Due date: 11:59 pm, Sunday, April 2, 2023 on Gradescope.

This is a double homework because of spring break, and you should be able to do problems 1- this week. You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

A proper subset of the problems will be selected for grading.

Reading assignment:

- For the week of March 13, please read sections 18.2 and 18.3.
- For the week of March 27, please continue reading section 18.3 and 19.1. We will also use the book Marsden-Hoffman, *Elementary Classical Analysis*, section 8.2 this week.

  You will find Marsden-Hoffman, section 8.2 in Canvas in *Files/Marsden-Hoffman/Ch 8: Integration and Lebesgue's Theorem.*

  Please read all of *1Marsden-Ch8LebesgueThm.pdf.* Please read pages 476-7 in the file *2Marsden-Ch8Proofs.pdf.* We will not prove Lebesgue's Theorem in class, but you are welcome to read the proof of that theorem (which starts on page 477). It is a cool generalization of the Jordan Integrability Theorem which we *will* discuss this week.

Here are some useful theorems. Theorems 1 and 2 are from Math 135 and Theorem 5 will be proven after break. Refer to them by number if you use them.

Theorem 1 *Let A and B be sets in $\mathbb{R}^n$. Then,* $\mathrm{bd}(A \cup B) \subset \mathrm{bd}(A) \cup \mathrm{bd}(B)$.

Theorem 2 *Let $A \subset \mathbb{R}^n$, then the closure of A,* $\mathrm{cl}(A)$, *is the smallest closed set containing A.*

Theorem 3 *If R is a generalized rectangle in $\mathbb{R}^n$, then R is a Jordan domain.*

Theorem 4 *The finite union of Jordan domains is a Jordan domain. That is, if $A_1, A_2, \ldots, A_m$ are Jordan domains then so is $\bigcup_{j=1}^m A_j$.*

Theorem 5 *Let A be a compact set (equiv. sequentially compact set) with measure zero. Then A has volume zero.*

Problems:

NOTE: you may not use the results in Marsden-Hoffman that aren't in Fitzpatrick to prove problems 1-6 (and you don't need to).

1. (10 points) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by $f(x,y) = xy^2$. Use the Archimedes-Riemann Theorem and the sequence of $\{\mathbb{P}_n\}$ of regular partitions $\mathbb{P}_n = (\{0, 1/n, 2/n, \ldots, 1\}, \{0, 1/n, 2/n, \ldots, 1\})$ of $[0,1] \times [0,1]$ to show f is integrable and find $\int_{\mathbb{I}} f$.

   HINT: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

2. (10 points) Let S_1 and S_2 be sets in $\mathbb{R}^n$ with Jordan content zero. Use the definition of Jordan content zero to prove that $S_1 \cup S_2$ has Jordan Content zero.

3. (10 points) Let A_1 and A_2 be Jordan domains. Prove that $A_1 \cup A_2$ also is also a Jordan domain.

   NOTE: you may not use Theorem 4 at the start of the test to prove this result. This is the first step in an induction proof of Theorem 4.

The problem set continues on the next page

4. (15 points) Let $f : [0,1]^2 \to \mathbb{R}$ be a bounded function that is continuous on the interior, $(0,1)^2$. Prove that f is integrable on $[0,1]^2$. The proof outlined below uses a couple of arguments similar to those in the proof of the Jordan Integrability Theorem.

   Assume $M > 0$ such that $-M \le f(\mathbf{x}) \le M$, $\forall \mathbf{x} \in [0,1]^2$. Recall the partition from group work 5: Let $\delta \in (0, 1/2)$, then $\mathbb{P}_\delta = (\{0, \delta, 1-\delta, 1\}, \{0, \delta, 1-\delta, 1\})$.

   (a) Why is there a partition $\mathbb{P}'$ of $[\delta, 1-\delta] \times [\delta, 1-\delta]$ such that $U(f, \mathbb{P}') - L(f, \mathbb{P}') < \epsilon/2$?

   (b) Let $\mathbb{P}^* = \mathbb{P}_\delta \overline{\cup} \mathbb{P}^*$. How can you choose δ such that $U(f, \mathbb{P}^*) - L(f, \mathbb{P}^*) < \epsilon$?
   HINT: divide the rectangles in $\mathbb{P}^*$ into those contained in $[\delta, 1-\delta]^2$ and those that meet this rectangle only on their boundary.

5. (15 points) Let A be a set of Jordan content zero in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded function. Prove that f is integrable and that $\int_A f = 0$. You may use the following steps or develop your own proof.

   Let $\mathbb{I}$ be a generalized rectangle such that $A \subset \mathbb{I}$, and let $\hat{f}$ be the zero extension of f to $\mathbb{I}$.

   Let $M > 0$ such that $-M \le \hat{f}(\mathbf{x}) \le M$ for all $\mathbf{x} \in \mathbb{I}$. Let $\epsilon > 0$.

   You may use any part of the problem, in a subsequent part, even if you are not sure how to prove it.

   (a) Explain why there is a finite set of generalized rectangles $\{R_1, R_2, \dots, R_N\}$ contained in $\mathbb{I}$ such that $A \subset \bigcup_{i=1}^N R_i$ and $\sum_{i=1}^N \mathrm{Vol}(R_i) < \frac{\epsilon}{2M}$.
   Now, let $F = \bigcup_{i=1}^N R_i$

   (b) Explain why the characteristic function $\mathbb{1}_F$ is integrable.

   (c) Show that $\int_{\mathbb{I}} M \mathbb{1}_F \le \frac{\epsilon}{2}$.

   (d) Explain why $-M\mathbb{1}_F(\mathbf{x}) \le \hat{f}(\mathbf{x}) \le M\mathbb{1}_F(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{I}$.

   (e) Show that $-\frac{\epsilon}{2} \le \underline{\int}_{\mathbb{I}} \hat{f} \le \overline{\int}_{\mathbb{I}} \hat{f} \le \frac{\epsilon}{2}$. HINT: you may assume that if g and h are bounded functions on $\mathbb{I}$ and $g \le h$ then $\underline{\int} g \le \underline{\int} h$ and $\overline{\int} g \le \overline{\int} h$.

   (f) Now explain why $\hat{f}$ is integrable on $\mathbb{I}$ and find $\int_{\mathbb{I}} \hat{f}$.

   (g) (1 point) Briefly explain why f is integrable on A and why $\int_A f = 0$.

6. (10 points) Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let $f : \mathbb{I} \to \mathbb{R}$ be integrable. Assume $\int f \ne 0$. Prove there is a nontrivial generalized rectangle $R \subset \mathbb{I}$ such that $f(\mathbf{x}) \ne 0$ for all $\mathbf{x} \in R$.
   SMALL HINT: WLOG, you may assume $\int f > 0$.

7. (10 points)

   (a) Let S be a bounded set with Jordan content zero. Prove that $\mathrm{cl}(S)$ has Jordan content zero.

   (b) If S has measure zero, does $\mathrm{cl}(S)$ have measure zero? Why or why not?

8. (10 points) Let A be a bounded subset of $\mathbb{R}^n$. Prove that A has volume if and only if $\mathrm{bd}(A)$ has Jordan content zero.

9. (10 points) Let $\mathbb{I}$ be a generalized rectangle and let f be an integrable function from $\mathbb{I}$ to $\mathbb{R}$. Recall that $D(f, \mathbb{I}) = \{\mathbf{x} \in \mathbb{I} \mid f \text{ is not continuous at } \mathbf{x}\}$ is the set of discontinuities of f on $\mathbb{I}$

   (a) Let $g(\mathbf{x})$ be the composite function $g(\mathbf{x}) = \sin(f(\mathbf{x}))$. Prove that $D(g, \mathbb{I}) \subset D(f, \mathbb{I})$. HINT: This is equivalent to showing that if f is continuous at $\mathbf{x}_0 \in \mathbb{I}$ then $g = \sin \circ f$ is continuous at $\mathbf{x}_0$.

   (b) Prove that g is integrable on $\mathbb{I}$.

Challenge problems are on the next page.

Here are optional challenge problems that will give you extra points if you successfully do them. Todd will grade them.

You may do up to three *of them for credit.*

1. (1 point) Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let A be a subset of $\mathbb{I}$ with Jordan content zero. Assume $f : \mathbb{I} \to \mathbb{R}$ is integrable and assume $g : \mathbb{I} \to \mathbb{R}$ is bounded and

$$g(\mathbf{x}) = f(\mathbf{x}), \ \forall \mathbf{x} \in \mathbb{I} \smallsetminus A.$$

Prove that g is integrable on $\mathbb{I}$ and that $\int_{\mathbb{I}} g = \int_{\mathbb{I}} f$. You may use any other homework problem on this assignment.

2. (1 points) Let $S_1, S_2, \ldots, S_m$ be a finite number of sets of Jordan content zero. Prove that $\bigcup_{j=1}^m S_j$ also has Jordan content zero.

3. (1 point) Let S be a subset of $\mathbb{R}^n$ with measure zero. Prove that $\operatorname{int}(S) = \varnothing$.

4. (1 points) Prove that if R is a generalized rectangle in $\mathbb{R}^n$, then R is a Jordan domain (that is, prove Theorem 3).