Tuesday, October 19

1. Suppose F is a differentiable function from $[0,1]$ into $[0,1]$, $x_* = F(x_*)$, and $|F'(x_*)| < 1$. We saw in class: If $x^{(0)}$ is chosen close enough to x_*, then the results of fixed point iteration, $x^{(1)}$, $x^{(2)}$, etc., will converge to x_*. Explain: If $x^{(k)} \neq x_*$ for all k, but $\lim_{k\to\infty} x^{(k)} = x_*$, then
$$\lim_{k\to\infty} \frac{x_* - x^{(k+1)}}{x_* - x^{(k)}} = F'(x_*).$$
Explain why this means that the size of $|F'(x_*))|$ determines how fast the $x^{(k)}$ approach the fixed point x_*.

2. Nonlinear equations, or systems of equations, in science and engineering are very often solved by a method called *Newton's method*, or variations on it. We'll study it in the simplest case here. Suppose that $f = f(x)$ is a differentiable function, and you would like to solve
$$f(x) = 0.$$
Make a guess, call it $x^{(0)}$. It's probably not going to be a solution. But compute now the x-intercept of the *tangent* to the graph of f at the point $(x^{(0)}, f(x^{(0)}))$, and call that $x^{(1)}$. Then obtain $x^{(2)}$ from $x^{(1)}$ the same way, then $x^{(3)}$ from $x^{(2)}$, and so on.

   (a) Draw a picture illustrating how $x^{(k+1)}$ is computed from $x^{(k)}$.

   (b) Derive a formula for $x^{(k+1)}$, given $x^{(k)}$. This formula will have the form
   $$x^{(k+1)} = F(x^{(k)}).$$
   (F is not f.)

   (c) Show that $f(x) = 0$ if and only if $x = F(x)$.

   (d) Show: If $f(x) = 0$ and $f'(x) \neq 0$, then $F'(x) = 0$.

   (e) In light of problem 1, do you expect the convergence of Newton's method to be fast or slow? Why?

   (f) Use Newton's method to solve $x^2 - 2 = 0$, starting with $x^{(0)} = 1$. What are $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(4)}$?

   (g) Watch this to see the most amazing things about Newton's method.