Part II, assignment IV

Graded

Student

Scott A. Fullenbaum

Total Points

24 / 25 pts

Question 1

14.2

5 / 5 pts

✔ **– 0 pts** Correct

– 0 pts Your notation implies that all rotations are conjugate, but I see from your proof what you meant.

– 1 pt Did not give proof

Question 2

14.3

5 / 5 pts

– 0 pts Correct

✔ **– 0 pts** Showed that if $[g]$ is a conjugacy class that $\varphi([g])$ is contained in a conjugacy class but not that it is an entire conjugacy class. That is if h is conjugate to $\varphi(g)$ then $h \in \varphi([g])$.

– 2.5 pts Significant problem with proof

Question 3

14.4

4 / 5 pts

– 0 pts Correct

– 1 pt Showed that **one** $g \in S_8$ such that $g(12345)(678)g^{-1} = (43786)(215)$ is odd not that **every** such g is odd

– 1 pt Slight logic issue

– 2.5 pts Did not show which permutations are conjugate in A_n

✔ **– 1 pt** The proof that "if $gag^{-1} = hah^{-1} = b$ so $g^{-1}ha(g^{-1}h)^{-1} = a$ but g is odd and h is even so... but $g^{-1}h$ is even!" Only works if you first show the existence of an odd g.

– 1 pt Did not find all the conjugacy classes of S_8

Question 4

If every element of a group has order 2, is the group abelian?

5 / 5 pts

✔ **– 0 pts** Correct

– 2.5 pts Significant issues with logic

Question 5

Prove that that multiplicative group R_p is cyclic when p is a prime.

5 / 5 pts

✔ **− 0 pts** Correct

− 0.5 pts Minor logical issue

− 2.5 pts Did not finish problem

− 2.5 pts Major logical issue

1 The bit about prime factorization is unnecessary

Question assigned to the following page: 1

Algebra HW PII #3

$sr = r^{-1}s$

14.2)

First if n is odd:

$\{e\}$

$$(r^m s) r^k (r^m s)^{-1} = r^m s r^k s r^{-m}$$
$$= r^m r^{-k} r^{-m} = r^{-k}$$

So $\{r^k, r^{-k}\}$ but up to $k \leq \frac{(n-1)}{2}$ to avoid repetitions.

For $r^k s$

$$(r^m s) r^k s (r^m s)^{-1}$$
$$= r^m s r^k r^{-m} = r^m s r^{k-m} = r^{2m-k} s$$

but as n is odd, this will generate all reflections so for odd n have:

$\{e\}$, $\{r^k, r^{-k}\}$ $k \leq \frac{(n-1)}{2}$, $\{r^k s \mid 0 \leq k \leq n-1\}$

If n is even:

$\{e\}$

For r^k $(r^m s) r^k (r^m s)^{-1} = r^m s r^k s r^m$
$$= r^{-k}$$

So $\{r^k, r^{-k}\}$ but as $n =$ even, $r^{-n/2} = r^{n/2}$ so gives $\{r^{n/2}\}$ and $\{r^k, r^{-k}\}$ $k \leq \frac{(n-1)}{2} - 1$

Now, for $r^k s$ have:

$$(r^m s) r^k s (r^m s)^{-1} = r^{2m-k} s$$

but as n even here only gets $r^{2i} s$ if m, k even now:

$$r^m (r^k s) r^{-m} = r^{m+k} r^m s = r^{2m+k} s$$

So if k is odd, then this gets the odd ones so they form a distinct class. So we get for even n:

$\{e\}$, $\{r^{n/2}\}$, $\{r^{\pm k/2}\}$ for $k \leq \frac{n}{2} - 1$ and $\{r^{2i} s \mid 0 \leq i \leq n/2 - 1\}$ and $\{r^{2i+1} s \mid 0 \leq i \leq n/2 - 1\}$

14.3)

Let x, y ~~be~~ be conjugate to y. So, $\exists g \in G$ s.t $gxg^{-1} = y$

Then: $\phi(gxg^{-1}) = \phi(y)$
$\phi(g)\phi(x)\phi(g^{-1}) = \phi(y)$

As isomorphism, get
$\phi(g)\phi(x)\phi(g)^{-1} = \phi(y)$

So $\phi(x)$ is conjugate to $\phi(y)$. Therefore isomorphisms preserve conjugacy, so they then preserve conjugate classes $\square$

14.4)

S_6 has following cycle structures
e, (5)(1), (4)(2), (4)(1)(1),
(3)(3), (3)(2)(1), (3)(1)(1)(1),
(2)(2)(2), (2)(2)(1)(1), (2)(2)(1)(1)(1)(1)
(6)

So S_6 has 11 conjugacy classes.

To show (123)(456) and (531)(264) conjugate in A_6 then, as $(531)(264) = (264)(531)$, we get

$g(123)(456)g^{-1} = (g(1), g(2), g(3), g(4)) \; g(5) g(6)$

So we want the g s.t.

$g(1) = 2$ $\quad g(4) = 5$
$g(2) = 6$ $\quad g(5) = 3$ $\Rightarrow$ (126)(345)
$g(3) = 4$ $\quad g(6) = 1$

So, now this shows (for $g = (126)(345)$ which is in A_6, (123)(456) is

conjugate to $(531)(264)$
This also shows $(123)(456)$ and $(531)(264)$ are conjugate in S_6, as pick same element

For $\overset{\alpha}{(12345)(678)}$ and $\overset{B}{(43786)(215)}$ not conjugate in A_8.

Suppose $\exists g \in A_8$ s.t $g\alpha g^{-1} = B$, equivalently $\exists h \in A_8$ s.t $hBh^{-1} = \alpha$

We get $h\alpha h^{-1} = g\alpha g^{-1}$ from this, so leads to:

$$g^{-1}h\alpha(g^{-1}h)^{-1} = \alpha$$

If $h \in A_8$ then h is even. Now, then g must be odd so $g^{-1}h$ is odd

So, we want odd permutation commuting w/ α.

So would be like saying

$$f(12345)(678)f^{-1} = (12345)(678)$$

This has a 3 and a 5 cycle.

So, f must have a 5 cycle of form $(12345)^n$ and a 3 cycle of form $(678)^n$. However, product of 3 and 5 cycle is even, so f is even.

Meaning no odd permutation commutes w/ α, so no odd permutation h where $g^{-1}h$ commutes w/ α, as given, so α and B not conjugate.

Questions assigned to the following page: 4 and 5

Extra 1) If every element of G is order 2, G is Abelian

pf) $x, y \in G$ then $(xy)^2 = e$

so $(xy) = (xy)^{-1}$

but $(xy)^{-1} = y^{-1}x^{-1}$, but as $|x|=2, |y|=2$

then $x^{-1} = x$, $y^{-1} = y$ so $(xy)^{-1} = yx$

In conclusion: $(xy) = yx$ So G is abelian.

Extra 2) R_p is cyclic

$R_p = \{1, 2, \ldots p-1\}$

Now we have LCM of orders of each element, called m. as $x < p$

For $x^m = 1 \pmod p \Rightarrow x^m - 1 = 0 \pmod p$

every element is a solution by the hint so $p-1 \leq m$.

Next, note that by Fermat's little thm, $x^{p-1} = 1 \pmod p$ $\forall x \in G$. So, the order of every element divides $p-1$.

So denote the set of orders $A = \{a_1, a_2 \ldots a_n\}$ and $a_i \mid p-1$

We can see $LCM(a_1, a_2 \ldots a_n) \leq p-1$ as shown on earlier hw.

Also, can see as get prime factorization of each a_i, so coprime and $LCM = \prod p^i$ but accounting for overlapping factorizations leads to the fact above. So, our LCM, $m \leq p-1$, Meaning $m = p-1$. As R_p is finite abelian, it then has an element of order $p-1$, So it is cyclic. $\square$