I am going to make some suggestions of things you could with your journal. You can't follow them all, and you need not follow any of them.

Thursday, September 23

1. **Non-uniqueness.** In class, we thought about an initial value problem of the form

$$\frac{dx}{dt} = f(x), \tag{1}$$
$$x(t_0) = x_0. \tag{2}$$

   Here $x = x(t) \in \mathbb{R}^n$, $f(x) \in \mathbb{R}^n$. We said: If f is "nice" in the sense that all components of f are differentiable with respect to all components of x, and those $n \times n$ derivatives are continuous, then there is an $\varepsilon > 0$ so that for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, there is exactly one function $x(t)$ satisfying (1) and (2).

   Let's try a function $f(x)$ that does *not* satisfy our assumptions: $n = 1$,

$$\begin{aligned} \frac{dx}{dt} &= x^{1/3}, \\ x(0) &= 0. \end{aligned}$$

   (Notice that $f(x) = x^{1/3}$ is not differentiable at $x = 0$.) Notice that $x(t) = 0$ is a solution. Find a non-zero solution using separation of variables.

   Then show that there are actually infinitely many solutions. (Hint: Just as x can wander away from 0 at time 0, it can also sit at 0 for some time, then suddenly wander away.)

2. **Existence and the contraction mapping principle.** If you've taken Real Analysis, then think about this. You can equivalently write (1) and (2) as follows:

$$x(t) = \int_{t_0}^{t} f(x(s))ds + x_0.$$

   First convince yourself that that's equivalent to (1) and (2). Then convince yourself that for small enough $\varepsilon > 0$, if you restrict t to $[t_0 - \varepsilon, t_0 + \varepsilon]$ and assume that f has derivatives and those derivatives are continuous, then the mapping from

$$\{x(t)\}_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]}$$

   to

$$\left\{ \int_{t_0}^{t} f(x(s))ds + x_0 \right\}_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]}$$

   is contracting. (You may find this easier if first you take n to be 1.) Now use the contraction mapping principle to prove existence of a solution $x(t)$, $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$.

3. **Real Covid modeling.** Read this paper and summarize it, see what you think about it. I noticed that there is surprising agreement between modeling and reality *for a while*. I am more puzzled by that than by the divergence between modeling and reality later on.

4. **Could the ultimate outcome really depend significantly on a single early super-spreader event?** Use the discrete model with the modification that we made when we introduced a time step Δt:

$$\begin{aligned}\frac{S_{k+1}-S_k}{\Delta t} &= -\alpha I_k S_k\\ \frac{I_{k+1}-I_k}{\Delta t} &= \alpha I_k S_k - \beta I_k\\ \frac{R_{k+1}-R_k}{\Delta t} &= \beta I_k\end{aligned}$$

   Starting with S_0, I_0, R_0, you can compute all S_k, I_k, R_k from here. What you get is an approximation to a solution of the ODEs. In fact, this method of solving the ODEs, approximately, is called *Euler's method*. (Euler proposed it in his Calculus book, in a paragraph of prose, in Latin of course.) The smaller Δt, the better the approximation to the solution of the ODEs.

   Suppose the population size N is 400, and $\alpha = 0.004$, so $\alpha N = 1.6$, so the epidemic is on the bad side of the Kermack-McKendrick threshold. (As has Covid been, for much of the time at least.) Compare

$$S_0 = 398, \quad I_0 = 2, \quad R_0 = 0$$

   and

$$S_0 = 390, \quad I_0 = 10, \quad R_0 = 0.$$

   (In the latter situation, early during the pandemic, there was a big mask-free wedding that got 8 people infected.) In each case, compute $\lim_{k\to\infty} R_k$ numerically. (Which Δt should you use? The answer is, a small one. The smaller you choose Δt, the better do you approximate the ODEs. Make it so small that making it yet smaller would have no significant impact on your result.)