Tufts University
Department of Mathematics

Math 135 **Homework 7**[1] **Fall 2022**

Readings for Problem Set 7

Notes: Power series
§10.1: The linear structure of $\mathbb{R}^n$ and the scalar product

Problem Set 7
(Due Wednesday, November 2, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (10 points) **Pointwise vs. uniform convergence**.
   Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{1-x^{4n}}{1+x^{4n}}$.
   (a) Find the function f that the sequence $\{f_n\}$ converges to pointwise on $\mathbb{R}$.
   (b) Does $\{f_n\}$ converge uniformly to f on $\mathbb{R}$? Why or why not?

2. (15 points) **Pointwise vs. uniform convergence**.
   Let $f_n(x) = e^{-nx}$.
   (a) Find the pointwise limit, f, of the sequence $\{f_n\}$ on $[0,\infty)$.
   (*Hint*: Consider the point $x = 0$ separately.)
   (b) Show that $\{f_n\}$ converges to this function f uniformly on $[1,\infty)$.
   (c) (2 points) Explain why $\{f_n\}$ does not converge to f uniformly on $[0,\infty)$.
   (d) (4 points) Does $\{f_n\}$ converge to f uniformly on $(0,\infty)$? Prove your answer.

3. (10 points) (**Domain of convergence of a power series**) §9.5, p. 262, # 1a,b,c.
   Determine the domain of convergence of each of the following power series:
   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k5^k}$ (b) $\displaystyle\sum_{k=1}^{\infty} k!x^k$ (c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k+1)!}$

4. (10 points) (**Polarization identity**) §10.1, p. 276: # 5.
   Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Prove that
   $$\langle \vec{u}, \vec{v} \rangle = \frac{\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2}{4}.$$
   This identity shows that the scalar product can be expressed in terms of the norm. It is called the ***polarization identity***.

5. (10 points) (**Absolute value of a sum vs. Euclidean length**) §10.1, p. 276: #7.
For a natural number n and real numbers $a_1, \ldots, a_n$, prove that
$$|a_1 + \cdots + a_n| \leq \sqrt{n}\sqrt{a_1^2 + \cdots + a_n^2}.$$
(Hint: Apply the Cauchy–Schwarz inequality to two cleverly chosen vectors.)

6. (10 points) (**Convexity of an open ball**) §10.1, p. 277: # 10.
Let $\vec{u}$ be a point in $\mathbb{R}^n$ and let r be a positive number. Suppose that the points $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are at a distance less than r from the point $\vec{u}$. Prove that if $0 \leq t \leq 1$, then the point $t\vec{v} + (1-t)\vec{w}$ is also at a distance less than r from $\vec{u}$. (What this problem means geometrically is that the open ball with center $\vec{u}$ and radius r is convex. Draw a picture to convince yourself of this.)

(End of Problem Set 7)