M136 HW 2

1 $\Rightarrow$ If f is continuous at x_0, then $\lim_{x \to x_0} f(x) = f(x_0)$

Since f is continuous at x_0, $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t if $\|x - x_0\| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$

As $|f(x) - f(x_0)| < \varepsilon$

$\lim_{x \to x_0} |f(x) - f(x_0)| < \lim_{x \to x^0} \varepsilon \quad \forall \varepsilon > 0$

So $\lim_{x \to x^0} |f(x) - f(x_0)| = 0$ and $\lim_{x \to x_0} f(x) - f(x_0) = 0$

and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x_0) \Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$

$\Leftarrow$ If $\lim_{x \to x_0} f(x) = f(x_0)$ then f is continuous at x_0

If f is continuous at x_0 then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\|x - x_0\| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$

Since $\lim_{x \to x_0} f(x) = f(x_0)$ then $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ s.t $\forall k \geq K$ $|f(x) - f(x_0)| < \varepsilon$

However, as $x \to x_0$ then $\forall \delta > 0 \; \exists x \in \mathbb{N}$ where $\|x - x_0\| \leq \delta$, and $x \neq x_0$ as it is a limit point, so we can see f satisfies the ε-δ criteria at x_0 and is continuous at x_0

2a) $D_p f(0,0) = \begin{cases} a^2/b & b \neq 0 \\ 0 & b = 0 \end{cases}$
$p = (a,b)$

b) To see if directional derivative is at $\forall x \in \mathbb{R}$, can evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and see if continuous everywhere. If not continuous everywhere, then can see if directional derivative everywhere.

$\frac{\partial f}{\partial x} = \frac{(x^4+y^2)(2xy) - x^2y(4x^3)}{(x^4+y^2)^2}$ continuous except at $(0,0)$

$\frac{\partial f}{\partial y} = \frac{(x^4+y^2)x^2 - x^2y(2y)}{(x^4+y^2)^2}$ continuous except at $(0,0)$

And we know $D_p f(0,0)$ from part a, So directional derivative is at all $x \in \mathbb{R}$.

c) Can evaluate $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ to get ∇f

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f((t,0)) - 0}{t} = \frac{0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f((0,t)) - f(0,0)}{t} = 0$$

So $\nabla f(0,0) = 0$ and $\langle \nabla f(0,0), h \rangle = 0$

$\frac{\partial f}{\partial h}((0,0)) = 0$ from part a, so

$$\frac{\partial f}{\partial h}((0,0)) = \langle \nabla f(0,0), h \rangle$$

and this satisfies the conclusion of the directional derivative theorem $\square$

2d) f isn't continuously differentiable on $\mathbb{R}^2$ as the directional derivatives at $(0,0)$ aren't equivalent so the partial at $(0,0)$ isn't defined and f isn't continuously differentiable on $\mathbb{R}^2$

3 a) $f \in C^2(\mathcal{O})$ so it's second order partials are continuous. Since 2^{nd} order is partial of the partials, then f's first partials are continuous, as it is continuously differentiable. So, $f \in C^1(\mathcal{O})$ and then from class, this means f is continuous.

b) Yes, as $f \in C^1(\mathcal{O})$ so it is continuously differentiable.

4 Since g is order K approximation of f

$$\lim_{x \to x_0} \frac{g(x)-f(x)}{\|x-x_0\|^K} = 0$$

let $\{x_J\} \to x_0$ but $x_J \neq x_0$

$$= \lim_{J \to \infty} \frac{g(x_J)-f(x_J)}{\|x_J-x_0\|^K} = 0$$

so $\|x_J - x_0\| < \delta$ for $\forall \delta > 0$

$$\to \lim_{J \to \infty} \left| \frac{g(x_J)-f(x_J)}{\|x_J-x_0\|^K} \right| = 0$$

$$\to \lim_{J \to \infty} \frac{|g(x_J)-f(x_J)|}{\|x_J-x_0\|^K} = 0$$

so $\forall \epsilon > 0$ $\exists K \in \mathbb{N}$ $\forall J \geq K$

So $\forall \epsilon > 0, \exists K \in \mathbb{N}$ s.t $\forall J \geq K$ $\left| \frac{|g(x_J)-f(x_J)|}{\|x_K-x_0\|^K} \right| < \epsilon$

So $\frac{|g(x_J)-f(x_J)|}{\|x_J-x_0\|^K} < \epsilon \to |g(x_J)-f(x_J)| < \epsilon\|x_J-x_0\|^K$

as $x_J \to x_0$ $\lim |g(x_J)-f(x_J)| < \lim \epsilon\|x_J-x_0\|^K$

for $\|x_J - x_0\| < \delta$ then $|g(x)-f(x)| < \epsilon\|x-x_0\|^K$.

$\underset{x}{\shortparallel}$