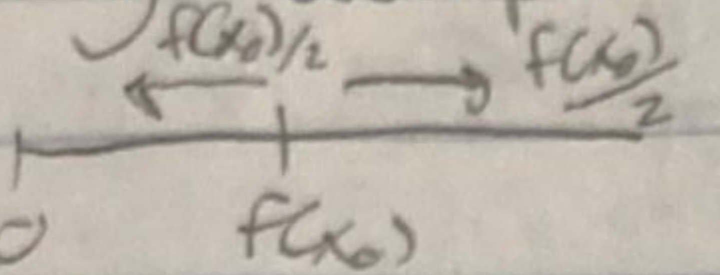
Math 135 HW 6

1) Since $f(x_0) > 0$ we want to choose ε s.t. $|f(x) - f(x_0)| < \varepsilon$ and $f(x) > 0$.
Suppose $\varepsilon = \frac{f(x_0)}{2}$. If $\varepsilon = \frac{f(x_0)}{2}$ then all $f(x)$ satisfying this inequality are positive.
To show:



So as $f(x)$ is continuous, it satisfies the ε-δ criteria: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in D$ $|x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon/2$
As $\varepsilon = \frac{f(x_0)}{2}$, which guarantees positivity of $f(x)$, $\exists \delta > 0$ s.t. if $|x - x_0| < \delta \to |f(x) - f(x_0)| < \frac{f(x_0)}{2}$.
Therefore a delta exists and the theorem is proven.

2) $|x_{n+1} - x_n| < \frac{1}{2^n}$ then $|x_{n+2} - x_{n+1}| < \frac{1}{2^{n+1}} + \ldots\ldots |x_{n+k+1} - x_{n+k}| < \frac{1}{2^{n+k}}$

$$\sum_{i=n}^{k} |x_{i+1} - x_i| < \sum_{i=n}^{k} \frac{1}{2^i}$$

where $k \in \mathbb{N} \geq 0$. This is a generalization and holds for all k.

Notice that $|x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \ldots + |x_{n+k+1} - x_{n+k}|$ is $\geq |(x_{n+1} - x_n) + (x_{n+2} - x_{n+1}) + \ldots\ldots + (x_{n+k+1} - x_{n+k})|$ by triangle inequality so:

$$|x_{n+k+1} - x_n| < \sum_{i=1}^{k} |x_{i+1} - x_i| < \sum_{i=1}^{k} \frac{1}{2^i}$$

$$|x_{n+k+1} - x_n| < \frac{1}{2^n} + \underbrace{\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} \cdots + \frac{1}{2^{n+k}}}_{< \frac{1}{2^n} \text{ by Geometric Sum}} < \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

$$|x_{n+k+1} - x_n| \leq \frac{2}{2^{n-1}}$$

A sequence is Cauchy if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N \in \mathbb{N}$, $|a_m - a_n| < \varepsilon$.
In our sequence above, could draw comparison that $m = n + k + 1$.
So let $\frac{1}{2^{n-1}} \leq \varepsilon$, then $1 - \log_2 \varepsilon < n$. So, for $m > n \geq 1 - \log_2 \varepsilon$
$|x_{n+m} - x_n| < \varepsilon$, and the sequence $\{x_n\}$ is Cauchy.
Since $\{x_n\}$ is Cauchy, by definition, it is convergent.

3 $f(x)=\sum_{n=0}^{\infty}\frac{a_n x^n}{n!}$ $|a_n|\le M$ and $x\in[-R,R]$

Using Weierstrass M-test first find M_K

$$\left|\frac{a_n x^n}{n!}\right| \le \frac{M|x^n|}{n!} \le \frac{MR^n}{n!} := M_n$$

To see if $\sum M_n$ converges, use the ratio test

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty}\left|\frac{MR^{n+1}}{(n+1)!}\cdot\frac{n!}{MR^n}\right| = \lim_{n\to\infty}\left|\frac{R}{n+1}\right| = 0$$

By RATFACE, M_n converges absolutely, so by Weierstrass M-test, $f(x)=\sum_{n=1}^{\infty}\frac{a_n x^n}{n!}$ converges uniformly on $[-R,R]$.

4 a) $\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$. $r\in(0,1)$

Using M-test, first find M_K

$$\left|\frac{(-1)^{n+1}x^n}{n}\right| = \left|\frac{x^n}{n}\right| \le \left|\frac{r^n}{n}\right|.$$ To see if $\sum M_n$ converges use ratio test:

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty}\left|\frac{r^{n+1}}{n+1}\cdot\frac{n}{r^n}\right| = \lim_{n\to\infty}\left|\frac{rn}{n+1}\right| = R\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$= r\cdot 1$ as $r\in(0,1)$, then $r<1$ so by RATFACE, M_n converges absolutely and by the Weierstrass M-test the power series for $\ln(1+x)$ converges uniformly on $[-r,r]$ where $r\in(0,1)$

b) From class, $\ln(1+x)$ is continuous on $[-r,r]$ where $r\in(0,1)$
Let $r_0\in(0,1)$ so, $\ln(1+x)$ is continuous on $[-r_0,r_0]$.
Define $r_1=\frac{r_0+1}{2}$, $r_1\in(0,1)$ and therefore $f(x)=\ln(1+x)$ is continuous on $[-r_1,r_1]$, this can be repeated so $r_K=\frac{r_{K-1}+1}{2}$, and $\lim_{K\to\infty} r_K=1$ as we half the distance between r_K and 1 each step. $f(x)$ is continuous at $[-r_K,r_K]$ and as $\lim r_K=1$, but $\nexists\, r_K=1$, $f(x)$ will be continuous on $(-1,1)$ $[-1,1]$

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5 Let $\epsilon > 0$, want to find $N \in \mathbb{N}$ s.t. $\forall n \geq N$ $|f_n + g_n - (f+g)| < \epsilon$

Scratch work:

$\left\{ \begin{array}{l} |f_n + g_n - (f+g)| = |(f_n - f) + (g_n - g)| \leq |f_n - f| + |g_n - g| \\ \text{Since } f_n \to f, \exists N_1 \in \mathbb{N} \text{ s.t. } \forall n \geq \quad \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{array} \right.$

Since $f_n \to f$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $|f_n - f| < \epsilon/2$

Since $g_n \to g$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$, $|g_n - g| < \epsilon/2$.

Choose $N = \max\{N_1, N_2\}$, $\forall n \geq N$.

So $|f_n + g_n - (f+g)| = |(f_n - f) + (g_n - g)| \leq |f_n - f| + |g_n - g| < \epsilon/2 + \epsilon/2 < \epsilon$

$\to < \epsilon/2 + \epsilon/2 < \epsilon$

Therefore $f_n + g_n$ converges uniformly to $f+g$ on $\mathbb{R}$.

6a) $\lim_{n \to \infty} \frac{\sin nx}{n} = 0$ on $\mathbb{R}$

To see if uniformly convergent, use Comparison test.

$|f_n(x) - f(x)| = |\frac{\sin nx}{n} - 0| \leq \frac{1}{n}$ as $|\sin nx| \leq 1$

$a_n = \frac{1}{n}$. $\lim_{n \to \infty} \frac{1}{n} = 0$, so by comparison test, $\frac{\sin x}{n}$ is uniformly convergent on $\mathbb{R}$.

b) $\lim_{n \to \infty} \frac{x}{nx+1} = 0$. Use comparison test. $\left|\frac{x/x}{\frac{nx+1}{x}}\right| = \left|\frac{1}{n + \frac{1}{x}}\right|$

$|f_n(x) - f(x)| = \left|\frac{x}{nx+1} - 0\right|$ as $x \in (0,1)$,

$\left|\frac{1}{n + \frac{1}{x}}\right| < \left|\frac{1}{n}\right| = \frac{1}{n} := a_n$ as $n + \frac{1}{x} > n$ as $x \in (0,1)$

$\lim_{n \to \infty} \frac{1}{n} = 0$ so as $a_n \to 0$ by comparison test, $\frac{x}{nx+1}$ converges uniformly to 0 on $(0,1)$.

c) $\lim_{n \to \infty} x^n = 0$. Use comparison test.

$|f_n(x) - f(x)| = |x^n - 0| = |x^n|$ as $x \in [0, .999]$, we can say $|x^n| \leq 0.9999^n := a_n$

$\lim_{n \to \infty} a_n = 0$, so by the comparison test, $f_n(x) = x^n$ converges uniformly to 0 on $[0, 0.999]$