Tufts University
Department of Mathematics

Math 136 | **First Exam**[1] | **Spring 2021**

This test will be available to download from Gradescope starting 12:01 a.m. Eastern Time on Friday, March 5.

You may take the test in any two hour period on Friday, Saturday, or Sunday. You will have 120 minutes **from the time you download the test** *to finish it, and you will have 10 additional minutes to upload it to Gradescope (for a total of two hours and 10 minutes).*

You need to finish uploading the test by 11:59 p.m. on Sunday, March 7. Therefore, if you download the test after 9:49 p.m. on Sunday, March 7, you will not have the full time.
Please let Todd know if you have any problems uploading. (If you have a problem, it helps to take a screenshot indicating the time you tried to upload.)

Instructions: This is an open-book exam: you may refer to class lecture slides and videos that are on Canvas, the book, or any other book or public internet source. However, your answer must be *in your own words*; a straight copy (whether copied by hand or cut and pasted) or a close copy (some words changed) is not acceptable. If you are using a source from outside the course, you must name it. If you find a statement equivalent to something you are asked to prove, you may not just quote that statement; you need to provide a proof in your own words. And of course, *during the exam, you may not consult anyone other than the staff of the course.* If you have a question about something on the exam, email Todd and Jue. Emailing both of us will help insure a timely response. Here are the email addresses:

Instructor:

- Todd Quinto: todd.quinto@tufts.edu

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When you upload your answers to Gradescope, please scan all your answers into one PDF file starting with your signed cover page and with the answers clearly numbered and in order. Please identify the page(s) for each problem as you upload to Gradescope.

Please sign the following pledge and submit the signed pledge with your answers:
The Tufts University statement on academic integrity holds that: "Academic integrity is the joint responsibility of faculty, students, and staff. Each member of the community is responsible for integrity in their own behavior and for contributing to an overall environment of integrity at the university." I accept this responsibility, affirm that I am an honest person who can be trusted with to do the right thing, and certify that the work I will do on this exam is mine alone.
I pledge that I have neither given nor received assistance on this exam and have used only the reference sources I have cited in my answers.

Signature____________________

The test problems start on the next page.

Here are some of the definitions you might find useful from Math 135 and 136. Just note the Definition or Theorem number if you use it.

Theorem 1. *Let $A \subset \mathbb{R}^n$ and let $\mathbf{F} : A \to \mathbb{R}^m$. Let $\bar{\mathbf{x}}_0$ be a limit point of A. then,*

$$\lim_{\bar{\mathbf{x}} \to \bar{\mathbf{x}}_0} \mathbf{F}(\bar{\mathbf{x}}) = \bar{\mathbf{0}} \quad \textit{if and only if} \quad \lim_{\bar{\mathbf{x}} \to \bar{\mathbf{x}}_0} \|\mathbf{F}(\bar{\mathbf{x}})\| = 0$$

Theorem 2 (The $\epsilon - \delta$ Condition for Continuity)**.** *Let $C \subset \mathbb{R}^n$ and let $\mathbf{F} : C \to \mathbb{R}^m$. Let $\bar{\mathbf{x}}_0 \in C$. Then, $\mathbf{F}$ is continuous at $\bar{\mathbf{x}}_0$ if and only if*

$$\forall \epsilon > 0 \ \exists \delta > 0, \ \textit{if } \bar{\mathbf{x}} \in C \ \textit{and} \ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_0\| < \delta \ \textit{then} \ \|\mathbf{F}(\bar{\mathbf{x}}) - \mathbf{F}(\bar{\mathbf{x}}_0)\| < \epsilon.$$

Theorem 3 (The $\epsilon - \delta$ Condition for *Uniform* Continuity)**.** *Let $A \subset \mathbb{R}^n$ and let $\mathbf{F} : A \to \mathbb{R}^m$. Then, $\mathbf{F}$ is uniformly continuous on A if and only if*

$$\forall \epsilon > 0 \ \exists \delta > 0, \ \textit{if } \bar{\mathbf{x}} \textit{ and } \bar{\mathbf{y}} \textit{ are in } A \textit{ and } \|\bar{\mathbf{x}} - \bar{\mathbf{y}}\| < \delta \ \textit{then} \ \|\mathbf{F}(\bar{\mathbf{x}}) - \mathbf{F}(\bar{\mathbf{y}})\| < \epsilon.$$

Definition 4. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$. Let $x_0 \in I$. Then f is differentiable at x_0 if $\displaystyle\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. In this case, this limit is denoted $f'(x_0)$, the first derivative of f at x_0.

Definition 5. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$. Let $\bar{\mathbf{x}}_0 \in \mathcal{O}$. Then, $\mathbf{F}$ is differentiable at $\bar{\mathbf{x}}_0$ if there is an $m \times n$ matrix A such that

$$\lim_{\bar{\mathbf{h}} \to \bar{\mathbf{0}}} \frac{\left\|\mathbf{F}(x_0 + \bar{\mathbf{h}}) - \left[\mathbf{F}(\bar{\mathbf{x}}_0) + A\bar{\mathbf{h}}\right]\right\|}{\|\bar{\mathbf{h}}\|} = 0$$

In this case, the derivative matrix of $\mathbf{F}$, $\mathbf{DF}(\bar{\mathbf{x}}_0)$, is equal to A and the differential $\mathbf{dF}(\bar{\mathbf{x}}_0)$ is the linear transformation with standard matrix A.

1. (25 points) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable ($\mathbf{F} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$) and $\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

   Assume $\mathbf{F}(1,1,1) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{DF}(1,1,1) = A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. You may assume A is invertible and $A^{-1} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

   (a) Find $\dfrac{\partial v}{\partial x}(1,1,1)$ and tell how you found it.

   (b) Can one use the Inverse Function Theorem to solve for (x,y,z) locally as a function of (u,v,w) near $(x,y,z) = (1,1,1)$ and $(u,v,w) = (2,2,2)$? Justify your answer using theorems.

   (c) Use the information from problem 1b to find $\dfrac{\partial y}{\partial w}(2,2,2)$ and tell how you found it.

2. (25 points) Let A be an $m \times n$ matrix and let $\bar{\mathbf{b}} \in \mathbb{R}^m$. Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $\mathbf{F}(\bar{\mathbf{x}}) = A\bar{\mathbf{x}} + \bar{\mathbf{b}}$. Use the definition of differentiability, Definition 5, to prove that $\mathbf{F}$ is differentiable on $\mathbb{R}^n$, and for all $\bar{\mathbf{x}}_0 \in \mathbb{R}^n$, $\mathbf{DF}(\bar{\mathbf{x}}_0) = A$.

The test continues on the other side of this page.

3. (25 points) Consider the following definition:

Definition 6. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$. Then f is *uniformly differentiable on I* if there is a function $h : I \to \mathbb{R}$ such that for each $\epsilon > 0$ there is a $\delta > 0$ such that for all x and y in I,

$$\text{if} \quad |y - x| < \delta \quad \text{then} \quad |f(y) - [f(x) + h(x)(y - x)]| \le \epsilon |y - x| \tag{1}$$

Note that condition (1) *(like uniform continuity) is valid for* every *x and y in I that are sufficiently close–the same δ works for each ϵ.*

(a) Let $x \in I$. Assume f is uniformly differentiable according to Definition 6 Let $x \in I$. Prove that f is differentiable at x and that $f'(x) = h(x)$, the function satisfying (1).

(b) Now, let $f : I \to \mathbb{R}$ be uniformly differentiable. You may assume the result of part 3a, namely that $f' : I \to \mathbb{R}$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that for all x and y in I:

$$\text{if} \quad |y - x| < \delta \quad \text{then} \quad \left|f(y) - \left[f(x) + f'(x)(y - x)\right]\right| \le \epsilon |y - x| \tag{2}$$

Prove that the derivative $f'(x)$ is continuous on I.
HINT: Write down (2) and then switch the roles of x and y in (2).

(c) (3 points extra credit) Prove that f' is uniformly continuous on I.

4. (25 points) You will prove a version of the Implicit Function Theorem (generalization of Dini's Theorem) for functions on $\mathbb{R}^3$ following the proof Todd gave in the video proof of Dini's Theorem on Canvas. You may look at the video to do this proof, and the steps in the proof are outlined here.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ and let $\bar{\mathbf{0}} = (0, 0, 0) \in \mathcal{O}$. Let $f : \mathcal{O} \to \mathbb{R}$ be continuously differentiable (i.e., $f \in C^1(\mathcal{O}, \mathbb{R})$). Assume $\dfrac{\partial f}{\partial z}(\bar{\mathbf{0}}) \neq 0$. Define $\mathbf{F}(x, y, z) = \begin{pmatrix} x \\ y \\ f(x, y, z) \end{pmatrix}$. Then $\mathbf{F} : \mathcal{O} \to \mathbb{R}^3$.

(a) Explain why $\mathbf{F}$ is continuously differentiable on $\mathcal{O}$ (i.e., $\mathbf{F} \in C^1(\mathcal{O}, \mathbb{R}^3)$).

(b) Calculate $\mathbf{DF}(x, y, z)$ for $(x, y, z) \in \mathcal{O}$ and explain why $\mathbf{DF}(0, 0, 0)$ is an invertible matrix.

(c) Explain why $\mathbf{F}$ satisfies the Inverse Function Theorem near $\bar{\mathbf{x}}_0 = (0, 0, 0)$.

For the rest of the problem you may assume there is an open neighborhood $\mathcal{U}$ of $\bar{\mathbf{0}}$ and an open neighborhood $\mathcal{V}$ of $\mathbf{F}(\bar{\mathbf{0}}) = \bar{\mathbf{0}}$ such that $\mathbf{F} : \mathcal{U} \to \mathcal{V}$ is a C^1 bijection and $\mathbf{F}^{-1} : \mathcal{V} \to \mathcal{U}$ is a C^1 bijection.

It is now more convenient to denote points in $\mathcal{V}$ by (x, y, z).

You may assume that, if $(x, y, z) \in \mathcal{V}$, then $\mathbf{F}^{-1}(x, y, z) = \begin{pmatrix} x \\ y \\ h(x, y, z) \end{pmatrix}$ for some $h \in C^1(\mathcal{V}, \mathbb{R})$.

Let $\epsilon > 0$ such that $(-\epsilon, \epsilon)^3 \subset \mathcal{V}$ and define

$$g(x, y) = h(x, y, 0) \quad \text{for} \quad (x, y) \in (-\epsilon, \epsilon)^2.$$

(d) Now, explain why $\mathbf{F} \circ \mathbf{F}^{-1}(x, y, 0) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$.

(e) Now, use the fact that $\mathbf{F} \circ \mathbf{F}^{-1}(x, y, 0) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ to explain why $f(x, y, g(x, y)) = 0$ for $(x, y) \in (-\epsilon, \epsilon)^2$.