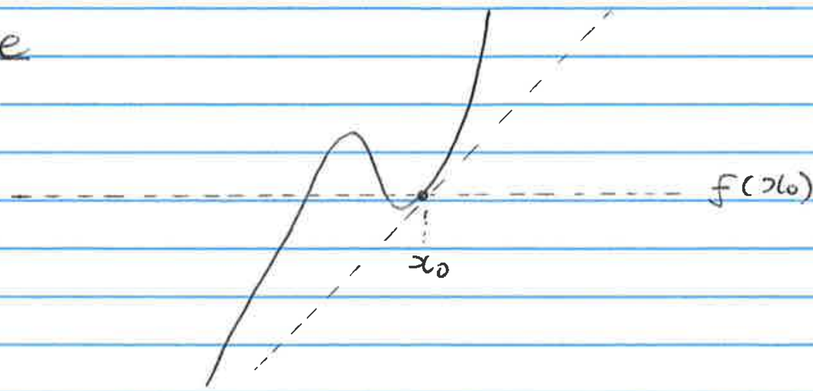
MATH 125 Lecture 5

Given a function $f(x)$, assume we know its value at some point $x = x_0$ i.e. $f(x_0)$ is known

What does this tell us about values of $f(x)$ near $x = x_0$?

Example



For points "near x_0", one idea is to approximate $f(x)$ by $f(x_0)$? Approximation by constant function

However, this does not take into account the fact that $f(x)$ might be increasing/decreasing near x_0. Let's use information about $f'(x_0)$. Recall that the slope of the tangent line passing through $(x_0, f(x_0))$ is $f'(x_0)$.

Exercise Find the equation of the tangent line passing through $(x_0, f(x_0))$

Solution Need equation of line: $y = mx + b$

$m = f'(x_0)$ $\therefore$ $y = f'(x_0)\, x + b$

plug $(x_0, f(x_0)) \Rightarrow f(x_0) = f'(x_0)\, x_0 + b$

We obtain $b = f(x_0) - x_0 f'(x_0)$

Therefore, the equation of the tangent line is

$$y = f'(x_0)\, x + f(x_0) - x_0 f'(x_0)$$

$$y = f'(x_0)(x - x_0) + f(x_0)$$

$f(x_0) \equiv$ constant approximation

$f(x_0) + f'(x_0)(x - x_0) \equiv$ Linear approximation

We need to make "near x_0" and "approximation" precise

Exercise Using the linear approximation, find a bound for $|f(x) - f(x_0)|$ if $|x - x_0| < \delta$

Solution

$$f(x) - f(x_0) = \cancel{f(x_0)} + f'(x_0)(x - x_0) - \cancel{f(x_0)}$$
$$= f'(x_0)(x - x_0)$$
$$|f(x) - f(x_0)| = |f'(x_0)(x - x_0)|$$
$$\leq |x - x_0| \, |f'(x_0)|$$
$$= \delta \, |f'(x_0)| \quad *$$

Interpret what * means

Taylor's theorem Let x and x_0 be real numbers. Let f be $(k+1)$ times continuously differentiable on the interval between x and x_0. Then there exists a number c between x and x_0 such that

$$\boxed{f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}}$$

$\equiv$ degree k Taylor polynomial

Taylor remainder

Exercise Find the degree 4 Taylor polynomial $P_4(x)$ for $f(x) = \sin(x)$ centered at $x_0 = 0$. Estimate the maximum possible error when using $P_4(x)$ to estimate $\sin(x)$ for $|x| \leq 10^{-4}$

Solution

$$f(x_0) = \sin(0) = 0$$
$$f'(x_0) = \cos(0) = 1$$
$$f''(x_0) = -\sin(0) = 0$$
$$f'''(x_0) = -\cos(0) = -1$$
$$f''''(x_0) = \sin(0) = 0$$

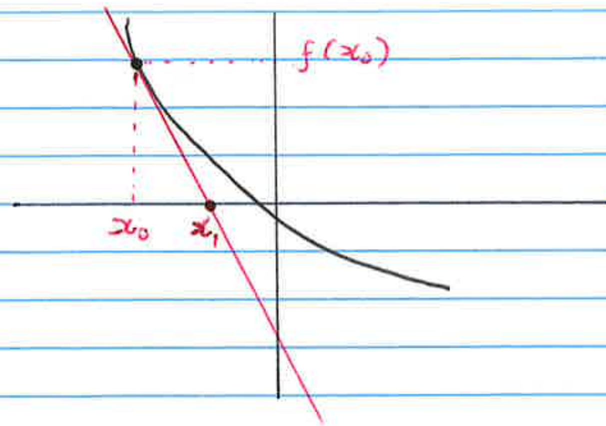
$$P_4(x) = x - \frac{1}{3}x^3$$

Remainder term:

$$\frac{f^5(c)}{5!}x^5 = \frac{x^5\cos(c)}{120}$$

$$\left|\frac{x^5\cos(c)}{120}\right| \leq \frac{10^{-20}}{120}$$

Newton's method



Approximate f locally by linear polynomial

Goal Find $f(x) = 0$

Guess or initial estimate x_0

Tangent line approximation $\equiv f(x_0) + f'(x_0)(x - x_0)$

set approximation to zero to find estimate of root

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Repeat the above process

Algorithm $x_0 \equiv$ initial guess

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{for } i = 0, 1, 2, \ldots$$

Convergence analysis

current guess: x_i

root: r

$$f(r) = f(x_i) + f'(x_i)(r - x_i) + \frac{(r - x_i)^2}{2} f''(c_i)$$

c_i is between x_i and r

Note $f(r) = 0$. Therefore, we have

$$0 = f(x_i) + f'(x_i)(r - x_i) + \frac{(r - x_i)^2}{2} f''(c_i)$$

$$-\frac{f(x_i)}{f'(x_i)} = (r - x_i) + \frac{(r - x_i)^2}{2} \frac{f''(c_i)}{f'(x_i)} \quad *$$

Remark: Goal is to relate error $|r - x_i|$ with $|r - x_{i+1}|$. Let's re-arrange $*$ for that reason

$$\underbrace{x_i - \frac{f(x_i)}{f'(x_i)}}_{x_{i+1}} - r = \frac{r - x_i}{2} f''(c_i)$$

$$x_{i+1} - r = e_i^2 \frac{f''(c_i)}{2f'(x_i)}$$

Take absolute value on both sides

$$|x_{i+1} - r| = e_i^2 \frac{|f''(c_i)|}{|2f'(x_i)|}$$

Hence, $\frac{e_{i+1}}{e_i^2} = \left| \frac{f''(c_i)}{2f'(x_i)} \right|$ (A)

Rough idea : $\frac{e_{i+1}}{e_i^2} = \left| \frac{f''(c_i)}{2f'(x_i)} \right| \approx \underbrace{\left| \frac{f''(r)}{2f'(r)} \right|}_{C}$ Quadratic convergence

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Does it converge?

f' is continuous and $f'(r) \neq 0$
choose $\varepsilon > 0$, sufficiently small, so that $f'(x) \neq 0$
for all x such that $|x - r| \leq \varepsilon$

$$I_\varepsilon = \{ x \mid |x - r| \leq \varepsilon \}$$

$$M(\varepsilon) = \frac{1}{2} \frac{\max_{x \in I_\varepsilon} |f''(x)|}{\min_{x \in I_\varepsilon} |f'(x)|}$$

Note $\lim_{\epsilon \to 0} M(\varepsilon) < \infty$. Witho that, define ε such that
$\varepsilon M(\varepsilon) < 1$.

Recall (A) Let $x_0 \in I_\epsilon$. Then $|x_1 - r| = |x_0 - r|^2 \left| \frac{f''(c_0)}{2f'(x_0)} \right|$

$$\leq e_0^2 M(\epsilon)$$
$$\leq |e_0| (\underbrace{\varepsilon M(\varepsilon)}_{\rho}) \leq |e_0| \rho$$

This implies that $|x_1 - r| \leq \varepsilon$ i.e $x_1 \in I_\epsilon$
continue this process $|x_2 - r| \leq |e_1| \rho \leq \rho^2 |e_0|$

$\therefore |e_n| \leq \rho^n |e_0|$ $\lim_{n \to \infty} e_n = 0 \Rightarrow$ convergence to r

Remarks

① Local quadratic convergence
$\Rightarrow$ Requires $f'(r) \neq 0$
$\Rightarrow$ $x^{(0)}$ sufficiently close to r

② Zeros with multiplicity m larger than 1
$(f'(r) = \dots = f^{(m-1)}(r) = 0)$

$\Rightarrow$ Linear convergence. To obtain quadratic convergence, consider the following modification

$$x^{(k+1)} = x^{(k)} - m \frac{f(x^{(k)})}{f'(x^{(k)})} \quad k \geq 0 \quad (f'(x^{(k)}) \neq 0)$$

This gives quadratic convergence

③ Stopping criterion $\quad |x^{(k)} - x^{(k-1)}| < \varepsilon$

$|f(x^{(k)})| < \varepsilon$

Secant method

Recall Newton's method

$$x^{(k+1)} = x^{(k)} - \frac{f(x_k)}{f'(x_k)}$$

What to do if $f'(x_k)$ is difficult to compute?

Do a linear approximation to $f'(x_k)$

$$f(x_k) \approx f(x_{k-1}) + f'(x_{k-1})(x_k - x_{k-1})$$

Expanding about x_{k-1}

$\longrightarrow$ $f(x_{k-1}) \approx f(x_k) + f'(x_k)(x_{k-1} - x)$

$$\Rightarrow \frac{f(x_{k-1}) - f(x_k)}{x_{k-1} - x_k} \approx f'(x_k)$$

$$\boxed{x^{(k+1)} = x^{(k)} - \left[\frac{f(x^{(k-1)}) - f(x^{(k)})}{x^{(k)} - x^{(k-1)}}\right]^{-1} f(x^{(k)})}$$

$$|x^{(k+1)} - r| \leq c\,|x^{(k)} - r|^{p} \qquad p = \frac{1+\sqrt{5}}{2} \qquad (\text{superlinear convergence})$$