Recall:

Definition 1. An **equivalence relation** $\sim$ on a set X is a relation $\sim$ on X such that

(1) (reflexivity) For all $x \in X$, $x \sim x$.
(2) (symmetry) For all $x, y \in X$, if $x \sim y$ then $y \sim x$.
(3) (transitivity) For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

(1) Let $\sim$ be the relation on $\mathbb{R}^2$ given by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $y_1 - x_1^2 = y_2 - x_2^2$.
 (a) Prove that $\sim$ is an equivalence relation.

 (b) What are the equivalence classes of $\sim$? Sketch and label the equivalence classes $[(0, 0)]$, $[(0, 1)]$, and $[(0, 2)]$ in $\mathbb{R}^2$.

Recall:

Definition 2. An **order relation** on a set X is a relation $\prec$ on X such that
(a) (comparability) If $x, y \in X$ and $x \neq y$, then $x \prec y$ or $y \prec x$.
(b) (anti-reflexivity) For all $x \in X$, we have $x \nprec x$.
(c) (transitivity) For all $x, y, z \in X$, if $x \prec y$ and $y \prec z$, then $x \prec z$.

Definition 3. If $(X, <_X)$ and $(Y, <_Y)$ are ordered sets, the **dictionary order** on $X \times Y$ is the order defined by

$$(x_1, y_1) < (x_2, y_2) \iff x_1 <_X x_2, \text{ or } x_1 = x_2 \text{ and } y_1 <_Y y_2.$$

(2) Prove that the dictionary order is an order relation.

Recall that the **cartesian product** of two sets X and Y is the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

To define products of more sets, we need to talk about tuples of more elements than two. We can do this using functions:

Definition 4. Let m be a positive integer. Given a set X, we define an m**-tuple** of elements of X to be a function

$$\mathbf{x} : \{1, \dots, m\} \to X.$$

Given an m-tuple $\mathbf{x}$, we often write x_i rather than $\mathbf{x}(i)$ and call it the ith **coordinate** of $\mathbf{x}$. We often denote the function $\mathbf{x}$ itself by the symbol

$$(x_1, \dots, x_m).$$

Definition 5. Given sets $A_1, \dots, A_m$, the **cartesian product** $A_1 \times \cdots \times A_m$ is the set of m-tuples

$$A_1 \times \cdots \times A_m = \{(x_1, \dots, x_m) \mid x_i \in A_i \text{ for each } i\}.$$

(We take $X = A_1 \cup \cdots \cup A_m$ so the definition of m-tuple makes sense here.)

These definitions extend easily to arbitrary products of sets.

Definition 6. Let I be a set. An I-tuple of elements of a set X is a function

$$\mathbf{x} : I \to X.$$

We write x_i rather than $\mathbf{x}(i)$ and call it the ith **coordinate** of $\mathbf{x}$. We often denote $\mathbf{x}$ itself by $(x_i)_{i \in I}$.

Given sets $\{A_i\}_{i \in I}$ indexed by a set I, the **cartesian product** $\prod_{i \in I} A_i$ is the set of I-tuples

$$\{(x_i)_{i \in I} \mid x_i \in A_i \text{ for each } i \in I\}.$$

(We take $X = \bigcup_{i \in I} A_i$.)

(3) Let $A_1 = \{1\}$, $A_2 = \{2\}$, $B_1 = \{3\}$, and $B_2 = \{4\}$. Compute $(A_1 \times A_2) \cup (B_1 \times B_2)$ and $(A_1 \cup B_1) \times (A_2 \cup B_2)$. How do the sets compare?

(4) Let $A_1 = \{1,2\}$, $A_2 = \{-1,-2\}$, and $A_3 = \{\pi, 2\pi\}$. Write out the elements of $A_1 \times A_2 \times A_3$, $A_1 \times (A_2 \times A_3)$, and $A_1 \times (A_3 \times A_2)$. Are these sets the same or different?

The answer to the last problem should feel annoying. Let's work our way towards another perspective on what a cartesian product is.

(5) Let $\pi_1 : \mathbb{Z} \times \mathbb{R} \to \mathbb{Z}$ and $\pi_2 : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be the functions given by

$$\pi_1(x,y) = x \quad \text{and} \quad \pi_2(x,y) = y.$$

(a) Let $f : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R}$ be the function defined as in the following table. Complete the rest of the table.

a	$f(a)$	$(\pi_1 \circ f)(a)$	$(\pi_2 \circ f)(a)$
1	$(3,4)$		
2	$(1,\pi)$		
3	$(-1,2\pi)$		

(b) There is a function $g : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R}$, some facts about which are recorded in the following table. Complete the rest of the table.

a	$g(a)$	$(\pi_1 \circ g)(a)$	$(\pi_2 \circ f)(a)$
1		2	$\sqrt{2}$
2		25	$\sqrt{3}$
3		-125	4

You should see that a function $f : \{1,2,3\} \to \mathbb{Z} \times \mathbb{R}$ is "the same" as a pair of functions $(f_1 : \{1,2,3\} \to \mathbb{Z}, f_2 : \{1,2,3\} \to \mathbb{R})$. That is, you can find such f_1 and f_2 from f and you can construct f from f_1 and f_2.

We state this property in general as follows.

Theorem 7 (The Universal Property of the Cartesian Product)**.** *Let X, Y be sets and let $P = X \times Y$. Write $\pi_1 : P \to X$ for the function $(x, y) \mapsto x$ and $\pi_2 : P \to Y$ for the function $(x, y) \mapsto y$. (These are called the **projection maps**.) Then for any set A and pair of functions $f_1 : A \to X$ and $f_2 : A \to Y$ there exists a unique function $f : A \to P$ so that the diagram*

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

commutes, i.e., $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

(6) Prove the theorem. (Hint: Think about your answers to (5).)

(7) Let X, Y be sets. Suppose that P' is a set and $\pi_1' : P' \to X$ and $\pi_2' : P' \to Y$ are functions that also have the universal property of the product, that is, for any set A and pair of functions $f_1 : A \to X$ and $f_2 : A \to Y$ there exists a unique function $f : A \to P'$ so that the diagram

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X & P' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & Y \end{tikzcd}$$

commutes, i.e., $f_1 = \pi_1' \circ f$ and $f_2 = \pi_2' \circ f$.

Prove that there is a bijection $P \to P'$. (Hint: Try $A = P$ and $A = P'$.)

(8) Let $\{X_i\}_{i\in I}$ be an arbitrary collection of sets. Let $P = \prod_{i\in I} A_i$ be the cartesian product. For each $i \in I$, let $\pi_i : P \to X_i$ be the projection map $(x_j)_{j\in I} \mapsto x_i$ taking tuples to their ith coordinate. Show that P has the following "Universal property of the product:"

Given a set A and functions $f_i : A \to X_i$ for each $i \in I$, there exists a unique function $f : A \to P$ so that the diagram

$$\begin{tikzcd} A \arrow[d, "f"] \arrow[dr, "f_i"] & \\ P \arrow[r, "\pi_i"] & X_i \end{tikzcd}$$

commutes for all i

(9) We have seen in problem (5) that $A_1 \times (A_2 \times A_3)$ is not quite the same set as $A_1 \times A_2 \times A_3$. However, the two sets are related by an easy-to-guess bijective function. See if you can find it and check that it is a bijection.

One perspective on where this function comes from is that both $A_1 \times (A_2 \times A_3)$ and $A_1 \times A_2 \times A_3$ have the universal property of the product of A_1, A_2, and A_3. Reasoning as in problem 7, there is a unique isomorphism between them that respects the projection maps. If you like, you can try to argue this way.