Homework 5

Early problem due on Gradescope at 11:59 pm on Tuesday, February 21st.
Due on Gradescope at 11:59 pm on Friday, February 24th.

(1) (Early problem)

(a) Let $X = \mathbb{R}$. Prove that the following set defines a topology on X:

$$\tau = \{(-x, x) \mid x \in \mathbb{R}_{>0}\} \cup \{\emptyset, \mathbb{R}\}.$$

(b) Does the following set define a topology on X?

$$\tau' = \{(a, b) \mid a, b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

Prove that it does or show why not.

(2) Let (X, τ_i) be a topological space with the indiscrete topology. Let (Y, τ_Y) be any topological space with the property that for any two points $y_1 \neq y_2$ of Y, there exists an open subset of Y containing one of the points y_1, y_2 but not the other. What are the continuous functions $f : (X, \tau_i) \to (Y, \tau_Y)$? Prove that your answer is correct.

(3) Recall that the Sierpinski space is the topological space (S, τ) with set of points $S = \{1, 2\}$ and topology $\tau = \{\emptyset, \{1\}, \{1, 2\}\}$. Prove that if X is any topological space, the set of continuous functions $f : X \to S$ is in bijection with the set of open subsets of X. That is, find a bijection

$$\phi_X : \{f : X \to S \mid f \text{ is continuous}\} \longrightarrow \left\{U \subseteq X \mid U \text{ is open }\right\}.$$

(4) Consider $\mathbb{Q}$, the set of rational numbers, as a subset of $\mathbb{R}$ with the usual topology. Prove or disprove: The subspace topology on $\mathbb{Q}$ is the discrete topology.

(5) Recall that a polynomial in the variables x, y is a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by a finite sum

$$f(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} c_{i,j} x^i y^j.$$

The *zero locus* of a set of polynomials $\{f_i(x, y)\}_{i \in I}$ is the set

$$Z(\{f_i\}) := \{(x, y) \in \mathbb{R}^2 \mid f_i(x, y) = 0 \text{ for all } i \in I\}.$$

A subset V of $\mathbb{R}^2$ is a (real) *algebraic variety* in $\mathbb{R}^2$ if it is the zero locus of some set of polynomials.

(a) Show that circles are algebraic varieties in $\mathbb{R}^2$.

(b) Show that

(i) $\emptyset, \mathbb{R}^2$ are algebraic varieties in $\mathbb{R}^2$;

(ii) If $\{V_i\}_{i \in I}$ are algebraic varieties in $\mathbb{R}^2$, then $\bigcap_{i \in I} V_i$ is an algebraic variety in $\mathbb{R}^2$ too;

(iii) If $V_1, \ldots, V_n$ are algebraic varieties, then $\bigcup_{i=1}^{n} V_i$ are algebraic varieties in $\mathbb{R}^2$ too. (Hint: to avoid frightening notation, prove this for $n = 2$, then use induction.)

(c) Conclude that the set

$$\tau_{\text{Zariski}} = \{U \subseteq \mathbb{R}^2 \mid U^c \text{ is an algebraic variety}\}$$

defines a topology on $\mathbb{R}^2$. This is called the *Zariski topology*.

(d) Show that algebraic varieties are closed subsets in the usual topology on $\mathbb{R}^2$. (Hint: polynomials are continuous functions in the usual topology.)

(e) Conclude that the usual topology on $\mathbb{R}^2$ is finer than the Zariski topology.

Remark. Algebraic geometry is the study of algebraic varieties (though usually we have $\mathbb{C}^n$ instead of $\mathbb{R}^2$). The Zariski topology is the preferred topology for algebraic geometers, because it plays well with polynomial functions and makes sense in any field (not just $\mathbb{C}$ or $\mathbb{R}$, but also weirder number systems like finite fields and the p-adic numbers...). Being "closed" has a strong meaning in the Zariski topology: it means that you can describe a subset using only polynomial equations.