Homework 9

Early problem due on Gradescope at 11:59 pm on Tuesday, April 4th.
Due on Gradescope at 11:59 pm on Friday, April 7th.

(1) (Early problem) Check the details of Example 3 on page 138 of the textbook.

(2) (a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.
(b) If $A \subseteq X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

(3) Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points (x, y) for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map that is neither open nor closed.

(4) (a) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows

$$(x_0, y_0) \sim (x_1, y_1) \quad \iff \quad x_0 + y_0^2 = x_1 + y_1^2.$$

Let $X/\!\sim$ be the corresponding quotient space. It is homeomorphic to a familiar space; what is it? (Hint: set $g((x, y)) = x + y^2$.)
(b) Repeat the first part for the equivalence relation

$$(x_0, y_0) \sim (x_1, y_1) \quad \iff \quad x_0^2 + y_0^2 = x_1^2 + y_1^2$$

(5) Find an example of a pair of topological spaces X and Y with respective open covers $\{U_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ so that U_i is homeomorphic to V_i for each $i \in I$ but X is not homeomorphic to Y. (It is interesting to compare this with part i) of the sheaf property from last Friday.)