1. (ε-δ criterion for continuity) §3.5, p. 73, #1 (only at $x=2$).
Define $f(x) = x^2$. Verify the ε-δ criterion for continuity at $x = 2$.

Solution. Let $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$|x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon.$$

Scratch work.

To find δ, we try to solve $|x^2-4| < \varepsilon$.

$$|x^2-4| = |x-2||x+2| < \delta\,|x+2| < \delta M,$$

where M is an upper bound for $|x+2|$. Then one can take $\delta = \frac{\varepsilon}{M}$, for then $|x^2-4| < \delta M = \frac{\varepsilon}{M} \cdot M = \varepsilon$. To find an upper bound for $|x+2|$, it is enough to bound $|x-2|$, since $|x+2| = |(x-2)+4| \le |x-2| + 4$.

Suppose $|x-2| < 1$. Then

$$|x+2| = |(x-2)+4| \le |x-2| + 4 \quad (\Delta \text{ ineq.})$$
$$< 1 + 4 = 5.$$

Thus, an upper bound for $|x+2|$ is 5 and we can take

$$\delta = \min\left(1, \frac{\varepsilon}{5}\right).$$

Back to the proof. Choose $\delta = \min(1, \frac{\varepsilon}{5})$. Then

$$|x-2| < \delta \le 1,$$

so
$$|x+2| = |(x-2)+4| \le |x-2| + 4 \quad (\Delta \text{ ineq.})$$
$$< 1+4 = 5,$$

and
$$|x^2-4| = |x-2|\,|x+2| < \delta \cdot 5 \le \frac{\varepsilon}{5} \cdot 5 = \varepsilon. \quad \square$$

2. (15 points) **ε-δ criterion for continuity**. §3.5, p. 73, #5.
Define $h(x) = 1/(1+x^2)$ for all $x \in \mathbb{R}$. Prove that the function $h: \mathbb{R} \to \mathbb{R}$ satisfies the ε-δ criterion on $\mathbb{R}$. (To prove the ε-δ condition on $\mathbb{R}$ means to verify the condition at every point a of $\mathbb{R}$.)

Proof. Let $\varepsilon > 0$ and $a \in \mathbb{R}$. We need to find $\delta > 0$ such that

$$|x-a| < \delta \Rightarrow \left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| < \varepsilon.$$

Scratch Work

$$\left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| = \left| \frac{1+a^2-(1+x^2)}{(1+x^2)(1+a^2)} \right| = \frac{|x^2-a^2|}{(1+x^2)(1+a^2)}$$

$$= |x-a| \frac{|x+a|}{(1+x^2)(1+a^2)}$$

$$\leq |x-a|\,|x+a| \quad (\text{Since } 1+x^2 \geq 1 \text{ and } 1+a^2 \geq 1).$$

$$< \delta |x+a| \quad (\text{Assume } |x-a| < \delta)$$

So it is enough to find an upper bound for $|x+a|$.

$$|x+a| = |(x-a)+2a| \leq |x-a| + |2a| \quad (\Delta \text{ ineq.})$$

If we choose $|x-a| < 1$, then $|x+a| \leq 1 + |2a|$.

So

$$\left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| \leq |x-a|\,|x+a|$$

$$< \delta (1+|2a|) < \varepsilon$$

$$\delta < \frac{\varepsilon}{1+|2a|}.$$

Proof Continued. Choose $\delta = \min\left(1, \frac{\varepsilon}{1+|2a|}\right)$. Suppose $|x-a| < \delta \leq 1$. Then

$$|x+a| = |(x-a)+2a| \leq |x-a| + |2a| \quad (\Delta \text{ ineq.})$$

$$< 1 + |2a|,$$

and

$$\left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| = |x-a| \frac{|x+a|}{(1+x^2)(1+a^2)} \quad (\text{algebra done in Scratch})$$

$$\le |x-a|\,|x+a| \qquad \left(\text{because } (1+x^2)(1+a^2) \ge 1\right)$$

$$< |x-a|\,(1+|2a|)$$

$$< \frac{\varepsilon}{1+|2a|}(1+|2a|) \qquad \left(\text{because } |x-a| < \delta \le \frac{\varepsilon}{1+|2a|}\right)$$

$$= \varepsilon .$$

This proves that for any $a \in \mathbb{R}$, with $\delta = \min\left(1, \frac{\varepsilon}{1+|2a|}\right)$,

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon .$$

Therefore, $f(x) = 1/(1+x^2)$ is continuous at $x = a$ for all $a \in \mathbb{R}$. $\square$

3. (Existence of a root) § 3.3, p.65, #3.

Prove that there is a solution of the equation

$$\frac{1}{\sqrt{x+x^2}} + x^2 - 2x = 0, \quad x > 0.$$

Let $f(x) = \frac{1}{\sqrt{x+x^2}} + x^2 - 2x$. Then

$$f(1) = \frac{1}{\sqrt{2}} + 1 - 2 = \frac{1}{\sqrt{2}} - 1 < 0 \quad \text{because } \sqrt{2} > 1.$$

$$f(2) = \frac{1}{\sqrt{6}} + 4 - 4 = \frac{1}{\sqrt{6}} > 0.$$

Since f is continuous on $[1,2]$, by the intermediate value theorem, there is an $x_0 \in (1,2)$ such that $f(x_0) = 0$.

4. (Fixed point theorem) Prove that every continuous function $f: [a,b] \to [a,b]$ has a fixed point.

<u>Solution</u> If $f(a) = a$ or $f(b) = b$, then f would have a fixed point at a or b. So we will assume that $f(a) \neq a$ and $f(b) \neq b$. Consider the function

$$g(x) = f(x) - x.$$

Then

$$g(a) = f(a) - a > 0 \quad \text{because } f(a) \in (a, b].$$

$$g(b) = f(b) - b < 0 \quad \text{because } f(b) \in [a, b).$$

Since g is continuous on $[a, b]$, by the intermediate value theorem there is an $x_0 \in (a, b)$ such that $g(x_0) = 0$.

Then $$f(x_0) = x_0.$$

So f has a fixed point at $x_0 \in [a, b]$. $\square$

5. (6 pts) (Limits) For any sequence a_n, prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$.

Proof. By the ε-N definition of a limit,

$\lim a_n = 0$ iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|a_n| = |a_n - 0| < \varepsilon,$$

and $\lim |a_n| = 0$ iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|a_n| = ||a_n| - 0| < \varepsilon.$$

Therefore, the two ε-N conditions are exactly the same. This shows that

$$\lim a_n = 0 \text{ iff } \lim |a_n| = 0. \qquad \square$$

6. (Uniform continuity) Define $f(x) = x^3$ for all x. Prove that $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous

Solution. Let $u_n = n + \frac{1}{n}$ and $v_n = n$.

Then $\lim u_n - v_n = \lim (n + \frac{1}{n}) - n = \lim\limits_{n \to \infty} \frac{1}{n} = 0$.

However, $\lim f(u_n) - f(v_n) = \lim (n + \frac{1}{n})^3 - n^3$

$= \lim n^3 + 3n^2 \cdot \frac{1}{n} + 3n \cdot \frac{1}{n^2} + \frac{1}{n^3} - n^3$

$= \lim\limits_{n \to \infty} 3n + \frac{3}{n} + \frac{1}{n^3}$

$= \lim\limits_{n \to \infty} 3n = +\infty$.

Hence, $f(x) = x^3$ is not uniformly continuous on $(-\infty, \infty)$. $\square$

7. (Lipschitz => uniform continuity) §3.4, p. 69, #11.

Suppose $f: D \to \mathbb{R}$ is Lipschitz, i.e., there is a constant $C \geq 0$ such that $\forall u, v \in D$,

$$|f(u) - f(v)| \leq C\, |u - v|.$$

Suppose $\{u_n\}$ and $\{v_n\}$ are sequences in D such that

$$\lim |u_n - v_n| = 0.$$

By the Lipschitz condition,

$$0 \leq |f(u_n) - f(v_n)| \leq C\, |u_n - v_n|.$$

Taking the limit as $n \to \infty$, by the sandwich lemma,

$$\lim |f(u_n) - f(v_n)| = 0.$$

Hence, f is uniformly continuous on D. $\square$

8. (14 points) **Uniformly continuous but not Lipschitz**. §3.5, p. 74, #7.
Define $f(x) = \sqrt{x}$ for $0 \le x \le 1$.
(a) Prove that the function $f\colon [0,1] \to \mathbb{R}$ is continuous.
(b) Use part (a) to show that $f\colon [0,1] \to \mathbb{R}$ is uniformly continuous.
(c) Show that $f\colon [0,1] \to \mathbb{R}$ is not Lipschitz.
(*Hint*:
$$\sqrt{x} - \sqrt{x_0} = (\sqrt{x} - \sqrt{x_0})\frac{\sqrt{x}+\sqrt{x_0}}{\sqrt{x}+\sqrt{x_0}}.)$$

Solution. (a) Fix $x_0 \in [0,1]$. Suppose $x_n \in [0,1]$ and $x_n \to x_0$. By the square root root, $\sqrt{x_n} \to \sqrt{x_0}$. This proves that $f(x) = \sqrt{x}$ is continuous at any $x_0 \in [0,1]$.

(b) By Theorem 3.17, a continuous function on $[a,b]$ (actually on any sequentially compact domain) is uniformly continuous. Since $f(x)=\sqrt{x}$ is continuous on $[0,1]$, it is uniformly continuous.

(c) Suppose $\exists\, C \ge 0$ such that for all $x, x_0 \in [0,1]$,
$$|\sqrt{x} - \sqrt{x_0}| \le C\,|x - x_0|$$
Then
$$|\sqrt{x} - \sqrt{x_0}| = \left|\frac{\cancel{x - x_0}}{\sqrt{x}+\sqrt{x_0}}\right| \le C\,\cancel{|x - x_0|}$$
$$\frac{1}{\sqrt{x}+\sqrt{x_0}} \le C.$$
This is not possible because if both x and x_0 approach 0,
$$\frac{1}{\sqrt{x}+\sqrt{x_0}} \to \infty. \qquad \square$$

A simpler proof.

Suppose $\exists\, C \ge 0$ such that for all $x, x_0 \in [0,1]$,
$$|\sqrt{x} - \sqrt{x_0}| \le C\,|x - x_0|$$
Pick $x_0 = 0$. Then $\sqrt{x} \le Cx$. For $x > 0$,
$$\frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x} \le C.$$
This is not possible because $1/\sqrt{x}$ is unbounded on $(0,1]$. $\square$

9. (ε-δ criterion for uniform continuity)
Prove that for a function $f: D \to \mathbb{R}$, TFAE:
(i) for two sequences $\{u_n\}$, $\{v_n\}$ in D,
$$\lim |u_n - v_n| = 0 \Rightarrow \lim |f(u_n) - f(v_n)| = 0.$$
(ii) $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall u, v \in D$,
$$|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon.$$

(i)$\Rightarrow$(ii) We prove the contrapositive.
Suppose (ii) is not true. Then
$\exists \varepsilon > 0$ such that $\forall \delta = \frac{1}{n} > 0$, $\exists u_n, v_n \in D$ s.t.
$$|u_n - v_n| < \frac{1}{n} \quad \text{and} \quad |f(u_n) - f(v_n)| \geq \varepsilon.$$
Taking the limit as $n \to \infty$, by the sandwich lemma,
$$\lim |u_n - v_n| = 0 \quad \text{and} \quad \lim |f(u_n) - f(v_n)| \geq \varepsilon.$$
Thus, $\lim |f(u_n) - f(v_n)| \neq 0$.

(ii)$\Rightarrow$(i)
Assume (ii). Let $\{u_n\}$, $\{v_n\}$ be two sequences in D such that $\lim |u_n - v_n| = 0$. Let $\varepsilon > 0$.
By (ii), $\exists \delta > 0$ such that $\forall u, v \in D$,
$$|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon.$$
Since $\lim |u_n - v_n| = 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,
$$|u_n - v_n| < \delta.$$
By (ii), $|f(u_n) - f(v_n)| < \varepsilon$.
This is precisely the condition for $\lim |f(u_n) - f(v_n)| = 0$.