# Final Exam, Math 65, Fall 2021, Friday Dec 17, 12-2p.m.

This exam is to be completed by you alone without any external help from anybody or anything. No books, notes or internet consultation. SHOW YOUR WORK and please include complete justification.

**Question 0.1.** Let $A, B$ be sets and $f : A \to B$ a function.

(a) Define what it means for $f$ to be one to one.
(b) Define what it means for $f$ to be onto.
(c) Give an example of a function from the set $A = \{1, 2, , \ldots, 10\}$ to itself that is one to one and not onto or show that no such function exists. Make sure that you check that your function satisfies the conditions requested.
(d) Give an example of a function from the set $\mathbb{Z}$ to itself that is one to one and not onto or show that no such function exists. Make sure that you check that your function satisfies the conditions requested.

**Question 0.2.** Recall that after fixing a natural number $n$, one can define an equivalence relation in the integers as follows $z_1, z_2 \in \mathbb{Z}$, then $z_1 \sim_n z_2 \iff \exists k \in \mathbb{Z}, z_1 - z_2 = nk$. The equivalence class of an $a \in \mathbb{Z}$ is denoted by $[a]_n$. You do not need to show that this is an equivalence relation. Consider the assignments

$$f: \begin{array}{ccc} \mathbb{Z}_5 & \to & \mathbb{Z}_{15} \\ [a]_5 & \to & [3a]_{15} \end{array}, \qquad g: \begin{array}{ccc} \mathbb{Z}_3 & \to & \mathbb{Z}_{15} \\ [a]_3 & \to & [3a]_{15} \end{array}$$

(a) Explain why $[2]_6 = [-3]_5$.
(b) Prove or disprove that f is a well defined function.
(c) Prove or disprove that $g$ is a well defined function.

**Question 0.3.** A coloring of a graph is an assignment of colors to the vertices. A graph is said to have chromatic number at most $n$ if there is a coloring of the vertices with $n$ colors so that adjacent vertices have different colors. It is said to have chromatic number exactly $n$ if there is no such coloring with $n - 1$ or fewer colors.

Show that a tree with two or more vertices has chromatic number 2. Hint: Use induction on the number of vertices.

**Question 0.4.** A simple graph has a total of 6 vertices, two of them of degree 1 and the other four of degree 2.

(a) How many edges does it have?
(b) If you know the graph is planar but do not know whether it is connected, can you tell in how many faces it divides the plane? Explain
(c) Sketch three non-isomorphic graphs satisfying the given conditions (and show they are not isomorphic) or show that this is impossible.

**Question 0.5.** (a) Write the definition of $(a_n)$ is a Cauchy sequence.
(b) Write the condition $(a_n)$ is not a Cauchy sequence so that the last equation in your sentence has a symbol "greater than" ($>$).
(c) Show that $a_n = \frac{(-1)^n(6n+1)}{3n-1}$ is not a Cauchy sequence.

Please scan your work and upload to gradescope. Thank you and have a great, well deserved break.

## 1. Model Answers

**Question 1.1.** Let $A, B$ be sets and $f : A \to B$ a function.

(a) Define what it means for $f$ to be one to one.
(b) Define what it means for $f$ to be onto.
(c) Give an example of a function from the set $A = \{1, 2, , \dots, 10\}$ to itself that is one to one and not onto or show that no such function exists. Make sure that you check that your function satisfies the conditions requested.
(d) Give an example of a function from the set $\mathbb{Z}$ to itself that is one to one and not onto or show that no such function exists. Make sure that you check that your function satisfies the conditions requested.

**Answer 1.** (a) The function $f$ is one to one if different elements have different images: for every $a, a' \in A$, $f(a) = f(a')$ implies $a = a'$..
(b) The function $f$ is onto if the image is the shole codomain: for every $b \in B$ there is some $a \in A$ such that $f(a) = b$.
(c) There is no function from the set $A = \{1, 2, , \dots, 10\}$ to itself that is one to one and not onto: the image of a one to one function is in a bijection with the domain. Hence, in this case, the image has 10 elements and needs to agree with the whole set $A$. This means that $f$ is also onto.
(d) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 2n$. The function is one to one: if $f(a) = f(a')$, from the definition of $f$, $2a = 2a'$ and therefore $a = a'$.
The function is not onto as the odd numbers are not in the image.

**Question 1.2.** Recall that after fixing a natural number $n$, one can define an equivalence relation in the integers as follows $z_1, z_2 \in \mathbb{Z}$, then $z_1 \sim_n z_2 \iff \exists k \in \mathbb{Z}, z_1 - z_2 = nk$. The equivalence class of an $a \in \mathbb{Z}$ is denoted by $[a]_n$. You do not need to show that this is an equivalence relation. Consider the assignments

$$f: \begin{array}{ccc} \mathbb{Z}_5 & \to & \mathbb{Z}_{15} \\ [a]_5 & \to & [3a]_{15} \end{array}, \qquad g: \begin{array}{ccc} \mathbb{Z}_3 & \to & \mathbb{Z}_{15} \\ [a]_3 & \to & [3a]_{15} \end{array}$$

(a) Explain why $[2]_6 = [-3]_5$.
(b) Prove or disprove that f is a well defined function.
(c) Prove or disprove that $g$ is a well defined function.

**Answer 2.** (a) By definition, $[2]_5 = [-3]_5$ if and only if there is some $k \in \mathbb{Z}$ such that $2 - (-3) = 5k$. In this case, $k = 1$ works.
(b) The first assignment is well defined: assume $[a]_5 = [b]_5$. By definition of the equivalence relation $\sim_5$, this means that there exists an integer $k \in \mathbb{Z}$ such that $a - b = 5k$. Then, $3a - 3b = 3(a - b) = 3 \times 5k = 15k$. Therefore, by definition of the equivalence relation $\sim_{15}$, $[a]_{15} = [b]_{15}$.
(c) The second assignment is not well defined as $[0]_3 = [3]_3$ while $[3 \times 0]_{15} = [0]_{15} \neq [9]_{15} = [3 \times 3]_{15}$

**Question 1.3.** A coloring of a graph is an assignment of colors to the vertices. A graph is said to have chromatic number at most $n$ if there is a coloring of the vertices with $n$ colors so that adjacent vertices have different colors. It is said to have chromatic number exactly $n$ if there is no such coloring with $n - 1$ or fewer colors.

Show that a tree with two or more vertices has chromatic number 2. Hint: Use induction on the number of vertices.

**Answer 3.** We prove this by induction on the number of vertices. If $n = 2$ as the tree is connected, the chromatic number cannot be 1. As there are two vertices, two colors is certainly enough.

Assume now that the result is correct for any tree up to $n$ vertices and take a tree $T$ with $n + 1$ vertices. We know that $T$ has a leaf $v$ joined to the rest of the tree by an edge $e$ with vertices $v, v_1$ and that removing $e, v$, we obtain a tree $T'$ with $n$ vertices. By induction assumption, $T'$ can be colored with 2 colors. Coloring $v$ with the color not used for $v_1$ and keeping the rest of the coloring of $T'$, we obtain a proper coloring of $T$.

Moreover, one color will never be enough for any graph with at least one edge, as the two vertices at the end of the edge must have different colors.

A non-inductive proof that two colors are enough could go as follows: Choose an arbitrary vertex $v_0$. For any other vertex $v$, the condition of tree means that there is a unique simple path from $v_0$ to $v$ say $v_0 e_1, v_1, e_2 \dots v_n = v$. If $n$ is even, color $v$ the same color as $v_0$, if odd the opposite. The uniqueness of the path from $v_0$ to any vertex guarantees that adjacent vertices have different colors.

**Question 1.4.** A simple graph has a total of 6 vertices, two of them of degree 1 and the other four of degree 2.

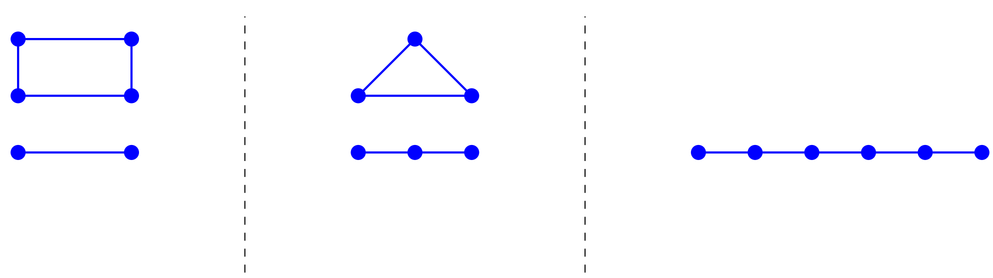
(a) How many edges does it have?
(b) If you know the graph is planar but do not know whether it is connected, can you tell in how many faces it divides the plane? Explain
(c) Sketch three non-isomorphic graphs satisfying the given conditions (and show they are not isomorphic) or show that this is impossible.

**Answer 4.** (a) By the hand shake theorem, the sum of degrees is twice the number of edges. So

$$2 \times 1 + 4 \times 2 = 10 = 2e \Rightarrow e = 5$$

The graph has 5 edges.

(b) A planar graph with $v$ vertices, $e$ edges $c$ components and dividing the plane if $f$ faces satisfies that $v - e + f = 1 + c$. Without knowing the number of components, we cannot tell the number of faces. In fact, in the example below, there is one connected graph with a single face and two others with two faces



(c) The three graphs above are not isomorphic. In the first graph , the two vertices of degree one are connected to each other and this is not true in any of the other two graphs. So the first graph is not isomorphic to any of the other two. The third graph is connected while the other two have two connected components, so the third graph cannot be isomorphic to any of the other two.

**Question 1.5.** (a) Write the definition of $(a_n)$ is a Cauchy sequence.

(b) Write the condition $(a_n)$ is not a Cauchy sequence so that the last equation in your sentence has a symbol "greater than" ($>$).

(c) Show that $a_n = \frac{(-1)^n(6n+1)}{3n-1}$ is not a Cauchy sequence.

**Answer 5.** (a) Choose $\epsilon \in \mathbb{Q}^+$

$$\exists m \in \mathbb{N}, \forall n_1, n_2 \geq m, |a_{n_1} - a_{n_2}| < \epsilon$$

(b) The negation of the condition of Cauchy sequence is obtained by negating the statement (a). Recall that the negation of a quantifier such as "for all $x, P(x)$ is "there exists some $x$ not $P(x)$". The negation of "exists $x, P(x)$ is "for all $x$ not $P(x)$". Therefore, the negation of $(a_n)$ being a Cauchy sequence is

$$\exists \epsilon_0 > 0,\ \forall m \in \mathbb{N}\ \exists n_1, n_2 \geq m,\ |a_{n_1} - a_{n_2}| > \epsilon$$

(c) Take $\epsilon_0 = 4$. Choose any $m \in \mathbb{N}$ and choose $n_1 > m$ such that $n_1$ is even, $n_2 = n_1 + 1$. For $n_1$ even, $a_{n_1} = \frac{(6n_1+1)}{3n_1-1} > \frac{(6n_1)}{3n_1} = 2$. Now $n_2 = n_1 + 1$ is odd and $a_{n_2} = -\frac{(6n_2+1)}{3n_2-1} < -2$. Hence, if $n_1$ is even , $n_2 = n_1 + 1$, then $|a_{n_1} - a_{n_1+1}| = a_{n_1} - a_{n_1+1} > 2 - (-2) = 4$ contradicting the condition for Cauchy sequence.