Tufts University
Department of Mathematics

Math 235 | **Homework #3 : Due on 10/6**[1] | **Fall, 2023**

All sections referred to Heil's Introduction to real analysis.

1. **Reading assignment**

   1.1 Sections 3.1–3.3

2. **Problems: The problems with bolded numbers will be collected for grading for both completeness and correctness. However, credit will also be given for attempts that are fully justified.**

   2.1 **These questions are independent and will be graded.**

   2.1.1 Let S be the set of irrationals in $[0,1]$. Prove that S has positive exterior Lebesgue measure but S contains no open intervals.

   2.1.2 Let $S = (0,1)^2$ be the open unit square in $\mathbb{R}^2$. Prove that there does not exits a collection of disjoint open ball $\{B_j\}_{j\in J}$ whose union is S.

   2.1.3 Using only the definition, find the Lebesgue measure of $A = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.

   2.1.4 Let $A \subset [0,1]$. Prove that A is Lebesgue measurable if and only if $|A|_e + |[0,1] \setminus A|_e = 1$.

   2.2 **This problem will be graded.** Let $\mathcal{M}$ be a nonempty collection of subsets of a set X. We say that $\mathcal{M}$ is a monotone class if given sets $E_n, F_n \in \mathcal{M}$ such that $E_1 \subset E_2 \subset \ldots$ and $F_1 \supset F_2 \supset \ldots$, we have both $\cup_n E_n \in \mathcal{M}$ and $\cap_n F_n \in \mathcal{M}$. Prove the following statements.

   2.2.1 Every $\sigma-$algebra on X is a monotone class.

   2.2.2 If $\mathcal{A}$ is a nonempty collection of subsets of X, then there exists a smallest monotone class $\mathcal{M}$ that contains $\mathcal{A}$.

   2.2.3 If $\mathcal{A}$ is a nonempty collection of subsets of X, if $\mathcal{M}$ is the smallest monotone class that contains $\mathcal{A}$, and if Σ is the $\sigma-$algebra generated by $\mathcal{A}$, then $\mathcal{A} \subset \mathcal{M} \subset \Sigma$. This inclusion can be proper.

   2.3 3.1.14, 3.1.15, **3.1.16,** 3.1.17, **3.1.18, 3.1.19.**