Tufts University
Department of Mathematics

Math 136 **Group work for the Week of February 13, 2023** [1] **Spring, 2023**

Due date: 11:59 pm, five days after I hand it out. So, if I hand it out on Monday, February 13, it will be due Saturday, February 18, and if I hand it out on Wednesday, February 15, it will be due Monday, February 20 on Gradescope.

Please enter the names of all group members into Gradescope when submitting.

Every week we plan to work in small groups to learn to write proofs and to solve problems. I will grade each group's work.

- *Scribe:* each week, someone in the group will volunteer to submit to Gradescope the group's answer along with group member names. This role should rotate through the group.
- *Respect:* when discussing problems, please make sure that everyone feels comfortable speaking and that all feedback is supportive and encouraging.
- Please keep your group name so you can rejoin the same group each week.

Problem: In this problem you will prove the Chain Rule for functions $f : I \to \mathbb{R}$ where I is an open subset of $\mathbb{R}$.

Theorem 1 (Chain Rule) *Let I be a neighborhood of $x_0 \in \mathbb{R}$ (an open set containing x_0) and let $f : I \to \mathbb{R}$ be differentiable at x_0. Let J be an open set in $\mathbb{R}$ and let $g : J \to \mathbb{R}$. Assume $f(I) \subset J$ and assume g is differentiable at $f(x_0)$. Then, the composition $g \circ f : I \to \mathbb{R}$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.*

NOTE: our proof that differentiability implies continuity for shows that if $f : I \to \mathbb{R}$ is differentiable at a point $x_0 \in I$ then f is continuous at x_0.

(a) First, assume $f : I \to \mathbb{R}$ is injective (one-to-one). Prove the chain rule for f and g.

(b) Where in your proof did you use that f is injective?

Now you will use a trick to prove the Chain Rule in the general case. Let $y_0 = f(x_0)$. Define $h : J \to \mathbb{R}$ by

$$h(y) = \begin{cases} \frac{g(y)-g(y_0)}{y-y_0} & y \neq y_0 \\ g'(y_0) & y = y_0 \end{cases}$$

(c) Explain why $g(y) - g(y_0) = h(y)[y - y_0]$ for all $y \in J$.

(d) Explain why

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = h(f(x)) \left[\frac{f(x) - f(x_0)}{x - x_0} \right] \quad \forall x \in I \setminus \{x_0\}.$$

(e) Use the result of part (d) to show $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.