Recall that the field axioms for $\mathbb{R}$ are as follows:

(1) (Associativity) For all $x, y, z \in \mathbb{R}$,
$$x + (y + z) = (x + y) + z \quad \text{and } x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$
(2) (Commutativity) For all $x, y \in \mathbb{R}$,
$$x + y = y + x \quad \text{and } x \cdot y = y \cdot x.$$
(3) (Identity elements) There exists a unique element of $\mathbb{R}$ called **zero**, denoted by 0, such that for all $x \in \mathbb{R}$ we have $x + 0 = x$.
There exists a unique element of $\mathbb{R}$ called **one**, different from 0, denoted by 1, such that for all $x \in \mathbb{R}$, $x \cdot 1 = x$.
(4) (Inverses) For each element $x \in \mathbb{R}$, there exists a unique element $y$ (called the **negative** of $x$ and usually denoted by $-x$) such that $x + y = 0$.
Similarly, for each element $x \in \mathbb{R} - \{0\}$, there exists a unique element $y$ (called the **reciprocal** of $x$ and usually denoted by $1/x$ or $x^{-1}$) such that $x \cdot y = 1$.
(5) (Distributivity) For all $x, y, z \in \mathbb{R}$,
$$x \cdot (y + z) = x \cdot y + x \cdot z \quad \text{and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

We take $x - y$ to be an abbreviation for $x + (-y)$ and $x/y$ to be an abbreviation for $x \cdot (1/y)$.
We have shown in class that

(P1) If $x + y = x$, then $y = 0$
(P2) $0 \cdot x = 0$
(P3) $-0 = 0$
(P4) $-(-x) = x$

(1) Using only the axioms (1)–(5) and properties proven in class, prove the following statements for all $x, y, z \in \mathbb{R}$:
  (a) (-1)x = -x

(b) x(-y) = -(xy) = (-x)y

(c) x(y - z) = xy - xz

(d) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$

(e) If $x \neq 0$, then $x/x = 1$

Recall that $\mathbb{R}$ also satisfies the following axioms related to ordering:

(6) For all $x, y, z \in \mathbb{R}$, if $x > y$, then $x + z > y + z$.
For all $x, y, z \in \mathbb{R}$, if $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$.
(7) The order relation $<$ has the least upper bound property.
(8) If $x < y$, there exists an element $z$ such that $x < z$ and $z < y$.

We have shown in class that

(P5) $x > y$ and $w > z$ implies $x + w > y + z$;
(P6) $x > 0$ and $y > 0$ implies $x + y > 0$ and $x \cdot y > 0$;
(P7) $x > 0 \iff -x < 0$

(2) Prove the following "Laws of inequalities"
(a) $x > y \iff -x < -y$

(b) $x > y$ and $z < 0$ implies $xz < yz$

(c) $x \neq 0$ implies $x^2 > 0$, where $x^2 = x \cdot x$

(3) Prove that every positive number has a square root as follows.
   (a) Show that if $x > 0$ and $0 < h < 1$, then
$$(x+h)^2 < x^2 + h(2x+1)$$
$$(x-h)^2 > x^2 - 2xh.$$

(b) Let $x > 0$. Show that if $x^2 < a$, then $(x+h)^2 < a$ for some $h > 0$. Similarly, show that if $a < x^2$, then $a < (x-h)^2$ for some $h > 0$.

(c) Given $a > 0$, let $B$ be the set of all real numbers $x$ such that $x^2 < a$. Show that $B$ is bounded above and contains at least one positive number. (Hint: it may help to consider the case that $a \geq 1$ separately from that case that $0 < a < 1$.)

(d) Let $b = \sup B$. Show that $b^2 = a$. (Hint: Suppose $b^2 < a$, then derive a contradiction. Then do the same when $b^2 > a$.)

(e) Show that if $b$ and $c$ are positive and $b^2 = c^2$, then $b = c$.