Tufts University
Department of Mathematics

Math 136 **Solutions Homework 4** [1] **Spring, 2023**

You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

Reading assignment:

- *By Wednesday, February 15*, please read Sections 15.2, 15.3, 16.1, and the statement of the General Inverse Function Theorem in section 16.3. Also please read section 17.1 and Dini's theorem.

  The first part of this homework covers Sections 15.2, 15.3, the Inverse Function Theorem and Dini's Theorem.

- *By Thursday, February 23* We now start integration! Please read Sections 6.1-3.

  The second part of this homework covers section 6.1.

Facts from linear algebra:
Let A be an $m \times n$ matrix.

I. If $m < n$ then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

II. If $m > n$ then $A\mathbf{x} = \mathbf{y}$ does not have a solution for all $\mathbf{y} \in \mathbb{R}^m$.

III. If $m = n$ (so A is a square matrix), then the following are equivalent

- A is an invertible matrix
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- $A\mathbf{x} = \mathbf{y}$ has a solution for every $\mathbf{y} \in \mathbb{R}^n$.

Problems:

1. (10 points) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $F(x,y,z) = \begin{bmatrix} x^2y \\ e^{xy} \\ \sin(x^2+yz) \end{bmatrix}$. Explain why $F \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ and find the derivative matrix $Df(x,y,z)$.

   Solution: You may assume polynomial functions in $\mathbb{R}^n$ have continuous derivatives of all orders on $\mathbb{R}^n$. you may assume the elementary functions, e^x, and trig functions all have continuous derivatives of all orders on their domains.

   Let $F = (F_1, F_2, F_3)^T$ where the T is the transpose, turning the row vector I wrote into a column vector. Then, F_2 is the composition $(x,y) \mapsto xy \mapsto e^{xy}$ and the composition of C^2 functions is C^2 by the Chain Rule. Therefore, the composition of C^2 functions is C^2 (try proving this!). Similarly F_3 is C^2. now, we calculate DF:

$$Df(x,y,z) = \begin{bmatrix} \nabla F_1(x,y,z) \\ \nabla F_2(x,y,z) \\ \nabla F_3(x,y,z) \end{bmatrix} = \begin{bmatrix} 2xy & x^2 & 0 \\ ye^{xy} & xe^{xy} & 0 \\ 2x\cos(x^2+yz) & z\cos(x^2+yz) & y\cos(x^2+yz) \end{bmatrix}$$

2. (15 points) Let f, g be twice continuously differentiable on $\mathbb{R}$ and define

$$F(x,t) = f(x+ct) + g(x-ct)$$

for $(x,t) \in \mathbb{R}^2$ and $c > 0$. Use the Chain Rule to show that F satisfies the *wave equation*:

$$c^2 \tfrac{\partial^2}{\partial x^2} F(x,y) = \tfrac{\partial^2}{\partial t^2} F(x,t).$$

If x is position and t is time, then the functions f and g are called travelling waves, g travelling to the right with speed c as t increases and f travelling to the left. Cool, eh!

Solution: We successively calculate partial derivatives of F using the Chain Rule each time since $F \in C^2(\mathbb{R}^2)$ since F is the sum of the composition of C^2 functions: $(x,t) \mapsto x+ct \mapsto f(x+ct)$ and $(x,t) \mapsto x-ct \mapsto g(x-ct)$. So:

$$\frac{\partial f}{\partial x}(x+ct) = f'(x+ct) \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x+t) = f''(x+ct)$$
$$\frac{\partial g}{\partial t}(x-ct) = -cg'(x-ct) \qquad \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)(x-ct) = c^2 g''(x-ct)$$

Using these expressions for second derivates, we see

$$c^2 \frac{\partial^2}{\partial x^2} F(x,t) = c^2\left(f''(x+ct) + g''(x-ct)\right) = c^2 f''(x+ct) + c^2 g''(x-ct) = \frac{\partial^2}{\partial t^2} F(x,t)$$

and F satisfies the wave equation.

3. (10 points) Problem # 5 on p. 428.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and let $c > 0$. We assume $f'(x) \geq c$ for all $x \in \mathbb{R}$. You need to prove $f : \mathbb{R} \to \mathbb{R}$ is bijective.

Injectivity (one-to-one): Since $f'(x) > 0$ for all $x \in \mathbb{R}$, f is strictly increasing by Corollary 4.21 (p. 105). Therefore, f is injective.

Surjectivity (onto): We have to use the fact that $f'(x) \geq c$ for all x, so f is increasing at a rate greater than or equal to c.

We will first show that values of f are arbitrarily large or arbitrarily small and then use the Intermediate Value Theorem.

Let $M > 0$ be arbitrary. We prove there is an $r > 0$ such that $f(r) \geq f(0) + M$ and $f(-r) \leq f(0) - M$.

Here goes! Let $r = M/c$. $r > 0$ as both M and c are positive. Since f is differentiable on $\mathbb{R}$ so continuous on $\mathbb{R}$, we use the MVT to conclude for some $d \in (0,r)$

$$f(r) - f(0) = rf'(d) \geq rc = M, \quad \text{therefore} \quad f(r) \geq M + f(0).$$

Now, using the analogous argument on the interval $[-r, 0]$, for some $s \in (-r, 0)$

$$f(-r) - f(0) = (-r)f'(s) \leq (-r)c = -M, \quad \text{therefore} \quad f(-r) \leq f(0) - M.$$

Since f is continuous on $\mathbb{R}$, we can use the IVT to conclude the interval $[f(-r), f(r)]$ is in the range of f. Therefore,

$$[f(0) - M, f(0) + M] \subset [f(-r), f(r)] \subset f(\mathbb{R}).$$

Since M is arbitrary, f is surjective.

4. (10 points) Problem # 9 on p. 428. The function in this problem shows that a global version of the inverse function theorem is not true in general: This problem gives an example of a function $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ such that $JF(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$, but $F : \mathbb{R}^2 \to \mathbb{R}^2$ is not injective on $\mathbb{R}^2$. In contrast if I is an open interval in $\mathbb{R}$, $f \in C^1(I, \mathbb{R})$, and f' is never zero then f *is* globally injective as you proved on homework.

   Solution:

   (a) $DF(x_0, y_0) = \begin{pmatrix} e^{x_0}\cos y_0 & -e^{x_0}\sin y_0 \\ e^{x_0}\sin y_0 & e^{x_0}\cos y_0 \end{pmatrix}$

   The determinant of this matrix is $e^{2x_0}\cos^2 y_0 + e^{2x_0}\sin^2 y_0 = e^{2x_0}$ It follows that the Inverse Function Theorem applies at every point (x_0, y_0)

   (b) $F(0,0) = F(0, 2\pi) = (1,0)$ so F is not injective.

   (c) No. The Inverse Function Theorem only guarantees that F is bijective (hence injective) locally

5. (20 points) Let B be an $m \times n$ matrix and let $\mathbf{a} \in \mathbb{R}^m$. Define $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(\mathbf{x}) = \mathbf{a} + B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. F is called an *affine function*

   (a) Find the derivative matrix DF and explain why F has continuous first derivatives (i.e., $F \in C^1(\mathbb{R}^n, \mathbb{R}^m)$).

   Solution: Let b_{ij} be the entry of B in the i^{th} row and j^{th} column and let F_i be the i^{th} component function of F, so $F = (F_1, \dots, F_n)$. Let $\mathbf{a} = (a_1, \dots, a_n)$. Then,

   $$F_i(\mathbf{x}) = a_i + (b_{i1}, b_{i2}, \dots, b_{in})\mathbf{x} = a_i + b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{ij}x_j + \cdots b_{in}x_n$$

   so $\frac{\partial F_i}{\partial x_j} = b_{ij}$. Therefore, $DF(\mathbf{x}) = B$. Since the entries of $DF(\mathbf{x}) = B$ are constant functions, they are continuously differentiable.

   Now assume $m = n$. In this case, $F : \mathbb{R}^n \to \mathbb{R}^n$ and therefore $\mathbf{a} \in \mathbb{R}^n$ and $B \in M_{n \times n}$.

   (b) Find conditions on B for which F is bijective (one-to-one and onto). **Solution:**

   Injectivity: Assume $F(\mathbf{x}) = F(\mathbf{y})$. Then, $\mathbf{a} + B\mathbf{x} = \mathbf{a} + B\mathbf{y}$ or $B(\mathbf{x} - \mathbf{y}) = \mathbf{0}$. Therefore, F is injective if and only if the homogenous equation $B\mathbf{w} = 0$ has only the trivial solution $\mathbf{w} = 0$ (so $\mathbf{x} - \mathbf{y}$ must be $\mathbf{0}$). Fact III at the start of the problem set asserts this holds IFF B is an invertible matrix.

   Surjectivity: F is surjective if and only if for all $\mathbf{z} \in \mathbb{R}^n$, there is a $\mathbf{x} \in \mathbb{R}^n$ such that

   $$F(\mathbf{x}) = \mathbf{a} + B\mathbf{x} = \mathbf{z}.$$

   This is equivalent to the equation

   $$B\mathbf{x} = \mathbf{z} - \mathbf{a}$$

   has a solution for all $\mathbf{z} \in \mathbb{R}^n$. By Fact III, this occurs if and only if B is an invertible matrix.

   So F is bijective if and only if B is an invertible matrix.

   Now, let B be an $m \times n$ matrix, and let $\mathbf{a} \in \mathbb{R}^m$. Define $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(\mathbf{x}) = \mathbf{a} + B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (c) Assume $m > n$. Prove that F is not surjective. **Solution:** By Fact II, there is some $\mathbf{y} \in \mathbb{R}^m$ such that $B\mathbf{x} = \mathbf{y}$ does not have a solution. Therefore, $F(\mathbf{x}) = \mathbf{y} + \mathbf{a}$ has no solution, and F is not surjective.

   (d) Assume $m < n$. Prove that F is not injective.

   Solution: By Fact I, $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions. So, assume $B\mathbf{x}_0 = \mathbf{0}$. Therefore, $F(\mathbf{x}_0) = F(\mathbf{0}) = \mathbf{a}$ and F is not injective.

   This explains why one can't expect an Inverse Function Theorem for $F \in C^1(\mathcal{O}, \mathbb{R}^m)$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and $m \neq n$.

6. (10 points) Problem #2 on p. 447. Note: The hypothesis of Dini's theorem fails at $(0,0)$ and there are two curves $g_1(x) = x$ and $g_2(x) = -x$ on the set $f(x,y) = 0$ that intersect at $(0,0)$. However, under the hypotheses of Dini's Theorem, there would be only one such curve (see class notes and the end of the statement of Dini's theorem in Fitzpatrick).

   Solution:

   (a) $\nabla f(x,y) = (2x(x^2-y^2)+2x(x^2+y^2-2), 2y(x^2-y^2)-2y(x^2+y^2-2)) = (4x(x^2-1), -4y(y^2-1))$
   It is clear that
   $\nabla f(0,0) = \nabla f(1,1) = \nabla f(1,-1) = \nabla f(-1,-1) = \nabla f(-1,1) = (0,0)$.
   Therefore, we cannot apply Dini's theorem.

   (b) The graph is the union of the circle centered at $(0,0)$ and radius $\sqrt{2}$ and the line $y = \pm x$. It can be seen that the near $(0,0) f(x,y) = 0$ if and only $(x,y) = (0,0)$. So the conclusion of Dini's theorem fail. You can draw the same conclusion at the other points.

   The next problems are on integration for $f : [a,b] \to \mathbb{R}$.

7. (10 points) Problem # 3 on p. 141.

   Solution: Let $f : [a,b] \to \mathbb{R}$ be integrable and assume $f(q) = 0$ for all rational numbers $q \in [a,b]$.

   Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$.

   We first show $L(f,P) \leq 0 \leq U(f,P)$. Since there is a rational number q_i in the i^{th} interval of P and $f(q_i) = 0$,

$$m_i(f) = \inf_{[x_{i-1},x_i]} f \leq f(q_i) = 0 \leq \sup_{[x_{i-1},x_i]} f = M_i(f).$$

   Therefore,

$$L(f,P) = \sum_{i=1}^{n} m_i(f)\Delta x_i \leq \sum_{i=1}^{n} f(q_i)\Delta x_i = 0 \leq \sum_{i=1}^{n} M_i(f)\Delta x_i = U(f,P).$$

   Since $L(f,P) \leq 0$ for all partitions P of $[a,b]$, the supremum over partitions is non-positive (0 is an upper bound so the least upper bound must be less than or equal to 0), $\underline{\int}_a^b f \leq 0$.

   Similarly, since $U(f,P) \geq 0$ for all partitions P of $[a,b]$, the infimum over all partitions is nonnegative, $\overline{\int}_a^b f \geq 0$.

8. (15 points)

   (a) Do Problem # 5 on p. 141.
   Solution: Let f and g be bounded functions on $[a,b]$ and assume $f(x) \leq g(x)$ for all $x \in [a,b]$. We show $\underline{\int}_a^b f \leq \underline{\int}_a^b g$ by first showing $L(f,P) \leq L(g,P)$ for every partition P of $[a,b]$.
   To answer part (a) of # 5 on p. 141, we let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ and let $I_j = [x_{j-1}, x_j]$ be the j^{th} subinterval of P. Since $f(x) \leq g(x)$ for all $x \in I_j$, $m_j(f) \leq m_j(g)$.
   (Here is the reason that $m_j(f) \leq m_j(g)$: $m_j(f) \leq f(x) \leq g(x)$ for all $x \in I_j$.
   Therefore $m_j(f)$ is a lower bound for the set of $g(x)$ for $x \in I_j$. Therefore, $m_j(f) \leq m_j(g)$ the greatest lower bound of g on I_j.)
   Now, by summing for $j = 1$ to n we see

$$L(f,P) = \sum_{j=1}^{n} m_j(f)\Delta x_j \leq \sum_{j=1}^{n} m_j(g)\Delta x_j = L(g,P).$$

To answer part (b) of # 5 on p. 141, we do a sup argument for the lower and upper integrals. As

$$\underline{\int_a^b} g = \sup\left\{L(g,P) \,\middle|\, P \text{ is a partition of } [a,b]\right\},$$

we have for each partition P of $[a,b]$, $L(f,P) \le L(g,P) \le \underline{\int_a^b} g$.

This shows that $\underline{\int_a^b} g$ is an upper bound for the set $\left\{L(f,P) \,\middle|\, P \text{ is a partition of } [a,b]\right\}$. Therefore, the least upper bound of this set, $\underline{\int_a^b} f$, is less than or equal to this upper bound. That is, $\underline{\int_a^b} f \le \underline{\int_a^b} g$.

(b) Use the result of part (a) to show that if f and g are integrable and $f(x) \le g(x)\ \forall x \in [a,b]$, then $\int_a^b g \le \int_a^b f$.

Solution: Now assume both f and g are integrable on $[a,b]$. This means that their integrals are equal to their lower integrals. Therefore,

$$\int_a^b f = \underline{\int_a^b} f \le \underline{\int_a^b} g = \int_a^b g.$$