Hypothesis testing and decision rules

Type I and Type II Errors

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Outline

Type I and Type II errors

Our decision	True state of nature: H_0 is true	True state of nature: H_1 is true
Fail to reject H_0	Correct decision	Type II error
Reject H_0	Type I error	Correct decision

- Analysis of Type I error for normal r.v.:

$$\begin{aligned} P(\text{Type I error}) =& P\left(\text{Reject } H_0 \mid H_0 \text{ is true}\right) \\ =& P\left(Z \geq z_\alpha \mid \mu = \mu_0\right) \\ =& P\left(\frac{X - \mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \mid \mu = \mu_0\right) \\ =& \alpha. \end{aligned}$$

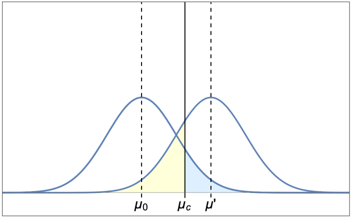
Type II errors

- Analysis of Type II errors for normal r.v.:

$$\begin{aligned}
P(\text{Type II error}) =& P\left(\text{Fail to reject } H_0 \mid H_1 \text{ is true}\right)\\
=& P\left(Z \leq z_\alpha \mid \mu = \mu' > \mu_0\right)\\
=& P\left(\frac{X - \mu'}{\sigma/\sqrt{n}} \leq z_\alpha \mid \mu = \mu'\right)\\
=& \beta.
\end{aligned}$$

- Note that β depends on the assumed mean $\mu' > \mu_0$.
- Note also that β depends on α.

Graphical depiction of Type I and Type II errors

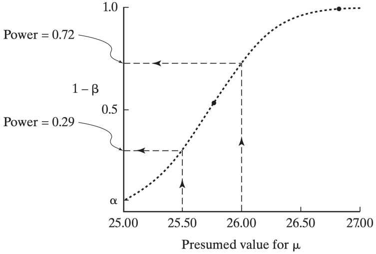


- Type I (blue): $\frac{\mu_c - \mu_0}{\sigma/\sqrt{n}} = +z_\alpha$ so $\mu_c = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$
- Type II (yellow): $\frac{\mu_c - \mu'}{\sigma/\sqrt{n}} = -z_\beta$ so $\mu_c = \mu' - z_\beta \frac{\sigma}{\sqrt{n}}$
- Eliminating μ_c yields relation between α, β, μ'

$$\boxed{(z_\alpha + z_\beta)\frac{\sigma}{\sqrt{n}} = \mu' - \mu_0}$$

- For fixed μ', there is a tradeoff between α and β.

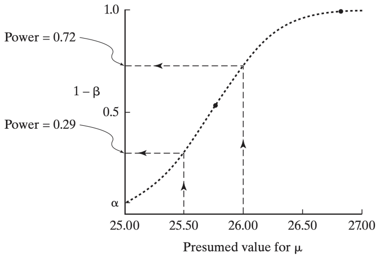
Power curves



From Larsen & Marx, Fig. 6.4.4, p. 362

- $1 - \beta$ is the probability that we reject H_0 when H_1 is true.
- This is called the *power of the test*
- Plot of $1 - \beta$ versus μ' is called a *power curve*
  - If $\mu' = 26$, easy to distinguish μ', μ, so power is 0.72
  - If $\mu' = 25.5$, difficult to distinguish μ', μ, so power is 0.29

Power curves



From Larsen & Marx, Fig. 6.4.4, p. 362

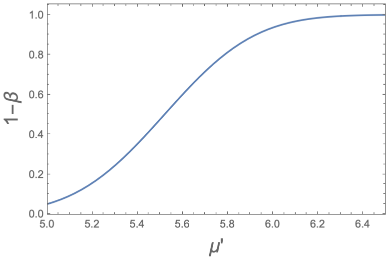
- Equation of power curve is, for fixed α,

$$(z_\alpha + z_\beta)\,\frac{\sigma}{\sqrt{n}} = \mu' - \mu_0$$

- As $\mu' \to \mu_0$, we have $z_\alpha + z_\beta \to 0$, so $1 - \beta = \alpha$.
- Vertical axis intercept of the power curve is $1 - \beta = \alpha$.

Power curves (continued)

- Return to $(z_\alpha + z_\beta)\frac{\sigma}{\sqrt{n}} = \mu' - \mu_0$
- Solve for $z_\beta = -z_\alpha + \frac{\sqrt{n}}{\sigma}(\mu' - \mu_0)$
- Fix $\alpha = 0.05$, $n = 10$, $\sigma = 1$, $\mu_0 = 5$.
- Plot of $1 - \beta$ vs. μ' results in:



Summary

- We have defined and studied Type I and Type II errors
- We have determined the key relation

$$\boxed{(z_\alpha + z_\beta)\frac{\sigma}{\sqrt{n}} = \mu' - \mu_0}$$

- For fixed μ', this illustrates a tradeoff between α and β.
- For fixed α, this describes the power curve, which is a plot of $1 - \beta$ versus μ'.