c. For the case $A\psi = x\psi$ and $B\psi = (\hbar/i)\partial\psi/\partial x$ (position and momentum), show that

$$\Delta^2(A,\psi)\Delta^2(B,\psi) \geq 4\hbar^2$$

(for $\|\psi\| = 1$). This is called the ***Heisenberg uncertainty principle.***

Theorem Proofs for Chapter 10

10.1.2 Theorem *The space $\mathcal{V}$ of continuous functions $f : [a,b] \to \mathbb{C}$ forms an inner product space if we define*

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,dx.$$

Proof The properties of the inner product follow from these computations:

i.

$$\begin{aligned}\langle af+bg,h\rangle &= \int_a^b [af(x)+bg(x)]\overline{h(x)}\,dx\\ &= a\int_a^b f(x)\overline{h(x)}\,dx + b\int_a^b g(x)\overline{h(x)}\,dx\\ &= a\langle f,g\rangle + b\langle f,g\rangle.\end{aligned}$$

ii.

$$\begin{aligned}\langle f,g\rangle &= \int_a^b f(x)\overline{g(x)}\,dx = \int_a^b \overline{\overline{f(x)}g(x)}\,dx\\ &= \overline{\int_a^b \overline{f(x)}g(x)\,dx} = \overline{\langle g,f\rangle}.\end{aligned}$$

iii. Note from **ii** that $\langle f,f\rangle = \overline{\langle f,f\rangle}$, and so $\langle f,f\rangle$ is real; thus $\langle f,f\rangle \geq 0$ makes sense. Here

$$\langle f,f\rangle = \int_a^b f(x)\overline{f(x)}\,dx = \int_a^b |f(x)|^2\,dx \geq 0,$$

since $|f(x)|^2 \geq 0$.

Finally, suppose $\langle f,f\rangle = 0$. If h is continuous and $h \geq 0$, then

$$\int_a^b h(x)\,dx = 0 \quad \text{implies} \quad h = 0$$

(see §8.4), and so $\langle f,f\rangle = 0$ implies $\int_a^b |f|^2\,dx = 0$ and hence $f = 0$. ■

10.1.3 Cauchy-Schwarz Inequality *Let f,g belong to the inner product space $\mathcal{V}$; then*

$$|\langle f,g\rangle| \leq \|f\|\,\|g\|.$$

Also, $\mathcal{V}$ with norm $\|f\|$ satisfies the axioms of a normed space and, with $d(f,g) = \|f-g\|$, the axioms of a metric space.

Proof First, let us prove the inequality when $\|g\| = 1$:

$$\begin{aligned}0 \leq \|f-\langle f,g\rangle g\|^2 &= \langle f-\langle f,g\rangle g, f-\langle f,g\rangle g\rangle\\ &= \langle f,f\rangle - \langle f,g\rangle\overline{\langle f,g\rangle} - \langle f,g\rangle\overline{\langle f,g\rangle} + \langle f,g\rangle\overline{\langle f,g\rangle}\langle g,g\rangle\\ &= \langle f,f\rangle - \langle f,g\rangle\overline{\langle f,g\rangle} = \|f\|^2 - |\langle f,g\rangle|^2.\end{aligned}$$

Thus $|\langle f,g\rangle|^2 \leq \|f\|^2$.

For the general case $|\langle f,g\rangle| \leq \|f\|\,\|g\|$, we can suppose $g \neq 0$, so that $\|g\| \neq 0$. Let $h = g/\|g\|$, and so that $\|h\| = 1$. Then $|\langle f,h\rangle| \leq \|f\|$. But $|\langle f,h\rangle| = |\langle f,g\rangle|/\|g\| \leq \|f\|$, and so we obtain the result. ■

This method is similar to that used to prove the Cauchy-Schwarz inequality in real inner product spaces (Chapter 1), except that now a bit more care is needed to keep track of complex conjugates. The reader should derive the other properties, taking special care with the triangle inequality $\|f+g\| \leq \|f\|+\|g\|$.

10.1.6 Theorem *Let $\mathcal{V} = L^2$ be the space of functions $f : [a,b] \to \mathbb{C}$ such that $|f|^2$ is integrable (that is, $\int_a^b |f(x)|^2\,dx < \infty$). Then the space $\mathcal{V}$ is an inner product space with inner product*

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

and norm

$$\|f\| = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$$

Proof First, if $||f|| = 0$, we have $\int_a^b |f(x)|^2\,dx = 0$, and so by Theorem **8.3.4ii**, Chapter 8, f is zero except possibly on a set of measure zero. Since we are identifying functions that agree except on a set of measure zero, $f = 0$. Finally, $\langle f, g\rangle$ satisfies all the other rules of an inner product space as shown in Theorem **10.1.2**. We need only show that $\langle f, g\rangle$ is finite (that is, that fg is integrable).

If we were to work only with bounded functions, fg would be integrable and bounded, as are both f and g (see Chapter 8). Since we allow improper integrals, f and g need not be bounded. If we split f and g into real and imaginary parts, and into positive and negative parts, we are reduced to the case in which f and g are real and positive (the reader is asked to carry out the details as an exercise). For each $M \geq 1$, define $(fg)_M$ as in Chapter 8. We want to show that

$$\lim_{M\to\infty} \int_a^b (fg)_M < \infty.$$

By examining cases, we see that

$$0 \leq (fg)_M \leq f_{\sqrt{M}} g_{\sqrt{M}} + f^2_{\sqrt{M}} + g^2_{\sqrt{M}}$$

and so

$$\begin{aligned}\int_a^b (fg)_M &\leq \langle f_{\sqrt{M}}, g_{\sqrt{M}}\rangle + ||f_{\sqrt{M}}||^2 + ||g_{\sqrt{M}}||^2 \\ &\leq ||f_{\sqrt{M}}||^2 ||g_{\sqrt{M}}||^2 + ||f_{\sqrt{M}}||^2 + ||g_{\sqrt{M}}||^2,\end{aligned}$$

by the Schwarz inequality. But $||f_{\sqrt{M}}|| \leq ||f||$ and $||g_{\sqrt{M}}|| \leq ||g||$, and so

$$\int_a^b (fg)_M \leq ||f||\,||g|| + ||f||^2 + ||g||^2 < \infty.$$

Hence we obtain the result (the limit exists as the integral increases with M; we needed only to show that it was bounded above).

Finally, for sectionally continuous functions, observe that they form a vector space (Exercise 9 at the end of the chapter) and are bounded (Exercise 11). Hence both functions, f and $|f|^2$, are integrable, since the set of discontinuities is finite (see Chapter 8). ■

10.2.1 Theorem *In an inner product space $\mathcal{V}$, suppose $f = \sum_{k=0}^{\infty} c_k\varphi_k$ for an orthonormal family $\varphi_0, \varphi_1, \ldots$ in $\mathcal{V}$ (convergence in the mean) and $f \in \mathcal{V}$. Then $c_k = \langle f, \varphi_k\rangle = \overline{\langle \varphi_k, f\rangle}$.*

Proof Let $s_n = \sum_{k=0}^{n} c_k\varphi_k$, so that $||f - s_n|| \to 0$. Fix i and choose $n \geq i$. Form

$$\langle f - s_n, \varphi_i\rangle = \langle f, \varphi_i\rangle - \langle s_n, \varphi_i\rangle.$$

This expression approaches zero as $n \to \infty$, since $|\langle f - s_n, \varphi_i\rangle| \leq ||f - s_n||$. For $n \geq i$, we have

$$\langle s_n, \varphi_i\rangle = \sum_{k=0}^{n}\langle c_k\varphi_k, \varphi_i\rangle = \sum_{k=0}^{n} c_k\langle\varphi_k, \varphi_i\rangle = \sum_{k=0}^{n} c_k\delta_{ki} = c_i.$$

Thus, $\langle f, \varphi_i\rangle - c_i \to 0$ as $n \to \infty$. Since this expression is *independent* of n, we have $\langle f, \varphi_i\rangle = c_i$. ■

10.2.3 Bessel's Inequality *Let $\varphi_0, \varphi_1, \ldots$ be an orthonormal system in an inner product space $\mathcal{V}$. For each $f \in \mathcal{V}$, the real series $\sum_{i=0}^{\infty} |\langle f, \varphi_i\rangle|^2$ converges and we have the inequality*

$$\sum_{i=0}^{\infty} |\langle f, \varphi_i\rangle|^2 \leq ||f||^2.$$

Proof Let $s_n = \sum_{i=0}^{n}\langle f, \varphi_i\rangle\varphi_i$. We first show that $f - s_n$ and s_n are orthogonal. To see this, it is enough to show that $f - s_n$ and φ_i, $1 \leq i \leq n$, are orthogonal (why?). Indeed, $\langle f - s_n, \varphi_i\rangle = \langle f, \varphi_i\rangle - \langle s_n, \varphi_i\rangle$ and $\langle s_n, \varphi_i\rangle = \langle f, \varphi_i\rangle$, since

$$\langle s_n, \varphi_i\rangle = \sum_{j=0}^{n}\langle\langle f, \varphi_j\rangle\varphi_j, \varphi_i\rangle = \sum_{j=0}^{n}\langle f, \varphi_j\rangle\delta_{ij} = \langle f, \varphi_i\rangle$$

(this is the same computation as in the proof of Theorem **10.2.1**). If g and h are orthogonal, then $||g + h||^2 = ||g||^2 + ||h||^2$ (Pythagoras' relation, Exercise 1, **§10.1**), and so

$$||f||^2 = ||f - s_n + s_n||^2 = ||s_n||^2 + ||f - s_n||^2;$$

hence $||s_n|| \leq ||f||^2$. Now

$$||s_n||^2 = \left\|\sum_{i=0}^{n}\langle f, \varphi_i\rangle\varphi_i\right\|^2 = \sum_{i=0}^{n}|\langle f, \varphi_i\rangle|^2||\varphi_i||^2,$$

since the φ_i are orthogonal, and therefore

$$||s_n||^2 = \sum_{i=0}^{n}|\langle f, \varphi_i\rangle|^2 \leq ||f||^2.$$

Thus the series $\sum_{i=0}^{\infty}|\langle f, \varphi_i\rangle|^2$ has partial sum $||s_n||^2$, which is an increasing sequence, since the terms of the series are ≥ 0 and the series is bounded above by $||f||^2$. Hence the series converges, with sum $\leq ||f||^2$. ■

10.2.4 Parseval's Theorem *Let $\mathcal{V}$ be an inner product space and $\varphi_0, \varphi_1, \dots$ an orthonormal system. Then $\varphi_0, \varphi_1, \dots$ is complete iff for each $f \in \mathcal{V}$, we have*

$$||f||^2 = \sum_{n=0}^{\infty} |\langle f, \varphi_n \rangle|^2.$$

Proof Let $s_n = \sum_{i=0}^{n} \langle f, \varphi_i \rangle \varphi_i$. In the proof of Bessel's inequality, it was shown that

$$||f||^2 = ||f - s_n||^2 + ||s_n||^2.$$

If $\varphi_0, \varphi_1, \dots$ is complete, then $s_n \to f$, and so $||f - s_n||^2 \to 0$. Therefore, letting $n \to \infty$ in $||s_n||^2 = \sum_{i=0}^{n} |\langle f, \varphi_i \rangle|^2$ gives $||f||^2 = \sum_{i=0}^{\infty} |\langle f, \varphi_i \rangle|^2$.

Conversely, if this relation holds, then $||f||^2 - ||s_n||^2 \to 0$ as $n \to \infty$, and so $||f - s_n||^2 \to 0$, that is, $s_n \to f$, which means that

$$f = \sum_{i=0}^{\infty} \langle f, \varphi_i \rangle \varphi_i. \quad \blacksquare$$

10.2.5 Best Mean Approximation Theorem *Let $\mathcal{V}$ be an inner product space and $\varphi_0, \varphi_1, \dots, \varphi_n$ a set of orthonormal vectors in $\mathcal{V}$. Then for each set of numbers $t_0, t_1, \dots, t_n$,*

$$\left\| f - \sum_{k=0}^{n} t_k \varphi_k \right\| \geq \left\| f - \sum_{k=0}^{n} \langle f, \varphi_k \rangle \varphi_k \right\|.$$

Equality holds iff $t_k = \langle f, \varphi_k \rangle$.

Proof Let $c_k = \langle f, \varphi_k \rangle$, $s_n = \sum_{i=0}^{n} c_i \varphi_i$, and $h_n = \sum_{i=0}^{n} t_i \varphi_i$. It is required to show that

$$||f - s_n||^2 \leq ||f - h_n||^2$$

with equality iff $t_k = c_k$. To see this, we show that

$$||f - h_n||^2 = ||f||^2 - \sum_{k=0}^{n} |c_k|^2 + \sum_{k=0}^{n} |c_k - t_k|^2 = ||f - s_n||^2 + \sum_{k=0}^{n} |c_k - t_k|^2,$$

which suffices to prove the theorem. To prove this equality, note that

$$||f - h_n||^2 = \langle f - h_n, f - h_n \rangle = \langle f, f \rangle - \langle f, h_n \rangle - \langle h_n, f \rangle + \langle h_n, h_n \rangle.$$

First,

$$\langle h_n, h_n \rangle = \sum_{i,j} \langle t_i \varphi_i, t_j \varphi_j \rangle = \sum_{i,j} t_i \overline{t_j} \delta_{ij} = \sum_{i=0}^{n} |t_i|^2.$$

Second,

$$\langle f, h_n \rangle = \left\langle f, \sum_{k=0}^{n} t_k \varphi_k \right\rangle = \sum_{k=0}^{n} c_k \overline{t_k}.$$

Thus

$$||f - h_n||^2 = ||f^2|| - \sum_{k=0}^{n} c_k \overline{t_k} - \sum_{k=0}^{n} t_k \overline{c_k} + \sum_{k=0}^{n} |t_k|^2 = ||f||^2 - \sum_{k=0}^{n} |c_k|^2 + \sum_{k=0}^{n} |c_k - t_k|^2,$$

as required. $\blacksquare$

10.3.1 Mean Completeness Theorem *The exponential and trigonometric systems on $[0, 2\pi]$ (or $[-\pi, \pi]$) are complete in the space $\mathcal{V} = L^2$ of functions $f : [0, 2\pi] \to \mathbb{C}$ with $\int_0^{2\pi} |f(x)|^2 \, dx < \infty$ (the integral may be improper).*

Proof[4] By our remarks in the text and Exercise 1, **§10.3**, it suffices to consider the exponential case. Two necessary points are contained in the following lemmas.

Lemma 1 *(Stone-Weierstrass theorem in a special case) Let $f : [0, 2\pi] \to \mathbb{C}$ be continuous and let $f(0) = f(2\pi)$ (periodicity). Then for any $\varepsilon > 0$ there is an n and there are constants c_i, $i = -n, \dots, -1, 0, 1, \dots, n$, such that if we form the function*

$$p_n(x) = c_0 + c_1 e^{ix} + c_2 e^{2ix} + \dots + c_n e^{nix} + c_{-1} e^{-ix} + c_{-2} e^{-2ix} + \dots + c_{-n} e^{-nix},$$

then

$$|f(x) - p_n(x)| < \varepsilon$$

for all $x \in [0, 2\pi]$.

[4] A proof due to Luxemburg and not relying on the Stone-Weierstrass theorem is outlined in Exercise 75 at chapter's end. Another proof, due to Lebesgue, is given in Exercise 76. Both proofs, however, rely on the converse of Example **10.2.7** (see Exercise 14 at the end of the chapter), which uses completeness of L^2, that is, the Lebesgue integral.

The Stone-Weierstrass theorem was proved in Chapter 5. See also Exercise 44b, Chapter 5.

Lemma 2 *Let $f : [0, 2\pi] \to \mathbb{C}$ be square integrable, and $\varepsilon > 0$. Then there is a continuous function $g : [0, 2\pi] \to \mathbb{C}$ with $g(0) = g(2\pi)$ such that*

$$||f - g||^2 = \int_0^{2\pi} |f(x) - g(x)|^2 \, dx < \varepsilon.$$

Proof First suppose that $f \geq 0$ and is bounded by M. Given $\varepsilon > 0$, choose a partition P of $[0, 2\pi]$ such that, setting $h = f^2$,

$$\left| \int_0^{2\pi} h - \sum_{i=1}^{n} h(c_i)(x_{i+1} - x_i) \right| < \frac{\varepsilon}{2},$$

and make a similar estimate for f. We can, by drawing a graph composed of straight lines, construct a continuous g such that g is constant $= f(c_i)$ on $[y_i, z_i]$, where $[y_i, z_i] \subset [x_i, x_{i+1}]$, $|y_i - x_i| < \varepsilon/8M^2n$, and $|x_{i+1} - z_i| < \varepsilon/8M^2n$, and such that g is bounded by M. Then

$$\int_0^{2\pi} |f - g|^2 \, dx = \int_0^{2\pi} (f^2 + g^2 - 2fg) \, dx < \frac{\varepsilon}{2} + 4M^2 \cdot \frac{\varepsilon}{8M^2n} \cdot n = \varepsilon,$$

by adding and subtracting the approximations for $\int f^2 = \int h$ and $\int f$ and using the definition of g. The details are left to the reader.

The general case may be dealt with by writing f as $f = f^+ - f^-$ (see Chapter 8), so that we can assume $f \geq 0$. Form f_M as in Chapter 8 and choose M large enough that $\int |f - f_M|^2 < \varepsilon/4$, which is possible by a corollary of the monotone convergence theorem (Chapter 8). Thus, we can choose g such that $\int |g - f_M|^2 < \varepsilon/4$, and so $\int |f - g|^2 < \varepsilon$, since

$$||f - g|| \leq ||f - f_M|| + ||g - f_M|| \leq \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}}{2} = \sqrt{\varepsilon}. \quad \blacksquare$$

To prove the theorem from these lemmas requires two steps.

Step 1 *Proof of the theorem for f continuous and periodic.*
Let

$$s_n = \sum_{-n}^{n} \langle f, \varphi_k \rangle \varphi_k, \quad \text{where } \varphi_k(x) = \frac{e^{ixk}}{\sqrt{2\pi}}.$$

For $\varepsilon > 0$, we must show there is an N such that $n \geq N$ implies $||f - s_n|| < \varepsilon$. It suffices to produce a single n, because, by Theorem **10.2.5**, $||f - s_{n+k}|| \leq ||f - s_n||$. Using Lemma **1**, choose p_n so that $|f(x) - p_n(x)| < \varepsilon/\sqrt{2\pi}$, and form the corresponding s_n. Now

$$||f - p_n||^2 = \int_0^{2\pi} |f(x) - p_n(x)|^2 \, dx \leq \int_0^{2\pi} \left(\frac{\varepsilon}{\sqrt{2\pi}} \right)^2 dx = \varepsilon^2.$$

Thus $||f - p_n|| < \varepsilon$. By Theorem **10.2.5**,

$$||f - s_n|| \leq ||f - p_n|| < \varepsilon,$$

since the Fourier series gives the best mean approximation to f. This proves Step **1**.

Step 2 *General case.*
In view of Lemma **2** and Step **1**, it suffices to prove the following. Here, $\mathcal{V}$ is the space of square integrable functions, but the lemma is stated in general terms.

Lemma 3 *Let $\mathcal{V}$ be an inner product space and let $\varphi_0, \varphi_1, \ldots$ be an orthonormal family. Suppose $f \in \mathcal{V}$ and $f_n \to f$. If*

$$f_n = \sum_{k=0}^{\infty} \langle f_n, \varphi_k \rangle \varphi_k$$

for each n, then

$$f = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k.$$

Proof Given $\varepsilon > 0$, choose N such that $k \geq N$ implies $||f_k - f|| = \varepsilon/3$. Choose M such that $n \geq M$ implies

$$\left\| \sum_{j=0}^{n} \langle f_N, \varphi_j \rangle \varphi_j - f_N \right\| < \frac{\varepsilon}{3}.$$

Using the triangle inequality,

$$\left\|\sum_{j=0}^{n}\langle f,\varphi_j\rangle\varphi_j - f\right\| \le \left\|\sum_{j=0}^{n}\langle f,\varphi_j\rangle\varphi_j - \sum_{j=0}^{n}\langle f_N,\varphi_j\rangle\varphi_j\right\| + \left\|\sum_{j=0}^{n}\langle f_N,\varphi_j\rangle\varphi_j - f_N\right\| + \|f_N - f\|.$$

By Bessel's inequality, the first term is $\le \|f - f_N\|$. Thus $n \ge M$ implies

$$\left\|\sum_{j=0}^{n}\langle f,\varphi_j\rangle\varphi_j - f\right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which proves our assertion. ■

10.3.2 Pointwise Convergence Theorem *Let $f : [0, 2\pi] \to \mathbb{R}$ (or $f : [-\pi,\pi] \to \mathbb{R}$) be sectionally continuous and have a jump discontinuity at x_0, and assume that $f'(x_0^+)$ and $f'(x_0^-)$ both exist. Then the Fourier series of f (in either exponential or trigonometric form) evaluated at x_0 converges to $[f(x_0^+) + f(x_0^-)]/2$. In particular, if f is differentiable at x_0, the Fourier series of f converges at x_0 to $f(x_0)$.*

Proof It is convenient to first prove the following special case:

Lemma 4 *Let $f : [-\pi,\pi] \to \mathbb{C}$ be square integrable and differentiable at x_0 (as usual, extend f so that it is periodic). Then the Fourier series of f at x_0 converges to $f(x_0)$.*

Proof (We follow a proof of P. Chernoff.) By translating and adding a constant, we can assume $x_0 = 0$ and $f(x_0) = 0$ (why?). Define a new function $g(x)$ by setting

$$g(x) = \frac{f(x)}{e^{ix} - 1} \text{ if } x \neq 0 \quad\text{and}\quad g(x) = \frac{f'(0)}{i} \text{ if } x = 0.$$

By the quotient rule of calculus, it follows that g is continuous at 0. Since $1/(e^{ix} - 1)$ is bounded in absolute value outside a neighborhood of 0, it follows that g is square integrable (why?).

Now $f(x) = (e^{ix} - 1)g(x)$. Let $c_n(f)$ be the nth Fourier coefficient of f and $c_n(g)$ that for g. By definition,

$$c_n(f) = c_{n-1}(g) - c_n(g).$$

Thus,

$$\sum_{n=-N}^{N} c_n(f) = c_{-N-1}(g) - c_N(g),$$

since we have a telescoping sum. Since $x_0 = 0$, $\sum_{-N}^{N} c_n(f)$ is the Nth partial sum at $x = 0$ of the Fourier series of f. But $c_N(g) \to 0$, by Bessel's inequality. Hence $\sum_{-N}^{N} c_n(f) \to 0 = f(x_0)$. ■

Actually, we do not need the fact that f is differentiable at x_0. If f "is Lipschitz" at x_0 (that is, if there is a constant M such that $|(f(x) - f(x_0))/(x - x_0)| \le M$ for $|x - x_0| < \delta$, $x \neq x_0$), we could obtain the same result by a similar proof (we only need g in the proof to be square integrable—or even just integrable). For example, if f is continuous and $f'(x_0^+)$ and $f'(x_0^-)$ exist, then this condition is satisfied (why?).

To prove the theorem from Lemma **4** and the preceding remarks, consider

$$h(x) = \begin{cases} f(x_0^-) & x < x_0 \\ f(x_0) & x = x_0 \\ f(x_0^+) & x > x_0. \end{cases}$$

Then h is a step function and we can compute its Fourier series explicitly (as in item 1, Table 10.5.4; see Example **10.5.3**). In particular, this series converges to $[f(x_0^-) + f(x_0^+)]/2$ at x_0. Now consider the function

$$k(x) = f(x) - h(x).$$

Then $k(x_0) = 0 = k(x_0^+) = k(x_0^-)$ and $k'(x_0^+)$, $k'(x_0^-)$ exist. Hence, by Lemma **4**, the Fourier series of k converges to 0 at x_0. Therefore, the Fourier series for f converges to $[f(x_0^+) + f(x_0^-)]/2$ at x_0. This proves the assertion. ■

Now we turn to the longer, classical proof of Theorem **10.3.2.** Later, it will be convenient to have this longer proof at hand, despite the fact that it is more complex than the one just given. First, we explain the basic idea behind the proof. Let $s_n(x)$ be the nth partial sum of the trigonometric Fourier series. We will write

$$s_n(x) = \int_0^{2\pi} f(\xi) D_n(x - \xi)\, d\xi$$

for a function D_n specified later (Lemma 9); we say that s_n is the ***convolution*** of f and D_n. We will prove that D_n has unit area and "concentrates" around 0; that is, it behaves like a Dirac delta function. As $n \to \infty$, the convolution will pick off the value of f at x. See Figure 10.P-1. For this reason, D_n is also called an ***approximate identity.***

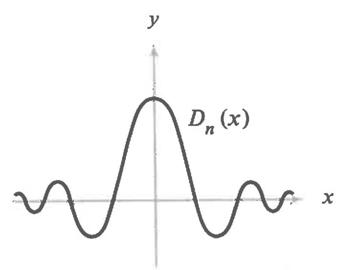

FIGURE 10.P-1 The graph of the approximate identity function $D_n(x)$

Before we can formalize these ideas, we need some preliminary results. The first lemma is a generalization of Example **10.2.9.**

Lemma 5 Riemann-Lebesgue Lemma *Suppose f is bounded and (Riemann) integrable on $[a,b]$. Then*

$$\lim_{\alpha\to\infty}\int_a^b f(x)\sin(\alpha x)\,dx = 0$$

(where the limit is taken through all real $\alpha > 0$).

Proof First, suppose f is a constant M. Then

$$\left|\int_a^b f(x)\sin\alpha x\,dx\right| = |M|\left|\int_a^b \sin\alpha x\,dx\right| = |M|\frac{|\cos(\alpha a)-\cos(\alpha b)|}{\alpha}$$
$$\le \left|\frac{2M}{\alpha}\right| \to 0 \text{ as } \alpha\to\infty.$$

Thus the result is true if f is a constant.

For the general case, given $\varepsilon > 0$ choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon/2$. Then

$$U(f,P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) \quad\text{and}\quad L(f,P) = \sum_{i=1}^n m_i(x_i - x_{i-1}),$$

where M_i is the maximum of f on $[x_{i-1}, x_i]$ and m_i is the minimum. Let m be the step function equal to m_i on $]x_{i-1}, x_i]$. Choose N so that

$$\int_a^b m(x)\sin\alpha x\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} m_i \sin\alpha x\,dx < \frac{\varepsilon}{2}$$

if $\alpha \ge N$, which is possible because m_i is constant and n is fixed and finite. By the triangle inequality, for $\alpha \ge N$,

$$\left|\int_a^b f(x)\sin\alpha x\,dx\right| \le \left|\int_a^b m(x)\sin\alpha x\,dx\right| + \left|\int_a^b [f(x)-m(x)]\sin\alpha x\,dx\right|$$
$$< \frac{\varepsilon}{2} + \int_a^b |M(x) - m(x)|\,dx,$$

where $M = M_i$ on $]x_{i-1}, x_i]$. (Here we have used $|\sin\alpha x| \le 1$.) But $M(x) - m(x) \ge 0$ and

$$\int_a^b [M(x) - m(x)]\,dx = U(f,P) - L(f,P) < \frac{\varepsilon}{2},$$

and so, for $\alpha > N$,

$$\left|\int_a^b f(x)\sin\alpha x\,dx\right| < \varepsilon. \quad \blacksquare$$

Lemma 6 *Suppose $g : [0,a] \to \mathbb{R}$ is sectionally continuous and $g'(0^+)$ exists. Then*

$$\lim_{k\to\infty}\int_0^a g(x)\frac{\sin kx dx}{x} = \frac{\pi}{2}g(0^+).$$

Proof Since

$$\int_0^a g(x)\frac{\sin kx dx}{x} = g(0^+)\int_0^a \frac{\sin kx dx}{x} + \int_0^a \frac{g(x)-g(0^+)}{x}\sin kx\,dx,$$

it suffices to show that

$$\int_0^a \frac{\sin kx dx}{x}\,dx \to \frac{\pi}{2} \quad \text{as } k\to\infty, \tag{1}$$

and

$$\int_0^a \frac{g(x)-g(0^+)}{x}\sin kx\,dx \to 0 \quad \text{as } k\to\infty. \tag{2}$$

To prove (1), note that

$$\int_0^a \frac{\sin kx}{x}\,dx = \int_0^{ka} \frac{\sin t}{t}\,dt,$$

which converges to $\pi/2$ as $k \to \infty$, since $\int_0^\infty [(\sin t)/t]\,dt = \pi/2$; see Example **8.5.6** and Exercise 29 at the end of this chapter.

To prove Equation (2), observe that $[g(x)-g(0^+)]/x$ is bounded and integrable (since, as $x \to 0$, this approaches a limit $g'(0^+)$). Therefore

$$\int_0^a \frac{g(x)-g(0^+)}{x} \sin kx\,dx \to 0$$

as $k \to \infty$, by Lemma **5**. ■

Lemma **5** is needed for α real and an arbitrary interval $[a,b]$. Note that this case does not follow from Example **10.2.9**.

Lemma 7 *Let g be sectionally continuous on $]a,b[$ and have a jump discontinuity at x_0. Suppose $g'(x_0^+)$ and $g'(x_0^-)$ exist. Then*

$$\lim_{k\to\infty} \int_a^b g(x)\frac{\sin k(x-x_0)}{x-x_0}\,dx = \frac{\pi[g(x_0^+)+g(x_0^-)]}{2}.$$

Proof Write the integral as a sum,

$$\int_a^b = \int_a^{x_0} + \int_{x_0}^b,$$

and note that

$$\int_a^{x_0} g(x)\frac{\sin[k(x-x_0)]}{x-x_0}\,dx = \int_0^{x_0-a} g(x_0-t)\frac{\sin kt}{t}\,dt$$

and

$$\int_{x_0}^b g(x)\frac{\sin[k(x-x_0)]}{x-x_0}\,dx = \int_0^{b-x_0} g(x_0-t)\frac{\sin kt}{t}\,dt.$$

Now let $k \to \infty$ and employ Lemma **6**. We get $\pi g(x_0^-)/2$ and $\pi g(x_0^+)/2$ for the limit of these two integrals, respectively, as $k \to \infty$. ■

Lemma 8 *Let $f : [0, 2\pi] \to \mathbb{R}$. Then the nth partial sum of the Fourier series of f may be written as*

$$s_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_0^{2\pi} f(t)\cos k(t-x)\,dt.$$

Proof This is clear if we remember that

$$\cos[k(t-x)] = \cos kt\cos kx + \sin kt\sin kx. \quad \blacksquare$$

Lemma 9 *Let $s_n(x)$ be the nth partial sum of the Fourier series of f. Then*

$$s_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)D_n(t-x)\,dt \quad \text{where} \quad D_n(u) = \frac{\sin[(n+1/2)u]}{\sin(u/2)}.$$

Proof This follows from Lemma **8** and the identity

$$\sum_{k=-n}^{n} e^{iku} = 1 + 2\sum_{k=1}^{n}\cos ku = \frac{\sin[(n+1/2)u]}{\sin(u/2)}$$

(Exercise 5, §**10.2**). $\blacksquare$

We are now ready to prove the pointwise convergence theorem. We must show that

$$s_n(x_0) \to \frac{f(x_0^+) + f(x_0^-)}{2}$$

as $n \to \infty$. We shall assume that $0 < x_0 < 2\pi$. The reader is asked to consider the cases $x_0 = 0$ and $x_0 = 2\pi$ separately. By Lemma **9**,

$$s_n(x_0) = \frac{1}{\pi}\int_0^{2\pi} g(t)\frac{\sin[(n+1/2)(t-x_0)]}{t-x_0}\,dt,$$

where

$$g(x) = f(t)\frac{(t-x_0)/2}{\sin[(t-x_0)/2]}, \quad x_0 \neq t.$$

By Lemma **7** (which is applicable by Exercise 41 at the end of this chapter), we have

$$s_n(x_0) \to \frac{g(x_0^+) + g(x_0^-)}{2}.$$

Now it is a simple matter to see that

$$g(x_0^+) = f(x_0^+) \quad \text{and} \quad g(x_0^-) = f(x_0^-),$$

and so the theorem is obtained. ■

10.4.2 Fejér's Theorem *Assume that f is piecewise continuous on $[0, 2\pi]$ and that $f(x_0^+)$ and $f(x_0^-)$ exist. Then the Fourier series of f converges $(C, 1)$ at x_0 to $[f(x_0^+) + f(x_0^-)]/2$. If f is continuous and $f(0) = f(2\pi)$, then the Fourier series converges $(C, 1)$ uniformly to f.*

Proof For notational reasons it is slightly more convenient to use $[-\pi, \pi]$ instead of $[0, 2\pi]$ for this proof; this does not affect the conclusions. Consider $s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$, the nth partial sum of the Fourier series of f. To discuss $(C, 1)$ summability, we let

$$\sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(x).$$

Using Lemma **9**,

$$\sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_k(t)\, dt;$$

that is,

$$\sigma_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) F_n(t)\, dt,$$

where

$$F_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t)$$

is the ***Fejér kernel.*** We shall need the following lemmas.

Lemma 10 $F_n(t) = \dfrac{1}{n+1} \dfrac{\sin^2[(n+1)t/2]}{\sin^2[t/2]}.$

Proof By the formula for D_n (Lemma **9**), we have

$$\begin{aligned}(n+1)F_n(t) &= \sum_{k=0}^{n} \frac{\sin[(k+1/2)t]}{\sin(t/2)} = \frac{1}{\sin(t/2)}\operatorname{Im}\left\{\sum_{k=0}^{n} e^{i(k+1/2)t}\right\} \\ &= \frac{1}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}\frac{e^{i(n+1)t}-1}{e^{it}-1}\right\} \\ &= \frac{1}{\sin(t/2)}\operatorname{Im}\left\{\frac{e^{i(n+1)t}-1}{e^{it/2}-e^{-it/2}}\right\} \\ &= \frac{1-\cos[(n+1)t]}{2\sin^2(t/2)} = \frac{\sin^2[1/2(n+1)t]}{\sin^2(t/2)}. \quad \blacksquare\end{aligned}$$

Lemma 11 *The Fejér kernel has the following properties:*

i. $F_n(t)$ *is* 2π*-periodic.*

ii. $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(t)\,dt = 1.$

iii. $F_n(t) \geq 0.$

iv. *For each fixed* $\delta > 0$, $\lim_{n\to\infty} \int_{\delta\leq|t|\leq\pi} F_n(t)\,dt = 0.$

Proof **i** and **ii** follow from the definition of F_n; **iii** follows from Lemma **10**. **iv**: For $\delta \leq |t| \leq \pi$ we have $1/(\sin^2 t/2) \leq 1/(\sin^2 \delta/2)$. Hence

$$0 \leq F_n(t) \leq \frac{1}{n+1}\,\frac{1}{\sin^2(\delta/2)}, \qquad \delta \leq |t| \leq \pi.$$

Since this converges to 0 uniformly as $n\to\infty$, the integral $\int_{\delta\leq|t|\leq\pi} F_n(t)\,dt \to 0.$ $\blacksquare$

Let us now prove Fejér's theorem. Using the same technique that was used in the proof of the pointwise convergence theorem (see the arguments following Lemma **4**), it suffices to prove the last part of the theorem. Thus, assume that f is continuous. We have

$$\sigma_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)\,dt.$$

Hence, by **ii** of Lemma **11**,

$$f(x) - \sigma_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - f(x-t))F_n(t)\,dt.$$

Accordingly, by **iii** of Lemma **11** (positivity of F_n),

$$|f(x) - \sigma_n(x)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - f(x-t)| F_n(t)\, dt.$$

Given $\varepsilon > 0$, we can use uniform continuity of f to find $\delta > 0$ such that $|f(x) - f(y)| \le \varepsilon$ if $|x - y| \le \delta$. Then

$$\begin{aligned} |f(x) - \sigma_n(x)| \le \frac{1}{2\pi} &\int_{|t| \le \delta} |f(x) - f(x-t)| F_n(t)\, dt \\ &+ \frac{1}{2\pi} \int_{\delta \le |t| \le \pi} |f(x) - f(x-t)| F_n(t)\, dt. \end{aligned}$$

The first integral on the right-hand side is bounded above by

$$\frac{1}{2\pi} \int_{|t| \le \delta} \varepsilon F_n(t)\, dt \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \varepsilon F_n(t)\, dt = \varepsilon.$$

The second integral is bounded by

$$\frac{1}{2\pi} \int_{\delta \le |t| \le \pi} 2M F_n(t)\, dt = \frac{M}{\pi} \int_{\delta \le |t| \le \pi} F_n(t)\, dt,$$

where $M = \sup_t |f(t)|$. By property **iv** of Lemma **11,** we may choose N so that if $n \ge N$, this last integral is $\le \varepsilon$. Thus, if $n \ge N$, $|f(x) - \sigma_n(x)| \le \varepsilon + \varepsilon = 2\varepsilon$. ■

One can prove that the Cesaro sums converge to an integrable function except possibly on a set of measure zero (see Hewitt and Stromberg, *Real and Abstract Analysis,* Springer-Verlag, p. 294).

10.5.1 Integration Theorem *Suppose $\int_{-\pi}^{\pi} |f(x)|^2\, dx < \infty$ and f has Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Then, letting $g(x) = \int_{-\pi}^{x} f(y)\, dy$, we have

$$\begin{aligned} g(x) &= \frac{a_0(x+\pi)}{2} + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{x} \cos ny\, dy + b_n \int_{-\pi}^{x} \sin ny\, dy \right) \\ &= \frac{a_0(x+\pi)}{2} + \sum_{n=1}^{\infty} \left\{ \frac{a_n}{n} \sin nx + \frac{b_n}{n} ((-1)^n - \cos\ nx) \right\} \end{aligned}$$

and the convergence is uniform for $-\pi \le x \le \pi$.

Proof We prepare the following lemma.

Lemma 12 *Suppose* $f_n : [a,b] \to \mathbb{R}$ *are such that* $\int_a^b |f_n(x)|^2\,dx < \infty$ *and* $f_n \to f$ *in mean. Let*

$$g_n(x) = \int_a^x f_n(y)\,dy \quad \text{and} \quad g(x) = \int_a^x f(y)\,dy.$$

Then $g_n \to g$ *uniformly on* $[a,b]$.

Proof By the Cauchy-Schwarz inequality,

$$\begin{aligned} |g_n(x) - g(x)|^2 &\le \left(\int_a^x |f_n(y) - f(y)|\,dy\right)^2 \\ &\le \left(\int_a^x |f_n(y) - f(y)|^2\,dy\right)(x-a) \le \|f_n - f\|^2(b-a), \end{aligned}$$

from which the result is obvious. ■

For the theorem, let $s_n(x)$ be the nth partial sum of the Fourier series and take $f_n = s_n$ in the lemma. We know that $f_n \to f$ in mean (Theorem 10.3.1), and so $g_n \to g$ uniformly. Here g_n is the partial sum of the integrated Fourier series, and so we have the result. ■

10.5.2 Gibbs' Phenomenon *Consider*

$$f(x) = \begin{cases} a & -\pi \le x < 0 \\ b & 0 \le x \le \pi, \end{cases}$$

and suppose $a < b$. *Let* $s_n(x)$ *be the nth partial sum of the trigonometric Fourier series. Then the maximum of* s_n *occurs at* $\pi/2n$ *and the minimum at* $-\pi/2n$ *and*

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{2n}\right) = \left(\frac{b-a}{2}\right)\left(\frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt + 1\right) + b \approx (b-a)(0.089) + b.$$

Similarly,

$$\lim_{n\to\infty} s_n\left(-\frac{\pi}{2n}\right) = \left(\frac{b-a}{2}\right)\left(-\frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt + 1\right) + a \approx a - (b-a)(0.089),$$

and the difference of these limits is

$$\left(\frac{b-a}{2}\right)\left(-\frac{4}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt + 1\right) \approx (b-a)(1.179).$$

Proof Let us first prove this for the special case $a = -1$, $b = 1$. We have seen that the Fourier series of

$$g(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

is

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)x]}{2n-1}.$$

Let

$$s_n(x) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin[(2k-1)x]}{2k-1}.$$

By differentiating, we see that s_n has its maximum at $x_n = \pi/2n$ (some details here are left to the reader). The value at this maximum is

$$s_n\left(\frac{\pi}{2n}\right) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin[(2k-1)\pi/2n]}{2k-1} = \frac{2}{\pi}\sum_{k=1}^{n}\frac{\sin[(2k-1)\pi/2n]}{(2k-1)\pi/2n}\left(\frac{\pi}{n}\right).$$

This sum is a Riemann sum for the function $(\sin y)/y$ on $[0, \pi]$ with partition $\{0, \pi/n, 2\pi/n, \ldots, \pi\}$. Hence, if we choose n even and let $n \to \infty$, this converges (see Chapter 8) to

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin y}{y}\,dy.$$

The case of the minimum of f for $x < 0$ holds, since f and s_n are both odd.

The numerical value of the integral is approximately 1.179 and is computed by numerical methods, as readers can verify on their calculators or home computers.

The general case for f follows by observing that its Fourier series has nth partial sum $(1/2)(b-a)(s_n+1)+a$ (why?). ■

10.6.1 Uniform Convergence Theorem *Suppose f is continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and f' is sectionally continuous with jump discontinuities. Then the (trigonometric or exponential) Fourier series of f converges to f absolutely and uniformly. A similar statement holds for f on $[0, 2\pi]$.*

Proof We can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx),$$

by Theorem **10.3.2.** The Fourier coefficients of f' are

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx, \quad \beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx,$$

and we have

$$\alpha_n = \frac{n}{\pi}f(x)\cos nx\Big|_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = nb_n,$$

since we can integrate by parts and since $f(\pi) = f(-\pi)$. Similarly, $\beta_n = -na_n$.

Some care is needed to justify integration by parts, since f' exists only in sections. But if integration by parts is applied on each section using the fact that f' has jump discontinuities only, and noting the continuity of f, then we get the results we obtained here.

Lemma 13 *Under the conditions of Theorem* **10.6.1,** *we have*

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sum_{n=1}^{\infty} |b_n| < \infty,$$

and $na_n \to 0$, $nb_n \to 0$.

Proof We know that $\sum_{n=1}^{\infty} \beta_n^2$ converges (by Bessel's inequality for f'). Letting $s_n = \sum_{k=1}^{n} |a_k|$,

$$s_n = \sum_{k=1}^{n} \frac{|\beta_k|}{k} = \sum_{k=1}^{n} \sqrt{\frac{\beta_k^2}{k^2}} \leq \left\{\left(\sum_{k=1}^{\infty} \beta_k^2\right)\left(\sum_{k=1}^{\infty} 1/k^2\right)\right\}^{1/2},$$

by the Schwarz inequality. Since this is bounded, so is s_n, and therefore it converges (an increasing sequence converges iff it is bounded). Thus $\sum_{n=1}^{\infty} |a_n|$ converges. Since $\beta_n \to 0$, we also have $na_n \to 0$. The case of the b_n's is similar. ■

It suffices to show that $a_0/2+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)$ converges uniformly, since the limit must be $f(x)$. It is enough, in turn, to show that $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is uniformly convergent. However,

$$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| = M_n,$$

and so, by the lemma, $\sum_{n=1}^{\infty} M_n$ converges. Hence, by the Weierstrass M test, the series converges uniformly and absolutely. ■

10.6.2 Differentiation Theorem *Let f be continuous on $[-\pi, \pi]$, let $f(-\pi) = f(\pi)$, and let f' be sectionally continuous with jump discontinuities. Suppose f'' exists at $x \in [-\pi, \pi]$. Then the Fourier series for*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

may be differentiated term by term at x to give

$$f'(x) = \sum_{n=1}^{\infty}(-na_n \sin nx + nb_n \cos nx).$$

Furthermore, this is the Fourier series of f'.

Proof The proof of Theorem **10.6.1** showed that the Fourier coefficients of f' are given by

$$\alpha_n = nb_n, \quad \beta_n = -na_n.$$

This remark suffices to prove the theorem, since if f'' exists, $f'(x)$ will be the sum of Fourier series. ■

10.7.1 Theorem *In the initial-displacement problem, suppose that f is twice differentiable. Then the solution to the initial-displacement problem is*

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l},$$

where the b_n are the half-interval sine coefficients,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx,$$

and f is extended to be odd periodic. (Twice differentiable means we are assuming that the extended f is twice differentiable.) (See Figure 10.7-2.)

Proof The series for $y(x,t)$ converges because $\sum_{n=1}^{\infty} b_n \sin(n\pi x/l)$ converges uniformly and absolutely to f (Theorem **10.6.1**). Let us show that

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} = \frac{1}{2}[f(x-ct) + f(x+ct)].$$

For this, note that

$$2\ \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} = \sin\frac{n\pi(x-ct)}{l} + \sin\frac{n\pi(x+ct)}{l},$$

so that

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} = \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi(x-ct)}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi(x+ct)}{l}$$

$$= \frac{1}{2}[f(x-ct)+f(x+ct)].$$

Now we verify that

$$y(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)]$$

satisfies all the conditions. First,

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}c^2[f''(x-ct+f''(x+ct)] = c^2\frac{\partial^2 y}{\partial x^2}.$$

Second, at $t=0$, $y(x,0)=f(x)$ and

$$\frac{\partial y}{\partial t}(x,0) = \frac{1}{2}c[-f'(x)+f'(x)+f'(x)] = 0.$$

Third, $y(0,t) = \frac{1}{2}[f(-ct)+f(ct)] = 0$, because f is odd (when extended); and

$$y(l,t) = \frac{1}{2}[f(l-ct)+f(l+ct)] = 0,$$

because $f(l-ct) = -f(ct-l) = -f(ct+l)$, since $f(x) = f(x+2l)$ by periodicity. ■

10.7.2 Theorem *If f is square integrable, then, for each $t>0$,*

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/l^2}\cos\frac{n\pi x}{l}$$

converges uniformly, is differentiable, and satisfies the heat equation and boundary conditions. At $t=0$, it equals f in the sense of convergence in the mean, and pointwise if f is of class C^1. As usual,

$$a_n = \frac{2}{l}\int_0^1 f(x)\cos\frac{n\pi x}{l}\,dx.$$

Proof To show that $T(x,t)$ satisfies the heat equation, what we must do is justify term-by-term differentiation in both x and t. For this we use Theorem **5.4.3.** What we must show is that the series of derivatives

$$-\sum_{n=1}^{\infty} \frac{a_n \pi^2 n^2}{l^2} e^{-n^2\pi^2 t/l^2} \cos \frac{n\pi x}{l}$$

(which represents both $\partial T/\partial t$ and $\partial^2 T/\partial x^2$) converges uniformly in t and in x, which we do by the Weierstrass M test. Since $|a_n|$ is bounded ($a_n \to 0$, in fact), we can omit the terms $a_n\pi^2/l^2$. Now in x, let $M_n = n^2 e^{-n^2\pi^2 t/l^2}$. By the ratio test, $\sum M_n < \infty$, and so the series converges uniformly in x.

Uniformly in t means uniformly for all $t \geq \varepsilon$, where $\varepsilon > 0$ is arbitrary but fixed. In this case we let $M_n = n^2 e^{-n^2\pi^2 \varepsilon/l^2}$ and note that $\sum M_n$ converges. (We cannot allow $t = 0$.) The rest of the theorem is obvious. ■

10.7.3 Theorem *In Theorem* **10.7.2,**

$$\lim_{t\to 0, t>0} T(x,t) = f(x)$$

in the sense of convergence in mean, and, converges uniformly (and pointwise) if f is continuous, with f' sectionally continuous. More generally, for any f, if the Fourier series of f converges at x to $f(x)$, then $T(x,t) \to f(x)$ as $t \to 0$.

Proof For the first part, it will suffice to show the following.

Lemma 14 *For each $t > 0$, suppose $f_t \in \mathcal{V}$, an inner product space, and $\varphi_0, \varphi_1, \ldots$ is a complete orthonormal basis. Let*

$$f_t = \sum_{n=1}^{\infty} c_n(t)\varphi_n, \quad f = \sum_{n=1}^{\infty} c_n\varphi_n.$$

If

$$\lim_{t\to 0} \sum_{n=1}^{\infty} |c_n(t) - c_n|^2 = 0,$$

then $f_t \to f$ (in mean).

Proof The result follows from Parseval's relation $||f_t - f||^2 = \sum_{n=1}^{\infty} |c_n(t) - c_n|^2$. ■

In the case of Theorem **10.7.3,** we must show that

$$\lim_{t\to 0}\sum_{n=1}^{\infty}|a_n|^2(1-e^{-n^2\pi^2t/l^2})^2=0.$$

To do this, it is enough to show that the function $g(t)=\sum_{n=1}^{\infty}|a_n|^2(1-e^{-n^2\pi^2t/l^2})^2$ is continuous in t, since $g(0)=0$. To show that $g(t)$ is continuous, we shall show that the series converges uniformly in t. To do this, Abel's test will be used. The form we need is the following:

Lemma 15 *Let $\sum_{n=1}^{\infty}c_n$ be a convergent series and $\varphi_n(t)$ a uniformly bounded, decreasing (respectively, increasing) sequence defined for $t\geq 0$. Then $g(t)=\sum_{n=1}^{\infty}c_n\varphi_n(t)$ converges uniformly in t. In particular, g is continuous and $g(0)=\lim_{t\to 0}g(t)$.*

See Theorem **5.9.1** for the proof. One deduces the increasing case from the decreasing case by considering $-g(t)$, instead of $g(t)$. In our case $c_n=|a_n|^2$ and $\varphi_n(t)=(1-e^{-n^2\pi^2t/l^2})^2$. Now $\varphi_n\leq\varphi_m$ if $n\leq m$, and $|\varphi_n(t)|\leq 1$. Thus, from the lemma and the fact that $\sum c_n$ converges, we have our result.

Now suppose f' is sectionally continuous. From the proof of Theorem **10.6.1,** $\sum_{n=1}^{\infty}|a_n|<\infty$. Thus, for a given x,

$$|f(x)-T(x,t)|\leq\sum_{n=1}^{\infty}|a_n|(1-e^{-n^2\pi^2t/l^2}).$$

By an argument like the preceding, the series on the right converges uniformly, and so we can let $t\to 0$ in each term to conclude that $T(x,t)\to f(x)$ as $t\to 0$. Indeed, note that the convergence is uniform in x because we have the bound $\sum_{n=1}^{\infty}|a_n|(1-e^{-n^2\pi^2t/l^2})$, which approaches 0 as $t\to 0$ and is independent of x.

Finally, suppose $\sum_{n=1}^{\infty}a_n\cos(n\pi x/l)$ converges for some fixed x. Then we wish to show that (for this x fixed)

$$\lim_{t\to 0}g(t)=\lim_{t\to 0}\sum_{n=1}^{\infty}a_ne^{-n^2\pi^2t/l}\cos\frac{n\pi x}{l}=0.$$

Here we cannot make the same estimate, because the factor $\cos(n\pi x/l)$ is needed for $\sum a_n\cos(n\pi x/l)$ to converge. However, Lemma **15** can be applied with $c_n=a_n\cos(n\pi x/l)$ and $\varphi_n(t)=e^{-n^2\pi^2t/l^2}$ to yield the conclusion, since the φ_n are decreasing and are bounded by 1. ■

From this proof we also conclude that

$$\lim_{t\to t_0}T(x,t)=T(x,t_0);$$

that is, T is continuous in t, in each of the three cases of Theorem **10.7.3.** Indeed, we already know that for $t > 0$, $T(x, t)$ is differentiable and hence continuous. However, $T(x, t)$ may not be differentiable at $t = 0$, but the proof just given does show that we have continuity at $t = 0$.

These methods using Abel's and Dirichlet's tests are important for establishing convergence in other problems (such as Laplace's equation), as we shall see in the next proof.

10.7.4 Theorem

i. *Given g_1, let $\varphi(x, y)$ be defined by*

$$\varphi(x, y) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi(b - y)}{a} \frac{\sin(n\pi x/a)}{\sinh(n\pi b/a)}, \tag{12}$$

Suppose g_1 is of class C^2 and $g_1(0) = g_1(a) = 0$. Then φ converges uniformly, and is the solution to the Dirichlet problem with $f_1 = f_2 = g_2 = 0$, and is continuous on the whole square, and $\nabla^2\phi = 0$ on the interior.

ii. *If each of f_1, f_2, g_1, g_2 is of class C^2 and vanishes at the corners of the rectangle, then the solution $\varphi(x, y)$ is the sum of four series like Equation (12), $\nabla^2\varphi = 0$ on the interior, and ∇ is continuous on the whole rectangle and assumes the given boundary values. Furthermore, φ is C^∞ on the interior.*

iii. *If f_1, f_2, g_1, g_2 are only square integrable, then the series for φ converges on the interior, $\nabla^2\varphi = 0$, and φ is C^∞. Also, φ takes on the boundary values in the sense of convergence in mean. This means, for example, that $\lim_{y \to 0} \varphi(x, y) = \varphi(x, 0) = g_1(x)$ with convergence in mean.*

Proof For simplicity, let us take the case $a = b = \pi$, the general case being obtained by a change of coordinates. To prove parts **i** and **ii** of the theorem, we show that $\varphi(x, y)$ converges uniformly in x and y and that we can differentiate twice, term by term, on the interior. In view of the preceding remarks, this suffices to prove the theorem. Part **ii** is a consequence of **i** and linearity; the boundary values are assumed because g_1 is represented by its Fourier series.

By Theorems **10.6.1** and **10.6.2,**

$$g_1(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad g_1'(x) = \sum_{n=1}^{\infty} nb_n \cos nx,$$

and these series converge uniformly and absolutely. Here we use the fact that $g_1(0) = g_1(\pi) = 0$.

To show that φ converges uniformly we use Abel's test, as in the proof of Theorem **10.7.3,** on the square $[0, \pi] \times [0, \pi]$. Thus we must show, since the series for g_1 converges uniformly, that $\varphi_n = (\sin n(\pi - y))/\sin n\pi$ is decreasing with n and these functions are uniformly bounded. If we can show that they are decreasing, then uniform boundedness follows, because $0 \le \varphi_n(y) \le \varphi_1(y)$ and φ_1 is bounded, since it is continuous. In fact, $\varphi_1 \le 1$ in this case.

To show that $\varphi_{n+1} \le \varphi_n$, fix y and consider the function $\psi(t) = (\sinh t(\pi - y))/\sinh t\pi$, $t > 0$. It suffices to show that $\psi'(t) \le 0$, for then ψ decreases as t increases and, in particular, $\psi(n+1) \le \psi(n)$. This is a special case of the following lemma.

Lemma 16 *For constants α, β satisfying $\beta > 0$, $\beta \ge \alpha$, and letting $\psi(t) = \sinh(\alpha t)/\sinh(\beta t)$, $\psi'(t) \le 0$ for $t \ge 0$.*

Proof

$$\begin{aligned}\sinh^2(\beta t)\ \psi'(t) &= \alpha \sinh(\beta t)\cosh(\alpha t) - \beta \sinh(\alpha t)\cosh(\beta t)\\ &= -\frac{\beta^2 - \alpha^2}{2}\left[\frac{\sinh[(\alpha+\beta)t]}{\alpha+\beta} - \frac{\sinh[(\beta-\alpha)t]}{\beta-\alpha}\right],\end{aligned}$$

using the identity $\sinh(u+v) = \sinh u \cosh v + \sinh v \cosh u$. If the term in brackets is ≥ 0, we are finished, since $\beta^2 - \alpha^2 > 0$. This is in fact true. To see it, let

$$\rho(t) = \frac{\sinh[(\alpha+\beta)t]}{\alpha+\beta} - \frac{\sinh[(\beta-\alpha)t]}{\beta-\alpha}.$$

Now $\rho(0) = 0$ and $\rho'(t) = \sinh[(\alpha+\beta)t] - \sinh[(\beta-\alpha)t] \ge 0$, since sinh is increasing. Hence $\rho(t) \ge 0$ for all $t \ge 0$. ∎

This establishes the first part of the proof, which says that the series for $\varphi(x, y)$ converges uniformly. For the differentiability part, we must show that

$$\lambda(x, y) = \sum_{n=1}^{\infty} n^2 b_n \sinh n(\pi - y)\ \frac{\sin nx}{\sinh(n\pi)}$$

converges uniformly (this is the second formal y derivative; the second x derivative is its negative).

It is important to realize that we can get uniform convergence only if we stay away from the boundary; in fact, for any $\varepsilon > 0$ we shall establish uniform convergence on $0 < \varepsilon \le y \le \pi$ and x arbitrary. With this extra restriction, the delicacy of Abel's test is no longer needed; the Weierstrass M test will do. We have $|b_n| \le M$. Let

$$M_n = n^2 M \frac{\sinh[n(\pi - \varepsilon)]}{\sinh(\pi n)}.$$

Then M_n bounds the terms in λ. But $2\sinh[n(\pi-\varepsilon)] < e^{n(\pi-\varepsilon)}$ and $2\sinh(n\pi) \ge e^{n\pi}(1 - e^{n\pi})$, from the definition of sinh. Thus

$$M_n \le Mn^2 \frac{e^{-n\varepsilon}}{1 - e^{-2\pi}}.$$

Since $\varepsilon > 0$, $\sum M_n$ converges, and so we have uniform convergence.

Note that we could use n^k instead of n^2 for any k and still have convergence; in fact, we can differentiate any number of times; that is, φ is C^∞ (a little thought shows that φ is analytic—see Example **6.8.7**).

The proof of part **iii** is routine. To show that $\nabla^2\varphi = 0$ and φ is a C^∞ function on the interior, the proof is similar to the preceding (all that was used was that the b_n are bounded). For convergence in mean, we proceed as in the proof of Theorem **10.7.3,** using Lemma **15.**

Worked Examples for Chapter 10

Example 10.1 *Let $f : [0,\pi] \to \mathbb{C}$ be a continuous function. Prove that the following inequality holds:*

$$\left|\int_0^\pi f(x)\ \sin x\, dx\right|^2 + \cdots + \left|\int_0^\pi f(x)\ \sin nx\, dx\right|^2 \le \frac{\pi}{2}\int_0^\pi |f(x)|^2\, dx.$$

Solution This follows from Bessel's inequality applied to the following (incomplete) orthonormal system on $[0, \pi]$:

$$\sqrt{2/\pi}\ \sin x, \ldots, \sqrt{2/\pi}\ \sin nx.$$

If we used an infinite sum, we would have equality by Parseval's theorem (see Table 10.5-3). ♦

Example 10.2 *Show that if $f_n \to f$ (in mean) and $g_n \to g$ (in mean) in an inner product space, then*

$$\langle f_n, g_n\rangle \to \langle f, g\rangle.$$