Lyapunov exponents

(a bit more accurately)

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Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, and consider

$$\frac{dx}{dt} = f(x).$$

Let $T > 0$, and consider two solutions $x(t)$ and $\tilde{x}(t)$ that are close to each other. The difference

$$u(t) = x(t) - \tilde{x}(t)$$

satisfies

$$\frac{du}{dt} = \frac{dx}{dt} - \frac{d\tilde{x}}{dt} = f(x(t)) - f(\tilde{x}(t)) \approx Df(x(t))(x(t) - \tilde{x}(t))$$

so it approximately satisfies

$$\frac{du}{dt} = J(t)u \tag{1}$$

where

$$J(t) = Df(x(t)).$$

The closer $x(t)$ and $\tilde{x}(t)$ are to each other, the more accurate is the approximation (1), and if $T > 0$ is fixed while $u(0)$ tends to 0, then (1) becomes exact in the limit. Note that this is true even for arbitrarily large (but fixed) T.

Equation (1) implies that $u(t)$ depends linearly on $u(0)$:

$$u(t) = M(t; x_0)u(0), \quad 0 \le t \le T,$$

for some time-dependent $n \times n$ matrix M. Notice that this matrix depends on x_0. We can calculate it numerically, simply by determining $u(t)$ when $u(0) = \varepsilon e_k$, $1 \le k \le n$, where $\varepsilon > 0$ is small and e_k is the k-th canonical basis vector of $\mathbb{R}^n$.

We are interested in how rapidly trajectories that are near each other initially will separate, and we are therefore interested in

$$\|M(t; x_0)\|_2 = \max \left\{ \frac{\|M(t; x_0)u_0\|_2}{\|u_0\|_2} \ : \ u_0 \in \mathbb{R}^n, \quad u_0 \ne 0 \right\}.$$

On the right-hand side of this equation, $\|\cdot\|_2$ denotes the Euclidean norm of vectors. The equation *defines* $\|M(t; x_0)\|_2$, and the quantity thus defined is called the *spectral norm* of $M(t; x_0)$.

If it is possible for initially nearby trajectories to separate exponentially fast, then one would expect that

$$\|M(t; x_0)\|_2 \approx e^{\lambda t} \quad \text{for some } \lambda > 0.$$

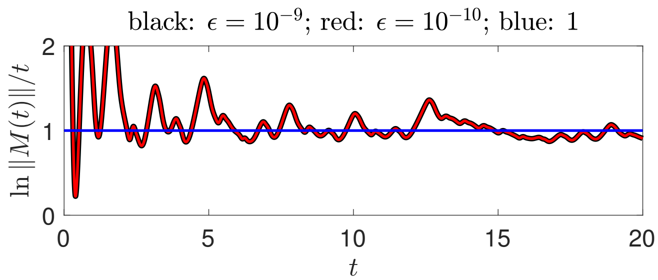
Solving for λ, we find

$$\lambda \approx \frac{\ln \|M(t; x_0)\|_2}{t}. \tag{2}$$

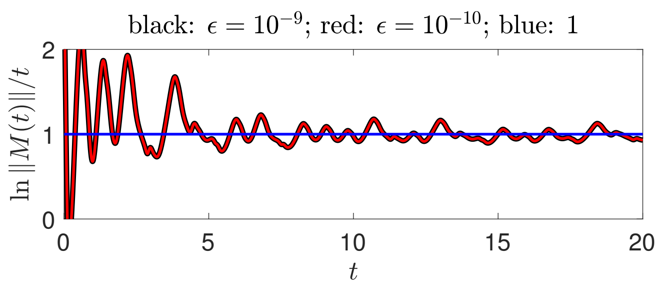
Let us try this for the Lorenz system:

$$\begin{aligned}\frac{dx}{dt} &= -10x+10y,\\ \frac{dy}{dt} &= -y+28x-xz,\\ \frac{dz}{dt} &= -\frac{8}{3}z+xy.\end{aligned}$$

We prescribe $x(-20)$, $y(-20)$, and $z(-20)$ at random, then solve up to time $t=0$, obtaining a random point $(x_0,y_0,z_0)=(x(0),y(0),z(0))$ that nearly lies on the Lorenz attractor. Starting with this point, we then compute $(x(t),y(t),z(t))$ for $0<t\le 20$, as well as three perturbed solutions obtained by replacing first x_0 by $x_0+\varepsilon$, then y_0 by $y_0+\varepsilon$, and finally z_0 by $z_0+\varepsilon$, where $\varepsilon>0$ is small. From these perturbed solutions, we obtain approximations for $M(t;x_0)$, $0\le t\le 20$. (Notice that $M(0,x_0)$ is the identity matrix.) We plot (2) as a function of t, once with $\varepsilon=10^{-9}$ and once with $\varepsilon=10^{-10}$. The result looks like this:



The fact that the black and red curves are nearly identical confirms that $\varepsilon=10^{-9}$ is small enough for $0\le t\le 20$. Starting with a different random initial condition, we get this:



Remarkably, the result looks very similar, especially for large t. In fact, *Oseledets' Theorem* implies that

$$\lambda = \lim_{t\to\infty} \frac{\ln \|M(t;x_0)\|_2}{t}$$

exists and is independent of x_0. Intuitively, the independence of x_0 holds because any one trajectory comes arbitrarily close to every point on the Lorenz attractor infinitely many times; this is, of course, not obvious. The quantity λ is the *Lyapunov exponent* of the Lorenz system. It is one single quantity that characterizes "how chaotic" the system is. As the plots suggest, it is (1) slightly smaller than 1, and (2) hard to compute with great accuracy. (The calculations above almost push the limit of what is possible on a computer with 16-digit arithmetic, if the straightforward approach to computing λ that I have outline here is used.)