Homework 6

Early problem due on Gradescope at 11:59 pm on Tuesday, March 7th.
Due on Gradescope at 11:59 pm on Friday, March 10th.

Definition 1. Let X be a topological space and $x \in X$ be a point. An *neighborhood of x* is an open subset U of X so that $x \in U$.

Given a basis $\mathcal{B}$ for the topology on X, we say a *basic open neighborhood of x* is a basis element $B \in \mathcal{B}$ so that $x \in B$.

(1) (Early problem) Suppose X and Y are topological spaces and $\mathcal{B}_X, \mathcal{B}_Y$ are bases for their respective topologies.
  (a) Show that $f : X \to Y$ is continuous if and only if for each $B \in \mathcal{B}_Y$, we have that $f^{-1}(B)$ is an open subset of X.
  (b) Show that $f : X \to Y$ is continuous if and only if for each $x \in X$ and each basic open neighborhood $B_{f(x)} \in \mathcal{B}_Y$ of $f(x)$, there is a basic open neighborhood $B_x \in \mathcal{B}_X$ of x so that $f(B_x) \subseteq B_{f(x)}$. (Possible hint: we saw in the optional problem on the first homework that $S \subseteq f^{-1}(T)$ if and only if $f(S) \subseteq T$.)
  (c) (Just think about this, no need to submit it.) Let $X, Y = \mathbb{R}$ and let

$$\mathcal{B}_X = \mathcal{B}_Y = \{(x - \epsilon, x + \epsilon) \mid x \in \mathbb{R}, \epsilon \in \mathbb{R}_{>0}\}$$

  be the usual basis for $\mathbb{R}$. Compare the statement of (b) with the ϵ-δ definition of continuity.

(2) Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is a neighborhood U of x such that $U \cap A$ is an open subset of X. Show that A is open in X.

(3) (a) If $\{\tau_\alpha\}_{\alpha \in A}$ is a family of topologies on X, show that $\bigcap_{\alpha \in A} \tau_\alpha$ is a topology on X. Is $\bigcup_{\alpha \in A} \tau_A$ a topology on X?
  (b) Let $\{\tau_\alpha\}_{\alpha \in A}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections τ_α, and a unique largest topology contained in all τ_α.
  (c) If $X = \{a, b, c\}$, let

$$\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \tau_2 = \{\emptyset, X, \{a\}, \{a, c\}\}.$$

  Find the smallest topology containing τ_1 and τ_2, and the largest topology contained in both τ_1 and τ_2.

(4) Show that if $\mathcal{B}$ is a basis for a topology on X, then the topology generated by $\mathcal{B}$ equals the intersection of all topologies on X that contain $\mathcal{B}$.

(5) (Part (c)) on next page!)
  (a) Find an example of a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ that is not the identity function. Prove that your example is a homeomorphism.
  (b) Find an example of two infinite subsets A, B of $\mathbb{R}$ that are not homeomorphic when given the subspace topology. Prove that they are not homeomorphic.

(c) Find an example of two finite topological spaces X, Y so that $|X| = |Y|$, but X and Y are not homeomorphic.