Math 65 HW4

1) if f is onto, then there is a $g: B \to A$ $f \circ g = I_B$
Since f is onto, $\exists a \in A$, $f(a) = b$, $\forall b \in B$. $g(b) = a$, as $g: B \to a$
$f \circ g = f(g(b)) \to f(a) = b = I_B$

Going from $f \circ g = I_B$, $f(g(b)) = I_B$ $g(b) = a$,
where $a \in A, b \in B$ $(f \circ g) b = I_B$
$I(B b) = I_B$

Onto as it maps $g(b)$ to f and A.
Therefore f is onto iff $g: B \to A$ exists such that
$f \circ g = I_B$

2a) If f is one-to-one, then $f(a_1, b_1) = f(a_2, b_2)$ $a_1 = a_2$ $b_1 = b_2$
$(a_1 + b_1, a_1 - b_1)(a_2 + b_2, a_2 - b_2)$
$a_1 + b_1 = a_2 + b_2$ $a_1 = a_2 + b_2 - b_1$
$a_1 - b_1 = a_2 - b_2$
$a_2 + b_2 - b_1 - b_1 = a_2 - b_2$
$-2b_1 = -2b_2$, $b_1 = b_2$

Since $b_1 = b_2$, $a_1 = a_2$, Meaning f is one-to-one.

b) If f is onto, $\forall (c, d) \in \mathbb{Z} \times \mathbb{Z}$ $f(a, b) = (c, d)$
$(a+b, a-b) = (c, d)$
$a + b = c$ $a - b = d$
$a = c - b$. $c - 2b = d$
$a = c - \left(\frac{c-d}{2}\right)$ $\frac{c-d}{2} = b$
$a = \frac{c}{2} + \frac{d}{2}$ $b = \frac{c}{2} - \frac{d}{2}$
$f\left(\frac{c}{2} + \frac{d}{2}, \frac{c}{2} - \frac{d}{2}\right)$ is not onto as $(c, d) \notin \mathbb{Z} \times \mathbb{Z}$
for $\forall (a, b)$

3 a) $f(f^{-1}(\varphi)) \subseteq \varphi$. $\exists x \in f^{-1}(\varphi)$, where $f(x) = \varphi$

$f(x) \in \varphi$, and by definition

$f(x) = y \in B$, therefore $f(f^{-1}(\varphi)) \subseteq \varphi$

b) Choose a $b \in B$ to find $a \in A$ such that $f(a) = b$. Have $q = \{b\}$. We know that

$f(f^{-1}(b)) = a \in A$ $\quad f(a) = \{b\} \Rightarrow \exists a \in A$ $f(a) = b$

c) Since f is onto, for $\forall y \in B$, $\exists x \in X$

$f(x) = \varphi$. Meaning $x \in f^{-1}(y)$ so $y = f(x)$

and $f(x) = f(f^{-1}(\varphi))$ as $f^{-1}(\varphi) = x$.

4 a) If f is a bijection, it is one-to-one and onto

To prove one to one, $f(x_1) = f(x_2)$, $x_1 = x_2$, $x_1, x_2 \in \mathbb{Z}$

$4x_1^3 + 13 = 4x_2^3 + 13$, $\quad x_1^3 = x_2^3$, $\quad x_1 = x_2$

To prove onto, for $\forall y \in \mathbb{Z}$, $\exists x \in \mathbb{Z}$, such that $f(x) = y$

$y = 0$, $\quad 0 = 4x^3 + 13$, $\quad -\frac{13}{4} = x^3$, $\quad x = \sqrt[3]{-\frac{13}{4}} \notin \mathbb{Z}$

therefore f is not onto and not a bijection.

b) To prove f is one to one $f(x_1) = f(x_2)$, $x_1 = x_2$, $x_1, x_2 \in \mathbb{R}$

$4x_1^3 + 13 = 4x_2^3 + 13$, $\quad x_1^3 = x_2^3$ $\quad x_1 = x_2$ $\quad$ f is one to one

To prove onto, $\forall y \in \mathbb{R}$, $\exists x \in \mathbb{R}$ where $f(x) = y$

$4x^3 + 13 = y$

$\sqrt[3]{x^3 = \frac{y-13}{4}}$ $\quad x = \sqrt[3]{\frac{y-13}{4}}$

$f\left(\sqrt[3]{\frac{y-13}{4}}\right)$ has a solution for all $y \in \mathbb{R}$, meaning f is onto

Therefore, $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 4x^3 + 13$ forms a bijection as it is onto and one-to-one

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5a) $f(A \cap B) \subseteq f(A) \cap f(B)$

Take an element $y \in f(A \cap B)$. We know some $x \in A \cap B$ where $y = f(x)$. Since $x \in A$, we have $y = f(x) \in A$, and $x \in B$, so $y = f(x) \in f(B)$.

Therefore, $y = f(x) \subseteq f(A) \cap f(B)$

Reverse example: $f(A) \cap f(B) \not\subseteq f(A \cap B)$.

For example, $f: \mathbb{R} \to \mathbb{R}$, where $A = \mathbb{R}^{\geq 0}$ $B = \mathbb{R}^{<0}$ and $f(x) = x^2$, $f(A \cap B) = \phi$, $f(A \cap B) = \phi$.

However $f(A) \cap f(B) = \mathbb{R} - \{0\}$ as $f(x) = x^2$ is an even function, as $f(-x) = f(x)$ $(-x)^2 = x^2$.

6.1) If h is one to one, then $h(a_1) = h(a_2)$ $a_1, a_2 \in A$ and $a_1 = a_2$

$h(a_1) = h(a_2)$

$g(f(a_1)) = g(f(a_2))$ f is one to one, $f(a_1) \in f(a_2)$, $a_1 = a_2$

$g(b_1) = g(b_2)$, g is one to one, so $b_1 = b_2$

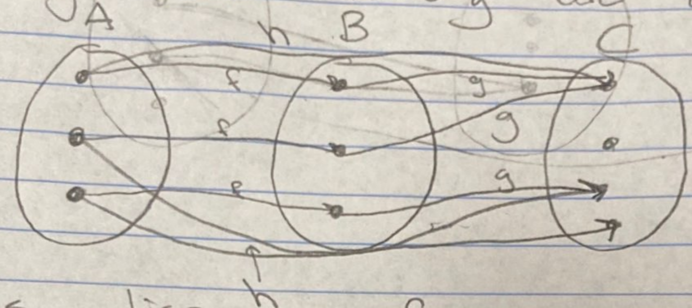
Therefore h is one-to-one

2) $\forall c \in C \ \exists a \in A \ h(a).\ A \to C$

$f(a) = b$ exists as f is onto, $b \in B$, $g(b) = c$ exists as g is onto, $h(a) = (g \circ f)(a)$, therefore $h(a)$ exists $\forall a \in A$ and h is onto.

6.3) f, g are one to one and onto. In parts 1, and 2 we proved if f, g are both one-to-one/onto, then h is one-to-one/onto. Therefore, h is one to one and onto and forms a bijection

(6.4) If h is one to one, that doesn't guarantee f and g are one-to-one.



This disproves f, g are one to one. As while f is one to one, g is not. We can make the map G as we are not told if h or g are onto, meaning there is no guarantee that a $\forall c \in C$, there is an $h(a)=c$, or a $g(b)=c$