Differential equation $y'(t) = f(t, y(t))$ ①

We consider numerical solutions to first order differential equations.

Initial value problem

Example $y'(t) = cy(1-y)$

This is the logistic equation

$y \equiv$ population as a ratio of the carrying capacity of habitat

When y is near 0, rate of change is small Ⓐ

When y is near 1, rate of change is small Ⓑ

Exercise Interpret Ⓐ and Ⓑ

To solve ①, we need the initial condition $y(0)$.

Initial value Problem

$$y' = f(t, y)$$
$$y(a) = y_a$$
$$t \text{ in } [a, b]$$

In many cases, finding exact solutions is hard. We resort to numerical methods.

Euler's method

$t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_n$

Let $h = t_i - t_{i-1}$

Assume we know $y(t_{i-1})$. How can we estimate $y(t_i)$?

Let's do Taylor expansion about t_{i-1}. Assume y'' exists and is continuous.

$$y(t_i) = y(t_{i-1}) + h\, y'(t_{i-1}) + \frac{h^2}{2} y''(\xi)$$

$$\xi \in (t_{i-1}, t_i)$$

$$y'(t_{i-1}) = \frac{y(t_i) - y(t_{i-1})}{h} + \frac{h}{2} y''(\xi)$$

Let's approximate $y'(t_{i-1})$ as $y'(t_{i-1}) \approx \dfrac{y(t_i) - y(t_{i-1})}{h}$

Note that $y'(t_{i-1}) = f(t_{i-1}, y(t_{i-1}))$

Therefore, we can approximate $y(t_i)$ as follows

$$y(t_i) \approx y(t_{i-1}) + h\, f(t_{i-1}, y(t_{i-1}))$$

This is the idea of Euler's method

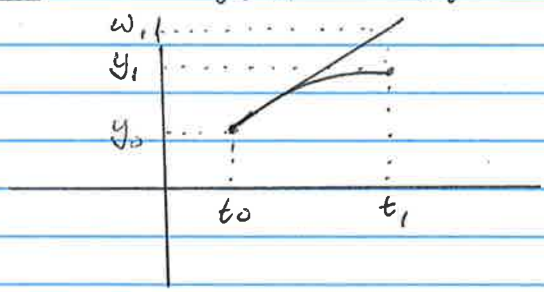
<u>Notation</u> $w_i \equiv$ Estimate of $y(t_i)$

$w_0 = y_0$

$w_1 = y_0 + h\, f(t_0, w_0)$

<u>Euler's method</u>

$$w_0 = y_0$$
$$w_{i+1} = w_i + h\, f(t_i, w_i)$$



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consider Taylor expansion of $y(t_{i-1})$ about $y(t_i)$

$$y(t_{i-1}) = y(t_i) - h\, y'(t_i) + \frac{h^2}{2} f''(\xi)$$
$$\xi \in (t_{i-1}, t_i)$$

As before, approximate $y'(t_i)$ as

$$y'(t_i) \approx \frac{y(t_i) - y(t_{i-1})}{h}$$

We can now approximate $y(t_i)$ in the following way:

$$y(t_i) \approx y(t_{i-1}) + h\, f(t_i, y(t_i))$$

<u>Backward Euler</u>

$$w_0 = y_0$$
$$w_{i+1} = w_i + h\, f(t_{i+1}, w_{i+1})$$

Note backward Euler is not implicit.

An import problem is under what condition an IVP admits a unique solution.

For this, we rely on Lipschitz continuity of $f(t, y)$

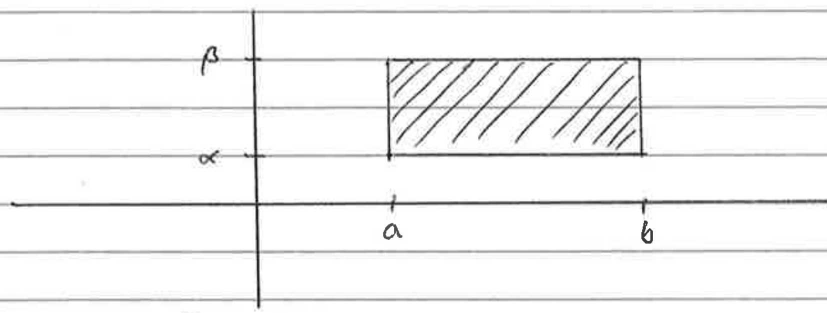
Def A function f is Lip. continuous in the variable y on $S = [a, b] \times [\alpha, \beta]$ if there exists a constant L such that

$$|f(t, y_1) - f(t, y_2)| \leq L\,|y_1 - y_2| \text{ for each } (t, y_1), (t, y_2) \in S$$

Example Find L for $f(t, y) = ty + t^2$ for $0 \leq t \leq 1$

$$|f(t, y_1) - f(t, y_2)| = |t y_1 - t y_2| \leq |t|\,|y_1 - y_2| \leq |y_1 - y_2|$$

Therefore, f is Lip. on $0 \leq t \leq 1$ and $-\infty \leq y \leq \infty$



* Assume f is continuously differentiable in y. For each fixed t, there is a c between y_1 and y_2 such that

$$\frac{f(t, y_1) - f(t, y_2)}{y_1 - y_2} = \frac{\partial f}{\partial y}(t, c)$$

$$L = \text{maximum of } \left|\frac{\partial f}{\partial y}(t, c)\right| \text{ on the set}$$

Here on $S = [a, b] \times [\alpha, \beta]$ and $\alpha < y_a < \beta$

Theorem Assume that $f(t, y)$ is Lip. continuous in y on S. Then there is $c \in (a, b)$ such that

$$y' = f(t, y)$$
$$y(a) = y_a$$
$$t \text{ in } [a, c]$$

has exactly one solution $y(t)$. Moreover, if f is Lip. continuous on $[a, b] \times [-\infty, \infty]$, then there exists exactly one solution on $[a, b]$.

Important Note solution only is guaranteed to exist on $[a, c]$

Example Show that there is a unique solution to the IVP $y' = 1 + t\sin(ty)$ $0 \le t \le 2$
$y(0) = 0$

Solution $\frac{\partial f}{\partial y}(t, y) = t^2 \cos(yt)$

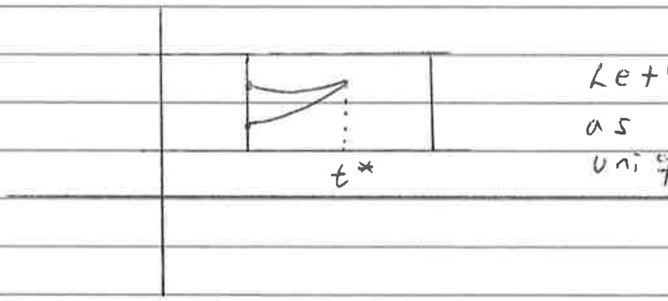
$|t^2 \cos(yt)| \le 4$ on $0 \le t \le 2$

$\therefore L = 4$

Therefore on $[0, 2] \times [-\infty, \infty]$, f is Lip. continuous in y. Unique solution exists.

* **Exercise** Do solution curves starting at different initial conditions cross?

Assume two solution curves intersect at $t^* \in (a, b)$



Let's use the intersection as initial value. Then, uniqueness theorem contradicts two solutions.

$Y(t) \equiv$ solution of IVP with $y_a = Y(a)$

$Z(t) \equiv$ solution of IVP with $y_a = Z(a)$

Assume f is Lip. in y on S

* CASE I $Y(a) = Z(a) \Longrightarrow Y(t) = Z(t)$. why?

$Y(t) \neq Z(t)$ in the interval $\Rightarrow$ why?

Define $u(t) = Y(t) - Z(t)$. WLOG, assume $u > 0$

$u'(t) = Y'(t) - Z'(t)$

$= f(t, Y(t)) - f(t, Z(t))$

Note $|f(t, Y(t)) - f(t, Z(t))| \le L\,|Y(t) - Z(t)| = L\,|u(t)|$

$u'(t) \le |u'(t)| \le L\,u(t)$ $= L\,u(t)$

$$\frac{|u'(t)|}{u(t)} \le L$$

$$\frac{u'(t)}{u(t)} \le L$$

$$[\ln(u(t))]' \le L$$

Apply mean value theorem to $f(t) = \ln(u(t))$

$$\frac{\ln(u(t)) - \ln(u(a))}{t-a} = f'(c) \le L$$

$$\ln \frac{u(t)}{u(a)} \le L(t-a)$$

$$u(t) \le u(a)\, e^{L(t-a)}$$

$$\Downarrow$$

$$\boxed{|Y(t) - Z(t)| \le |Y(a) - Z(a)|\, e^{L(t-a)}} \quad *$$

Exercise Interpret $*$

Error analysis

Global truncation error $\equiv g_i = |w_i - y_i|$

Local truncation error (LTE) $\equiv e_{i+1} = |w_{i+1} - z(t_{i+1})|$

$z(t_{i+1})$ is a solution for $y' = f(t, y)$, $y(t_i) = w_i$, t in $[t_i, t_{i+1}]$

Assume y'' is continuous

$$y(t_i + h) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2} y''(c) \qquad t_i \le c \le t_{i+1}$$

Note for LTE, $y(t_i) = w_i$

$$y(t_i + h) = w_i + h f(t_i, w_i) + \frac{h^2}{2} y''(c) \quad (2)$$

Euler $\quad w_{i+1} = w_i + h f(t_i, w_i) \quad (3)$

$$(2) - (3) \qquad y(t_{i+1}) - w_{i+1} = \frac{h^2}{2} y''(c)$$

$$|e_{i+1}| = \frac{h^2}{2} \| y''(c) \|$$

Assume $\| y''(c) \| \le M$ on $[a, b]$

$$|e_{i+1}| \le \frac{M h^2}{2}$$

Next, we analyze global error.

- $y(a) = y_a$

  $g_0 = |w_0 - y_0| = 0$

- $g_1 = |w_1 - y_1| = e_1$ why?

- Next consider g_2

$$\left.\begin{array}{l} y' = f(t, y) \\ y(t_1) = w_1 \\ t \text{ in } [t_1, t_2] \end{array}\right. \xRightarrow{\text{solution}} z(t_2)$$

$$e_2 = |w_2 - z(t_2)|$$

$$\begin{aligned} g_2 = |w_2 - y_2| &= |w_2 - z(t_2) + z(t_2) - y_2| \\ &\le \underbrace{|w_2 - z(t_2)|}_{e_2} + |z(t_2) - y_2| \\ &= e_2 \end{aligned}$$

Need to understand this

What is the difference between $z(t_2)$ and y_2? The initial conditions.

$\Rightarrow$ use our previous result

$$|z(t_2) - y_2| \le |w_1 - y_1|\, e^{Lh} = g_1 e^{Lh}$$

$$g_2 \le e_2 + e^{Lh} e_1$$

<u>Exercise</u>   Analyze g_3

$$\begin{aligned} g_3 = |w_3 - y_3| &= |w_3 - z(t_3) + z(t_3) - y_3| \\ &\le |w_3 - z(t_3)| + |z(t_3) - y_3| \\ &= e_3 + \underbrace{|z(t_3) - y_3|}_{\le |w_2 - y_2| e^{Lh}} \\ &\le e_3 + |w_2 - y_2|\, e^{Lh} \\ &= e_3 + g_2 e^{Lh} \\ &= e_3 + e^{Lh} e_2 + e^{2Lh} e_1 \end{aligned}$$

$$g_i = |w_i - y_i| \le e_i + e^{Lh} e_{i-1} + e^{2Lh} e_{i-2} + \dots + e^{(i-1)Lh} e_1$$

Assume $e_i \le c h^{k+1}$ for some c and k

$$\begin{aligned} g_i &\le c h^{k+1} \left(1 + e^{Lh} + \dots + (e^{(i-1)Lh})\right) \\ &= c h^{k+1} \frac{1 - e^{iLh}}{1 - e^{Lh}} \\ &\le c h^{k+1} \frac{e^{L(t_i - a)} - 1}{Lh} = \frac{c h^k}{L} \left(e^{L(t_i - a)} - 1\right) \end{aligned}$$

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$