Math 70 HW 2

Question 1 By theorem 3 of chapter 1, when $A\vec{x}=\vec{b}$, the equation has the same solution as $x_1a_1+\dots x_4a_4=b$, which has the same solution as the augmented matrix $[a_1, a_2, a_3, a_4, b]$. If this system is consistent then $x_1a_1+x_2a_2+x_3a_3+x_4a_4=b$, which is a linear combination of the columns of A.

Question 2 Let $A_{mn} = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$

$m\times n, n\times 1$

$Ax = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ To multiply, the # columns of A must equal the # rows of x. Assuming this condition is met, $Ax = \begin{bmatrix} a_{11}\cdot 0 \\ \vdots \\ a_{m1}\cdot 0 \end{bmatrix}$ as the matrix resulting from Ax will have dimensions $m\times 1$. Further simplifying Ax, we get $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, meaning $\vec{x}=\vec{0}$ is not in the solution set of Ax.

Question 3 If W is linearly dependent, then by the definition of linear dependence, $c_1\vec{v}+c_2\vec{w}+c_3(2\vec{v}-3\vec{w})=0$, has a nontrivial solution. The equation breaks into $\vec{v}(c_1+2c_3)+\vec{w}(c_2-3c_3)=\vec{0}$, and for this to be true $c_1+2c_3=0$, and $c_2-3c_3=0$. From these equations, $c_1=-2c_3$ and $c_2=3c_3$, meaning c_3 is a free variable. Since c_3 is free, there are infinitely many values of c_1, c_2, and c_3 that solve the equation, thus W is linear dependent, as a nontrivial solution exists.

Question 4 Since $\vec{u}$ and $\vec{v}$ are linearly independent then for $\vec{u}$ and $\vec{v}$, then $x_1\vec{u} + x_2\vec{v} = \vec{0}$, only has the trivial solution of $x_1 = 0$ and $x_2 = 0$.

2x2 2x1

Since u and v are linearly independent, u and v must have basic variables and a pivot in every column. Therefore, the reduced echelon form of $A\vec{x} = \vec{u}$ is $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ meaning that $x_1 = 0$ and $x_2 = 0$, giving us the trivial solution.

Let $\vec{v}'$ be a solution to $Ax = \vec{v}$, thus $A\vec{v}' = \vec{v}$
Let $\vec{v}''$ be a solution to $Ax = \vec{v}$, thus $A\vec{v}'' = \vec{v}$
This means that $A\vec{v}'' - A\vec{v}' = \vec{v} - \vec{v}$
$A(\vec{v}'' - \vec{v}') = \vec{0}$.
The unique solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, meaning $\vec{v}'' - \vec{v}' = \vec{0}$, and $\vec{v}'' = \vec{v}'$. Since the two vectors are equal, $A\vec{x} = \vec{v}$ can only have the unique solution $\vec{v}'$.

To prove $\vec{u}'$ and $\vec{v}'$ are linearly independent, suppose they are linearly dependent, meaning $\vec{v}'$ is a multiple of $\vec{u}'$. Thus $\vec{v}' = c\vec{u}'$ where c is a constant. Then $A\vec{u}' = cA\vec{v}'$, by multiplying by A.
Since $A\vec{u}' = \vec{u}$ and $A\vec{v}' = \vec{v}$, $\vec{u} = c\vec{v}$.
Since $\vec{v}$ is a multiple of $\vec{u}$, $\vec{u}$ and $\vec{v}$ are linearly dependent.

Since the contrapositive is true ($\vec{u}'$ and $\vec{v}'$ are linearly dependent, then $\vec{u}$ and $\vec{v}$ are linearly dependent), the original statement is true. Thus if $\vec{u}$ and $\vec{v}$ are linearly independent, then $\vec{u}'$ and $\vec{v}'$ are linearly independent.

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