Math 166 HW6

1

$$r = \frac{30\sum_{i=1}^{39} x_i y_i - \left(\sum_{i=1}^{39} x_i\right)\sum_{i=1}^{39} y_i}{\sqrt{30\sum_{i=1}^{39} x_i^2 - \left(\sum_{i=1}^{39} x_i\right)^2}\sqrt{30\sum_{i=1}^{39} y_i^2 - \left(\sum_{i=1}^{39} y_i\right)^2}}$$

$$r = \frac{30(7,807.36) - 1,300.69 \cdot 323}{\sqrt{30 \cdot 86754.6939^2 - 1300.69^2}\sqrt{30 \cdot 11,881 - 323^2}}$$

$r = -0.388$

$r = -0.388$ which indicates a weak negative correlation between bonuses and performance

2 Like from previous question, we have the equation for r listed above. Following the formula, we get that

$$r = \frac{36(7051.2633) - 994.77 \cdot 254.69}{\sqrt{36(28462.1047) - 994.77^2}\sqrt{36(1816.1417) - 254.69^2}}$$

$\boxed{r = 0.1148 \text{ is the } xy \text{ correlation}}$

3

$$E(X^2Y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y^2 f_{x,y}(x,y)\,dx\,dy$$

Change of variables $\xi = (x-\mu_x) + K(y-\mu_y)$

$\eta = (y-\mu_y) + \lambda(x-\mu_x)$

$$\begin{bmatrix} 1 & K \\ \lambda & 1 \end{bmatrix}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

Want X, Y in terms of η, ξ

Can multiply by inverse to get $\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$

So have $\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} = \frac{1}{1-\lambda k} \begin{bmatrix} 1 & -k \\ -\lambda & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$

From inverse of 2x2

$$\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} = \frac{1}{1-\lambda k} \begin{bmatrix} \xi - k\eta \\ -\lambda \xi + \eta \end{bmatrix}$$

$$x = \frac{1}{1-\lambda k}(\xi - k\eta) + \mu_x$$

$$y = \frac{1}{1-\lambda k}(-\lambda\xi + \eta) + \mu_y$$

Also need to get Jacobian to transform

and $|J| = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & k \\ \lambda & 1 \end{vmatrix} = 1-\lambda k$

Then have to transform $f_{x,y}(x,y)$

From notes, if we know that

$$f_{x,y}(x,y) = \frac{\sqrt{ac-b^2}}{\pi} \exp\left\{-[a(x-\mu_x)^2] + 2b(x-\mu_x)(y-\mu_y) + c(y-\mu_y)^2\right\}$$

Let $\mu = \frac{b^2}{a+c \pm 2\sqrt{ac-b^2}}$, then substituting in our values derived earlier for x, y, we get $f_{\xi\eta}(\xi,\eta) = \frac{\mu}{\pi}\exp[-\mu(\xi^2+\eta^2)]$

Via painful simplification

Our bounds are still $-\infty, \infty$ for η, and ξ

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3 So the integral is now: (let $\xi = x$, $\eta = y$)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{1}{1-\lambda k}(x-ky)+\mu_x\right]^2 \left[\frac{1}{1-\lambda k}(y-\lambda x)+\mu_y\right]^2 \cdot \frac{\nu}{\pi} \exp[-\nu(x^2+y^2)]\, dx\,dy$$

$$\left[\frac{1}{1-\lambda k}(x-ky)+\mu_x\right]\left[\frac{1}{1-\lambda k}(x-ky)+\mu_x\right]$$

$$= \left[\frac{(x-ky)}{1-\lambda k}\right]^2 + \frac{2\mu_x}{1-\lambda k}(x-ky) + \mu_x^2$$ and symmetric for other

and have:

$$\left[\frac{(y-\lambda x)}{1-\lambda k}\right]^2 + \frac{2\mu_y(y-\lambda x)}{1-\lambda k} + \mu_y^2$$

let $\frac{x-ky}{1-\lambda k} = a$ and $\frac{y-\lambda x}{1-\lambda k} = b$

So $(a^2 + 2\mu_x + \mu_x^2)(b^2 + 2\mu_y + \mu_y^2)$

$$= a^2b^2 + 2a^2\mu_y + a^2\mu_y^2 + 2\mu_x b^2 + 4\mu_x\mu_y + 2\mu_x\mu_y^2$$
$$+ \mu_x^2 b^2 + 2\mu_x^2\mu_y + \mu_x\mu_y^2$$

$$= \frac{\nu}{\pi}\iint_{\mathbb{R}^2}\left[a^2b^2 + 2a^2\mu_y + a^2\mu_y^2 + 2\mu_x b^2 + 4\mu_x\mu_y + 2\mu_x\mu_y^2 + \mu_x^2 b^2 + 2\mu_x^2\mu_y + \mu_x\mu_y^2\right]\exp[-\nu(x^2+y^2)]\,dx\,dy$$

I couldn't think of a method to easily simplify and solve this integral. Wolfram Alpha couldn't solve, and I think integration by parts for x, and y would solve. However, there really isn't a way to do it simply. To solve this, I could break the integral into 9 other integrals, but only the three w/o a, b would be easily solvable, as it can be treated as solving Gaussian integral twice

4 $Z_{\alpha} = \frac{K - \frac{n}{2}}{\sqrt{\frac{n}{4}}}$ where K is number of observations where $y_i > \mu_0$, as $\mu_0 = 13$ would be the number ws ≤ 20

as $\alpha = 0.05$, $1.64 = \frac{K - 11}{\sqrt{\frac{22}{4}}}$, so $K = 14.85$
$n = 22$

For power, want find $P(K \ge 14.85 \mid \mu = 11)$

We can first find $P(Y > 10 \mid \mu = 11) = P\left(\frac{Y-11}{6} > \frac{10-11}{6} \mid \mu = 11\right)$
$= P(Z > -0.17)$
$= 0.5675$

So when $\mu = 11$, Y is binomial w/ $n = 22$ and $p = 0.5675$

So we can standardise from above.

$P(Y_+ \ge 14.85 \mid \mu = 11) = P\left(\frac{Y_+ - 22(0.5675)}{\sqrt{22(0.5675)(0.4325)}} > \frac{14.85 - 22(0.5675)}{\sqrt{22(0.5675)(0.4325)}}\right)$

$= P(Z > 1.02) = 1 - 0.8461$
≈ 0.1539

where $Z = \frac{Y_+ - 22(0.5675)}{\sqrt{22p(1-p)}}$

$\boxed{\text{Power of the test is } 0.1539\text{, and for } H_0: \mu = 10,\ H_1: \mu > 10\text{, we will reject } H_0 \text{ if } \frac{K-11}{\sqrt{\frac{22}{4}}} > Z_{0.05}\text{, or } K > 14.85}$

5 For $n = 5$ we get the pdf as:

$= \frac{1}{32}\left(1 + e^{t} + e^{2t} + 2e^{3t} + 2e^{4t} + 3e^{5t} + 3e^{6t} + 3e^{7t} + 3e^{8t} + 3e^{9t} + 3e^{10t} + 2e^{11t} + 2e^{12t} + e^{13t} + e^{14t} + e^{15t}\right)$

The possible α level for $P(W \ge 15) = \frac{1}{32}$
for $P(W \ge 14) = \frac{2}{32}$, $P(W \ge 13) = \frac{3}{32}$ and so on by taking the sum of the coefficients.