MATH235 HOMEWORK 4 SOLUTION

- 3.2.19. Let $E \subseteq \mathbb{R}$ be a measurable set that is contained in an interval I, and assume that $f : I \to \mathbb{C}$ is a measurable function that is differentiable at each point in E, i.e.,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \text{ exists and is a scalar for all } x \in E.$$

Show that f' is a measurable function on E.

Proof. Consider

$$g_n = \frac{f(x+1/n) - f(x)}{1/n}$$

is a measurable function on I. Moreover, $\lim_{n\to\infty} g_n$ exists on E, hence $\lim_{n\to\infty} g_n(x) = f'(x)$ is measurable $\forall x \in E$. $\square$

- 3.2.20. Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is measurable, and $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is a bijection such that φ^{-1} is Lipschitz. Prove that $f \circ \varphi$ is measurable.

Proof. Let $a \in \mathbb{R}$, Consider $\{f \circ \varphi > a\} = (f \circ \varphi)^{-1}(a, \infty) = \varphi^{-1}(f^{-1}(a, \infty))$. Notice that f is measurable and φ^{-1} is Lipschitz therefore it maps measurable sets to measurable sets. $\square$

- 3.2.21 Assume that E is a measurable subset of $\mathbb{R}^d$ such that $|E| < \infty$.

(a) Suppose that $f : E \to \mathbb{R}$ is measurable. Prove that for each $\varepsilon > 0$, there is a closed set $F \subseteq E$ such that $|E\backslash F| < \varepsilon$ and f is bounded on F.

(b) Let f_n be a measurable function on E for each $n \in \mathbb{N}$. Suppose that for all $x \in E$ we have

$$M_x = \sup_{n\in\mathbb{N}} |f_n(x)| < \infty.$$

Prove that for each $\varepsilon > 0$, there exists a closed set $F \subseteq E$ and a finite constant M such that $|E\backslash F| < \varepsilon$ and $|f_n(x)| \le M$ for all $x \in F$ and $n \in \mathbb{N}$.

Proof. (a).Let $\epsilon > 0$, consider $E = \cup_{n=1}^{\infty}\{|f| \le n\} = \cup_{n=1}^{\infty} E_n$, where $E_1 \subseteq E_2 \cdots$. By continuity of measure we have $\lim_{n\to\infty} |E_n| = |E|$, which tells $\exists n \ge 1$ such that $|E| - |E_n| = |E \setminus E_n| = \frac{\epsilon}{2}$ because $|E| < \infty$. Also, notice that there exists some $F_n \subseteq E_n, F_n$ closed and $|E_n \setminus F_n| < \frac{\epsilon}{2}$. Combining all the information above we have $|E \setminus F_n| < \epsilon$, as desired.

(b). Consider $f(x) = M_x < \infty$, notice that $f : E \to \mathbb{R}$ is measurable therefore from (a) there exists $F \subseteq E$ closed and $|E \setminus F| < \epsilon$ and f is bounded on F for all $\epsilon > 0$. Hence $\forall x \in F, \forall n \ge 1, \exists M$ such that $|f_n(x)| \le M$. $\square$

- 3.3.9 For each $a \in \mathbb{R}$, let $f_a = \chi_{[a,a+1]}$. Prove that $\{f_a\}_{a\in\mathbb{R}}$ is an uncountable set of functions in $L^\infty(\mathbb{R})$ such that $\|f_a - f_b\|_\infty = 1$ for all real numbers $a \neq b$.

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Proof. Notice that $||f_a||_\infty = 1 \forall a \in \mathbb{R}$. If $a \neq b$ and $[a, a+1] \cap [b, b+1] = \emptyset$ then

$$f_a(x) - f_b(x) = \begin{cases} 1, & x \in [a, a+1] \\ -1, & x \in [b, b+1] \\ 0, & x \notin [a, a+1] \cup [b, b+1] \end{cases}$$

which gives $||f_a(x) - f_b(x)||_\infty = 1$. If $a \neq b$ and $[a, a+1] \cap [b, b+1] \neq \emptyset$ then

$$f_a(x) - f_b(x) = \begin{cases} 1, & x \in [a, a+1] \setminus [b, b+1] \\ -1, & x \in [b, b+1] \setminus [a, a+1] \\ 0, & x \in [a, a+1] \cap [b, b+1] \\ 0, & x \notin [a, a+1] \cup [b, b+1] \end{cases}$$

which also gives $||f_a(x) - f_b(x)||_\infty = 1$. □

- 3.4.6 (a) Exhibit a sequence of functions that converges almost uniformly but does not converge in L^∞-norm.

  (b) Exhibit a sequence of functions that converges pointwise a.e. but does not converge almost uniformly.

  Proof. (a). Consider $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = \chi_{[0, \frac{1}{n}]}(x)$, then f_n converges to 0 pointwise a.e. on $[0, 1]$ and by Egorov's Theorem it converges almost uniformly. However its L^∞ norm converges to 1.
  (b). Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \chi_{[n, n+1]}(x)$, then f_n converges to 0 pointwise a.e. on $\mathbb{R}$ but on any set $E \subset \mathbb{R}$, $|E| > 0$ one has $\sup_{x \in E} |f_n(x)| = 1$. □