Tufts University
Department of Mathematics

Due date: 11:59 pm, Sunday, March 12, 2023 on Gradescope.

You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

A proper subset of the problems will be selected for grading.

Reading assignment: For this week, please read sections 18.1 and 18.2, integration in $\mathbb{R}^n$ and the statement that continuous functions are integrable on rectangles. This homework focusses on integration in $\mathbb{R}$

Problems:

1. (20 points) Let $a < b$ and $c \in (a, b)$. Assume $f : [a, b] \to \mathbb{R}$ is bounded and integrable on both $[a, c]$ and on $[c, b]$. Prove that f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$. Note that this is the converse of the theorem on additivity of the integral over intervals (Theorem 6.12) from the book.

   Solution: Let $\{P_n^{[a,c]}\}$ be an Archimedean sequence of partitions for f on $[a, c]$ and let $\{P_n^{[c,b]}\}$ be an Archimedean sequence of partitions for f on $[c, b]$. Then $\lim_{n\to\infty} U(f, P_n^{[a,c]}) - L(f, P_n^{[a,c]}) = 0$ and $\lim_{n\to\infty} U(f, P_n^{[c,b]}) - L(f, P_n^{[c,b]}) = 0$

   Now, $P_n^{[a,b]} = P_n^{[a,c]} \cup P_n^{[c,b]}$ is a partition of $[a, b]$ and

   $$L(P_n^{[a,b]}, f) = L(P_n^{[a,c]}, f) + L(P_n^{[c,b]}, f), \quad U(P_n^{[a,b]}, f) = U(P_n^{[a,c]}, f) + U(P_n^{[c,b]}, f) \tag{1}$$

   because the partition intervals of $P_n^{[a,b]}$ are the union of the partition intervals of $P_n^{[a,c]}$ and those of $P_n^{[c,b]}$ (and there is no overlap except at c). Putting this together, we see

   $$U(P_n^{[a,b]}, f) - L(P_n^{[a,b]}, f) = \left[U(P_n^{[a,c]}, f) - L(P_n^{[a,c]}, f)\right] + \left[U(P_n^{[c,b]}, f) - L(P_n^{[c,b]}, f)\right].$$

   Since $\{P_n^{[a,c]}\}$ and $\{P_n^{[c,b]}\}$ are Archimedean sequences on their intervals, the two terms on the right in braces converge to 0 as $n \to \infty$. Therefore, the difference on the left converges to 0. This shows f is integrable on $[a, b]$.

   We use (1) to calculate the integral on $[a, b]$:

   $$\int_a^c f + \int_c^b f = \lim_{n\to\infty} \left(L(f, P_n^{[a,c]}) + L(f, P_n^{[c,b]})\right) = \lim_{n\to\infty} L(f, P_n^{[a,b]}) = \int_a^b f.$$

   This finishes the proof.

2. (20 points) Let $f : [a, b] \to \mathbb{R}$ be bounded.

   (a) Assume f is continuous on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ *and* assume $f(x_0) > 0$ for some $x_0 \in [a, b]$. Prove that $\int_a^b f > 0$.

   (b) Is the conclusion in part (a) true if one assumes f is integrable on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ *and* assume $f(x_0) > 0$ for some $x_0 \in [a, b]$? If so, prove it, and if not, provide a counterexample

Solution:

(a) Since $f : [a, b] \to \mathbb{R}$ is continuous and $f(x_0) > 0$, there is a nontrivial closed interval $I = [x_1, x_2] \subset [a, b]$ that contains x_0 on which $f(x) > 0$ for all $x \in I$.
Since f is continuous, f has a minimum on I and since f is strictly greater than zero on I, this minimum $m_1(f) > 0$.
(If you were given the version of this theorem that states $f(x) \geq f(x_0)/2$ for $x \in I$, then you can use that without the EVT argument above.)
Now, let $P = \{a, x_1, x_2, b\}$. Since $\underline{\int}_a^b f$ is the supremum of the lower sums,

$$\int_a^b f = \underline{\int}_a^b f \geq L(f, P) = m_1(f)\Delta x_1 + m_2(f)\Delta x_2 + m_3(f)\Delta x_3 \geq 0 + \frac{f(x_0)}{2}\Delta x_2 + 0 > 0.$$

Here we use that $f(x) \geq 0$ on $[a, b]$ and that $f(x) \geq f(x_0)/2$ for $x \in [x_1, x_2]$. This shows $\int_a^b f > 0$.

(b) Conclusion is not true without f's continuity. Consider the function defined in problem 3(a) and check the solution of 3(a).

3. (20 points)Let f and g be bounded functions from $[a, b]$ to $\mathbb{R}$. Assume g is integrable and $f = g$ except at a finite number of points in $[a, b]$. In this problem, you will prove that f is integrable and $\int_a^b f = \int_a^b g$.

   (a) First prove that a function that is zero except at a has integral zero: Let $r \in \mathbb{R}$ constants. Define $f : [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} r & x = a \\ 0 & x \in (a, b] \end{cases}$. Prove f is integrable and $\int_a^b f = 0$. A similar proof can be used to prove $\int_a^b f = 0$ if f is equal to zero except at b.

   (b) Now, assume f is zero except for a finite number of points in $[a, b]$. Explain why $\int_a^b f = 0$. HINT: Problem 1 and part (a) could be helpful in one proof.

   (c) Let f and g be functions from $[a, b]$ to $\mathbb{R}$. Assume g is integrable, and assume $f = g$ except at a finite number of points. Prove that f is integrable on $[a, b]$ and $\int_a^b f = \int_a^b g$.

Solution:

(a) WLOG, we assume $r > 0$, consider the regular partition $P_n = \{a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \ldots, \frac{(n-1)(b-a)}{n}, b\}$
On each interval $\Delta_i = [a + \frac{(i-1)(b-a)}{n}, a + \frac{i(b-a)}{n}]$,
for $m_i = min\{f(x)|x \in \Delta_i\}, M_i = max\{f(x)|x \in \Delta_i\}$,except $M_0 = r$, other m_i, M_i's are 0
Hence $0 = \sum_{i=1}^{n}(m_i\frac{b-a}{n}) = L(f, P_n) \leq \underline{\int}_a^b f \leq \overline{\int}_a^b f \leq U(f, P_n) = \sum_{i=1}^{n}(M_i\frac{b-a}{n}) = \frac{(b-a)r}{n}$
Since it's seen that $|U(f, P_n) - L(f, P_n)| \to 0$ as $n \to \infty$, f is integrable and $\int_a^b f = \lim_{n\to\infty}(U(f, P_n) = 0)$

(b) You can assume f is zero except at $a_1, ...a_k$ in [a,b].And we set $a_0 = a, a_{k+1} = b$
by problem 1 we have: $\int_a^b f = \sum_{i=1}^{k+1} \int_{a_{i-1}}^{a_i} f$.
since on each $[a_{i-1}, a_i]$, f is zero except for one point, so by (a) part we have $\int_{a_{i-1}}^{a_i} f = 0$ for each i, thus $\int_a^b f = 0$

(c) Define $h = f - g$, since f=g except at a finite number of points, then h is zero except for a finite number of points. We can apply part (b) conclusion that h is integrable and $\int_a^b h = 0$. By linearity of Integral(Thm. 6.15 of Fitzpatrick),
we have f=g+h is integrable and $\int_a^b f = \int_a^b g + \int_a^b h = \int_a^b g$.

4. (20 points) Let $f : [0, 2] \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$ for $x \in [0, 2]$. Prove that f is integrable on $[0, 2]$ and find the integral using the definition of the integral or by using the Archimedean Riemann Theorem.

HINT: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Solution: Following the example 6.11 of Fitzpatrick, we use regular partition of $[0, 2]$, $x_i = \frac{2i}{n}, i = 0, 1, 2, ..., n$

$M_i = \sup\{f(x)|x \in [x_{i-1}, x_i]\} = (\frac{4i^2}{n^2} + 1)$,$m_i = \inf\{f(x)|x \in [x_{i-1}, x_i]\} = (\frac{4i^2-4i+2}{n^2} + 1)$,

$U(f, P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n}((\frac{4i^2}{n^2} + 1)\frac{2}{n}) = \sum_{i=1}^{n} \frac{8i^2}{n^3} = 2 + \frac{8n(n+1)(2n+1)}{6n^3} + 2$ by using hint.

$L(f, P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n}((\frac{4i^2 - 4i + 2}{n^2} + 1)\frac{2}{n}) = \sum_{i=1}^{n} \frac{8i^2}{n^3} - 8\frac{\sum_{i=1}^{n} i}{n^3} + \frac{\sum_{i=1}^{n} 4}{n^3} = \frac{8n(n+1)(2n+1)}{6n^3} - 8\frac{n(n+1)}{2n^3} + \frac{4}{n^2} + 2$ by using hint.

By taking limit for $n \to \infty$, $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = 2 + \frac{8}{3} = \frac{14}{3}$. So by Archimedean Riemann Theorem, f is integrable and its integral is $\frac{14}{3}$

5. (20 points) Let f be an integrable function from $[a, b]$ to $\mathbb{R}$. In this problem, you will show that the absolute value of f, $|f|$, is integrable and $\left|\int_a^b f\right| \le \int_a^b |f|$.

(a) Show $|f|$ is integrable
You may assume that for every partition P of $[a, b]$,

$$0 \le U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P). \tag{2}$$

(b) Now show $\left|\int_a^b f\right| \le \int_a^b |f|$.

Solution:

(a) By hint for every partition P of $[a, b]$,$0 \le U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P)$
By Archimedean Riemann Theorem, since f is integrable on $[a, b]$, $\exists$ a sequence of partitions P_n such that $lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$
Then by sandwiches theorem, $lim_{n\to\infty}(U(|f|, P_n) - L(|f|, P_n)) = 0$. Thus by Archimedean Riemann Theorem again, $|f|$ is integrable on $[a, b]$.

(b) See Fitzpatrick Thm. 6.15 and Corollary 6.16.
Since for all points $x \in [a, b]$, $-|f(x)| \le f(x) \le |f(x)|$,
the reasoning by the linearity of Integral(Thm. 6.15) and Corollary 6.16.

Here are optional challenge problems that will give you extra points if you successfully do them. Todd will grade them.

1. (3 points extra credit) Prove the inequality (2) for arbitrary $f : [a,b] \to \mathbb{R}$.

   In one proof, it helps to first prove for any set $I \subset [a,b]$ and all $x \in I$, $y \in I$ that $f(x) - f(y) \le \sup_I f - \inf_I f$ and that $f(y) - f(x) \le \sup_I f - \inf_I f$.

   Answer: Let I be a nonempty subset of $[a,b]$ and let x and y be in I. Then $f(y) \le \sup_I f$ and $f(x) \ge \inf_I f$ by definition of sup and inf. If we subtract this, we see $f(y) - f(x) \le \sup_I f - \inf_I f$ and switching x and y in this argument, we see $f(x) - f(y) \le \sup_I f - \inf_I f$. Therefore $|f(y) - f(x)| \le \sup_I f - \inf_I f$. Now using the "reverse" triangle inequality,

   $$|f(y)| - |f(x)| \le |f(y) - f(x)| \le \sup_I f - \inf_I f$$

   Fix $x \in I$ and rearrange the above inequality, then for every $y \in I$, $|f(y)| \le |f(x)| + \sup_I f - \inf_I f$. Therefore, $|f(x)| + \sup_I f - \inf_I f$ is an upper bound for $\{|f(y)| \mid y \in I\}$. So $\sup_I |f| \le |f(x)| + \sup_I f - \inf_I f$. Using a similar argument for $x \in I$, one proves that

   $$\sup_I |f| - \inf_I |f| \le \sup_I f - \inf_I f.$$

   Now, let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. By letting $I = [x_{i-1}, x_i]$ in this inequality, one sees

   $$0 \le \sum_{i=1}^n \sup_{[x_{i-1},x_i]} |f|\Delta x_i - \sum_{i=1}^n \inf_{[x_{i-1},x_i]} |f|\Delta x_i \le \sum_{i=1}^n \sup_{[x_{i-1},x_i]} f\Delta x_i - \sum_{i=1}^n \inf_{[x_{i-1},x_i]} f\Delta x_i$$

   and this proves the result.

2. (2 points extra credit) We know if f and g are integrable that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. Prove or find counterexamples to the following.

   (a) If f and g are arbitrary bounded functions is it true that $\overline{\int}_a^b (f+g) = \overline{\int}_a^b f + \overline{\int}_a^b g$?

   Answer: No! (Since it *is* true for integrable functions, we should consider a non-integrable function.)

   Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet salt and pepper function, $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$.

   Then, $\overline{\int}_0^1 f = 1$ and by a similar argument $\overline{\int}_0^1 (-f) = 0$.

   So, $1 = \overline{\int}_0^1 f + \overline{\int}_0^1 (-f)$ but $\overline{\int}_0^1 (f + -f) = \overline{\int}_0^1 0 = 0$.

   (b) If $f$and g are arbitrary bounded functions is it true that $\underline{\int}_a^b (f+g) = \underline{\int}_a^b f + \underline{\int}_a^b g$?

   Answer: No (and this is sort of a freebee :-))! Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet salt and pepper function, $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$. Then, $\underline{\int}_0^1 f = 0$ and by a similar argument $\underline{\int}_0^1 (-f) = -1$.

   So, $-1 = \underline{\int}_0^1 f + \underline{\int}_0^1 (-f)$ but $\underline{\int}_0^1 (f + -f) = \underline{\int}_0^1 0 = 0$.