Math 0235 **Fall 2020** **(Take Home) Midterm Exam**

1 Instructions

- Complete all the three problems in Section 2 and only two problems of your choice from the problems listed in Section 3.
- You may discuss the problems with peers. You must, however, write up your own solutions.
- Show work and be rigorous within reason.
- List all the references you might use.
- The exam is due by 10:30 a.m. on Monday October 26, 2020.
- If you need hints, clarifications, etc..., do not hesitate to come and talk to me.
- Good Luck!

2 Complete all the three problems in this section

Problem 1: (4 points) (The different parts of this problem are independent).

(1.1) Let m_* be the Lebesgue outer measure on $\mathbb{R}$. Assume that $A, B \subset \mathbb{R}$ are such that there exists $\epsilon > 0$ with

$$\inf\{|a - b| \, a \in A, b \in B\} \geq \epsilon.$$

Prove that $m_*(A \cup B) = m_*(A) + m_*(B)$.

(1.2) For $A, B \subset \mathbb{R}$, let $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Assume that A is a Lebesgue measurable set and $B \subset \mathbb{R}$ such that $m_*(A\Delta B) = 0$. Show that B is Lebesgue measurable and that $m(A) = m(B)$.

(1.3) Let $E \subset \mathbb{R}$ be Lebesgue measurable with $0 < m(E) < \infty$. Given a sequence of measurable sets $\{A_n\}_{n=1}^{\infty}$ with $A_n \subset E$ and $\lim_{n\to\infty} m(A_n) = m(E)$, prove that there exists a subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ such that $m(\cap_{k=1}^{\infty} A_{n_k}) > 0$.

(1.4) Suppose that $E \subset \mathbb{R}$ such that $m_*(E) > 0$. Prove that there exists $S \subset E$ such that S is not Lebesgue measurable.

Problem 2 (4 points) Assume $E \subset \mathbb{R}^d$ be a Lebesgue measurable set such that $m(E) < \infty$.

(2.1) Suppose that $f : E \to \mathbb{R}$ is Lebesgue measurable. Prove that for each $\epsilon > 0$ there exists a closed set $F \subset E$ such that $m(E \setminus F) < \epsilon$ and f is bounded on F.

(2.2) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions defined on E Suppose that for each $x \in E$ we have $M_x = sup_{n\geq 1}|f_n(x)| < \infty$. Prove that for each $\epsilon > 0$, there exist a closed set $F \subset E$ and a finite constant M such that $m(E \setminus F) < \epsilon$ and $|f_n(x)| \leq M$ for all $x \in F$ and $n \geq 1$.

Problem 3 (4 points) (The different parts of this problem are independent).

(3.1) Let $f : \mathbb{R}^d \to [0, \infty]$ be a Lebesgue measurable function. Prove that $f = \sum_{n=0}^{\infty} a_n 1_{A_n}$ where $a_n \geq 0$ and A_n is Lebesgue measurable.

(3.2) Let m denote the Lebesgue measure restricted to the interval $[0, 1]$, and let $\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}$ be two sequences of Lebesgue measurable functions defined from $[0, 1]$ into $\mathbb{R}$. Assume that $\sum_{n=1}^{\infty} m(\{x \in [0, 1] : f_n(x) \neq g_n(x)\}) < \infty$. Prove that there exists a Lebesgue measurable set $A \subset [0, 1]$, with $m(A) = 0$, and such that for all $x \in [0, 1] \setminus A$, there exists an integer k such that $f_n(x) = g_n(x)$ for all $n > k$.

(3.3) Let m denote the Lebesgue measure restricted to the interval $[0, 1]$, and let f_n be a sequence of nonnegative Lebesgue measurable functions on $[0, 1]$. Show that f_n converges to zero in measure if and only if

$$\lim_{n\to\infty} \int_0^1 \tfrac{f_n}{1+f_n}\, dm = 0.$$

3 Complete two problems of your choice from this group

Problem 4 (4 points) Let $\mathcal{B}$ denote the Borel $\sigma-$algebra on $\mathbb{R}$. Let $\mu : \mathcal{B} \to [0, \infty]$ be a measure such that for each $x \in \mathbb{R}$ and each $A \in \mathcal{B}$, $A + x \in \mathcal{B}$ and $\mu(x + A) = \mu(A)$. Set $\alpha = \mu((0, 1])$.

(4.1) Prove that $\mu((0, x]) = \alpha m((0, x]) = \alpha x$ for each $x \in \mathbb{R}$, and that $\mu((a, b]) = \alpha m((a, b]) = \alpha(b - a)$ for all $-\infty \leq a < b \leq \infty$, where m is the Lebesgue measure.

(4.2) Prove that μ is a $\sigma-$finite measure, that is $\mathbb{R} = \cup_{n=1}^{\infty} X_n$, $X_n \in \mathcal{B}$ are disjoint and $\mu(X_n) < \infty$ for all $n \geq 1$. Finally, conclude that $\mu(A) = \alpha m(A)$ for each $A \in \mathcal{B}$.

Problem 5 (4 points) Let m be the Lebesgue measure on $[0, 1]$. For any two measurable subsets E, F of $[0, 1]$, define a relation $\mathcal{R}$ by

$$E\,\mathcal{R}\,F \quad \Longleftrightarrow \quad m(E\Delta F) = 0.$$

(5-1) Show that $\mathcal{R}$ is an equivalence relation, i.e., that $\mathcal{R}$ is reflexive, symmetric and transitive.

(5-2) For $E, F \in \mathcal{L}$ define $\rho(E, F) = m(E\Delta F)$. Show that for all $E, F, G \in \mathcal{L}$, $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ and conclude that ρ is a metric on the space $\mathcal{L}/\mathcal{R}$ of equivalence classes.

Problem 6 (4 points) Let $\phi : [0, 2\pi) \to \mathbb{C}$ be defined by $\phi(\theta) = e^{i\theta}$. Let $S^1 = \phi([0, 2\pi))$ be the range of ϕ (note S^1 is just the unit circle in the complex plane.) Denote by $\mathcal{B}([0, 2\pi))$ the Borel σ-algebra on $[0, 2\pi)$, and define

$$\widetilde{\mathcal{B}}(S^1) = \{B \subset S^1 : \phi^{-1}(B) \in \mathcal{B}([0, 2\pi)).\}$$

(6-1) Show that $\widetilde{\mathcal{B}}(S^1)$ is a σ-algebra of subsets of S^1.

(6-2) Define the non-negative function m_1 on $\widetilde{\mathcal{B}}(S^1)$ by $m_1(B) = \frac{1}{2\pi} m(\phi^{-1}(B))$ where m is the Lebesgue measure on $[0, 2\pi)$. Show that m_1 is a measure on $\widetilde{\mathcal{B}}(S^1)$.

(6-3) For $x = e^{i\theta} \in S^1$, let ψ_x be a function defined on S^1 by $\psi_x(y) = x \cdot y$, where $x \cdot y$ is the usual product of complex numbers. Note that ψ_x is a continuous bijection map whose inverse is also continuous. Show that for all $B \in \widetilde{\mathcal{B}}(S^1)$, $\psi_x^{-1}(B) = \phi(\phi^{-1}(B) + \theta)$, and conclude that ψ_x is measurable (with respect to $\widetilde{\mathcal{B}}(S^1)$).

(6-4) Show that $m_1(\psi_x^{-1}(B)) = m_1(B)$ for all $B \in \widetilde{\mathcal{B}}(S^1)$.