# Homework 2

Early problem due on Gradescope at 8 pm on Tuesday, January 31st.
Due on Gradescope at 8 pm on Friday, February 3rd.

(1) (Early problem) Define two points $(x_0, y_0)$ and $(x_1, y_1)$ of the plane to be equivalent if $y_0^2 + x_0^2 = y_1^2 + x_1^2$. Check that this is an equivalence relation and describe the equivalence classes.

(2) Here is a "proof" that every relation $C$ that is both symmetric and transitive is also reflexive: "Since $C$ is symmetric, $aCb$ implies $bCa$. Since $C$ is transitive, $aCb$ and $bCa$ together imply $aCa$, as desired." Find the flaw in this argument.

(3) Let $f : X \to Y$ be a surjective function. Let us define a relation on $X$ by setting $x_1 \sim x_2$ if

$$f(x_1) = f(x_2).$$

(a) Show that this is an equivalence relation.
(b) Let $X/\sim$ be the set of equivalence classes. Show that there is a bijective correspondence of $X/\sim$ with $Y$.

(4) Let $\prec$ be the relation on $\mathbb{R}$ given by $x \prec y$ if $x^2 < y^2$, or if $x^2 = y^2$ and $x < y$. (We gave this as an example in class. It's Example 7 in the text.) Prove that $\prec$ is an order relation.

(5) Prove the following theorem:

**Theorem 1.** *If an ordered set $A$ has the least upper bound property, then it has the greatest lower bound property.*

(6) Let $A = A_1 \times A_2 \times \cdots$ and $B = B_1 \times B_2 \times \cdots$.
(a) Show that if $B_i \subseteq A_i$ for all $i$, then $B \subseteq A$. (Strictly speaking, if we are given a function mapping the index set $\mathbb{Z}_{>0}$ into the union of the sets $B_i$, we must change its range before it can be considered as a function mapping $\mathbb{Z}_{>0}$ into the union of the sets $A_i$. We shall ignore this technicality when dealing with cartesian products.)
(b) Show that the converse of (a) holds if each $B_i$ is non-empty.
(c) Show that if $A$ is nonempty, each $A_i$ is non-empty. Does the converse hold?
(d) What is the relation between the set $A \cup B$ and the cartesian product of the sets $A_i \cup B_i$? (i.e., is there a containment in one direction or an equality? Do you need to assume any sets are non-empty to get that equality or containment?) What is the relation between the set $A \cap B$ and the cartesian product of the sets $A_i \cap B_i$?