Are the following functions continuous or not continuous? Sketch an argument using the ϵ-δ definition of continuity and an argument using the open set definition of continuity.

(1) $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

(2) $g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(3) Prove or give a counterexample: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then whenever $U \subseteq \mathbb{R}$ is open, $f(U) \subseteq \mathbb{R}$ is open.

(4) Prove or give a counterexample: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then whenever $Z \subseteq \mathbb{R}$ is closed, $f(Z) \subseteq \mathbb{R}$ is closed.

We will show the following on homework:

Theorem 1. *If $f_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ are continuous, then the function*

$$\begin{aligned} f_1 \times f_2 : \mathbb{R}^n &\to \mathbb{R}^{m_1+m_2} \\ \mathbf{x} &\mapsto (f_1(\mathbf{x}), f_2(\mathbf{x})) \end{aligned}$$

is continuous.

(5) Let's show that subtraction is continuous without ϵ's and δ's.

(a) Show that the constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto -1$ is continuous using the open sets definition of continuity.

(b) Show that the identity function $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is continuous using the open sets definition of continuity.

(c) Prove that

$$\begin{aligned} - : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto x - y \end{aligned}$$

is continuous by writing it as a composition of continuous functions.

(d) Recall that the graph of a function $f : X \to Y$ is the subset

$$\mathrm{graph}(f) = \{(x, y) \in X \times Y \mid f(x) = y\}.$$

Is the graph of $\mathrm{id}_{\mathbb{R}}$ is a closed subset of $\mathbb{R}^2$? Is it an open subset of $\mathbb{R}^2$?

(6) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then for each closed subset Z of $\mathbb{R}^m$, $f^{-1}(Z)$ is a closed subset of $\mathbb{R}^n$. (Hint: $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.)

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Prove that the function $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto y - f(x)$ is continuous.

(b) Use part (a) to show that the graph of f is a closed subset of $\mathbb{R}^2$.

(8) Consider the function

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

again. We will show that the graph of g is not closed. (And therefore g is not continuous!)

(a) For each $y \in [-1, 1]$, show that $(0, y)$ is a limit point of graph(g). (Hint: what are the x's so that $\sin(1/x) = y$?)