Math 171 HW

2) $\forall x \in A$, $\exists u_x$ where u_x is a neighborhood of X. As $u_x \cap A$ is open,

$$\bigcup_{x \in X} u_x \cap A = \left(\bigcup_{x \in X} u_x\right) \cap A = \text{open}$$

Claim $\bigcup_{x \in X} u_x \cap A = A$

Let $x \in \left(\bigcup u_x\right) \cap A$, then $x \in A$ so $\bigcup_{x \in X} u_x \cap A \subseteq A$

Let $x \in A$, then $\exists u_x$ s.t. $u_x \cap A$ is open

so $x \in \bigcup_{x \in X} u_x \cap A$ and $A \subseteq \bigcup u_x \cap A$.

So then $\bigcup_{x \in X} u_x \cap A = A$ so A is open

3a) Show $\bigcap_{\alpha \in A} \tau_\alpha$ is a topology

i) As τ_α is a topology $\forall \alpha \in A$ then $\emptyset, X \in \tau_\alpha$ $\forall \alpha \in A$ so $\emptyset, X \in \bigcap_{\alpha \in A} \tau_\alpha$

ii) $\forall i \in I$ let $u_i \in \bigcap_{\alpha \in A} \tau_\alpha$

Show $\bigcup_{i \in I} u_i \in \bigcap_{\alpha \in A} \tau_\alpha$

As $u_i \in \bigcap_{\alpha \in A} \tau_\alpha$ $\forall i$, then it follows that $u_i \in \tau_\alpha$ $\forall \alpha \in A$ so $\bigcup u_i \in \bigcap_{\alpha \in A} \tau_\alpha$

iii) Let $u_1 \ldots u_n \in \bigcap \tau_\alpha$ show $\bigcap_{i=1}^{n} u_i \in \bigcap_{\alpha \in A} \tau_\alpha$. As $u_i \in \tau_\alpha$ $\forall \alpha \in A$, it follows that $\bigcap_{i=1}^{n} u_i \in \bigcap_{\alpha \in A} \tau_\alpha$. Therefore, $\bigcap \tau_\alpha$ is a topology on X.

$\bigcup \tau_\alpha$ isn't a topology. Consider $X = \{1, 2, 3\}$ $\tau_1 = \{\emptyset, X, \{1, 2\}\}$ $\tau_2 = \{\emptyset, X, \{2, 3\}\}$

Both are clearly topologies on X.

$\tau_1 \cup \tau_2 = \{\emptyset, X, \{1, 2\}, \{2, 3\}\}$. But $U = \{1, 2\}$ and $V = \{2, 3\}$ are open and $U \cap V = \{2\}$ which isn't open, so fails finite intersection condition.

3b) Unique largest topology contained in all τ_α is $\bigcap_{\alpha \in A} \tau_\alpha$

To show, suppose it's not the largest and $\exists$ a topology τ' s.t. $\bigcap \tau_\alpha \subseteq \tau'$ but $\tau' \not\subseteq \bigcap \tau_\alpha$. Then $\exists\, u \in \tau'$ s.t. $u \notin \bigcap \tau_\alpha$ so $\exists\, \tau_\alpha$ s.t. $u \notin \tau_\alpha$ and meaning the topology τ' isn't contained in all τ_α so such a topolgy τ' doesn't exist and $\bigcap \tau_\alpha$ is the unique largest topology

The unique smallest topology is
$\tau = \bigcap \{\tau \mid \tau \supseteq \tau_\alpha \ \forall \alpha \in A\}$,
To see, assume $\exists\, \tau'$ s.t. $\tau_\alpha \subseteq \tau'$ $\forall \alpha \in A$. Then $\tau \subseteq \tau'$ as τ is in the intersection, making it the smallest topology τ.

3c) Smallest: $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$
Largest: $\{\emptyset, X, \{a\}\}$

4) Let τ_B be topology generated by $\mathcal{B}$ and $\{\tau_\alpha\}_{\alpha \in A}$ a family of topologies s.t. $\mathcal{B} \subseteq \tau_\alpha$ $\forall \alpha \in A$. We need to show $\tau_B = \bigcap_{\alpha \in A} \tau_\alpha$

First $\bigcap_{\alpha \in A} \tau_\alpha \subseteq \tau_B$ is trivial as $\tau_B \in \{\tau_\alpha\}_{\alpha \in A}$ as $B \subseteq \tau_B$ since τ_B is generated by $\mathcal{B}$.

To show $\tau_B \subseteq \bigcap_{\alpha \in A} \tau_\alpha$. Let $U \in \tau_B$ then $\forall x \in U\ \exists\, B_x \in \mathcal{B}$ s.t. $B_x \subseteq U$ and $U = \bigcup_{x \in U} B_x$. Since $\mathcal{B} \subseteq \tau_\alpha$ $\forall \alpha \in A$, then

$$\bigcup_{x \in U} B_x \subseteq \bigcup_{x \in X} B_x \subseteq \bigcap \tau_\alpha \text{ so}$$

As over all of X

So $U \subseteq \bigcap \tau_\alpha \rightarrow \tau_B \subseteq \bigcap \tau_\alpha$ and $\tau_B = \bigcap_{\alpha \in A} \tau_\alpha$ $\square$

5a) $f:\mathbb{R}\to\mathbb{R}$ $\qquad f(x)=2x$ is
$x\to 2x$ $\qquad$ homeomorphism
$f^{-1}:\mathbb{R}\to\mathbb{R}$
$y\to y/2$

We can clearly see f and f^{-1} are continuous functions as polynomials are continuous, and that $f(f^{-1}(y))=y$ and $f^{-1}(f(x))=x$ So they are inverses.
Therefore $f(x)=2x$ is a homeomorphism on $\mathbb{R}\to\mathbb{R}$

b) Consider $A=\mathbb{Q}$ and $B=\mathbb{Q}^c$ (irrationals)
We can only construct a homeomorphism between sets if there is a bijection, meaning that $|A|=|B|$. However $\mathbb{Q}$ is countable and $\mathbb{Q}^c$ is not, so as they don't have the same cardinality then you cannot construct a bijection and therefore not a homeomorphism between them. (All assume $\mathbb{Q}, \mathbb{Q}^c$ in subspace topology)

c) $X=\{1,2,3\}$ and $Y=\{1,2,3\}$
$|X|=3=|Y|$
Consider $(X, \tau_{discrete})$ and $(Y, \tau_{indiscrete})$
$f: X\to Y$ is always continuous.
$f^{-1}: Y\to X$. However, so for $f^{-1}: Y\to X$
Consider $V\subseteq Y$ is closed in Y, then $f^{-1}(V)=U$ where $U\subseteq X$ is closed
and for $V\subseteq Y$ where $V\neq\emptyset, X$ then V is closed, but U is always open, as X is on discrete topology, so $f^{-1}(\text{closed})\neq \text{closed}$
and X and Y are not homeomorphic.