Solutions

Consider the relation on $\mathbb{R}^3 - \{(0,0,0)\}$ defined by

$(x, y, z) \sim (x', y', z') \iff$ there exists $c \in \mathbb{R} - \{0\}$ such that $(cx, cy, cz) = (x', y', z')$.

(1) Verify that $\sim$ is an equivalence relation.

$(x,y,z) \sim (x,y,z)$ by taking $c = 1$

$(x,y,z) \sim (x',y',z') \Rightarrow \exists c$ s.t. $c(x,y,z) = (x',y',z')$

$\Rightarrow \exists c$ s.t. $\frac{1}{c}(x',y',z') = (x,y,z)$

$\Rightarrow (x',y',z') \sim (x,y,z)$

$(x,y,z) \sim (x',y',z')$ and $(x',y',z') \sim (x'',y'',z'')$ $\Rightarrow \exists c,d \in \mathbb{R}-\{0\}$ s.t. $c(x,y,z) = (x',y',z')$, $d(x',y',z') = (x'',y'',z'')$

$\Rightarrow \exists c,d \in \mathbb{R}-\{0\}$ s.t. $cd(x,y,z) = (x'',y'',z'')$

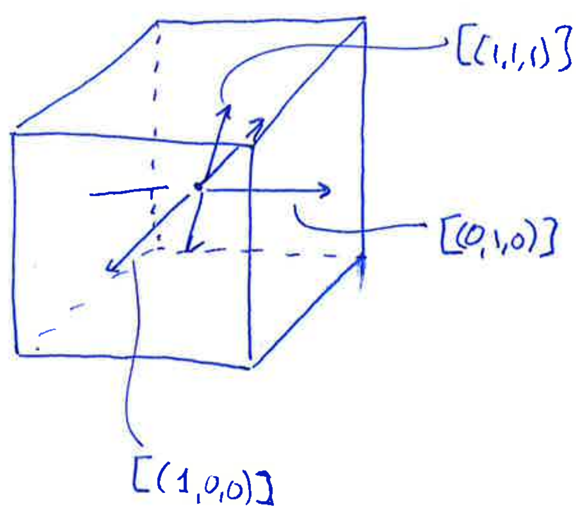
$\Rightarrow (x,y,z) \sim (x'',y'',z'')$

Definition 1. Denote the equivalence class of $(x, y, z) \in \mathbb{R}^3 - \{0\}$ by $[x : y : z]$. The set of equivalence classes of $\sim$ is called the **real projective plane** and denoted

$$\begin{aligned} \mathbb{RP}^2 &:= (\mathbb{R}^3 - \{0\})/\sim \\ &= \{[x : y : z] \mid (x, y, z) \in \mathbb{R}^3 - \{0\}\}. \end{aligned}$$

(In general, n-dimensional real projective space $\mathbb{RP}^n$ is $(\mathbb{R}^{n+1} - \{0\})/\sim$ where two $n+1$-dimensional vectors are equivalent if and only if one is a non-zero scalar multiple of the other.)

(2) What is the equivalence class of $(1, 0, 0)$ as a subset of $\mathbb{R}^3 - \{0\}$? What about $(1, 1, 1)$? $(0, 1, 0)$?



all are lines thru the origin

We consider $\mathbb{RP}^2$ as a topological space by giving it the quotient topology. Write

$$p : \mathbb{R}^3 - \{0\} \to \mathbb{RP}^2$$
$$(x, y, z) \mapsto [x : y : z]$$

for the quotient map.

(1) Denote by $D(z)$ the subset of $\mathbb{RP}^2$ defined by

$$D(z) = \{[x : y : z] \in \mathbb{RP}^2 \mid z \neq 0\}$$

(This is well-defined since the third coordinate of (x, y, z) is 0 if and only if the third coordinate of (cx, cy, cz) is 0 for a nonzero number c.)

Show that $D(z)$ is an open subset of $\mathbb{RP}^2$.

$p^{-1}(D(z)) = \{(x, y, z) \in \mathbb{R}^3 - \{0\} \mid z \neq 0\}$

is open in $\mathbb{R}^3 - \{0\}$.

$D(z)$ is open by definition of quotient topology.

(2) Write

$$\tilde{D}(z) := p^{-1}(D(z)) = \{(x, y, z) \in \mathbb{R}^3 - \{0\} \mid z \neq 0\}.$$

Show that the restriction $p_z : \tilde{D}(z) \to D(z)$ of p to $\tilde{D}(z)$ is a quotient map.

Let $U \subseteq D(z)$. U is open in $D(z)$

$\iff$ U is open in $\mathbb{RP}^2$

$\iff$ $p^{-1}(U)$ is open in $\mathbb{R}^3 - \{0\}$

$\iff$ $p_z^{-1}(U) = p^{-1}(U)$ is open in $\tilde{D}(z)$.

(3) Show that

$$\varphi : \mathbb{R}^2 \to D(z)$$
$$(x, y) \mapsto [x : y : 1]$$

is a homeomorphism with inverse

$$\psi : D(z) \to \mathbb{R}^2$$
$$[x : y : z] \mapsto \left(\frac{x}{z}, \frac{y}{z}\right).$$

(Hint: To show φ is continuous, write it as a composite of functions $\mathbb{R}^2 \to \tilde{D}(z) \to D(z)$. To show ψ is continuous, use the universal property of quotients.)

Let $\tilde{\varphi} : \mathbb{R}^2 \to \cancel{\mathbb{R}^3 - \{0\}}\ \tilde{D}(z)$
$(x,y) \mapsto (x,y,1)$.

$\varphi = p \circ \tilde{\varphi}$ so φ is continuous.

To use univ. prop. of quotients, we check $\psi \circ p$ is continuous and ψ is well-defined.

(well-def'd) suppose $[x:y:z] = [x':y':z']$ then $\exists c \in \mathbb{R} - \{0\}$ s.t. $c(x,y,z) = (x', y', z')$

$$\Rightarrow \left(\frac{x}{z}, \frac{y}{z}\right) = \left(\frac{cx}{cz}, \frac{cy}{cz}\right) = \left(\frac{x'}{z'}, \frac{y'}{z'}\right).$$

($\psi \circ p$ continuous) $\psi \circ p$ is $\tilde{D}(z) \to \mathbb{R}^2$
$(x,y,z) \mapsto \left(\frac{x}{z}, \frac{y}{z}\right)$ which is clearly continuous.

$\therefore$ ψ is continuous.

Lastly we check ψ, φ are inverses:

$\psi \circ \varphi$: $(x,y) \mapsto [x:y:1] \mapsto \left(\frac{x}{1}, \frac{y}{1}\right) = (x,y)$

$\varphi \circ \psi$: $[x:y:z] \mapsto \left(\frac{x}{z}, \frac{y}{z}\right) \mapsto \left[\frac{x}{z} : \frac{y}{z} : 1\right] = [x:y:z]$.

Remark 2. One can define similar open subsets $D(x)$ and $D(y)$ which are also homeomorphic to $\mathbb{R}^2$. Therefore $\mathbb{RP}^2$ has an open cover by open subsets homeomorphic to $\mathbb{R}^2$. We say that $\mathbb{RP}^2$ is a **topological manifold**.

(4) Any line L in $\mathbb{R}^2$ can be written in the form

$$L = \{(x,y) \in \mathbb{R}^2 \mid ax + by + c = 0\}$$

where $a, b, c \in \mathbb{R}$ and at least one of a, b is nonzero. Consider the set

$$\overline{L} = \{[x : y : z] \in \mathbb{RP}^2 \mid ax + by + cz = 0\}.$$

Show that $\overline{L}$ is well-defined and that $\varphi(L) = \overline{L} \cap D(z)$. How many points are in $\overline{L}$ but not L? (The notation comes from the fact that $\overline{L}$ is the closure of $\varphi(L)$ in $\mathbb{RP}^2$.)

well-defined: s'ps $[x:y:z]$ satisfies $ax+by+cz=0$.

Let $\gamma \in \mathbb{R}-\{0\}$. Then $[\gamma x : \gamma y : \gamma z]$ satisfies

$$a(\gamma x) + b(\gamma y) + c(\gamma z) = \gamma(ax+by+cz) = 0.$$

$$\varphi(L) = \{[x:y:1] \mid ax+by+c=0\}$$

$$\overline{L} \cap D(z) = \{[x:y:z] \mid ax+by+cz=0,\ z \neq 0\}$$

$$= \{[\tfrac{x}{z} : \tfrac{y}{z} : 1] \mid a(\tfrac{x}{z}) + b(\tfrac{y}{z}) + c = 0,\ z \neq 0\}$$

$$= \{[x:y:1] \mid ax+by+c=0\}$$

Points in $\overline{L}$ but not L are those w/ $z=0$: $[x:y:0]$ s.t. $ax+by=0$.

This is just one point: all such x, y s are scalar multiples of each other

(5) Let L_1 be the line in $\mathbb{R}^2$ defined by $x = 0$ and L_2 the line defined by $x = 1$. Form $\overline{L_1}$ and $\overline{L_2}$ as in the previous problem. Compute and compare $L_1 \cap L_2$ with $\overline{L_1} \cap \overline{L_2}$.

$L_1 \cap L_2 = \emptyset$ since L_1 and L_2 are parallel lines

$$\overline{L_1} = \{[x:y:z] \mid x=0\}$$

$$\overline{L_2} = \{[x:y:z] \mid x-z=0\}$$

$[0:1:0]$ is the unique point of intersection

(6) In fact, any algebraic variety in $\mathbb{R}^2$ extends in a nice way to $\mathbb{RP}^2$. Let $f(x,y) = \sum_{i,j} c_{ij}x^iy^j$ be a polynomial of degree d, i.e., the largest value of $i+j$ for which $c_{ij} \neq 0$ is d. Write

$$V = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; f(x,y) = \sum_{i,j} c_{ij}x^iy^j = 0 \right\}$$

for the variety it defines. Consider the set

$$\overline{V} = \left\{ [x:y:z] \in \mathbb{RP}^2 \;\middle|\; \sum_{i,j} c_{ij}x^iy^jz^{d-i-j} = 0 \right\}.$$

Show that $\overline{V}$ is well-defined and $\varphi(V) = \overline{V} \cap D(z)$.

$$\overline{V} \cap D(z) = \{ [x:y:z] \mid \sum c_{ij} x^i y^j z^{d-i-j} = 0,\ z \neq 0 \}$$

$\downarrow$ divide by z

$$= \{ [x:y:1] \mid \sum c_{ij} x^i y^j 1^{d-i-j} = 0 \}$$

$$= \varphi(V).$$

$[x:y:z] \in \overline{V}$

$\iff \sum c_{ij} x^i y^j z^{d-i-j} = 0$

$\iff c^d \left(\sum c_{ij} x^i y^j z^{d-i-j} \right) = 0 \qquad (c \neq 0)$

$\iff \sum c_{ij} (cx)^i (cy)^j (cz)^{d-i-j} = 0$

$\iff [cx:cy:cz] \in \overline{V}$