Monday, January 30

Friday, January 27, 2023 11:19

TA Help session 10:30 Fridays, Math library, JCC 574
Student hours with Todd 1:30-3:00 my office JCC 575 (end of hall)

Student hours will start on Friday at 2:00 and we can continue to 3:30 (because of AWM panel and lunch)

MATHEMATICAL CONTEST IN MODELING: February 16-20, 2023. TEAMS OF THREE UNDERGRADS
https://www.contest.comap.com/undergraduate/contests/

DIRECTED READING PROGRAM: grad student and undergrad read a math book or article and learn about it together
A list of projects and descriptions can be found here: https://drive.google.com/file/d/1ffyVId43yPtFP-9GiODrtHf3ZlJ2Nc2S/view?usp=sharing

Application: https://forms.gle/P46BCsEKvdnzftLo9

Save the date! AWM Panel & Lunch with Malena Espanol
Friday February 3rd at 1pm in JCC 501

Malena Espanol is an assistant professor in the school of Mathematical and Statistical Sciences at Arizona State University. She earned a Ph.D. in math from Tufts in 2009. The Tufts AWM chapter is excited to host Dr. Espanol for a Q&A over lunch! Everyone in the Tufts community is welcome to join.

Please RSVP at https://tufts.qualtrics.com/jfe/form/SV_0cR5K8g15jJQ7eC

Review
Def. directional deriv $\Theta \subset \mathbb{R}^n$ op.

Review

Defn directional deriv $\mathcal{O} \subset \mathbb{R}^n$ open

$f: \mathcal{O} \to \mathbb{R}$ $\quad \bar{x} \in \mathcal{O} \quad \bar{h} \in \mathbb{R}^n \setminus \{0\}$

$$\frac{\partial f}{\partial \bar{h}}(\bar{x}) = D_{\bar{h}} f(\bar{x}) = \lim_{t \to 0} \frac{f(\bar{x} + t\bar{h}) - f(\bar{x})}{t}$$

$$= \frac{d}{dt} f(\bar{x} + t\bar{h}) \Big|_{t=0}$$

if it $\exists$

std basis of $\mathbb{R}^n$

$\bar{e}_j = (0, \dots, 0, 1, 0, \dots, 0) \quad j \in \{1, \dots, n\}$

$$\frac{\partial f}{\partial x_j} = D_{\bar{e}_j} f$$

dir deriv of f in dir $\bar{e}_j$

eg $\quad f: \mathbb{R}^2 \to \mathbb{R} \quad f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x,y) \neq \bar{0} \\ 0 & (x,y) = \bar{0} \end{cases}$

recall f not cont at $\bar{0}$

let $\bar{h} = (a,b)$ find $\dfrac{\partial f}{\partial \bar{h}}(0,0)$ if it $\exists$

$\bar{h} \neq (0,0)$

Use def

$$D_{\bar{h}} f(0,0) = \lim_{t \to 0} \frac{f((0,0) + t(a,b)) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(ta, tb) - 0}{t} = \lim_{t \to 0} \frac{1}{t} \frac{(ta)^2 (tb)}{(ta)^4 + (tb)^2}$$

$$= t^4 a^4 + t^2 b^2$$

$$= \lim_{t \to 0} \frac{1}{\cancel{t}} \left(\frac{\cancel{t^3} a^2 b}{\cancel{t^2} (t^2 a^4 + b^2)} \right)$$

$$= \lim_{t \to 0} \frac{a^2 b}{t^2 a^4 + b^2} = \frac{a^2}{b}. \qquad \bar{h} = (a,b) \neq (0,0)$$

consider if $b = 0$ get $\lim\limits_{t \to 0} 0 \quad \therefore$

consider if $b=0$ get $\lim_{t\to 0} 0$ ∴

$$D_{\bar h} f(0,0) = \begin{cases} \frac{a^2}{b} & b \neq 0 \\ 0 & b = 0 \end{cases}$$

$\bar h = (a,b)$

Note: f has 1st partials everywhere but f is not cont at $(0,0)$

Defn $f : \mathcal{O} \to \mathbb{R}$, $\mathcal{O}$ open in $\mathbb{R}^n$
f is continuously differentiable $f \in C^1(\mathcal{O})$ if f has all 1st partial deriv's on $\mathcal{O}$ and they are cont on $\mathcal{O}$

Thm $\mathcal{O} \subset \mathbb{R}^n$ open, if $f \in C^1(\mathcal{O})$ then f is cont.

eg $f(x,y) = \sin(xy)$ is in $C^1(\mathbb{R}^2)$ (cont. diff)

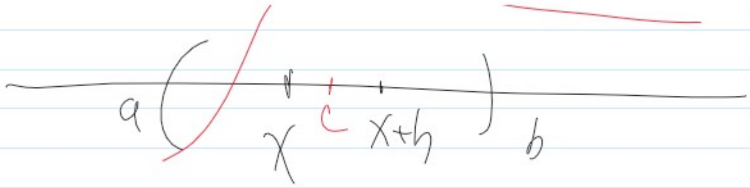
as $\frac{\partial f}{\partial x} = y \cos xy$, $\frac{\partial f}{\partial y} = x \cos xy$ both cont.

eg $f(x,y)$ in last ex has all 1st partials everywhere but is not C^1 as f is discont. at $(0,0)$

Mean Value thm in $\mathbb{R}^n$

recall MVT in $\mathbb{R}$ $f : (a,b) \to \mathbb{R}$

$x \in (a,b)$ $h \neq 0$ $x+h$



$f(x+h) - f(x) = f'(c)\,h$ for some c between x and $x+h$

gen to $\mathbb{R}^n$ $f: \mathcal{O} \to \mathbb{R}$

$\bar{x} \in \mathcal{O}$ $\bar{h} \in \mathbb{R}^n$ $\bar{x}+\bar{h} \in \mathcal{O}$ ...?

write $f(\bar{x}+\bar{h}) - f(\bar{x}) = \text{deriv} \cdot \bar{h}$

??

Mean value Proposition

$r > 0$ $\bar{x} \in \mathbb{R}^n$ assume $f: B_r(\bar{x}) \to \mathbb{R}$

has all 1^{st} partial deriv assume

$0 < \|\bar{h}\| < r$ $\bar{h} \in \mathbb{R}^n$

then $\exists$ points $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$

in $B_r(\bar{x})$ s.t.

$$f(\bar{x}+\bar{h}) - f(\bar{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{z}_i)\, h_i$$

ball radius $\|h\|$ and $\|\bar{z}_i - \bar{x}\| < \|h\|$ $\bar{h} = (h_1, h_2, \ldots h_n)$

pf Big idea use 1 din MVT in each variable of in $\mathbb{R}^2$

$\bar{x} = (x_1, x_2)$ $\bar{h} = (h_1, h_2)$

$\bar{x}+\bar{h}$

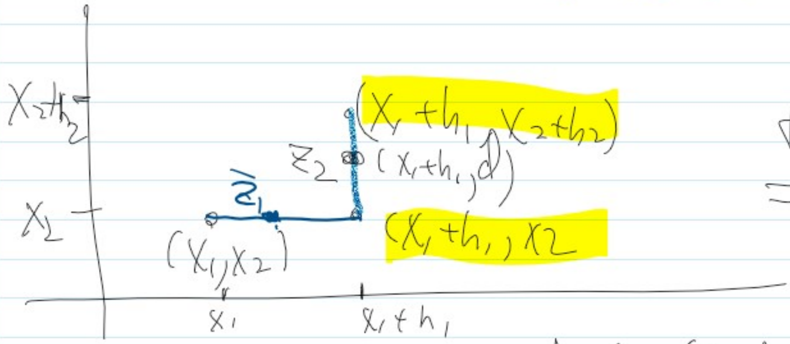
$f(x_1+h_1, x_2+h_2) - f(x_1+h_1, x_2)$

$+ f(x_1+h_1, x_2) - f(x_1, x_2)$

1 din do MVT in x_2 variable

do 1 din MVT in x_1

2D 1d MVT in X_1



$f(X_1+h_1, X_2+h_2) - f(X_1+h_1, X_2)$

$= \frac{\partial f}{\partial X_2}(X_1+h_1, d)(h_2)$

some d between X_2, X_2+h_2

Let $(X_1+h_1, d) = \bar{z}_2$

$\therefore$ $f(X_1+h_1, X_2+h_2) - f(X_1+h_1, X_2)$

$= \frac{\partial f}{\partial X_2}(\bar{z}_2)h_2$

Similarly $f(X_1+h_1, X_2) - f(X_1, X_2) = \frac{\partial f}{\partial X_1}(\bar{z}_1)\ h_1$

$\therefore$ $f(X_1+h_1, X_2+h_2) - f(X_1, X_2) = \frac{\partial f}{\partial X_1}(\bar{z}_1)h_1 + \frac{\partial f}{\partial X_2}(\bar{z}_2)h_2$

and $\|\bar{z}_j - \bar{X}\| < \|h\|$ $\quad j = 1, 2$

<u>Defn</u> $f : \mathcal{O} \subset \mathbb{R}$ $\quad \mathcal{O} \subset \mathbb{R}^n$ open

f has all 1st partial derivs on $\mathcal{O}$

define <u>gradient of f</u> to be

$$\nabla f(\bar{X}) = \left(\frac{\partial f}{\partial X_1}(\bar{X}), \frac{\partial f}{\partial X_2}(\bar{X}), \cdots \frac{\partial f}{\partial X_n}(\bar{X})\right)$$

for $\bar{X} \in \mathcal{O}$

ex $\nabla(X^2 y + \sin(xy))$

$= (2xy + y\cos(xy),\ x^2 + x\cos(xy))$

<u>Thm</u> $\mathcal{O}$ Open in $\mathbb{R}^n$ $f : \mathcal{O} \to \mathbb{R}$

continuously diff ; $f \in C^1(\mathcal{O})$

continuously "diff" ; $f \in C^1(\theta)$

then if $\bar{h} \in \mathbb{R}^n \sim \{0\}$

$$D_{\bar{h}} f(\bar{x}) = \nabla f(\bar{x}) \cdot \bar{h}$$

$\bar{x} \in \theta$