Math 166 HW 3

1 $Y_{min} = P(Y_{min} < y) = 1 - P(Y_{min} \geq y)$

From class, can get pdf:

pdf $= n[1-F_Y(y)]^{n-1} f_Y(y)$

$F_Y(y) = \frac{y}{\theta}$, $f_Y = \frac{1}{\theta}$

$F_Y(y) = \frac{n}{\theta}\left(1-\frac{y}{\theta}\right)^{n-1}$

$-\frac{dy}{\theta} = du$

$1 - \frac{y}{\theta} = u$

$(1-u)\theta = y$

$E(Y_{min}) = \frac{n}{\theta}\int_0^\theta \left(1-\frac{y}{\theta}\right)^{n-1} y\,dy$

$= -n\int_1^0 \theta(1-u)u^{n-1}\,du$

$= n\theta\int_0^1 u^{n-1} - u^n\,du = n\theta\left(\frac{u^n}{n} - \frac{u^{n+1}}{n+1}\right)\Big|_0^1$

$= n\theta\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= \frac{\theta}{n+1}$

Since $E(\hat{\theta}) = \theta$ to be unbiased, multiply by $n+1$ so the unbiased estimator is: $(n+1)Y_{min}$

2 $\hat{\theta} = Y^2$ $\quad F_Y(y) = \frac{y}{\theta}$ $\quad f_Y(y) = \frac{1}{\theta}$

$\theta = Y$

$E(\hat{\theta}) = E(Y^2) = \int_0^\theta \frac{1}{\theta} y^2\,dy = \frac{y^3}{3\theta} = \frac{\theta^2}{3}$

which is biased,

but if $\hat{\theta} = 3Y^2$, then $E(3y^2) = \theta^2$

$3y^2$ is unbiased estimator for θ^2

3 $Var(Y_{max}) = E(Y_{max}^2) - E(Y_{max})^2$

$f_{Y_{max}} = n\left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}$ From class and textbook

$E[Y_{max}] = \frac{n}{\theta^n}\int_0^\theta y^{n-1} \cdot y\,dy = \frac{n}{n+1} \cdot \theta$

$E[Y_{max}^2] = \frac{n}{\theta^n}\int_0^\theta y^{n+1}\,dy = \frac{n}{}\theta^2$

$$Var(Y_{max}) = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2$$

$$= \frac{n\theta^2(n+1)^2 - n\theta^2(n+2)}{(n+1)^2(n+2)} \quad \text{for } n=5$$

$$Var(Y_{max}) = \frac{5}{252}\theta^2$$

$$Var(Y_{min}) = E(Y_{min}^2) - E(Y_{min})^2$$

$$f_{Y_{min}} = \frac{n}{\theta}\left(1-\frac{y}{\theta}\right)^{n-1}$$

$$E(Y_{min}^2) = \frac{n}{\theta}\int_0^\theta \left(1-\frac{y}{\theta}\right)^{n-1} y^2\,dy \qquad 1-\frac{y}{\theta} = u,\ -\frac{dy}{\theta} = du,\ (1-u)\theta = y$$

$$= n\int_0^1 ((1-u)\theta)^2 (u)^{n-1}\,du$$

$$= n\theta^2\int_0^1 (1-u)^2 u^{n-1}\,du \qquad n=5$$

$$= 5\theta^2\int_0^1 (1-u)^2 u^4\,du$$

$$= 5\theta^2\int_0^1 u^4 - 2u^5 + u^6\,du$$

$$= 5\theta^2\left(\frac{u^5}{5} - \frac{u^6}{3} + \frac{u^7}{7}\right)\Big|_0^1$$

$$= 5\theta^2\left(\frac{1}{7} - \frac{1}{3} + \frac{1}{5}\right) = \frac{5\theta^2}{105} = \frac{\theta^2}{21}$$

$E(Y_{min})$ for $n=5$, $\Rightarrow f_{Y_{min}} = \frac{5}{\theta}\left(1-\frac{y}{\theta}\right)^4$ from earlier

$$E(Y_{min}) = \frac{5}{\theta}\int_0^\theta y\left(1-\frac{y}{\theta}\right)^4 dy = \theta\int_0^1 (1-u)^5\,du = \theta\cdot\frac{1}{6} = \frac{\theta}{6}$$

$$Var(Y_{min}) = \frac{\theta^2}{21} - \frac{\theta^2}{36} = \frac{5}{252}\theta^2$$

$$\frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)} = \frac{\frac{36}{25}Var(Y_1)}{36\,Var(Y_2)} = \frac{1}{25} < 1,$$

$\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$

4 Since unbiased $E(\hat{\lambda}) = \lambda$, and for a Poisson distribution X, $Var(X) = \lambda$

relative efficiency $= \dfrac{Var(\hat{\lambda}_2)}{Var(\hat{\lambda}_1)} = \dfrac{Var\left(\frac{X_1 + \dots + X_n}{n}\right)}{Var(X_1)}$

$= \dfrac{\frac{1}{n^2} Var(X_1 + \dots + X_n)}{Var(X_1)}$

$= \dfrac{1}{n^2}\left(\dfrac{Var X_1 + \dots + Var X_n}{Var(X_1)}\right)$

$= \dfrac{1}{n^2} \cdot \dfrac{\cancel{n\lambda}}{\lambda} = \boxed{\dfrac{1}{n}}$

5) Cramer-Rao bound $= \left\{ n E\left(\frac{\partial \ln \frac{1}{\theta}}{\partial \theta} \right)^2 \right\}^{-1}$

$$= \left\{ -n E\left(\frac{1}{\theta^2} \right) \right\}^{-1} = \frac{-\theta^2}{n}$$

This pdf won't work under Cramer-Rao as it isn't continuous, which is why the minimum variance is negative

From Q_3, $g_{max} = \frac{ny^{n-1}}{\theta^n}$, $\hat{\theta} = (n+1)y^{n-1}$

$E(\hat{\theta}) = \theta$, as unbiased

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5 $E(\hat{y}^2) = \int_0^{\theta} \frac{(n+1)ny^{n-1}}{n\theta^n} y^2 \, dy$

$$= \frac{(n+1)}{n\theta^n} \int_0^{\theta} y^{n+1} dy = \frac{(n+1)}{\theta^n}\frac{y^{n+2}}{n+2}\Big|_0^{\theta} = \frac{n+1}{n+2}\theta^2$$

$Var(c\hat{\theta}) = \frac{n+1}{n+2}\theta^2$; but applying Cramér-Rao band yields $Var = -\frac{\theta^2}{n}$, b/c the band can't be applied as f_y isn't continuous for all y.

6 a) $\hat{\theta} = \frac{1}{r}\bar{Y} = \frac{1}{r}\left(\frac{Y_1+Y_2+\dots Y_n}{n}\right)$ $E(\hat{\theta}) = \frac{1}{r}E\left(\frac{Y_1+Y_2+\dots+Y_n}{n}\right)$

$= \frac{1}{nr}(E(Y_1)+\dots+E(Y_n))$

As $Y_1 \dots Y_n$ are identically dependently distributed, becomes $\frac{1}{nr}nE(Y) = \frac{E(Y)}{r}$

$$E(Y) = \int_0^{\infty} \frac{1}{(r-1)!\theta^r} y^r e^{-y/\theta} dy = \frac{1}{(r-1)!\theta^r}\int_0^{\infty} y^r e^{-y/\theta} dy$$

This is gamma function w/ parameters r and $\frac{1}{\theta}$ From pg. 269 can use parameters and follow formula to get $E(Y) = r\theta$

$E(\hat{\theta}) = \frac{1}{r}\cdot r\theta = \theta$, so $\hat{\theta}$ is unbiased

b) Cramér-Rao Band $= \left\{-nE\left[\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2}\right]\right\}^{-1}$

$\frac{\partial}{\partial\theta}\ln f_Y(Y;\theta) = \ln y^{r-1} + \ln e^{-\frac{y}{\theta}} - \ln(r-1)! - \ln\theta^r$

$= \frac{y}{\theta^2} - \frac{r}{\theta} = \frac{y - r\theta}{\theta^2}$

$\frac{\partial}{\partial\theta}\frac{y}{\theta^2} - \frac{r}{\theta} = \frac{-2y}{\theta^3} + \frac{r}{\theta^2}$ $\left\{-nE\left[\frac{-2y}{\theta^3} + \frac{r}{\theta^2}\right]\right\}^{-1}$

$= \left(-n\left(E\left(\frac{-2y}{\theta^3}\right) + E\left(\frac{r}{\theta^2}\right)\right)\right)^{-1} = \left[-n\left(-\frac{2r\theta}{\theta^3} + \frac{r}{\theta^2}\right)\right]^{-1}$

$-\frac{2}{\theta^3}E(y)$

$= \left[\frac{nr}{\theta^2}\right]^{-1} = \frac{\theta^2}{nr}$

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Cramér-Rao bound for φ is $\frac{\theta^2}{nr}$

Variance of $\frac{1}{r}\bar{\varphi} = \frac{1}{r}\left(\frac{\varphi_1+\varphi_2+\dots+\varphi_n}{n}\right)$

$$Var\left(\frac{\bar{\varphi}}{r}\right) = \frac{1}{r^2}Var(\bar{\varphi}) = \frac{1}{r^2}\left(Var\left(\frac{\varphi_1+\varphi_2+\dots+\varphi_n}{n}\right)\right)$$

$$= \frac{1}{n^2r^2}(Var\,\varphi_1 + Var\,\varphi_2 + \dots + Var\,\varphi_n)$$

$$= \frac{1}{n^2r^2}\,n\,Var(\varphi) = \frac{Var(\varphi)}{nr^2}$$

$Var(\varphi)$, factoring out $\frac{1}{(r-1)!\,\theta^r}$, have Gamma function with parameters r and $1/\theta$, so from pg. 269 $Var(\varphi) = r/(\frac{1}{\theta})^2 = \theta^2 r$

So $Var\,\hat{\theta} = \frac{1}{nr^2}\cdot\theta^2 r = \frac{\theta^2}{nr}$ which is equivalent to the Cramér-Rao bound

So $\hat{\theta}$ is a minimum variance estimator