Homework 7

Graded

Student

Scott A. Fullenbaum

Total Points

20 / 20 pts

Question 1

Volume between cylinders and cones

2 / 2 pts

✔ **− 0 pts** Completed

− 2 pts Incomplete

Question 2

Average temperature in a ball

5 / 5 pts

✔ **− 0 pts** Correct

− 2 pts Calculation error in integral

− 2 pts Did not average over the volume of the sphere

− 0.5 pts No constant of proportionality present

Question 3

Charge density between sphere and cones

5 / 5 pts

– 0 pts Correct

– 0.5 pts Cylindrical: Incorrect bounds for r

– 0.5 pts Cylindrical: Incorrect bounds for z

– 1 pt Cylindrical: Bounds and order of integration do not match (draw a picture!)

– 0.5 pts Spherical: Incorrect bounds for ρ

– 0.5 pts Spherical: Incorrect bounds for ϕ.

– 0.5 pts Spherical: incorrect integrand

– 0.5 pts The given cone is not restricted to $z \geq 0$ (i.e. the region is the entire sphere minus two cones)

– 0.25 pts Answer should be simplified and/or exact. Using a right triangle with side lengths 1 and 3 one can show that $\cos(\tan^{-1}(1/3)) = 3/\sqrt{10}$ (or $\cos(\sin^{-1}(1/\sqrt(10)))$)

– 0.5 pts Minor algebraic errors

– 1 pt Algebraic errors

– 0.25 pts Order of integration flipped

– 0.5 pts Insufficient work shown.

– 2 pts Incorrect region.

– 1 pt Conceptual errors

– 0.5 pts Cylindrical: incorrect integrand

– 0.25 pts Notational errors

Question 4

Mass of a parallelepiped using change of variables

5 / 5 pts

– 0 pts Fully correct.

– 0.5 pts Some type of issue with transformation matrix either incorrect derivatives, incorrect bounds for given coordinate change, or the matrix, whose determinant is taken, is not correct for the transformation in the problem (regardless of final answer). Going from (x,y,z) to (u,v,w) the Jacobian determinant must be J(u,v,w) found by solving for x,y,z in terms of u,v,w.

– 1 pt No work indicating that a Jacobian was considered.

– 1 pt Incorrect answer.

– 3.5 pts Incorrect approach and answer.

– 2 pts Problem indicated to solve using a change of variables.

– 5 pts No work submitted.

Not volume.

Question 5

Change of variables: diamond

2 / 2 pts

✔ **– 0 pts** Completed

– 2 pts Incomplete

Question 6

Change of variables: ellipse

1 / 1 pt

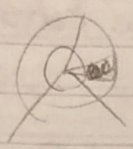
✔ **– 0 pts** Completed

– 1 pt Incomplete

Question assigned to the following page: 1

Math HW 7

1



$0 \le \theta \le 2\pi$

$\frac{\pi}{6} \le \phi \le \frac{\pi}{3}$

$\csc\phi \le \rho \le 2\csc\phi$

$r = \rho\sin\phi$

$1 = \rho\sin\phi$

$\csc\phi = \rho$

$$V = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc\phi}^{2\csc\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$V = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\rho^3}{3}\sin\phi \Big|_{\csc\phi}^{2\csc\phi} d\phi = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\csc^3\phi}{3}\sin\phi - \frac{\csc^3\phi}{3}\sin\phi \, d\phi$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8}{3}\csc^2\phi - \frac{\csc^2\phi}{3} \, d\phi$$

$$= 2\pi\left(-\frac{8}{3}\cot\phi + \frac{\cot\phi}{3}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 2\pi\left(-\frac{7}{3}\cot\phi\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$\frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} \quad \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \quad \frac{\sin\frac{\pi}{6}}{\frac{\sqrt{3}}{2}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \quad \frac{1}{2}\cdot\frac{2}{\sqrt{3}}$

$$= 2\pi\left(-\frac{7}{3}\cot\frac{\pi}{3} + \frac{7}{3}\cot\frac{\pi}{6}\right)$$

$$= 2\pi\left(-\frac{7}{3}\left(\frac{1}{\sqrt{3}}\right) + \frac{7}{3}\left(\sqrt{3}\right)\right)$$

$$= 2\pi\left(\frac{-7\sqrt{3}}{3\sqrt{3}} + \frac{7\sqrt{3}}{3\sqrt{3}}\right)$$

$$= 2\pi\left(\frac{-7+21}{3\sqrt{3}}\right) = \frac{28\pi}{3\sqrt{3}}$$

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Question assigned to the following page: 2

2) $T(\rho) = k\sqrt{\rho}$ where k is a constant

$$\text{avg temp} = \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{a} k\rho^{5/2} \sin\phi \, d\rho \, d\phi \, d\theta}{V}$$

a is radius of sphere

$$= \frac{2\pi \int_0^{\pi} \frac{2}{7} k\rho^{7/2} \sin\phi \Big|_0^a d\phi}{\frac{4}{3}\pi a^3}$$

$$= \frac{6}{4a^3} \int_0^{\pi} \frac{2}{7} k a^{7/2} \sin\phi \, d\phi$$

$$= \frac{3}{2a^3} \int_0^{\pi} \frac{2}{7} k a^{7/2} \sin\phi \, d\phi$$

$$= \frac{3a^{7/2}k}{7a^3} \int_0^{\pi} \sin\phi \, d\phi$$

$$= \frac{3k\sqrt{a}}{7} \cdot -\cos\phi \Big|_0^{\pi} = \frac{6k\sqrt{a}}{7}$$

The average temperature is $\frac{6k\sqrt{a}}{7}$, where k is a constant, and a is the radius of the sphere

$\lim_{a\to\infty} \frac{6k\sqrt{a}}{7} = \infty$, meaning as the ball gets infinitely large, the average temperature also increases to ∞

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Question assigned to the following page: 3

3 $f(x,y,z) = \frac{1}{x^2+y^2+z^2} = \frac{1}{\rho^2}$

sphere $\rho^2 = 40$

cone $z^2 = 9r^2$

$\rho^2 = 40$ $\quad \rho^2\cos^2\phi = 9\rho^2\sin^2\phi$

$\cot^2\phi = 9, \quad \cot\phi = \frac{1}{3}$

$\tan^{-1}\phi = \frac{1}{3}$

$r = \rho\sin\phi$

$z = \rho\cos\phi$

$0 \le \rho \le \sqrt{40}$

$\tan^{-1}\frac{1}{3} \le \phi \le \pi - \tan^{-1}\frac{1}{3}$

$0 \le \theta \le 2\pi$

So for spherical

$$\text{Charge} = \int_0^{2\pi}\int_{\tan^{-1}\frac{1}{3}}^{\pi-\tan^{-1}\frac{1}{3}}\int_0^{\sqrt{40}} \sin\phi \, d\rho \, d\phi \, d\theta$$

For cylindrical

$r^2+z^2 = 40, \quad z^2 = 9r^2, \quad z = \sqrt{40-r^2}$

$$\int_0^{2\pi}\int_{-6}^{6}\int_{z/3}^{\sqrt{40-z^2}} \frac{r}{r^2+z^2} \, dr \, dz \, d\theta$$

$z^2 = 9r^2, \quad z = \pm 3r, \quad r = \frac{z}{3}$

$10r^2 = 40$

$r = \pm 2$, if $r = 2$, $z = \pm 6$

Solving w/ spherical

$$\text{Charge} = \int_0^{2\pi}\int_{\tan^{-1}\frac{1}{3}}^{\pi-\tan^{-1}\frac{1}{3}}\int_0^{\sqrt{40}} \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi\sqrt{40}\int_{\tan^{-1}\frac{1}{3}}^{\pi-\tan^{-1}\frac{1}{3}} \sin\phi \, d\phi$$

40
^
2 20
^
2 10
^
2 5

$$= 4\pi\sqrt{10}\int_{\tan^{-1}\frac{1}{3}}^{\pi-\tan^{-1}\frac{1}{3}} \sin\phi \, d\phi$$

Question assigned to the following page: 3

3 $4\pi\sqrt{10}\int_{\tan^{-1}\frac{1}{3}}^{\pi-\tan^{-1}\frac{1}{3}} \sin\phi \, d\theta = 4\pi\sqrt{10}\left(-\cos\phi\right)\Big|_{\tan^{-1}\frac{1}{3}}^{\pi-\tan^{-1}\frac{1}{3}}$

$= 4\pi\sqrt{10}\left(-\cos\left(\pi-\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}\frac{1}{3}\right)\right)$

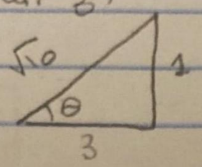
$\cos(\pi-x) = -\cos x$ $\quad = 4\pi\sqrt{10}\left(\cos\left(\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}\frac{1}{3}\right)\right)$

$= 8\pi\sqrt{10}\left(\cos\left(\tan^{-1}\frac{1}{3}\right)\right)$

$= 8\pi\sqrt{10} \cdot \frac{3}{\sqrt{10}} = 24\pi$

$\cos\left(\tan^{-1}\frac{1}{3}\right)$

$=$



Spherical coordinates:

$$\int_0^{2\pi}\int_{\tan^{-1}\frac{1}{3}}^{\pi-\tan^{-1}\frac{1}{3}}\int_0^{\sqrt{40}} \sin\phi \, d\rho \, d\phi \, d\theta$$

Cylindrical coordinate integral:

$$\int_0^{2\pi}\int_{-6}^{6}\int_{2/3}^{\sqrt{40-z^2}} \frac{r}{r^2+z^2} \, dr \, dz \, d\theta$$

The Charge is equal to 24π

Question assigned to the following page: 4

4 $0 \le x+y+z \le 10$, $0 \le y+z \le 5$, $0 \le z \le 2$

$x+y+z=u$, $y+z=v$, $z=w$

$0 \le u \le 10$, $0 \le v \le 5$, $0 \le w \le 2$

$x=u-v$, $y=v-w$, $z=w$

$z+y=v-z$, $y=v-w$

$x=u-y-z$ $-w$

$=u-(v-w)-w$

$=u-v$

$$J(u,v,w) = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J(u,v,w) = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$J(u,v,w) = 1$

$$V = \int_0^{10}\int_0^{2}\int_0^{5} wv \, dv\,dw\,du$$

$$V = 10\int_0^2 \frac{wv^2}{2}\Big|_0^5 dw = 10\int_0^2 \frac{25w}{2}dw = 10\left(\frac{25w^2}{4}\right)\Big|_0^2$$

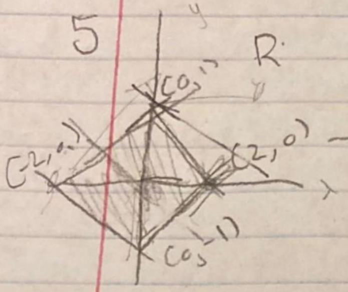
$= 10(25)$

$= 250$

The volume of the parallelepiped is 250

Question assigned to the following page: 5

5



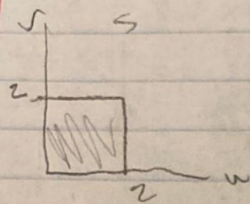
$$I = \iint_R 9x^2 + 36xy + 36y^2 \, dx\, dy$$

$u = x + 2y$, $u - 2y = x$ such that

$-\; v = x - 2y$ $\quad + v + 2y = x = x + 2y$

$u - v = 4y$ $\quad \frac{u+v}{2} = x$

$\frac{u-v}{4} = y$ $\quad \frac{u+v}{2} = x$



$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}$$

$$I = \frac{1}{4}\int_0^2\int_0^2 \left(9\left(\frac{u+v}{2}\right)^2 + 36\left(\frac{u+v}{2}\right)\left(\frac{u-v}{4}\right) + 36\left(\frac{u-v}{4}\right)^2\right)\frac{1}{4}\, du\, dv$$

$$I = \int_0^2\int_0^2 \frac{1}{4}\left(\frac{9}{4}(u^2 + 2uv + v^2) + \frac{9}{2}(u^2 - v^2) + \frac{9}{4}(u^2 - 2uv + v^2)\right) du\, dv$$

$$I = \int_0^2\int_0^2 \frac{1}{4}\left(\frac{9}{2}u^2 + \frac{9}{2}v^2 + \frac{9}{2}u^2 - \frac{9}{2}v^2\right) du\, dv$$

$$I = \int_0^2\int_0^2 \frac{9u^2}{4}\, du\, dv$$

$$= \frac{9}{2}\int_0^2 u^2\, du = \frac{9}{2}\cdot\frac{u^3}{3}\Big|_0^2 = \frac{9}{2}\cdot\frac{8}{3} = 12$$

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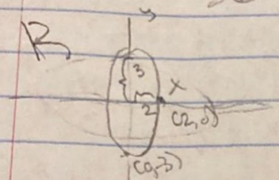
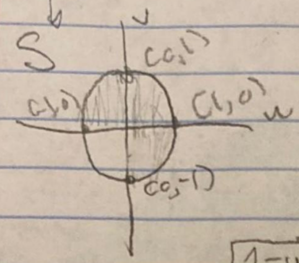
Question assigned to the following page: 6

6 $\quad 9x^2+4y^2=36$ $\;/36$ $\qquad \frac{x^2}{4}+\frac{y^2}{9}=1$



$\frac{x}{2}=u \quad \frac{y}{3}=v, \; u^2+v^2=1$

$x=2u, \; y=3v$



$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$I = 2\int_{-1}^{1}\int_{0}^{\sqrt{1-u^2}} 6(2u)(3v)\,dv\,du$$

$$I = 2\int_{-1}^{1} 9v^2u \Big|_{0}^{\sqrt{1-u^2}} du = \int_{-1}^{1} 9(1-u^2)u\,du,$$

$= 2\int_{-1}^{1} 9u - 9u^3\,du$, ; if $f(u)=9u-9u^3$, $f(-u)=9u^3-9u$, meaning f is odd.

$2\int_{-a}^{a} f(u)\,du = 0$ if $f(u)$ is odd, meaning $2\int_{-1}^{1} 9u - 9u^3\,du = 0$

$\iint_R xy\,dA$ where R is interior of ellipse $9x^2+4y^2=36$ is 0