# Homework 3

Early problem due on Gradescope at 8 pm on Wednesday, February 8th.
Due on Gradescope at 8 pm on Friday, February 10th.

**Definition 1.** (Universal property of the quotient, 2.0.)

Let $X$ be a set and $\sim$ an equivalence relation on $X$. Say a function $f : X \to B$ *respects the equivalence relation* if for all $x_1, x_2 \in X$, we have that $x_1 \sim x_2$ implies $f(x_1) = f(x_2)$. We say a set $Q$ together with a function $\pi : X \to Q$ *has the universal property of the quotient* if:

(a) For each set $B$ and each function $\overline{f} : Q \to B$, the composite function $f = \overline{f} \circ \pi$ respects the equivalence relation.

(b) For each set $B$ and each function $f : X \to B$ that respects the equivalence relation, there exists a unique function $\overline{f} : Q \to B$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & B \\ Q \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

commutes.

(1) (Early Problem) Prove that if $\pi : X \to Q$ and $\pi' : X \to Q'$ have the universal property of the quotient (2.0), then there is a unique bijection $f : Q \to Q'$ such that $f \circ \pi = \pi'$.

(For those using LaTeXfor their homework, look into the tikz-cd package.)

(2) Let $X_1, X_2$ be sets. Note that there are "inclusion" functions $j_1 : X_1 \cap X_2 \to X_1$, $j_2 : X_1 \cap X_2 \to X_2$, $i_1 : X_1 \to X_1 \cup X_2$ and $i_2 : X_2 \to X_1 \cup X_2$, all defined by the rule $x \mapsto x$, and the following diagram commutes:

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "i_2"] \\ X_1 \arrow[r, "i_1"] & X_1 \cup X_2. \end{tikzcd}$$

Prove that $X_1 \cup X_2$ has the following universal property:

For all sets $B$ and functions $f_1 : X_1 \to B$ and $f_2 : X_2 \to B$ such that

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"] & B. \end{tikzcd}$$

commutes, there exists a unique function $f : X_1 \cup X_2 \to B$ such that

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "i_2"] \arrow[ddr, bend left, "f_2"] & \\
X_1 \arrow[r, "i_1"] \arrow[drr, bend right, "f_1"'] & X_1 \cup X_2 \arrow[dr, dashed, "f"] & \\
& & B
\end{tikzcd}$$

commutes. (This is a special case of the "Universal property of the pushout.")

(3) Prove the following "laws of algebra" for all $x, y, z, w \in \mathbb{R}$ using only axioms (1)–(5) and "laws of algebra" proven in class and recitation.
  (a) $x/1 = x$
  (b) $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$
  (c) If $y, z \neq 0$, then $(1/y)(1/z) = 1/(yz)$
  (d) If $y, z \neq 0$, then $(x/y)(w/z) = (wx)/(yz)$
  (e) $x \neq 0 \implies 1/\mathrm{x} \neq 0$
  (f) If $w, z \neq 0$, then $1/(w/z) = z/w$
  (g) If $z \neq 0$, then $(xy)/z = x(y/z)$

(4) Prove the following "laws of inequalities" for all $x, y, z, w \in \mathbb{R}$, using axioms (1)–(6), the results of the previous exercise, and "laws of algebra and inequalities" proven in class and recitation.
  (a) $-1 < 0 < 1$
  (b) $xy > 0 \iff x$ and $y$ are both positive or both negative.
  (c) $x > 0 \implies 1/x > 0$
  (d) $x > y > 0 \implies 1/x < 1/y$.

(5) Let $a \in \mathbb{R}$. Define inductively

$$a^1 = a,$$
$$a^{n+1} = a^n \cdot a$$

for $n \in \mathbb{Z}_{\geq 0}$. Show that for $n, m \in \mathbb{Z}_{>0}$ and $a, b \in \mathbb{R}$:
  (a) $a^n \cdot a^m = a^{n+m}$
  (b) $(a^n)^m = a^{nm}$
  (c) $a^m b^m = (ab)^m$

These are called the *laws of exponents*. (Hint: for fixed $n$, prove the formulas by induction on $m$.)

(6) (Optional) Suppose $X_1, X_2, Y$ are sets and $a_1 : Y \to X_1$, $a_2 : Y \to X_2$ are functions. A set $P$ together with functions $i_1 : X_1 \to P$ and $i_2 : X_2 \to P$ such that

$$\begin{tikzcd} Y \arrow[r, "a_2"] \arrow[d, "a_1"] & X_2 \arrow[d, "i_2"] \\ X_1 \arrow[r, "i_1"] & P. \end{tikzcd}$$

is said to be a *pushout* if it satisfies the following universal property:

For all sets $B$ and functions $f_1 : X_1 \to B$ and $f_2 : X_2 \to B$ such that

$$\begin{tikzcd} Y \arrow[r, "a_2"] \arrow[d, "a_1"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"] & B. \end{tikzcd}$$

commutes, there exists a unique function $f : P \to B$ such that

$$\begin{tikzcd} Y \arrow[r, "a_2"] \arrow[d, "a_1"] & X_2 \arrow[d, "i_2"] \arrow[ddr, bend left, "f_2"] & \\ X_1 \arrow[r, "i_1"] \arrow[drr, bend right, "f_1"] & P \arrow[dr, dashed, "f"] & \\ & & B \end{tikzcd}$$

commutes.

Find a set $P$ and functions $i_1, i_2$ so that $P$ is a pushout. (Perhaps as a hint, the usual notation for the pushout is $X_1 \sqcup_Y X_2$.)