Scott Fullenbaum
Math 87 HW 5
Due 3/03/22

---

1. Suppose you're standing on a street with buildings labelled by the integers (specifically, you're in front of the building labelled 0, and suppose that the indices are increasing to the right). Suppose that every minute you flip a coin. If the coin is heads you walk right and if the coin is tails you walk left.

   (a) Explain why your position (i.e. the building you're in front of) as a function of time can be modeled.
   Position as a function of time can be modeled because we determine the probability of a person being at any location at any time, as we know the probability of the person going to the left or to the right.

   (b) Is the distance from where you started as a function of time a Markov Chain?
   Yes, because we can represent each building as a node on a directed graph, since the system transitions from state to state at each position. Furthermore, the probability of being at any given building will add to one. As a result, we have a transition diagram which means we have a Markov Chain.

   (c) Now suppose that every minute you flip two coins. If both are heads, you move right, if both are tails you move left and otherwise you stay put. Is your distance from where you started a Markov chain in this scenario? How do you expect this to compare to the process described in part 2? Yes, the process is still a Markov chain, but with an additional potential state, as from the definition in class, a Markov chain doesn't require movement. I would expect the second state to stay closer to the middle, as there is a 0.5 chance that you stay in the same location every time.

   (d) For both experiments, compute the probability that you are standing on an odd number for minute 0,1,2,3,4.

   For the experiment where you move left or right, the probability of being odd is: t= 0: P(odd) = 0 as position is 0
   t= 1: P(odd) = 1 as only possible positions are -1, 1
   t = 2: P(odd) = 0 as only possible positions are -2, 0, 2
   t = 3: P(odd) = 1 as only possible positions are -3, -1, 1, 3
   t = 4: P(odd) = 0 as only possible positions are -4, -2, 0, 2, 4

   This makes sense as when you move left to right or vice versa, you go from even to odd or odd to even, meaning that you're either guaranteed to be on odd or even.

   For the second experiment, the probability of being on an odd building is 0.5 at any point. To prove, pick a position, n. If n is odd, there is a 0.5 chance we go the left or right, meaning we end on an even position, or a 0.5 chance we remain on the same building which is odd. The same can be shown for an even starting position. So moving from any previous state, there is a 0.5 chance that we will end on an odd number.

2. On Planet X, the weather is strangely predictable: The weather is always either sunny, rainy, foggy or snowy. If it rains today, its sunny tomorrow. If it is sunny today, its rainy tomorrow. If its foggy today, its not sunny tomorrow. Finally, the weather is never the same two days in a row. Apart from these rules, the weather is completely random, in that if e.g. its foggy

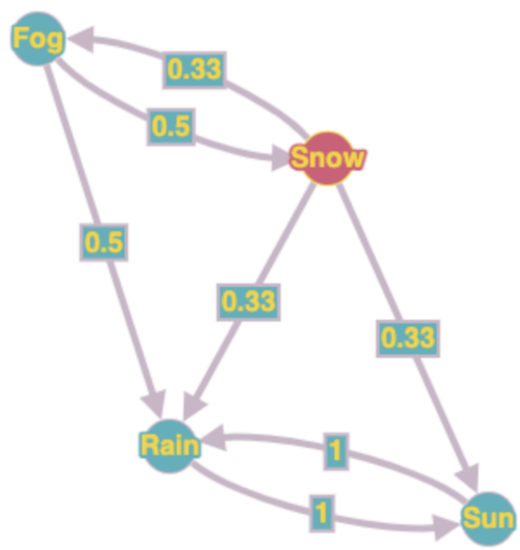
today it is equally likely to be either rainy or snowy tomorrow. You live on Planet X and are trying to figure out what to wear this week, so you'd like to develop a model for the weather.

(a) Explain why the weather can be modeled as a Markov chain. Write out the transition matrix, and draw the corresponding finite state machine.
The weather can be modeled as a Markov chain as we can represent the state as a finite state machine, and then show how we transition from state to state.

	Rains today	Sunny today	Foggy today	Snowy today
Rainy tomorrow	0	1	$\frac{1}{2}$	$\frac{1}{3}$
Sunny tomorrow	1	0	0	$\frac{1}{3}$
Foggy tomorrow	0	0	0	$\frac{1}{3}$
Snowy tomorrow	0	0	$\frac{1}{2}$	0

Table 1: Transition Matrix

Here is a graph of the finite state machine.



(b) Check whether the conditions for the Perron-Frobenius theorem is satisfied for this problem (aperiodic and strongly connected). Explain your reasoning.
The transition diagram is not strongly connected. For example, I cannot draw a path going from rain to snow. The transition diagram is aperiodic, as the greatest common denominator among all paths is 1.

(c) Do you expect power iteration to be effective for computing the greatest eigenvector of your transition matrix?
No, because as seen above, the transition diagram does not fulfill the conditions of the Perro-Frobenius theorem, meaning power iteration will not work.

(d) Find the eigenvalue decomposition for the transition matrix, and the associated eigenvectors. Explain why these values confirm your answer to part 2.
When solved, we get the following eigenvalues: $\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, 1, -1$ which results in the fol-

lowing eigenvectors: $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\begin{bmatrix}-0.437\\0.253\\-0.817\\1\end{bmatrix}$, and $\begin{bmatrix}-0.763\\-1.053\\0.816\\1\end{bmatrix}$

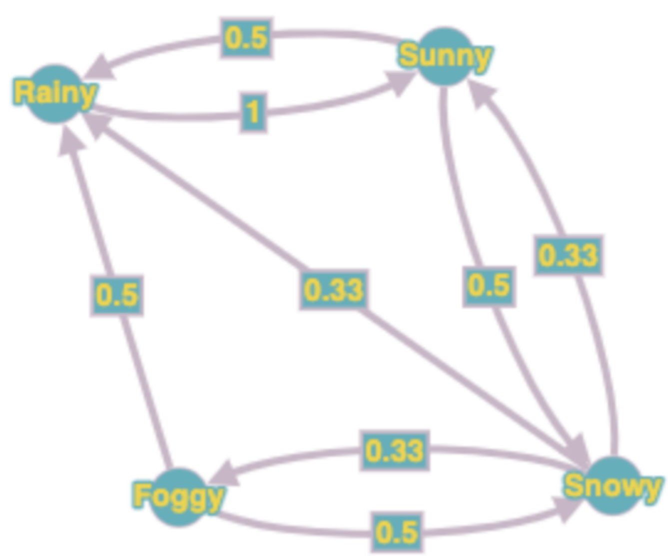
(e) Suppose that the "weather rules" change so that if its sunny today, it is equally likely to be snowy or rainy tomorrow. Write out the new transition matrix, associated finite state machine, and determine whether the conditions for the Perron-Frobenius are satisfied. Compute the eigenvalue decomposition and compare to the previous set of eigenvalues.

	Rains today	Sunny today	Foggy today	Snowy today
Rainy tomorrow	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$
Sunny tomorrow	1	0	0	$\frac{1}{3}$
Foggy tomorrow	0	0	0	$\frac{1}{3}$
Snowy tomorrow	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 2: Updated Transition Matrix

Here, we can see that the greatest common denominator among cycles is 1, as we can have a path of length 3 going from sunny to snowy to rainy, then back to sunny, and various paths of length of 2 and 4. This means the graph is aperiodic, and it is also strongly connected, as any node is accessible from any node, so it satisfies the Perron-Frobenius theorem conditions.

Transition diagram:



We can solve for eigenvalues and get $\lambda = 1$, -0.789, -0.211, 0. The respective eigenvectors are $\begin{bmatrix}1.33\\1.67\\0.33\\1\end{bmatrix}$, $\begin{bmatrix}0.578\\-1.155\\-0.423\\1\end{bmatrix}$, $\begin{bmatrix}\\-0.578\\1.155\\1\end{bmatrix}$, and $\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$