MATH235 HOMEWORK 8 SOLUTION

- 5.1.7

Proof. (1). By construction φ is continuous and non-decreasing. Notice that x is also continuous and strictly increasing, then $g(x)$ is continuous and strictly increasing. Since g is defined on a closed interval, it is strictly monotone. The inverse h is also a continuous bijection and suppose $x, y \in [0,2]$ with $x > y$, then we have

$$x = g(g^{-1}(x)) > g(g^{-1}(y)) = y.$$

Since g is strictly increasing, we obtain $g^{-1}(x) > g^{-1}(y)$, which gives strictly increasing.

(2). Notice that C is a compact set. Under continuous function compact sets remains compact, hence $g(C)$ is a compact subset of $[0,2]$. Notice that φ is a constant function on $[0,1] \setminus C$, hence $g([0,1] \setminus C)$ is in the form $x + c_n$ for constants c_n. Since $|[0,1] \setminus C| = 1$, $|g([0,1] \setminus C)| = 1$. Since g is a bijection, we have $|g(C)| = |[0,2]| - |g([0,1] \setminus C)| = 1$.

(3). Since $N \subseteq g(C)$, we have $h(N) \subseteq (h \circ g)(C) = C$, hence $A \subseteq C \subset [0,1]$ and by monotonicity of Lebesgue measure $|A| = 0$. Hence A is Lebesgue measurable (sets with Lebesgue meausre 0 is Lebesgue measurable).

(4). Consider the set $\{f \circ h > 0\}$, if $f \circ h$ is measurable then the set should also be measurable, but the set is A, which is not measurable. Therefore $f \circ h$ is not measurable. $\square$

- 5.2.21

Proof. If $m^*(A) = \infty$, then there is nothing to prove, assume otherwise.

Suppose $m^*(A) < \infty$, then there exists an open set O such that $\forall \varepsilon > 0$, $A \subseteq O$ and $|O \setminus A| < \varepsilon$.

Since O is an open set, O is union of countable disjoint interval (a_i, b_i) for $i = \{1, 2, ..., n\}$ and $(a_i, b_i) \in E$.

It is given that f is Lipschitz on E. And $a_i, b_i \in E$.

Define partition

$$\Gamma_i = \{a_i = x_1, x_2, ..., x_m = b_i\},$$

then we have

$$|f(x_{j+1}) - f(x_j)| < K_i(x_{j+1} - x_j).$$

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Compute that

$$S_{\Gamma_i} = \sum_{j=1}^{m-1} |f(x_j) - f(x_{j-1})| \le \sum_{j=1}^{m-1} K_i |x_{j+1} - x_j|$$
$$= K_i \sum_{j=1}^{m-1} |x_{j+1} - x_j|$$
$$= K_i(b_i - a_i).$$

Therefore, as (a_i, b_i) are disjoint, we obtain that

$$V[f; a_i, b_i] \le K_i(b_i - a_i).$$

Thus,

$$|f(A)|_e = \sum_{i=1}^{n} V[f, a_i, b_i] \le \sum_{i=1}^{n} K_i(b_i - a_i)$$
$$\le \max(K_i) \sum_{i=1}^{n} (b_i - a_i)$$
$$= K|A|_e,$$

where $K = \max(K_i)$.

Hence, we have proved that $|f(A)|_e \le K|A|_e$ for every set $A \subseteq E$. □

- 5.2.22

Proof. y symmetry property of f, it suffices to consider f restricted to $[0, 1]$.

(a) " $\Rightarrow$ " It is clear that f is continuous at $x = 0$. Compute that

$$f'(x) = ax^{a-1} \sin \frac{1}{x^b} - bx^{a-b-1} \cos \frac{1}{x^b}, \quad \forall x \in (0, 1].$$

By Lemma 5.2.9, it suffices to show that $f' \in L^1$. If $a > b$, then we have that

$$\int_0^1 |f'(x)|\, dx = \int_0^1 \left| ax^{a-1} \sin \frac{1}{x^b} - bx^{a-b-1} \cos \frac{1}{x^b} \right| dx$$
$$\le \int_0^1 \left| ax^{a-1} \sin \frac{1}{x^b} \right| dx + \int_0^1 \left| bx^{a-b-1} \cos \frac{1}{x^b} \right| dx$$
$$\le \int_0^1 \left| ax^{a-1} \right| dx + \int_0^1 \left| bx^{a-b-1} \right| dx$$
$$= \int_0^1 ax^{a-1} dx + \int_0^1 bx^{a-b-1} dx \le 1 + \frac{b}{a-b} < \infty.$$

" $\Leftarrow$ " Consider the partition defined by

$$\mathcal{P}_m = \{x_n\}_{n=1}^m = \left\{ \left(\frac{2}{(2n+1)\pi} \right)^{\frac{1}{b}} \right\}_{n=1}^m .$$

Observe that
$$\sin\frac{1}{x_n^b}=\begin{cases}1 & n \text{ even}\\ -1 & n \text{ odd}\end{cases}$$
and hence
$$f\left(x_n\right)=\begin{cases}x_n^a & n \text{ even}\\ -x_n^a & n \text{ odd}\end{cases}$$
Now,
$$\begin{aligned}\sum_{n=1}^{m}\left|f\left(x_n\right)-f\left(x_{n-1}\right)\right| &= \sum_{n=1}^{m}\left|(-1)^n\left(x_n^a+x_{n-1}^a\right)\right|\\ &= \sum_{n=1}^{m} x_n^a+x_{n-1}^a = 2\sum_{n=1}^{m-1} x_n^a + x_m + x_0\\ &\geq \sum_{n=1}^{m-1} x_n^a = \sum_{n=1}^{m-1}\left(\frac{2}{(2n+1)\pi}\right)^{\frac{a}{b}}.\end{aligned}$$
Note that the hidden power series
$$\lim_{m\to\infty}\sum_{n=1}^{m-1}\left(\frac{2}{(2n+1)\pi}\right)^{\frac{a}{b}}<\infty$$
converges iff. $a>b$.

(b) Assume $0\leq x<y\leq 1$. If $x=0$, the estimate is simple, we have
$$|f(y)-f(0)|\leq y^b\leq y^\alpha$$
Now we follow the hint. Consider $0<x<y\leq 1$ and set $h=y-x$. If $x^{b+1}<h$, note that we have inequalities
$$x^b<h^\alpha \quad \text{and } y^b=(x+h)^b<\left(h^{\frac{1}{b+1}}+h\right)^b<Ch^\alpha$$
where C is a real constant depending on b. Then
$$|f(y)-f(x)|\leq|f(y)|+|f(x)|\leq y^b+x^b<(1+C)h^\alpha$$
If $x^{b+1}\geq h$, MVT yields a $t\in(x,y)$, such that
$$|f(y)-f(x)|=h\left|f'(t)\right|\leq\frac{2bh}{t}\leq\frac{2bh}{x}\leq 2bh^\alpha.$$
(c) For any $0<\alpha<1$, set $a=b=\frac{\alpha}{1-\alpha}$ in the definition of f. Then, by (a) and (b), we have $f\in C^\alpha[-1,1]$ but f does not have bounded variation.

□

- 5.4.5

Proof. Let $I\subseteq\mathbb{R}$ be an interval and f is monotonically increasing on I. We can consider $I_k=I\cap[k,k+1]$ and each I_k is a bounded interval in $\mathbb{R}$. Take any I_k and assume it has endpoints a_k,b_k. Then $f'(x)$ exists a.e. on (a_k,b_k). Then we deduce that $f'(x)$ exists a.e. on $I\setminus Z$, $Z=\{a_k,b_k\}_{k\in\mathbb{Z}}$. Since Z has measure 0, we have the desired result. □

- 5.4.6

Proof. Assume $D^+ f \geq \delta > 0$ on (a, b). Since f is continuous it achieves its maximum on any closed interval $[x, y] \subset (a, b)$. Suppose f achieves its maximum at some x_0 where $x \leq x_0 < y$, then $f(s) \leq f(x_0)$ for all $s \in [x, y]$. Then we have

$$D^+ f(x_0) = \limsup_{t \to x_0^+} \frac{f(t) - f(x_0)}{t - x_0} \leq 0,$$

which is a contradiction. Now consider $D^+ f(x) \geq 0$ on (a, b). Consider $g(x) = f(x) + \delta x$ for any $\delta > 0$. Then we have

$$D^+ g(x) = D^+ f(x) + \delta.$$

From previous claim, $g(x)$ is monotonically increasing on (a, b). Then we know if $x, y \in (a, b)$ then $g(x) \geq g(y)$, which also gives

$$f(y) - f(x) \geq \delta(x - y).$$

Setting $\delta \to 0^+$ we have f is monotonically increasing on (a, b). □

- 5.5.19

Proof. (a). There exists a G_δ set $H \supset A$ such that $|A|_e = |H| > 0$. Then we have

$$D_A(x) = \lim_{r \to 0} \frac{|A \cap B_r(x)|_e}{|B_r(x)|} = \lim_{r \to 0} \frac{|H \cap B_r(x)|}{|B_r(x)|} = D_H(x) = 1$$

for a.e. x. Notice that there exists some $Z \subset \mathbb{R}^d$ with measure 0 and $H = A \cup Z$, we conclude $D_A = 1$ for a.e. x.
(b) . Suppose A is measurable, then for a.e. $x \notin A$ we have

$$D_A(x) = \lim_{r \to 0} \frac{|A \cap B_r(x)|}{|B_r(x)|} = \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_B \chi_A = \chi_A = 0.$$

Suppose $D_A(x) = 0$ for a.e. $x \notin A$. Consider $Z \subseteq A^C$ satisfies $|Z| = 0$ and for all $x \in A^C \setminus Z$, $D_A(x) = 0$. We claim that $A^C \setminus Z$ is measurable. Assume $x \in A^C \setminus Z$, then

$$\lim_{r \to 0} \frac{|A \cap B_r(x)|_e}{|B_r(x)|} = 0.$$

Let $\epsilon > 0$, there exists $A \subseteq U$ open set such that $|A|_e \leq |U| \leq |A|_e + \epsilon$. Then we have

$$\lim_{r \to 0} \frac{|A \cap B_r(x)|_e}{|B_r(x)|} \leq \lim_{r \to 0} \frac{|U \cap B_r(x)|}{|B_r(x)|} \leq \lim_{r \to 0} \frac{|A \cap B_r(x)|_e}{|B_r(x)|} + \epsilon,$$

which gives $D_U(x) \leq D_A(x) + \epsilon$ and therefore $D_U(x) = 0$ for a.e. $x \notin A$. Hence $A^C \setminus Z$ is measurable which gives A^C and A are measurable.
Consider a set where $D_E(x)$ does not exist. Define $E_n = (\frac{1}{2n!}, \frac{1}{(2n-1)!})$ and $E = \cup E_n$. Consider two sequences $\{a_n = \frac{1}{(2n-1)!}\}$ and $\{b_n = \frac{1}{(2n)!}\}$. Then

$$\frac{|E \cap B_{a_n}(x)|_e}{|B_{a_n}(x)|} = \frac{1/(2n)! - 1/(2n+1)!}{2/(2(n-1)!)} > 0$$

$$\frac{|E \cap B_{b_n}(x)|_e}{|B_{b_n}(x)|} = \frac{1/(2n-1)!}{2/(2(n-1)!)} \to 0$$

Hence $D_E(x)$ does not exist. On the other hand, consider a set where $D_E(x)$ attains any $\alpha \in (0, 1)$. Defince $\theta = 2\pi\alpha$ and consider E to be a slice of circle in $\mathbb{R}^2$ with θ radians. □

- 5.5.20

  Proof. Suppose $E \subseteq [0,1]$ is measurable and there exists some $\delta > 0$ such that $|E \cap [a,b]|_e \geq \delta(b-a)$ for all a, b. Then
  $$D_E(x) = \lim_{r\to 0} \frac{|E \cap B_r(x)|_e}{|B_r(x)|} = \lim_{r\to 0} \frac{|E \cap (x-r, x+r)|}{2r} \geq \delta > 0.$$
  Hence, $D_E = 0$ for a.e. $x \notin E$. Then we have $|[0,1] \setminus E| = 0$ which gives $|E| = 1$. $\square$

- 5.5.21

  Proof. Suppose $f \in L^1[0,1]$ is such that $\int_E f = 0$ for all measurable $E \in [0,1]$ with $|E| = \lambda$. Consider $x \leq y \in [0,1]$ be two Lebesgue points of f and $\delta = \min\{y-x, 1-y, \lambda, 1-\lambda\}$. We claim that there exists $F \subseteq ([x, x+\delta] \cup [y, y+\delta])^C$ with $|F| = \lambda - \delta$. Consider $E_1 = [x, x+\delta] \cup F$ and $E_2 = [y, y+\delta] \cup F$. Then
  $$\int_{E_1} f = \int_{[x,x+\delta]} f + \int_F f = 0 = \int_{E_2} f = \int_{[y,y+\delta]} f + \int_F f$$
  which implies
  $$\int_x^{x+\delta} f = \int_y^{y+\delta} f.$$
  Since x, y are Lebesgue points,
  $$f(x) = \lim_{\delta\to 0^+} \int_x^{x+\delta} f = \lim_{\delta \to 0^+} \int_y^{y+\delta} f = f(y).$$
  This tells that $f(x) = K$ for some constant K a.e. on $x \in [0,1]$. By the assumption $\int_E f = 0$ for all measurable $E \in [0,1]$ with $|E| = \lambda$, we have $K = 0$. $\square$