MATH 135 | Exam 2 | November 21, 2022

(100 points) | 12 noon–1:20 p.m.

• **Carefully PRINT your full name:**

• **CIRCLE your section:** Section 1 (Tu) Section 2 (Tu) Section 3 (Hasselblatt)

**Instructions: No books, notes, calculators, or external help** from any person or device are allowed. Except in the true-false questions or when instructed otherwise, justify all of your steps.
Write only in the space provided and do not attach any extra page.

• **Please sign the following pledge:**
*I pledge that in this exam I have neither given nor received assistance or cheated in any other way.*

**Signature:** ______________________________

1. (16 points) Circle either True or False. You do not need to justify your choice.

   (a) **True** / **False:** Every continuous function on a closed set has a maximum.

   (b) **True** / **False:** Uniform convergence implies pointwise convergence, but the converse is not true.

   (c) **True** / **False:** Suppose $A \subset B \subset \mathbb{R}^n$. If $A$ is closed and $B$ is sequentially compact, then $A$ is sequentially compact.

   (d) **True** / **False:** If $A, B \subset \mathbb{R}^n$, then $\text{int}A \cup \text{int}B = \text{int}(A \cup B)$.

   (e) **True** / **False:** If $A, B \subset \mathbb{R}^n$, then $\text{bd}A \cup \text{bd}B = \text{bd}(A \cup B)$.

   (f) **True** / **False:** If the sets $A_i$ are open in $\mathbb{R}^n$ for all $i \in \mathbb{N}$, then the intersection $\bigcap_{i=1}^{\infty} A_i$ is open in $\mathbb{R}^n$.

   (g) **True** / **False:** Let $A$ be a subset of $\mathbb{R}^n$ and let $F\colon A \to \mathbb{R}^m$ be continuous. If $A$ is sequentially compact, then $F(A)$ is sequentially compact.

   (h) **True** / **False:** Let $A$ be a subset of $\mathbb{R}^n$ and let $F\colon A \to \mathbb{R}^m$ be continuous. If $A$ is closed in $\mathbb{R}^n$, then $F(A)$ is closed in $\mathbb{R}^m$.

2. (10 points) Fill in the blanks (no words). You do not need to justify your answer.
   Let $A = [0,1] \setminus \mathbb{Q}$, where $\mathbb{Q}$ denotes the rationals. Thus, $A$ is the set of irrationals in $[0,1]$.

   (a) $\text{int}A =$ ______________________________

   (b) $\text{bd}A =$ ______________________________

   (c) $\text{ext}A =$ ______________________________

   (d) $\text{cl}A =$ ______________________________

   (e) $\mathbb{R} \setminus A =$ ______________________________

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3. (10 points) Let $A \subset \mathbb{R}^n$.

   (a) State the definition (using sequences) of **continuity** of a mapping $F\colon A \to \mathbb{R}^m$ at a point $\mathbf{u} \in A$ and on the whole domain $A$.
   $F\colon A \to \mathbb{R}^m$ is **continuous at** $u \in A$ if

   $F\colon A \to \mathbb{R}^m$ is **continuous on** $A$ if

   (b) State the $\epsilon$-$\delta$ criterion for **uniform continuity** of a mapping $F\colon A \to \mathbb{R}^m$. (Note that it is **continuity** in (a), but **uniform continuity** in (b).)

4. (14 points)

(a) State the definition of **uniform convergence** in $\mathbb{R}$: Let $f_n$ and $f$ be functions from $\mathbb{R}$ to $\mathbb{R}$. Then $f_n$ converges uniformly to $f$ on $\mathbb{R}$ if

(b) Let $\{f_n\}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$. Assume that $f_n \to f$ uniformly on $\mathbb{R}$. Prove using the definition of uniform convergence that $3f_n + 2 \to 3f + 2$ uniformly on $\mathbb{R}$.

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5. (10 points)

   (a) State the definition of a Cauchy sequence in $\mathbb{R}$: a sequence $\{x_n\}$ in $\mathbb{R}$ is *Cauchy* if

   (b) Suppose $|x_{n+k} - x_n| < 1/n$ for all $n, k \in \mathbb{N}$. Prove that $\{x_n\}$ converges. (You may use whatever theorem about convergence you think is appropriate.)

6. (10 points) Determine the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{kx^k}{3^k}$.

7. (10 points) Prove that $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly on $\mathbb{R}$. (*Hint*. You may assume the $p$-test: $\sum_{n=1}^{\infty} 1/n^p$ converges for $p > 1$.)

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8. (10 points) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^6 + y^6 = 1\}$. Prove that $A$ is sequentially compact.

9. (10 points) Define

$$\mathcal{O} = \{u \in \mathbb{R}^n \mid \|u\| > 1\}.$$

Prove that $\mathcal{O}$ is open in $\mathbb{R}^n$ by showing that every point in $\mathcal{O}$ is an interior point of $\mathcal{O}$. (*Hint*. Pictures may help but do not suffice.)

***Bonus question (3 points, of which 1 for correct spelling—so write clearly!):***
Name a mathematician who made important contributions to real analysis.
[No credit for naming Fitzpatrick, Sandwich, or Tufts faculty.]