Math 166 HW 9

1a) If $y = ae^{bx}$, then $(x, \ln y)$ have a linear relation.

$$b = \frac{n\sum_{i=1}^{n} x_i \ln y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \ln y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Calculations done via calculator

$$b = \frac{1379.74 - 35(41.357)}{1690 - 1225}$$

$$b = -0.146$$

$$\ln a = \frac{\sum_{i=1}^{n} \ln y_i - b \sum_{i=1}^{n} x_i}{n} = e^{4.6465} = 104.14 = a$$

$$\boxed{\text{So exponential curve of best fit is } y = 104.14e^{-0.146x}}$$

b) Half life is length it takes for half of a substance to dissapear, so want to generally solve for x when $\frac{a}{2} = ae^{bx}$, $\frac{1}{2} = e^{bx}$

$0.5 = e^{-0.146x}$, $x = 4.748$ days

$\boxed{{}^{195}Au \text{ has a half life of } 4.748 \text{ days}}$

2 $y = a + \frac{b}{x}$, $(\frac{1}{x}, y)$ has a linear relation)

We can obtain the following equations for a, b:

$$b = \frac{n\sum_{i=1}^{n} \frac{y_i}{x_i} - \sum_{i=1}^{n} \frac{1}{x_i} \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} \left(\frac{1}{x_i}\right)^2 - \left(\sum_{i=1}^{n} \left(\frac{1}{x_i}\right)\right)^2}$$

and $a = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{b}{n}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)$

$$b = \frac{3049.375 - 8.02(169.7)}{\cancel{149.145}\ 64.267}$$

$b = 19.827$

We then get $a = 1.536$

$y(x) = 1.536 + \frac{19.827}{x}$

$y\left(\frac{1}{4}\right) = 80.844$

3 $\sum_{i=1}^{n}(y_i - \hat{B}_0 - \hat{B}_1 x_i)^2 = \sum_{i=1}^{m}\left[y_i - (\bar{y} - \hat{B}_1\bar{x}) - \hat{B}_1 x_i\right]^2$

$= \sum_{i=1}^{n}\left[(y_i - \bar{y}) - \hat{B}_1(x_i - \bar{x})\right]^2$

$= \sum_{i=1}^{n}(y_i - \bar{y})^2 + \hat{B}_1^2\sum_{i=1}^{n}(x_i - \bar{x})^2 - 2\hat{B}_1\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$= \sum_{i=1}^{n}(y_i - \bar{y})^2 + \hat{B}_1\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 - 2\hat{B}_1\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$= \sum_{i=1}^{n}(y_i - \bar{y})^2 - \hat{B}_1\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$= \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 - \hat{B}_1\left[\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i\right)\right]$

$= \sum_{i=1}^{n} y_i^2 - \hat{B}_1\sum_{i=1}^{n} x_i y_i - n\bar{y}^2 + \frac{1}{n}\hat{B}_1\sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i$

$= \sum_{i=1}^{n} y_i^2 - \hat{B}_1\sum_{i=1}^{n} x_i y_i - (\bar{y} - \hat{B}_1\bar{x})\sum_{i=1}^{n} y_i$

$= \sum_{i=1}^{n} y_i^2 - \hat{B}_1\sum_{i=1}^{n} x_i y_i - \hat{B}_0\sum y_i$ $\boxed{}$

4 $\bar{\varphi} = \frac{y_1 + y_2 + \dots + y_n}{n}$ $\bar{X} = \frac{x_1 + x_2 + \dots + x_n}{n}$

$$E(\bar{\varphi}) = \sum_{i=1}^{n} E(\varphi_i \mid x_i) = \frac{1}{n}\sum_{i=1}^{n} B_0 + B_1 x_i$$

$$= \frac{1}{n} n B_0 + \frac{B_1}{n}\sum_{i=1}^{n} x_i$$

$$E(\bar{\varphi}) = B_0 + \frac{B_1}{n} \cdot n\bar{X}$$

$$E(\bar{\varphi}) = B_0 + B_1 \bar{X}$$

5 a) We assume $x, y > 0$

If $c < 0$, then $\lim_{y\to\infty} e^{-cy} = \infty$, If the equation were to hold then it's reasonable to assume that then as $y \to \infty$, e^{a+bx} must reach some similarly large value < 0, However, $\forall x \in \mathbb{R}^+$, $e^{a+bx} > 0$, meaning the equation will not hold true if $c < 0$, therefore c must be ≥ 0. $c \neq 0$ as the equation becomes $e^{a+bx} + 1 = 1$, $e^{a+bx} = 0$, which is never true. As a result c must be positive

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5b) $e^{a+bx} + e^{-cy} = 1$

$a + bx = \ln(1 - e^{-cy})$

So we have a linear relationship between $(x_i, \ln(1-e^{-cy_i}))$.

Let z_i refer to $\ln(1-e^{-cy_i})$ for writing this out to preserve space

We can calculate that, from def of LSR

$$b = \frac{n\sum_{i=1}^{n} x_i z_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i z_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\overline{xz} - \bar{x}\bar{z}}{\overline{x^2} - (\bar{x})^2}$$

and $a = \frac{1}{n}\sum_{i=1}^{n} z_i - \frac{1}{n}\sum_{i=1}^{n} x_i \cdot b$, $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$, $\bar{x} = \frac{1}{n}\sum x_i$

The LSR line must contain $(\bar{x}, \bar{y})$, so we can then ~~rewrite~~ ~~the~~ equation and now get:

$$e^{a+b\bar{x}} + e^{-c\bar{y}} = 1$$

We get the following equation by plugging in a, b

$$\exp\left[(\bar{z} - \bar{x})\left(\frac{\overline{xz} - \bar{x}\bar{z}}{\overline{x^2} - (\bar{x})^2}\right) + \left(\frac{\overline{xz} - \bar{x}\bar{z}}{\overline{x^2} - (\bar{x})^2}\right)\bar{x}\right] + e^{-c\bar{y}} = 1$$

And $\bar{z} = \sum_{i}^{n} z_i = \ln(1 - e^{-cy_i})$