MATH235 HOMEWORK 6 SOLUTION

- 4.5.17. Show that if $f \in L^1(\mathbb{R})$, then its indefinite integral $F(x) = \int_0^x f(t)dt$ is uniformly continuous on $\mathbb{R}$.

  Proof. Given $f \in L^1(\mathbb{R})$, there exists a simple function $\phi = \sum_{k=1}^N c_k \chi_{E_k}$ such that $||f - \phi||_{L^1} < \epsilon/3$. Consider $M = \max\{c_1, \cdots, c_k\}$. Choose δ such that for any $|x - y| < \delta < \epsilon/(3M), \forall \epsilon > 0$, then we have

$$|F(x) - F(y)| = |\int_0^x f(t)dt - \int_0^y f(t)dt|$$

$$\leq |\int_0^x f(t)dt - \int_0^x \phi(t)dt| + |\int_0^x \phi(t)dt - \int_0^y \phi(t)dt| + |\int_0^y f(t)dt - \int_0^y \phi(t)dt|$$

$$\leq \frac{3}{\epsilon} + M\delta + \frac{3}{\epsilon} = \epsilon$$

  which gives uniform continuous. □

- 4.5.22 Show that the conclusion of the Dominated Convergence Theorem continues to hold if we replace the hypothesis $f_n \to f$ a.e. with $f_n \overset{\text{m}}{\to} f$.

  Proof. Choose an arbitrary subsequence $\{f_{n_k}\}$ of $\{f_n\}$. We know both $\{f_{n_k}\}$ and $\{f_n\}$ converge to f in measure. Then $\{f_{n_k}\}$ has a subsequence $\{f_{n'_k}\}$ that converges to f a.e. By dominated convergence theorem $||f - f_{n'_k}||_{L^1} \to 0$. By uniqueness of convergence (Exercise 1.1.22) we must have $||f - f_n||_{L^1} \to 0$. □

- 4.5.26. Let E be a measurable subset of $\mathbb{R}^d$ such that $|E| < \infty$. Prove that $\lim_{h \to 0} |E \cap (E + h)| = |E|$.

  Proof. $|E| < \infty$ implies that $\int_{\mathbb{R}} \chi_E < \infty$, $\chi_E \in L^1$. By Steinhaus Theorem, if $|E| < \infty$, then the function

$$f(h) = |E \cap (E + h)|$$

  is continuous on $\mathbb{R}$. For this question, our goal is to show that f is continuous at $h = 0$. Choose $\{h_n\}$ to be a sequence such that

$$\lim_{n \to \infty} h_n = 0.$$

  We aim to show that

$$\lim_{n \to \infty} f(h_n) = f(0).$$

  Note that

$$f(h_n) = \int_{\mathbb{R}} \chi_{E \cap (E + h_n)}.$$

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Combine Dominated Convergence Theorem, we obtain

$$\begin{aligned}\lim_{n\to\infty} f(h_n) &= \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{E\cap(E+h)} \\ &= \int_{\mathbb{R}} \lim_{n\to\infty} \chi_{E\cap(E+h)} \\ &= \int_{\mathbb{R}} \chi_E = f(0).\end{aligned}$$

Therefore, function $(h) = |E \cap (E+h)|$ is continuous on at $h = 0$.
Hence, we have proved that $\lim_{h\to 0} |E \cap (E+h)| = |E|$.

□

- 4.5.27. This problem will establish a Generalized Dominated Convergence Theorem. Let E be a measurable subset of $\mathbb{R}^d$. Assume that:
  (a) $f_n, g_n, f, g \in L^1(E)$,
  (b) $f_n \to f$ pointwise a.e.,
  (c) $g_n \to g$ pointwise a.e.,
  (d) $|f_n| \le g_n$ a.e., and
  (e) $\int_E g_n \to \int g$.
  Prove that $\int_E f_n \to \int_E f$ and $\|f - f_n\|_1 \to 0$

  Proof. Notice that $-g_n \le f_n \le g_n$ gives $f_n + g_n \ge 0$ a.e. We have

  $$\int f + g \le \lim_{n\to\infty} \int f_n + g_n \le \lim_{n\to\infty} \int f_n + \int g$$

  which gives

  $$\int f \le \lim_{n\to\infty} \int f_n$$

  Where we applied Fatou's lemma on $f_n + g_n$ and dominated convergence theorem on g_n. Similarly, consider $g_n - f_n \ge 0$ a.e., we have

  $$\int g - f \le \lim_{n\to\infty} \int f_n - g_n \le \int g - \lim_{n\to\infty} \int f_n$$

  which gives

  $$\int f \ge \lim_{n\to\infty} \int f_n$$

  Hence we conclude $\int f_n \to \int f$ and $\|f - f_n\|_1 \to 0$ □

- 4.6.12. Let $Q = [0,1]^2$, and let $Q_1, Q_2, \ldots$ be an infinite sequence of nonoverlapping squares centered on the main diagonal of Q, as shown in Figure 4.6. Subdivide each square Q_n into four equal subsquares, and let $f = 1/|Q_n|$ on the lower left and upper right subsquares of Q_n, and $f = -1/|Q_n|$ on the lower right and upper left subsquares. Set $f = 0$ everywhere else. Prove that

  $$\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = \int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = 0,$$

  but $\iint_Q |f(x,y)|(dxdy) = \infty$. Use this to show that $\iint_Q f(x,y)(dxdy)$, the Lebesgue integral of f on Q, is undefined.

Proof. a. $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = \int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = 0.$

Focus on $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy$ firstly. Fix y, we do not need to concern about $Q_1, Q_2, \ldots, Q_{n-1}$ since the integral with respect to x over there is 0 by definition of f. As for Q_n, the negative part is equal to the positive part because each square is subdivided into four equal subsquares which means the symmetry. So the integral with respect to x over the part related to Q_n is also 0. Thus, we have

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = \int_0^1 0dy = 0.$$

Similarly, we can obtain that

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = 0.$$

Therefore, we have proved that $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = \int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = 0.$

b. $\iint_Q |f(x,y)|(dxdy) = \infty$. Since $Q_1, Q_2, \ldots$ is an infinite sequence of nonoverlapping squares, we attain that

$$\begin{aligned}\iint_Q |f(x,y)|(dxdy) &= 0 + \sum_{n=1}^{\infty} \iint_{Q_n} \frac{1}{|Q_n|}(dxdy) \\ &= \sum_{n=1}^{\infty} \frac{|Q_n|}{|Q_n|} = \sum_{n=1}^{\infty} 1 = \infty.\end{aligned}$$

c. $\iint_Q f(x,y)(dxdy)$ undefined. According to b, we have for the positive part

$$\begin{aligned}\iint_Q f^+(x,y)(dxdy) &= 0 + \sum_{n=1}^{\infty} \iint_{Q_n} \frac{1}{|Q_n|}(dxdy) \\ &= \sum_{n=1}^{\infty} \frac{|Q_n|}{2|Q_n|} = \sum_{n=1}^{\infty} \frac{1}{2} = \infty,\end{aligned}$$

and for the negative part

$$\begin{aligned}\iint_Q f^-(x,y)(dxdy) &= 0 + \sum_{n=1}^{\infty} \iint_{Q_n} -\frac{1}{|Q_n|}(dxdy) \\ &= \sum_{n=1}^{\infty} -\frac{|Q_n|}{2|Q_n|} = \sum_{n=1}^{\infty} -\frac{1}{2} = -\infty,\end{aligned}$$

Based on all the claims above, we conclude that

$$\begin{aligned}\iint_Q f(x,y)(dxdy) &= \iint_Q f^+(x,y)(dxdy) + \iint_Q f^-(x,y)(dxdy) \\ &= \infty - \infty,\end{aligned}$$

which is undefined. ☐

- 4.6.20. Given $f \in L^1[0,1]$, define

$$g(x) = \int_x^1 \frac{f(t)}{t} dt, \quad 0 < x \leq 1.$$

Show that g is defined a.e. on $[0,1]$, $g \in L^1[0,1]$, and $\int_0^1 g(x)dx = \int_0^1 f(x)dx$.

Proof. Consider

$$g(x) = \int_0^1 \chi_{(x,1)} \frac{f(t)}{t} dt = \int_0^1 F(x,t)dt$$

where $F(x,t)$ is measurable on $[0,1] \times [0,1]$. Then we have,

$$\int_0^1 \int_0^1 F(x,t)dxdt = \int_0^1 (\int_0^1 \chi_{(x,1)} \frac{|f(t)|}{t} dx)dt = \int_0^1 \frac{|f(t)|}{t} (\int_0^1 \chi_{(x,1)} dx)dt = \int_0^1 \frac{|f(t)|}{t} \cdot tdt < \infty$$

Hence we know $g \in L^1[0,1]$. Also, by Tonelli's theorem,

$$\int_0^1 g(x)dx \int_0^1 \int_x^1 \frac{f(t)}{t} dtdx = \int_0^1 \int_0^t \frac{f(t)}{t} dxdt = \int_0^1 f(t)dt$$

Hence we conclude

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx$$

$\square$

- 4.4.22. Suppose that $f \in L^1[a,b]$ satisfies $\int_a^x f(t)dt = 0$ for all $x \in [a,b]$. Prove that $f = 0$ a.e.

  Proof. For any closed interval $[c,d] \subseteq [a,b]$, we have

  $$\int_{[c,d]} f = \int_{[a,d]} f - \int_{[a,c]} f = 0.$$

  Since any open set $U \subseteq [a,b]$ can be written as a union of disjoint open intervals, say $U = \cup [c_n, d_n]$, then

  $$\int_U f = \int_{\cup [c_n,d_n]} f = \sum_i^n \int_{[c_i,d_i]} f = 0.$$

  It is then easy to see that for any closed $K \subseteq [a,b]$, we have

  $$\int_K f = \int_{[a,b]} f - \int_{[a,b]\setminus K} f dm = 0.$$

  as the complement of a closed set is an open set. Now consider any F_σ set $F = \cup F_n = \bigcup_{k=1}^n K_k$ for some closed sets K_k. Then $\{F_n\}$ is a nesting increasing sequence of closed sets with limit F. Since $f \in L^1[a,b]$, by Lebesgue dominated convergence theorem,

  $$\int_F f = \lim_{n\to\infty} \int_{F_n} f = 0.$$

  Recall that any Lebeasgue measurable set $E \subseteq [a,b]$ can be written as $E = F \cup Z$, for a F_σ set F and Z a set of measure 0. Hence

  $$\int_E f = \int_F f = 0.$$

  Thus we conclude that $f = 0$ a.e. on $[a,b]$. □

- 4.5.32

  Proof. Since $\frac{\partial f}{\partial x} f(x,y)$ exists and bounded, we use $M = \sup_{x,y} |\frac{\partial f}{\partial x} f(x,y)|$. For fixed x, consider $f_n(x,y) = \frac{f(x+1/n,y)-f(x,y)}{1/n}$. Since $f(x,y)$ is measurable as a function of y for fixed x, it follows that f_n is also measurable as a function of y. Since $\frac{\partial f}{\partial x} f(x,y)$ exists, $\lim_{n\to\infty} f_n(x,y) = \frac{\partial f}{\partial x} f(x,y)$ is also measurable.

  For the second part of the claim, since $\frac{\partial f}{\partial x} f(x,y)$ bounded, we know $\int_0^1 \frac{\partial f}{\partial x} f(x,y)dy$ is well defined. Now consider $F(x) = \int_0^1 f(x,y)dy$ and the limit expression $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$. Notice that this limit is exactly $\frac{d}{dx}\int_0^1 f(x,y)dy$. By bounded convergence theorem, we can swap limit and integral, which gives the desired result. □