Tufts University
Department of Mathematics

Math 135 **Homework 6 v. 2**[1] **Fall 2022**

Readings for the weeks of October 10 and 17, 2022

§9.1: Cauchy sequences, tests for convergence of series
The convergence tests for series (pp. 230–239) is essentially review of Calc II.
§9.2: Pointwise convergence for sequences of functions (you may omit Examples 9.23, 9.24, 9.25, which involve differentiation and integration.)
§9.3: Uniform convergence of sequences of functions
§9.4: Uniform limit of functions (up to and not including Th. 9.34, p. 252)
Notes: The Comparison Test and the Weierstrass M-Test

Theorem 1 (Comparison test for real series)**.** *Let* $\{a_k\}$ *and* $\{b_k\}$ *be real sequences and assume* $0 \le a_k \le b_k$ *for all* $n \in \mathbb{N}$*. If* $\sum_{k=1}^{\infty} b_k$ *converges then* $\sum_{k=1}^{\infty} a_k$ *converges. (Therefore, if* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.)*

Theorem 2 (Divergence test)**.** *If* $\sum_{k=1}^{\infty} a_k$ *converges then* $\lim_{k\to\infty} a_k = 0$*. (Therefore, if the limit of the sequence* $\{a_k\}$ *is not zero (i.e., is a nonzero number or does not exist), then* $\sum_{k=1}^{\infty} a_k$ *diverges.)*

Theorem 3 (The p test)**.** *The p-series* $\sum_{k=1}^{\infty} 1/k^p$ *converges for* $p > 1$ *and diverges for* $p \le 1$*.*

Theorem 4 (Geometric series)**.** *The geometric series* $\sum_{k=0}^{\infty} ar^k$ *converges to* $a/(1-r)$ *for* $|r| < 1$ *and diverges otherwise.*

Theorem 5 (Ratio test)**.** *For the series* $\sum_{k=1}^{\infty} a_k$*, suppose that*

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \ell.$$

(i) *If* $\ell < 1$*, the series converges absolutely.*
(ii) *If* $\ell > 1$*, the series diverges.*

Theorem 6 (Comparison test for uniform convergence)**.** *Let* $\{f_n\}$ *be a sequence of functions from* D *to* $\mathbb{R}$*. If there is a sequence of constants* a_n *such that for all* $x \in D$*,* $|f_n(x) - f(x)| \le a_n$ *and* $\lim a_n = 0$*, then* f_n *converges to* f *uniformly on* D*.*

Theorem 7 (Weierstrass M-test)**.** *Let* $\{f_n\}$ *be a sequence of functions from* D *to* $\mathbb{R}$*. If the series* $\sum M_n$ *of constants converges and* $|f_n(x)| \le M_n$ *for all* n *and all* $x \in D$*, then the series* $\sum f_n(x)$ *converges uniformly on* D*.*

Problem Set 6
(Due Wednesday, October 26, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (10 points) **Elephant under the rug theorem**.
   This is a problem on old material, but it's a good one. Let $D \subset \mathbb{R}$. Suppose $f \colon D \to \mathbb{R}$ is continuous and $f(x_0) > 0$ at some point $x_0 \in D$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in D$ with $|x - x_0| < \delta$. (If an elephant gets under your rug, it will lift the rug up, not only where it is, but around it.)

2. (15 points) **Cauchy convergence criterion**.
   Suppose $\{x_n\}$ is a sequence of points in $\mathbb{R}$ satisfying the condition on the distance between *successive* points
   $$|x_{n+1} - x_n| < \frac{1}{2^n}.$$
   Show that the sequence $\{x_n\}$ converges.
   (*Hint*: If you decide to show that $|x_{n+k} - x_n|$ is small for large n and $k \in \mathbb{N}$, you end up with something related to a series, it can help to use the Cauchy Criterion given in Theorem 9.17 of the book for that series.)

3. (10 points) **Uniform convergence**.
   Assume $\{a_n\}$ is a bounded sequence of numbers and M is an upper bound, $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $r > 0$. Prove that the series
   $$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$
   converges uniformly on $[-r, r]$ to a continuous function.

4. (15 points) **Continuity of** ln.
   In Calculus II, you showed, for $x \in (-1, 1]$, that
   $$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}\ .$$
   (a) Show that this series converges uniformly on the interval $[-r, r]$ for any $r \in (0, 1)$. (You may use any series convergence test that is in our book.)
   (b) Prove that $\ln(1+x)$ is a continuous function on $(-1, 1)$.

5. (10 points) **Sum of uniformly convergent sequences**.
   Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions from $\mathbb{R}$ to $\mathbb{R}$. Assume $f_n \to f$ uniformly on $\mathbb{R}$ and $g_n \to g$ uniformly on $\mathbb{R}$. Prove that $f_n + g_n$ converges to $f + g$ uniformly on $\mathbb{R}$.

6. (15 points) **Uniform convergence**.
For each of the following sequences, find the limit as $n \to \infty$ and determine whether it converges pointwise or uniformly. Show your reasoning.

(a) $\dfrac{\sin x}{n}$ on $\mathbb{R}$ (b) $\dfrac{x}{nx+1}$ on $(0,1)$ (c) x^n on $[0, 0.999]$

(End of Problem Set 6)