A Quick Review of Order Statistics

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Outline

Cumulative distribution functions

- If $f_Y(y)$ is the *probability density function (PDF)* of random variable Y, we define the *cumulative distribution function (CDF)* of Y as

$$F_Y(y) := \text{Prob}(Y < y) = \int_{-\infty}^{y} dz\ f_Y(z).$$

- Note that, by the *Fundamental Theorem of Calculus*, it follows immediately that the PDF is given by

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

- Sometimes it is easier to figure out the CDF, and then differentiate to obtain the PDF.

Order statistics for $Y_{\max}$

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- Suppose that $\vec{y} = \{Y_1, Y_2, \ldots, Y_n\}$ are n i.i.d. random variables with by the continuous PDF $f_Y(y)$. What is the PDF of $Y_{\max} = \max_j Y_j$?
- Note that if $Y_{\max} < y$, then it must be that $Y_j < y$ for $j = 1, 2, \ldots, n$.

$$
\begin{aligned}
F_{Y_{\max}}(y) =& \text{Prob}(Y_{\max} < y) \\
=& \text{Prob}(Y_1 < y)\,\text{Prob}(Y_2 < y) \cdots \text{Prob}(Y_n < y) \\
=& F_{Y_1}(y) F_{Y_2}(y) \cdots F_{Y_n}(y) \\
=& \left[F_Y(y)\right]^n .
\end{aligned}
$$

- It follows that $f_{Y_{\max}}(y) = \frac{d}{dy} F_{Y_{\max}}(y)$, so

$$
\boxed{f_{Y_{\max}}(y) = n\left[F_Y(y)\right]^{n-1} f_Y(y).}
$$

Order statistics for Y_{min}

- Suppose that $\vec{y} = \{Y_1, Y_2, \ldots, Y_n\}$ are n i.i.d. random variables with by the continuous PDF $f_Y(y)$. What is the PDF of $Y_{\text{min}} = \min Y_j$?
- Note that if $Y_{\text{min}} < y$, then it must be that $Y_j > y$ for $j = 1, 2, \ldots, n$.

$$
\begin{aligned}
F_{Y_{\text{min}}}(y) =& \text{Prob}(Y_{\text{min}} < y) = 1 - \text{Prob}(Y_{\text{min}} > y)\\
=& 1 - \text{Prob}(Y_1 > y)\,\text{Prob}(Y_2 > y) \cdots \text{Prob}(Y_n > y)\\
=& 1 - [1 - F_{Y_1}(y)]\,[1 - F_{Y_2}(y)] \cdots [1 - F_{Y_n}(y)]\\
=& 1 - [1 - F_Y(y)]^n .
\end{aligned}
$$

- It follows that $f_{Y_{\text{max}}}(y) = \frac{d}{dy}\left\{1 - [1 - F_Y(y)]^n\right\}$, so

$$
\boxed{f_{Y_{\text{min}}}(y) = n\,[1 - F_Y(y)]^{n-1}\, f_Y(y).}
$$

Summary

- We reviewed the topic of *order statistics*.
- This is normally covered in probability courses.
- It is useful to us in assessing *unbiasedness* of certain estimators.