MATH235 HOMEWORK 6 SOLUTION

- 4.2.14. Let f be a continuous, nonnegative function on the interval $[a, b]$. Prove that the Riemann integral of f on $[a, b]$ coincides with its Lebesgue integral $\int_a^b f(x)dx$

Proof. f is Riemann integrable since it is a continuous function on a closed interval. In particular, it is measurable and bounded, then f is Lebeague integrable.

For each n, consider the partition

$$\mathcal{P}_n = \left\{ x_0 = a, x_1 = a + \frac{b-a}{n}, \cdots, x_k = a + \frac{k(b-a)}{n}, \cdots, x_n = b \right\},$$

and define

$$m_k = \min_{x \in [x_{k-1}, x_k]} f(x).$$

Then the Riemann integral, by definition, should be the lower Darboux-Riemann sum,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n} m_k.$$

Define for each $n \in \mathbb{N}$, the simple function

$$s_n = \sum_{k=1}^n m_k \chi_{[x_{k-1}, x_k)} + f(b)\chi_{\{b\}}.$$

Note that $s_n(x) \to f(x)$ for all $x \in [a, b]$, and $\{s_n\}$ is an increasing sequence of nonnegative simple functions. By MCT, we have

$$\int_{[a,b]} f dm = \int_{[a,b]} \lim s_n dm = \lim_{n\to\infty} \int_{[a,b]} s_n dm = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n} m_k.$$

Therefore, the Riemann integral and the Lebesgue integral of f have the same value.

□

- 4.2.17. Let $f : E \to [0, \infty]$ be a nonnegative, measurable function defined on a measurable set $E \subseteq \mathbb{R}^d$. This problem will quantify the idea that the integral of f equals "the area of the region under its graph."

(a) The graph of f is

$$\Gamma_f = \{(x, f(x)) : x \in E, f(x) < \infty\}.$$

Show that $|\Gamma_f| = 0$

(b) The region under the graph of f is the set R_f that consists of all points $(x, y) \in \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ such that $x \in E$ and y satisfies

$$\begin{cases} 0 \le y \le f(x), & \text{if } f(x) < \infty \\ 0 \le y < \infty, & \text{if } f(x) = \infty \end{cases}$$

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Show that R_f is a measurable subset of $\mathbb{R}^{d+1}$, and its Lebesgue measure is

$$|R_f| = \int_E f(x)dx$$

Proof. (a) Consider any $A \subseteq E$ and $f = \chi_A$, then we have

$$\Gamma_f = \{(x, f(x)), x \in A, f(x) < \infty\} = \{(x, 1), x \in A\} \cup \{(x, 0), x \notin A\} = A \times \{1\} \cup A^c \times \{0\}$$

Notice that in this case, $|\Gamma_f| = 0$. Now consider $f = \sum_{n=1}^{N} c_k \chi_{Ek}$, we have $\Gamma_f = \cup_{n=1}^{N} E_k \times \{c_k\}$ and $|\Gamma_f| = 0$. Then we consider a sequence of nonnegative monotone simple function $\{\phi_n\}$ that converges to f, then by monotone convergence theorem we know for any f defined a priori, $|\Gamma_f| = \lim_{n\to\infty} |\Gamma_{\phi_n}| = 0$.
(b) Consider an increasing sequence of nonnegative simple functions $\{\phi_n\}$ converges to f. Then we have $\cup_{n=1}^{\infty} R_{\phi_n} \cup \Gamma_f = R_f$. By subadditivity of Lebesgue measure, we have $| \cup R_{\phi_n}| \leq |R_f|$. On the other hand, $|R_f| \leq | \cup R_{\phi_n}| + |\Gamma_f| = | \cup R_{\phi_n}| + 0 = | \cup R_{\phi_n}|$. Therefore

$$|R_f| = | \cup R_{\phi_n}| = \lim_{n\to\infty} |R_{\phi_n}| = \int_E \lim_{n\to\infty} \phi_n = \int_E f$$

Where the second last equality is from monotone convergence theorem.
4.3.9.Assume that $f : \mathbb{R}^d \to \overline{\mathbb{F}}$ is measurable. Show that if $\int_{\mathbb{R}^d} f$ exists, then for each point $a \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} f(x-a)dx = \int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} f(a-x)dx$$

Proof. Given a measurable set $E \subseteq \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \chi_E(x-a)dx = \int_{\mathbb{R}^d} \chi_{E+a}(x)dx = |E+a| = |E| = \int_{\mathbb{R}^d} \chi_E(x)dx$$

Hence the integral of a characteristic function is invariant under translations. Taking linear combinations, this fact extends to simple functions. Given a nonnegative function $f : \mathbb{R}^d \to [0, \infty]$, there exist simple functions ϕ_n that increase pointwise to f. The functions $\phi_n(x-a)$ increase pointwise to $f(x-a)$, so by applying the Monotone Convergence Theorem we see that

$$\int_{\mathbb{R}^d} f(x-a)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi_n(x-a)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi_n(x)dx = \int_{\mathbb{R}^d} f(x)dx$$

Now suppose that $f : \mathbb{R}^d \to [-\infty, \infty]$ is an arbitrary extended real-valued function whose integral exists. Then the integrals of $f^+$and f^-both exist, with at most one of these being infinite. Applying the translation-invariance proved for nonnegative functions, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x-a)dx &= \int_{\mathbb{R}^d} f^+(x-a)dx - \int_{\mathbb{R}^d} f^-(x-a)dx \\
&= \int_{\mathbb{R}^d} f^+(x)dx - \int_{\mathbb{R}^d} f^-(x)dx \\
&= \int_{\mathbb{R}^d} f(x)dx.
\end{aligned}$$

Finally, if f is complex-valued then we write $f = f_r + if_i$ and use the fact that the integrals of f_r and f_i are invariant under translations.

The proof for invariance under reflection is similar, starting with the calculation
$$\int_{\mathbb{R}^d} \chi_E(-x)dx = \int_{\mathbb{R}^d} \chi_{-E}(x)dx = |-E| = |E| = \int_{\mathbb{R}^d} \chi_E(x)dx.$$
This equality then extends by cases to generic functions.

□

4.4.17 (a) Suppose that $f, g : E \to [-\infty, \infty]$ are measurable functions, where E is a measurable subset of $\mathbb{R}^d$. Prove that if f is integrable and $f \leq g$ a.e., then $g - f$ is measurable and $\int_E (g - f) = \int_E g - \int_E f$.
(b) Show that the Monotone Convergence Theorem and Fatou's Lemma remain valid if we replace the assumption $f_n \geq 0$ with $f_n \geq g$ a.e., where g is an integrable function on E. However, this can fail if g is not integrable.

Proof. (a). Since $f \leq g$ a.e., we have $f^+ - f^- \leq g^+ - g^-$. Notice that $f^+ \leq g^+$ and $g^- \leq f^-$, which implies $\int g^- \leq \int f^- < \infty$ and therefore $\int g$ exists and $g - f$ is measurable. Now consider if $\int g^+ < \infty$, then we have $\int g$ is finite and therefore $\int g - f = \int g - \int f$. Now suppose $\int g^+ = \infty$, then we have $\int g = \infty$. Suppose for the sake of contradiction we have $\int g - f < \infty$, then we have
$$\int g = \int g - f + \int f < \infty$$
Hence we have $\int g - f = \infty$ and the conclusion holds.
(b). Notice that $f_n - g \geq 0$ a.e. from definition. Apply monotone convergence theorem on $\{f_n - g\}$ we have
$$\lim_{n\to\infty} \int f_n - g = \int f - g$$
From part (a) we know
$$\lim_{n\to\infty} \int (f_n - g) = \lim_{n\to\infty} (\int f_n - \int g)$$
Notice that $\int g$ is not dependent on n, thus
$$\int f - g + \int g = \int f = \lim_{n\to\infty} \int f_n$$
where monotone convergence theorem remains true. Similar proof follows for Fatou's lemma. □

4.4.19. Prove that if $f \in L^1(\mathbb{R})$ is differentiable at $x = 0$ and $f(0) = 0$, then $\int_{-\infty}^{\infty} \frac{f(x)}{x} dx$ exists.

Proof. $\int_{-\infty}^{\infty} \frac{f(x)}{x} dx$ exists implies that $\int_{-\infty}^{\infty} \frac{f(x)}{x} dx < \infty$.
Write
$$\int_{-\infty}^{\infty} \frac{f(x)}{x} dx = \int_{-\infty}^{-b} \frac{f(x)}{x} dx + \int_{-b}^{0} \frac{f(x)}{x} dx + \int_{0}^{a} \frac{f(x)}{x} dx + \int_{a}^{\infty} \frac{f(x)}{x} dx,$$
where $a, b \in \mathbb{R}^+$.
Since for all $x > A$, where A is a scalar, we have $\frac{1}{x} < \frac{1}{A}$. Thus,
$$\int_{a}^{\infty} \frac{f(x)}{x} dx \leq \int_{a}^{\infty} \frac{f(x)}{A} dx < \infty.$$

Similarly, we can obtain
$$\int_{-\infty}^{-b} \frac{f(x)}{x} dx < \infty.$$
Then since $f \in L^1(\mathbb{R})$ is differentiable at $x = 0$, which means $f'(0) < \infty$.
By definition of derivative,
$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} < \infty.$$
Therefore, $\forall \epsilon > 0$, $\exists \delta > 0$, s.t.
$$\left| \frac{f(x)}{x} - f'(0) \right| < \epsilon.$$
Choose $\epsilon = 1$ and $\delta = a$ then we have
$$\left| \frac{f(x)}{x} - f'(0) \right| < 1,$$
and

$$\begin{aligned}
\int_0^a \frac{f(x)}{x} dx = \int_0^\delta \frac{f(x)}{x} dx &\leq \int_0^\delta \left| \frac{f(x)}{x} \right| dx \\
&\leq \int_0^\delta \left| \frac{f(x)}{x} - f'(0) + f'(0) \right| dx \\
&\leq \int_0^\delta \left(\left| \frac{f(x)}{x} - f'(0) \right| + |f'(0)| \right) dx \\
&< \int_0^\delta (1 + |f'(0)|) \, dx < \infty.
\end{aligned}$$

Similarly, we can obtain that
$$\int_0^{-b} \frac{f(x)}{x} dx < \infty.$$
Hence, we have proved that
$$\int_{-\infty}^{\infty} \frac{f(x)}{x} dx = \int_{-\infty}^{-b} \frac{f(x)}{x} dx + \int_{-b}^{0} \frac{f(x)}{x} dx + \int_0^a \frac{f(x)}{x} dx + \int_a^\infty \frac{f(x)}{x} dx < \infty.$$
$\square$

4.4.21. Given a measurable set $E \subseteq \mathbb{R}^d$, prove the following statements.
(a) If $f \in L^1(E)$ and $g \in L^\infty(E)$, then $fg \in L^1(E)$.
(b) If $|E| > 0$, then $L^1(E)$ is not closed under products, i.e., there exist functions $f, g \in L^1(E)$ such that $fg \notin L^1(E)$.
(c) If f, g are measurable functions on E such that $|f|^2$ and $|g|^2$ each belong to $L^1(E)$, then $fg \in L^1(E)$.

Proof. (a). Notice that $|fg| \leq |f| ||g||_\infty$ a.e., therefore
$$\int_E |fg| \leq ||g||_\infty \int_E |f| < \infty$$
Hence $fg \in L^1(E)$.
(b). For $d = 1$, pick some $h \in \mathbb{R}$ such that $|(E + h) \cap (0, 1)| > 0$ and consider

$f(x) = g(x) = \frac{1}{\sqrt{x}}$. For $d > 1$, consider some $h \in \mathbb{R}$ such that $|(E+h) \cap (0,1)^n| > 0$ and choose $f(x) = g(x) = \frac{1}{\sqrt{x_1 \cdots x_d}}$ where $x = (x_1, \cdots, x_d)$.

(c). Notice that $0 \le (|f| + |g|)^2 = |f|^2 + 2|f||g| + |g|^2$, which gives

$$|f||g| \le \frac{|f|^2 + |g|^2}{2}$$

hence $fg \in L^1(E)$. $\square$