MATH 125 Lecture 7

Given data points $(x_i, y_i)_{i=1}^{n}$ we are interested in estimating the underlying function $y = f(x)$

Setup - collecting experimental data
- There is a formula for $f(x)$ but it is difficult to evaluate

Example Given $(0,1)$ $(1,6)$ $(-1,2)$ there is a parabola that passes through the points.

Let $p(x) = ax^2 + bx + c$

$p(0) = c = 1$

$p(1) = a + b + c = 6 \implies a + b = 5 \implies 2a = 6 \Rightarrow a = 3$

$p(-1) = a - b + c = 2 \qquad a - b = 1 \implies b = 2$

Degree 2 interpolating polynomial $\equiv 3x^2 + 2x + 1$

---

polynomial interpolation

We are given $(n+1)$ data points

(x_0, y_0) (x_1, y_1) ... (x_n, y_n)

$(x_i)_{i=1}^{n} \equiv$ nodes

$(y_i)_{i=1}^{n} \equiv$ values

We seek a polynomial p of lowest possible degree for which $p(x_i) = y_i$ $(0 \le i \le n)$

- such a polynomial is said to interpolate the data

Theorem 1 If $x_0, x_1, \ldots x_n$ are distinct real numbers, then for any values $y_0, y_1, \ldots y_n$, there is a unique polynomial p_n of degree $\le n$ such that $p_n(x_i) = y_i$ $(0 \le i \le n)$
(at most n)

Proof Uniqueness

For contradiction, assume there were two polynomials p_n and q_n

$\Rightarrow (p_n - q_n)(x_i) = 0$ for $0 \le i \le n$ *

$P_n - q_n$ is at most degree n
$\Rightarrow$ can have at most n zeros
However, using $*$, $P_n - q_n$ has $n+1$ zeros (Note: the x_i's are distinct)
$\therefore P_n - q_n = 0 \Rightarrow P_n = q_n$

Existence

i) $n = 0$ choose a constant function such that
$P_0(x_0) = y_0$

suppose we have obtained P_{k-1} of degree $\leq k-1$ with $P_{k-1}(x_i) = y_i$ for $0 \leq i \leq k-1$. Let's construct P_k as follows:

$$P_k(x) = P_{k-1}(x) + c(x-x_0)(x-x_1)\cdots(x-x_{k-1})$$

$*$ $P_k(x)$ is polynomial of degree at most k. Why?

$*$ P_k interpolates the data P_{k-1} interpolates

$$P_k(x_i) = P_{k-1}(x_i) = y_i \quad (0 \leq i \leq k-1)$$

$*$ To find c, set $P_k(x_k) = y_k$

$$P_k(x_k) = P_{k-1}(x_k) + c(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})$$

$$c = \frac{P_k(x_k) - P_{k-1}(x_k)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})}$$

Exercise Why is c always defined?

Lagrange polynomial interpolation

Goal: Express $p(x)$ as

$$p(x) = \underbrace{y_0 \, l_0(x) + y_1 \, l_1(x) + \cdots + y_n \, l_n(x)}_{\text{Depend on nodes}} = \sum_{k=0}^{n} y_k l_k(x)$$

Let's consider $p(x_j)$

$$p(x_j) = \sum_{k=0}^{n} y_k l_k(x_j) = y_j$$

$$l_k(x_j) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

Let's consider l_0

$$l_0(x_0) = 1$$
$$l_0(x_k) = 0 \quad \text{for } k \neq 0$$

$$l_0(x) = c\,(x-x_1)(x-x_2)\cdots(x-x_n) = c\prod_{j=1}^{n}(x-x_j)$$

Note $\underbrace{l_0(x_0)}_{=1} = c\prod_{j=1}^{n}(x_0-x_j) \Rightarrow c = \prod_{j=1}^{n}(x_0-x_j)^{-1}$

Therefore, $l_i(x) = \prod_{\substack{j=0\\ j\neq i}} \dfrac{x-x_j}{x_i-x_j} \quad (0\le i\le n)$

Exercise Given the data $(2, 1.5)$ and $(5, 4)$, find the linear interpolation function

solution $l_0(x) = \dfrac{(x-x_1)\cancel{(x-x_1)}}{x_0-x_1}$ $\quad l_1(x) = \dfrac{(x-x_0)}{(x_1-x_0)}$

$l_0(x) = \dfrac{x-5}{-3} = \dfrac{5-x}{3}$ $\quad l_1(x) = \dfrac{x-2}{3}$

$$p(x) = \frac{3}{2}\left(\frac{5-x}{3}\right) + 4\left(\frac{x-2}{3}\right)$$

Error in interpolation

Theorem $f\in C^{n+1}[a,b]$. Let p be the polynomial of degree at most n that interpolates f at $n+1$ distinct points $x_0 \ldots x_n$ in $[a,b]$. To each x in $[a,b]$, there is a point ξ_x in (a,b) such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)\prod_{i=0}^{n}(x-x_i)$$

proof * simple case: If x is one of the nodes, what is $f(x)-p(x)$?

Let x be some arbitrary point

$$w(t) = \prod_{i=0}^{n}(t-x_i) \qquad \phi \equiv f - p - \lambda w$$

where $\lambda\in\mathbb{R}$ such that $\phi(\lambda)=0$. It follows that

$$\lambda = \frac{f(x) - p(x)}{\omega(x)}$$

$\phi \in C^{n+1}[a,b]$

Note that ϕ vanishes at $x, x_0, x_1, \dots x_n$ *

Rolle's theorem: If f is continuous on $[a,b]$, if f' exists on (a,b), and if $f(a) = f(b)$ then $f'(\xi) = 0$ for some ξ in the interval (a,b)

Applying Rolle's theorem to *

ϕ' has atleast $n+1$ distinct zeros in (a,b)

Repeating this argument

ϕ'' has at least n distinct zeros in (a,b)

$\vdots$

$\phi^{(n+1)}$ has atleast one zero ξ_x in (a,b)

$$\begin{aligned}
\phi^{(n+1)} &= f^{(n+1)} - p^{(n+1)} - \lambda\,\omega^{(n+1)} \\
&= f^{(n+1)} - \lambda\,\omega^{(n+1)} \\
&= f^{(n+1)} - \lambda \underline{\frac{d^{(n+1)}}{dt^{(n+1)}}\left[\prod_{i=0}^{n}(t-x_i)\right]}
\end{aligned}$$

Highest power: t^{n+1}

$$\begin{aligned}
&= f^{(n+1)} - \lambda (n+1)! \\
&= f^{(n+1)} - \left[\frac{f(x) - p(x)}{\omega(x)}\right](n+1)!
\end{aligned}$$

(Let's evaluate this at ξ_x)

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)\,\omega(x)$$

Exercise If $f(x) = \sin(x)$ is approximated by a polynomial of degree 9 that interpolates f at ten points in the interval $[0,1]$, how large is the error on this interval?

Solution

$$|\sin(x) - p(x)| \le \frac{1}{10!} \quad \left(\text{since } |f^{(10)}(\xi_x)| \le 1 \text{ and } \prod_{i=0}^{9}|x - x_i| \le 1\right)$$

Discussion

- Equispaced nodes

$$x_i = x_{i-1} + h \; ; \quad i = 1, \dots n \qquad h > 0 \; , \; x_0 \in \mathbb{R}$$

$$\left| \prod_{i=0}^{n} (x - x_i) \right| \leq n! \, \frac{h^{n+1}}{4} \qquad \text{if} \quad x \in (x_0, x_n)$$

Therefore,

$$\max_{x \in I} |E_n f(x)| \qquad E_n f(x) = f(x) - p(x)$$

$$\leq \frac{\max_{x \in I} |f^{(n+1)}(x)|}{4(n+1)} \, h^{n+1}$$

Note $\lim_{n \to \infty} \frac{h^{(n+1)}}{4(n+1)} = 0$. However error may not always go to zero as $n \to \infty$.

Example $\quad f(x) = \frac{1}{1+x^2} \qquad I = [-5, 5]$

$\max_{x \in I} |f^{(n+1)}(x)|$ grows faster than $\frac{h^{(n+1)}}{4(n+1)}$

Runge phenomenon

---

stability $\qquad f(x_i) \qquad \hat{f}(x_i)$

$$\max_{x \in I} \left| \sum_{i=0}^{n} f(x_i) \, l_i(x) - \sum_{i=0}^{n} \hat{f}(x_i) \, l_i(x) \right|$$

$$\leq \underbrace{\max_{x \in I} \sum_{i=0}^{n} |l_i(x)|}_{\text{Lebesgue constant} = \Lambda_n(x)} \cdot \max_{0 \leq i \leq n} |f(x_i) - \hat{f}(x_i)|$$

For lagrange interpolation $\Lambda_n(x) \simeq \frac{2^{n+1}}{en(\log n + \gamma)}$

$e \approx 2.71834$

$\gamma \simeq 0.547721$

stability depends on n.