Math 165 HW 7

1a) $P(X=1 \cap Y=2)$ Since they are independent,
$P(X=1 \cap Y=2) = P(X=1)\,P(Y=2)$
For X, $\lambda=1$, for y, $\lambda=2$ $\longrightarrow$ Use Poisson

$P(x=1) = \frac{1^1}{1!}e^{-1}$ $\quad P(y=2) = \frac{2^2}{2!}e^{-2}$

$e^{-1} \cdot 2e^{-2} = 2e^{-3} = \boxed{.0996}$

b) $P(\frac{x+y}{2} \geq 1) = P(X+Y \geq 2)$
Let $Z = X+Y$ from class, for the sum of 2 independent Poisson distributions, the parameter of the new distribution is the sum of the other 2 parameters, meaning, $\lambda_z = \lambda_x + \lambda_y$, $\lambda_z = 3$

$P(Z \geq 2) = 1 - P(Z=0) - P(Z=1)$

$P(Z=0) = \frac{3^0}{0!}e^{-3}$ $\quad P(Z=1) = \frac{3^1}{1!}e^{-3}$

$P(Z \geq 2) = 1 - e^{-3} - 3e^{-3} = \boxed{.8008}$

c) $P(X=1 \mid \frac{x+y}{2}=2) = P(x=1 \mid x+y=4) = \frac{P(x=1 \cap x+y=4)}{P(x+y=4)}$

if $x=1$ and $x+y=4$, $y=3$ so it becomes

$\frac{P(x=1 \cap y=3)}{P(x+y=4)} = \frac{P(x=1)\,P(y=3)}{P(x+y=4)}$ $\leftarrow$ from A

$P(x=1) = \frac{1^1}{1!}e^{-1}$ $\quad P(y=3) = \frac{2^3}{3!}e^{-2}$ $\quad P(x+y=4) = \frac{(3)^4}{4!}e^{-3}$

Our equation becomes:

$$\frac{e^{-1} \cdot \frac{4}{3}e^{-2}}{\frac{e^{-3}}{4!} \cdot 3^4} = \frac{32}{81} = \boxed{0.3951}$$

2a) X = number of chocolate chips per cookie.
Let's assume the average of 2 choc. chips and 1 marshmellow per in^3 holds. Let's also assume each cubic inch is independent from each other inch and follows a Poisson distribution.

Want $P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

The distribution is poisson with parameter $\lambda = 2 \cdot 3 = 6$

$$P(X \leq 4) = \frac{6^0}{0!} e^{-6} + \frac{6^1}{1!} e^{-6} + \frac{6^2}{2!} e^{-6} + \frac{6^3}{3!} e^{-6} + \frac{6^4}{4!} e^{-6}$$

$$= e^{-6}(1 + 6 + 18 + 36 + 54) = \frac{115}{e^6} = \boxed{0.2851}$$

b) Let M_i = number of marshmellows in cookie 1, 2, 3 where $i = 1, 2,$ or 3 respectively, saying the same for C_i = number of chocolate chips. M_i and C_i are independent and are Poisson. Consider the poisson variable $N_i := C_i + M_i$. To solve now for

P(at most one cookie has no marshmellows and choc. chips)

$N_i = 0$ if the cookie is missing both. Could also have no cookies missing both.

$= P(N_i > 0 \text{ for all } i) + P(N_1 = 0, N_2 \geq 0, N_3 \geq 0) +$
$P(N_1 > 0, N_2 = 0, N_3 \geq 0) + P(N_1 > 0, N_2 \geq 0, N_3 = 0)$

N_1 is poisson with parameter 6 as cookie 1 has 2 in^3

N_2, N_3 are poisson with parameter 9 as cookies 2, 3 have 3 in^3

N_1, N_2, N_3 are also all independent as C_i and M_i are independent

Answer is: Using Poisson formula of $P(N_i > 0) = (1 - P(N_i = 0))$

$P(N_i > 0) = (1 - P(N_i \neq 0))$ for each i $= (1 - e^{-6})(1 - e^{-9})(1 - e^{-9})$

$P(N_1 = 0, N_2 \geq 0, N_3 \geq 0) = e^{-6}(1 - e^{-9})^2$

$P(N_1 \geq 0, N_2 = 0, N_3 \geq 0) = P(N_1 \geq 0, N_2 \geq 0, N_3 = 0) = (1 - e^{-6})(1 - e^{-9})e^{-9}$

Sum is $(1 - e^{-6})(1 - e^{-9})^2 + e^{-6}(1 - e^{-9})^2 + 2(1 - e^{-6})(1 - e^{-9})e^{-9}$

$= \boxed{.999999373}$

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3 a) $Var(X) = E[X(X-1)] + E[X] - (E[X])^2$
$= E[X^2 - X] + E[X] - (E[X])^2$

By additivity of expected value:
$= E[X^2] - \cancel{E[X]} + \cancel{E[X]} - (E[X])^2$
$Var(X) = E[X^2] - (E[X])^2$

b) For Poisson distribution:
$E[X(X-1)] = \sum_{k=0}^{\infty} k(k-1) e^{-\mu} \frac{\mu^k}{k!}$ → by definition of $E(x)$
$= e^{-\mu} \sum_{k=0}^{\infty} \frac{k(k-1)\mu^k}{k(k-1)(k-2)!}$

multiply inside by μ^{-2} and outside by μ^2 gives:
$= e^{-\mu}\mu^2 \sum_{k=0}^{\infty} \frac{\mu^{k-2}}{(k-2)!} = e^{-\mu}\mu^2 \cdot e^{\mu} = \mu^2$

$E(x) = \mu$ and $[E(x)]^2 = \mu^2$
So $Var(X) = \mu^2 + \mu - \mu^2 = \boxed{\mu}$

4 a) We expect an average 1.8 hot dogs per 5 minutes, are 12 5 minute intervals in an hour, so average is $1.8 \times 12 = \boxed{21.6 \text{ hot dogs per hour}}$

b) From a, we know average per hour is 21.6, so can multiply by 8 hr day to give us
$\boxed{172.8 \text{ hotdogs}}$

c) We want the smallest H = number of hotdogs where H is a Poisson random variable such that $P(H \le J) \ge 0.5$ $\lambda_{15} = 3\lambda_5 = 3(1.8) =$
H has parameter 5.48 and is represented by the sum:
$P(H \le J) = \sum_{n=0}^{J} \frac{(5.4)^n}{n!} e^{-5.4}$ Via testing values of J, the first value of J with $P(H \le J) \ge 0.5$ is $J = 5$ means

$\boxed{\text{Barry needs 5 hotdogs on hand}}$

5 a) Since $f(x)$ is a density function

$\int_1^{\infty} \frac{c}{x^4} = 1$, to solve: $\lim_{a\to\infty} \int_1^a \frac{c}{x^4} dx = 1$

integral is $\lim_{a\to\infty} \frac{cx^{-4+1}}{-4+1} \Big|_1^a = \lim_{a\to\infty} \frac{-c}{3a^3} + \frac{c}{3} = 1$, $\frac{c}{3} = 1$ $c=3$

$\boxed{C=3}$

b) $E(x) = \int_1^{\infty} x f(x) dx$

$E(x) = \int_1^{\infty} x\left(\frac{3}{x^4}\right) dx = \int_1^{\infty} \frac{3}{x^3} dx = \frac{3x^{-2}}{-2} \Big|_1^{\infty} = \boxed{\frac{3}{2}}$

c) $Var(x) = E[x^2] - (E[x])^2$

$E[x^2] = \int_1^{\infty} x^2 \cdot \frac{3}{x^4} dx = \int_1^{\infty} \frac{3}{x^2} dx = 3$

$Var(x) = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \boxed{\frac{3}{4}}$

6 a) We know $P(X \leq 0) = \frac{1}{3}$ and $P(X \leq 1) = \frac{2}{3}$

X is normal with mean μ and SD σ. We can standardize X using the formula

$Z = \frac{X-\mu}{\sigma}$, for $x=0$, $Z = \frac{-\mu}{\sigma}$ $P(Z \leq \frac{-\mu}{\sigma}) = \frac{1}{3}$

and $P(X \leq 1) \rightarrow P(Z \leq \frac{1-\mu}{\sigma}) \leq \frac{2}{3}$

So on the normal table, no value of $\frac{1}{3}$, but we can find when value is $2/3$, and since area above that Z is $1/3$, by symmetry, $-Z$ is the value we are looking for.

So, for $P(Z \leq \frac{-\mu}{\sigma}) = \frac{1}{3}$ $Z \doteq .43$ closest to $2/3$,

so $z = -.43$ has $P(Z \leq -0.43) = \frac{1}{3}$

For $P(Z \leq \frac{1-\mu}{\sigma}) = 2/3$, we know it's 0.43.

$\frac{-\mu}{\sigma} = -0.43$ $\frac{1-\mu}{\sigma} = 0.43$, $\boxed{\mu = 0.5, \sigma = 1.163}$

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6b) If $P(X \le 1) = 3/4$, can standardize $P(X \le 0)$ and $P(X \le 1)$ using similar logic from part A. where $Z = \frac{X-\mu}{\sigma}$

$$P(X \le 0) \longrightarrow P(Z \le \frac{-\mu}{\sigma}) = \frac{1}{3}$$
$$P(X \le 1) \longrightarrow P(Z \le \frac{1-\mu}{\sigma}) = 3/4$$

From part a, we know $\frac{-\mu}{\sigma} = -0.43$.
Looking at the normal table, closest probability to $3/4$ is $Z = 0.67$ yields the system

$\frac{-\mu}{\sigma} = -0.43$ $\frac{1-\mu}{\sigma} = 0.67$ which has solutions $\boxed{\mu = 0.391, \sigma = 0.909}$

7a) Standardize measurements of 9.784 and 9.8 as is approximately normal distribution

$Z_{9.784} = \frac{9.784 - 9.78}{.0031}$ $Z_{9.8} = \frac{9.8 - 9.78}{.0031}$

$Z_{9.784} = 1.29$ $Z_{9.8} = 6.45$

Want to find $P(1.29 \le Z \le 6.45)$ where Z standard normal distribution.

$$P(1.29 < Z < 6.45) = \Phi(6.45) - \Phi(1.29)$$

Normal table covers up to $Z = 3.5$, where $\Phi(3.5) = .9998$ so $\Phi(6.45)$ is between .9998 and 1, but since so high, let's assumes it's approximately 1, so our $P \approx 1 - \Phi(1.29) \approx 1 - 0.9015 = .0985$

Probability measurement between 9.784 and 9.8 is .0985

b) Standardize 9.7794, $Z_{9.7794} = \frac{9.7784 - 9.78}{.0031}$

$P(Z \le -0.19) = \Phi(-0.19)$ $Z_{9.7794} = -0.19$

$\Phi(-0.19) = 1 - \Phi(0.19) = 1 - .5753 = .4247$

42.47% of measurements are less than 9.7794

7c) Want Z on normal table such that $\Phi(z) = 0.9$ there is no Z where this is true on the table, but it falls between $Z = 1.28$ and $Z = 1.29$. For this sake, let's say $Z = 1.29$ as don't know exact value.

$\boxed{\text{Measurement associated with } Z = 1.29 \text{ is } 9.784\text{cm from part A.}}$

8a) $P(X \le x)$ = P(distance from center to X is $\le x$)

First, we need to find the cdf.

$$\text{cdf} = \frac{\text{Volume(sphere w/ radius } x)}{\text{volume(sphere w/ radius } R)}$$

$$\text{cdf} = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi R^3} = \frac{x^3}{R^3}, \text{ so the pdf} = \frac{3x^2}{R^3}$$

$\text{cdf} = \frac{x^3}{R^3}$, meaning $\boxed{P(X \le x) = \frac{x^3}{R^3}}$

b) $P(X > SR)$. From A, we know the pdf is $\frac{3x^2}{R^3}$ and the cdf is $\frac{x^3}{R^3}$.

To find $P(X > SR)$, we know is bounded by $S = 1$, so we can write our equation as $P(SR < X < R)$, and we can use our pdf to solve this

$$P(SR < X < R) = \int_{SR}^{R} \frac{3x^2}{R^3}\,dx = \left.\frac{x^3}{R^3}\right|_{SR}^{R}$$

$$= \frac{R^3}{R^3} - \frac{(SR)^3}{R^3}$$

$$P(SR < X < R) = 1 - S^3$$

$$\boxed{P(X > SR) = 1 - S^3}$$