Math 145 Practicum Questions

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Question 1. Consider the two lines k and ℓ on the hexagon below. The symmetry group of the regular hexagon D_6 is generated by rotation of 60° and a reflection. Does it matter which of k or ℓ is used for that reflection?

Question 2. Let k, ℓ be two lines in the plane, r the reflection across k and s the reflection across ℓ. Find the order of rs in each of the following cases:

(a) k and ℓ intersect at angle $\frac{p}{q}\pi$ radians where p, q are positive integers

(b) k and ℓ intersect at any other angle

(c) k and ℓ are parallel

Question 3. (a) Fix a positive integer n and let $G_n = \{z \in \mathbb{C} \,|\, z^n = 1\}$. Show that G_n forms a group under multiplication of complex numbers.

(b) Now let n vary; show G is a group under multiplication complex numbers where:

$$G = \bigcup_{n=1}^{\infty} \{z \in \mathbb{C} \,|\, z^n = 1\}$$

Question 4. (a) Which of the following sets form a group under multiplication modulo 14:

- $\{1, 3, 5\}$
- $\{1, 3, 5, 7\}$
- $\{1, 7, 13\}$
- $\{1, 9, 11, 13\}$

(b) Show that if a subset of $\{1, 2, \ldots, 21\}$ contains an even number or the number 11 then it cannot form a group under multiplication modulo 22.

Question 5. Prove that if p is a prime number then the set $\{1, 2, \ldots, n-1\}$ with multiplication modulo p is a group.

Question 6. Let G be a group and $x, y, g \in G$

(a) Show that x and gxg^{-1} have the same order.

(b) Show that xy and yx have the same order.

Question 7. Let $G = \{x \in \mathbb{Q} \,|\, 0 \leqslant x < 1\}$ and define

$$x +_1 y = \begin{cases} x+y, & 0 \leqslant x+y < 1 \\ x+y-1, & x+y \geqslant 1 \end{cases}$$

Show G with $+_1$ is a infinite group and every element of G has finite order.

Question 8. (a) Check that rs and r^2s together generate D_n.

(b) Find all subgroups of D_4 and D_5.

Question 9. Draw a diagonal in a regular hexagon. List those plane symmetries of the hexagon which leave the diagonal fixed, and those which send the diagonal to itself. Show that both collections of symmetries are subgroups of the group of all plane symmetries of the hexagon.

Question 10. Let f be an isometry from the real line to itself which sends the integers to integers.

(a) Show that when f has no fixed points, f is a translation by an integer distance.

(b) Show that when f leaves exactly one point fixed, this point is either an integer or lies midway between two integers and f is a reflection across this fixed point.

(c) Show that when f fixes more than one point that it is the identity map.

Question 11. Express each of the following elements of S_6 as a product of disjoint cyclic permutations, and as a product of transpositions.

(a)
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 1 & 6 & 4 \end{bmatrix}$$

(b) $(1235)(2134)$

(c) $(135)(234)(246)(345)$

Question 12. Show that the elements of S_5 which send the numbers $1, 3, 5$ among themselves form a subgroup of S_5. What is the order of this subgroup? Then do the same for the elements of S_5 sending $2, 4$ among themselves.

Question 13 (Challenging)**.** Show that:

(a) When n is odd (123) and $(12\cdots n)$ generate A_n.

(b) When n is even (123) and $(23\cdots n)$ generate A_n.

Question 14. Label the vertices of the tetrahedron $\{1,2,3,4\}$. Check that each of the following are subgroups of the rotational symmetry group of the tetrahedron, and write as

(a) All rotational symmetries fixing the vertex 1 and the midpoint of vertices $2,3,4$.

(b) All rotational symmetries fixing the midpoint of vertices $1,2$ and the midpoint of vertices $3,4$.

(c) All rotational symmetries fixing vertex 4 and the midpoint of vertices $2,3$.

Show that each of these collections form a subgroup of the group of rotational symmetries of the tetrahedron.

Question 15 (6.7 Part 1)**.** For each of the following permutations $p \in S_n$ find two order-2 permutations $\alpha, \beta \in S_8$ such that $p = \alpha\beta$:

(a) $p = (1234)$

(b) $p = (12345)$

(c) $p = (123456)$

(d) $p = (1234567)$

(e) $p = (12345678)$

Is there a pattern or strategy that you can use for longer and longer cycles?

Question 16 (6.7 Part 2)**.** Show that if $p, q, r \in S_n$ are disjoint cycles and each is the product of two order-2 permutations then pqr is also the product of two order-2 permutations.

Together with the previous question this allows you to do Problem 6.7 from the book, that if $n \geqslant 4$ then every element of S_n is the product of two order-2 permutations.

Question 17. Let $P(x_1, \ldots, x_n) = (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_n)(x_2 - x_3) \cdots (x_{n-1} - x_n)$ and for $\alpha \in S_n$, $\alpha P(x_1, \ldots, x_n) = P(x_{\alpha(1)}, \ldots, x_{\alpha(n)}$. Show that if $\alpha, \beta, \gamma \in S_n$ satisfy $\alpha\beta = \gamma$, then $\alpha\beta P = \gamma P$. This will show that using P is a well-defined way of defining whether a permutation is even or odd.

Question 18. Show that S_n is generated by $\{(12), (23), \ldots, (n-1, n)\}$. Then show that it is generated by $\{(12), (123 \cdots n)\}$.

In this question I accidentally assigned something already proven in the book. I recommend trying to prove it yourself even if you've already read about it in the book, as practice. As a challenge you can try instead:

Find a generating set for the symmetry group of the cube.

Hint: You will need at least one reflection, let's say through the plane P. If S is your proposed generating set, then every symmetry needs to be written as a word in S, in particular every reflection. Check that if Q is the plane of a reflection that there is a word w in S such that $w(P) = Q$. If not, then S does not generate the whole symmetry group.