Math 165 HW 8

1a) T = r.v for how much of atom is left
Want $P(T \geq 5)$ and half life = 1,
meaning $\frac{1}{2} = e^{-\lambda}$, $\lambda = \ln 2$, so $P(T \geq 5) = e^{-\ln 2 \cdot 5} = \left(\frac{1}{2}\right)^5$
$= \boxed{\frac{1}{32}}$

b) Using function from a, $.1 = e^{-\ln 2 \cdot t}$
$\frac{-\ln 10}{-\ln 2} = t$, $t = \frac{\ln 10}{\ln 2} \approx \boxed{3.32 \text{ years}}$

c) $1 = 1024 e^{-\ln 2 t}$
$\frac{1}{1024} = \left(\frac{1}{2}\right)^t$, $\rightarrow 2^{-10} = \left(\frac{1}{2}\right)^t$, $t = 10$
$\boxed{10 \text{ years}}$

d) Probability an atom lasts longer than 10 years is $\frac{1}{1024}$, so probability it's gone is $\frac{1023}{1024}$,
this 1024 times gives answer
$= \left(\frac{1023}{1024}\right)^{1024} = \boxed{0.3679}$

2a) mean = 10, $\frac{1}{\lambda} = 10$, $\lambda = \frac{1}{10}$ T = lifetime
$P(T > 20) = e^{-\frac{1}{10}(20)}$ from $e^{-\lambda t}$ $= e^{-2} = \boxed{0.135}$

b) median is $\frac{\ln 2}{\lambda}$, which is $10 \ln 2 = \boxed{6.9315}$

c) $SD = \frac{1}{\lambda} = \boxed{10}$

d) W_i = lifetime of i^{th} component. Want
$\frac{1}{100}\sum$ Density is $\frac{1}{10} e^{-\frac{1}{10}t} \cdot \frac{1}{10} t$ from definition of density function
To get sum want
$P(T > 11) = P(N_t \leq r) = \sum_{k=1}^{r} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ $t = 11$, $r = 1$

$= \boxed{\sum_{k=1}^{100} \frac{e^{-1.1}(1.1)^k}{k!}}$

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2e) We can approach e, using $\bar{w} = \frac{1}{2}\sum_{i=1}^{2} w_i$
but $\sum \bar{w}_i = 22$, and $\lambda = \frac{1}{10}$, $r = 2$ so have gamma dist.
and Integral becomes

$$P(\bar{w} > 11) = \int_{22}^{\infty} \frac{e^{-\frac{t}{10}}}{10} \cdot \frac{\left(\frac{t}{10}\right)^2}{2!} dt$$

$$= \boxed{0.6227}$$

3 If T_3 is when 3rd hit happens, we have gamma distribution with parameters $r = 3$ and $\lambda = 1$
Density function is:

$$f(t) = \frac{1^3 \cdot t^2}{\Gamma(3)} e^{-1t} = \frac{1}{2} t^2 \cdot e^{-t}$$

To find $P(2 \le T_3 \le 4) = \int_2^4 f(t)\,dt$

$$= \int_2^4 \frac{1}{2} t^2 e^{-t} dt$$

$$= \frac{1}{2} \int_2^4 t^2 e^{-t} dt$$

D	I
t^2	e^{-t}
$2t$	$-e^{-t}$
2	e^{-t}
0	$-e^{-t}$

$$= \frac{1}{2}\left(-t^2 e^{-t} - 2te^{-t} - 2e^{-t}\right)\Big|_2^4$$

$$= \frac{1}{2}\left(-e^{-t}(t^2 + 2t + 2)\right)\Big|_2^4$$

$$= \frac{1}{2}\left(-e^{-4}(26) + e^{-2}(10)\right)$$

$$= -e^{-4} \cdot 13 + e^{-2} \cdot 5 = \boxed{0.4386}$$

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4 a) M_{100} = transistor with mean 100hrs
M_{200} = transistor w/mean 200hrs

$P(X \geq 200) = P(X \geq 200 \mid M_{100})P(M_{100}) + P(X \geq 200 \mid M_{200})P(M_{200})$

$P(X \geq 200 \mid M_{100}) = e^{-\lambda t}, \ t = 200, \lambda = \frac{1}{100} = e^{-2}$

$P(X \geq 200 \mid M_{100}) = e^{-\lambda t}, \ t = 100, \lambda = \frac{1}{100} = e^{-1}$

$P(X \geq 200) = e^{-2}\left(\frac{1}{3}\right) + e^{-1}\left(\frac{2}{3}\right) = \boxed{0.29}$

b) $E(X) = \sum_{i=1}^{n} P(X_i)E(X_i)$

So $E(X) = \frac{1}{3}(100) + \frac{2}{3}(200) = \boxed{166.67\text{hrs}}$

c) $Var(X) = E(X^2) - (E(X))^2$

$E(X^2)$ we can find it using integral and weights so

$E(X^2) = \frac{1}{3}\int_0^\infty x^2 \cdot \frac{1}{100} e^{\frac{-x}{100}} dx + \frac{2}{3}\int_0^\infty x^2 \cdot \frac{1}{200} e^{\frac{-x}{200}}$

$E(X^2) = 60{,}000$ → solved using calculator, normally use integration by parts

$Var(X) = 60{,}000 - (166.67)^2$

$\boxed{Var(X) = 32{,}222.\overline{2}}$

5 a) $P(X \geq \frac{1}{2})$, since $F(x)$ is cdf at the at $x = \frac{1}{2}$, $(\frac{1}{8})$ of area below on $[0, \frac{1}{2}]$, meaning

$P(X \geq \frac{1}{2}) = 1 - \frac{1}{8} = \boxed{\frac{7}{8}}$

b) pdf $= \frac{d}{dx}$ cdf, $\boxed{\text{pdf} = 3x^2}$

c) $E(X) = \int_0^1 x \cdot 3x^2 = \int_0^1 3x^3 dx = \frac{3x^4}{4}\Big|_0^1 = \boxed{\frac{3}{4}}$

6 a) $y = \sqrt{t}$, meaning $\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$f_T(t) = \lambda e^{-\lambda t}$

$$f_Y(y) = \frac{f_T(t)}{\frac{dy}{dt}} = \frac{\lambda e^{-\lambda t}}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}\,\lambda e^{-\lambda t} \qquad t = y^2$$

$$\boxed{f_Y(y) = 2y\,\lambda e^{-\lambda y^2}}$$

b) $$E(y) = \int_0^\infty 2y^2 \lambda e^{-\lambda y^2}\,dy$$

$$= 2\lambda \int_0^\infty 2y^{2} e^{-\lambda y^2}\,dy \qquad y^2 = t, \quad 2y\,dy = dt$$

$$= 2\lambda \cdot \frac{1}{2} \int_0^\infty \sqrt{t}\, e^{-\lambda t}\,dt$$

$$= \lambda \underbrace{\int_0^\infty t^{\frac{3}{2}-1} e^{-\lambda t}\,dt}_{\frac{\Gamma(n)}{\lambda^n} \text{ and } n = \frac{3}{2}}$$

$$= \lambda \left(\frac{\Gamma(\frac{3}{2})}{\lambda^{3/2}}\right) = \frac{\Gamma(\frac{3}{2})}{\sqrt{\lambda}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{\sqrt{\lambda}}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$$

So $E(y) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$ for $\lambda = 3$, this becomes $\boxed{E(y) = 0.51}$