1. Math 61, Fall 20, Review Sheet for the final exam

Question 1.1. *Let T be the set of all Tufts students Denote with $K(x, y)$ the statement " student x knows student y"*

(a) Use quantifiers and connections to express the statement "For any two students at Tufts, there is always a third student that knows both".

(b) Write the negation of the above statement so that any negative sign appears next to a "$K(-, -)$".

Question 1.2. *To which of the statements below is $\neg(p \wedge q)$ equivalent to? Justify your answer.*

(a) $(\neg p) \wedge (\neg q)$

(b) $(\neg p) \vee (\neg q)$

(c) $p \to \neg q$

(d) $\neg p \to q$

Question 1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\forall x, y \in \mathbb{R}\ f(xy) = xf(y) + yf(x)$*

(a) Show that $f(1) = 0$.

(b) Show that if $u \in \mathbb{R}, n \in \mathbb{N}$, then $f(u^n) = nu^{n-1}f(u)$.

Question 1.4. *A car has a license plate which has three letters (chosen from among 26) followed by three digits (each of which is one of $0, 1, 2, \ldots 9$). The three letters include the ordered consecutive pair AS and the three digits include each of the digits 5 and 2 exactly once in some order. How many license plates fit this description?*

Question 1.5. *(a) Prove algebraically that if $k < n$ are natural numbers, then*

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

(b) Give a combinatorial argument to prove the equation in (a).

Question 1.6. *Let $n_1, n_2, \ldots, n_k$ be natural numbers such that $n = n_1 + n_2 + \cdots + n_k$. Show that*

$$\sum_{j=1}^{k} \binom{n_j}{2} \leq \binom{n}{2}$$

Hint: think of each side of the equation as counting some edges in a graph.

Question 1.7. *A forest is a not necessarily connected graph with no simple closed loops. A certain forest has 50 vertices and 40 edges. How many connected components does it have?*

Question 1.8. *If a connected planar graph has 10 vertices of degree 4, 12 vertices of degree 3 and 3 of degree 2, how many edges does it have? In how many regions does it divide the plane?*

Question 1.9. *Let G be a connected graph. A Hamilton circuit is a circuit (closed path) that contains all vertices exactly once, except for the first and last vertices that are the same.*

(a) Show that if G has a Hamilton circuit, then $\deg(v) \geq 2$ for all vertices v.

(b) Show that if G has a Hamilton circuit but no Euler circuit then deg$(v) \geq 3$ for at least <u>two</u> vertices in G.

(c) Show that $K_n, n \geq 3$ has $(n-1)!$ Hamilton circuits.

Question 1.10. *Show that if a tree T has a vertex v of degree k, then T has at least k vertices of degree 1.*

Question 1.11. *Suppose that every vertex of a graph G has degree at most k. Prove that the chromatic number of G satisfies $CN(G) \leq k+1$. Show that this bound is the best possible by exhibiting (for every k) a graph with maximum degree k and chromatic number $k+1$. Hint: Use induction on the number of vertices.*

Question 1.12. *Your friend is an avid tennis player.The fraction of the games that she wins against each of her four opponents is .6, .5, .45, .4 respectively. Suppose that your friend plays 30 matches against each of the first two, and 20 matches against each of the second two.*
(a) What is the probability that your friend wins one of her matches.
(b) Given that your friend won a particular match, what is the probability that she was playing the first opponent.
(c) Given that your friend lost a particular match, what is the probability that she was playing the fourth opponent (assume there aren't any ties).

Question 1.13. *(a) Define what it means for a set to be countable.*
(b) Show that $\mathbb{Z} \times \mathbb{N}$ is countable

Question 1.14. *(a) Consider the assignment* $f: \begin{array}{ccc} \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \\ (\frac{a}{b}, \frac{c}{d}) & \to & \frac{a+b}{c+d} \end{array}$ *Prove or disprove that this is a well defined function.*
(b) Consider the assignment $f: \begin{array}{ccc} \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \\ (\frac{a}{b}, \frac{c}{d}) & \to & \frac{ad+bc}{bd} \end{array}$ *Prove or disprove that this is a well defined function.*

Question 1.15. *Let $f : A \to B$ be a function. Show that f is onto if and only if there exists a function $g : B \to A$ such that $f \circ g = I_B$.*

Question 1.16. *For a natural number a written in decimal expression, we will denote its digits with $a_n a_{n-1} \ldots a_1 a_0$. For example, for the number 367, $n = 2, a_2 = 3, a_1 = 6, a_0 = 7$.*
(a) Show that $a = a_n a_{n-1} \ldots a_1 a_0$ is divisible by 3 if and only if $a_n + a_{n-1} + \cdots + a_1 + a_0$ is divisible by 3.
(b) Show that $a = a_n a_{n-1} \ldots a_1 a_0$ is divisible by 9 if and only if $a_n + a_{n-1} + \cdots + a_1 + a_0$ is divisible by 9.

Question 1.17. *(a) Show that the sequence $a_n = \frac{4n+5}{n+1} \in \mathbb{Q}$ satisfies $\lim_{n\to\infty} a_n = 4$.*
(b) Show that the sequence $a_n = \frac{(-1)^n(4n+5)}{n+1} \in \mathbb{Q}$ does not have a limit.

Question 1.18. *(a) Define subtraction of real numbers by $[(a_n)] - [(b_n)] = [(a_n - b_n)]$. Show that this gives a well defined operation.*
(b) Prove the associative property for addition of real numbers.