MATH235 HOMEWORK 5 SOLUTION

- 3.5.12 Define f by
$$f(x) = \begin{cases} 1 & x \in (-1,1) \\ 0 & x = \pm 1 \\ -1 & x \in \mathbb{R}, x \notin [-1,1] \end{cases}$$
We claim that $f_n \to f$ pointwise a.e. and in measure, but not uniformly. The convergence is not uniform, as f_n for each n is continuous while f is not on $\mathbb{R}$.
- 3.5.13

Proof. (a). Consider the set $Z_\epsilon = \{x \in E : |f(x) - g(x)| > \epsilon\}$. Using triangle inequality we have
$$|f(x) - g(x)| \leq |f(x) - f_n(x)| + |g(x) - g_n(x)|.$$
Consider the set A_n and B_n defined by $A_n = \{x \in E : |f(x) - f_n(x)| > \epsilon\}$ and $B_n = \{x \in E : |g_n(x) - g(x)| > \epsilon\}$. It follows that $Z_\epsilon \subseteq A \cup B$. By convergence in measure, we know $\lim_{n\to\infty} |A_n| = \lim_{n\to\infty} |B_n| = 0$. Hence $|Z_\epsilon|$ goes to 0 which implies $f = g$ a.e.

(b). It suffices to prove for any $\epsilon > 0$ the measure of the following set goes to 0:
$$A = \{x \in E : |f(x) + g(x) - (f_n(x) + g_n(x))| > \epsilon\}$$
By triangle inequality, we can consider $A \subseteq A_1 \cup A_2$ where
$$A_1 = \{x \in E : |f(x) - f_n(x)| > \epsilon\}$$
$$A_2 = \{x \in E : |g(x) - g_n(x)| > \epsilon\}$$
By convergence in measure assumption, measure of A_1 and A_2 goes to 0, therefore $|A| \to 0$.

(c).Let $\{f_{n_k} g_{n_k}\}$ denotes an arbitrary subsequence of $\{f_n g_n\}$. Since $f_n \to f$ in measure, for the subsequence f_{n_k}, there exists a subsequence $\{f_{n_{k_l}}\}$ converges to f a.e. and similarly for g we have a further subsequence $\{g_{n_{k_{l_p}}}\}$ converges to g a.e. Hence $\{f_{n_{k_{l_p}}} g_{n_{k_{l_p}}}\}$ converges to fg a.e. and $f_n g_n \to fg$ in measure.

(d). Consider $g(x) = 1/x, f(x) = x$ and $g_n(x) = g(x)\chi_{[-n,n]}, f(x) = x$ defined on $\mathbb{R} \setminus \{0\}$. Then we have $|g(x) - g_n(x)| \leq 1/n$ for all x and n, which gives two convergence in measure sequences. However, $fg = 1$ and $f_n g_n = \chi_{[-n,n]}$ while $\lim_{n\to\infty} |\{f_n g_n(x) - x > \epsilon\}| = 0$ is not true for arbitrarily small ϵ.

(e). Consider $\frac{|f_n(x) - f(x)|}{|f(x) f_n(x)|} = |\frac{1}{f(x)} - \frac{1}{f_n(x)}|$ and assume $|f_n|, |f| \geq \delta$. ($|f| \geq \delta$ can be derived using a subsequence argument). Then we have
$$|\{|\frac{1}{f(x)} - \frac{1}{f_n(x)}| \geq \epsilon\}| \leq |\{|f_n - f| \geq \epsilon\delta^2\}| \to 0$$

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and $1/f_n \to 1/f$ in measure.

$\square$

- 3.5.15

  Proof. (a). Assume φ is uniformly continuous. Then for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in E. |x - y| < \delta$ implies $|\varphi(x) - \varphi(y)| < \epsilon$. Then for some $\delta' > 0$, we have
  $$\{|\varphi \circ f - \varphi \circ f_n| > \epsilon\} \subseteq \{|f - f_n| > \delta'\}$$
  and since the right hand side of the above expression goes to 0, we have our desired result. The counterexample can be $\varphi = x^2$ and $f_n = x - 1/n$ on $\mathbb{R}$.

  (b). We can always find a subsequence $\{f_{n_k}\}$ such that converges a.e. to f. Then since φ is continuous we have $\varphi \circ f_{n_k}$ converges to $\varphi \circ f$ a.e. and therefore we have $\varphi \circ f_n$ converges in measure. The counterexample we need is same as above. $\square$

- 4.1.12. Let E be a measurable subset of $\mathbb{R}^d$. Suppose that f and g are measurable functions on E such that $0 \leq f \leq g$ and $\int_E f < \infty$. Prove that $g - f$ is measurable, $0 \leq \int_E (g - f) \leq \infty$, and, as extended real numbers,
  $$\int_E (g - f) = \int_E g - \int_E f$$
  Proof. Since f and g are measurable, $g - f$ is also measurable. Since $f \leq g$, we have $0 \leq \int g - f \leq \infty$. If $\int_E g < \infty$, notice that $g = (g - f) + f$, where all two terms in the right hand side are positive. Hence we have $\int_E g = \int_E g - f + \int_E f$ which proves the claim. If $\int_E g = \infty$, then we have $\int_E g - \int_E f = \infty$ as $\int_E f < \infty$, which also proves the claim. $\square$

- 4.2.11. Assume $E \subseteq \mathbb{R}^d$ and $f : E \to [0, \infty]$ are measurable, and $\int_E f < \infty$. Given $\varepsilon > 0$, prove that there exists a measurable set $A \subseteq E$ such that $|A| < \infty$ and $\int_A f \geq \int_E f - \varepsilon$

  Proof. Since $\int_E f < \infty$, we know $f < \infty$ a.e. on E, therefore $|\{f = \infty\}| = 0$. Consider $f_n = f\chi_{B_n(0)}$ which converges to f pointwise a.e. For all $\epsilon > 0$, there exists some $N_0 \geq 1$ such that $\int_E f - \int_{B_{N_0(0)}} f < \epsilon$. Switch sides of the inequality and set $A = B_{N_0(0)}$ give the desired result. $\square$