1. (10 points) (**Product of sequentially compact sets**) §11.2, p. 304, # 8.
Let A and B be sequentially compact subsets of $\mathbb{R}$. Prove that $A \times B$ is sequentially compact in $\mathbb{R}^2$. (The Cartesian product $A \times B$ is the set $\{(a,b) \in \mathbb{R}^2 \mid a \in A \text{ and } b \in B\}$.)

Proof. Let $u_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix}$ be a sequence in $A \times B$.
Since A is sequentially compact, $\{a_k\}$ has a convergent subsequence $\{a_{k_i}\}_i$ say $a_{k_i} \to a_0$.
Since B is sequentially compact, $\{b_{k_i}\}$ has a convergent subsequence $\{b_{k_{i_j}}, 1 \le j \le 2n\}$,

$b_{k_{i_j}} \to b_0$ in B.

Then $u_{k_i} = \begin{bmatrix} a_{k_{i_j}} \\ b_{k_{i_j}} \end{bmatrix} \to \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \in A \times B$.

Hence, $A \times B$ is sequentially compact. $\square$

2. (15 points) (**Continuity in terms of closed sets**)
   (a) Prove the lemma: Let X and Y be two sets. For any map $F: X \to Y$ and $C \subset Y$, $F^{-1}(Y \setminus C) = X \setminus F^{-1}(C)$. ("The inverse image preserves the complement.") (*Hint*: The easiest way to proceed is to write down a sequence of equivalent statements starting with $x \in F^{-1}(Y \setminus C)$ iff $\cdots$.)
   (b) Let $F : \mathbb{R}^n \to \mathbb{R}^m$. Prove that F is continuous on domain $\mathbb{R}^n$ if and only if for every closed subset C in $\mathbb{R}^m$, the inverse image $F^{-1}(C)$ is closed in $\mathbb{R}^n$.

(a) $x \in F^{-1}(Y \setminus C)$

iff $F(x) \in Y \setminus C$ (def of $F^{-1}(\ )$)

iff $F(x) \notin C$ (def of complement in Y)

iff $x \notin F^{-1}(C)$ (def of $F^{-1}(\ )$)

iff $x \in X \setminus F^{-1}(C)$. (def of complement in X).

(b) ($\Rightarrow$)

Suppose $F: \mathbb{R}^n \to \mathbb{R}^m$ is continuous and C is closed in $\mathbb{R}^m$. Then $V := \mathbb{R}^m \setminus C$ is open in $\mathbb{R}^m$. By Th. 11.12, $F^{-1}(V) = F^{-1}(\mathbb{R}^m \setminus C) = \mathbb{R}^n \setminus F^{-1}(C)$ is open in $\mathbb{R}^n$. By the complementing characterization (Th. 10.16), $F^{-1}(C)$ is closed in $\mathbb{R}^n$.

(b) ($\Leftarrow$)

Conversely, suppose the inverse image $F^{-1}(C)$ of a closed set in $\mathbb{R}^m$ is closed in $\mathbb{R}^n$. Let V be an open set in $\mathbb{R}^m$. Then $\mathbb{R}^n \setminus V$ is closed and so $F^{-1}(\mathbb{R}^m \setminus V) = \mathbb{R}^n \setminus F^{-1}(V)$ is closed. By the complementing characterization (Th. 10.16), $F^{-1}(V)$ is open in $\mathbb{R}^n$. By Th. 11.12, $F: \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

$\square$

3. (20 points) (**Distance from a point to a set**)

(a) (5 points) Let $\vec{v} \in \mathbb{R}^n$. Define $D : \mathbb{R}^n \to \mathbb{R}$ by $D(\vec{u}) = \text{dist}(\vec{u}, \vec{v})$ for $\vec{u} \in \mathbb{R}^n$. Prove D is continuous.

Proof. $D(u) = d(u,v) = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$ is a composition of sums and products of continuous functions. By Th. 11.3 and 11.4, $D(u)$ is continuous. □

(b) (10 points) §11.2, p. 304, # 9.

Let A be a sequentially compact subset of $\mathbb{R}^n$ and let $\vec{v}$ be a point in $\mathbb{R}^n \setminus A$. Prove that there is a point $\vec{u}_0$ in the set A such that

$$\text{dist}(\vec{u}_0, \vec{v}) \le \text{dist}(\vec{u}, \vec{v}) \quad \text{for all points } \vec{u} \in A.$$

Is this point unique? (Note that the proof works even if $\vec{v} \in A$.)

The function $D: A \to \mathbb{R}$, $D(u) = d(u,v)$, is continuous by (a). Since A is sequentially compact, by the extreme value theorem D has a minimum at some point $u_0 \in A$. This means

$$d(u_0, v) \le d(u,v) \qquad \text{for all } u \in A.$$

The point u_0 is not unique. For example, if A is the unit circle center at 0 and $v = 0$, then every point of A is at minimum distance from v. □

(c) (5 points) Is the conclusion of (b) true if A is closed (and not necessarily sequentially compact)? Why or why not? (*Hint*: Consider the closed ball $\bar{B}_r(\vec{v})$ of radius r centered at $\vec{v}$. Let the radius be large enough so that $\bar{B}_r(\vec{v}) \cap A$ is nonempty. Explain why the point of A at minimum distance from $\vec{v}$ must be in $\bar{B}_r(\vec{v}) \cap A$. What kind of set if $\bar{B}_r(\vec{v}) \cap A$?)

Yes. Let a be a point of A and $r = d(a,v)$. Then the closed ball $\bar{B}(v,r)$ of center v and radius r contains a. We can restrict our attention to $A \cap \bar{B}(v,r)$, since all the points of A not in $\bar{B}(v,r)$ will have distance $> r$ from v. Since A is closed and $\bar{B}(v,r)$ is closed, $A \cap \bar{B}(v,r)$ is closed and bounded. Therefore, $A \cap \bar{B}(v,r)$ is sequentially compact. Therefore, the continuous function $D(u) = d(u,v)$ has a minimum $u_0 \in A \cap \bar{B}(v,r)$. This point also satisfies

$$d(u_0, v) \le d(u,v) \qquad \text{for all } u \in A$$

since if $u \in A \cap \bar{B}(v,r)$, then $d(u_0,v) \le d(u,v)$ and if $u \in A \setminus \bar{B}(v,r)$, then $d(u_0,v) \le r < d(u,v)$. □

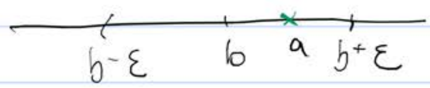
4. (10 points) (**Boundary of a set in $\mathbb{R}$**)

Let A be a bounded and nonempty subset of $\mathbb{R}$.

(a) Prove that $\inf(A) \in \mathrm{bd}(A)$.

(b) Now assume that A is closed. Prove that $\inf(A) \in A$.

(a) Let $b = \inf A$. For any $\varepsilon > 0$, the ball $B(b, \varepsilon) = (b-\varepsilon, b+\varepsilon)$ contains a point $b - \frac{\varepsilon}{2} \notin A$ and a point $a \in A$ such that $b \leq a < b + \varepsilon$.



Therefore, $b \in \mathrm{bd} A$.

(b) Now suppose A closed. By Prop. 10.19 (ii) A is closed iff $\mathrm{bd} A \subseteq A$. Therefore, $\inf A \in \mathrm{bd} A \subseteq A$. $\square$

5. (20 points) **True or False.**

If a statement is true, prove it, and if it is false, come up with a counterexample.

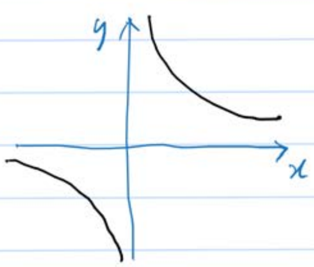
(a) $\{(x,y) \in \mathbb{R}^2 \,|\, x^4 + y^4 = 1\}$ is sequentially compact.

True. The set $A = f^{-1}(\{0\})$, where $f(x,y) = x^4 + y^4 - 1$ is continuous, is the inverse image of a closed set, so it is closed.

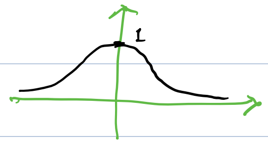
A is bounded because if $(x,y) \in A$, then $x^4 \leq 1 \Rightarrow |x| \leq 1$ $y^4 \leq 1 \Rightarrow |y| \leq 1$.

(b) Let $A \subset \mathbb{R}^n$ and assume $F : A \to \mathbb{R}^m$ is continuous. If A is closed, then the image $F(A)$ is closed.

False. Let $A = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ be the hyperbola. It is a closed set because $A = G^{-1}(\{1\})$ is the inverse image of the closed set $\{1\}$ in $\mathbb{R}$ under the continuous map $G(x,y) = xy$. However, $F(x,y) = x$ is continuous, but $F(A) = \mathbb{R} \setminus \{0\}$ is not closed.



Another counterexample. $F: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x^2+1}$ is



continuous and has graph on the right. Then $A = \mathbb{R}$ is closed in $\mathbb{R}$, but $F(A) = (0,1]$ is not closed. $\square$

(c) Let $A \subset \mathbb{R}^n$ and assume $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on domain $\mathbb{R}^n$. If A is bounded, then its image $F(A)$ is bounded.

True. Suppose $\|u\| \leq M$ for all $u \in A$.
Then $A \subseteq \bar{B}_M(0)$ the closed ball of center 0 and radius M. Since $\bar{B}_M(0)$ is closed and bounded, it is sequentially compact. Therefore, $F(\bar{B}_M(0))$ is sequentially compact by Th. 11.20. Thus, $F(\bar{B}_M(0))$ is closed and bounded, so $F(A) \subseteq F(\bar{B}_M(0))$ is also bounded.

Alternative proof. Since A is bounded, $cl A$ is also bounded (A bounded $\Rightarrow$ $A \subset$ closed ball $\bar{B}_M(0)$ $\Rightarrow$ $cl A \subset \bar{B}_M(0)$ because the closure of A is the smallest closed set containing A). Thus, $cl A$ is closed and bounded, hence sequentially compact. By Th. 11.20, $F(cl A)$ is sequentially compact and therefore bounded.

(d) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and C is sequentially compact in $\mathbb{R}^m$, then the inverse image $f^{-1}(C)$ is sequentially compact in $\mathbb{R}^n$.

False. The constant function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a$, is continuous and $C = \{a\}$ is closed and bounded and therefore sequentially compact, but $f^{-1}(C) = f^{-1}(\{a\}) = \mathbb{R}$ is not sequentially compact, because $\mathbb{R}$ is not bounded. $\square$

6. (10 points) (**Sequentially compact**) §11.2, p. 304, # 7.
Suppose that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(\vec{u}) \geq \|\vec{u}\|$ for every point $\vec{u}$ in $\mathbb{R}^n$. Prove that $f^{-1}([0,1])$ is sequentially compact.

Proof.

Let $A = f^{-1}([0,1])$. If $u \in A$, then $f(u) \in [0,1]$, so

$$\|u\| \leq f(u) \leq 1.$$

Thus, A is bounded. Since $A = f^{-1}([0,1])$ is the inverse image of a closed set under a continuous map, A is also closed. Being closed and bounded, A is compact. □

7. (10 points) (**Cubes versus balls**)
Define $K(u,r)$ to be the cube with center u and radius r in $\mathbb{R}^n$:

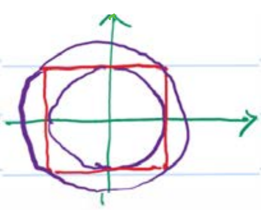
$$K(u,r) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_i - u_i| < r \text{ for all } i\}.$$

Prove that

$$B(0,r) \subset K(0,r) \subseteq B(0, \sqrt{n}\, r).$$

(*Hint*: Draw a picture of the 2-dimensional case when the cube is a square. Do the problem in $\mathbb{R}^2$ first.) Notation. $B(0,r) = B_r(0)$

Proof.



Let $x \in B(0,r)$. Then $\|x\| < r$.
So $|x_i| \leq \|x\| < r$ for all $i = 1, \ldots, n$.
Thus, $x \in K(0,r)$.

Suppose $x \in K(0,r)$.
Then $|x_i| < r$ for all i.
So $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2} < \sqrt{n r^2} = \sqrt{n}\, r$.
Thus, $x \in B(0, \sqrt{n}\, r)$.

This proves the two inclusions. □