Last real assignment :(

Graded

Student

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Total Points

35 / 35 pts

Question 1

18.1 — **5** / 5 pts

✔ **– 0 pts** Correct

– 0 pts The rotation fixing the midpoint of two edges has 7 orbits of edges for 2^7 colorings in its fixed set.

– 0 pts Arithmetic error

– 2.5 pts Did not use a group action to figure out which colorings are equivalent.

– 0 pts There are 8 rotations that fix two vertices (4 clockwise and 4 anti-clockwise) each with 4 orbits of edges, so this term in the sum should be $8 \cdot 2^4$

Question 2

18.5 — **5** / 5 pts

✔ **– 0 pts** Correct

– 0 pts You could also flip the bracelet over so the group should be D_5. But for the $\mathbb{Z}_5$ action you looked at you got the correct number.

– 0 pts Arithmetic error

Question 3

20.1 — **5** / 5 pts

– 0 pts Correct

✔ **– 0 pts** Minor issue

– 1 pt You need more explanation about why the Sylow $$p_$-subgroup is normal (specifically you need to point that there is only one Sylow p-subgroup for that p).

I could not read parts of your submission. Be more careful about lining up the paper when you scan it.

1 of size 125

Question 4

20.4 | **5** / 5 pts

✔ **– 0 pts** Correct

– 2.5 pts You need to explain why all groups of order 1225 are abelian.

– 1 pt You need to explain why the group is a direct product of its Sylow subgroups, in particular why the Sylow subgroups commute with each other.

– 1 pt The Sylow subgroups have orders 25 and 49, not 5 and 7

Question 5

20.11 | **5** / 5 pts

✔ **– 0 pts** Correct

– 0 pts Unclear reasoning

– 0 pts Did not demonstrate action-ness

– 2 pts Incorrect reasoning.

Question 6

21.2 | **5** / 5 pts

✔ **– 0 pts** Correct

– 0 pts Missed one or more abelian groups of order 100

– 0 pts Incorrect torsion coefficients

Question 7

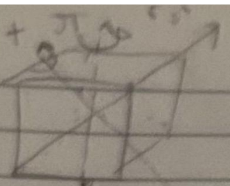
21.6 | **5** / 5 pts

✔ **– 0 pts** Correct

– 0 pts Problems with proof

At first I though you stared your proof with the word "Scoff" which would be the funniest way to start a proof.

So $\frac{12}{3}$ is 3 cycles
2^4



18.1)

So the cube has 12 edges and rotational group, s. $|G|=24$

So by book elements are r, r^2, s, t w/ conjugacy classes of size 6, 3, 8, 6 elements respectively

For r: Top 4 edges can be any, but same, as same for sides/bottom. So is $2\times2\times2$ choices.

For r^2: Opposites must have same color So $2\times2\times2\times2\times2\times2$ combinations

For s: The edges get sent in a 3 cycle? as fixes 2 vertices and must send among themselves.

So as 12 edges, 4 have 4 3-cycles so # colorings is 2^4.

For t So two edges are fixed and can then think abt. as sending it kind of to opposite edges. So will have 5 order 2 and 7 orbits

So 2^7 ways to color

hat means #orbits:

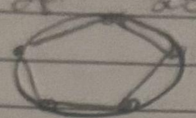
$$= \frac{1}{24}\left(2^{12} + 6\cdot 2^3 + 3\cdot 2^6 + 6\cdot 2^7 + 8\cdot 2^4\right)$$

$= 218$ unique orbits $\square$

Final Algebra HW

18.5)

So as have 5 points on circle can think of as D_5 w/edges colored as:



So works out

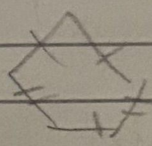
The conjugacy classes of D_5 are $\{e\}$, $\{r, r^4\}$ $\{r^2, r^3\}$, and $\{s, sr, sr^2, sr^3, sr^4\}$

Now, each edge is given a color either red blue, yellow.

So, what fixes:

e : 3^5 options as is fixed

r : edges all need same color, so 3



r^2 : Same as r, so 3 options

s : Opposite edges need same color, one can be whatever, so 3^3

$|D_5| = 10$ So by counting thm:

$$\#\text{unique orbits} = \frac{1}{10}\left(3^5 + 2(3) + 2(3) + 5(3^3)\right)$$

$$= \frac{1}{10}(390) = 39$$

So 39 unique orbits and 39 ways to get the bracelet. $\square$

20.1)

So $126/7 = 18$ and $m = 1$ so there is a subgroup of size 7.

N.t.s only 1 subgroup of size 7.

subgroups $\equiv 1 \bmod 7$, so is $\{1, 15\}$

not 15 as not factor of 18, so 1 subgroup of size 7 so it is normal

Now for $|G| = 1000$ $1000 = 2^3 5^3$
so for $p=5$, $m=3$ $\# \equiv 1 \bmod 5$ and factor of 125
Then $n_5 = \{1, 6, 11, 16, 21, 26, 31, 36 \ldots\}$.
Clearly the only number here that 125 is divisible by is 1
Also so $n_5 = 1$, unique normal subgroup of size 5,
So the group of order 1000 isn't simple. $\square$

20.4) 1,225

$$1225 = 5^2 7^2$$

1,225
5   245
5   49
7   7

So first if $|G| = 1225$, G has unique subgroups of size 5 and 7
As $n_5 = \{1, 6, \ldots\}$ but none factor of 49 other than 1   1mod5
$n_7 = \{1, 8, 15, 22 \ldots\}$ none factor of 25 other than 1
So G has normal subgroups of size 5 and 7
Now groups of order p^2 are abelian, so as normal, then PQ commute so G is abelian.

Therefore, by thm of finitely gen. abelian groups:
one of:
$$G \cong \mathbb{Z}_{1225}, \mathbb{Z}_5 \times \mathbb{Z}_{245}, \mathbb{Z}_7 \times \mathbb{Z}_{175}, \mathbb{Z}_{35} \times \mathbb{Z}_{35}$$

20.11)
To show this is an action:
We w.t.s.
$$e \cdot xH = exH = xH$$
and show under composition:
$$g(h(xH)) = (ghxH) = gh(xH)$$ so group action on H!

$\Rightarrow$ H normal so $Hx = xH$
$h(xH) = hxH = hHx = Hx = xH$
So: $g(xH) = \{xH\}$ and forward direction holds as arbitrarily one element.

$\Leftarrow$ $h \in H$, $x \in G$ $h(xH) = xH$, by def, so
$h \in xHx^{-1}$, and $H = xHx^{-1}$ meaning that H is normal $\square$

21.2)
By Cauchy's thm $\exists a \in G, b \in G$ s.t. $|a| = 5$, $|b| = 2$.
It follows $\langle a \rangle \cap \langle b \rangle = \{e\}$ so the element $a+b$ has order 10.
As: $k(a+b) = 0 \Leftrightarrow ka + kb = 0$
$\Rightarrow ka = -kb$
So, essentially want $2 | k$ and $5 | k$ and smallest number to do so is 10. So $a+b$ is order 10

For second part, by Cauchy's thm, order of elements is:
1, 2, 4, 5, 10, 20, 25, 50, 100
As none greater than 10, restricts to
1, 2, 4, 5, 10;
There can't be an element $|g| = 4$ as by logic of above, if element h of order 10 then $|g+h| = 20$
So possible order is 1, 2, 5, 10.
So $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_{10} \times \mathbb{Z}_5 \cong \mathbb{Z}_{10} \times \mathbb{Z}_{10}$
So Torsion coefficient is 10.

Question assigned to the following page: 7

21.6) So off the bat, as $G \times A \cong G \times B$, $|G||A| = |G||B|$ so $|A| = |B|$ as $|G| < \infty$

As G is abelian say $G \cong \mathbb{Z}/m_1 \times \mathbb{Z}/m_2 \cdots \times \mathbb{Z}/m_n$ in form where $m_i | m_{i+1}$

$\mathbb{Z}/m_1 \times \mathbb{Z}/m_2 \times \cdots \mathbb{Z}/m_n \times A = \mathbb{Z}/m_1 \times \mathbb{Z}/m_2 \times \cdots \times \mathbb{Z}/m_n \times B$

Now as ~~A~~ A is abelian; also B abelian
$A \cong \mathbb{Z}/k_1 \times \mathbb{Z}/k_2 \times \cdots \mathbb{Z}/k_n$ but $\prod k_i = \prod l_i$
$B \cong \mathbb{Z}/l_1 \times \mathbb{Z}/l_2 \times \cdots \mathbb{Z}/l_s$
Now by lemma that clearly $C \times D \cong D \times C$, can inductively do this to permute ordering for finite products.
As $G \times A \cong G \times B$ then:
$\cancel{\mathbb{Z}/m_1 \times} \prod \mathbb{Z}/m_i \times \prod \mathbb{Z}/k_i \cong \prod \mathbb{Z}/m_i \times \prod \mathbb{Z}/l_i$

So, we construct this isomorphism by mapping $\mathbb{Z}/m_i$ in $G \times A$ to $\mathbb{Z}/m_i$ in $G \times B$.
→ as $G \times A \cong G \times B$
So, each $\mathbb{Z}/k_i$ must map to some $\mathbb{Z}/l_i$. However, as A, B just abelian, can be multiple. and get this map by reducing A and B, to ~~the~~ direct products of their coefficients then having its A → torsion → B essentially we have shown the basis for $G \times A$ is the same as the basis for $G \times B$ as have map on generators of each cyclic group.

However, as G is same, then this means If we "ignore it" then it follows have map from A to B and $A \cong B$