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Maximum Likelihood Estimation:

The Poisson Distribution

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Outline

The Poisson distribution

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- Sample space S is nonnegative integers.
- Poisson random variable $X \in S = \{0, 1, 2, \ldots\}$
- Poisson probability distribution

$$\boxed{p_X(k) = \text{Prob}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

- This probability distribution has one parameter, λ.
- Normalization follows from Maclaurin series for e^{λ}

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{+\lambda} = 1.$$

Moments of the Poisson distribution

- Poisson probability distribution

$$\boxed{p_X(k) = \text{Prob}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

- Every application of $\lambda\frac{d}{d\lambda}$ to λ^k introduces factor of k.
- We have $\left(\lambda\frac{d}{d\lambda}\right)\lambda^k = k\lambda^k$, and hence $\left(\lambda\frac{d}{d\lambda}\right)^m\lambda^k = k^m\lambda^k$.
- Since $\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{\lambda}$, it follows $\sum_{k=0}^{\infty}k^m\frac{\lambda^k}{k!} = \left(\lambda\frac{d}{d\lambda}\right)^m e^{\lambda}$
- Multiplying by $e^{-\lambda}$ then yields

$$\boxed{E(X^m) = \sum_{k=0}^{\infty}k^m p_X(k) = \sum_{k=0}^{\infty}k^m\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\left(\lambda\frac{d}{d\lambda}\right)^m e^{\lambda}.}$$

- Right-hand side straightforward to calculate.

Moments of the Poisson distribution

- Mean (first moment):

$$E(X) = \sum_{k=0}^{\infty} p_X(k) k = e^{-\lambda} \left(\lambda \frac{d}{d\lambda} \right)^1 e^{\lambda} = \lambda$$

- Mean square (second moment):

$$E(X^2) = \sum_{k=0}^{\infty} p_X(k) k^2 = e^{-\lambda} \left(\lambda \frac{d}{d\lambda} \right)^2 e^{\lambda} = \lambda^2 + \lambda$$

- Variance:

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = (\lambda^2 + \lambda) - (\lambda)^2 = \lambda$$

- Note mean and variance both equal to parameter λ.

Estimation with the Poisson distribution

- Suppose we are given $n = 50$ Poisson-distributed samples:

$$3, 4, 3, 5, 3, 4, 5, 0, 5, 5, 5, 4, 2, 2, 3, 2, 2, 5, 5, 5, 5, 6, 5, 2, 5,$$
$$2, 2, 0, 6, 3, 6, 7, 2, 4, 5, 1, 3, 3, 5, 1, 7, 2, 0, 8, 4, 5, 8, 2, 3, 5$$

- Can we estimate λ using maximum likelihood estimation?

Defining likelihood for n independent samples

- Given results of experiment $\vec{k} = \langle k_1, k_2, \ldots \rangle$
- Define the *likelihood function*

$$L(\lambda; \vec{k}) := \mathrm{Prob}(\vec{X} = \vec{k}) = \prod_{j=1}^{n} p_{X_j}(k_j) = \prod_{j=1}^{n} e^{-\lambda} \frac{\lambda^{k_j}}{k_j!}$$

- Easier to maximize *log likelihood*

$$\ln L(\lambda; \vec{k}) = \sum_{j=1}^{n} \left[-\lambda + k_j \ln \lambda - \ln\left(k_j!\right)\right] = -n\lambda + n\overline{k} \ln \lambda - \sum_{j=1}^{n} \ln\left(k_j!\right)$$

where $\overline{k} := \frac{1}{n} \sum_{j=1}^{n} k_j$ is the *average* of the k_j.

Maximizing the likelihood for n samples

- Calculate the *log likelihood*

$$\ln L(\lambda; \vec{k}) = -n\lambda + n\overline{k} \ln \lambda - \sum_{j=1}^{n} \ln\left(k_j!\right)$$

- *Log likelihood* maximized for

$$0 = \frac{d}{d\lambda} \ln L(\lambda; \vec{k}) = -n + \frac{n\overline{k}}{\lambda}$$

- Result is $\lambda = \lambda_e$, where the *estimate* is given by

$$\boxed{\lambda_e = \overline{k} = \frac{1}{n} \sum_{j=1}^{n} k_j}$$

MLE for Poisson-distributed data

- *Maximum likelihood estimator* for Poisson distribution is

$$\boxed{\hat{\lambda}(\vec{k}) = \frac{1}{n}\sum_{j=1}^{n} k_j}$$

- For the 50 points shown earlier, $\hat{\lambda}(\vec{k}) = 3.78$.
- Actual value of λ used to sample the points was 4.

$$3, 4, 3, 5, 3, 4, 5, 0, 5, 5, 5, 4, 2, 2, 3, 2, 2, 5, 5, 5, 5, 6, 5, 2, 5,$$
$$2, 2, 0, 6, 3, 6, 7, 2, 4, 5, 1, 3, 3, 5, 1, 7, 2, 0, 8, 4, 5, 8, 2, 3, 5$$

Discussion

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- So far, these results are not terribly surprising.
- The mean of the Bernoulli trials is p.
- The mean of the Poisson distribution is λ.
- Both MLE analyses estimate parameter by sample mean.

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Summary

- We have...
  - Reviewed properties of the *Poisson distribution*
  - Calculated *moments* $E(X^m)$ of the Poisson distribution
  - Defined *likelihood function* for the Poisson distribution
  - Maximized likelihood to find the *estimate* λ_e
  - Derived the *estimator* $\hat{\lambda}(\vec{k})$