# Math 135 HW 7

1 a) $\lim\limits_{n\to\infty} \frac{1-x^{4n}/x^{4n}}{1+x^{4n}/x^{4n}} = \lim\limits_{n\to\infty} \frac{\frac{1}{x^{4n}}-1}{\frac{1}{x^{4n}}+1} = \frac{\lim\limits_{n\to\infty}\frac{1}{x^{4n}} - \lim 1}{\lim\limits_{n\to\infty}\frac{1}{x^{4n}}+\lim 1} = -1 = f(x)$

$\{f_n\}$ converges to $f(x) = -1$ pointwise. $\square$

b) No, If $x \in (-1,1)$ $f_n(x) \to 1$ as $x^{4n} \to 0$
But $f_n(1) = 0$ and $f_n(-1) = 0$, and for $|x| > 1$, $f_n(x) = -1$

So $f(x) = \begin{cases} 1 & |x|<1 \\ 0 & |x|=1 \\ -1 & |x|>1 \end{cases}$ This isn't continuous on $\mathbb{R}$, therefore convergence isn't uniform.

2a) At $x=0$ $f_n(0) = 1$ $\lim\limits_{n\to\infty} 1 = 1$
At $x>0$, $f_n(x) = e^{-nx}$ $\lim\limits_{n\to\infty} e^{-nx} = 0$

$\lim f_n(x) = f(x) = \begin{cases} 1 & x=0 \\ 0 & x>0 \end{cases}$

b) Using the comparison test to see if $f_n \to 0$
$|e^{-nx} - 0| = \left|\frac{1}{e^{nx}}\right|$ as $x \geq 1$ $\left|\frac{1}{e^{nx}}\right| \leq \left|\frac{1}{e^n}\right| := a_n$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left|\frac{1}{e^n}\right| = \lim \left(\frac{1}{e}\right)^n = 0$ so $f_n$ converges uniformly on $[1,\infty)$.

c) Because $f(x) = \begin{cases} 1 & x=0 \\ 0 & x>0 \end{cases}$ The limit isn't continuous, so $f_n$ cannot converge uniformly on $[0,\infty)$

d) No. If $\{f_n\}$ is uniformly continuous on $D = (0,\infty)$ then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall x \in D$ and $\forall n \geq N$, $|f_n(x) - f(x)| < \epsilon$
So, to disprove, we need and $\epsilon$ that violates this condition
Suppose $\epsilon < 1$
$|e^{-nx} - 0| < \epsilon = e^{-nx} < \epsilon$ $-nx < \ln\epsilon$ $x > \frac{\ln\epsilon}{-n}$ since $x \in (0,\infty)$.
but, by this definition this is true if $0 \leq \frac{\ln\epsilon}{-n} < x$, and as $\epsilon < 1$, $\frac{\ln\epsilon}{-n} > 0$, so this won't be satisfied for $x$ between 0 and $\frac{\ln\epsilon}{-n}$, meaning $f_n$ is not converging uniformly on $(0,1)$

3 a) Using ratio test: $\sum_{k=1}^{\infty} \frac{x^k}{k5^k}$ D = domain of convergence

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)5^{k+1}} \cdot \frac{k5^k}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x}{5} \cdot \frac{k}{k+1} \right| = \left|\frac{x}{5}\right| \lim_{k\to\infty} \frac{k}{k+1} < 1$$

$\lim_{k\to\infty} \frac{k}{k+1} = 1$, so $\left|\frac{x}{5}\right| < 1$ and $|x| < 5$. At bounderies

$x = 5$, sum $= \sum_{k=1}^{\infty} \frac{1}{k}$ which is the harmonic series and diverges

At $x = -5$, sum $= \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, which by the alternating series test, as $\frac{1}{k} \to 0$, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges

$$\boxed{D = [-5, 5)}$$ $\square$

b) Using ratio test. $\sum_{k=1}^{\infty} k!\, x^k$ D = Domain of convergence

$$\lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{k!\, x^k} \right| = \lim_{k\to\infty} |(k+1)x| = x \lim_{k\to\infty} |k+1| \to \infty$$

So $R = 0$ and $\boxed{D = [0]}$

~~D =~~

c) Using ratio test: $\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k+1)!}$ D = Domain of convergence

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2(k+1)-1}}{(2(k+1)+1)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k-1}} \right| = \lim_{k\to\infty} \left| \frac{x^{2k+1}}{x^{2k-1}} \cdot \frac{(2k+1)!}{(2k+3)!} \right|$$

$$= \lim_{k\to\infty} \left| x^2 \cdot \frac{1}{(2k+3)(2k+2)} \right| = x^2 \lim_{k\to\infty} \frac{1}{(2k+3)(2k+2)} < 1$$

$= x^2 \cdot 0 < 1$, so $x^2 \in \mathbb{R}$

$$\boxed{D = (-\infty, \infty)}$$ $\square$

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4 $||\vec{w}+\vec{v}||^2 = \langle \vec{w}+\vec{v}, \vec{w}+\vec{v} \rangle = \langle \vec{w},\vec{w} \rangle + 2\langle \vec{w},\vec{v} \rangle + \langle \vec{v},\vec{v} \rangle$

$||\vec{w}-\vec{v}||^2 = \langle \vec{w}-\vec{v}, \vec{w}-\vec{v} \rangle = \langle \vec{w},\vec{w} \rangle + \langle \vec{w},-\vec{v} \rangle + \langle -\vec{w},\vec{v} \rangle + \langle -\vec{v},-\vec{v} \rangle$

$= \langle \vec{w},\vec{w} \rangle - 2\langle \vec{w},\vec{v} \rangle + \langle \vec{v},\vec{v} \rangle$

Subtracting these.

$||\vec{w}+\vec{v}||^2 - ||\vec{w}-\vec{v}||^2 = \langle \vec{w},\vec{w} \rangle + 2\langle \vec{w},\vec{v} \rangle + \langle \vec{v},\vec{v} \rangle - (\langle \vec{w},\vec{w} \rangle - 2\langle \vec{w},\vec{v} \rangle + \langle \vec{v},\vec{v} \rangle)$

$= \frac{4\langle \vec{w},\vec{v} \rangle}{4} = \langle \vec{w},\vec{v} \rangle$ $\square$

5 Let $a = (a_1, a_2, a_3 \ldots a_n)$

Let $b = (1_1, 1_2, 1_3 \ldots 1_n)$ (All ones)

By Cauchy Schwarz $|\langle a,b \rangle| \leq ||a|| \, ||b||$.

$|\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} b_i^2} \cdot \sqrt{\sum_{i=1}^{n} a_i^2}$

$||$

$|a_1+a_2+a_3+\ldots+a_n|$ as $b_i = 1 \ \forall i$ $\sqrt{\sum b_i^2} = \sqrt{\underbrace{1+1+\ldots+1}_{n \text{ times}}} = \sqrt{n}$

$\sqrt{\sum a_i^2} = \sqrt{a_1^2+a_2^2+a_3^2+\ldots+a_n^2}$

So $|a_1+a_2+a_3+\ldots+a_n| \leq \sqrt{n} \cdot \sqrt{a_1^2+a_2^2+a_3^2+\ldots+a_n^2}$ $\square$

6 If $\text{dist}(\vec{u},\vec{v}) < r$ and $\text{dist}(\vec{u},\vec{w}) < r$ then

$||\vec{u}-\vec{v}|| < r$ and $||\vec{u}-\vec{w}|| < r$

Want $\text{dist}(t\vec{v}+(1-t)\vec{w}, \vec{u}) < r$

$||t\vec{v}+(1-t)\vec{w}-\vec{u}|| < r$ $\quad t\vec{v}+(1-t)\vec{w}-\vec{u} = t(\vec{v}-\vec{w})+(1-t)(\vec{w}-\vec{u})$

$= ||t(\vec{v}-\vec{w})+(1-t)(\vec{w}-\vec{u})|| < r$ by triangle inequality,

$\leq ||t(\vec{v}-\vec{w})|| + ||(1-t)(\vec{w}-\vec{u})||$

$= t||\vec{v}-\vec{u}|| + (1-t)||\vec{w}-\vec{u}|| < t \cdot r + (1-t)r = r$ $\square$