Math 166 HW 1

1 $\int_0^\infty f_x(x)\,dx = 1 \rightarrow \int_0^\infty \frac{C(B)x^2}{1+e^{Bx}}\,dx = 1$

$C(B)\int_0^\infty \frac{x^2}{1+e^{Bx}}\,dx \Rightarrow$ $Bx = z,\ x = \frac{z}{B},\ Bdx = dz$ $\frac{C(B)}{B^3}\int_0^\infty \frac{(\frac{z}{B})^2}{1+e^z}\,dz = \frac{C(B)}{B^3}\int_0^\infty \frac{z^2}{1+e^z}\,dz$

$\rightarrow \frac{C(B)}{B^3}\cdot\frac{3}{2}\zeta(3) = 1,\quad C(B) = \frac{2B^3}{3\zeta(3)}$

$$\boxed{f_x(x) = \frac{2B^3x^2}{3\zeta(3)(1+e^{Bx})}}$$

2 $L(B) = \prod_{j=1}^{n} \frac{2B^3x_j^2}{3\zeta(3)(1+e^{Bx})}$ $\leftarrow$ MLE

$\ln L(B) = \sum_{j=1}^{n} \ln\left(\frac{2B^3x_j^2}{3\zeta(3)(1+e^{Bx})}\right)$ $\leftarrow$ Log likelihood

$\ln L(B) = \sum_{j=1}^{n} \ln(2B^3x_j^2) - \sum_{j=1}^{n} \ln(3\zeta(3)(1+e^{Bx_j}))$

$\ln L(B) = \sum_{j=1}^{n} \ln(x_j^2) + \sum_{j=1}^{n} \ln(2B^3) - \sum_{j=1}^{n} \ln(3\zeta(3)) - \sum_{j=1}^{n} \ln(1+e^{Bx})$

$\frac{d}{dB}\ln L(B) = \sum_{j=1}^{n} \ln(x_j^2) + n\ln(2B^3) - n\ln(3\zeta(3)) - \sum_{j=1}^{n} \ln(1+e^{Bx_j})$

Set to 0 to maximize $B = B_e$

$0 = 0 + \frac{n}{2B_e^3}\cdot 6B_e^2 - 0 - \sum_{j=1}^{n} \frac{x_j e^{B_e x_j}}{1+e^{B_e x_j}}$

$$\boxed{0 = \frac{3n}{B_e} - \sum_{j=1}^{n} \frac{x_j e^{Bx_j}}{1+e^{Bx_j}}}$$

B_e satisfies the above equation

3 $\mu = 1^{st}$ moment, have 1 parameter β,
$E(x) = \mu$.

$$I = E(x) = \int_0^\infty x f_x(x)\,dx = \int_0^\infty \frac{2\beta^3 x^3}{3\zeta(3)(1+e^{\beta x})}\,dx = \frac{2\beta^3}{3\zeta(3)}\int_0^\infty \frac{x^3}{1+e^{\beta x}}\,dx$$

Let $\frac{d}{dx}\beta x = z$, $\rightarrow \beta\,dx = dz$, $\rightarrow \frac{2\beta^3}{3\beta\zeta(3)}\int_0^\infty \frac{(\frac{z}{\beta})^3}{1+e^z}\,dz = I$

$$I = \frac{2\beta^2}{3\zeta(3)}\int_0^\infty \frac{\frac{z^3}{\beta^3}}{1+e^z}\,dz = \frac{2}{3\beta\zeta(3)}\int_0^\infty \frac{z^3\,dz}{1+e^z} = \frac{2}{3\beta\zeta(3)} \cdot \frac{7\pi^4}{120}$$

$$\mu = \frac{7\pi^4}{180\beta\zeta(3)}$$

$\mu = E(x)$, and $\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$

Call β_m when $\mu = \bar{x}$, so $\frac{7\pi^4}{180\beta_m\zeta(3)} = \bar{x}$

$$\boxed{\beta_m = \frac{7\pi^4}{180\zeta(3)\bar{x}} \quad \text{where } \bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j}$$

4 if $\forall j$, $x_j = 1$, looking at β_e,

$$0 = \frac{3n}{\beta_e} - \sum_{j=1}^{n}\frac{e^{\beta}}{1+e^{\beta}}, \rightarrow \frac{ne^{\beta_e}}{1+e^{\beta_e}} = \frac{3n}{\beta_e} \quad \frac{\beta_e e^{\beta_e}}{1+e^{\beta_e}} = 3$$

Using a graphing tool, $\beta_e = 3.131$

For β_m, since $\forall j$, $x_j = 1$, $\bar{x} = 1$, so $\beta_m = \frac{7\pi^4}{180\zeta(3)} = 3.152$

$\beta_e = 3.131$
$\beta_m = 3.152$
Looking at the graph for $\beta = 3$ and $\beta = 3.5$, and knowing the graph for β_e and β_m lies between them, the graphs for $\beta = 3$ and $\beta = 3.5$ peak around 1, so the graphs for β_e and β_m should peak around 1 as well, so having an average of 1 for the samples is reasonable