MATH 42 HOMEWORK 9

This homework is due at 11:59 p.m. (Eastern Time) on Tuesday, November 24. Scan the completed homework and upload it **as one pdf file** to Gradescope. The Canvas module "Written Assignments" has instructions for how to upload the assignment to Gradescope. This assignment covers §17.4−17.6.

Be sure to show work (integration by parts, substitutions, etc.) when calculating integrals. Unless stated in the problem, it is insufficient to simply respond with a numerical evaluation of definite integrals or an antiderivative of a non-standard integrand.

The provided solutions are meant to assist understanding; the solutions are not a grading rubric. Refer to the final pages of this pdf for plots generated in *Mathematica*.

(1) Use Green's Theorem to:
   (a) Find the area of the region bounded by $\mathbf{r}(t) = \langle t(1-t^2), 1-t^2\rangle$ for $t \in [-1,1]$. Plot the curve and be cognizant of the orientation.
   (b) Find the area of the astroid $x^{2/3} + y^{2/3} = 4$. Plot the curve. Hint: Show that the parameterization $x = 8\cos^3 t, y = 8\sin^3 t$ for $t \in [0, 2\pi)$ works.
   (c) Evaluate $\oint_C 2x\,\mathrm{d}y - 3y\,\mathrm{d}x$ where C is the square with vertices $(0,2), (2,0), (-2,0)$, and $(0,-2)$, oriented counterclockwise.

Important: If you used any curve plotting software tools to help you solve the problem above, please list them.

Solution: First part: Take the curve's parameterization in (a) from 1 to −1 to obtain the counterclockwise orientation

$$\frac{1}{2}\oint x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2}\int_1^{-1} t\left(1-t^2\right)(-2t) - \left(1-t^2\right)\left(1-3t^2\right)\mathrm{d}t = \frac{8}{15}.$$

Second part: Using the parameterization given the area is

$$\frac{1}{2}\int_0^{2\pi} 8\cos^3 t(24\sin^2 t\cos t) + 8\sin^3 t(24\cos^2 t\sin t)\,\mathrm{d}t = 96\int_0^{2\pi}\sin^2 t\cos^2 t\,\mathrm{d}t = 24\pi,$$

where the identity $\cos^2 t\sin^2 t = \frac{1}{8}(1-\cos(4t))$ easily gives the final integral.

Third part: Instead of evaluating the line integral of $\mathbf{F} = \langle -3y, 2x\rangle$, by Green's theorem compute the integral over the region

$$\iint_R 5\,\mathrm{d}A = 40.$$

Since the integrand is a constant, the integral is invariant under rotations of the region so rotate to obtain an iterated integral that is trivial (or simply recognize that the area of the rotated square is still 8).

(2) Without finding a potential function, calculate $\int_C e^x\cos y\,\mathrm{d}x - e^x\sin y\,\mathrm{d}y$ where C is the curve (not closed) from $(\ln 2, 0)$ to $(0,1)$ to $(-\ln 2, 0)$. Hint: Consider the straight-line closing of the curve and apply Green's Theorem.

Solution: Close the tent curve C by using a straight line C' from $(\ln 2, 0)$ to $(-\ln 2, 0)$ given by $\mathbf{r}(t) = \langle t, 0\rangle$ for $t \in [-\ln 2, \ln 2]$. The region enclosed by the union of C and C' is simply connected. Apply Green's Theorem, it is clear that the integrand of the double integral is $0 = -e^x \sin y - (-e^x \sin y)$ so it must be that $\int_C e^x \cos y \, \mathrm{d}x - e^x \sin y \, \mathrm{d}y + \int_{C'} e^x \cos y \, \mathrm{d}x - e^x \sin y \, \mathrm{d}y = 0$. The negation of the second term is $-\int_{-\ln 2}^{\ln 2} e^t \, \mathrm{d}t = -3/2$, with simplifications because $y = 0$ along C'.

(3) A three-dimensional *central force*, such as gravity or Coulomb attraction, may be written as $\mathbf{F}(r) = f(r)\mathbf{r} = f(r)\langle x, y, z\rangle$, where f is some scalar-valued function of the magnitude r. Show that the curl of any central force is irrotational, $\nabla \times \mathbf{F} = \vec{0}$.

Solution: Directly compute the first term of the curl; the other two terms follow by symmetry. Hence

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{i}} = \frac{\partial f(r)z}{\partial y} - \frac{\partial f(r)y}{\partial z} = z\frac{\partial f(r)}{\partial y} - y\frac{\partial f(r)}{\partial z}.$$

Recall that $r = \sqrt{x^2 + y^2 + z^2}$. The chain rule is applied to calculate $\frac{\partial f(r)}{\partial y} = \frac{\partial r}{\partial y}\frac{\mathrm{d}f(r)}{\mathrm{d}r} = \frac{y}{r} f'(r)$, and analogous results hold for the other derivatives. Hence the first component simplifies to $z(y/r)f'(r) - y(z/r)f'(r) = 0$.

(4) Let $f(x, y, z)$ be a scalar-valued function and let $\mathbf{A}$ and let $\mathbf{B}$ be vector fields on $\mathbb{R}^3$. Prove the following identities:
(a) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$.
(b) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$.
(c) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$.

Solution: First components are computed and the rest follow from symmetry. Let $\mathbf{A} = \langle A_x, A_y, A_z\rangle$ and likewise for $\mathbf{B}$. First, $\partial/\partial x(fA_x) = f\partial/\partial x A_x + A_x \partial/\partial x f$; summing and "undoing" the dot products gives the desired result. Second, $\partial/\partial x(A_y B_z - A_z B_y) = A_y \partial/\partial x B_z + B_z \partial/\partial x A_y - A_z \partial/\partial x B_y - B_y \partial/\partial x A_z$; collecting many such terms for the other components gives the desired result once the dot products are "undone" and the curl calculations recognized. Third, $(\partial/\partial y f A_z - \partial/\partial z f A_y)\hat{\mathbf{i}} = (f\partial/\partial y A_z + A_z \partial/\partial y f - f\partial/\partial z A_y - A_y \partial/\partial z f)\hat{\mathbf{i}} = (f(\nabla \times \mathbf{A}) \cdot \hat{\mathbf{i}} - \mathbf{A} \times (\nabla f) \cdot \hat{\mathbf{i}})\hat{\mathbf{i}}$.

(5) Compute the surface integral $\iint_S xyz \, \mathrm{d}S$, where S is the portion of the plane $z + y = 6$ that lies inside the cylinder $x^2 + y^2 = 4$.

Solution: Recognize that a clever symmetry argument could promptly give the answer of 0. To perform the surface integral, first parameterize the surface as $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 6 - u \sin v\rangle$ for $u \in [0, 2]$ and $v \in [0, 2\pi)$. The scaling term $|\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v| = |\langle 0, u, u\rangle| = u\sqrt{2}$. The integral is then $\sqrt{2} \int_{u=0}^{2} \int_{v=0}^{2\pi} u^3 \cos v \sin v(6 - \sin v) \, \mathrm{d}v \, \mathrm{d}u = 0$.

(6) Let the vector $\mathbf{r}$ denote the vector between the origin and (x, y, z). Let p be a real number and $|\mathbf{r}| = r$. Where possible, simplify work by performing the calculation for one coordinate and arguing that the other two coordinates follow a similar pattern.
(a) Show that $\nabla(1/r^p) = \frac{-p\mathbf{r}}{r^{p+2}}$.
(b) Show that $\nabla \cdot (\mathbf{r}/r^p) = \frac{3-p}{r^p}$.

(c) Show that $\nabla \cdot \nabla(1/r^p) = \frac{p(p-1)}{r^{p+2}}$. Hint: Use the previous parts.
(d) Sketch the vector field $-\nabla(1/r) = \hat{\mathbf{r}}/r^2$ and show that its divergence, $\nabla \cdot (\hat{\mathbf{r}}/r^2)$, is zero.
(e) Remark on the vector field drawn compared to the divergence calculated.
(f) Compute, $\iint_{\text{sphere}} r^{-2}\,\mathrm{d}S$, the surface integral over an origin-centered sphere of radius R. Is the value obtained the same for any $R > 0$?

See below for some remarks on the importance of this computation.[1]

Solution: Through this solution, first components are computed and the rest follow from symmetry. The chain rule reigns throughout this problem. First, $\partial(r^{-p})/\partial x = (\partial(r^{-p})/\partial r)(\partial r/\partial x) = -p(1/r^{p+1})(x/r)$ where as before $\partial r/\partial x = x/r$ was used. The vector comprising the three components may be succinctly written as $\nabla(1/r^p) = \frac{-p\mathbf{r}}{r^{p+2}}$ with $\mathbf{r} = \langle x, y, z\rangle$. Second, $\partial(x/r^p)/\partial x = r^{-p} + x\partial(r^{-p})/\partial x = r^{-p} - x^2 p(1/r^{p+2})$, using the chain rule and the previous result. Summing the three terms corresponding to each coordinate, $\nabla \cdot (\mathbf{r}/r^p) = \frac{3-p}{r^p}$ comes from the simplification of $r^2 = x^2 + y^2 + z^2$. Using these results $\nabla \cdot \nabla(1/r^p) = (-p)\nabla \cdot \frac{\mathbf{r}}{r^{p+2}} = \frac{-p(3-(p+2))}{r^{p+2}} = \frac{p(p-1)}{r^{p+2}}$. And $\nabla \cdot (\hat{\mathbf{r}}/r^2) = \nabla \cdot (\mathbf{r}/r^3) = 0$.

This vector field radiates outward from the origin, and the magnitude of the vectors decreases quadratically with distance from origin. It seems surprising, absent further mathematics, that the divergence appears to be zero everywhere for such a field.

The surface integral over any origin-centered sphere of the negative gradient of r^{-1} reduces to $\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} r^{-2} r^2 \sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi$, regardless of the radius of the sphere.

[1]The calculation in part (c) tells us the divergence of a vector field given by the gradient of the scalar field $1/r^p$. For $p = 1$, the divergence of the vector field is zero almost everywhere (except at the origin) yet the surface integral over any sphere is non-zero and constant. This suggests that something exciting happens at the origin. One example of what we have encountered is a single point in space (with no dimensional extent) that is endowed with electric charge, referred to as a *point charge*.

The potential function for an electric point charge q at the origin is $\phi = (4\pi\epsilon_0)^{-1}(q/r)$ and the force $\mathbf{F} = -\nabla\phi \propto r^{-2}$. Likewise $\nabla \cdot \mathbf{F} = \rho/\epsilon_0$ where $\rho(x, y, z)$ is the electric charge density – so everywhere, except for the exact location of the point charge, the divergence is zero. This indicates that a point charge is a source/sink of electric field.

Recall that all of these computations were done assuming three-dimensional space: The answer to part (c) disappears only when $p = 1$ so there is something special about nature's choice to have many forces follow inverse-square laws. One view is that this phenomenon is a geometric manifestation of the dimensions of space in which our world resides. Perhaps some physical laws would need modification if more complicated understandings of the universe's geometry were considered…

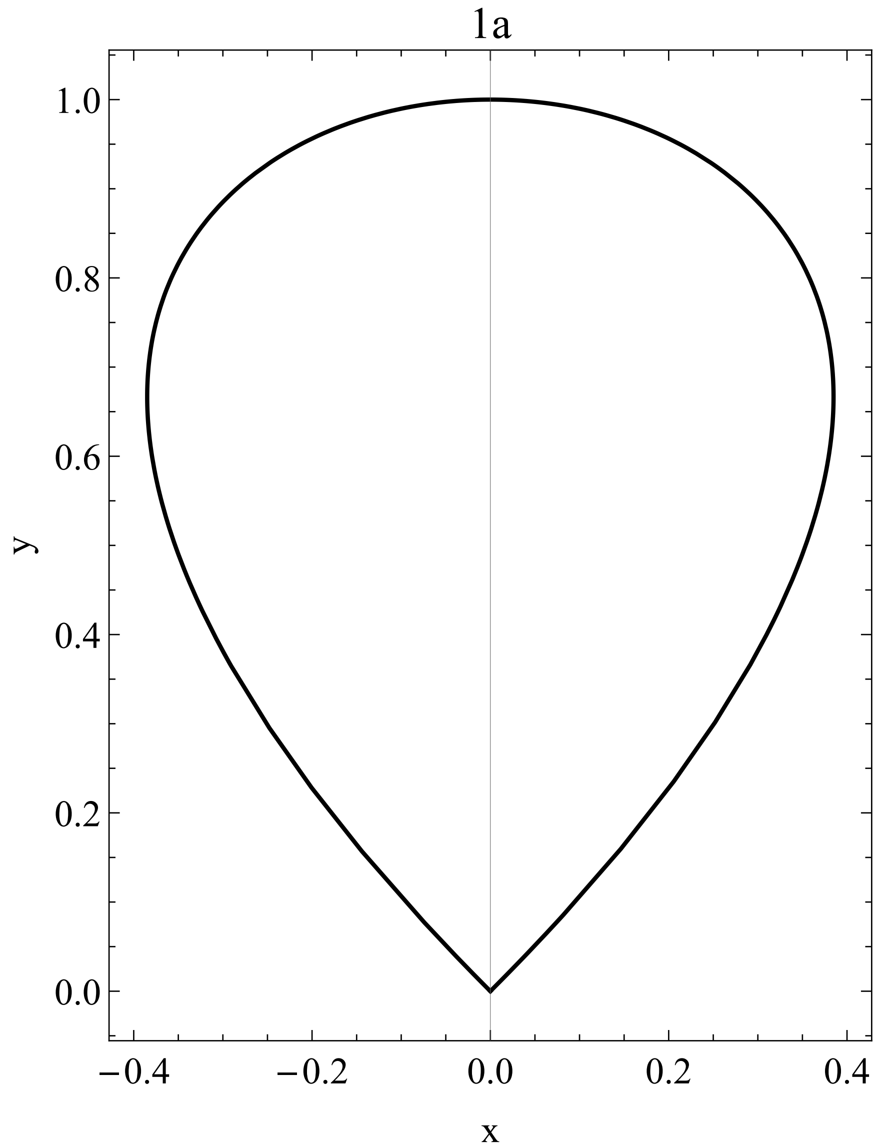

Figure 1. Plot for problem 1 part (a).

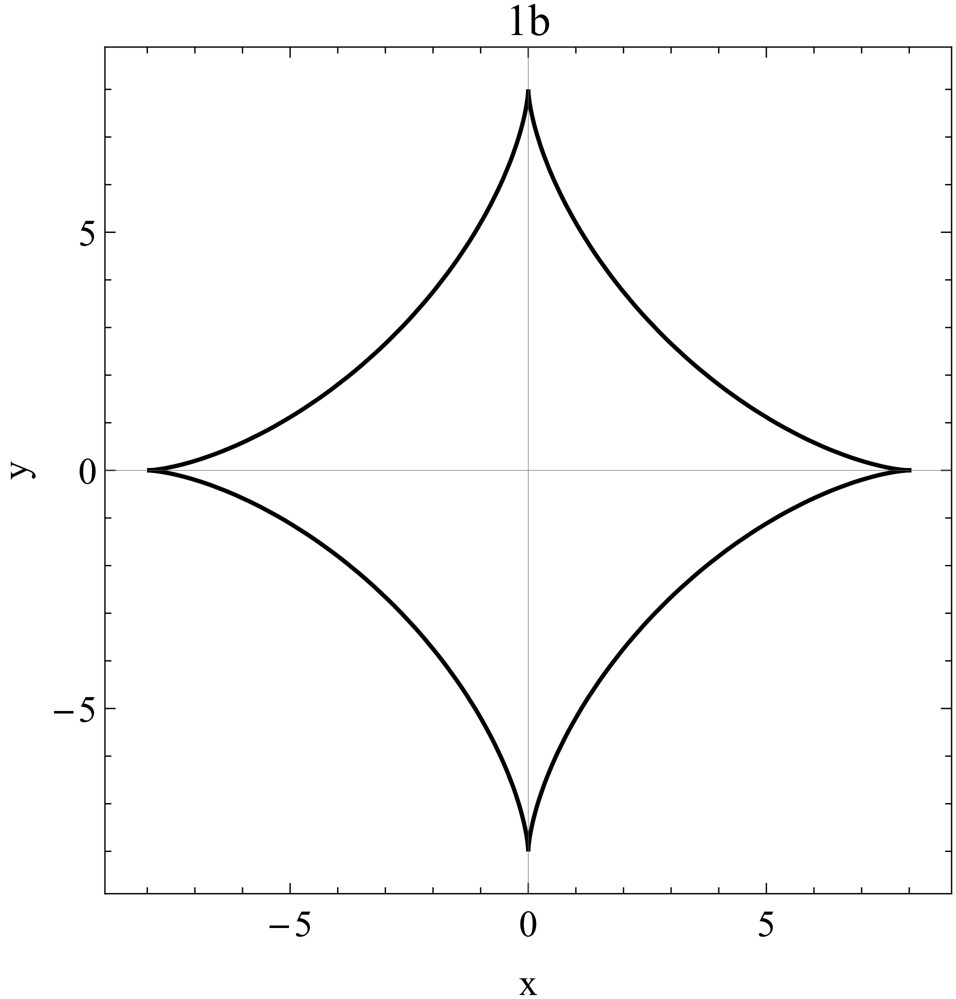

FIGURE 2. Plot for problem 1 part (b).

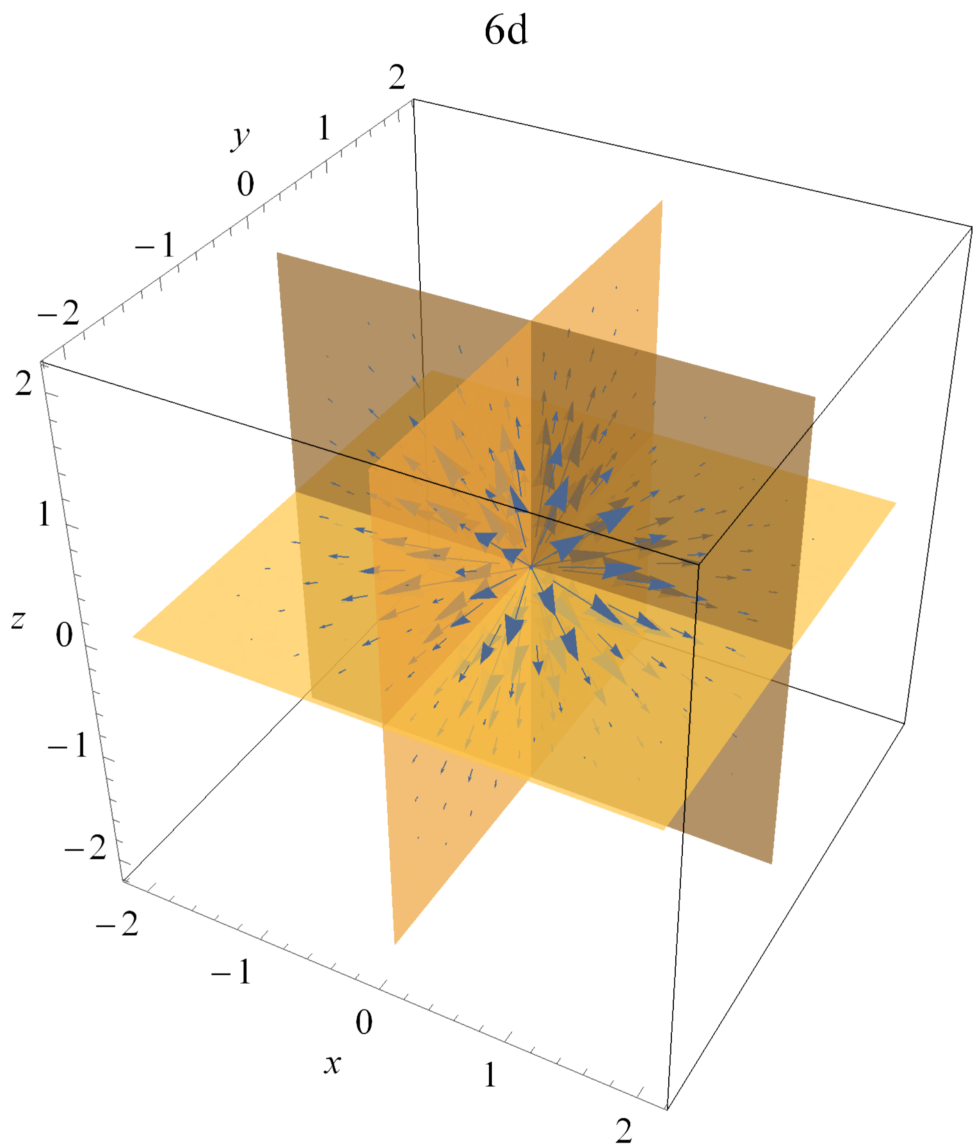

FIGURE 3. Vector field plot for problem 6 part (d).