Math 166 HW 2

1a) $\mu_e = \frac{1}{n}\sum_{j=1}^{n} x_j$ for a normal distribution

Since $fn = 1$ and $(1-f)n = -1$,

$\mu_e = \frac{1}{n}(fn - (1-f)n) = 2f-1$

$\sigma_E = \sqrt{V_E}$ $\quad V_E = \frac{1}{n}\sum_{j=1}^{n}(x_j - \mu_e)^2$ x_j

$fn = 1$, and $(1-f)n$ meas. $= -1$,

So $V_e = \frac{1}{n}\left[(1-(2f-1))^2 fn - (1-f)n(-1-(2f-1))^2\right]$

$V_e = (2-2f)^2 f - (1-f)(-2f)^2$

$V_e = (4f - 8f^2 + 4f^3 + 4f^2 - 4f^3) = 4f - 4f^2$

$\sigma_E = \sqrt{4f-4f^2} = \boxed{2\sqrt{f(1-f)} = \sigma_E \quad 2f-1 = \mu_e}$

b) Want to find $P(\mu > 0) = 1-\alpha$

Since have normal distribution, $Z = \frac{\mu_e - \mu}{\sigma/\sqrt{n}}$ $\quad \frac{Z\sqrt{n}}{\sigma_1} = \mu_e - \mu$

Get $\mu = \mu_e - \frac{Z\sqrt{n}}{\sigma_E}$, want $\mu > 0$, so $0 \le \mu - \frac{Z\sqrt{n}}{\sigma_E}$,

$P\left(Z \le \frac{\sqrt{n}\mu_e}{\sigma_E}\right) = 1-\alpha$, $P\left(Z \ge \frac{\sqrt{n}\mu_e}{\sigma_E}\right) = \alpha$.

$\downarrow$ Z_α

$Z_\alpha = \frac{\sqrt{n}\mu_e}{\sigma_E}$

$n = \left(\frac{Z_\alpha^2 \sigma_E^2}{\mu_e^2}\right) = \boxed{\frac{4Z_\alpha^2(f-f^2)}{(2f-1)^2} \le n}$

c) Using 1.b, $f = 0.51$ $\quad Z_{.05} = -1.645$ $\quad (1-\alpha) \to (1-\alpha) = .95$, $\alpha = 0.05$

$$\frac{4(-1.645)^2(.51 - .51^2)}{(2(0.51)-1)^2} \le n$$

$6762.31 \le n$, so

n must be greater than or equal to 6763

2a)

Normalization. $\int_a^\infty \frac{1}{\mu-a} e^{-\left(\frac{y-a}{\mu-a}\right)} dy$ $\frac{y-a}{\mu-a} = z, \frac{dy}{\mu-a} = z$

$= \int_0^\infty e^{-z} dz = 1$

Theoretical mean $= \int_a^\infty \frac{y}{\mu-a} e^{-\left(\frac{y-a}{\mu-a}\right)} dy$

D	I
y	$e^{-\left(\frac{y-a}{\mu-a}\right)}$
-1	$-(\mu-a)e^{-\left(\frac{y-a}{\mu-a}\right)}$
0	$(\mu-a)^2 e^{-\left(\frac{y-a}{\mu-a}\right)}$

$\frac{1}{\mu-a}\left(-y(\mu-a)e^{-\left(\frac{y-a}{\mu-a}\right)} - (\mu-a)^2 e^{-\left(\frac{y-a}{\mu-a}\right)}\right)\Big|_a^\infty$

$= \frac{1}{\cancel{\mu-a}}\left(-a(\cancel{\mu-a}) + (\mu-a)^{\cancel{2}}\right) = (-a-(\mu-a)) = \mu = \text{mean}$

$Var(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \int_a^\infty \frac{y^2}{\mu-a} e^{-\left(\frac{y-a}{\mu-a}\right)} dy$

D	I
y^2	$e^{-\left(\frac{y-a}{\mu-a}\right)}$
$-2y$	$-(\mu-a)e^{-\left(\frac{y-a}{\mu-a}\right)}$
2	$(\mu-a)^2 e^{-\left(\frac{y-a}{\mu-a}\right)}$
0	$-(\mu-a)^3 e^{-\left(\frac{y-a}{\mu-a}\right)}$

$E(Y^2) = \frac{1}{\mu-a}\left[-y^2(\mu-a)e^{-\frac{y-a}{\mu-a}} - 2y(\mu-a)^2 e^{-\frac{y-a}{\mu-a}} - 2(\mu-a)^3 e^{-\left(\frac{y-a}{\mu-a}\right)}\right]$

$= \frac{1}{\mu-a}\left[-\left(-a^2(\mu-a) - 2a(\mu-a)^2 - 2(\mu-a)^3\right)\right]$

$= a^2 + 2a(\mu-a) + 2(\mu-a)^2$

$= a^2 + 2a\mu - \cancel{2a^2} + 2\mu^2 - 4a\mu + \cancel{2}a^2$

$= a^2 - 2a\mu + 2\mu^2$

$[E(Y)]^2 = \mu^2$,

so $Var(Y) = (a^2 - 2a\mu + 2\mu^2) - \mu^2$

$Var(Y) = (a-\mu)^2$

$SD(Y) = \pm(a-\mu) = \mu - a$, as $\mu \geq a$

$\boxed{\begin{array}{l}\text{Normalization} \\ \text{Mean: } \mu \\ \text{SD: } \mu - a\end{array}}$

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2b) $L(\hat{\mu}, \hat{a}) = \prod_{j=1}^{n} \frac{1}{\mu - a} e^{-\left[\frac{y_j - a}{\mu - a}\right]}$

$\ln L(\hat{\mu}, \hat{a}) = \sum_{j=1}^{n} \ln\left(\frac{1}{\mu - a} e^{-\frac{y_j - a}{\mu - a}}\right)$

$\ln L(\hat{\mu}, \hat{a}) = \sum_{j=1}^{n} -\frac{y_j - a}{\mu - a} - n\ln(\mu - a)$

Take partial derivatives and set to 0

$\frac{\partial L}{\partial \mu} = \sum_{j=1}^{n} \frac{y_j - a}{(\mu - a)^2} - \frac{n}{\mu - a} = 0$

$= \sum_{j=1}^{n} \frac{y_j}{(\mu - a)^2} - \frac{na}{(\mu - a)^2} - \frac{n}{\mu - a} = 0$

$= \frac{1}{\mu - a} \sum_{j=1}^{n} y_j = n + \frac{an}{\mu - a} \quad \mu - a =$

$\sum_{j=1}^{n} y_j = n\mu - \cancel{an} + \cancel{an}, \quad \frac{1}{n}\sum_{j=1}^{n} y_j = \hat{\mu}_{mle}$

$\frac{\partial L}{\partial a} = \sum_{j=1}^{n} -\frac{((a - \mu) - a + y_j)}{(\mu - a)^2} + \frac{n}{\mu - a}$

$= \sum_{j=1}^{n} \frac{\mu - y_j}{(\mu - a)^2} + \frac{n}{\mu - a} = 0$

Above isn't really solvable, and since $a < \mu$, $a_{mle} \leq \mu_{mle}$ and $y \geq a$. Looking at likelihood, max $\hat{a}_{mle}$ when $\hat{a}_{mle} - y_j = 0$, as e^{-x} is at max on $[0, \infty)$ at $x = 0$, so a best estimate $\hat{a}_{mle} = \min\{y_j\}$

$$\boxed{\begin{aligned} \hat{\mu}_{mle} &= \bar{y} \\ \hat{a}_{mle} &= \min\{y_j\} \end{aligned}}$$

2c) From definition of a confidence interval with average $\bar{y}$, SD σ, and n samples is

$$\left(\bar{y} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{y} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Plugging in with a mean $\hat{\mu}_{mle}$ and since $\sigma = \hat{\beta} - \hat{\alpha}$, $\sigma = \hat{\mu} - \min\{y_i\}$

$$\boxed{\left(\hat{\mu}_{mle} - Z_{\alpha/2}\frac{\hat{\mu}_{mle} - \min\{y_i\}}{\sqrt{n}},\ \hat{\mu}_{mle} + Z_{\alpha/2}\frac{\hat{\mu}_{mle} - \min\{y_i\}}{\sqrt{n}}\right)}$$

3a) $\left(\bar{y} - 1.64\frac{\sigma}{\sqrt{n}},\ \bar{y} + 2.33\frac{\sigma}{\sqrt{n}}\right)$

$Z_{1.64} = 0.0505$ $\quad Z_{2.33} = .9901$

Interval $= 100(.9901 - 0.0505) = \boxed{93.96\%}$

b) $\left(-\infty,\ \bar{y} + 2.58\frac{\sigma}{\sqrt{n}}\right)$

$Z_{-\infty} = 0$ $\quad Z_{2.58} = .9951$

Confidence $= 100(.9951) = \boxed{99.51\%}$

c) $\left(\bar{y} - 1.64\frac{\sigma}{\sqrt{n}},\ \bar{y}\right)$

$Z_{1.64} = .0505$ $\quad Z_0 = 0.5$

Confidence $= 100(.5 - .0505) = \boxed{44.95\%}$

5.3.15a)

We can use the formula from the textbook that to have an interval $\leq d$, need sample size n, $n = \frac{Z_{\alpha/2}^2}{4d^2}$

Have 99% confidence, so $\alpha = .001$, and $d = 0.01$ as length of interval want ≤ 0.02

$n = \frac{Z_{.005}^2}{4(.01)^2} = \frac{(-2.58)^2}{4(0.01)^2} = 16641$, so need at least 16641 people to have 99% confidence w/ length ≤ 0.02.

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