Math 70 Wksht 7

1 a) For T to be a linear transformation, $T(x+y) = T(x) + T(y)$, and $T(c\vec{u}) = cT(\vec{u})$, where $c \in \mathbb{R}$

First, prove for $T(A)$, that $T(A+B) = T(A) + T(B)$. Let A and B be matrices both of size $m \times n$ so

$$A = \begin{bmatrix} a_{1} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$m \times 3 \quad 3 \times 1$

$$T(A+B) = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

For this to be multipliable, $n = 3$ for a and b and $m = 3$

$$= \begin{bmatrix} a_{11}+b_{11} + 3(a_{12}+b_{12}) & 0 \\ a_{21}+b_{21} + 3(a_{22}+b_{22}) & 0 \\ \vdots & \\ a_{m1}+b_{m1} + 3(a_{m2}+b_{m2}) & \end{bmatrix}$$

$$T(A) = A\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \qquad T(B) = B\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$ where $n = 3$

$$= \begin{bmatrix} a_{11} + 3a_{12} \\ a_{21} + 3a_{22} \\ \vdots \\ a_{m1} + 3a_{m2} \end{bmatrix} + \begin{bmatrix} b_{11} + 3b_{12} \\ b_{21} + 3b_{22} \\ \vdots \\ b_{m1} + 3b_{m2} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} + 3(a_{12}+b_{12}) \\ a_{21}+b_{21} + 3(a_{22}+b_{22}) \\ \\ a_{m1}+b_{m1} + 3(a_{m2}+b_{m2}) \end{bmatrix}$$

QED

To prove $T(cU) = cT(U)$, Let U be a matrix,

$$U = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{m1} & \cdots & u_{mn} \end{bmatrix}, \quad T(cu) = \begin{bmatrix} cu_{11} & \cdots & cu_{1n} \\ \vdots & & \\ cu_{m1} & \cdots & cu_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$ u has $n = 3$

$$T(cu) = \begin{bmatrix} cu_{11} + 3cu_{12} \\ cu_{21} + 3cu_{22} \\ cu_{m1} + cu_{m2} \end{bmatrix}, \quad cT(u) = \begin{bmatrix} u_{11} + 3u_{12} \\ u_{21} + 3u_{22} \\ u_{m1} + 3u_{m2} \end{bmatrix} = \begin{bmatrix} cu_{11} + 3cu_{12} \\ cu_{21} + 3cu_{22} \\ cu_{m1} + 3cu_{m2} \end{bmatrix}$$

Meaning $T(cu) = cT(u)$ and $T(A+B) = T(A) + T(B)$, so T is linear

1b) Null space is set where $A\begin{bmatrix}1\\3\\0\end{bmatrix} = \vec{0} \in \mathbb{R}^3$

$$\begin{array}{ccc} a_1 & a_2 & a_3 \end{array}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix}1\\3\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$r_2 = r_2 + r_1$

$r_3 = r_3 + r_1$

$$\begin{bmatrix} a_{11} + 3a_{12} \\ a_{21} + 3a_{22} \\ a_{31} + 3a_{32} \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \begin{cases} a_{11} + 3a_{12} = 0, & a_{11} = -3a_{12} \\ a_{21} + 3a_{22} = 0 & a_{21} = -3a_{22} \\ a_{31} + 3a_{32} = 0 & a_{31} = -3a_{32} \end{cases}$$

$$T(A) = \begin{bmatrix} -3a_{12} & a_{12} & a_{13} \\ -3a_{22} & a_{22} & a_{23} \\ -3a_{32} & a_{32} & a_{33} \end{bmatrix}$$ remove variables

1c) basis = $\begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 1 & 0 \end{bmatrix},$

1c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1c) range = col(T(A))

$$T(A) = \left.\begin{bmatrix} a_{11} + 3a_{12} \\ a_{21} + 3a_{22} \\ a_{31} + 3a_{32} \end{bmatrix}\right\}$$ as derived in part b

$$\text{col}(T(A)) = \text{span}\left\{\begin{bmatrix} a_{11} + 3a_{12} \\ a_{21} + 3a_{22} \\ a_{31} + 3a_{32} \end{bmatrix}\right\}, \text{ where } a_{11} \dots a_{32} \in \mathbb{R}$$

This vector can have any value in $\mathbb{R}^3$, as the variables are not dependent on each other. This means that col(T(A)) spans $\mathbb{R}^3$, and the basis for $\mathbb{R}^3$ is:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

2a) Nul(A) is solutions for $A\vec{x} = \vec{0}$, and since $A \sim B$

$$\left[\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & -2 & 0 \\ 0 & 3 & 9 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & -2 & 0 \\ 0 & 1 & 3 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r_1 = r_1 + 2r_2$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & -2 & 6 & 0 \\ 0 & 1 & 3 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_3 - 2x_4 + 6x_5 = 0, \quad x_1 = -2x_3 + 2x_4 - 6x_5$

$x_2 + 3x_3 - 4x_4 + 4x_5 = 0, \quad x_2 = -3x_3 + 4x_4 - 4x_5$

$$\begin{bmatrix} -2x_3 + 2x_4 - 6x_5 \\ -3x_3 + 4x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul(A) is $\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

2b) RREF of A is

$$\begin{bmatrix} 1 & 0 & 2 & -2 & 6 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Col(A) = Span{column vectors}
Looking at span, it is clear that all vectors can be written as linear combination of first 2 column vectors

Basis for Col(A): $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

2(c) If $Nul(C) = \{0\}$, then the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$ is the only solution.
This means by the definition of one-to-one, where $A\vec{x} = \vec{0}$ has only the trivial solution, C is one-to-one.
For $T(x)$ to be onto, $C\vec{x} = \vec{b}$ has at least 1 solution, where $\vec{b} \in \mathbb{R}^n$, and C needs a pivot in every row, which isn't guaranteed, since all we know C's columns of C are linearly independent, which just guarantees a pivot in each column.

3(a) Since T is one-to-one, the columns of T are linearly independent. Additionally, since T is linear, $T(x+y) = T(x) + T(y)$.
For B' to be a basis, $\{T(b_1), \ldots T(b_n)\}$ must be linear independent, and B' must span $\mathbb{R}^n$
To prove B' is linearly independent,
$c_1 T(b_1) + c_2 T(b_2) + \ldots c_n T(b_n) = \vec{0}$, $c_1 \ldots c_n = 0$
To prove, by property $c_1 T(b_1) = T(c_1 b_1)$, we get

$T(c_1 b_1) + T(c_2 b_2) + \ldots T(c_n b_n) = \vec{0}$
$T(c_1 \vec{b}_1 + c_2 \vec{b}_2 + \ldots c_n \vec{b}_n) = \vec{0}$
Since $\{\vec{b}_1 \ldots b_n\}$ is a basis for $\mathbb{R}^n$, the equation $c_1 \vec{b}_1 + \ldots + c_n \vec{b}_n = \vec{0}$ has only solution $c_1 \ldots c_n = 0$ as it's linearly independent. Therefore, $c_1 T(b_1) + \ldots c_n T(b_n)$, $c_1 \ldots c_n = 0$ and is linearly independent.

To prove $\{T(\vec{b}_1) \ldots T(\vec{b}_n)\}$ spans $\mathbb{R}^n$ for $c_1 \ldots c_n \in \mathbb{R}$,
$c_1 T(b_1) + c_2 T(b_2) + \ldots c_n T(\vec{b_n}) = \vec{w}$ where $\vec{w} \in \mathbb{R}^n$, need to show the system is consistent.
We can rearrange the equation as: $T(c_1 \vec{b}_1 + c_2 \vec{b}_2 + \ldots c_n \vec{b}_n) = \vec{w}$
We know T is linear, thus $T(x) = \vec{u}$ has at most 1 solution.

3 b) ~~m~~ $m>n$ $T: \mathbb{R}^n \to \mathbb{R}^m$, different size
By the definition of a basis, the basis of $\mathbb{R}^c$ where $c \in \mathbb{R}$, has exactly c vectors. This means the basis for $\mathbb{R}^n$ has n vectors, and the basis for $\mathbb{R}^m$ has m vectors. B' contains n elements, but since $m>n$, $m \neq n$, meaning B' is not a basis for $\mathbb{R}^m$.

3 c) $m<n$ for $T: \mathbb{R}^n \to \mathbb{R}^m$, By the definition of a basis, the basis of $\mathbb{R}^n$ has n vectors and the basis for $\mathbb{R}^m$ has m vectors. Since B' contains n vectors and $m<n$, B' cannot be a basis for $\mathbb{R}^m$, as it has n vectors

4. $W = \{\varphi \in M_{2\times 2} \mid \varphi A = A\varphi\}$, meaning

$\varphi \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \varphi$, $\varphi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & 5b \\ c & 5d \end{bmatrix} = \begin{bmatrix} a & b \\ 5c & 5d \end{bmatrix} - a = a, \ 5b = b$$

$d = d, \ c = 5c$

This means $b, c = 0$, and $a = a$, $d = d$

Therefore, $\varphi = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

The basis for W is the span of φ.,

So basis of W is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

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