Homework 10 solution

We begin by recalling a Lemma from a Canvas announcement mid-week. I've added a proof.

Lemma 1. *$\mathbb{S}^1$ has a basis by open sets of the form*

$$U_{(a,b)} = \{(\cos(\theta), \sin(\theta)) \mid a < \theta < b\}$$

where $a, b \in \mathbb{R}$ and $a < b$. We call these "open arcs."

Proof. Our strategy is to show that the continuous function $f : \mathbb{R} \to \mathbb{S}^1$ given by $f(t) = (\cos(t), \sin(t))$ is a local homeomorphism.

Write $U_{x>0}, U_{y>0}, U_{x<0}, U_{y<0}$ for the open subsets of $\mathbb{S}^1$ satisfying the subscripted conditions.

For each integer $n \in \mathbb{Z}$, let

$$\begin{aligned}
g_{n,x>0} &: U_{x>0} \to (-\pi/2 + 2\pi n, \pi/2 + 2\pi n) \\
&\quad (x, y) \mapsto \arcsin(y) + 2\pi n \\
g_{n,y>0} &: U_{y>0} \to (2\pi n, \pi + 2\pi n) \\
&\quad (x, y) \mapsto \arccos(x) + 2\pi n \\
g_{n,x<0} &: U_{x<0} \to (\pi/2 + 2\pi n, 3\pi/2 + 2\pi n) \\
&\quad (x, y) \mapsto \pi/2 - \arcsin(y) + 2\pi n \\
g_{n,y<0} &: U_{y<0} \to (\pi + 2\pi n, 2\pi + 2\pi n) \\
&\quad (x, y) \mapsto \pi - \arccos(x) + 2\pi n.
\end{aligned}$$

Observe that each function g is continuous and f composed with each g is the identity on the appropriate set. It follows that the restrictions

$$\begin{aligned}
f|_{(-\pi/2+2\pi n, \pi/2+2\pi n)} &: (-\pi/2 + 2\pi n, \pi/2 + 2\pi n) \to U_{x>0} \\
f|_{(2\pi n, \pi+2\pi n)} &: (2\pi n, \pi + 2\pi n) \to U_{y>0} \\
f|_{(\pi/2+2\pi n, 3\pi/2+2\pi n)} &: (\pi/2 + 2\pi n, 3\pi/2 + 2\pi n) \to U_{x<0} \\
f|_{(\pi+2\pi n, 2\pi+2\pi n)} &: (\pi + 2\pi n, 2\pi + 2\pi n) \to U_{y<0}
\end{aligned}$$

are homeomorphisms. In particular for any open subset U of $\mathbb{R}$ contained in the domain of one of these restrictions, $f(U)$ is open in $\mathbb{S}^1$. It follows that $U_{(a,b)}$ is open, since

$$\begin{aligned}
U_{(a,b)} &= f((a, b)) \\
&= f\left(\bigcup_{n\in\mathbb{Z}} \left((a, b) \cap \left(0 + \frac{\pi}{2}n, \pi + \frac{\pi}{2}n\right)\right)\right) \\
&= \bigcup_{n\in\mathbb{Z}} f\left((a, b) \cap \left(0 + \frac{\pi}{2}n, \pi + \frac{\pi}{2}n\right)\right)
\end{aligned}$$

is a union of open sets. Therefore all "open arcs" are open in $\mathbb{S}^1$.

To see that these form a basis, observe that if $U \subseteq \mathbb{S}^1$ is open in $\mathbb{S}^1$, then $f^{-1}(U)$ can be written as a union of open intervals $\bigcup_{i\in I}(a_i, b_i)$, so

$$U = f(f^{-1}(U)) = f\left(\bigcup_{i\in I}(a_i, b_i)\right) = \bigcup_{i\in I} U_{(a_i,b_i)}.$$

□

Problem 1. The n-sphere is the subspace

$$\mathbb{S}^n := \{\vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1\}$$

of $\mathbb{R}^{n+1}$ with the subspace topology.

Prove that $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is homeomorphic to $[0, 1]/\sim$ where $\sim$ is the equivalence relation where $0 \sim 1$, $1 \sim 0$, and $x \sim x$ for all $x \in [0, 1]$.

Solution. Let $g : [0, 1] \to \mathbb{S}^1$ be the continuous function defined by $g(t) = (\cos(2\pi t), \sin(2\pi t))$.

Observe that $g(t_1) = g(t_2)$ if and only if $t_1 \sim t_2$. By the universal property of quotients, there is a continuous function $f : [0, 1]/\sim \to \mathbb{S}^1$ given by $f([t]) = g(t)$.

By Corollary 22.3, f is a homeomorphism if and only if g is a quotient map. To that end, suppose that $U \subseteq \mathbb{S}^1$ is a subset so that $V = g^{-1}(U) \subseteq [0, 1]$ is open. Let $p \in U$. The pre-image of p is either a single point $x \in (0, 1)$ or the pre-image of p is $\{0, 1\}$. We will show that in either case there is an open arc N_p so that $p \in N_p \subseteq U$.

If $g^{-1}(p)$ is a single point $x \in (0, 1)$, then, since V is open and $x \in V \cap (0, 1)$, there exists $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq V$. Then $g((x - \epsilon, x + \epsilon)) \subseteq g(V) = U$ is an open arc N_p around p.

If $g^{-1}(p) = \{0, 1\} \subseteq V$, then, since V is open there exists an $\epsilon > 0$ so that $[0, \epsilon)$ and $(1 - \epsilon, 1] \subseteq V$. Then $g([0, \epsilon) \cup (1 - \epsilon, 1]) \subseteq g(V) = U$ is an open arc N_p around p.

Then $U = \bigcup_{p\in U} N_p$ is open, as desired.

Problem 2. Let $U_1 = \{1, 2\} \times (0, 3)$, $U_2 = \{3, 4\} \times (0, 3)$. Let U_{12} be the open subset $\{1, 2\} \times ((0, 1) \cup (2, 3))$ of U_1 and let U_{21} be the open subset $\{3, 4\} \times ((0, 1) \cup (2, 3))$ of U_2.

(1) Find a homeomorphism $\phi_{12} : U_{12} \to U_{21}$ so that the space X obtained by gluing U_1 to U_2 with ϕ_{12} is homeomorphic to a disjoint union of two copies of $\mathbb{S}^1$. You do not need to prove that X is homeomorphic to two copies of $\mathbb{S}^1$.

(2) Find a homeomorphism $\psi_{12} : U_{12} \to U_{21}$ so that the space Y obtained by gluing U_1 to U_2 with ψ_{12} is homeomorphic to $\mathbb{S}^1$. You do not need to prove that Y is homeomorphic to $\mathbb{S}^1$.

Solution. (1) There are many ways to proceed. In any case, we should form the two circles by gluing each circle out of two of the given line segments, one from U_1 and one from U_2. Let's glue the left line of U_1 to the left line of U_2 on both ends and the right line of U_1 to the right line of U_2 at both ends. So more specifically, let's set up ϕ_{12} to send the bottom end of the left line of U_1 to the bottom end of the left line of U_2; the top end of the left line of U_1 to the top end of the left line of U_2; the bottom end of the right line of U_1 to the bottom end of the right line of U_2; and the top end of the right line of U_1 to the top end of the right line of U_2. We'll have to reverse the orientation on the overlap so that we get a circle shape at the end rather than a Y-shape.

A function ϕ_{12} that achieves this is:

$$\phi_{12} : U_{12} \to U_{21}$$
$$\phi_{12}(x,y) = \begin{cases} (3, 1-y) & \text{if } x = 1 \text{ and } 0 < y < 1 \\ (3, 5-y) & \text{if } x = 1 \text{ and } 2 < y < 3 \\ (4, 1-y) & \text{if } x = 2 \text{ and } 0 < y < 1 \\ (4, 5-y) & \text{if } x = 2 \text{ and } 2 < y < 3. \end{cases}$$

(2) We should form one circle by gluing together 4 line segments in sequence. We have to glue segments of U_1 to segments of U_2, so we'll have to alternate segments from U_1 with segments from U_2. One way to make this work is as follows: Glue the top end of the left line of U_1 to the bottom end of the left line of U_2; the top end of the left line of U_2 to the bottom end of the right line of U_1; the top end of the right line of U_1 to the bottom of the right line of U_2; and finally the top end of the right end of U_2 back to the bottom end of the left line of U_1.

A function ψ_{12} that achieves this is:

$$\psi_{12} : U_{12} \to U_{21}$$
$$\psi_{12}(x,y) = \begin{cases} (4, y+2) & \text{if } x = 1 \text{ and } 0 < y < 1 \\ (3, y-2) & \text{if } x = 1 \text{ and } 2 < y < 3 \\ (3, y+2) & \text{if } x = 2 \text{ and } 0 < y < 1 \\ (4, y-2) & \text{if } x = 2 \text{ and } 2 < y < 3. \end{cases}$$

Problem 3. Consider the equivalence relation $\sim$ on $\mathbb{S}^1$ defined by $\vec{x} \sim \vec{y}$ if and only if $\vec{x} = \pm\vec{y}$. Find a homeomorphism of $\mathbb{S}^1/\sim$ with a familiar space and prove that your map is a homeomorphism.

Solution. We'd like to use the multiply-the-angle-by-two map, i.e, we'd like to send $(\cos(\theta), \sin(\theta)) \mapsto (\cos(2\theta), \sin(2\theta))$. This is a bit tricky to check for well-definedness and continuity, but applying double angle formulas yields

$$(\cos(\theta), \sin(\theta)) \mapsto (\cos^2(\theta) - \sin^2(\theta), 2\cos(\theta)\sin(\theta)).$$

This suggests the more-obviously continuous function below.

Let $g : \mathbb{S}^1 \to^1$ be the function defined by $g(x,y) = (x^2 - y^2, 2xy)$. To see that this is well-defined (i.e., actually takes points on the unit circle to points on the unit circle), note that when $x^2 + y^2 = 1$,

$$\begin{aligned} (x^2 - y^2)^2 + (2xy)^2 &= x^4 - 2x^2y^2 + y^4 + 4x^2y^2 \\ &= x^4 + 2x^2y^2 + y^4 \\ &= (x^2 + y^2)^2 \\ &= 1. \end{aligned}$$

The function is clearly continuous, since it is defined by a polynomial function in each coordinate.

Next we'd like to show that given points $(x, y), (x', y') \in \mathbb{S}^1$, we have $g(x, y) = g(x', y')$ if and only if $(x, y) = \pm(x', y')$. This is clear from the geometric interpretation of g, but let's give a direct algebraic proof for those interested. The implication $(x, y) = \pm(x', y') \implies g(x, y) = g(x', y')$ is immediate since the formula for g is homogeneous of degree 2. Conversely suppose that $g(x, y) = g(x', y')$. Then $x^2 - y^2 = (x')^2 - (y')^2$ and $2x'y' = 2xy$. Since $x^2 + y^2 = (x')^2 + (y')^2 = 1$,

$$\begin{aligned} & x^2 - y^2 = (x')^2 - (y')^2 \\ \implies & 1 - 2y^2 = 1 - 2(y')^2 \\ \implies & y^2 = (y')^2 \\ \implies & y = \pm y'. \end{aligned}$$

Symmetrically, $x = \pm x'$. Now since $2xy = 2x'y'$ we must have either $x = x'$ and $y = y'$ or $x = -x'$ and $y = -y'$, as desired.

Now by Corollary 22.3, all that remains is to show that g is a quotient map. Suppose that $U \subseteq \mathbb{S}^1$ is a subset such that $V = g^{-1}(U) \subseteq \mathbb{S}^1$ is open. Since open arcs form a basis for the topology on $\mathbb{S}^1$, we may write $V = \bigcup_{i \in I} U_{(a_i, b_i)}$. Now,

$$U = g(V) = g\left(\bigcup_{i \in I} U_{(a_i, b_i)}\right) = \bigcup_{i \in I} g(U_{(a_i, b_i)}).$$

Now

$$\begin{aligned} g(U_{(a_i, b_i)}) &= \{g(\cos(\theta), \sin(\theta)) \mid a_i < \theta < b_i\} \\ &= \{(\cos(2\theta), \sin(2\theta)) \mid a_i < \theta < b_i\} \\ &= \{(\cos(\theta, \sin(\theta)) \mid 2a_i < \theta < 2b_i\} \\ &= U_{(2a_i, 2b_i)}, \end{aligned}$$

so $U = \bigcup_{i \in I} U_{(2a_i, 2b_i)}$. Since U is a union of open sets, it is open. We conclude that g is a quotient map, and therefore $\mathbb{S}^1 \cong \mathbb{S}^1/\sim$, as desired.

Problem 4. Let τ and τ' be two topologies on X. If τ' is finer than τ, what does connectedness of X in one topology imply about connectedness in the other?

Solution. If (X, τ) is connected, we cannot conclude that (X, τ') is also connected. For example, if $|X| > 2$, τ' is the discrete topology, and τ is the indiscrete topology, then (X, τ) is connected but (X, τ') is not connected.

We claim that (X, τ') connected implies that (X, τ) connected. To that end, suppose that (X, τ') is connected and suppose for contradiction U, V are a separation of (X, τ). Then U and V are open in τ' and remain disjoint and nonempty with union X. But then U, V form a separation of (X, τ'), a contradiction. It follows that connectedness of (X, τ') implies connectedness of (X, τ), as desired.

Problem 5. Let X be a topological space. For each integer $n \geq 1$, let A_n be a connected subspace of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup_{n \in \mathbb{Z}_{>0}} A_n$ is connected.

Solution. Here's a nice triangular-matrix-y trick that reduces the problem to Theorem 23.3.

For each positive integer n, let $B_n = \bigcup_{i=1}^{n} A_n$. Since $A_1 \cap A_2 \neq \emptyset$ we can choose a point $x \in A_1 \cap A_2$. Then $x \in A_1$, so $x \in B_n$ for all n.

Next we claim that B_n is connected for all $n \in \mathbb{Z}_{>0}$. Since $B_1 = A_1$, B_1 is connected.

Suppose for induction that B_n is connected. Then $B_{n+1} = B_n \cup A_{n+1}$ is a union of connected sets with nonempty intersection, since $B_n \cap A_{n+1} \supseteq A_n \cap A_{n+1} \neq \emptyset$. By Theorem 23.3, B_{n+1} is connected. It follows by induction that B_n is connected for all $n \in \mathbb{Z}_{>0}$.

Now

$$\bigcup_{n \in \mathbb{Z}_{>0}} A_n = \bigcup_{n \in \mathbb{Z}_{>0}} B_n$$

is a union of connected sets with the point x in common. By Theorem 23.3 again, $\bigcup_{n \in \mathbb{Z}_{>0}} A_n$ is connected.

Problem 6. Recall that the lower limit topology is the topology on $\mathbb{R}$ generated by the basis

$$\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}\}.$$

(See Recitation 6 or Section 13 of the text.) Denote $\mathbb{R}$ with the lower limit topology by $\mathbb{R}_\ell$. Is $\mathbb{R}_\ell$ connected or disconnected? Justify your answer.

Solution. Recall from Recitation 6 that the lower limit topology is finer than the usual topology on $\mathbb{R}$. In particular, $U = (-\infty, 0)$ and $V = [0, \infty) = \bigcup_{n \in \mathbb{Z}_{>0}} [0, n)$ are non-empty, disjoint open subsets of $\mathbb{R}_\ell$ with union $\mathbb{R}_\ell$. Therefore $\mathbb{R}_\ell$ is disconnected.