# MATH 125 Lecture 9

## Proof of Chebyshev's theorem

$P_{n+1}(x) \equiv$ a monic polynomial of degree $n+1$
For any $x \in [-1, 1]$, assume $|P_{n+1}(x)| < \frac{1}{2^n}$ ⊛

Note that $T_{n+1}(x)$ alternates between $-1$ and $1$ a total of $(n+2)$ times

* At these $(n+2)$ points, $P_{n+1} - \frac{T_{n+1}}{2^n}$ is alternatively positive and negative. Why?

* We note that at one of these points $z$

$$P_{n+1}(z) - \frac{1}{2^n} T_{n+1}(z)$$

$$= P_{n+1}(z) - \frac{1}{2^n} \qquad\qquad P_{n+1}(z) + \frac{1}{2^n}$$

$$< 0 \qquad\qquad\qquad\qquad > 0$$

Using intermediate value theorem, $P_{n+1} - \frac{1}{2^n} T_{n+1}$ has atleast $n+1$ zeros.

* However, $P_{n+1} - \frac{1}{2^n} T_{n+1}$ is at most degree $n \cancel{+1}$. This (since it is monic for $P_{n+1}$ & $T_{n+1}$) leads to a contradiction.

$\Rightarrow$ What if $P_{n+1} - \frac{1}{2^n} T_{n+1} = 0$. Then $P_{n+1} = \frac{1}{2^n} T_{n+1}$

In that case, $|P_{n+1}(x)| = \left|\frac{1}{2^n} T_{n+1}(x)\right| = \frac{1}{2^n}$. This contradicts definition of $P_{n+1}$ (see *)

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Note that $T_{n+1}$ is a polynomial with $n+1$ roots $x_0 \dots x_n$
Therefore, $T_{n+1}(x) = c(x - x_0)(x - x_1) \dots (x - x_n)$
Let's require $T_{n+1}(x) = 1 \Rightarrow c = 1$.
Leading coefficient of

How about the stability?

$$\Lambda_n(x) < \frac{2}{\pi}\left(\log(n+1) + \gamma + \log\frac{8}{\pi}\right) + \frac{\pi}{72(n+1)^2}$$

$\gamma \equiv$ Euler constant

This is much less sensitive to perturbation errors compared to Lagrange interpolation.

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* Chebyshev nodes are optimal when $f$ is a smooth explicit function

* However there are many functions in applications that don't have smoothness or we only have access to discrete data.

## Piecewise linear interpolation

$x_0, x_1, \dots x_n$

$I_i = [x_i \dots x_{i+1}]$

Given : $(x_i, f(x_i))$ $(x_{i+1}, f(x_{i+1}))$

Linear interpolant : $\Pi_1 f(x) \equiv f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)}(x - x_i) \quad x \in I_i$

Main Result: If $f \in C^2(I)$, where $I = [x_0, x_n]$ then

$$\max_{x\in I} |f(x) - \Pi_1 f(x)| \le \frac{H^2}{8} \max_{x \in I} |f''(x)|$$

$H \equiv$ Maximum length of intervals $I_i$

$\Pi_1 f(x) \to f(x)$ as $H \to 0$

proof: Recall proof of Lagrange interpolation

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

set $n=1$

$$f(x) - p(x) = \frac{1}{2!} f''(\xi_x) \prod_{i=0}^{2}(x - x_i)$$

$$|f(x) - p(x)| \le \max_{x \in I} |f''(x)| \frac{H^2}{8}$$

Applications : Fitting time-series data . fplot in MATLAB

$(x_0, y_0) \quad (x_1, y_1) \quad (x_2, y_2) \quad \dots \quad (x_n, y_n)$

$P_n(x) = a_0 + a_1 x + \dots \; a_n x^n$

$$\begin{aligned} P_n(x_0) &= y_0 \\ P_n(x_1) &= y_1 \\ &\vdots \\ P_n(x_n) &= y_n \end{aligned} \implies \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} & x_0^n \\ \vdots & & & & & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & x_n^n \end{bmatrix}}_{V = \text{ Vandermonde matrix}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

Exercise: show that $V$ is invertible (assuming distinct points

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condition number of matrix

$$\frac{\|A(x+\Delta x) - Ax\|}{\|Ax\|} = \frac{\|A\Delta x\|}{\|Ax\|} \le \frac{\|A\| \, \|\Delta x\|}{\|Ax\|} \le \left(\frac{\|A\| \, \|x\|}{\|Ax\|}\right) \frac{\|\Delta x\|}{\|x\|} \equiv \kappa(x) \frac{\|\Delta x\|}{\|x\|}$$

Assume $A$ is invertible.

$$\|x\| = \|A^{-1}Ax\| \le \|A^{-1}\| \, \|Ax\|$$

$$\frac{\|A\| \, \|x\|}{\|Ax\|} \le \frac{\|A\| \, \|A^{-1}\| \, \cancel{\|Ax\|}}{\cancel{\|Ax\|}} = \|A\| \, \|A^{-1}\| = \kappa(A)$$

It turns out that $V$ is highly ill-conditioned as a function of $n$