Practice with Abstract Vector Spaces

Math 70

October 22, 2020

Sometimes it can hard to wrap our heads around these abstract vector spaces (with polynomials, functions, or matrices). So let's practice and try to get comfortable.

(1) Consider two different polynomial vector spaces $\mathbb{P}_2$ and $\mathbb{P}_4$.

(a) Is $\mathbb{P}_2$ a vector subspace of $\mathbb{P}_4$? Yes!

(b) Find a spanning (or generating) set for $\mathbb{P}_2$. $\{1, t, t^2\}$

(c) Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_4$ described by

$$T(a_0 + a_1 t + a_2 t^2) = (a_1 + a_2) + a_1 t + 3a_0 t^4.$$

Describe the kernel and range of T. What is a spanning set for the range?

Kernel: $a_1 + a_2 = 0$, $a_1 = 0$
$3a_0 = 0$
$\Rightarrow a_0 = 0,\ a_1 = 0,\ a_2 = 0$
So kernel $= \{0\}$.

Range: Span$\{1, t, t^4\}$

(2) Consider the vector space of finitely supported signals $\mathbb{S}_f$, that is all bi-infinite sequences $\{y_k\}$, where only finitely many of the y_k are nonzero. Consider the map $T : \mathbb{S}_f \to \mathbb{R}$, where

$$T(\{y_k\}) = \sum_{i=-\infty}^{\infty} y_k.$$

(a) Is T a linear transformation? Yes

(b) What is the kernel of T? Do you think it would be possible to find a spanning set for the kernel? Kernel $= \{ \{y_k\} : \sum y_k = 0 \}$ ← sum of all nonzero y_k is equal to zero | The spanning set will be infinitely generated

(3) Consider the transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ described by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix}.$$

Is this transformation linear? Yes! (Taking derivatives is a linear transformation)

(4) Consider the space V of all linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$.

(a) Is V is vector space? Why or why not? Yes! Define $(T_1 + T_2)(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x})$

(b) Using what we know about standard matrices for linear transformations, can you think of another way to describe V?

All linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ can be described by their standard matrices (2×2 matrices $\to$ in $M_{2\times 2}$)

$M_{2\times 2}$ = space of all 2×2 matrices is also a vector space and is "the same" as V.

↑ isomorphic