1. (10 points) (**Logic**)
   (a) Let A and B be logical statements. Use truth tables to show that the statement "$\sim (A \Rightarrow B)$" is equivalent to "$A \wedge (\sim B)$."
   (b) Find the negation of

   "For some $x \in \mathbb{R}$, $x^2 > 5$ and $x < -3$,"

   equivalently,

   "$\exists x \in \mathbb{R}$ such that $(x^2 > 5) \wedge (x < -3)$."

(a)

A	B	$A \Rightarrow B$	$\sim B$	$A \wedge (\sim B)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	T	F

opposite truth values

Since $A \Rightarrow B$ and $A \wedge (\sim B)$ have opposite truth values, they are negations of each other. Thus, $\sim (A \Rightarrow B)$ is equivalent to $A \wedge (\sim B)$.

(b) $\forall x \in \mathbb{R}$, $x^2 \leq 5$ or $x \geq -3$.

2. (10 points) *Let $x \in \mathbb{R}$. Consider the statement:* **"If $x \neq 0$, then $\exists \varepsilon > 0$ such that $|x| \geq \varepsilon$."**

(a) *Write the contrapositive of this statement without using the word "not.":*

If $\forall \varepsilon > 0 \ |x| < \varepsilon$, then $x = 0$.

(b) *Write the converse of this statement:*

If $\exists \varepsilon > 0 \ .\ni. \ |x| \geq \varepsilon$, then $x \neq 0$

(c) *Let $x \in \mathbb{R}$. Prove the if and only if statement, $\forall \varepsilon > 0$, $|x| < \varepsilon$* **iff** *$x = 0$ by proving both implications:*[1]

$\Rightarrow$ **("only if"):** We need to prove

If $\forall \varepsilon > 0$, $|x| < \varepsilon$, then $x = 0$

It is easier to prove the (logically equivalent) contrapositive statement

If $x \neq 0$, then $\exists \varepsilon > 0$ such that $|x| \geq \varepsilon$.

Proof:

If $x \neq 0$, then $|x| > 0$.
Let $\varepsilon = \frac{|x|}{2}$.
Then multiplying both sides of the inequality $1 > \frac{1}{2}$ by the positive number $|x|$, we obtain

$$|x| > \frac{1}{2}|x| = \varepsilon.$$

$\Leftarrow$ **("if"):** We need to show

If $x = 0$, then $\forall \varepsilon > 0$, $|x| < \varepsilon$

Proof: If $x = 0$ then $|x| = 0$ as well, so obviously whenever $\varepsilon > 0$, we have in particular $|x| = 0 < \varepsilon$, or $\forall \varepsilon > 0, |x| < \varepsilon$

3. (10 points) **(Inf and sup)** Fitzpatrick, §1.1, p. 11, #13. Suppose that S is a nonempty set of real numbers that is bounded. Prove that $\inf S \leq \sup S$.

Proof: Let $a = \inf(S)$ and $b = \sup(S)$. Since S is nonempty we can pick some $s \in S$. Then by definition, $a \leq s$ and $s \leq b$. By transitivity, $a \leq b$. $\square$

4. (10 points) (**Irrationality**) Prove that $\sqrt{10}$ is not a rational number. You may use the unique factorization property of integers.

Proof by contradiction. Assume that $\sqrt{10}$ is rational.

This means $\sqrt{10} = m/n$, where $m, n \in \mathbb{Z}$ and $n \neq 0$.

By cancelling out common factors, we may assume m/n in lowest terms. Since $\sqrt{10}$ is positive, we may assume $m, n \in \mathbb{N}$.

(If m, n are both negative, the negative signs cancel out.)

Square both sides to get $10 = m^2/n^2$, or

$$10n^2 = m^2. \qquad (1)$$

Factor m, n into prime factors: $m = q_1 \cdots q_s$, $n = p_1 \cdots p_r$. Then

$$10n^2 = m^2 \Rightarrow 2 \cdot 5\, p_1^2 \cdots p_r^2 = q_1^2 \cdots q_s^2.$$

By UF (unique factorization), 2 must be equal to a factor q_i on the right. Then 2 is a factor of m, so $m = 2k$ for some $k \in \mathbb{N}$. Eq. (1) becomes

$$10n^2 = m^2 = (2k)^2 = 4k^2$$

or $$5n^2 = 2k^2.$$

By UF again, since 2 is a factor of RHS, it is a factor of LHS $5n^2$ and so 2 is a factor of n. Then both m and n have 2 as a factor, contradicting the fact that m/n is in lowest terms. Therefore, the initial hypothesis is false and $\sqrt{10}$ is not rational. $\square$

5. (10 points) (**Mathematical induction**) Prove by mathematical induction the summation formula for geometric series: For $r \in \mathbb{R}$, $r \neq 1$: $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$.

Proof. **Initial Case:** For $n = 0$, the statement is that

$$r^0 = \frac{1-r^1}{1-r} .$$

Both sides equal a.

Inductive Step: Suppose for some n we know that

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} .$$

Then

$$\begin{aligned}
\sum_{k=0}^{n+1} r^k &= \sum_{k=0}^{n} r^k + r^{n+1} \\
&= \frac{1-r^{n+1}}{1-r} + r^{n+1} \\
&= \frac{1-r^{n+1}}{1-r} + \frac{r^{n+1}(1-r)}{1-r} \\
&= \left(\frac{1}{1-r}\right)\left(1-r^{n+1}+(r^{n+1}-r^{n+2})\right) \\
&= \frac{1-r^{n+2}}{1-r} .
\end{aligned}$$

$\square$

6. (15 points) (**Subsets, equality of sets**) Let $f: A \to B$ and let C and D be subsets of B.

(a) (7 points) Prove or provide a counterexample that $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$. To prove this, you need to show every element of $f^{-1}(C \cap D)$ is in $f^{-1}(C) \cap f^{-1}(D)$. To provide a counterexample, you give a *specific* A, B, $f: A \to B$ and subsets C and D of B, such that the assertion is not true for this specific A, B, C, D and f.

(b) (7 points) Prove or provide a counterexample to the assertion $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$ and prove your answer. To prove the assertion, you need to show every element of $f^{-1}(C) \cap f^{-1}(D)$ is in $f^{-1}(C \cap D)$. To provide a counterexample, you give a *specific* A, B, $f: A \to B$ and subsets C and D of B, such that the assertion is not true for this specific A, B, C, D and f.

(c) (1 point) Does $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ in general?

Proof. (a) $x \in f^{-1}(C \cap D)$ means $f(x) \in C \cap D$, in other words, $f(x) \in C$ and $f(x) \in D$, which in turn means $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. But this is precisely the statement that $x \in f^{-1}(C) \cap f^{-1}(D)$.

(b) We run the preceding reasoning backwards: $x \in f^{-1}(C) \cap f^{-1}(D)$ means $f(x) \in C$ and $f(x) \in D$, which is the same as $f(x) \in C \cap D$. This is the definition of $x \in f^{-1}(C \cap D)$.

(c) From (a), we get $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$.

From (b), we get $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$

Thus, equality $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ holds. $\square$

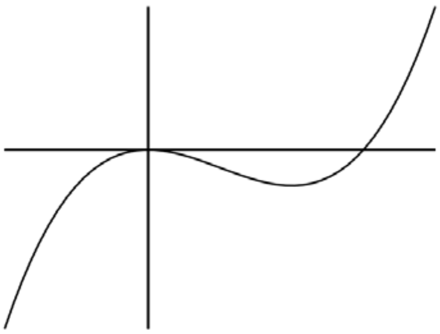
7. (10 points) (**Injection, surjection, image, preimage**) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - 3x^2$. You may, for purposes of this problem, use what you know from Calc I-II to determine the location of maxima and minima and intervals of monotonicity. Use the definition of f and the shape of the graph of f to answer the questions.
   (a) Is f injective? Why or why not?
   (b) Is f surjective? Why or why not?
   (c) What is $f([0,3/2])$?
   (d) what is $f^{-1}([0,\infty))$?

Solution

Since $f'(x) = 6x^2 - 6x = 6x(1x-1)$, we see that there are two critical points, $x = 0$ and $x = 1$; the function is strictly increasing on $(-\infty, 0]$ and $[1, \infty)$, and strictly decreasing on $[0, 1]$. Also,

$$\begin{aligned} &f(x) \to -\infty \text{ as } x \to -\infty \\ &f(0) = 0 \\ &f(1) = -1 \\ f&\left(\frac{3}{2}\right) = 0 \text{ (since } f(x) = x^2(2x-3)\text{)} \\ &f(x) \to \infty \text{ as } x \to \infty \end{aligned}$$

From this we conclude that the graph of $f(x)$ looks like this:



(a) *Is f injective? Why or why not?*

No: for $-1 < y < 0$, the equation $y = f(x)$ has three distinct solutions: one each in $(-1, 0)$, $(0, 1)$, and $(1, \frac{3}{2})$.

(b) *Is f surjective? Why or why not?*

Yes, since f is continuous and goes to $\pm\infty$ as x does.

(c) *What is $f([0, 3/2])$?*

On each of the intervals $[0, 1]$ and $\left[1, \frac{3}{2}\right]$, $f(x)$ is strictly monotone; since $f(0) = f(\frac{3}{2}) = 0$ and $f(1) = -1$, $f\left(\left[0, \frac{3}{2}\right]\right) = [-1, 0]$.

(d) Clearly, $f(x) \in [0, \infty)$ for $x \in \left[\frac{3}{2}, \infty\right)$. But also, $f(0) = 0$. Since no other points have nonnegative image,

$$f^{-1}([0,\infty)) = \{0\} \cup \left[\frac{3}{2}, \infty\right).$$

$\square$