Math 136 **Real Analysis II**[1] **Spring 2023**
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1. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$.
   (a) State the definition of $f'(a)$ for $a \in I$.

   Answer: The function $f : I \to \mathbb{R}$ is differentiable at $a \in I$ if $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(a)$ exists. In this case, $f'(a)$ is the derivative of f at a.

   (b) Let $a \in \mathbb{R}$. Use the definition of the derivative to compute $f'(a)$ for $f(x) = x^3$.

   Answer: By definition, if the limit exists

   $$f'(a) = \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h} = \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h} = \lim_{h\to 0} \left(3a^2 + 3ah + h^2\right) = 3a^2.$$

   So, $f'(a) = 3a^2$ for $a \in \mathbb{R}$.

   (c) Assume $f : I \to \mathbb{R}$ is differentiable on I. Use the definition of derivative to prove that f is continuous on I.

   Answer: Here we are referring to a general $f : I \to \mathbb{R}$. Assume f is differentiable then $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ exists. Furthermore, $\lim_{h\to 0} h = 0$ as the identity function is continuos. Therefore, since the product of limits is the limit of the product:

   $$\left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) = \lim_{h\to 0} h \left(\frac{f(x+h) - f(x)}{h}\right) = \lim_{h\to 0} \left(f(x+h) - f(x)\right) = 0f'(x) = 0$$

   Then, adding the constant $f(x)$ to both sides, we see $\lim_{h\to 0} f(x+h) = f(x)$ and f is continuous at x.

2. Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$ and let $G \in C^1(\mathbb{R}, \mathbb{R}^2)$. Let $h(t) = f(G(t))$ for $t \in \mathbb{R}$. Find $\frac{dh}{dt}(t)$. (Note, here ∇f is viewed as a row vector so you can multiply it by DG.)

   Answer: For $(x,y) \in \mathbb{R}^2$, since f is C^1, f is differentiable and $\mathbf{D}f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$. To take the derivative, we need to label the component functions of g, so assume $g(t) = \begin{pmatrix} g_1(t) \\ g_2(g) \end{pmatrix}$. Then, since g is C^1, g_1 and g_2 are differentiable and $\mathbf{D}g(t) = \begin{pmatrix} g_1'(t) \\ g_2'(t) \end{pmatrix}$. Now, since f and g are C^1, we use the Chain Rule to get

   $$\begin{aligned} h'(t) = \mathbf{D}f(g(t))\mathbf{D}g(t) &= \left(\frac{\partial f}{\partial x}(g_1(t), g_2(t)), \frac{\partial f}{\partial y}(g_1(t), g_2(t))\right)\begin{pmatrix} g_1'(t) \\ g_2'(t) \end{pmatrix} \\ &= \frac{\partial f}{\partial x}(g_1(t), g_2(t))g_1'(t) + \frac{\partial f}{\partial y}(g_1(t), g_2(t))g_2'(t) \end{aligned}$$

3. Define $g(x,y) = \dfrac{x^2 y}{x^2 + y^4}$ on domain $\mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\}$. Calculate $\lim_{(x,y)\to \mathbf{0}} g(x,y)$ and prove your result or explain why this limit does not exist.

   Answer: I haven't used the $\epsilon - \delta$ condition enough in class, so let's do it here! My spidey sense says that the limit is zero. The reason is that one can note that $\frac{x^2}{x^2+y^2} \le 1$ so this allows us to bound $\left|\frac{x^2 y}{x^2+y^4}\right|$ above by $|y|$.

   Let $\epsilon > 0$ let $\delta = \epsilon$. Then, $\delta > 0$ because $\epsilon > 0$.

   Let $(x,y) \in \mathbb{R}^2$ satisfy $0 < \|(x,y)\| = \|(x,y) - \mathbf{0}\| < \delta$. Then,

   $$\left|\frac{x^2 y}{x^2 + y^4} - 0\right| = \left|\frac{x^2 y}{x^2 + y^4}\right| \le |y| \frac{x^2}{x^2 + y^4} \le |y| \le \|(x,y)\| < \delta = \epsilon.$$

This prove that $\lim_{(x,y)\to\mathbf{0}} \frac{x^2y}{x^2+y^4} = 0$.

4. Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by $f(x,y) = x^2+\cos(xy)+y$.
   (a) Note that $f(0,1)=2$. Explain why there is an open interval I containing 0 and a function $g\in C^1(I,\mathbb{R})$ that satisfies $g(0)=1$ and $f(x,g(x))=2$ for all $x\in I$. Find $g'(0)$. Justify your answers.
   Answer: We use Dini's Theorem. First, f is in $C^1(\mathbb{R}^2,\mathbb{R})$ because it is the sum of compositions of C^1 functions. Secondly, $\frac{\partial f}{\partial y}(x,y) = -x\sin(xy)+1$ and so $\frac{\partial f}{\partial y}(0,1)=1$. Therefore, Dini's theorem can be used. Therefore, there is an open interval I containing $x=0$ and a function $g\in C^1(I,\mathbb{R})$ such that $g(0)=1$ and for all $x\in I$, $f(x,g(x))=2$. Now, by Dini's theorem we know that $g'(0) = \frac{\frac{\partial f}{\partial x}(0,1)}{\frac{\partial f}{\partial y}(0,1)} = \frac{0}{1} = 0$.
   (b) Explain why $f:\mathbb{R}^2\to\mathbb{R}$ is not injective (one-to-one).
   Answer: we have that $f(x,g(x))=2$ for all $x\in I$ and so if $x_1\in I$ and $x_1\neq x_0$, then $f(x_1,g(x_1)) = 2 = f(x_0,g(x_0))$ but $(x_1,g(x_1))\neq(x_0,g(x_0))$. Therefore, f is not injective.
   This can be part of a simple proof that if $\mathcal{O}$ is open in $\mathbb{R}^2$ and $f\in C^1(\mathcal{O},\mathbb{R})$ then $f:\mathcal{O}\to\mathbb{R}^2$ is not injective.
   However, an even cooler proof shows that if $f:\mathcal{O}\to\mathbb{R}$ is continuous, then f is not injective.

5. let $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$ be continuously differentiable (C^1). Assume $\mathbf{F}(1,2,3)=(4,5,6)$ and

$$\mathbf{DF}(1,2,3) = A = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}. \text{ You may assume that } A^{-1} = \begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}.$$

   (a) Does $\mathbf{F}$ satisfy the Inverse Function Theorem at $(1,2,3)$? Justify your answer.
   Answer: Since $\mathbf{F}$ is continuously differentiable and $\mathbf{DF}(1,2,3)$ is an invertible matrix as indicated in the problem, $\mathbf{F}$ satisfies the Inverse Function Theorem. So, there is a neighborhood U of $(1,2,3)$ and a neighborhood V of $(4,5,6)$ such that $\mathbf{F}:U\to V$ is bijective and $\mathbf{F}^{-1}$ is C^1.
   Assume the domain of $\mathbf{F}$ is given coordinates (x,y,z) and the target has coordinates (u,v,w), that is $\mathbf{F}(x,y,z) = \begin{pmatrix}u\\v\\w\end{pmatrix}$.
   (b) Find $\frac{\partial u}{\partial x}(1,2,3)$.
   Answer: Since u is the first variable in the target of $\mathbf{F}$ and x is the first variable in the domain of $\mathbf{F}$, $\frac{\partial u}{\partial x}(1,2,3)$ is the $1,1$ entry of $\mathbf{DF}(1,2,3)$ and that is 1 so $\frac{\partial u}{\partial x}(1,2,3)=1$.
   (c) Find $\frac{\partial x}{\partial u}(4,5,6)$. Recall that $\mathbf{F}(1,2,3)=(4,5,6)$.
   Answer: Since u is the first variable in the domain of $\mathbf{F}^{-1}$ and x is the first variable in the target of $\mathbf{F}^{-1}$, $\frac{\partial x}{\partial u}(4,5,6)$ is the $1,1$ entry of $\mathbf{DF}^{-1}(4,5,6)$ and that is 1 so $\frac{\partial x}{\partial u}(4,5,6)=1$.

6. Let $F(x,y) = \begin{pmatrix}F_1(x,y)\\F_2(x,y)\\F_3(x,y)\end{pmatrix}$ be a C^1 function from $\mathbb{R}^2$ to $\mathbb{R}^3$ and assume $DF(1,1) = \begin{pmatrix}1&2\\1&1\\2&2\end{pmatrix}$. Prove that there is an open neighborhood $\mathcal{U}$ of $(1,1)$ such that $F:\mathcal{U}\to\mathbb{R}^3$ is injective.

   HINT: first consider $\tilde{F}:\mathbb{R}^2\to\mathbb{R}^2$ defined by $\tilde{F}(x,y) = \begin{pmatrix}F_1(x,y)\\F_2(x,y)\end{pmatrix}$.

   Answer: Since F is continuously differentiable, the component functions F_1 and F_2 are continuously differentiable. Therefore $\tilde{F}\in C^1(\mathbb{R}^2,\mathbb{R}^2)$. Since the rows of DF are the gradients of the component functions, $\nabla F_1(1,2) = (1,2)$ $\nabla F_2(1,1) = (1,1)$. Therefore $D\tilde{F}(1,1) = \begin{pmatrix}1&2\\1&1\end{pmatrix}$. Since this matrix is invertible (it's determinant, -1, is not zero) and $\tilde{F}$ is continuously differentiable, we can use the Inverse Function Theorem to show there is a neighborhood $\mathcal{U}$ of $(1,1)$ and a neighborhood $\mathcal{V}$ of $\tilde{F}(1,1)$ such that $\tilde{F}:\mathcal{U}\to\mathcal{V}$ is bijective. Therefore, $\tilde{F}:\mathcal{U}\to\mathbb{R}^2$ is injective.

We now show $F : \mathcal{U} \to \mathbb{R}^3$ is injective. Let (x_1, y_1) and (x_2, y_2) be points in $\mathcal{U}$ and assume

$$F(x_1,y_1) = \begin{pmatrix} F_1(x_1,y_1) \\ F_2(x_1,y_1) \\ F_3(x_1,y_1) \end{pmatrix} = \begin{pmatrix} F_1(x_2,y_2) \\ F_2(x_2,y_2) \\ F_3(x_2,y_2) \end{pmatrix} = F(x_2,y_2).$$

Therefore, $F_1(x_1,y_1) = F_1(x_2,y_2)$ and $F_2(x_1,y_1) = F_2(x_2,y_2)$ so $\tilde{F}(x_1,y_1) = \tilde{F}(x_2,y_2)$. Since $\tilde{F}$ is injective, $(x_1,y_1) = (x_2,y_2)$. This shows F is injective on the domain $\mathcal{U}$.

Recall the definition of differentiability:

Definition 1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $\mathbf{x}_0 \in \mathcal{O}$. Let $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$. Let B be an $m \times n$ matrix. Then, $\mathbf{F}$ is differentiable at $\mathbf{x}_0$ if*

$$(1) \qquad \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{F}(\mathbf{x}_0+\mathbf{h}) - [\mathbf{F}(\mathbf{x}_0) + B\mathbf{h}]\|}{\|\mathbf{h}\|} = 0 \quad \textit{equivalently} \quad \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|\mathbf{F}(\mathbf{x}) - [\mathbf{F}(\mathbf{x}_0) + B(\mathbf{x}-\mathbf{x}_0)]\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0$$

The function $\mathbf{F}$ is differentiable on $\mathcal{O}$ if $\mathbf{F}$ is differentiable at all points in $\mathcal{O}$.

Recall that, if $\mathbf{F}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$ then $\mathbf{F}$ has all first partial derivatives at $\mathbf{x}_0$ and $DF(\mathbf{x}_0) = B$, the matrix in the definition of derivative.

Also, note that if $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ then we write $\nabla f(\mathbf{x}_0)$ for the vector $\mathbf{b}$ such that

$$(2) \qquad \lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{x}_0+\mathbf{h}) - [f(\mathbf{x}_0) + \langle \mathbf{b},\mathbf{h}\rangle]|}{\|\mathbf{h}\|} = 0.$$

If $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ according to Definition (1), then the $1 \times n$ matrix $B = (b_1, b_2, \dots, b_n)$ in (1), and the vector $\nabla f(\mathbf{x}_0) = \mathbf{b}$ in (2) is the column vector B^T. Since, $B\mathbf{h} = \langle \mathbf{b},\mathbf{h}\rangle$, the limit in (1) is zero *iff* the limit in (2) is zero. This explains the equivalence of the two definitions of derivative–(1), and (2) if $f : \mathcal{O} \to \mathbb{R}$.

For the following problems, the Sandwich Theorem for functions is really useful and you may use it. I discussed it in the help session and in class on Monday.

Theorem 2. *Sandwich Theorem Let f, g, and h be functions from a set $A \subset \mathbb{R}^n$ to $\mathbb{R}$ and assume $\mathbf{x}_0$ is a limit point of A and $L \in \mathbb{R}$. Assume that*

$$(3) \qquad \forall \mathbf{x} \in A, \quad f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x}), \quad \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}_0) = L = \lim_{\mathbf{x}\to\mathbf{x}_0} h(\mathbf{x}).$$

Then, $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = L$.

This can simplify the estimates in the limits in the next two problems.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2 + 2xy + 3$. Let $(x,y) \in \mathbb{R}^2$. Use the definition of derivative to show that f is differentiable at (x,y) and that $\nabla f(x,y) = (2x + 2y, 2x)$. *Sorry for the misprint on the original version!*

You may use the following limits

$$0 = \lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2+k^2}} =$$

NOTE: you may use the Sandwich Theorem for limits of functions. I put a proof at the end of this review sheet. If you don't want to do this, you just apply the Sandwich Theorem for sequences to an arbitrary sequence $\mathbf{x}_k \to \mathbf{x}_0$

Answer: We make the guess that $\mathbf{b} = \nabla f(x_0, y_0) = (2x_0 + 2y_0, 2x_0)$ as suggested (with the misprint corrected). We let $\mathbf{h} = (h,k)$, and we let $\mathbf{x}_0 = (x_0, y_0)$. A calculation shows that

$$f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \langle (2x_0+2y_0, 2x_0), (h,k)\rangle = h^2 + 2hk$$

, so

$$0 \le \frac{|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - \langle (2x_0+2y_0,2x_0),(h,k)\rangle|}{\sqrt{h^2+k^2}} \le \frac{h^2}{\sqrt{h^2+k^2}} + \frac{|2hk|}{\sqrt{h^2+k^2}}.$$

Now we use the Sandwich theorem for functions and the limits given in the problem to show the middle term is trapped between 0 on the left and the right-hand term which converges to 0 as $(h,k)\to(0,0)$ by the limits we were given. Therefore the limit of the middle term is zero and f is differentiable and $\nabla f(x_0,y_0)=(2x_0+2y_0,2x_0)$.

8. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $f:\mathcal{O}\to\mathbb{R}$ be differentiable at $\mathbf{x}_0\in\mathcal{O}$ and $\nabla f(\mathbf{x}_0)=\mathbf{b}$ is a vector in $\mathbb{R}^n$. Use the definition of derivative to prove that f is continuous at $\mathbf{x}_0$.

   Answer: We assume $\lim_{\mathbf{x}\to\mathbf{x}_0}\frac{|f(\mathbf{x})-[f(\mathbf{x}_0)+\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle]|}{\|\mathbf{x}-\mathbf{x}_0\|}=0$, and we show $\lim_{\mathbf{x}\to\mathbf{x}_0}|f(\mathbf{x})-f(\mathbf{x}_0)|=0$. By the triangle inequality

$$0\le|f(\mathbf{x})-f(\mathbf{x}_0)|\le|f(\mathbf{x})-[f(\mathbf{x}_0)+\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle]|+|\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle|\,. \tag{4}$$

   First observe that $|f(\mathbf{x})-[f(\mathbf{x}_0)+\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle]|\to 0$ as $\mathbf{x}\to\mathbf{x}_0$ for the following reason:

$$\begin{aligned}0&=\lim_{\mathbf{x}\to\mathbf{x}_0}\left(\frac{|f(\mathbf{x})-[f(\mathbf{x}_0)+\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle]|}{\|\mathbf{x}-\mathbf{x}_0\|}\right)\lim_{\mathbf{x}\to\mathbf{x}_0}\|\mathbf{x}-\mathbf{x}_0\|\\&=\lim_{\mathbf{x}\to\mathbf{x}_0}\left(\frac{|f(\mathbf{x})-[f(\mathbf{x}_0)+\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle]|}{\|\mathbf{x}-\mathbf{x}_0\|}\|\mathbf{x}-\mathbf{x}_0\|\right)\\&=\lim_{\mathbf{x}\to\mathbf{x}_0}|f(\mathbf{x})-[f(\mathbf{x}_0)+\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle]|\end{aligned}$$

   (Note that we know the limits in the first line of this last set of inequalities exist, so we can combine them.) Here we use that $\lim_{\mathbf{x}\to\mathbf{x}_0}\|\mathbf{x}-\mathbf{x}_0\|=0$ (the proof is similar to the proof from class that $\lim_{\mathbf{x}\to\mathbf{0}}\|\mathbf{x}\|=0$). This shows the limit in red exists and is zero so the second to last term in (4) goes to zero as $\mathbf{x}_0\to\mathbf{x}_0$

   We now show that $\lim_{\mathbf{x}\to\mathbf{x}_0}|\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle|=0$. By Cauchy Schwartz Bunyakovsky,

$$|\langle\mathbf{b},(\mathbf{x}-\mathbf{x}_0)\rangle|\le\|\mathbf{b}\|\,\|\mathbf{x}-\mathbf{x}_0\|\,.$$

   Therefore, $\|\mathbf{b}\|\,\|\mathbf{x}-\mathbf{x}_0\|\to 0$ as $\mathbf{x}\to\mathbf{x}_0$, and the last term in (4) goes to zero as $\mathbf{x}\to\mathbf{x}_0$.

   Putting these together, we see the left-hand term in (4) is zero and the right hand term goes to zero as $\mathbf{x}\to\mathbf{x}_0$. Therefore the middle term does and so f is continuous at $\mathbf{x}_0$!

9. Explain why the graph of $z=\sin(xy)$ has a tangent plane at $(\pi,1,0)$ and find an equation of that tangent plane.

   Answer: Because $(x,y)\to xy$ and $t\mapsto\sin(t)$ are both C^1, their composition is C^1 by the chain rule. Therefore, the graph of $z=\sin(xy)$ does have a tangent plane at $(\pi,1,0)$. To find it, we recall the formula of the tangent plane to $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$ is

$$z=f(x_0,y_0)+\langle\nabla f(x_0,y_0),(x-x_0,y-y_0\rangle\,.$$

   In this case, the equation of the tangent plane becomes

$$z=-(x-\pi)-\pi(y-1)=-x-\pi y+2\pi.$$

The Proof of the Sandwich Theorem

Proof of Sandwich Theorem. Let $\{\mathbf{x}_k\}$ be an arbitrary sequence in $A\smallsetminus\{\mathbf{x}_0\}$ that converges to $\mathbf{x}_0$. By the assumptions in (3) $f(\mathbf{x}_k)\to L$, $h(\mathbf{x}_k)\to L$ and for all $k\in\mathbb{N}$, $f(\mathbf{x}_k)\le g(\mathbf{x}_k)\le h(\mathbf{x}_k)$. So, by the Sandwich Theorem for sequences, $\lim_{k\to\infty}g(\mathbf{x}_k)\to L$.

This shows that for every sequence $\{\mathbf{x}_k\}$ in $A\smallsetminus\{\mathbf{x}_0\}$ that converges to $\mathbf{x}_0$ that $g(\mathbf{x}_k)\to L$. Therefore, $\lim_{\mathbf{x}\to\mathbf{x}_0}g(\mathbf{x})=L$. $\square$