Math 136 **Real Analysis II** **Spring 2022**

Exam 1 Review

Some of the big ideas about differentiation:

- Topics from last term that you are responsible for:
  - Continuity and various ways to check continuity, IVT, EVT, and other theorems, definitions, and ideas that we have used in class and homework.
- Differentiation for functions $f : I \to \mathbb{R}$ where I is an open interval (or open set) in $\mathbb{R}$
  - Definition of Derivative
  - Mean Value Theorem
  - Proof that differentiable functions are continuous
  - Calculations of derivatives using the definition
- Definition of limit point for $\mathcal{O} \subset \mathbb{R}^n$ and for limit of a function $F : \mathcal{O} \to \mathbb{R}^m$. Know how to check continuity using limits.
- Differentiation for functions $F : \mathcal{O} \to \mathbb{R}^m$ where $\mathcal{O}$ is an open set in $\mathbb{R}^n$
  - Definitions of $C^1(\mathcal{O}, \mathbb{R}^m)$ and $C^2(\mathcal{O}, \mathbb{R}^m)$.
  - Definition of DF and, if $f : \mathcal{O} \to \mathbb{R}$, definition of ∇f.
  - First order approximations and its relationship to differentiability and continuous differentiability.
  - Stating and using the Directional Derivative Theorem and the MVT for $f \in C^1(\mathcal{O}, \mathbb{R})$.
  - Using the definition of derivative to calculate derivatives and prove theorems such as if $f : \mathcal{O} \to \mathbb{R}$ is differentiable then f is continuous and the vector $\mathbf{b}$ in the definition is equal to $\nabla f(\mathbf{x}_0)$.
  - Using the Inverse Function Theorem. We will provide a statement of the Inverse Function Theorem on the test.
  - Using Dini's Theorem. We will provide a statement of Dini's Theorem on the test.

Review Problems for Exam 1

1. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$.
   (a) State the definition of $f'(a)$, for $a \in I$.

   (b) Let $a \in \mathbb{R}$. Use the definition of the derivative to compute $f'(a)$ for $f(x) = x^3$.

   (c) Assume $f : I \to \mathbb{R}$ is differentiable on I. Use the definition of derivative to prove that f is continuous on I.

2. Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$ and let $G \in C^1(\mathbb{R}, \mathbb{R}^2)$. Let $h(t) = f(G(t))$ for $t \in \mathbb{R}$. Find $\frac{dh}{dt}(t)$. (Note, here ∇f is viewed as a row vector so you can multiply it by DG.)

3. Define $g(x, y) = \dfrac{x^2 y}{x^2 + y^4}$ on domain $\mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\}$. Calculate $\lim_{(x,y)\to\mathbf{0}} g(x, y)$ and prove your result or explain why this limit does not exist.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + \cos(xy) + y$.
   (a) Note that $f(0, 1) = 2$. Explain why there is an open interval I containing 0 and a function $g \in C^1(I, \mathbb{R})$ that satisfies $g(0) = 1$ and $f(x, g(x)) = 2$ for all $x \in I$. Find $g'(0)$. Justify your answers.
   (b) Explain why $f : \mathbb{R}^2 \to \mathbb{R}$ is not injective (one-to-one).

5. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable (C^1). Assume $\mathbf{F}(1,2,3) = (4,5,6)$ and
$$\mathbf{DF}(1,2,3) = A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \text{ You may assume that } A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$
   (a) Does $\mathbf{F}$ satisfy the Inverse Function Theorem at $(1,2,3)$? Justify your answer.
   Assume the domain of $\mathbf{F}$ is given coordinates (x,y,z) and the target has coordinates (u,v,w) that is $\mathbf{F}(x,y,z) = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$.
   (b) Find $\frac{\partial u}{\partial x}(1,2,3)$.
   (c) Find $\frac{\partial x}{\partial u}(4,5,6)$. Recall that $\mathbf{F}(1,2,3) = (4,5,6)$.

6. Let $F(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \\ F_3(x,y) \end{pmatrix}$ be a C^1 function from $\mathbb{R}^2$ to $\mathbb{R}^3$ and assume $Df(1,1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$. Prove that there is an open neighborhood $\mathcal{U}$ of $(1,1)$ such that $F : \mathcal{U} \to \mathbb{R}^3$ is injective.
   HINT: first consider $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tilde{F}(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix}$.

Recall the definition of differentiability

Definition 1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $\mathbf{x}_0 \in \mathcal{O}$. Let $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$. Let B be an $m \times n$ matrix. Then, $\mathbf{F}$ is differentiable at $\mathbf{x}_0$if*

$$(1) \qquad \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{F}(\mathbf{x}_0 + \mathbf{h}) - [\mathbf{F}(\mathbf{x}_0) + B\mathbf{h}]\|}{\|\mathbf{h}\|} = 0 \quad \textit{equivalently} \quad \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|\mathbf{F}(\mathbf{x}) - [\mathbf{F}(\mathbf{x}_0) + B(\mathbf{x} - \mathbf{x}_0)]\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

The function $\mathbf{F}$ is differentiable on $\mathcal{O}$ if $\mathbf{F}$ is differentiable at all points in $\mathcal{O}$.

Recall that, if $\mathbf{F}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$ then $\mathbf{F}$ has all first partial derivatives at $\mathbf{x}_0$ and $DF(\mathbf{x}_0) = B$, the matrix in the definition of derivative.

Also, note that if $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ then we write $\nabla f(\mathbf{x}_0)$ for the vector $\mathbf{b}$ such that

$$(2) \qquad \lim_{\mathbf{h}\to\mathbf{0}} \frac{|F(\mathbf{x}_0 + \mathbf{h}) - [f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{h} \rangle]|}{\|\mathbf{h}\|} = 0.$$

If $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ according to Definition (1), then the matrix $B = (b_1, b_2, \ldots, b_n)$ in (1) is $1 \times n$, and the vector $\nabla f(\mathbf{x}_0) = \mathbf{b}$ in (2) is the column vector B^T. This is true because $B\mathbf{h} = \langle \mathbf{b}, \mathbf{h} \rangle$.

For the following problems, the Sandwich Theorem for functions is really useful and you may use it. I discussed it in the help session and in class on Monday.

Theorem 2. *Sandwich Theorem Let f, g, and h be functions from a set $A \subset \mathbb{R}^n$ to $\mathbb{R}$ and assume $\mathbf{x}_0$ is a limit point of A and $L \in \mathbb{R}$. Assume that*

$$(3) \qquad \forall \mathbf{x} \in A, \quad f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x}), \quad \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}_0) = L = \lim_{\mathbf{x}\to\mathbf{x}_0} h(\mathbf{x}).$$

Then, $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = L$.

This can simplify the estimates in the limits in the next two problems. Its proof will be on the answer sheet for the review problems

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2 + 2xy + 3$. Let $(x,y) \in \mathbb{R}^2$. Use the definition of derivative to show that f is differentiable at (x,y) and that $\nabla f(x,y) = (2x + 2y, 2x)$. (sorry for the misprint in ∇f!)

You may use the following limits

$$0 = \lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2+k^2}} =$$

8. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in \mathcal{O}$ and $\nabla f(\mathbf{x}_0) = \mathbf{b}$ is a vector in $\mathbb{R}^n$. Use the definition of derivative to prove that f is continuous at $\mathbf{x}_0$.

9. Explain why the graph of $z = \sin(xy)$ has a tangent plane at $(\pi, 1, 0)$ and find an equation of that tangent plane.