Homework 3

Early problem due on Gradescope at 8 pm on Wednesday, February 8th.

Due on Gradescope at 8 pm on Friday, February 10th.

Definition 1. (Universal property of the quotient, 2.0.)

Let X be a set and \sim an equivalence relation on X. Say a function $f: X \to B$ respects the equivalence relation if for all $x_1, x_2 \in X$, we have that $x_1 \sim x_2$ implies $f(x_1) = f(x_2)$. We say a set Q together with a function $\pi: X \to Q$ has the universal property of the quotient if:

- (a) For each set B and each function $\overline{f}:Q\to B$, the composite function $f=\overline{f}\circ\pi$ respects the equivalence relation.
- (b) For each set B and each function $f: X \to B$ that respects the equivalence relation, there exists a unique function $\overline{f}: Q \to B$ such that



commutes.

(1) (Early Problem) Prove that if $\pi: X \to Q$ and $\pi': X \to Q'$ have the universal property of the quotient (2.0), then there is a unique bijection $f: Q \to Q'$ such that $f \circ \pi = \pi'$.

(For those using LATEX for their homework, look into the tikz-cd package.)

(2) Let X_1, X_2 be sets. Note that there are "inclusion" functions $j_1: X_1 \cap X_2 \to X_1$, $j_2: X_1 \cap X_2 \to X_2$, $i_1: X_1 \to X_1 \cup X_2$ and $i_2: X_2 \to X_1 \cup X_2$, all defined by the rule $x \mapsto x$, and the following diagram commutes:

$$X_{1} \cap X_{2} \xrightarrow{j_{2}} X_{2}$$

$$\downarrow^{j_{1}} \qquad \downarrow^{i_{2}}$$

$$X_{1} \xrightarrow{i_{1}} X_{1} \cup X_{2}.$$

Prove that $X_1 \cup X_2$ has the following universal property:

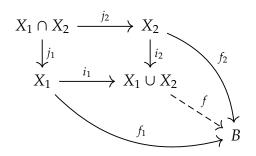
For all sets *B* and functions $f_1: X_1 \to B$ and $f_2: X_2 \to B$ such that

$$X_1 \cap X_2 \xrightarrow{j_2} X_2$$

$$\downarrow^{j_1} \qquad \downarrow^{f_2}$$

$$X_1 \xrightarrow{f_1} B.$$

commutes, there exists a unique function $f: X_1 \cup X_2 \rightarrow B$ such that



commutes. (This is a special case of the "Universal property of the pushout.")

- (3) Prove the following "laws of algebra" for all x, y, z, $w \in \mathbb{R}$ using only axioms (1)–(5) and "laws of algebra" proven in class and recitation.
 - (a) x/1 = x
 - (b) $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$
 - (c) If $y, z \neq 0$, then (1/y)(1/z) = 1/(yz)
 - (d) If $y, z \neq 0$, then (x/y)(w/z) = (wx)/(yz)
 - (e) $x \neq 0 \implies 1/x \neq 0$
 - (f) If $w, z \neq 0$, then 1/(w/z) = z/w
 - (g) If $z \neq 0$, then (xy)/z = x(y/z)
- (4) Prove the following "laws of inequalities" for all x, y, z, $w \in \mathbb{R}$, using axioms (1)–(6), the results of the previous exercise, and "laws of algebra and inequalities" proven in class and recitation.
 - (a) -1 < 0 < 1
 - (b) $xy > 0 \iff x$ and y are both positive or both negative.
 - (c) $x > 0 \implies 1/x > 0$
 - (d) $x > y > 0 \implies 1/x < 1/y$.
- (5) Let $a \in \mathbb{R}$. Define inductively

$$a^1 = a,$$

$$a^{n+1} = a^n \cdot a$$

for $n \in \mathbb{Z}_{\geq 0}$. Show that for $n, m \in \mathbb{Z}_{> 0}$ and $a, b \in \mathbb{R}$:

- (a) $a^n \cdot a^m = a^{n+m}$
- (b) $(a^n)^m = a^{nm}$
- (c) $a^m b^m = (ab)^m$

These are called the *laws of exponents*. (Hint: for fixed n, prove the formulas by induction on m.)

(6) (Optional) Suppose X_1, X_2, Y are sets and $a_1 : Y \to X_1, a_2 : Y \to X_2$ are functions. A set P together with functions $i_1 : X_1 \to P$ and $i_2 : X_2 \to P$ such that

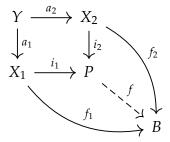
$$\begin{array}{ccc}
Y & \xrightarrow{a_2} & X_2 \\
\downarrow^{a_1} & & \downarrow^{i_2} \\
X_1 & \xrightarrow{i_1} & P.
\end{array}$$

is said to be a *pushout* if it satisfies the following universal property:

For all sets *B* and functions $f_1: X_1 \to B$ and $f_2: X_2 \to B$ such that

$$\begin{array}{ccc}
Y & \xrightarrow{a_2} & X_2 \\
\downarrow^{a_1} & & \downarrow^{f_2} \\
X_1 & \xrightarrow{f_1} & B.
\end{array}$$

commutes, there exists a unique function $f: P \rightarrow B$ such that



commutes.

Find a set P and functions i_1 , i_2 so that P is a pushout. (Perhaps as a hint, the usual notation for the pushout is $X_1 \sqcup_Y X_2$.)