Due date: 11:59 pm, Friday, April 21, 2023 on Gradescope. You can do all problems after Wednesday, April 12.

You are encouraged to work on problems with other Math 136 students and to talk with Todd and our TA Wentao, but your answers should be in your own words.

A proper subset of the problems will be selected for grading.

- **Reading assignment for the week of April 10:** Please read the material in sections 10.1 and 10.2 of Marsden-Hoffman as well as the proofs of theorems.

- **Reading assignment for the weeks of April 17 and April 24:** On Friday, April 21, we might touch on pointwise convergence of Fourier Series, section 10.3. Then, we learn about the heat equation (this material starts on p. 598 of the book).

To address some questions, here is the definition of *complex inner product space.*

Definition 1 (Complex (Hermetian) Inner Product Space) *Let V be a vector space over the complex numbers and let $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{C}$ be a function.*

Then, $\langle\cdot,\cdot\rangle$ is an inner product and $(V,\langle\cdot,\cdot\rangle)$ is an inner product space if the following properties hold. Let $\mathbf{u},\mathbf{v},\mathbf{w}$ be arbitrary vectors in V and let c be an arbitrary complex number.

(a) Linearity in first argument: $\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{w}\rangle + \langle\mathbf{v},\mathbf{w}\rangle$ and $\langle c\mathbf{u},\mathbf{w}\rangle = c\langle\mathbf{u},\mathbf{w}\rangle$.

(b) Conjugate symmetry: $\langle\mathbf{v},\mathbf{w}\rangle = \overline{\langle\mathbf{w},\mathbf{v}\rangle}$.

(c) Positive definiteness: $\langle\mathbf{v},\mathbf{v}\rangle \geq 0$ and $\langle\mathbf{v},\mathbf{v}\rangle = 0$ if and only if $\mathbf{v}=\mathbf{0}$.

If $\langle\cdot,\cdot\rangle$ satisfies all of the properties above *except* $\langle\mathbf{v},\mathbf{v}\rangle = 0$ iff $\mathbf{v} = 0$, then it is called a *positive semidefinite* inner product. For $(V,\langle\cdot,\cdot\rangle)$ to be an inner product space, the inner product must be positive definite.

The product for $\mathcal{L}^2([a,b],\mathbb{C})$ is only positive semidefinite, so $\mathcal{L}^2([a,b],\mathbb{C})$ is not an inner product space.

Problems:

1. (30 points) Let $(V,\langle\cdot,\cdot\rangle)$ be a complex inner product space.

   Let $h\in V$ and define the function $\Lambda : V\to\mathbb{C}$ by $\Lambda(f) = \langle f,h\rangle$ for all $f\in V$.

   (a) Prove that Λ is linear, that is if f and g are in V and $c\in\mathbb{C}$ then you need to prove: $\Lambda(f+g) = \Lambda(f)+\Lambda(g)$, and $\Lambda(cf) = c\Lambda(f)$.

   (b) Show that Λ is continuous. HINT: It is easiest to prove Λ is Lipschitz, I.e., show there is a constant $c>0$ such that $|\Lambda(f-g)|\leq c\|f-g\|$ for f and g in V.

2. (30 points) In this problem, you will show a couple of the properties of ℓ^2, the set of all complex sequences $\mathbf{a} = (a_0,a_1,a_2,\dots)$ such that $\|\mathbf{a}\| = \sqrt{\sum_{j=0}^{\infty}|a_j|^2} < \infty$ (i.e., the sum $\sum_{j=0}^{\infty}|a_j|^2$ converges).

   Let $\mathbf{a} = (a_0,a_1,a_2,\dots)$ and $\mathbf{b} = (b_0,b_1,b_2,\dots)$ be vectors in ℓ^2.

   (a) Prove that $\mathbf{a}+\mathbf{b}\in\ell^2$. That is, show that the $\sum_{j=0}^{\infty}|a_j+b_j|^2$ converges absolutely.

   HINT: Let a and b be complex numbers. Then, $|a+b|^2\leq 2(|a|^2+|b|^2)$.

   (b) Prove that $\langle a,b\rangle = \sum_{k=0}^{\infty}a_k\overline{b_k}$ converges absolutely. This shows that $\langle\mathbf{a},\mathbf{b}\rangle\in\mathbb{C}$ so $\langle\cdot,\cdot\rangle : \ell^2\times\ell^2\to\mathbb{C}$.

   HINT: Let a and b be complex numbers. Then, $|ab|\leq\frac{1}{2}(|a|^2+|b|^2)$. *# 3 is on the next page.*

3. (40 points) Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous, and assume $\frac{\partial f}{\partial y}(x,y)$ is defined on $(a,b) \times (c,d)$ and extends to a continuous function on $[a,b] \times [c,d]$.

   For all $y \in [c,d]$, define $F(y) = \int_{x=a}^{b} f(x,y)\,dx$. You will prove that F is differentiable on (c,d) and

$$\frac{d}{dy} \int_{x=a}^{b} f(x,y)\,dx = \frac{dF}{dy}(y) = \int_{x=a}^{b} \frac{\partial f}{\partial y}(x,y)\,dx. \tag{1}$$

   That is, you can bring the derivative with respect to y inside this integral with respect to x.

   First, define $g(y) = \displaystyle\int_{x=a}^{b} \frac{\partial f}{\partial y}(x,y)\,dx$ for $y \in (c,d)$.

   You may assume g is continuous for $y \in [c,d]$ (the proof was problem 1 on HW 7). Now, use the following steps to show identity (1) .

   (a) (short answer) Let $y \in [c,d]$. Explain why $\displaystyle\int_{t=c}^{y} g(t)\,dt = \int_{t=c}^{y} \int_{x=a}^{b} \frac{\partial f}{\partial y}(x,t)\,dx\,dt.$

   (b) Justify why $\displaystyle\int_{t=c}^{y} \int_{x=a}^{b} \frac{\partial f}{\partial y}(x,t)\,dx\,dt = \int_{x=a}^{b} \int_{t=c}^{y} \frac{\partial f}{\partial y}(x,t)\,dt\,dx$ for each $y \in [c,d]$.

   (c) Now, justify why $\displaystyle\int_{t=c}^{y} g(t)\,dt = \int_{x=a}^{b} [f(x,y) - f(x,c)]\,dx$ for each $y \in [c,d]$.

   (d) Use the result of part (c) to prove (1).

Here is an optional extra-credit challenge problem. Todd will grade it.

1. (2 points) Prove the inequalities in the hints for both parts of problem 2.