1. Math 65, Exam 2, Fall 2021.

SHOW ALL YOUR WORK Proofs should be written neatly. You must state the definitions and results you use and include all details to get full credit. Each completely correct question is worth 20 points

Question 1. We have four coins, two are fair and come heads half the time and the other two are loaded and come heads 25% of the time.

(a) Define what it means for two events A, B to be independent.
(b) If you pick a coin randomly what is the probability that it is a fair coin?
(c) If you pick a coin randomly and toss it, what is the probability that it comes up heads?
(d) Are the events"a randomly picked coin is fair" and " the toss of a coin comes up heads' independent?

Question 2. Write your answer to this question using factorials and/or choose numbers but not multi-choose numbers. Carefully explain how you obtain your answers.

(a) Thirty identical food lots are to be given to three food pantries. In how many ways can this be done?
(b) Thirty identical food lots are to be given to three food pantries so that the first food pantry gets at most 10. In how many ways can this be done?
(c) Thirty identical food lots are to be given to three food pantries so that the first and second food pantries get each at most 10. In how many ways can this be done?

Question 3. Let $(X, \preceq)$ be a poset.

(a) Give the definition for $a \in X$ is a maximal element.
(b) Give the definition for $b \in X$ is a maximum element.
(c) Prove or disprove: if $(X, \preceq)$ is a **finite** poset and $x \in X$ an arbitrary element, then there exists $y \in X$ such that y is a maximum and $x \preceq y$. Provide all details in your proof.
(d) Prove or disprove: if$(X, \preceq)$ is a **finite** poset and $x \in X$ an arbitrary element, then there exists $y \in X$ such that y is maximal and $x \preceq y$. Provide all details in your proof.

Question 4. Let S be the set of functions from the real line $\mathbb{R}$ to itself. Given $f_1, f_2 \in S$, that is $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, we will say that $f_1 \sim f_2 \Leftrightarrow f_1(0) = f_2(0)$.

(a) Show that $\sim$ is an equivalence relation.
In the two parts below, we denote by **B be the set of equivalence classes by** $\sim$.
(b) Define the assignment G_1 on B with outputs in $\mathbb{R}$ given by $G_1([f]) = f(0)$. Is G_1 a well defined function $G_1 : B \to \mathbb{R}$?
(c) Define the assignment G_2 on B with outputs in $\mathbb{R}$ given by $G_2([f]) = f(1)$. Is G_2 a well defined function $G_2 : B \to \mathbb{R}$?

Question 5. Show either algebraically or using a combinatorial argument that

$$\binom{n+2}{3} = \binom{n}{3} + 2\binom{n}{2} + \binom{n}{1}$$

If you prove it both ways correctly, you will receive two bonus points. Hint: Think of two of the $n + 2$ elements as special.

Post-test Instructions If possible scan and upload the test to gradescope. Make sure that you uploaded everything and matched questions to pages If you cannot scan or upload, it is OK too, just give me the hard copy and let me know.

Question 6. We have four coins, two are fair and come heads half the time and the other two are loaded and come heads 25% of the time.

(a) Define what it means for two events A, B to be independent.
(b) If you pick a coin randomly what is the probability that it is a fair coin?
(c) If you pick a coin randomly and toss it, what is the probability that it comes up heads?
(d) Are the events"a randomly picked coin is fair" and " the toss of a coin comes up heads' independent?

Answer 1. With a single toss of the coin, we consider the following events
F the coin is fair
L the coin is loaded
H the coin comes up heads
T the coin comes up tails
We are given the following information

$$P(F) = \frac{2}{4}, P(L) = \frac{2}{4}, P(H|F) = \frac{1}{2}, P(H|L) = \frac{1}{4}$$

(a) Two events A, B are said to be independent if $P(A|B) = P(A)$. As $P(A|B) = \frac{P(A\cap B)}{P(B)}$, two events are independent if and only if

$$P(A|B) = P(A) \iff P(A) = \frac{P(A \cap B)}{P(B)} \iff P(A\cap B) = P(A)P(B) \iff P(B|A) = P(B)$$

(b) There are four coins, two of which are fair. The probability that a random picked coin is fair is $P(F) = \frac{2}{4}$,
(c) We compute the probability that a randomly picked coin comes up heads

$$P(H) = P(H \cap F) + P(H \cap L) = P(H|F)P(F) + P(H|L)P(L) = \frac{1}{2}\frac{1}{2} + \frac{1}{4}\frac{1}{2} = \frac{3}{8}$$

(d) The two events are not independent as

$$P(H|F) = \frac{1}{2} \neq \frac{3}{8} = P(H)$$

Question 7. Write your answer to this question using factorials and/or choose numbers but not multi-choose numbers. Carefully explain how you obtain your answers.

(a) Thirty identical food lots are to be given to three food pantries. In how many ways can this be done?
(b) Thirty identical food lots are to be given to three food pantries so that the first food pantry gets at most 10. In how many ways can this be done?
(c) Thirty identical food lots are to be given to three food pantries so that the first and second food pantries get each at most 10. In how many ways can this be done?

Answer 2. (a) As the food packages are identical, the order does not matter, the only thing that is important is how many each charity gets. We want to find non-negative numbers x_1, x_2, x_3 such that

$$x_1 + x_2 + x_3 = 30$$

This is like choosing 30 objects of three different kinds. The number of options is

$$\left(\!\!\binom{3}{30}\!\!\right) = \binom{32}{2}.$$

(b) This time, we want to find non-negative numbers x_1, x_2, x_3 such that

$$x_1 + x_2 + x_3 = 30, \;\; x_1 \leq 10$$

We compute first the number of options of numbers x_1, x_2, x_3 such that

$$x_1 + x_2 + x_3 = 30, \;\; x_1 \geq 11$$

Then

$$(x_1 - 11) + x_2 + x_3 = 19, \;\; x_1 - 11 \geq 0, x_2 \geq 0, x_3 \geq 0$$

The number of options is

$$\left(\!\!\binom{3}{19}\!\!\right) = \binom{21}{2}.$$

The number we are interested in is the difference between the number in (a) and this number, namely

$$\binom{32}{2} - \binom{21}{2}.$$

(c) The bad options are the ones in which the first panty gets at least 11 (there are $\binom{21}{2}$ of them) or the second pantry gets at leas 11 (there are also $\binom{21}{2}$ of them). These two sets are not disjoint, they intersect in the set where they both gets at least 11 and therefore only the 8 remaining packages are to be shared. The number of such options is

$$\left(\!\!\binom{3}{8}\!\!\right) = \binom{10}{2}.$$

Then, the number of bad options is

$$\binom{21}{2} + \binom{21}{2} - \binom{10}{2}$$

The allowable options is the complement

$$\binom{32}{2} - \binom{21}{2} - \binom{21}{2} + \binom{10}{2}.$$

Question 8. Let $(X, \preceq)$ be a poset.

(a) Give the definition for $a \in X$ is a maximal element.
(b) Give the definition for $b \in X$ is a maximum element.
(c) Prove or disprove: if $(X, \preceq)$ is a finite poset and $x \in X$ an arbitrary element, then there exists $y \in X$ such that y is a maximum and $x \preceq y$. Provide all details in your proof.
(d) Prove or disprove: if$(X, \preceq)$ is a finite poset and $x \in X$ an arbitrary element, then there exists $y \in X$ such that y is maximal and $x \preceq y$. Provide all details in your proof.

Answer 3. (a) An element $x \in X$ is maximal if there is nothing strictly above it, that is if any $y \in X$ satisfies $x \preceq y$, then $x = y$.
(b) An element $x \in X$ is a maximum if it is above everything else, that is $\forall y \in X, \;\; y \preceq x$.

(c) This is false, a finite poset may not have a maximum. For example, take the set $\{2, 3\}$ of natural numbers ordered by divisibility. The two elements are not related, therefore there is no maximum.

(d) Assume that $(X, \preceq)$ is a finite poset and $x \in X$ is an arbitrary element. If x is maximal, take $y = x$. If x is not maximal, by definition of maximal, there exists $x_1 \neq x$ such that $x \preceq x_1$. If x_1 is maximal, take $y = x_1$. If x_1 is not maximal, by definition of maximal, there exists $x_2 \neq x_1$ such that $x_1 \preceq x_2$. As the set is finite, the process must stop and the only way to stop is by reaching a maximal. Then we have

$$x \preceq x_1 \preceq x_2 \cdots \preceq x_n = y$$

where y is maximal and by the transitive property $x \preceq y$.

Question 9. Let S be the set of functions from the real line $\mathbb{R}$ to itself. Given $f_1, f_2 \in S$, that is $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, we will say that $f_1 \sim f_2 \Leftrightarrow f_1(0) = f_2(0)$.

(a) Show that $\sim$ is an equivalence relation.
In the two parts below, we denote by **B be the set of equivalence classes by** $\sim$.

(b) Define the assignment G_1 on B with outputs in $\mathbb{R}$ given by $G_1([f]) = f(0)$. Is G_1 a well defined function $G_1 : B \to \mathbb{R}$?

(c) Define the assignment G_2 on B with outputs in $\mathbb{R}$ given by $G_2([f]) = f(1)$. Is G_2 a well defined function $G_2 : B \to \mathbb{R}$?

Answer 4. (1) We need to show that $\sim$ is reflexive symmetric and transitive:

- $f \in S$, $f \sim f \Leftrightarrow f(0) = f(0)$ which is true. Hence $\sim$ is reflexive.
- Symmetric, let $f_1, f_2 \in S$, be such that $f_1 \sim f_2$. By definition of $\sim$, this means that $f_1(0) = f_2(0)$. Then also $f_2(0) = f_1(0)$. By definition of $\sim$, $f_2 \sim f_1$. So, $\sim$ is symmetric.
- Transitive: let $f_1, f_2, f_3 \in S$, be such that $f_1 \sim f_2$, $f_2 \sim f_3$. By definition of $\sim$, this means that $f_1(0) = f_2(0)$, $f_2(0) = f_3(0)$. Then also $f_1(0) = f_3(0)$. By definition of $\sim$, this implies that $f_1 \sim f_3$ and $\sim$ is transitive.

(2) Assume that $[f_1] = [f_2]$. By definition of $\sim$, this means that $f_1(0) = f_2(0)$. Defining $G_1([f]) = f(0)$, we would have

$$G_1([f_1]) = f_1(0) = f_2(0) = G_1([f_2])$$

So, G_1 is well defined on equivalence classes

(3) Define $f_1 : \mathbb{R} \to \mathbb{R}$ as the constant function 0, that is $f_1(r) = 0\ \forall r \in \mathbb{R}$. Define $f_2 : \mathbb{R} \to \mathbb{R}$ by $f_2(r) = r^2$. Then, $f_1(0) = 0 = 0^2 = f_2(0)$. Therefore $f_1 \sim f_2$ and $[f_1] = [f_2]$. On the other hand

$$G_2([f_1]) = f_1(1) = 0 \neq 1 = f_2(1) = G_2([f_2])$$

So, G_2 is not a well defined function on equivalence classes

Question 10. Show either algebraically or using a combinatorial argument that

$$\binom{n+2}{3} = \binom{n}{3} + 2\binom{n}{2} + \binom{n}{1}$$

If you prove it both ways correctly, you will receive two bonus points. Hint: Think of two of the $n + 2$ elements as special.

Answer 5. If we think of our set of $n+2$ elements as a set with n elements with 2 extra ones and we want to pick 3, we have the following options

- Choose all 3 among the n regular ones. There are $\binom{n}{3}$ options.
- Choose 2 among the n regular ones and the remaining one one of the two special ones. There are $\binom{n}{2}$ options for choosing the first 2, and 2 options for the remaining one, so $2\binom{n}{2}$ options in all.
- Choose one among the n regular ones and the remaining two the two special ones. There are $\binom{n}{1}$ options for choosing the first one, and only one options for the remaining two elements, so $\binom{n}{1}$ options in all.

Adding the total number of options confirms the given equation.

For the algebraic proof, we use that $\binom{m}{k}+\binom{m}{k+1}=\binom{m+1}{k+1}$. Take $m=n+1, k=2$, this gives

$$\binom{n+2}{3}=\binom{n+1}{2}+\binom{n+1}{3}$$

With $m=n$ and $k=2,1$ respectively, we obtain

$$\binom{n+1}{3}=\binom{n}{2}+\binom{n}{3}, \quad \binom{n+1}{2}=\binom{n}{1}+\binom{n}{2}$$

Substituting in the previous equation, we have

$$\binom{n+2}{3}=\binom{n}{1}+\binom{n}{2}+\binom{n}{2}+\binom{n}{3}=\binom{n}{1}+2\binom{n}{2}+\binom{n}{3}$$