Math 70 HW 8

1 a) null(T) contains all vectors such that $T(\vec{v})=0$, where $\vec{v}\in\mathbb{R}^n$.
Since B is a basis for V, $\vec{v}=c_1\vec{b_1}+c_2\vec{b_2}+\dots c_n\vec{b_n}$
where $c\in\mathbb{R}$
$T(\vec{v})=T(c_1\vec{b_1}+\dots c_n\vec{b_n})=c_1T(\vec{b_1})+c_2T(\vec{b_2})+\dots c_nT(\vec{b_n})$
Since $T(\vec{b_j})=\lambda_j\vec{b_j}$, that gives us
$c_1\lambda_1\vec{b_1}+c_2\lambda_2\vec{b_2}+\dots c_n\lambda_n\vec{b_n}=\vec{0}$, $\lambda_1\dots\lambda_n\neq 0$,
and since $b_1\dots b_n$ forms a basis for V, $b_1\dots b_n\neq\vec{0}$.
This means $c_1\dots c_n=0$, and the only $\vec{v}\in$ null(T) is $\vec{0}$,
this means dim(null(T)) = $\{0\}$

b) By the rank-nullity theorem, for any transformation $T: V\to W$, rank(T) + nullity(T) = dim V.
Since b has n vectors, dim V = n, additionally,
nullity(T) = dim(null(T)) = 0,
rank(T) + 0 = n, meaning dimension of range of T is n.

$\boxed{x}$

2 a) $A=[T]_B=[[T\vec{b_1}]_B\ [T\vec{b_2}]_B\ [T\vec{b_3}]_B]$

$[T(\vec{b_1})]_B=\begin{bmatrix}1\\0\\0\end{bmatrix}$ $[T(\vec{b_2})]_B=\begin{bmatrix}1\\\frac{1}{2}\\0\end{bmatrix}$ $[Tb_3]_B=\begin{bmatrix}1\\0\\\frac{1}{2}\end{bmatrix}$

$$A=\begin{bmatrix}1&1&1\\0&\frac{1}{2}&0\\0&0&\frac{1}{2}\end{bmatrix}$$

b) $A\vec{x}=\lambda\vec{x}$, $(A-\lambda I)\vec{x}=\vec{0}$

$$(A-\lambda I)=\begin{bmatrix}1-\lambda&1&1\\0&\frac{1}{2}-\lambda&0\\0&0&\frac{1}{2}-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\vec{0}$$

To find λ, since $A-\lambda I$ is triangular, we need to find
$\det(A-\lambda I)=(1-\lambda)(\frac{1}{2}-\lambda)^2$, $\lambda=1$, $\lambda=\frac{1}{2}$

For the eigenspace of $\lambda_1 = 1$.

$$A - 1I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left[\vec{x} \right] = \vec{0}.$$

This means the basis for the eigenspace of $\lambda_1 = 1$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

For eigenspace of $\lambda_2 = \frac{1}{2}$

$$A - \frac{1}{2}I = \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\frac{1}{2}x_1 + x_2 + x_3 = 0$, $x_1 = -2x_2 - 2x_3$,

$$\vec{x} = \begin{bmatrix} -2x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$, Meaning B for $\lambda = \frac{1}{2}$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for e-space of $\lambda_1 = 1$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Basis for e-space of $\lambda_2 = \frac{1}{2}$ $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

2c) For $\vec{v} \in \mathbb{R}^3$, to be an eigen vector of A,

$A\vec{x} = \lambda_1 \vec{x}$, $A\vec{x} = \lambda_2 \vec{x}$

For λ_1, $A\vec{x} = \vec{x}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ \frac{1}{2}x_2 \\ \frac{1}{2}x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$, Meaning $x_2 = 0$, $x_3 = 0$, $x_1 = x_1$

All e-vectors of $\lambda_1 = 1$ are of form $c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, where $c \in \mathbb{R}$.

For λ_2, $A\vec{x} = \frac{1}{2}\vec{x}$

2c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ \frac{x_2}{2} \\ \frac{x_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{2} \\ \frac{x_2}{2} \\ \frac{x_3}{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ \frac{x_2}{2} \\ \frac{x_3}{2} \end{bmatrix} - \begin{bmatrix} \frac{x_1}{2} \\ \frac{x_2}{2} \\ \frac{x_3}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{2} + x_2 + x_3 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{x_1}{2} + x_2 + x_3 = 0$, $x_1 = -2x_2 - 2x_3$

$$\vec{x} = \begin{bmatrix} -2x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

all e-vectors for $\lambda_2 = \frac{1}{2}$ are within span$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

For B', let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

For B' to be a basis of $\mathbb{R}^3$, v_1, v_2, v_3 must be linearly independent and span $\mathbb{R}^3$.

To prove linear independence:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \vec{0}, \quad c_1 \ldots c_3 = \vec{0}$$

$$\begin{bmatrix} c_1 - 2c_2 - 2c_3 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ $c_2 = 0, c_3 = 0, c_1 = 0$

B' is linearly independent

To prove it spans $\mathbb{R}^3$: $\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

which forms $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which is the standard basis for $\mathbb{R}^3$, meaning B' spans $\mathbb{R}^3$.

$$B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2 d) For $j=1$, $v_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[w_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = w_1 = 1$

$[w_2]_B = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = w_2 = -2+t$

$[w_3]_B = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = w_3 = -2+t^2$

$$\boxed{\begin{aligned} w_1 &= 1 \\ w_2 &= -2+t \\ w_3 &= -2+t^2 \end{aligned}}$$

2 e) For $A = PDP^{-1}$, ~~by~~ by the diagnolization theorem, the columns of P are linearly independent eigen vectors of A and D has diagonal entries of e-values of A that align with P.
As shown in part c, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are linearly indepent e-vectors of A, so

$P = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ To find P^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{r_1+2r_2+2r_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$P^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ To find D, we know $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an e-vector $\lambda_1 = 1$, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is an e-vector of $\lambda_2 = \frac{1}{2}$
$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is an e-vector of $\lambda_2 = \frac{1}{2}$. Means $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 f) $A^4 = [T^4]_B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

Since A is diagonal,

$A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$

$A^4 = PD^2P^{-1} \cdot PD^2P^{-1}$

$A^4 = PD^4P^{-1}$

$$A^4 = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & (\frac{1}{2})^4 & 0 \\ 0 & 0 & (\frac{1}{2})^4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As $D^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$

$-2 \times \frac{1}{16}$

$$A^4 = \begin{bmatrix} 1 & -\frac{1}{8} & -\frac{1}{8} \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{15}{8} & \frac{15}{8} \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$$

$T^4 = A^4 [T]_B$

$\frac{30}{4}$

$\frac{12}{4} - \frac{15}{}$

Since $\begin{bmatrix} 1 & \frac{15}{8} & \frac{15}{8} \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{8} \\ -\frac{1}{4} \end{bmatrix}$ $\boxed{T^4 = -\frac{3}{4} + \frac{1}{8}t - \frac{1}{4}t^2}$

$T^4 = -\frac{33}{4} \quad \frac{1}{8}t - \frac{1}{4}t^2$

Optional: For $T^k(3+2t-4t^2)$, $A^k = [T^k]_B$ Ignore!

$$A^k = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\frac{1}{2})^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & -2(\frac{1}{2})^k & -2(\frac{1}{2})^k \\ 0 & (\frac{1}{2})^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 2-2(\frac{1}{2})^k & 2-2(\frac{1}{2})^k \\ 0 & (\frac{1}{2})^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{bmatrix}$$ let $2-2(\frac{1}{2})^k = K'$

$A^k = [T^k]_B = (a_0 + (2-K')a_1 + (2-K')a_2 + (\frac{1}{2})^k a_1 t + (\frac{1}{2})^k a_2 t^2$

$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)$

Plugging in $a_0 = 3$, $a_1 = 2$, $a_2 = -4$ ~~Ignore!~~ gives us $T^k(3+2t-4t^2)$:

~~$3 + (2-k')\cdot 2 + (2-k')\cdot -4 + \left(\frac{1}{2}\right)^{k-1} t + \left(\frac{1}{2}\right)^{k-2} t^2$~~

$T^k(3+2t-4t^2) = 3 + 4 - 2k' - 8 + 4k' + \left(\frac{1}{2}\right)^{k-1} t + \left(\frac{1}{2}\right)^{k-2} t^2$

$= -1 - 6k' + \left(\frac{1}{2}\right)^{k-1} t + \left(\frac{1}{2}\right)^{k-2} t^2$

where $k' = 2 - 2\left(\frac{1}{2}\right)^k$

Generalizing t to T^k

3 $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$ For a to be diagonalizable, it must have 3 linearly independent eigenvectors.

To find eigen values, must find $\det(A-\lambda I) = 0$

$A - \lambda I = \begin{bmatrix} 1-\lambda & a & b \\ 0 & 1-\lambda & c \\ 0 & 0 & 2-\lambda \end{bmatrix}$ $\det(A-\lambda I) = ((1-\lambda)^2 (2-\lambda)) = 0$ when $\lambda_1 = 1$, $\lambda_2 = 2$.

To find eigenvectors of $\lambda_1 = 1$

$c + db = 0$
$-db$
$c = -db$

$A - I = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} = \vec{0} \rightarrow \left[\begin{matrix} ax_2 + bx_3 = 0 \\ cx_3 = 0 \\ x_3 = 0 \end{matrix} \right.$, $ax_2 = 0$, since, don't know $-\frac{b}{a}x_3$

$a, x_2 = 0$

$\vec{X} = \begin{bmatrix} x_1 \\ \cancel{x_2} \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$

For $\lambda_2 = 2$

$A - 2I = \begin{bmatrix} -1 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix} = \vec{0}$ $-x_1 + ax_2 + bx_3 = 0$. $x_1 = ax_2 + bx_3$

$-x_2 + cx_3 = 0$ $x_2 = cx_3$

$\vec{X} = \begin{bmatrix} acx_3 + bx_3 \\ cx_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} ac+b \\ c \\ 1 \end{bmatrix}$

3 (cont) However, we currently only have 2 dependent e-vectors. To get a third, a, b, or c must have such a value that can introduce a 3rd eigen-vector. Let a = 0, this transforms A - I into

$$\begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} = 0, \text{ meaning } x_3 = 0 \text{ and}$$

$$\text{null}(A-I) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

This gives us our three e-vectors of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} b \\ c \\ 1 \end{bmatrix}$, as $a = 0$. To verify independence

$$r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r_3 \begin{bmatrix} b \\ c \\ 1 \end{bmatrix} = \vec{0}, \quad r_1, r_2, r_3 \neq 0.$$

$$\begin{bmatrix} r_1 & 0 & r_3 b \\ 0 & r_2 & r_3 c \\ 0 & 0 & r_3 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad r_3 = 0, \text{ meaning } r_2, r_1 = 0.$$

Therefore the 3 eigen vectors are linearly independent for $b, c \in \mathbb{R}$.

This means for A to be diagnolizable, $a = 0$, but b and c can be any value.

4 If AB is diagnolizable, then $AB = PDP^{-1}$, where D is diagonal. Since A is invertible $A \cdot A^{-1} = I_n$. Since A is invertible, $A^{-1}(AB = PDP^{-1})A \to B = A^{-1}PDP^{-1}$

This means $BA = (A^{-1}PDP^{-1}A)$. Restructing the product as $BA = (A^{-1}P)D(P^{-1}A)$.

For BA to be diagnalizable $A^{-1}P$ and $P^{-1}A$ must be inverses. To prove:

$A^{-1}P \cdot P^{-1}A = I_n \to A^{-1} \cdot I \cdot A = I_n$, $A^{-1}A = I_n$, $I_n = I_n$

$P^{-1}A \cdot A^{-1}P = I \to P^{-1} \cdot I_n \cdot P = I_n$, $P^{-1}P = I_n$, $I_n = I_n$

This means that $P^{-1}A$ and $A^{-1}P$ are invertible, and that BA is diagnolizable.