EE 159 HW4

1 $\Psi(\phi(x)) = \dfrac{E\left(\frac{Ax+b}{c^Tx+d}\right)+f}{g^T\left(\frac{Ax+b}{c^Tx+d}\right)+h}$

$$= \frac{\frac{EAx+Eb}{c^Tx+d} + f\left(\frac{c^Tx+d}{c^Tx+d}\right)}{\frac{g^TAx+g^Tb}{c^Tx+d} + h\left(\frac{c^Tx+d}{c^Tx+d}\right)}$$

$$= \frac{EAx+Eb+fc^Tx+fd}{g^TAx+g^Tb+hc^Tx+hd}$$

$$= \frac{EAx+fc^Tx+Eb+fd}{g^TAx+hc^Tx+g^Tb+hd}$$

as $d \in \mathbb{R}$

$$= \frac{(EA+fc^T)x+(Eb+df)}{(g^TA+hc^T)x+(g^Tb+dh)}$$

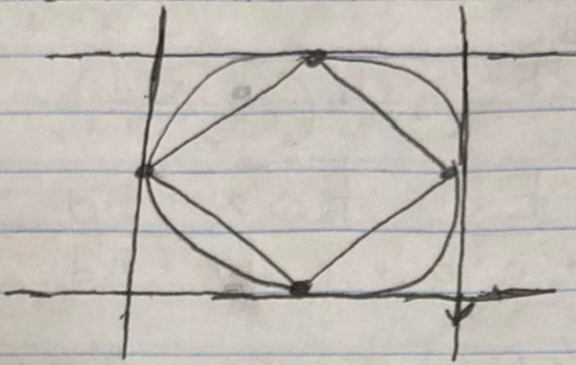
So $\Psi(\phi(x))$ is associated with the matrix: $\begin{bmatrix} EA+fc^T & Eb+df \\ g^TA+hc^T & g^Tb+dh \end{bmatrix}$

And note $\begin{bmatrix} E & f \\ g^T & h \end{bmatrix}\begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$

$$= \begin{bmatrix} EA+fc^T & Eb+fd \\ g^TA+hc^T & g^Tb+hd \end{bmatrix}$$

So Γ is linear fractional function as it can be written in the form and the matrix associated with it is the product of $\begin{bmatrix} E & f \\ g^T & h \end{bmatrix}\begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$

2) As C is closed $\{x_1 \dots x_n\} \in C$
As $P_{inner} = \text{conv}\{x_1 \dots x_n\}$ and C is convex then by definition $P_{inner} \subseteq C$.
For $x \in C$, $a_i^T(x - x_i) \leq 0$ for all i,
So $x \in P_{outer}$. Therefore $C \subseteq P_{outer}$



example with 4 points

3) Since $x^T D x \leq 0$ $\sum_{k,j} x_k x_j D_{kj} \leq 0$
According to textbook, a convex cone is a cone s.t. for any $x_1, x_2 \in C$, $\forall \theta_1, \theta_2 \geq 0$ $\theta_1 x_1 + \theta_2 x_2 \in C$. Applying this definition to D, let $D_1, D_2 \in S^n$ and are Euclidean distance matrices.

$$x^T(\theta_1 D_1 + \theta_2 D_2)x \quad w/ \quad 1^T x = 0$$
$$= (\theta_1 x^T D_1 + \theta_2 x^T D_2)x = \theta_1 x^T D_1 x + \theta_2 x^T D_2 x$$

As D_1, D_2 are Euclidean distance matrices, $x^T D_1 x \leq 0$ and $x^T D_2 x \leq 0$ and as $\theta_1 \geq 0$, $\theta_2 \geq 0$ then $\theta_1 x^T D_1 x \leq 0$ and $\theta_2 x^T D_2 x \leq 0$
So, $\theta_1 x^T D_1 x + \theta_2 x^T D_2 x \leq 0$ and the set of Euclidean distance matrices is a convex cone.

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4a) The first order condition states f is convex iff $f(z) \geq \nabla f(x)^T(z-x) + f(x)$ on dom f

$\text{Epi} f = \{(z,t) \mid f(z) \leq t\}$

So $t \geq f(z) \geq \nabla f(x)^T(z-x) + f(x)$

Define $t \geq \nabla f(x)^T(z-x) + f(x)$

$0 \geq \nabla f(x)^T(z-x) + f(x) - t$

Want hyperplane at $(x_0, f(x_0))$ so

$0 \geq \nabla f(x_0)^T(z-x_0) + f(x_0) - t$

and as supporting, $0 = \nabla f(x_0)^T(x-x_0) + f(x_0) - t$

Can change to this as want epigraph at $(x_0, f(x_0))$.

Define a vector $\begin{bmatrix} x \\ t \end{bmatrix}$ it can be seen then

$$0 = \begin{bmatrix} \nabla f(x_0) \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} - \begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^T \begin{bmatrix} x_0 \\ f(x_0) \end{bmatrix}$$

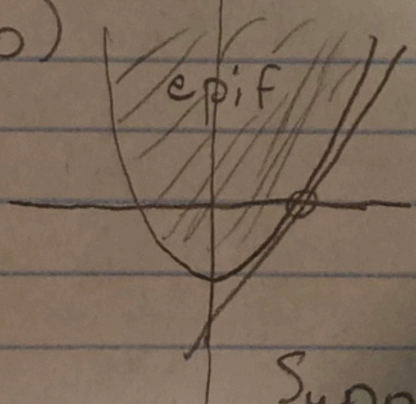
This is our equation just rewritten.

So $$0 = \begin{bmatrix} \nabla f(x_0) \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} x \\ t \end{bmatrix} - \begin{bmatrix} x_0 \\ f(x_0) \end{bmatrix} \right)$$

Using the substitutions in the problem, this becomes $0 = a^T(z-b)$

But as we also know this is satisfying $f(z) - t \geq \ldots$ can say this is $y = a^T(z-b)$ which is the supporting hyperplane.

4b)



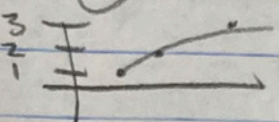
Using the equation and $t=0$

get $$y = \begin{bmatrix} 2 \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} x \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$y = 2(x-1) = 2x-2$

Supporting hyperplane is $y = 2x-2$

5a) The sublevel sets appear to be convex so at a minimum it is quasiconvex. Furthermore it doesn't appear to be quasiconcave or concave as the superlevel sets aren't convex. As for convexity, we can see that $\Delta f(x)$ decrease as $f(x)$ increases something like



which isn't convex. This is seen on right side of graph.

b) The sublevel sets aren't convex so f isn't convex nor is it quasiconvex. However, looking at specific sublevels, they do appear to be concave from the drawing, so it can potentially also be quasiconcave