taken from [Higham, 2004]. The call options were traded on the London International Financial Futures and Options Exchange (LIFFE) and reported in the Financial Times on August 22 2001. The data is for the FTSE 100 index which is an average of 100 equity shares quoted on the London Stock Exchange. The expiry date for these options was December 2001. The initial asset price was 5420.3. Let the interest rate be $r = 0.05$ and the duration of the option be $4/12$. Using your proposed method in (h), compute implied volatility for the following eight different exercise prices and option prices.

Exercise price	Option price
5125	475
5225	405
5325	340
5425	280.5
5525	226
5625	179.5
5725	139
5825	105

(k) Briefly comment on your result in (j).

Solution:

For details of implementation, refer to main code `impied_volatility.m` and associated scripts in the `project1_soln` folder.

(a) First, note that d_1 and d_2 can equivalently be written as follows

$$d_1 = \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}$$
$$d_2 = \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}.$$

We now consider $\lim_{\sigma\to\infty} d_1$ and $\lim_{\sigma\to\infty} d_2$.

$$\lim_{\sigma\to\infty} d_1 = \lim_{\sigma\to\infty} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to\infty} \frac{1}{2}\sigma\sqrt{T-t} = \infty$$
$$\lim_{\sigma\to\infty} d_2 = \lim_{\sigma\to\infty} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to\infty} -\frac{1}{2}\sigma\sqrt{T-t} = -\infty.$$

It follows that $\lim_{\sigma\to\infty} \Phi(d_1) = 1$ and $\lim_{\sigma\to\infty} \Phi(d_2) = 0$. Therefore, $C(\sigma)$ approaches S as $\sigma \to \infty$.

(b) $S - Ke^{-r(T-t)} > 0$ implies that $\frac{S}{K} > e^{-r(T-t)}$. Taking logarithm on both sides, we obtain $\ln(\frac{S}{K}) + r(T-t) > 0$. We now consider $\lim_{\sigma\to 0^+} d_1$ and $\lim_{\sigma\to 0^+} d_2$.

$$\lim_{\sigma\to 0^+} d_1 = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} = \infty$$
$$\lim_{\sigma\to 0^+} d_2 = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} = \infty,$$

where we have used the fact that the expression in red is positive. It follows that $\lim_{\sigma\to 0^+} \Phi(d_1) = 1$ and $\lim_{\sigma\to 0^+} \Phi(d_2) = 1$. Therefore, $C(\sigma)$ approaches $S - Ke^{-r(T-t)}$ as $\sigma \to 0^+$.

(c) $S - Ke^{-r(T-t)} < 0$ implies that $\frac{S}{K} < e^{-r(T-t)}$. Taking logarithm on both sides, we obtain $\ln(\frac{S}{K}) + r(T-t) < 0$. We now consider $\lim_{\sigma\to 0^+} d_1$ and $\lim_{\sigma\to 0^+} d_2$.

$$\lim_{\sigma\to 0^+} d_1 = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} = -\infty$$
$$\lim_{\sigma\to 0^+} d_2 = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} = -\infty,$$

where we have used the fact that the expression in red is negative. It follows that $\lim_{\sigma\to 0^+} \Phi(d_1) = 0$ and $\lim_{\sigma\to 0^+} \Phi(d_2) = 0$. Therefore, $C(\sigma)$ approaches 0 as $\sigma \to 0^+$.

(d) $S - Ke^{-r(T-t)} = 0$ implies that $\frac{S}{K} = e^{-r(T-t)}$. Taking logarithm on both sides, we obtain $\ln(\frac{S}{K}) + r(T-t) = 0$. We now consider $\lim_{\sigma\to 0^+} d_1$ and $\lim_{\sigma\to 0^+} d_2$.

$$\lim_{\sigma\to 0^+} d_1 = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} = 0$$
$$\lim_{\sigma\to 0^+} d_2 = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} = \lim_{\sigma\to 0^+} \frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} = 0,$$

where we have used the fact that the expression in red is zero. It follows that $\lim_{\sigma\to 0^+}\Phi(d_1)=\frac{1}{2}$ and $\lim_{\sigma\to 0^+}\Phi(d_2)=\frac{1}{2}$. Therefore, $C(\sigma)$ approaches $\frac{1}{2}(S-Ke^{-r(T-t)})$ as $\sigma\to 0^+$. By assumption, $S-Ke^{-r(T-t)}=0$. Therefore, $C(\sigma)$ approaches 0 as $\sigma\to 0^+$.

(e) We can summarize the results in (b), (c) and (d) as follows

$$\lim_{\sigma\to 0^+} C(\sigma)=\max(S-Ke^{-r(T-t)},0)$$

(f) Note that $N'(d_1)>0$. Therefore, $C(\sigma)$ is monotonically increasing on the interval $[0,\infty)$. In parts (a)-(d), we have established the limits as $\sigma\to\infty$ and $\sigma\to 0$. Therefore, using the monotonic property of $C(\sigma)$ and the fact that S and $T-t$ are both positive quantities, we conclude that

$$\max(S-Ke^{-r(T-t)},0)<C^*<S.$$

(g) Since $c'(\sigma)>0$ on $[0,\infty)$ and S and $T-t$ are both positive quantities , $c''(\sigma)=0$ if $d_1d_2=0$. We consider the product d_1d_2 as follows.

$$\begin{aligned}
d_1d_2 &= \left(\frac{\ln(\frac{S}{K})+r(T-t)}{\sigma\sqrt{T-t}}+\frac{1}{2}\sigma\sqrt{T-t}\right)\cdot\left(\frac{\ln(\frac{S}{K})+r(T-t)}{\sigma\sqrt{T-t}}-\frac{1}{2}\sigma\sqrt{T-t}\right)\\
&= \left(\frac{\ln(\frac{S}{K})+r(T-t)}{\sigma\sqrt{T-t}}\right)^2-\frac{1}{4}\sigma^2(T-t)\\
&= \frac{\left(\ln(\frac{S}{K})+r(T-t)\right)^2}{\sigma^2(T-t)}-\frac{1}{4}\sigma^2(T-t).
\end{aligned}$$

Therefore, $d_1d_2=0$ implies

$$\left(\ln\left(\frac{S}{K}\right)+r(T-t)\right)^2=\frac{1}{4}\sigma^4(T-t)^2\longrightarrow\sigma^2=2\left|\frac{\ln\left(\frac{S}{K}\right)+r(T-t)}{T-t}\right|.$$

Therefore, σ^* is given by

$$\sigma^*=\sqrt{\left|\frac{\ln\left(\frac{S}{K}\right)+r(T-t)}{T-t}\right|}.$$

This concludes uniqueness of σ^*. It remains to show that σ^* is a critical point at which $c'(\sigma)$ achieves its maximum value.

Showing that σ^* is a global maxima. We use second derivative test and consider $c'''(\sigma)$. We note that

$$c'''(\sigma)=\frac{d_1d_2}{\sigma}c''(\sigma)+\left(\frac{d_1d_2}{\sigma}\right)'c'(\sigma).$$

To show that σ^* is a maxima, we need to certify that $c'''(\sigma^*)<0$. Since σ^* is a maxima of $c'(\sigma)$, $c''(\sigma^)=0$. Therefore, in the above equation, the expression in red vanishes at σ^*. It suffices to check sign of the second term, $\left(\frac{d_1d_2}{\sigma}\right)'c'(\sigma)$.

$$\begin{aligned}
\left(\frac{d_1d_2}{\sigma}\right)'c'(\sigma) &= \frac{1}{\sigma}(d_1d_2)'c'(\sigma)-\frac{1}{\sigma^2}(d_1d_2)c'(\sigma)\\
&= \frac{1}{\sigma}(d_1d_2)'c'(\sigma)-\frac{1}{\sigma}\left(\frac{1}{\sigma}d_1d_2c'(\sigma)\right)\\
&= \frac{1}{\sigma}(d_1d_2)'c'(\sigma)-\frac{1}{\sigma}c''(\sigma)
\end{aligned}$$

We note that the second term in the last equation vanishes at σ^*. With that, it suffices to consider the sign of the first term $\frac{1}{\sigma}(d_1 d_2)' c'(\sigma)$.

$$\frac{1}{\sigma}(d_1 d_2)' c'(\sigma) = \frac{1}{\sigma}[d_1' d_2 + d_2' d_1] c'(\sigma)$$

To simplify the above equation further, we consider d_1'.

$$\begin{aligned}
d_1' &= \left(\frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \right)' \\
&= \left(-\frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma^2\sqrt{T-t}} + \frac{1}{2}\sqrt{T-t} \right) \\
&= -\frac{1}{\sigma}\left(\frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma^2\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} \right) \\
&= -\frac{1}{\sigma} d_2
\end{aligned}$$

It then follows that

$$\begin{aligned}
\frac{1}{\sigma}(d_1 d_2)' c'(\sigma) &= \frac{1}{\sigma}[d_1' d_2 + d_2' d_1] c'(\sigma) \\
&= \frac{1}{\sigma}[-\frac{1}{\sigma} d_2^2 + (d_1' - \sqrt{T-t}) d_1] c'(\sigma) \\
&= -\frac{1}{\sigma^2} d_2^2 c'(\sigma) + \frac{1}{\sigma}\left(-\frac{1}{\sigma} d_1 d_2 c'(\sigma) \right) - \frac{1}{\sigma}\sqrt{T-t} d_1 c'(\sigma) \\
&= -\frac{1}{\sigma^2} d_2^2 c'(\sigma) + \frac{1}{\sigma} c''(\sigma) - \frac{1}{\sigma}\sqrt{T-t} d_1 c'(\sigma) \\
&= -\frac{1}{\sigma^2} d_2^2 c'(\sigma) + \frac{1}{\sigma} c''(\sigma) - \frac{1}{\sigma^2}\left(\ln\left(\frac{S}{K}\right) + r(T-t) \right) c'(\sigma) - \frac{1}{2\sigma}(T-t) c'(\sigma)
\end{aligned}$$

In the last equation, the last term is negative and the second term vanishes when evaluated at σ^*. It then remains to check if the combination of the first term and third term is negative at σ^*.

$$\begin{aligned}
& -\frac{1}{\sigma^2} d_2^2 c'(\sigma) - \frac{1}{\sigma^2}\left(\ln\left(\frac{S}{K}\right) + r(T-t) \right) c'(\sigma) \\
&= -\frac{1}{\sigma^2} c'(\sigma)\left[d_2^2 + \left(\ln\left(\frac{S}{K}\right) + r(T-t) \right) \right] \\
&= -\frac{1}{\sigma^2} c'(\sigma)\left[\frac{\left(\ln\left(\frac{S}{K}\right) + r(T-t)\right)^2}{\sigma^2(T-t)} + \frac{1}{4}\sigma^2(T-t) - \ln\left(\frac{S}{K}\right) + r(T-t) + \left(\ln\left(\frac{S}{K}\right) + r(T-t) \right) \right] \\
&= -\frac{1}{\sigma^2} c'(\sigma)\left[\frac{\left(\ln\left(\frac{S}{K}\right) + r(T-t)\right)^2}{\sigma^2(T-t)} + \frac{1}{4}\sigma^2(T-t) \right]
\end{aligned}$$

We note that the last term is negative for all values of σ and hence at σ^*. Therefore, σ^* is a maxima as desired.

(h) We can write $C(\sigma) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = C^*$ as

$$C(\sigma) - C^* = 0 \to S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) - C^* = 0,$$

which is a root finding problem. We could employ any of the root finding methods such as bisection, Newton or secant method.

(i) Let $f(\sigma) = C(\sigma) - C^*$. From Newton's method, $\sigma^{(1)}$ is given by

$$\sigma^{(1)} = \sigma^{(0)} - \frac{f(\sigma^{(0)})}{f'(\sigma^{(0)})}$$

Let σ_T be the true implied volatility that we want to estimate. Then, $f(\sigma_T)$. We now consider the error made in the first step.

$$\begin{aligned}\sigma^{(1)} - \sigma_T &= \sigma^{(0)} - \sigma_T - \frac{f(\sigma^{(1)}) - \sigma_T}{f'(\sigma^{(0)})} \\ &= \sigma^{(0)} - \sigma_T - \frac{(\sigma^{(0)} - \sigma_T)f'(\eta_0)}{f'(\sigma^{(0)})},\end{aligned}$$

where we have used the mean value theorem to obtain the last equation with $\eta_0 \in (\sigma^{(0)}, \sigma_T)$. Let $e_0 = \sigma^{(0)} - \sigma_T$ and $e_1 = \sigma^{(1)} - \sigma_T$ denote respectively the error made initially and at the first iteration. From the above equation, we have

$$\frac{e_1}{e_0} = \frac{\sigma^{(1)} - \sigma_T}{\sigma^{(0)} - \sigma_T} = 1 - \frac{f'(\eta_0)}{f'(\sigma^{(0)})}$$

First, using vega, we know that $f'(\sigma)$ is non-negative. Second, using the result in (g), $f'(\sigma)$ attains the maximum value at $\sigma^* = \sigma^{(0}$. Therefore, we conclude that $0 < \frac{e_1}{e_0} < 1$. We now consider two cases.

$\sigma^{(0)} < \sigma_T$: From the above error analysis, $\sigma_0 < \sigma_1 < \sigma_T$. We note that at $\sigma^{(0)} = \sigma^*$, $C'(\sigma)$ attains its maximum. Therefore, for $\sigma^0 < \sigma < \sigma_T$, $C'(\sigma)$ is decreasing. Using this knowledge, let's consider the second iteration of Newton's method.

$$\sigma^{(2)} = \sigma^{(1)} - \frac{f(\sigma^{(1)})}{f'(\sigma^{(1)})}$$

As before, invoking the mean value theorem, we can write relate the error e_2 to e_1.

$$\frac{e_2}{e_1} = \frac{\sigma^{(2)} - \sigma_T}{\sigma^{(1)} - \sigma_T} = 1 - \frac{f'(\eta_1)}{f'(\sigma^{(1)})},$$

where $\eta_1 \in (\sigma^{(1)}, \sigma_T)$. Using the fact that $C'(\sigma)$ is decreasing on $\sigma^0 < \sigma < \sigma_T$, we have

$$0 < \frac{e_2}{e_1} < 1$$

Arguing similarly, we conclude that $0 < \frac{e_{k+1}}{e_k} < 1$. Since the error is monotonically decreasing and bounded below, the iterates converge to σ_T. Using the same proof used in class for the convergence of Newton's method, it follows that Newton's method converges quadratically to the implied volatility. For the second case, $\sigma^{(0)} > \sigma_T$, a very similar argument as above can be made.

(j) Using Newton's method, we compute the implied volatility. For details of implementation, refer to the codes in `project1_soln` folder on Canvas. We summarize the main parameters

$$\begin{aligned}S &= 5420.3 \\ t &= 0 \\ T &= \frac{1}{3} \\ r &= 0.05\end{aligned}$$

Exercise price	Option price	Implied volatility
5125	475	0.19804118
5225	405	0.19598711
5325	340	0.19334690
5425	280.5	0.19028906
5525	226	0.18619330
5625	179.5	0.18329490
5725	139	0.17993714
5825	105	0.17660689

The table shows numerically estimated implied volatility for different combination of exercise price and quoted option price.

(k) If Black-Scholes was a perfect model, then all the implied volatility would be the same for all exercise prices. Black-Scholes makes different assumptions in deriving the model and hence the slight variation in volatility. In addition, the implied volatility is higher for options that start in-the-money than for options starting out-of-the-money.

References

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