Solutions

Recitation 2: Bigger cartesian products ~~and fibered products~~ — Math 171, Spring 2023

Recall:

Definition 1. An **equivalence relation** $\sim$ on a set X is a relation $\sim$ on X such that

(1) (reflexivity) For all $x \in X$, $x \sim x$.
(2) (symmetry) For all $x, y \in X$, if $x \sim y$ then $y \sim x$.
(3) (transitivity) For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

(1) Let $\sim$ be the relation on $\mathbb{R}^2$ given by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $y_1 - x_1^2 = y_2 - x_2^2$.

(a) Prove that $\sim$ is an equivalence relation.

reflexivity: $y - x^2 = y - x^2$ for all $(x,y) \in \mathbb{R}^2$

$\Rightarrow (x,y) \sim (x,y)$ for all $(x,y) \in \mathbb{R}^2$

symmetry: suppose $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$ and $(x_1,y_1) \sim (x_2,y_2)$

Then $y_1 - x_1^2 = y_2 - x_2^2$

$\Rightarrow y_2 - x_2^2 = y_1 - x_1^2$

$\Rightarrow (x_1,y_1) \sim (x_2,y_2)$

transitivity: s'pose $(x_1,y_1), (x_2,y_2), (x_3,y_3) \in \mathbb{R}^2$ and $(x_1,y_1) \sim (x_2,y_2)$ and $(x_2,y_2) \sim (x_3,y_3)$.

$\Rightarrow y_1 - x_1^2 = y_2 - x_2^2$ and $y_2 - x_2^2 = y_3 - x_3^2$

$\Rightarrow y_1 - x_1^2 = y_3 - x_3^2$

$\Rightarrow (x_1,y_1) \sim (x_3,y_3)$

(b) What are the equivalence classes of $\sim$? Sketch and label the equivalence classes $[(0,0)]$, $[(0,1)]$, and $[(0,2)]$ in $\mathbb{R}^2$.

$$[(x_0,y_0)] = \{(x,y) \in \mathbb{R}^2 \mid y_0 - x_0^2 = y - x^2\}$$

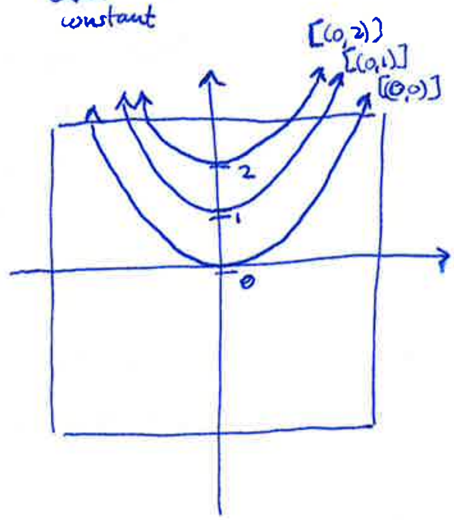
$$= \{(x,y) \in \mathbb{R}^2 \mid y = x^2 + \underbrace{(y_0 - x_0^2)}_{\text{constant}}\}$$

↑ parabolas!

$[(0,0)] = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 = 0\}$

$[(0,1)] = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 = 1\}$

$[(0,2)] = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 = 2\}$



Recall:

Definition 2. An **order relation** on a set X is a relation $\prec$ on X such that
(a) (comparability) If $x, y \in X$ and $x \neq y$, then $x \prec y$ or $y \prec x$.
(b) (anti-reflexivity) For all $x \in X$, we have $x \not\prec x$.
(c) (transitivity) For all $x, y, z \in X$, if $x \prec y$ and $y \prec z$, then $x \prec z$.

Definition 3. If $(X, <_X)$ and $(Y, <_Y)$ are ordered sets, the **dictionary order** on $X \times Y$ is the order defined by

$$(x_1, y_1) < (x_2, y_2) \iff x_1 <_X x_2, \text{ or } x_1 = x_2 \text{ and } y_1 <_Y y_2.$$

(2) Prove that the dictionary order is an order relation.

comparability: suppose $(x_1,y_1) \in \mathbb{R}^2$, $(x_2,y_2) \in \mathbb{R}^2$, $(x_1,y_1) \neq (x_2,y_2)$.
Then $x_1 \neq x_2$ or $y_1 \neq y_2$.

If $x_1 \neq x_2$, then $x_1 <_X x_2$ or $x_1 >_X x_2$, so $(x_1,y_1) < (x_2,y_2)$ or $(x_1,y_1) > (x_2,y_2)$, respectively.

If $y_1 \neq y_2$ and $x_1 = x_2$, then $y_1 <_Y y_2$ or $y_1 >_Y y_2$, so $(x_1,y_1) < (x_2,y_2)$ or $(x_1,y_1) > (x_2,y_2)$, respectively.

anti-reflexivity: if $x_1 = x_2$ and $y_1 = y_2$, then we do not have $x_1 <_X x_2$ or $x_1 >_X x_2$ and we do not have $y_1 <_Y y_2$ or $y_1 >_Y y_2$, so $(x_1,y_1) \not< (x_1,y_1)$.

transitivity: S'pose $(x_1,y_1), (x_2,y_2), (x_3,y_3) \in \mathbb{R}^2$, $(x_1,y_1) < (x_2,y_2)$ and $(x_2,y_2) < (x_3,y_3)$.
Then $x_1 <_X x_2$, or $x_1 = x_2$ and $y_1 <_Y y_2$
and $x_2 <_X x_3$, or $x_2 = x_3$ and $y_1 <_Y y_2$.

If $x_1 <_X x_2$ or $x_2 <_X x_3$, we have $x_1 <_X x_3$.
(3 cases: $x_1 <_X x_2$ and $x_2 <_X x_3 \Rightarrow x_1 <_X x_3$
$x_1 <_X x_2$ and $x_2 = x_3 \Rightarrow x_1 <_X x_3$
$x_1 = x_2$ and $x_2 <_X x_3 \Rightarrow x_1 <_X x_3$)
So $(x_1,y_1) < (x_3,y_3)$.

If $x_1 = x_2$ and $x_2 = x_3$, we must have $x_1 = x_3$, $y_1 <_Y y_2$ and $y_2 <_Y y_3$, so $(x_1,y_1) < (x_3,y_3)$.

Recall that the **cartesian product** of two sets X and Y is the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

To define products of more sets, we need to talk about tuples of more elements than two. We can do this using functions:

Definition 4. Let m be a positive integer. Given a set X, we define an **m-tuple** of elements of X to be a function

$$\mathbf{x} : \{1, \dots, m\} \to X.$$

Given an m-tuple $\mathbf{x}$, we often write x_i rather than $\mathbf{x}(i)$ and call it the ith **coordinate** of $\mathbf{x}$. We often denote the function $\mathbf{x}$ itself by the symbol

$$(x_1, \dots, x_m).$$

Definition 5. Given sets $A_1, \dots, A_m$, the **cartesian product** $A_1 \times \cdots \times A_m$ is the set of m-tuples

$$A_1 \times \cdots \times A_m = \{(x_1, \dots, x_m) \mid x_i \in A_i \text{ for each } i\}.$$

(We take $X = A_1 \cup \cdots \cup A_m$ so the definition of m-tuple makes sense here.)

These definitions extend easily to arbitrary products of sets.

Definition 6. Let I be a set. An I-tuple of elements of a set X is a function

$$\mathbf{x} : I \to X.$$

We write x_i rather than $\mathbf{x}(i)$ and call it the ith **coordinate** of $\mathbf{x}$. We often denote $\mathbf{x}$ itself by $(x_i)_{i \in I}$.

Given sets $\{A_i\}_{i \in I}$ indexed by a set I, the **cartesian product** $\prod_{i \in I} A_i$ is the set of I-tuples

$$\{(x_i)_{i \in I} \mid x_i \in A_i \text{ for each } i \in I\}.$$

(We take $X = \bigcup_{i \in I} A_i$.)

(3) Let $A_1 = \{1\}$, $A_2 = \{2\}$, $B_1 = \{3\}$, and $B_2 = \{4\}$. Compute $(A_1 \times A_2) \cup (B_1 \times B_2)$ and $(A_1 \cup B_1) \times (A_2 \cup B_2)$. How do the sets compare?

$$(A_1 \times A_2) \cup (B_1 \times B_2) = \{(1,2)\} \cup \{(3,4)\} = \{(1,2), (3,4)\}$$

$$(A_1 \cup B_1) \times (A_2 \cup B_2) = \{1, 3\} \times \{2, 4\} = \{(1,2), (1,4), (3,2), (3,4)\}.$$

We have $A_1 \times A_2 \cup B_1 \times B_2 \subsetneq (A_1 \cup B_1) \times (A_2 \cup B_2)$.

(4) Let $A_1 = \{1,2\}$, $A_2 = \{-1,-2\}$, and $A_3 = \{\pi, 2\pi\}$. Write out the elements of $A_1 \times A_2 \times A_3$, $A_1 \times (A_2 \times A_3)$, and $A_1 \times (A_3 \times A_2)$. Are these sets the same or different?

$$A_1 \times A_2 \times A_3 = \{(1,-1,\pi), (1,-1,2\pi), (1,-2,\pi), (1,-2,2\pi), (2,-1,\pi), (2,-1,2\pi), (2,-2,\pi), (2,-2,2\pi)\}$$

$$A_1 \times (A_2 \times A_3) = \{1,2\} \times \{(-1,\pi), (-1,2\pi), (-2,\pi), (-2,2\pi)\}$$
$$= \{(1,(-1,\pi)), (1,(-1,2\pi)), (1,(-2,\pi)), (1,(-2,2\pi)), (2,(-1,\pi)), (2,(-1,2\pi)), (2,(-2,\pi)), (2,(-2,2\pi))\}.$$

$$A_1 \times (A_3 \times A_2) = \{1,2\} \times \{(\pi,-1), (\pi,-2), (2\pi,-1), (2\pi,-2)\}$$
$$= \{(1,(\pi,-1)), (1,(\pi,-2)), (1,(2\pi,-1)), (1,(2\pi,-2)), (2,(\pi,-1)), (2,(\pi,-2)), (2,(2\pi,-1)), (2,(2\pi,-2))\}$$

They are all different sets.

The answer to the last problem should feel annoying. Let's work our way towards another perspective on what a cartesian product is.

(5) Let $\pi_1 : \mathbb{Z} \times \mathbb{R} \to \mathbb{Z}$ and $\pi_2 : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be the functions given by

$$\pi_1(x,y) = x \quad \text{and} \quad \pi_2(x,y) = y.$$

(a) Let $f : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R}$ be the function defined as in the following table. Complete the rest of the table.

a	$f(a)$	$(\pi_1 \circ f)(a)$	$(\pi_2 \circ f)(a)$
1	$(3,4)$	3	4
2	$(1,\pi)$	1	π
3	$(-1,2\pi)$	-1	2π

(b) There is a function $g : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R}$, some facts about which are recorded in the following table. Complete the rest of the table.

a	$g(a)$	$(\pi_1 \circ g)(a)$	$(\pi_2 \circ f)(a)$
1	$(2,\sqrt{2})$	2	$\sqrt{2}$
2	$(25,\sqrt{3})$	25	$\sqrt{3}$
3	$(-125,4)$	-125	4

You should see that a function $f : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R}$ is "the same" as a pair of functions $(f_1 : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R}, f_2 : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R})$. That is, you can find such f_1 and f_2 from f and you can construct f from f_1 and f_2.

We state this property in general as follows.

Theorem 7 (The Universal Property of the Cartesian Product). *Let X, Y be sets and let $P = X \times Y$. Write $\pi_1 : P \to X$ for the function $(x,y) \mapsto x$ and $\pi_2 : P \to Y$ for the function $(x,y) \mapsto y$. (These are called the* ***projection maps****.) Then for any set A and pair of functions $f_1 : A \to X$ and $f_2 : A \to Y$ there exists a unique function $f : A \to P$ so that the diagram*

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

commutes, i.e., $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

(6) Prove the theorem. (Hint: Think about your answers to (5).)

Suppose $f_1 : A \to X$ and $f_2 : A \to Y$ are functions.

Suppose $f : A \to P$ is a function s.t. $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$.

Let $a \in A$ be an arbitrary element. Write $(b_1, b_2) = f(a)$.

Then $\pi_1(f(a)) = \pi_1((b_1, b_2)) = b_1$. On the other hand, $\pi_1(f(a)) = f_1(a)$, so $b_1 = f_1(a)$. Similarly, $b_2 = f_2(a)$.

We conclude f is the function $f(a) = (f_1(a), f_2(a))$.

That is, if there is such a function, it is uniquely determined.

Conversely, if $f(a) = (f_1(a), f_2(a))$ for all $a \in A$, then

$$(\pi_1 \circ f)(a) = \pi_1(f_1(a), f_2(a)) = f_1(a) \quad \text{and} \quad (\pi_2 \circ f)(a) = \pi_2(f_1(a), f_2(a)) = f_2(a)$$

So this function makes the diagram commute.

Altogether, there exists a unique $f : A \to P$ making the diagram commute, as desired. $\square$

(7) Let X, Y be sets. Suppose that P' is a set and $\pi_1' : P' \to X$ and $\pi_2' : P' \to Y$ are functions that also have the universal property of the product, that is, for any set A and pair of functions $f_1 : A \to X$ and $f_2 : A \to Y$ there exists a unique function $f : A \to P'$ so that the diagram

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X & P' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & Y \end{tikzcd}$$

commutes, i.e., $f_1 = \pi_1' \circ f$ and $f_2 = \pi_2' \circ f$.

Prove that there is a bijection $P \to P'$. (Hint: Try $A = P$ and $A = P'$.)

Plugging in $A = P$, there is a unique map $f : P \to P'$ s.t.

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[d, "f"] \arrow[dr, "\pi_2"] & \\ X & P' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & Y \end{tikzcd}$$

commutes. Plugging in $A = P'$, there is a unique map $g : P' \to P$ s.t.

$$\begin{tikzcd} & P' \arrow[dl, "\pi_1'"'] \arrow[d, "g"] \arrow[dr, "\pi_2'"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

The composites give maps so that

$$\begin{tikzcd} & P' \arrow[dl, "\pi_1'"'] \arrow[d, "f \circ g"] \arrow[dr, "\pi_2'"] & \\ X & P' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[d, "g \circ f"] \arrow[dr, "\pi_2"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

commute. But $id_{P'}$, id_P also make the respective diagrams commute. By the universal property of products' uniqueness, $f \circ g = id_{P'}$ and $g \circ f = id_P$. We conclude $f : P \to P'$ is invertible, hence bijective.

$\square$

(8) Let $\{X_i\}_{i\in I}$ be an arbitrary collection of sets. Let $P = \prod_{i\in I} A_i$ be the cartesian product. For each $i \in I$, let $\pi_i : P \to X_i$ be the projection map $(x_j)_{j\in I} \mapsto x_i$ taking tuples to their ith coordinate. Show that P has the following "Universal property of the product:"

Given a set A and functions $f_i : A \to X_i$ for each $i \in I$, there exists a unique function $f : A \to P$ so that the diagram

$$\begin{tikzcd} A \arrow[d, "f"] \arrow[dr, "f_i"] & \\ P \arrow[r, "\pi_i"'] & X_i \end{tikzcd}$$

commutes for all i.

Suppose A is a set and $f_i : A \to X_i$ is a function for each $i \in I$. Suppose $f : A \to P$ is a function so that $\pi_i \circ f = f_i$ for all $i \in I$. Let $a \in A$. Then, writing $f(a) = (b_i)_{i\in I}$, we have

$$b_i = \pi_i(f(a)) = f_i(a)$$

for all i, so $f(a) = (f_i(a))_{i\in I}$. It follows that any function $f : A \to P$ making the diagrams commute is of the form $f(a) = (f_i(a))_{i\in I}$ for all $a \in A$.

Conversely suppose $f : A \to P$ is the function $f(a) = (f_i(a))_{i\in I}$. Then

$$f_i(a) = \pi_i((f_i(a))_{i\in I}) = \pi_i(f(a))$$

for all $a \in A$, so f makes the diagrams commute.

Therefore there is a unique function $f : A \to P$ making the diagrams commute, as desired. $\square$

(9) We have seen in problem (5) that $A_1 \times (A_2 \times A_3)$ is not quite the same set as $A_1 \times A_2 \times A_3$. However, the two sets are related by an easy-to-guess bijective function. See if you can find it and check that it is a bijection.

One perspective on where this function comes from is that both $A_1 \times (A_2 \times A_3)$ and $A_1 \times A_2 \times A_3$ have the universal property of the product of A_1, A_2, and A_3. Reasoning as in problem 7, there is a unique isomorphism between them that respects the projection maps. If you like, you can try to argue this way.

The function
$$f: A_1 \times (A_2 \times A_3) \longrightarrow A_1 \times A_2 \times A_3$$
$$(a_1, (a_2, a_3)) \longmapsto (a_1, a_2, a_3)$$
is the required bijection. It is injective since
$$(a_1, (a_2, a_3)) = (b_1, (b_2, b_3)) \iff a_1 = b_1,\ a_2 = b_2, \text{ and } a_3 = b_3$$
$$\iff (a_1, a_2, a_3) = (b_1, b_2, b_3).$$

It is surjective since, given $(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$, a pre-image is $(a_1, (a_2, a_3))$.

The way in which $A_1 \times (A_2 \times A_3)$ has the universal property of the product of A_1, A_2, A_3 is a "diagram chase":

1. Start with the ~~black ink~~ pencil:

$$\begin{tikzcd}
& A \arrow[ddl, bend right, "f_1"'] \arrow[d, dashed, "f"] \arrow[ddrr, dashed, bend left, "f_{23}"] \arrow[dddr, bend left, "f_2"] \arrow[dddrrr, bend left, "f_3"] & & & \\
& A_1 \times (A_2 \times A_3) \arrow[dl, "\pi_1"] \arrow[drr, "\pi_{23}"] & & & \\
A_1 & & & A_2 \times A_3 \arrow[dl, "\pi_2"'] \arrow[dr] & \\
& & A_2 & & A_3
\end{tikzcd}$$

2. Since $A_2 \times A_3$ has the univ. prop. for the product of A_2 and A_3, $\exists$ a unique $f_{23}: A \to A_2 \times A_3$

3. Since $A_1 \times (A_2 \times A_3)$ has the univ. prop. of the product for A_1 and $A_2 \times A_3$, and we have functions f_1 and f_{23}, we get a unique function $f: A \to A_1 \times (A_2 \times A_3)$

For how to imitate 7 to see $A_1 \times (A_2 \times A_3) \simeq A_1 \times A_2 \times A_3$, ask me in office hours.