Review Problems

1. Consider the following iterative method to solve a linear system $Ax = b$.

$$x^{(k+1)} = x^{(k)} + \omega\,(b - Ax^{(k)}) \qquad k = 0, 1, 2, \ldots$$ and ω is a scalar parameter.

(a) Prove that the iterative method is linear

(b) Find an explicit form of the iteration matrix

(c) For any initial estimate $x^{(0)}$, what condition ensures the convergence of the iterative method to the exact solution?

(d) Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If $\omega = \frac{1}{4}$, does the iterative method converge to the true solution of $Ax = b$?

solution

(a) Let x be the exact solution to $Ax = b$.

$$x - x^{(k+1)} = x - x^{(k)} - \omega A(x - x^{(k)}) \qquad ①$$

To obtain ①, we used the following two equations

$$x = x + \omega(b - Ax) \qquad Ⓐ$$

$$x^{(k+1)} = x^{(k)} + \omega(b - Ax^{(k)}) \qquad Ⓑ$$

Note that ① is equivalent to

$$e^{(k+1)} = (I - \omega A)\,e^{(k)}$$

$\therefore$ The method is linear

(b) Iteration matrix $\equiv I - \omega A$

(c) Require $\rho(I - \omega A) < 1$

Eigenvalues of A : $\lambda_1 \ldots \lambda_n$

Eigenvalues of $I - \omega A$: $1 - \omega\lambda_1, \ldots, 1 - \omega\lambda_n$

The condition is $|1 - \omega\lambda_j| < 1$ for $1 \le j \le n$

(d) Note $A\vec{1} = 4\vec{1}$ where $\vec{1}$ is a vector of ones

Also note $\lambda_1 + \lambda_2 + \lambda_3 = 8 = \text{Trace}(A)$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 12 \quad \det(A)$

Let $\lambda_3 = 1$. $\lambda_1 + \lambda_2 = 7$, $\lambda_1 \cdot \lambda_2 = 12$ $\Rightarrow \lambda_1 = 4$; $\lambda_2 = 3$

Therefore, the eigenvalues of A are 1, 3 and 4.

Eigenvalues of M: $3/4$, $1/4$, 0. $\therefore$ Iteration converges

2. Consider the integral $\int_0^1 \frac{1}{x^2+e^x} dx$.

We are given that $|f(x)| \le 1$, $|f'(x)| \le 1.1$, $|f''(x)| \le 1.1$, $|f'''(x)| < 11$ and $|f''''(x)| \le 34$

ⓐ use composite Simpson ($n=6$) to approximate the integral

Solution $h = \frac{1}{6}$

$$\int_0^1 f(x)\,dx \approx \frac{h}{3}\left[f(0) + 4f(1/6) + 2f\left(\tfrac{2}{6}\right) + 4f\left(\tfrac{3}{6}\right) + 2f\left(\tfrac{4}{6}\right) + 4f\left(\tfrac{5}{6}\right) + f(1)\right]$$

$$\approx 0.5657$$

ⓑ Find bounds on n and h such that the composite Simpson's rule has error less than 10^{-6}

Solution $|E| = \left|\frac{b-a}{180} h^4 f^4(\eta)\right| \quad \eta \in (0,1)$

$|f^4(\eta)| \le 34$

$\frac{1}{180} h^4 (34) < 10^{-6} \implies h \approx 0.0480 \quad n = \frac{1}{h} > 20.8$

Therefore, $n \ge 22$ (Note: Simpson requires even # of subintervals)

ⓒ Consider the quadrature rule

$$\int_0^1 f(x)\,dx = w_1 f(0) + w_2 f'(x_2)$$

ⓐ Find values of w_1, w_2 and x_2 so that this rule has the highest possible degree of accuracy

Let $f(x) = a + bx + cx^2$

$$\int_0^1 f(x)\,dx = ax + \frac{b}{2}x^2 + \frac{cx^3}{3}\Big|_0^1 = a + \frac{b}{2} + \frac{c}{3} \quad ⓒ$$

$$\int_0^1 f(x)\,dx = w_1 a + w_2(b + 2c x_2) \quad ⓓ$$

Equate ⓒ and ⓓ to get $w_1 = 1$
$w_2 = 1/2$
$x_2 = 1/3$

3. Derive a second order Taylor method to solve the following initial value problem.
$y' = te^y, \ y(0) = 1$

solution $\quad y(t_i+h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{6} y'''(\eta)$

Note $y'(t_i) = t_i e^{y(t_i)} = f(t, y)$

Consider y''

$$y'' = \frac{d}{dt}\left[f(t, y)\right]$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'$$

$$= e^y + te^y (y')$$

$$= e^y + te^y \cdot te^y = \underline{\underline{e^y + t^2 e^{2y}}}$$

$$w_{i+1} = w_i + h(te^y) + \frac{h^2}{2}\left(e^y + t^2 e^{2y}\right)$$

Local truncation error $\equiv O(h^3)$

Global error $\equiv O(h^2)$

4. Consider data points (x_0, y_0) (x_1, y_1) (x_2, y_2) ... (x_n, y_n) where all the $x_0, x_1, \ldots x_n$ are positive. We propose fitting the data points to the following model:

$$y = a + b \ln(x)$$

Using least squares, determine a and b.

solution

$$\underbrace{\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}}_{y} \approx \underbrace{\begin{pmatrix} 1 & \ln(x_0) \\ 1 & \vdots \\ \vdots & \vdots \\ 1 & \ln(x_n) \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{c}$$

Need to solve $(A^T A)c = A^T y$

$$A^T A = \begin{pmatrix} n & \ln(x_0) + \cdots \ln(x_n) \\ \ln(x_0) + \cdots \ln(x_n) & [\ln(x_0)]^2 + \cdots [\ln(x_n)]^2 \end{pmatrix}$$

$$\begin{bmatrix} n & \ln(x_0)+\dots+\ln(x_n) \\ \ln(x_0)+\dots\ln(x_n) & [\ln(x_0)]^2+\dots+[\ln(x_n)]^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} y_0+y_1+\dots y_n \\ y_0\ln(x_0)+\dots+y_n\ln(x_n) \end{pmatrix}$$

$$(\text{Row } 2) - \frac{(\ln(x_0)+\dots\ln(x_n))}{n} \cdot \text{row } 1$$

$$\begin{pmatrix} n & \ln(x_0)+\dots+\ln(x_n) \\ 0 & [\ln(x_0)]^2+\dots+[\ln(x_n)]^2 - \frac{[\ln(x_0)+\dots\ln(x_n)]^2}{n} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} y_0+y_1+\dots+y_n \\ y_0\ln(x_0)+y_1\ln(x_1)+\dots y_n\ln(x_n) - \frac{(y_0+\dots y_1)(\ln(x_0)+\dots\ln(x_n))}{n} \end{pmatrix}$$

Therefore, $$\boxed{b = \frac{n\sum_{i=0}^{n} y_i\ln(x_i) - \left(\sum_{i=0}^{n} y_i\right)\left(\sum_{i=0}^{n}\ln(x_i)\right)}{n\sum_{i=0}^{n}[\ln(x_i)]^2 - \left[\sum_{i=0}^{n}\ln(x)\right]^2}}$$

$$na + b\sum_{i=0}^{n}\ln(x_i) = \sum_{i=0}^{n} y_i$$

$$\boxed{a = \frac{\sum_{i=0}^{n} y_i - b\sum_{i=0}^{n}\ln(x_i)}{n}}$$

(5) Given the LU decomposition of a matrix, propose an efficient way to compute

i) $x^T A^{-1} x \quad x \in \mathbb{R}^n, \; A \in \mathbb{R}^{n\times n}$

ii) $\det(A)$

solution i) $x^T A^{-1} x$ Let $z = A^{-1}x \Rightarrow Az = x$

$LUz = x$

$O(n^2)$ to solve z

$x^T z = O(n)$

$\therefore O(n^2)$

ii) $\det(A) = \det(L)\,\det(U)$

lower triangular $O(n)$

upper triangular $O(n)$

$\therefore O(n)$

⑥ Consider the problem

$$\min \|x\|_2^2$$
$$A^T A x = A^T b$$

i) prove that the optimal solution x is in $\text{Null}(A)^\perp$

solution $x = x_1 + x_2$ $\quad x_1 \in \text{Null}(A)$, $x_2 \in \text{Null}(A)^\perp$ (orthogonal decomposition theorem)

$$A^T A x = A^T A x_1 + A^T A x_2$$
$$= A^T A x_2 \implies A^T A x_2 = b$$
$$A^T A x = A^T b$$
$$\|x\|_2^2 = \|x_1 + x_2\|_2^2$$
$$= \|x_1\|^2 + \|x_2\|^2$$
$$\geq \|x_2\|_2^2$$

$\therefore$ optimal solution is in $\text{Null}(A)^\perp$

ii) prove that the solution is unique

solution In (i) we established that $x^* \in \text{Null}(A)^\perp$
For contradiction, assume two solutions $x_1 \in \text{Null}(A)^\perp$ and $x_2 \in \text{Null}(A)^\perp$ which are optimal. Then

$$A^T A x_1 = A^T b \implies A^T A(x_1 - x_2) = \vec{0}$$
$$A^T A x_2 = A^T b \qquad (x_1 - x_2) \in \text{Null}(A^T A)$$
$$\Downarrow$$
$$(x_1 - x_2) \in \text{Null}(A)$$

However, $(x_1 - x_2) \in \text{Null}(A)^\perp$
This leads to a contradiction

$\therefore$ unique solution

(7) Show that $y' = \dfrac{4t^3 y}{1+t^4}$ $\quad 0 \le t \le 1$

$y(0) = 1$

has a unique solution.

solution $\left|\dfrac{\partial f}{\partial y}(t,y)\right| = \left|\dfrac{4t^3}{1+t^4}\right| \le 4$ for $0 \le t \le 1$ $\quad L = 4$

$f(t,y)$ is Lipschitz continuous in y on $[0,1] \times [-\infty, \infty]$.
Therefore, exactly one solution in $[0,1]$.

Extra ↓

To make the bound sharper, we can argue as follows. Note, this is not necessary for uniqueness of the IVP. Any finite L will work.

Note $\dfrac{d}{dt}\left(\dfrac{4t^3}{1+t^4}\right) = -\dfrac{4t^2(t^4-3)}{(1+t^4)^2}$ This is monotonically increasing on $[0,1]$

The maximum is achieved at $t=1 \Rightarrow \dfrac{4 \cdot 1^3}{1+1} = \dfrac{4}{2} = 2$