Math 125 **Problem Set 2** **Due: Thursday 10/6 at 11:59 p.m. EST**

Instruction: Read the assignment policy. For problems 1(d) and 5, include a printout your code with your homework submission. You should submit your assignment on Gradescope.

1. This question concerns the condition number of a problem.

(a) Let $f(x) = \ln(x)$. Find the condition number for $x > 0$. Discuss for what values of x the problem is ill-conditioned i.e. the condition number is very large.

(b) What is the condition number for $f(x) = \frac{x}{x-1}$ at x? Where is it ill-conditioned?

(c) Finding the roots of polynomials is typically a ill-conditioned problem. Consider a polynomial $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ of degree n. Let x_j denote the j-th root of $P(x)$. Find the condition number of x_j with respect to perturbations of a single coefficient a_i.

(d) The Wilkinson polynomial is defined as $P(x) = (x-1)(x-2)\dots(x-19)(x-20)$. Consider the expanded version of the polynomial

$$\begin{aligned} P(x) = {} & x^{20} - 210x^{19} + 20615\,x^{18} - 1256850\,x^{17} + 53327946\,x^{16} - 1672280820x^{15} \\ & + 40171771630\,x^{14} - 756111184500\,x^{13} + 11310276995381\,x^{12} \\ & - 135585182899530\,x^{11} + 1307535010540395\,x^{10} - 10142299865511450\,x^{9} \\ & + 63030812099294896\,x^{8} - 311333643161390640\,x^{7} + 1206647803780373360\,x^{6} \\ & - 3599979517947607200\,x^{5} + 8037811822645051776\,x^{4} - 12870931245150988800\,x^{3} \\ & + 13803759753640704000\,x^{2} - 8752948036761600000\,x^{1} + 2432902008176640000 \end{aligned}$$

Using a numerical solver of your choice, find all the roots after the perturbation of the coefficient x^{19} from -210 to $-210 - 2^{-23}$. Find the relative error of each computed root and present your result in a tabular form. Using your result in (c), explain why computing certain roots are more ill-conditioned than others.

2. Let $\tilde{f}$ be a backward stable algorithm to compute problem f. Prove that

$$\frac{|\tilde{f}(x) - f(x)|}{|f(x)|} = O(\kappa(x)\epsilon),$$

where $\kappa(x)$ denotes the relative condition number for computing $f(x)$ and ϵ denotes machine precision. Interpret the result in brief words.

3. Assume that $f(x)$ has two continuous derivatives, monotonically increasing, convex and has a root. A function is convex if $f''(x) > 0$ for all x. Prove that the root is unique and Newton's method will converge to it from any initial point.

4. In this problem, we use Newton's method to do division.

(a) Show how the reciprocal $\frac{1}{x}$ can be computed iteratively using Newton's method. Find an iterative formula in a way that requires at most two multiplications, one addition or subtraction, and no divisions.

(b) Take x_k to be the estimate of $\frac{1}{x}$ during the k-th iteration of Newton's method. If we define $\epsilon_k \equiv x\, x_k - 1$, show that $\epsilon_{k+1} = -\epsilon_k^2$.

(c) Approximately how many iterations of Newton's method are needed to compute $\frac{1}{x}$ within d binary decimal points? Write your answer in terms of ϵ_0 and d, and assume $|\epsilon_0| < 1$.

(d) Is this method always convergent regardless of the initial guess of $\frac{1}{x}$?

5. This question concerns root finding using the bisection, Newton and secant methods.

(a) Implement Newton's method. For each of the following equations, use your implementation to approximate the root to eight correct decimal places.

(1) $x^5 + x = 1$

(2) $\sin(x) = 6x + 5$

(3) $\ln(x) + x^2 = 3$

(b) Implement the secant Method. Use your implementation with starting guesses $x_0 = 0, x_1 = 1$ to find the root of $f(x) = x^3 + x - 1$.

(c) Implement the bisection method. Use your implementation to find the root of the equation $e^x = \sin(x)$ closest to 0.

Remark: For each of the above problems, your result should include a printout of the iterates.

6. Determine whether a fixed point iteration of $f(x)$ is locally convergent to the given fixed point r.

(a) $f(x) = (2x-1)^{\frac{1}{3}}, r = 1$

(b) $f(x) = \dfrac{x^3+1}{2}, r = 1$

(c) $f(x) = \sin(x) + x, r = 0$

7. Which of the following three fixed point iterations converge to $\sqrt{5}$? Rank the ones that converge from fastest to slowest.

(a) $x \rightarrow \dfrac{4}{5}x + \dfrac{1}{x}$

(b) $x \rightarrow \dfrac{x}{2} + \dfrac{5}{2x}$

(c) $x \rightarrow \dfrac{x+5}{x+1}$

8. Extra Credit: Assume that $f(x)$ is twice continuously differentiable for all x in some interval I. Assume that $f(r) = 0$ and x_0, x_1 are sufficiently close to the root r. Prove that the secant method converges to the root. What is the order of convergence?