Tufts University
Department of Mathematics

Math 235 | **Midterm Exam**[1] | **Fall, 2023**

This is an open notes exam and you are allowed to consult your class notes and Heil's textbook.

Please upload the test paper with your answers to Gradescope by 1:30 p.m. on Wednesday, October 18. When you upload your answers, please scan all your answers into one PDF file starting with your signed cover page and with the answers clearly numbered and in order. Please identify the page(s) for each problem as you upload to Gradescope.

Please sign the following pledge and submit the signed pledge with your answers:

The Tufts University statement on academic integrity holds that: "Academic integrity is the joint responsibility of faculty, students, and staff. Each member of the community is responsible for integrity in their own behavior and for contributing to an overall environment of integrity at the university." I accept this responsibility, affirm that I am an honest person who can be trusted with to do the right thing, and certify that the work I will do on this exam is mine alone.

I pledge that I have used only the reference sources I have cited in my answers.

Signature____________________

The test problems start on the next page.

Problem 1. (10 Points) Let $E = \{(x, 4x), x \in [0, 2]\}$ and $F = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le 4x\}$ be two subsets of $\mathbb{R}^2$. Carefully justify why E and F are Lebesgue measurable and find $|E \cup F|$.

E is the graph of the continuous function $x \mapsto 4x$ on $[0,2]$ so it is a closed subset of $\mathbb{R}^2$ and measurable

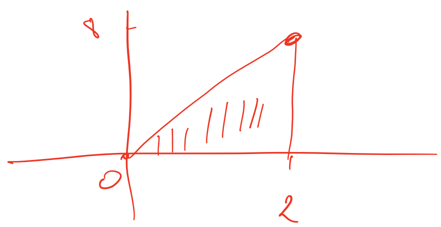
F is also closed in $\mathbb{R}^2$: let $\{(x_n, y_n)\}$ a sequence in F such that $\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0)$. We need to show that $(x_0, y_0) \in F$

$x_n \in [0,2]$ and $x_n \to x_0$ it follows that $x_0 \in [0,2]$

$y_n \le 4x_n$ and $y_n \to y_0 \Rightarrow y_0 \le 4x_0 \Rightarrow (x_0, y_0) \in F$

So F is measurable. Note that $E \subseteq F \Rightarrow$

$E \cup F = F$



$|E \cup F| = \frac{2 \cdot 8}{2} = 8$

Let $\varepsilon > 0$, Divide $[0,2]$ into n intervals of len ε

For each such subinterval $I_k = [x_{k-1}, x_k]$ consider $J_k = [4x_{k-1}, 4x_k]$

$\bigcup_{k=1}^{n} I_k \times J_k \supset E$ $\quad |E| \le \sum_{k=1}^{n} |I_k|\,|J_k| \le 4\varepsilon \sum_{k=1}^{n} |I_k| \le 8\varepsilon$

$|E| = 0$

Problem 2. (6 + 4 points) The two parts of this problem are independent.

2.1 Let E be measurable subset of $\mathbb{R}^d$ and $f : E \to [-\infty, \infty]$ be such that $f^{-1}((r, \infty])$ is Lebesgue measurable for each $r \in \mathbb{Q}$. Prove that f is a measurable function on E.

2.2 Let $E \subset \mathbb{R}$ be a measurable set and $f : E \to [-\infty, \infty]$ be a measurable function.

$$\text{Define } g(x) = \begin{cases} 1 \text{ if } f(x) > 0 \\ 0 \text{ if } f(x) = 0 \\ -1 \text{ if } f(x) < 0 \end{cases}$$

Using only the definition, prove that g is a measurable function.

Let $a \in \mathbb{R}$ $\quad \{f > a\} = \bigcup_{\substack{r \in \mathbb{Q} \\ a < r}} \{f > r\}$.

$g = \chi_{\{f>0\}} - \chi_{\{f<0\}} + 0 \cdot \chi_{\{f=0\}}$.

f being measurable, $\{f > 0\}$, $\{f < 0\}$ and $\{f = 0\}$ are all measurable so g is a simple funct. so it is measurable

Problem 3. (4 + 6 Points) Let $E \subset \mathbb{R}^d$ be measurable and $\{f_n\}$ be a sequence of measurable functions defined on E such that $\sum_{n=1}^{\infty} |\{n^4|f_n| > 1\}| < \infty$.

3.1 Prove that $A \subset E$ with $|A| = 0$ such that for all $x \in E \setminus A$, there exists $k \geq 1$ with $|f_n(x)| \leq \frac{1}{n^4}$ for all $n \geq k$.

3.2 Conclude that the series $\sum_{n=1}^{\infty} f_n$ converges for a.e. $x \in E$.

Let $A_n = \{n^4|f_n| > 1\} = \{|f_n| > \frac{1}{n^4}\}$

A_n is measurable.

Set $A = \limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. A is measurable

$\forall n \geq 1$ $|A| \leq |\bigcup_{k=n}^{\infty} A_k| \leq \sum_{k=n}^{\infty} |A_k| \xrightarrow[n \to \infty]{} 0 \Rightarrow |A| = 0$

$\forall x \in E - A \Rightarrow \exists k \geq 1 : x \notin \bigcup_{n=k}^{\infty} A_n \Rightarrow$

$\forall n \geq k$ $x \notin A_n \Rightarrow n^4 |f_n(x)| \leq 1$

$\forall x \in E \setminus A, \exists k \geq 1 : \forall n \geq k$ $|f_n(x)| \leq \frac{1}{n^4}$

Now let $x \in E \setminus A$ and choose $k \geq 1$:

$\forall n \geq k$ $|f_n(x)| \leq \frac{1}{n^4} \Rightarrow$

$\sum_{n=k}^{\infty} |f_n(x)| \leq \sum_{n=k}^{\infty} \frac{1}{n^4}$

by the comparison lemma $\sum_{n=1}^{\infty} |f_n(x)|$ is convergence $\Rightarrow$ $\sum f_n(x)$ converges

Problem 4. (5 + 5 Points) Define the inner Lebesgue measure of a subset $A \subset \mathbb{R}^d$ by

$$|A|_i = \sup\{|F| : \ F \text{ closed and } F \subset A\}.$$

Prove the following statements.

4.1 If A is Lebesgue measurable, then $|A| = |A|_e = |A|_i$.
4.2 If $|A|_e < \infty$ and $|A|_e = |A|_i$, then A is Lebesgue measurable.

4-1 A Lebesgue measurable : $|A| = |A|_e$. We also have trivially that $|A|_i \leq |A|$

Let $\varepsilon > 0$, $\exists$ $F \subseteq A$ closed : $|A \setminus F| < \varepsilon$

$$|A| = |(A \setminus F) \cup F| = |A \setminus F| + |F| < \varepsilon + |F| < \varepsilon + |A|_i$$

$\Rightarrow$ $|A| \leq |A|_i$ $\square$

4-2 If $|A|_e < \infty$ and $|A|_e = |A|_i$ $\Rightarrow$ A is measurable

$\forall$ $k \geq 1$ $\exists$ U_k open $U_k \supset A$ $|A|_e \leq |U_k| < |A|_e + \frac{1}{k}$

$\exists$ $F_k \subseteq A$ closed $|A|_i - \frac{1}{k} \leq |F_k|$

$G = \bigcap_k U_k$ is a G_δ set $A \subset G$

$H = \bigcup_k F_k$ is a F_σ set $H \subset A$

$H \subseteq A \subseteq G$ Note $|H| \leq |A|_e < \infty$ $|G| \leq |U_k| < |A|_e + \frac{1}{k} < \infty$ $\forall k$.

$|G| \leq |A|_e$

$$|G \setminus A|_e \leq |G \setminus H| = |G| - |H| = |A|_e - |A|_e = 0$$

$\square$

Problem 5. (10 Points) Let $f : [0,1] \times [0,1] \to \mathbb{C}$ be a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in [0,1]$. Let $0 < \epsilon, \delta < 1$ and define

$$E_{\epsilon,\delta} = \{x \in [0,1] : |f(x,y) - f(x,0)| \le \epsilon \text{ for all } y < \delta\}$$

Prove that E is measurable.

let $r < \delta$ be a rational number.

$E_{\varepsilon,r} = \{ x \in [0,1] : |f(x,r) - f(x,0)| \le \varepsilon \}$

$x \mapsto f(x,0)$ is measurable by assumption

and so is $x \mapsto f(x,r)$. So

$x \mapsto f(x,r) - f(x,0)$ is measurable

$\Rightarrow$ $x \mapsto |f(x,r) - f(x,0)|$ is also measurable

let's call this fct $g_r(x)$

$E_{\varepsilon,r} = \{ g \le \varepsilon \}$ which is a measurable subset of $[0,1]$

Claim $E_{\varepsilon,\delta} = \bigcap_{\substack{r \in \mathbb{Q} \\ r < \delta}} E_{\varepsilon,r}$

clearly $\left\{ \begin{array}{l} \forall r < \delta \\ r \in \mathbb{Q} \end{array} \right.$ $E_{\varepsilon,r} \subseteq E_{\varepsilon,\delta} \Rightarrow \bigcap_{\substack{r \in \mathbb{Q} \\ r < \delta}} E_{\varepsilon,r} \subseteq E_{\varepsilon,\delta}$

let $x \in E_{\varepsilon,\delta}$ $\Rightarrow$ $|f(x,y) - f(x,0)| \le \varepsilon$ for all $y < \delta$

for $x \in [0,1]$ the function $h_x : [0,1] \to \mathbb{C}$, $y \mapsto f(x,y)$ is continuous.

So continuous at 0. $\exists\ \delta' > 0$ ($\delta' < \delta$) such $\forall\ y \in [0,\delta')$

$|f(x,y) - f(x,0)| \le \varepsilon \Rightarrow |f(x,r) - f(x,0)| \le \varepsilon\ \forall\ r \in \mathbb{Q}\ r < \delta' < \delta$

$\Rightarrow \quad x \in E_{\varepsilon, r} \qquad \forall\, r < \delta \quad r \in \mathbb{Q}$.

(Extra page for work)

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