MATH 125 Lecture 6

Fixed Point iteration

Definition The real number r is a fixed point of the function g if $g(r) = r$

Example $x^{(1)} = \cos(1)$

$x^{(2)} = \cos(\cos(1)) = \cos(x^{(1)})$

$x^{(3)} = \cos(x^{(2)})$ and so on. This tends to the value $0.73908513 \equiv \alpha$

Note the fixed point iteration: $x^{(k+1)} = \cos(x^{(k)})$ $k = 0, 1, \ldots$

$x^{(0)} = 1$

$$\Downarrow$$

$\alpha = \cos(\alpha)$ where $\alpha = \lim_{k\to\infty} x^{(k)}$ $\times$

Generally, we have

$$\boxed{\begin{array}{l} x_0 = \text{initial guess} \\ x_{i+1} = g(x_i) \quad \text{for } i = 0, 1, 2, \ldots \end{array}}$$

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$x_3 = g(x_2)$ ...

If the x_i converge and g is continuous

$$g(r) = g\left(\lim_{i\to\infty} x_i\right) = \lim_{i\to\infty} g(x_i) = \lim_{i\to\infty} x_{i+1} = r$$

* We can turn root finding problems to fixed point problems

Example $x^3 + x - 1 = 0 \implies x = 1 - x^3$ (A)

$g(x) = 1 - x^3$

$\longrightarrow x = \sqrt[3]{1-x}$

$g(x) = \sqrt[3]{1-x}$ (B)

OR $3x^3 + x = 1 + 2x^3$

$(3x^2+1)x = 1 + 2x^3$

$x = \dfrac{1+2x^3}{1+3x^2}$ $g(x) = \dfrac{1+2x^3}{1+3x^2}$ (C)

(A) $g(x) = 1 - x^3$ with $x_0 = 0.5$ does not converge. It alternates between 0 and 1.

(B) $g(x) = \sqrt[3]{1-x}$ with $x_0 = 0.5$. converges to 0.6823

(C) $g(x) = \dfrac{1+2x^3}{1+3x^2}$ converges just in 4 iterations to 0.6823

Why are (A), (B) and (C) different?

contractive mapping theorem

F = contractive mapping

Let C be a closed set. If $F: C \to C$, then F has a unique fixed point. Moreover, the fixed point iteration scheme converges from any starting point x_0.

Proof uniqueness

Assume different fixed points r_1 and r_2

$|r_1 - r_2| = |F(r_1) - F(r_2)|$

> A function F is contractive if there exists $L<1$ such that $|F(x) - F(y)| \le L|x-y|$ $\forall x, y \in \text{domain}(F)$

Therefore, $|F(r_1) - F(r_2)| \le L|r_1 - r_2|$
$< |r_1 - r_2|$

$\therefore |r_1 - r_2| < |r_1 - r_2| \Rightarrow r_1 = r_2$

convergence

$0 \le |x^{(k+1)} - r| = |F(x^{(k)}) - F(r)|$
$\le L|\cancel{F}(x^{(k)}) - r|$
$\le L^2 |x^{(k-1)} - r| \dots \le L^{k+1} |x^{(0)} - r|$

Therefore, $\dfrac{|x^{(k)} - r|}{|x^{(0)} - r|} \le L^k$

$\lim_{k\to\infty} |x^{(k)} - r| = 0 \Rightarrow x^{(k)}$ converges to r

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Exercise why is there a fixed point in C in the above proof?

Solution Note $C = [a, b]$ $F(a) \in C \Rightarrow F(a) - a \ge 0$
$F(b) \in C \Rightarrow F(b) - b \le 0$

Apply IVT, there is a zero for $F(x) - x$ $\square$

Theorem Let r be a fixed point of a continuously differentiable function g in an interval around r. If $|g'(r)| < 1$, then there exists $\delta > 0$ such that $\{x^{(k)}\}$ converges to r provided $|x^{(0)} - r| < \delta$.

proof $x_{i+1} - r = g(x_i) - g(r)$

$= g'(c_i)(x_i - r)$ c_i is between x_i and r

$$|x_{i+1} - r| = |g'(c_i)|\,|x_i - r|$$

$$e_{i+1} = |g'(c_i)|\, e_i$$

Note $g'(x)$ is continuous. Let $S = |g'(r)|$

$S < 1$

There is a neighborhood around r such that

$|g'(x)| < \frac{S+1}{2}$

$$\boxed{\begin{aligned} |g'(x)| &= |g'(r) + g'(x) - g'(r)| \\ &\le |g'(r)| + |g'(x) - g'(r)| \\ &\le S + \left(\frac{1-S}{2}\right) \end{aligned}}$$

$$e_{i+1} \le \left(\frac{S+1}{2}\right) e_i$$ (assuming x_i is in that neighborhood)

similar to before, $\lim_{i\to\infty} x_i = r$

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i} = |g'(r)| = S$$

Multivariable problems

$$f: \mathbb{R}^n \to \mathbb{R}^m$$

Jacobian matrix : $(Df)_{ij} = \frac{\partial f_i}{\partial x_j}$

Example $f(x,y) = \begin{pmatrix} 5x \\ -xy^2 \\ x + 7y \end{pmatrix}$

$$Df(x,y) = \begin{pmatrix} 5 & 0 \\ -y^2 & -2xy \\ 1 & 7 \end{pmatrix}$$

First order approximation : $f(\vec{x}) \approx f(\vec{x}_k) + Df(\vec{x}_k)\cdot(\vec{x} - \vec{x}_k)$ set $f(\vec{x}) = 0$

$$\underline{Df(\vec{x}_k)}\cdot(\vec{x} - \vec{x}_k) = \underline{-f(\vec{x}_k)}$$

Linear system

If Df is invertible

$$\vec{x}_{k+1} = \vec{x}_k - [Df(x_k)]^{-1} f(x_k)$$

If it is not invertible, use pseudoinverse (To be discussed later

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Directional derivative of f in direction v $\equiv Df \cdot \vec{v}$

$$J_k \cdot (x_k - x_{k-1}) = f(x_k) - f(x_{k-1})$$

$$\underset{J_k}{\text{minimize}} \ \|J_k - J_{k-1}\|_F^2$$

$$J_k \cdot (x_k - x_{k-1}) = f(x_k) - f(x_{k-1})$$