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Math 87 HW 2
Due 2/09/22

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1. A bakery wants to sell forty five Valentine's Day gift bags. They have decided to offer two types of bags: Bags of type A will contain four of cupcakes and two cookies, and bags of type B will contain two cupcakes and five cookies. Baskets of type A will be sold for \$12 and baskets of type B will be sold for \$16. The bakery has 90 cookies and 115 cupcakes in total.

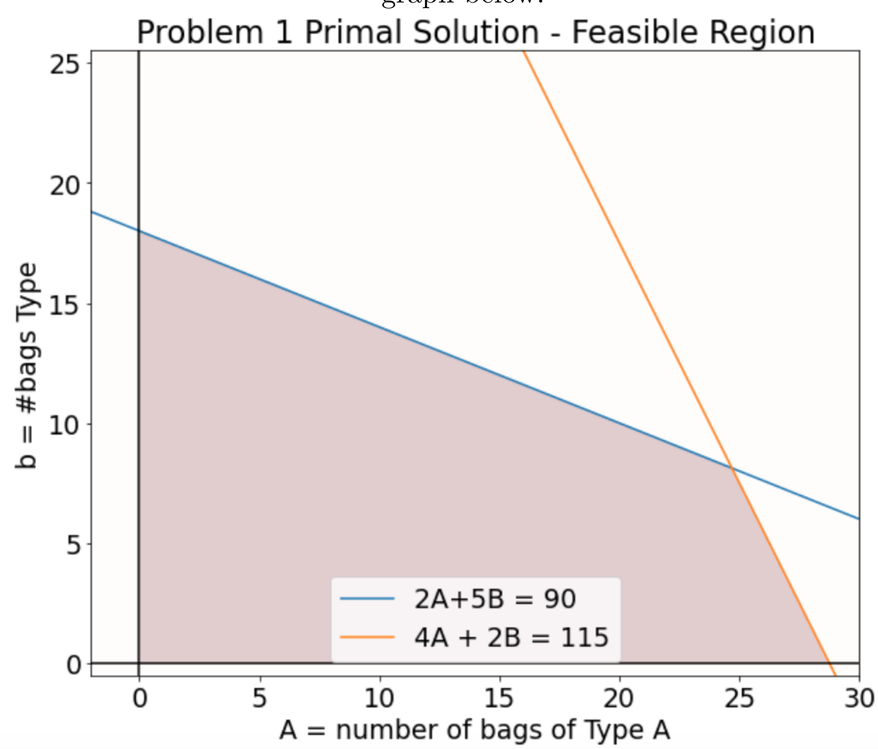
   a) Write the bakery's optimization problem as a primal problem. Solve this to determine how many baskets of both types should be made. If a fractional solution is obtained, round down to whole number solutions. What is the maximum profit? You may solve this by drawing the feasible region or using python.

   Let profit be $P(A, B)$, and the goal is to maximize the objective function $P(A, B) = 12A + 16B$, we have the following constraints:

$$2A + 5B \leq 90 \tag{1}$$

$$4A + 2B \leq 115 \tag{2}$$

(a) The overlapping region (A is on x axis, B is on y axis) is the feasible region in the graph below.



The corners of the feasible region are at the points p_1 =(24.688, 8.125), p_2 = (28.75, 0), p_3 = (0, 18), and p_4 = (0, 0) Since told to round down to nearest whole number, p_1 = (24, 8) and $p_2 = (28, 0)$, and total profit under each point is the following:

$$P(p_1) = 416, P(p_2) = 336, P(p_3) = 288, P(p_4) = 0$$

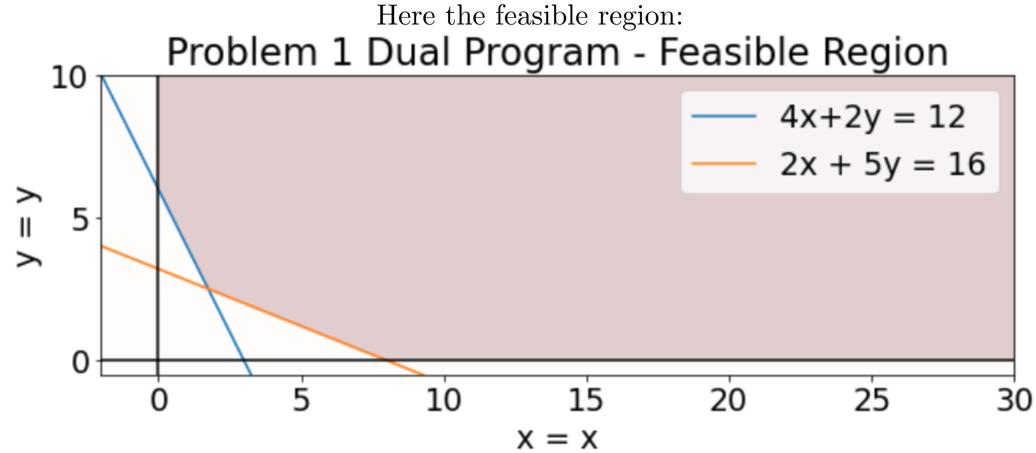
The bakery maximizes profit if they produce at p_1 meaning they should make 24 baskets of Type A and 8 baskets of Type B.

(b) Write down and solve the dual linear program. You may solve this by drawing the feasible region or using python.

To get the dual linear program, we want to take max cookies and cupcakes, which is 115 and 90 respectively, and minimize the function $P_2(x, y) = 115x + 90y$. The new constraints are:

$$4x + 2y \geq 12 \text{ and } 2x + 5y \geq 16$$

Here the feasible region:



The points that border the edge of the feasible region are (6, 0), (2.5,1.75), and (0, 8). Rounding down and evaluating them under the $P_2(x, y)$, gives us $P_2(6, 0) = 690$,$P_2(1, 2) = 295$, and $P_2(0, 8) = 720$ so the minimum is \$295 at x = 1, y = 2

2. You've decided to build a doomsday shelter under your house. You have a barrel which can store seven gallons of food, and you decide to fill it with rice and dried beans. You estimate that each gallon of beans will provides enough nutrition for approximately 9 days of meals, whereas each gallon of rice only provides around 5 days (Are these at all realistic? Asking for my doomsday shelter). Each gallon of beans costs \$12 and each gallon of rice costs \$5. You have \$60 to spend, and would like to calculate how many gallons of rice and beans to buy in order to maximize the number of days your food stores will last (fractional purchases are allowed).

   a) Write this problem as a dual linear program.
   I derived the primal equation in part b first, then got the dual program. Let x_1 and x_2 be the variables. The objective function being minimized is $7x_1 + 60x_2$. The constraints for the dual linear program are:

$$x_1 + 12x_2 \geq 9 \tag{3}$$

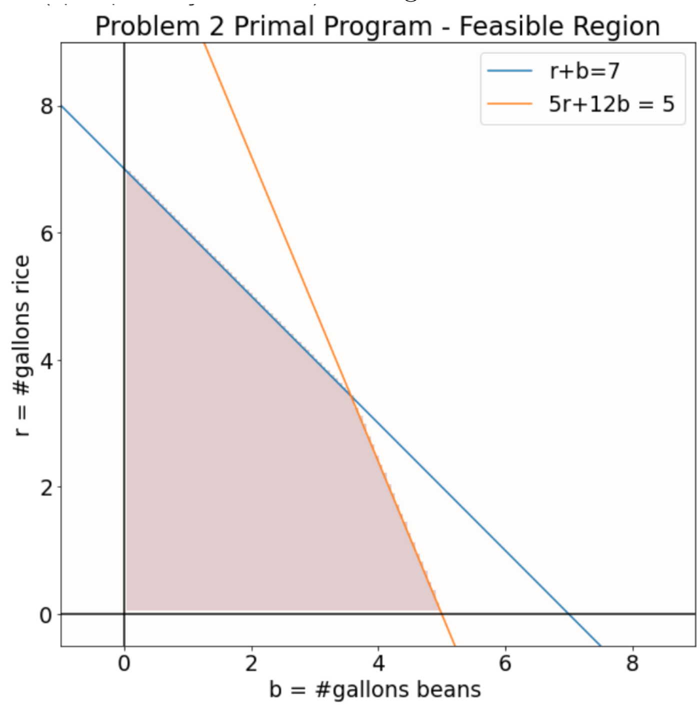
$$x_1 + 5x_2 \geq 5 \tag{4}$$

(b) Find the solutions to both the primal and the dual linear programs by plotting the feasible sets. Confirm that both the strong duality theorem and complementary slackness are satisfied. Write out the dual prices for each of your primal constraints.
For the primal equation the objective function is $S(r, b) = 9b + 5r$. First, let's write out the primal constraints:

$$r + b \leq 7 \tag{5}$$

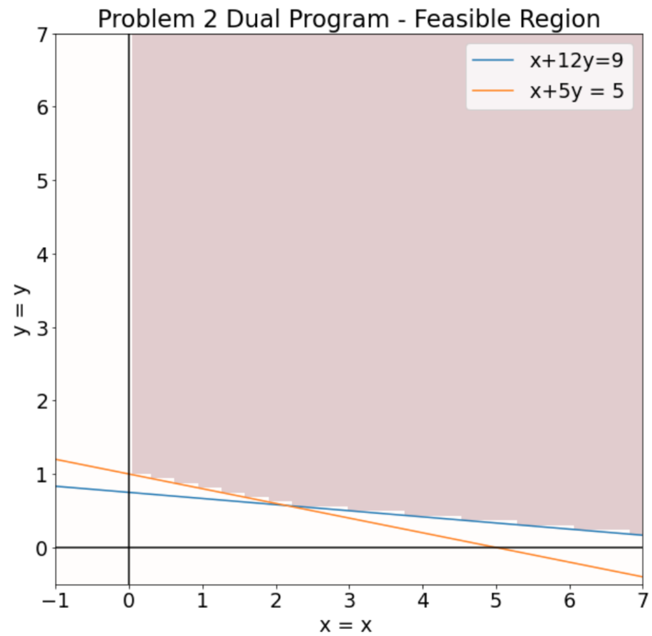
$$12b + 5r \leq 60 \tag{6}$$

feasible region



We can find the intersection of the boundaries to be (b, r) = $(3.571, 3.429)$, $(5, 0)$, $(0, 7)$, $(0, 0)$ and the maximum under S(b, r) is 49.284 at $(3.571, 3.429)$.

For dual program, using constraints from part a, get the following feasible region: (x_1 on x axis, x_2 on y axis)



The corners of the region are (0, 1), (2.143, 0.571), (9, 0) and have the minimum of 49.284 at $(2.1429, 0.5714)$

The solution satisfies strong duality as the maximal value under the primal constraints, is equivalent to the minimum under the dual program.

We also have to show complementary slackness. If we have a linear program defined as $Ax \leq b$ and also have a dual program, with feasible points x and y respectively, x and y are optimal if these two conditions are satisfied: $(b - Ax)^T \cdot y = 0$ and $(y^T A - c) \cdot x = 0$

We can plug in values and get that the variables above are:
A $= \begin{bmatrix} 1 & 1 \\ 12 & 5 \end{bmatrix}$, b $= \begin{bmatrix} 7 \\ 60 \end{bmatrix}$, x $= \begin{bmatrix} 3.571 \\ 3.429 \end{bmatrix}$, y $= \begin{bmatrix} 2.1429 \\ 0.5174 \end{bmatrix}$, and c $= \begin{bmatrix} 9 \\ 5 \end{bmatrix}$
For both of these, more precise values for x and y were used. If not, you get answers slightly above 0, which can be attributed to lack of precision when finding the optimal solution.
$(b - Ax)^T \cdot y = 0 = 0$ $(y^T A - c) \cdot x = 0 = 0$
Lastly, the dual price for constraint $r + b \leq 7$ is 2.1429 and the dual price for the constraint $12b + 5r \leq 60$ is 0.5714.

(c) Suppose you can increase the size of your barrel to hold an additional c gallons of food. Does the dual price for this modified constraint provide an accurate prediction of the increase in the primal objective function (i.e. the number of days of nutrition)? Answer this question for c = 1, 2, 4, 6.

We are maximizing the objective function 9b+5r, and as derived in part 2, the dual price is 2.1429 and the max for c =0 is 49.284:

c	New constraint	Dual price	Dual price max days survived	Actual max days survived	diff
1	$r + b \leq 8$	2.1429	51.433	51.428	0.005
2	$r + b \leq 9$	2.1429	53.5759	53.572	0.0039
4	$r + b \leq 11$	2.1429	57.8556	57.856	0.0004
6	$r + b \leq 13$	2.1429	62.1414	60	2.1414

Using the dual price to predict the new maximum of the object function is very close to the actual answer. It is also possible that some solutions could be equivalent to the predicted value, but the calculations don't show this due to lack of numerical precision. Using the dual price from $c = 0$ breaks down at $c = 6$, which makes sense, as for the other values of c, storage was the limiting factor to how much rice could be bought, but for c = 6, the limiting factor is price, as we have 13 gallons of storage, but can only buy 12 gallons of rice.