Tufts University
Department of Mathematics

Math 136 **Group work for the Week of March 27, 2023** [1] **Spring, 2023**

Due date: 11:59 pm, Monday, April 3, 2023

Roughly every week we will work in small groups to learn to write proofs and to solve problems. The problems will be long enough that you might need to talk with your group outside of class. Todd will grade the group work.

- *Scribe:* each week, someone in the group will volunteer to submit to Gradescope the group's answer *and* to enter all the group member names in Gradescope when uploading. This role should rotate through the group.
- *Respect:* when discussing problems, please make sure that everyone feels comfortable speaking and that all feedback is supportive and encouraging. For example:
  - Please make sure everyone has a chance to talk.
  - To start the discussion, you could have everyone suggest topics, theorems and ideas that relate to the question.
  - If you talk at the board please make sure everyone has a piece of chalk so they can add to the discussion.

In this group work, we will explore the concept of measure zero

Definition 1 (Measure Zero) *A subset S of $\mathbb{R}^n$ is said to have* measure zero *if for each $\epsilon > 0$ there is a finite or countably infinite collection of generalized rectangles $\{\mathbb{I}_j\}_{j=1}^N$ where $N \in \mathbb{N}$ if the collection is finite or $N = \infty$ if the collection is countably infinite, such that*

$$\text{(i.)}\ S \subset \bigcup_{j=1}^{N} \mathbb{I}_j \qquad \textit{and} \qquad \text{(ii.)}\ \sum_{j=1}^{N} \operatorname{Vol}(\mathbb{I}_j) < \epsilon.$$

In class, we proved that if $D(f,\mathbb{I})$ is a set of Jordan content zero then f is integrable on $\mathbb{I}$. The converse of this statement is not true. However we will now explore the concept of *measure zero* (defined below). It is used in the following wonderful theorem.

Theorem 2 (Lebesgue's Theorem) *Let f be a bounded function from the generalized rectangle $\mathbb{I}$ to $\mathbb{R}$. Then f is integrable* if and only if $D(f,\mathbb{I}) = \left\{\mathbf{x} \in \mathbb{I} \,\middle|\, f \text{ is discontinuous at } \mathbf{x}\right\}$ *has measure zero.*

Problems:

1. (4 points) Use the definition of measure zero to show that the set of natural numbers $\mathbb{N}$ is measure zero in $\mathbb{R}$.
2. (6 points) Is there a bounded set of measure zero that does not have volume? Prove your answer.