Math 135 HW 4

1 Since S is sequentially compact then for $\{a_n\} \in S$, $\{a_{n_k}\} \to a \in S$. Since $\lim a_n = \lim a_{n_k}$, then $a_n \to a$. Since a_n converges, then it is bounded, so a_n is a bounded set.

2 Let $\{a_n\} \in S$. $\{a_{n_k}\} \in S$ and $\{a_{n_k}\} \to a \in S$. Since a_n also converges to a, and $a \in S$, then S is closed as it satisfies the definition for a closed set.

3 Let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|\sqrt{a_n} - \sqrt{a}| < \varepsilon$

Note there are 2 cases.

Case 1 $a = 0$

$|\sqrt{a_n} - 0| < \varepsilon$, $(\sqrt{a_n} < \varepsilon)^2$ $a_n < \varepsilon^2$ since $a_n \to a$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $a_n < \varepsilon^2$

Case 2: $a \neq 0$

Scratch work

$|\sqrt{a_n} - \sqrt{a}| < \varepsilon$ Note since $a_n \to a$, $\sqrt{a_n} > \sqrt{\frac{a}{2}}$, $\sqrt{a_n} + \sqrt{a} > \sqrt{\frac{a}{2}} + \sqrt{a}$

$$\left| \frac{\sqrt{a_n} - \sqrt{a} \cdot (\sqrt{a_n} + \sqrt{a})}{\sqrt{a_n} + \sqrt{a}} \right| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \left| \frac{a_n - a}{\sqrt{\frac{a}{2}} + \sqrt{a}} \right| < \varepsilon$$

pf)

Let $\varepsilon > 0$, and $\varepsilon_1 = \varepsilon\left(\sqrt{\frac{a}{2}} + \sqrt{a}\right) > 0$, since $a_n \to a$ $\exists N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon_1 = \left(\sqrt{\frac{a}{2}} + \sqrt{a}\right)\varepsilon$

$$\text{Then } \forall n \geq N,\ |\sqrt{a_n} - \sqrt{a}| < \frac{|a_n - a|}{\sqrt{\frac{a}{2}} + \sqrt{a}} < \frac{\left(\sqrt{a/2} + \sqrt{a}\right)\varepsilon}{\left(\sqrt{\frac{a}{2}} + \sqrt{a}\right)} \leq \varepsilon$$

So $\lim \sqrt{a_n} = \sqrt{a}$ $\square$

4) $f(x)$ is continuous nowhere. For $\forall x_0 \in \mathbb{R}$ there are 2 cases.

Case 1. $x_0 \in \mathbb{Q}$ suppose $X_n \to X_0$. $\{x_n\}$ can be a sequence of irrational numbers s.t. $\lim x_n = x_0$

$X_0 = \lim X_n$
$g(X_0) = g(X_n)$
$X_0^2 = -X_0^2$, as $X_n \to X_0$
This isn't true, so not continuous for $X_0 \in \mathbb{Q}$

Case 2
$X_0 \notin \mathbb{Q}$ Let $X_n \to X_0$ $\{X_n\}$ can be a sequence of rational numbers s.t. $X_0 = \lim X_n$

$X_0 = \lim X_n$
$g(X_0) = g(X_n)$
$-X_0^2 = X_0^2$ as $X_n \to X_0$
Not true, so not continuous when $X_0 \notin \mathbb{Q}$
Since g isn't continuous when $X \in \mathbb{Q} \cup \mathbb{Q}^c$,
$g(x)$ is discontinuous everywhere.

5 a) No, $S \subset \mathbb{R}$ is sequentially compact iff S is closed and bounded. (Proved $\Rightarrow$ in Qs 1 and 2, other way in class.)
$\mathbb{Q} \cap [0,1)$ is not closed or bounded.
- To show not bounded, let $a_n = 1 - \frac{1}{n} \in \mathbb{Q} \cap [0,1]$.
$\lim a_n = 1 \notin \mathbb{Q} \cap [0,1)$. The rationals aren't closed, so $\mathbb{Q} \cap [0,1)$ isn't closed.

b) $\mathbb{Q} \cap [0,1]$ isn't closed, so it cannot be sequentially compact.

c) $[0,1] \cup [2,3]$ is closed and bounded,
Any $a_n \in [0,1] \cup [2,3]$ will converge to some $a \in [0,1] \cup [2,3]$
Also the set is clearly bounded.

d) -

d) $\mathbb{Z} \cap [1, 100]$ is sequentially compact.
The set is banded by 1 below, and 100 100 above and the integers are closed so $\mathbb{Z} \cap [1, 00]$ is closed and banded meaning it is sequentially compact.

e) $[0, \infty)$ isn't sequentially compact.
Let $a_n = n$, any subsequence of a_n would be unbanded and fail to converge, So $[0, \infty)$ isn't Sequentially compact.

6) Let $x_0 \in Q^c$, $\exists$ some sequence $\{x_n\}$ where $x_n \in Q^c$
If g is continuous: s.t. $x_n \to x_0$
$x_0 = \lim x_n$
$f(x_0) = g(x_n)$
$0 = g(x_n)$ meaning $g(x_n) = 0$, and since $x_n \in Q^c$, then guarantee continuity of $g(x)$, then $\forall x \notin Q$ $g(x) = 0$
So $g(x) = 0$ $\forall x \in Q \cup Q^c \to \forall x \in \mathbb{R}$. $\square$

7 a) $\{\frac{1}{n}\}$ has infinitely many peaks.
$\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2} > \frac{1}{n+3} + \dots + \frac{1}{n+k}$, $k \in \mathbb{N}$.
So every term is greater than every term after it, so every term of $\{\frac{1}{n}\}$ is a peak, meaning it has infinitely many peak indices

b) $\{(-1)^n n\} = \{-1, 2, -3, 4 \dots\}$
This sequence has no peaks. If n is odd, then $(-1)^n n < 0$. However, for the following term, $(-1)^{n+1}(n+1)$, n+1 is even so $(-1)^{n+1}(n+1) > 0$ So all odd indices aren't peaks.
For every even index, $(-1)^n n < (-1)^{n+2}(n+2)$, So no even index can be a peak and $\{(-1)^n n\}$ has no peak indices.

7c) $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4} \dots\right\}$

Every even index is a peak.

Let n be odd. $\frac{(-1)^n}{n} < 0$. The $(n+1)^{th}$ index is even so $\frac{(-1)^{n+1}}{n+1} > 0$ and therefore $\frac{(-1)^n}{n} < \frac{(-1)^{n+1}}{n+1}$

as $-\frac{1}{n} < \frac{1}{n+1}$, so an odd index can't be a peak index.

Let n be even, as $\frac{(-1)^n}{n} > 0$, it is greater than every odd index. $\frac{(-1)^n}{n} > \frac{(-1)^{n+2}}{n+2}$, $\frac{1}{n} > \frac{1}{n+2}$

$\frac{1}{n} > \frac{1}{n+2} > \frac{1}{n+4} + \dots > \frac{1}{n+2k}$, $k \in \mathbb{N}$

So any even index is greater than every even index after it.

Therefore, every even index of $\left\{\frac{(-1)^n}{n}\right\}$ (meaning n is even), is a peak index, as it is greater than every term after it.