Homework 11

Early problem due on Gradescope at 11:59 pm on Wednesday, April 26th.
Due on Gradescope at 11:59 pm on Friday, April 28th.

(1) (Early problem) Let τ and τ' be two topologies on a set X; suppose that τ' is finer than τ. What does compactness of X under one of these topologies imply about compactness under the other?

(2) (a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact
(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} - A$ is countable or all of $\mathbb{R}$, is $[0, 1]$ a compact subspace?

(3) Show that if X is compact Hausdorff under both the topology τ and the topology τ', then either τ and τ' are equal or they are not comparable.

(4) Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map, i.e., if Z is a closed subset of $X \times Y$, then $\pi_1(Z)$ is a closed subset of X. (Hint: for each $x_0 \in X - \pi(Z)$, use the tube lemma to find an open neighborhood around x_0 contained in $X - \pi(Z)$.)