1 a) If $f'(x)=0$ $\forall x \in [a,b]$ then by MVT $\exists c \in [a,b]$ s.t. $f'(c)=0=\frac{f(b)-f(a)}{b-a}$ so $f(b)=f(a)$ but this holds $\forall c,d \in [a,b]$, meaning f is constant.

b) Let $\forall x, \in [a,b]$, by MVT, as $f'(x)=3$, $\exists c \in [a,b]$ s.t. $3=\frac{f(x)-f(a)}{x-a}$, $3(x-a)=f(x)-f(a)$ so $3(x-a)+f(a)=f(x)$

2 a) Since $f'(x)\neq 0$, then either $f'(x)>0$ or $f'(x)<0$. To show, assume this is false, then WLOG $x_0<x_1$ and $f'(x_0)>0$ and $f'(x_1)<0$. As f' is continuous, then by IVT $\exists c \in (x_0,x_1)$ s.t. $f'(c)=0$ which is a contradiction. Therefore $f'(x)>0$ or $f'(x)<0$, meaning f is strictly increasing or decreasing on (a,b) and is therefore injective.

b) As an interval on $\mathbb{R}$ is a connected set, then $f:(a,b)\to\mathbb{R}$ is also connected, and as f goes to $\mathbb{R}$, then the image of f is also an interval.
To show I is open, use proof by contradiction. Assume I isn't open. WLOG, assume I is of the form $(c,d]$. $\exists x_0 \in (a,b)$ s.t. $f(x_0)=d$, and this can't be at an end point as the domain is open. Therefore, f reaches a maximum value at x_0, and since it's not an end point of the domain, $f'(x_0)=0$, contradicting our assumption meaning I is open. The proof is the same w/ unbounded I.
As I is an interval, f maps onto I.

2c) $\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$ as $y = f(x)$ and let $y_0 = f(x_0)$ $y, y_0 \in I$

$= \lim_{x \to x_0} \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)}$

$= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))} = (f^{-1})'(y_0)$

We don't divide by zero as $f'(x) \neq 0$

To show continuity we know for a continuous function f, as long as it is nonzero, $\frac{1}{f}$ is continuous. $f'(x)$ is continuous, so $\frac{1}{f'(f^{-1}(y_0))}$ is continuous

3 $\Rightarrow$ If A is closed then A contains all its limit points.

A set $A \subset \mathbb{R}^n$ is closed if for $\{x_k\} \in A$ if $\lim x_k = a$, then $a \in A$. Proof by contradiction: Assume a limit point, x_0, $\notin A$. Then $\exists \{x_k\} \in A$ where $\lim x_k = x_0$ but $x_k \neq x_0 \; \forall k$. However A is closed so $\lim x_k \to a \in A$ and as $x_0 \notin A$ then $\nexists \{x_k\} \to x_0$ so this is a contradiction and a limit point cannot exist outside of A.

$\Leftarrow$ A contains all it's limit points then it's closed

Let $\{x_k\} \in A$. $x_k \to x_0$ where x_0 is a limit point. Since every limit point is in A, and the $\{x_k\} \to$ to a point in A, then A is closed.

4a) Define $\{(x_k, y_k)\} \in \mathbb{R}^2 \setminus \{0,0\}$ where $(x_k, y_k) \to (0,0)$. So the limit becomes

$\frac{1}{3} \lim_{k \to \infty} \frac{\cos(x_k^3 + y_k^6)}{x_k^3 + y_k^6}$ let $z_k = x_k^3 + y_k^6$ and $\lim z_k = 0$ as $x_k, y_k \to 0$

$= \frac{1}{3} \lim_{k \to \infty} \frac{\cos(z_k)}{z_k} =$ Doesn't exist as $z_k \to 0$

4 b) $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+3y^2}$ Let $\{(x_k,y_k)\} \in \mathbb{R}^2 \setminus \{0,0\}$

where $\{x_k, y_k\} \Rightarrow (0,0)$

Let $\{x_k, y_k\} = \{\frac{1}{k}, 0\} \to (0,0)$

$$= \lim_{k\to\infty} \frac{x_k^3 y_k}{x_k^6+3y_k^2} = \lim_{k\to\infty} \frac{\frac{1}{k^3}\cdot 0}{\frac{1}{k^6}+\frac{3}{k^2}} = 0$$

Now let $x_k = x_k$ and $y_k = x_k^3 \to (x_k, x_k^3) \to (0,0)$

$$\lim_{k\to\infty} \frac{x_k^3 \cdot x_k^3}{x_k^6+3x_k^6} = \lim_{x\to\infty} \frac{x_k^6}{4x_k^6} = \frac{1}{4}$$

As $0 \neq \frac{1}{4}$ and the 2 limits aren't equal, it doesn't exist $\square$

5 When $f'(x)=0$, f can have a relative min/max. For any 2 points x_0, x_1 where $f'(x_0)=0$ and $f'(x_1)=0$. there exists at most 1 point where $f(x)=0$, if $\nexists x_2 \in (x_0,x_1)$ s.t $f'(x_2)=0$. This is because on (x_0,x_1) $f(x)$ is strictly increasing or decreasing as $\nexists$ a point where $f'(x)=0$ for $x\in(x_0,x_1)$, so $f'(x)>0$ or $f'(x)<0$, and if WLOG, $f(x_0)>0$ and $f(x_1)<0$ by IVT $\exists c \in (x_0,x_1)$ s.t. $f(c)=0$. Note this is assuming the maximum amount of zeros of $f(x)$. So as $f'(x)=0$ has $n-1$ solutions, there are $n-2$ intervals where each endpoint is a relative min/max, assuming these intervals are disjoint. Beyond this interval, $f(x)=0$ at most 2 times. Let x_{n-1} denote the 'last' relative min/max and WLOG it is assumed it is a max where $f(x_{n+1})>0$. If $f(x)$ is decreasing for $x>x_{n+1}$ then a solution to $f(x)=0$ exists. The same can be said for the other end, and can be shown w/ either mins or maxs. Therefore, at most, $f(x)=0$ has $n-2+2$ solutions, or n solutions $\square$

6) $|f(u) - f(v)| \leq c(u-v)^2$ let $\{u_k\}$ be

$|f(u_k) - f(v)| \leq c(u_k - v)^2$ a sequence and

$u_k \to v \neq V$

$\to$ By definition of absolute value,

$$-c(u_k - v)^2 \leq f(u_k) - f(v) \leq c(u_k - v)^2$$

$$\to \lim_{k\to\infty}\left(-c(u_k - v) \leq \frac{f(u_k) - f(v)}{(u_k - v)} \leq c(u_k - v)\right)$$

$$= \lim_{k\to\infty} -c|u_k - v| \leq \lim_{k\to\infty} \frac{f(u_k) - f(v)}{u_k - v} \leq \lim_{k\to\infty} c|u_k - v|$$

As $u_k \to v$ then $\lim -c|u_k - v| = 0$

$$= \lim_{k\to\infty} 0 \leq \lim_{k\to\infty} \frac{f(u_k) - f(v)}{u_k - v} \leq 0$$

as $u_k \to v$ this is saying

$0 \leq f'(v) \leq 0$ so $f'(v) = 0 \ \forall v \in \mathbb{R}$

and therefore as the derivative is zero everywhere, it is constant.

7 Let $\{x_k\} \in \mathbb{R}^n \setminus \{x_0\}$ be a sequence that $\to x_0$, but $x_k \neq x_0$

$$\underset{x \to x^0}{0} \leq |f(x_k) g(x_k)| \leq \underset{x_k \to \infty}{c|f(x_k)|}$$

$$\lim_{k\to\infty} 0 \leq \lim_{k\to\infty} |f(x_k) g(x_k)| \leq \lim_{k\to\infty} c|f(x_k)|$$

$$0 \leq \lim_{k\to\infty} |f(x_k) g(x_k)| \leq 0$$

So $\lim_{k\to\infty} |f(x_k) g(x_k)| = 0$ so $\lim_{k\to\infty} f(x_k) g(x_k) = 0$

and $\lim_{x\to x_0} f(x) g(x) = 0$. This doesn't hold if g is unbounded let $g(x_k) = \frac{1}{f(x_k)}$ as $f(x_k) \to 0$ so $g(x_k)$ is unbounded

$\lim_{k\to\infty} f(x_k) \cdot \frac{1}{f(x_k)} = \lim 1 = 1 \neq 0$ so it doesn't hold for unbounded g.