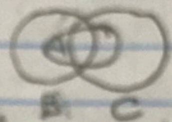
Math 171 Hw 1

1a) $\exists a \in A$, it is true that $a^2 \notin B$
b) $\forall a \in A$, it is true that $a^2 \notin B$
c) $\exists a \in A$, it is true that $a^2 \in B$
d) $\forall a \in A$, it is true that $a^2 \notin B$

2a) $\Rightarrow$ holds as if $A \subseteq B$ and $A \subseteq C$, as $B \subseteq B \cup C$ and $C \subseteq B \cup C$, $A \subseteq B \cup C$
$\Leftarrow$ Doesn't hold, let $B \cap C = \emptyset$ then $A \not\subseteq B$ and $A \not\subseteq C$ as the two sets are disjoint, so A can only be in one

b) $\Rightarrow$ holds if $A \subseteq B$ or $A \subseteq C$ then $A \subseteq B \cup C$ as $B \subseteq B \cup C$ and $C \subseteq B \cup C$
$\not\Leftarrow$ Doesn't hold, ~~as~~ if $A \subseteq B \cup C$, then A can overlap like this:



c) $\Rightarrow$ holds if $A \subseteq B$ and $A \subseteq C$ then A must be in $B \cap C$ so $A \subseteq B \cap C$
$\not\Leftarrow$ holds, if $A \subseteq B \cap C$ then $A \subseteq B$ and $A \subseteq C$ as $B \cap C \subseteq C$ and $B \cap C \subseteq B$

d) $\Rightarrow$ Doesn't hold, if $B \cap C = \emptyset$ but $A \subseteq B$, assuming $A \neq \emptyset$ then $A \not\subseteq \emptyset$
$\Leftarrow$ Yes, if $A \subseteq B \cap C$ then $A \subseteq B$ and $A \subseteq C$ so either 1 or must be true.

e) It's $A-(A-B) \subseteq B$, as if $B \not\subseteq A$ then $A-(A-B) = A \cap B$ as $A-(A-B)$ is elements in B that are in A, which doesn't guarantee all of B, but a subset $(A \subseteq B) \subseteq A-(A-B) \subseteq B$

f) It's $A-(B-A) \supseteq A-B$ as $B-A$ has nothing in A, so $A-(B-A) = A$ and $A \supseteq A-B$.

2g) $P(A) \cup P(B) \subseteq P(A \cup B)$ as $P(A)$ and $P(B)$ contain all subsets of A and B, but $P(A \cup B)$ has subsets of A and B, and subsets containing elements in both sets.

3a) $\bigcup_{i \in I} B_i = \{a \mid \exists i \in I \text{ s.t. } a \in B_i\}$

Note $x \notin B_i \longleftrightarrow \neg(\exists i \in I \text{ s.t. } x \in B_i)$

$x \notin B_i \longleftrightarrow \forall i \in I \; x \in B_i$

So if $x \in A$ and $\forall i \in I \; x \notin B_i$

$\forall i \in I \; x \in A$ and $x \notin B_i$

which is equivalent to $\forall i \in I \; x \in A - B_i$

$\Longleftrightarrow$ so $x \in \bigcap_{i \in I} A - B_i$

And $A \setminus \bigcup_{i \in I} B_i = \bigcap_{i \in I} A \setminus B_i$ $\square$

b) $\bigcap_{i \in I} B_i = \{a \mid \forall i \in I, a \in B_i\}$

Note $x \notin B_i \longleftrightarrow \exists i \in I, \text{ s.t. } a \notin B_i$

So if $x \in A$ and $\exists i \in I \; x \notin B_i$

$\exists i \in I \; x \in A$ and $x \notin B_i$

or $\exists i \in I \; x \in A - B_i$

$\longleftrightarrow \bigcup_{i \in I} (A - B_i)$ so $A - \bigcap_{i \in I} B_i = \bigcup_{i \in I} (A - B_i)$

4a) $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$

So $f^{-1}(\bigcup_{i \in I} B_i) = \{x \in X \mid f(x) \in \bigcup_{i \in I} B_i\}$

$= \{x \in X \mid \exists i \in I \text{ s.t. } f(x) \in B_i\}$

$= \bigcup_{i \in I} \{x \in X \mid f(x) \in B_i\}$

$= \bigcup_{i \in I} f^{-1}(B_i)$

M 171 HW 1

b) $f^{-1}(\bigcap_{i\in I} B_i) \Leftrightarrow \{x \in X \mid f(x) \in \bigcap_{i\in I} B_i\}$

$\Leftrightarrow \{x \in X \mid \forall i \in I, f(x) \in B_i\}$

$\Leftrightarrow \bigcap_{i\in I} \{x \in X \mid f(x) \in B_i\}$

$f^{-1}(\bigcap_{i\in I} B_i) \Leftrightarrow \bigcap_{i\in I} f^{-1}(B_i)$ $\square$

5 a) 8 functions go from $S \to T$ as each $s \in S$ goes to 4 or 5 so $2\times2\times2 = 8$
There are 9 functions that go from $T \to S$ as $\{4,5\}$ each have 3 choices and $3\times3=9$

b) No function $S \to T$ is injective as $|S| > |T|$
However, 6 are surjective, as 2 map all to $\{4\}$ or all to $\{5\}$ and the rest go to $\{4,5\}$ so 6 are surjective

c) 6 are injective as 3 functions $T \to S$ will have only 1 element, and the rest will have each $t \in T$ going to a different $s \in S$.
0 functions $T \to S$ are surjective as $|T| < |S|$ and each $t \in T$ can only go to 1 $s \in S$ so one element in $s \in S$ will not be in $f(T)$.

6 a) Proof by contrapositive. If f isn't injective $\Rightarrow$ $g \circ f$ isn't injective.
If f isn't injective $\exists x_1, x_2 \in X$ s.t. $f(x_1) = f(x_2)$ but $x_1 \neq x_2$. Since $f(x_1) = f(x_2)$, $g(f(x_1)) = g(f(x_2))$ but $x_1 \neq x_2$ so $g \circ f$ isn't injective.
As the contrapositive is true, if $g \circ f$ is injective, then f is injective $\square$

6b) If $g \circ f$ is surjective, then $\forall z \in Z$ $\exists x \in X$ s.t. $g(f(x)) = z$. Since $f: X \to Y$ $f(x) = y$ where $y \in Y$, therefore $g(y) = z$, so $\forall z \in Z$ $\exists y \in Y$ s.t. $g(y) = z$ and g is surjective

7 Since $f: X \to Y$ is surjective $\forall y \in Y$ $\exists x \in X$ s.t. $f(x) = y$.
Want to show $f^{-1}: P(Y) \to P(X)$ is injective.
Use proof by contradiction on pre-image.
If $\exists B_1, B_2 \subseteq P(Y)$ where $B_1 \neq B_2$ then $f^{-1}(B_1) \neq f^{-1}(B_2)$

$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ for B_1, B_2
$B_1, B_2 \subseteq P(Y)$ and WLOG let $z \in B_1 - B_2$.
$\exists x \in X$ s.t. $f(x) = z$. as f is surjective.
$x \in f^{-1}(B_1)$ but $x \notin f^{-1}(B_2)$ as $z \notin B_2$ and f is a surjective
Therefore for $B_1 \neq B_2 \Rightarrow f^{-1}(B_1) \neq f^{-1}(B_2)$
Then the contrapositive is true, meaning $f^{-1}: P(Y) \to P(X)$ is injective.