Math 70 HW 7

1 a) Let $S' = \{v_1+v_2, v_1-v_3, v_2+v_3\}$

For S' to be an independent set,

$c_1(v_1+v_2) + c_2(v_1-v_3) + c_3(v_2+v_3) = 0$, $c_1, c_2, c_3 = 0$

$c_1v_1 + c_1v_2 + c_2v_1 - c_2v_3 + c_3v_2 + c_3v_3 = 0$

$(c_1+c_2)v_1 + (c_1+c_3)v_2 + (c_3-c_2)v_3 = 0$

Need to solve for when $c_1+c_2=0$, $c_1+c_3=0$, $c_3-c_2=0$ yields the matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

c_3 is a free variable, and since $c_3 \neq 0$, S' is not an independent set $\boxtimes$

1 b) $S' = \{v_1+v_2, v_2, v_1-v_3\}$

For S' to be an independent set

$c_1(v_1+v_2) + c_2v_2 + c_3(v_1-v_3) = \vec{0}$, $c_1, c_2, c_3 = 0$

$c_1v_1 + c_1v_2 + c_2v_2 + c_3v_1 - c_3v_3 = \vec{0}$

$(c_1+c_3)v_1 + (c_1+c_2)v_2 - c_3v_3 = \vec{0}$. Since S is independent, let's set $c_1+c_3=0$, $c_1+c_2=0$, $-c_3=0$,

This means $c_3=0$, $c_1=0$, $c_2=0$. Therefore, the set $S' = \{v_1+v_2, v_2, v_1-v_3\}$ is linearly independent set. $\boxtimes$

2 a) Since B is the basis for V, $V = \text{span}\{B\}$. Thus, all vectors in B can be written as a linear combination of $b_1 \dots b_n$.

Since T is onto, all vectors $\vec{w} \in W$, can be written as $T(\vec{v})$, where $\vec{v} \in V$, and $\vec{v} = c_1b_1 + c_2b_2 + \dots c_nb_n$ gives $T(c_1b_1 + c_2b_2 + \dots c_nb_n) = c_1T(b_1) + c_2T(b_2) + \dots c_nT(b_n) = \vec{w}$

$c_1T(b_1) + c_2T(b_2) + \dots c_nT(b_n)$ is a linear combination for $T(b_1) \dots T(b_n)$

This means all vectors $\vec{w} \in W$ are a linear combination of $\{T(b_1), T(b_2) \dots T(b_n)\}$, and therefore, the set B' spans W, as all $\vec{w} \in W$, are a linear combination of B'. $\boxtimes$

2 b) Assume $\dim(W) > \dim(V)$. Since the basis for V, B, has n vectors, $\dim(V) = n$, $\dim(W) > n$, so W has a basis with more than n vectors. This means a set with n or less vectors cannot span W. This contradicts part A, as we showed B', a set with n vectors, spans W. Therefore, $\dim(W) \leq \dim(V)$ $\boxtimes$

3 a) $h_1(0) = 1$, $h_1(1) = 0$, $h_1(2) = 0$
$h_2(0) = 0$, $h_2(1) = 1$, $h_2(2) = 0$
$h_3(0) = 0$, $h_3(1) = 0$, $h_3(2) = 1$

3 b) If B is a basis for $\mathbb{P}_2$, then regardless of t, c_1, c_2, c_3 will equal 0.

$c_1 h_1(t) + c_2 h_2(t) + c_3 h_3(t) = 0$

$c_1 h_1(0) + c_2 h_2(0) + c_3 h_3(0) = 0$, gives us $c_1 = 0$. To solve for c_2,

$c_1 h_1(1) + c_2 h_2(1) + c_3 h_3(1) = 0$, gives us $c_2 = 0$. To solve for c_3, let

$c_1 h_1(2) + c_2 h_2(2) + c_3 h_3(2) = 0$, gives us $c_3 = 0$

This means $c_1, c_2, c_3 = 0$, meaning B is linearly independent. Additionally, since the highest degree from $h_1 \dots h_3$ is 2, B spans $\mathbb{P}_2$. Therefore, B is a basis for $\mathbb{P}_2$. $\boxtimes$

Scanned with CamScanner

3 c) If $p(t)^2 - t$ is an element of the space where $h_1(t), h_2(t), h_3(t)$ are the basises, then $p(t) = c_1h_1(t) + c_2h_2(t) + c_3h_3(t)$, by the definition of a basis this means that.

$c_1h_1(0) + c_2h_2(0) + c_3h_3(0) = p(0)$, $p(0) = 1$
$c_1 = 1$

$c_1h_1(1) + c_2h_2(1) + c_3h_3(1) = p(1)$, $p(1) = 1$
$c_2 = 1$

$c_1h_1(2) + c_2h_2(2) + c_3h_3(2) = p(2)$, $p(2) = 3$
$c_3 = 3$ $\quad c_3h_3(2) = 3$

Since $c_1 = 1$, $c_2 = 1$, $c_3 = 3$ for when $p(t)$ is a linear combination of $h_1(t), h_2(t)$, and $h_3(t)$, the coordinate of $p(t)$ relative to B is

$$\boxed{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}$$

4 $\text{rank}(A) = \dim(\text{of col}(A))$, since A is $n \times n$, and rank$=n$
dim of $\text{col}(A) = n$

By the invertible matrix theorem, for a matrix $A_{n \times n}$, A is invertible when $\text{Rank}(A) = n$. Additionally, if A is invertible, A^T is invertible. Since A is invertible, A^T is also invertible. To prove $A^T \cdot A$ is invertible, we first know for any two invertible matrices,
$(AB)^{-1} = (B^{-1}A^{-1})$

$A^TA(A^TA)^{-1} = A^TA((A^{-1}) \cdot (A^T)^{-1}) = A^T(AA^{-1}) \cdot (A^T)^{-1}$

Since $AA^{-1} = I_n$, $= A^T \cdot I_n \cdot (A^T)^{-1} = A^T \cdot (A^T)^{-1} = I_n$ $\quad A \cdot A^{-1} = I_n$.

Since the equation reduces to I_n, A^TA has an inverse and is an invertible matrix