1. Math 65, Exam I, Fall 2021.

SHOW ALL YOUR WORK Proofs should be written neatly. You must state the definitions and results you use and include all details to get full credit.

Please sign your exam to confirm that you completed your work without books, notes or external help.

Question 1.1. *(20 points) Show that $(P \Rightarrow Q) \Rightarrow R$ and $(P \wedge \neg Q) \vee R$ are two logically equivalent propositions. Hint: Show they have the same truth table.*

Question 1.2. *(20 points)*

(a) *Give an example of three sets A, B, C such that $C \subseteq A \cup B$ and $C \nsubseteq A$, and $C \nsubseteq B$ or show that no such sets exist.*
(b) *Give an example of sets A, B a function $f : A \to B$ and subsets $B_1, B_2 \subseteq B$ such that $f^{-1}(B_1 \cap B_2) \neq f^{-1}(B_1) \cap f^{-1}(B_2)$ or show that no such sets and function exist.*

Question 1.3. *(20 points)*

(a) *Give a proof or counterexample for each of the two statements below.*
(b) *Write the negation of each statement so that the negation appears as a $\neq$ sign in the equation.*
   - $\exists x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ *such that* $2x + y = 3y + 1$
   - $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}$ *such that* $xy = 0$

Question 1.4. *(20 points) Prove that $\forall n \in \mathbb{N}, n < 2^n$.*

Question 1.5. *(20 points) Consider the function*

$$\begin{array}{rccc} g: & \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \times \mathbb{Z} \\ & (a, b) & \to & (2a + b, b) \end{array}$$

(a) *Define what it means for a function $f : A \to B$ to be one-to-one or injective.*
(b) *For the g given above, is this function injective (one-to-one)?*
(c) *Define what it means for a function $f : A \to B$ to be onto or surjective.*
(d) *For the g given above, is this function onto?*

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2. Answers

Question. Show that $(P \Rightarrow Q) \Rightarrow R$ and $(P \wedge \neg Q) \vee R$ are two logically equivalent propositions. Hint: Show they have the same truth table.

Answer 1. We will write each step in the sentence $(\neg(P \vee Q)) \Rightarrow R$ and $P \vee (Q \vee R)$ and check that the truth tables for both are the same:

P	Q	R	P⇒ Q	(P⇒Q)⇒R	¬Q	P∧¬Q	(P∧¬Q)∨ R)
T	T	T	T	*T*	F	F	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	F	T
F	F	F	T	F	T	F	F

Question. (a) Give an example of three sets A, B, C such that $C \subseteq A \cup B$ and $C \nsubseteq A$, and $C \nsubseteq B$ or show that no such sets exist.

(b) Give an example of sets A, B a function $f : A \to B$ and subsets $B_1, B_2 \subseteq B$ such that $f^{-1}(B_1 \cap B_2) \neq f^{-1}(B_1) \cap f^{-1}(B_2)$ or show that no such sets and function exist.

Answer 2. (a) Take

$$A\{1, 2\},\ B = \{3, 4\},\ C = \{1, 3\},$$

Then $C = \{1, 3\} \subseteq A \cup B = \{1, 2, 3, 4\}$ and $C = \{1, 3\} \nsubseteq A = \{1, 2\}$, and $C = \{1, 3\}, \nsubseteq B = \{3, 4\}$..

(b) For all sets A, B, function $f : A \to B$ and subsets $B_1, B_2 \subseteq B$ it is always true that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$. We can prove this as follows: let $a \in f^{-1}(B_1 \cap B_2)$. By definition of inverse image, this means that $f(a) \in B_1 \cap B_2$. Therefore.again by definition of inverse image $a \in f^{-1}(B_1), a \in f^{-1}(B_2)$. Then, by definition of intersection, $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$.

Conversely, assume that $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$. By definition of intersection, this means that $a \in f^{-1}(B_1), a \in f^{-1}(B_2)$. By definition of inverse image, this is equivalent to saying that $f(a) \in B_1,\ f(a) \in B_2$. Therefore, $f(a) \in B_1 \cap B_2$.

Question. (a) Give a proof or counterexample for each of the two statements below.

(b) Write the negation of each statement so that the negation appears as a $\neq$ sign in the equation.

- $\exists x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ such that $2x + y = 3y + 1$
- $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ such that $xy = 0$

Answer 3. (a)
- $\exists x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ such that $2x + y = 3y + 1$. The equation can be written as $2x - 2y = 1$ or $2(x - y) = 1$. This equation is never satisfied as 1 is odd. Therefore, the statement $\exists x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ such that $2x + y = 3y + 1$ is false.
- $\forall x \in \mathbb{Z}\ \ \exists y \in \mathbb{Z}$ such that $xy = 0$ is a true statement as for any integer x, choosing $y = 0$, we obtain $xy = 0$.

(b) We already wrote the two negations above. Remember that the negation of " for all x, $p(x)$" is " ∃. some x such that no $p(x)$"

- The negation of the first statement is true (as the statement was false):

$$\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z} \quad 2x + y \neq 3y + 1$$

- The negation of the second statement is false (as the statement was true):

$$\exists x \in \mathbb{Z} \text{ such that } \forall y \in \mathbb{Z} \quad xy \neq 0$$

Question. Prove that $\forall n \in \mathbb{N}, n < 2^n$.

Answer 4. We want to prove by induction that $\forall n \in \mathbb{N}, n < 2^n$. We first check two base cases $n = 0$: $n = 0 < 2^0 = 1$, this works.

For $n = 1$: $n = 1 < 2^1 = 2$, this works.

Assume now that $n < 2^n$ and $n \geq 1$. We want to show the corresponding result for $n+1$, namely $n + 1 < 2^{n+1}$.

Using the assumption for n, we obtain $n + 1 < 2^n + 1$. Therefore, using that $1 \leq n$ and the assumption $n < 2^n$:

$$n + 1 < 2^n + 1 \leq 2^n + n < 2^n + 2^n = 2 \times 2^n = 2^{n+1}$$

So the result for $n + 1$ holds and by the principle of induction, it holds for every n..

Question. Consider the function

$$\begin{array}{rccc} f: & \mathbb{Z} \times \mathbb{Z} & \rightarrow & \mathbb{Z} \times \mathbb{Z} \\ & (a, b) & \rightarrow & (2a + b, b) \end{array}$$

(a) Define what it means for a function $f : A \rightarrow B$ to be one-to-one or injective.
(b) For the g given above, is this function injective (one-to-one)?
(c) Define what it means for a function $f : A \rightarrow B$ to be onto or surjective.
(d) For the g given above, is this function onto?

Answer 5. (a) The function f is one to one if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.
(b) We want to show that g is injective. Assume that $g(a, b) = g(c, d)$. Then $(2a + b, b) = (2c + d, d)$. Hence, $2a + b = 2c + d, b = d$. As $b = d, 2a = 2c$. Hence, $a = c, b = d$ and therefore $(a, b) = (c, d)$ proving the injectivity.
(c) The function f is onto if for all $b \in B$, there is at least one a in A such that $f(a) = b$.
(d) The map is onto if and only if for each $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ there exists $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $g((a, b)) = (x, y)$. From the definition of g, this is equivalent to $2a + b = x, b = y$ or $2a = y - x$. This equation does not have a solution if $y - x$ is odd. So, he function is not onto .