Tufts University
Department of Mathematics

Math 136 | **Homework 2-New Version** [1] | **Spring, 202**

Due date: 11:59 pm, Sunday, February 5, 2023 on Gradescope.

You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

Reading assignment: Read Sections 14.1, 14.2, 14.3 by Wednesday, February 1. This homework covers Sections 13.2, 13.3, and 14.1.

Problems:

1 (20 points) Let $A \subset \mathbb{R}^n$ and let $\mathbf{x}_*$ be a limit point of A that is in A. Let $f : A \to \mathbb{R}^m$. Prove that f is continuous at $\mathbf{x}_*$ if and only if $\lim_{\mathbf{x}\to\mathbf{x}_*} f(x) = f(\mathbf{x}_*)$. [HINT: one proof uses the $\epsilon - \delta$ condition for limit in Theorem 13.7 and the $\epsilon - \delta$ condition for continuity in Theorem 11.11.]

2 (35 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x,y) \neq \mathbf{0} \\ 0 & (x,y) = \mathbf{0} \end{cases}$.

(a) Let $\mathbf{p} = (a,b)$ Calculate $D_{\mathbf{p}} f(0,0)$. This shows that f has all directional derivatives at $\mathbf{0}$. (NOTE: If we do this in class, just quote the result.)

(b) Does f have all directional derivatives at all point $\mathbf{x} \in \mathbb{R}^2$? Why or why not?

(c) Does f satisfy the conclusion of the Directional Derivative Theorem (Theorem 13.6 in Fitzpatrick) at $\mathbf{x}_0 = \mathbf{0}$? That is for $\mathbf{h} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ does $\frac{\partial f}{\partial \mathbf{h}}((0,0)) = \langle \nabla f(0,0), \mathbf{h}\rangle$? Why or why not?

(d) Is f continuously differentiable on $\mathbb{R}^2$? Why or why not?

3 (20 points) Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$. Assume $f \in C^2(\mathcal{O})$; that is, f has all first and second order partial derivatives on $\mathcal{O}$ and the second order partial derivatives are continuous on $\mathcal{O}$.

(a) Prove that f is continuous on $\mathcal{O}$.

(b) ~~Is f differentiable on $\mathcal{O}$? Why or why not?~~ You don't need to answer this part.

(c) Is f continuously differentiable on $\mathcal{O}$? Why or why not?

4 (25 points) Let f and g be functions from $\mathcal{O}$ to $\mathbb{R}$ and assume g is a k^{th} order approximation to f at $\mathbf{x}_0 \in \mathcal{O}$. Prove the following

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } \mathbf{x} \in \mathcal{O} \setminus \{\mathbf{x}_0\} \text{ and } \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ then}$$
$$|f(\mathbf{x}) - g(\mathbf{x})| \leq \epsilon \|\mathbf{x} - \mathbf{x}_0\|^k .$$

[HINT: Theorem 13.7 on p. 352 of Fitzpatrick would be helpful.]
This shows that as $\mathbf{x} \to \mathbf{x}_0$, $f(\mathbf{x}) - g(\mathbf{x})$ goes to zero (eventually) faster than any multiple of $\|\mathbf{x} - \mathbf{x}_0\|^k$.