Math 125 Hw4

1) Let $f, g, h \in C[0,1]$ Broke into 4 parts

a) $\langle f+g, h\rangle = \langle f,h\rangle + \langle g,h\rangle$

$$\int_0^1 (f(x)+g(x))h(x)\,dx = \int_0^1 f(x)h(x) + f(x)g(x)\,dx$$

$$= \int_0^1 f(x)h(x)\,dx + \int_0^1 h(x)g(x)\,dx = \langle f,h\rangle + \langle g,h\rangle$$

b) $\langle cf, g\rangle = c\langle f, g\rangle$

$$\int_0^1 c\,f(x)g(x)\,dx = c\int_0^1 f(x)g(x)\,dx = c\langle f,g\rangle \quad \square$$

c) $\langle f,g\rangle = \langle g,f\rangle$

$$\int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx \quad \square$$

d) $\langle f,f\rangle \geq 0$ and $=0$ iff $f=0$

$$\int_0^1 f(x)f(x)\,dx = \int_0^1 [f(x)]^2\,dx \geq 0 \text{ as } (f(x))^2 \geq 0$$

$\langle f,f\rangle = 0 \rightarrow f=0$

$\Rightarrow$ $\int_0^1 [f(x)]^2\,dx = 0$, but $[f(x)]^2 \geq 0$ and doesn't change signs, so only 0 when $f=0$

$\Leftarrow$ If $f=0$ $\int_0^1 0\,dx = 0$ so $\langle f,f\rangle = 0$ $\square$

2a) Symmetric
$$G^T = \begin{bmatrix} \langle f_1,f_1\rangle & \langle f_2,f_1\rangle & \langle f_3,f_1\rangle \\ \langle f_1,f_2\rangle & \langle f_2,f_2\rangle & \langle f_3,f_2\rangle \\ \langle f_1,f_3\rangle & \langle f_2,f_3\rangle & \langle f_3,f_3\rangle \end{bmatrix} = G$$

By dot product rules

Positive definite means $z^T G z > 0$

$$z^T G z = \sum_{i,j} G_{ij} z_{ij} = \sum_{i,j} \langle f_i, f_j\rangle z_i z_j > 0$$

$$\langle f_1,f_1\rangle z_1^2 + \langle f_2,f_1\rangle z_2 z_1 + \langle f_3,f_1\rangle z_3 z_1 + \dots > 0$$

$$= \langle f_1,f_1\rangle z_1^2 + \langle f_1,f_2\rangle (z_1 z_2)^2 + \langle f_1,f_3\rangle (z_1 z_3)^2 + \dots$$ repeat until all 9 values

So becomes/written as $\langle z_1 f_1 + z_2 f_2 + z_3 f_3, z_1 f_1 + z_2 f_2 + z_3 f_3 \rangle$
- Done by rearranging full expansion. As for any f, $\langle f, f \rangle \geq 0$, and $z_1 f_1 + z_2 f_2 + z_3 f_3 \neq 0$ as z is nonzero and $\{f_1, f_2, f_3\}$ form a basis.
There fore G is positive definite.

b) $G\vec{c} = y$

$$\begin{bmatrix} \langle f_1, f_1 \rangle & \langle f_1, f_2 \rangle & \langle f_1, f_3 \rangle \\ \langle f_2, f_1 \rangle & \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle \\ \langle f_3, f_1 \rangle & \langle f_2, f_3 \rangle & \langle f_3, f_3 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \langle g, f_1 \rangle \\ \langle g, f_2 \rangle \\ \langle g, f_3 \rangle \end{bmatrix}$$

WLOG, let's show row 1.

$$G\vec{c} = \begin{bmatrix} c_1 \langle f_1, f_1 \rangle + c_2 \langle f_2, f_1 \rangle + c_3 \langle f_1, f_3 \rangle \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle g, f_1 \rangle \\ \vdots \\ \vdots \end{bmatrix}$$

$$\langle g, f_1 \rangle = \langle c_1 f_1 + c_2 f_2 + c_3 f_3, f_1 \rangle$$
$$= c_1 \langle f_1, f_1 \rangle + c_2 \langle f_2, f_1 \rangle + c_3 \langle f_3, f_1 \rangle$$

This is repeatable for rows 2 and 3, so $G\vec{c} = y$ is a solution
$\vec{c}$ is unique as by def. of positive definite all eigenvalues > 0, so G is invertible, and invertibility guarantees uniqueness so $\vec{c}$ is unique.

c) Want prove $h = \hat{h}$
so $h - \hat{h} \perp W$ and as $\{f_1, f_2, f_3\}$ are a basis:

$h - \hat{h} \perp f_1$ $\quad h - \hat{h} \perp f_2$ $\quad h - \hat{h} \perp f_3$

$\langle h - \hat{h}, f_1 \rangle = 0$ $\quad \langle h, f_1 \rangle = \langle \hat{h}, f_1 \rangle$

$\langle h - \hat{h}, f_2 \rangle = 0$ $\quad \langle h, f_2 \rangle = \langle \hat{h}, f_2 \rangle$

$\langle h - \hat{h}, f_3 \rangle = 0$ $\quad \langle h, f_3 \rangle = \langle \hat{h}, f_3 \rangle$.

as $\hat{h} \in W$ $\quad \hat{h} = c_1 f_1 + c_2 f_3 + c_3 f_3$

$\langle \hat{h}, f_1 \rangle = c_1 \langle f_1, f_1 \rangle + c_2 \langle f_1, f_2 \rangle + c_3 \langle f_1, f_3 \rangle$

$\langle \hat{h}, f_2 \rangle = c_1 \langle f_2, f_1 \rangle + c_2 \langle f_2, f_2 \rangle + c_3 \langle f_2, f_3 \rangle$

$\langle \hat{h}, f_3 \rangle = c_1 \langle f_3, f_1 \rangle + c_2 \langle f_2, f_3 \rangle + c_3 \langle f_3, f_3 \rangle$

2 (con) That equation becomes:

$$\begin{bmatrix} \langle f_1, f_1 \rangle & \langle f_1, f_2 \rangle & \langle f_1, f_3 \rangle \\ \langle f_2, f_1 \rangle & \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle \\ \langle f_3, f_1 \rangle & \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \langle \hat{h}, f_1 \rangle \\ \langle \hat{h}, f_2 \rangle \\ \langle \hat{h}, f_3 \rangle \end{bmatrix}$$

or $G\vec{c} = \begin{pmatrix} \langle \hat{h}, f_1 \rangle \\ \langle \hat{h}, f_2 \rangle \\ \langle \hat{h}, f_3 \rangle \end{pmatrix}$ So $\hat{h}$ is solution to this system

3 a/b Check code for specifics

- Linear: $y = a + bx$ Quadratic: $y = ax^2 + bx + c$

4 a) For least squares, want to minimize $Ax = b$

So have/want orthogonal projection, $\hat{b}$, and $\hat{b} \in \text{range}(A)$

So $b - \hat{b} \perp a_1 \longrightarrow \langle b - \hat{b}, a_1 \rangle = 0 \quad \langle b, a_1 \rangle = \langle \hat{b}, a_1 \rangle$

$b - \hat{b} \perp a_2 \longrightarrow \langle b - \hat{b}, a_2 \rangle = 0 \quad \langle b, a_2 \rangle = \langle \hat{b}, a_2 \rangle$

$\vdots$

$b - \hat{b} \perp a_n \quad \langle b - \hat{b}, a_n \rangle = 0 \quad \langle b, a_n \rangle = \langle \hat{b}, a_n \rangle$

$\langle b, a_n \rangle \; \forall a$, as this is a system and $\langle a_n, b \rangle = \langle b, a_n \rangle$

$A^T \leftarrow \begin{pmatrix} -a_1^T- \\ \vdots \\ -a_n^T- \end{pmatrix} \cdot W^T W \cdot b$ as $\langle x, y \rangle = \langle x^T, W^T W y \rangle$

b is the same, across all elements.

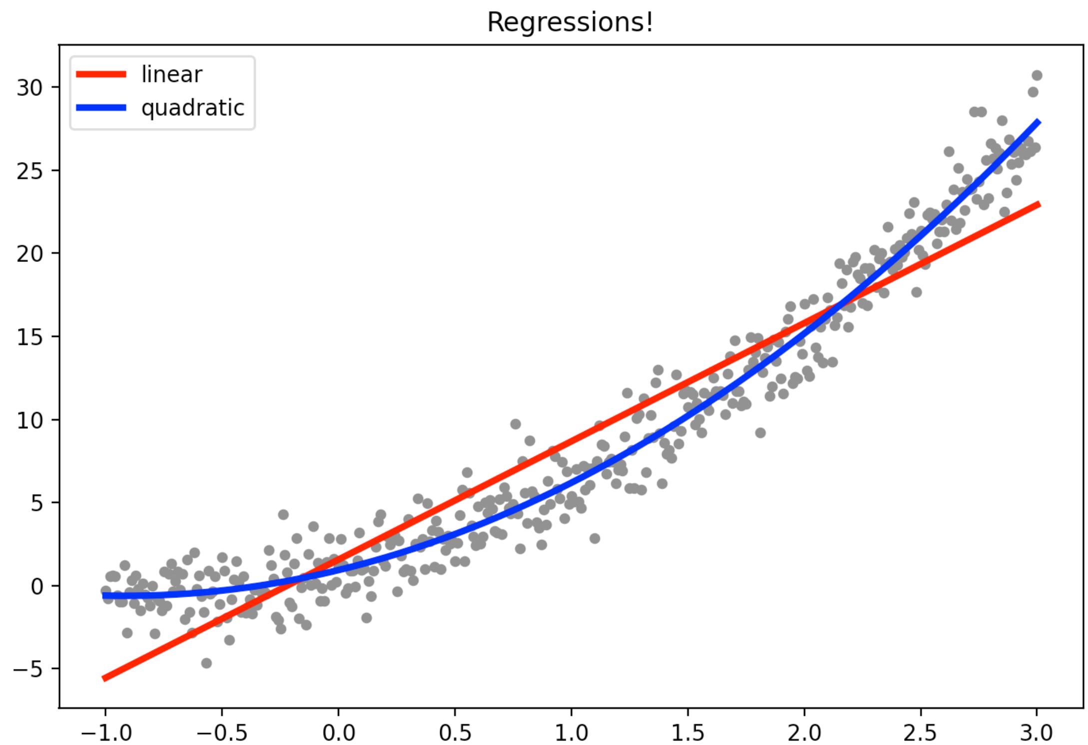
$\langle b \rangle = A^T W^T W b$

$\langle \hat{b}, a_n \rangle = \langle a_n, \hat{b} \rangle = A^T W^T W \hat{b}$ using similar logic

So $A^T W^T W b = A^T W^T W \hat{b}$ as $\hat{b} \in \text{range}(A)$

$\rightarrow A^T W^T W b = A^T W^T W A x \quad \square$

Question 3)
Here are regressions for part a and b:
Line of best fit is: y = 1.567 + 7.12x
Quadratic curve of best fit is: y = 0.934+3.395x+1.86x^2



Code)

```
import csv
import numpy as np
import matplotlib.pyplot as plt

x = np.zeros(401) #Setup
x2 = np.zeros(401)
y = np.zeros(401)
count = 0
fig, ax = plt.subplots()

#Reads in file
with open('least_squares_data.csv') as csv_file:
    csv_reader = csv.reader(csv_file, delimiter=',')
    for row in csv_reader:
        x[count] = float(row[0])
        x2[count] = float(row[0])**2
```

```
        y[count] = float(row[1])
        count += 1

#Linear regression
def linear(x, y):
    A1 = np.ones(401) #Sets up matrix equation
    supA = np.vstack((A1, x))
    A = np.transpose(supA)
    bigA = np.matmul(supA, A)
    YA = np.matmul(supA, y)
    return np.linalg.solve(bigA, YA) #Returns coefficients

#Quadratic regression
def quad(x, x2, y):
    A1 = np.ones(401) #Sets up matrix equation
    supA = np.vstack((A1, x, x2))
    A = np.transpose(supA)
    bigA = np.matmul(supA, A)
    YA = np.matmul(supA, y)
    return np.linalg.solve(bigA, YA) #Returns coefficients

line = linear(x, y)
quadr = quad(x, x2, y)

plt.title('Regressions!')
x_space = np.linspace(-1, 3, 200)
plt.scatter(x, y, color = 'gray', s = 15)
plt.plot(x_space, line[0] + line[1]*x_space, color = 'red',
linewidth=3, label = 'linear')
plt.plot(x_space, quadr[0] + x_space*quadr[1] + x_space**2*quadr[2],
color ='blue', linewidth=3, label='quadratic')
leg = ax.legend(loc = 'upper left')
plt.show()
```

$Ax=\lambda x$

4b) $A^TW^TWAx = \begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}^2\begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$= \begin{bmatrix}1&2&1\\0&2&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}Ax$

$= \begin{bmatrix}1&4&1\\0&4&2\end{bmatrix}\begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

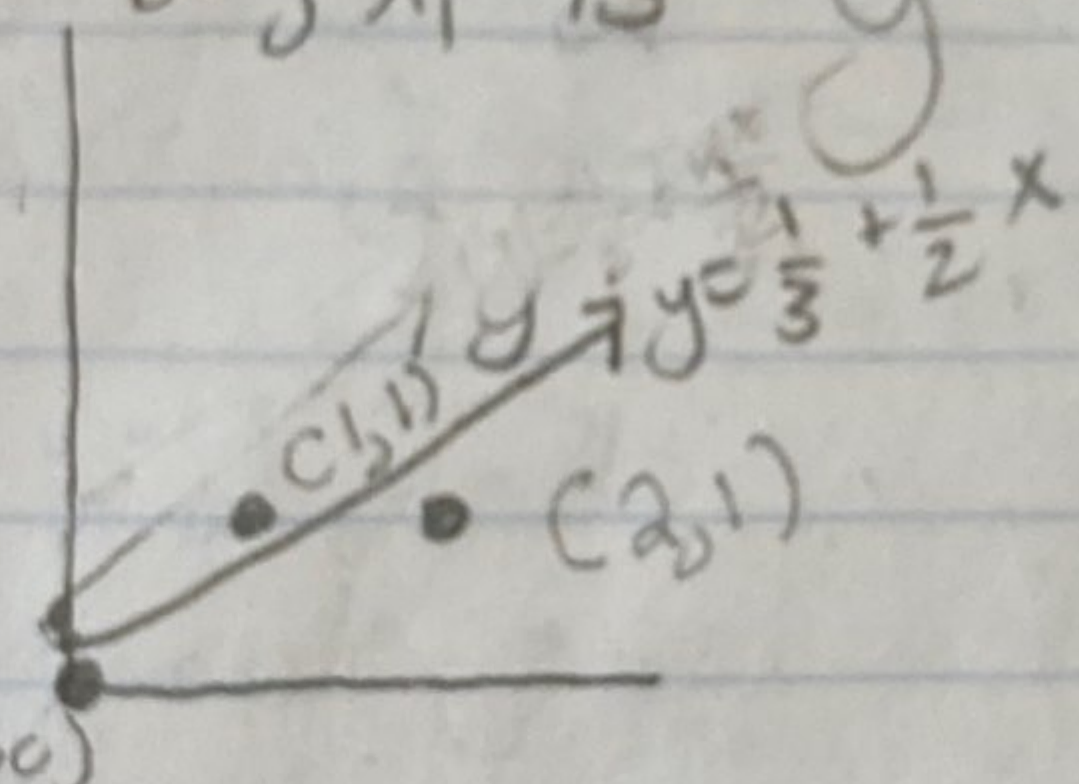
$= \begin{bmatrix}6&6\\6&8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = A^TW^TWb$

$= \begin{bmatrix}1&4&1\\0&4&2\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$

$\begin{bmatrix}6&6\\6&8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$

$\left[\begin{array}{cc:c}6&6&5\\6&8&6\end{array}\right] \xrightarrow{r_2-r_1} \left[\begin{array}{cc:c}6&6&5\\0&2&1\end{array}\right]$

$x_2 = \frac{1}{2},\ x_1 = \frac{1}{3}$ $\quad y = \frac{1}{3} + \frac{x}{2}$ $\quad 6x_1+3=5$, $6x_1=2$



c) Assume W is not invertible, therefore W^T isn't invertible.

$\langle x, y\rangle = \langle y, x\rangle$

$x^TW^TWy = (y^TW^TWx)^T = x^TW^TWy$

So $\langle x,y\rangle = \langle y,x\rangle$ iff $\langle x,y\rangle^T = \langle x,y\rangle$ which is true if $\langle x,y\rangle$ is diagonal or constant values. However, as W isn't invertible, W is not reducable to I, so this can't be guaranteed.

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5 a) $\int_{-1}^{1} \frac{\cos(i \arccos^{-1} x)\cos(J \arccos^{-1} x)}{\sqrt{1-x^2}}\,dx$ $w = \cos^{-1} x$
$dw = -\frac{1}{\sqrt{1-x^2}}$

$= -\int_{\pi}^{0} \cos(iw)\cos(Jw)\,dw = \int_{0}^{\pi} \cos(iw)\cos(iJ)\,du$ by prod of cosines law →

$\to \frac{1}{2}\int_{0}^{\pi} \cos(iw + uJ) + \cos(iw - uJ)\,du =$ assume $i \neq J$

$= \frac{1}{2}\left(\frac{1}{w+J}\sin(u(i+J)) + \frac{1}{w-J}\sin(u(i-J))\right)\Big|_0^{\pi}$ $\sin 0 = 0$
$\sin k\pi$ where $k \in \mathbb{Z}$
and $w-J \in \mathbb{Z}$

$= \frac{1}{2} \cdot 0 = 0$

as $\langle T_i(x), T_J(x)\rangle = 0$ for $i \neq J$
the set is orthogonal. $\square$

b) $T_0(x) = 1$ $T_1(x) = x$ $T_2(x) = 2x^2 - 1$

Want orthogonal projection of $f(x) = \sin x$, $\hat{f}$

$\hat{f} = C_0 T_0 + C_1 T_1 + C_2 T_2$ as that is the basis

$C_0 = \frac{\langle f, T_0\rangle}{\langle T_0, T_0\rangle}$ $C_1 = \frac{\langle f, T_1\rangle}{\langle T_1, T_1\rangle}$ $C_2 = \frac{\langle f, T_2\rangle}{\langle T_2, T_2\rangle}$

$\langle f, T_0\rangle = \int_{-1}^{1} \frac{\sin x}{\sqrt{1-x^2}}\,dx$ as $\sin x$ is odd, $\int_{-1}^{1} \frac{\sin x}{\sqrt{1-x^2}}\,dx = 0$

$\langle f, T_1\rangle = \int_{-1}^{1} \frac{x \sin x}{\sqrt{1-x^2}}\,dx = 1.382$ [Wolfram Alpha]

$1 = \sin w$ $-1 = \sin w$

$\langle T_1, T_1\rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx$ $x = \sin w$ $dx = \cos w\,dw$

$= \int_{-\pi/2}^{\pi/2} \frac{\sin^2 u}{\cos w} \cdot \cos w = \int_{-\pi/2}^{\pi/2} \sin^2 u\,du$

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$\sin u \cdot \sin u$

$$\int_{-\pi/2}^{\pi/2} \sin^2 u \, du = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1-\cos 2u)\,du = \int_{-\pi/2}^{\pi/2} \frac{du}{2} - \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos 2u \, du$$

$$= \frac{u}{2}\Big|_{-\frac{\pi}{2}}^{\pi/2} - \frac{1}{4}\sin(2u)\Big|_{-\pi/2}^{\pi/2}$$

$$\langle T_1, T_1 \rangle = \frac{\pi}{2}$$

$$\langle f, T_2 \rangle = \int_{-1}^{1} \frac{(2x^2-1)\sin x}{\sqrt{1-x^2}}\,dx = 0 \text{ as this is odd}$$

So $\hat{f} = C_0 T_0 + C_1 T_1 + C_2 T_2$ $\quad T_1 = x$

$$\hat{f} = \frac{1.382}{\pi/2} \cdot x \Rightarrow \hat{f} \approx .882x$$

Python attached, under $E = \max_{x \in [-1,1]} |f(x) - \hat{f}(x)| = 0.0385$

Image also attached

Question 5b plot)