Friday, April 21!

Wednesday, April 12, 2023 20:05

Student Hours for final:

Monday, April 24, 11:00-12:00 noon

Wednesday, April 26, 11:00-12:00

Thursday, April 27, 1:30-3:00+

Thursday, April 27, 4;30-5:45, Review Session, Nelson Auditorium, Anderson Hall SEC

Friday, April 28, 1:30-2:30+ in the math lounge

Our last class is Wednesday, April 26. There is NO class on Monday May 1.

You will have 6 days to do the final from the time you pick it up. The final will be available on Gradescope on Sunday evening, April 23, and you can pick it up any time between Sunday evening and Friday evening April 28. You will need to return it by 11:59 p.m. on the sixth day after you pick it up.

NOTE that I will be away starting April 29, and I would encourage you to plan to have questions next week, before I leave. I will not be able to answer many questions after I do leave.

The test will be something like a HW assignment, emphasizing later material. There might be a problem on Fourier series, Bessel's inequality and Parseval's Theorem (covered today), and the heat equation (covered Monday)

Hint for #3 on HW 8 $\frac{\partial}{\partial y} f(x,t) = \frac{\partial}{\partial t} f(x,t)$ as both represent the derivative of f with respect to the second variable

The orthonormal set $\Phi = \{\varphi_0, \varphi_1, \varphi_2 \ldots\}$ in the inner product space $(V, \langle \cdot,\cdot \rangle$ is ***complete*** if for every $f \in V$, $f = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k$

This sum is called the *Fourier series of f with respect to the orthonormal set* Φ and the $\langle f, \varphi_k \rangle$ *are called the Φ − Fourier coefficients of f.*

The Normalized Fourier Sine and Cosine system is complete: in $L^2([0,2\pi], \mathbb{C})$

$$\Phi = \left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \frac{\cos(3x)}{\sqrt{\pi}}, \frac{\sin(3x)}{\sqrt{\pi}}, \ldots\right\}$$

$f, g \in L^2$

$\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx$

$\|f\|_{L^2} = \sqrt{\int_0^{2\pi} |f(x)|^2 dx}$

$$\forall f \in L^2([0,2\pi], \mathbb{C}), \qquad f(x) = \frac{A_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \left(A_k \frac{\cos(kx)}{\sqrt{\pi}} + B_k \frac{\sin(kx)}{\sqrt{\pi}}\right) \textit{ where convergence is in } L^2$$

And where $A_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)dx$, $A_k = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x)\cos(kx)\,dx$, $B_k = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x)\sin(kx)\,dx$, *for* $k = 1,2, \ldots$

$A_k = \langle f, \frac{\cos kx}{\sqrt{\pi}} \rangle$, $B_k = \langle f, \frac{\sin(kx)}{\sqrt{\pi}} \rangle$

$A_0 = \langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)dx$

trig-plot

standard Fourier sine & cosine series

$f \in L^2([0,2\pi], \mathbb{C})$

$f = a_0 + \sum (a_n \cos(nx) + b_n \sin(nx))$

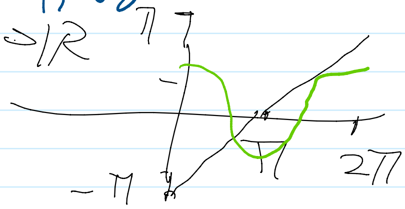
$f \in L^2([0,2\pi], \mathbb{C})$

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$ $\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

ex $f(x) = x-\pi$ $\quad f : [0, 2\pi] \to \mathbb{R}$



find std Fourier series of f.

Soln $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 0$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \underbrace{f(x)}_{\text{odd about } \pi} \cos nx\,dx = 0$$

even about π

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (x-\pi)\sin nx\,dx \qquad u = x-\pi \quad dv = \sin nx\,dx$$

$$= \frac{-2}{n} \qquad x - \pi = \underset{a_0}{0} + \sum_{n=1}^{\infty}\left(0 \cos nx + \frac{-2}{n}\sin nx\right)$$

$$x - \pi = \sum_{n=1}^{\infty} \frac{-2}{n}\sin nx \quad \text{for } x \in [0, 2\pi]$$

where convergence is in L^2

Relation between norm of $f \in V$ and norm of its Φ Fourier series in ℓ^2

Lemma $(V, \langle , \rangle)$ ip space $\Phi = \{\phi_0, \phi_1, \phi_2 \cdots \phi_N\}$ finite o.n. set in V

let $f \in V$ and assume $f = \sum_{n=0}^{N} \langle f, \phi_n \rangle \phi_n$

(ie f is a linear comb of Φ)

then $\|f\|_V^2 = \sum_{n=0}^{N} |\langle f, \phi_n \rangle|^2$

pf $f = \sum^{N} \langle f, \phi_n \rangle \phi_n$

pf $f = \sum_{n=0}^{N} \langle f, \phi_n \rangle \phi_n$

$$\|f\|_V^2 = \langle f, f \rangle = \left\langle \sum_{n=0}^{N} \langle f, \phi_n \rangle \phi_n , \sum_{m=0}^{N} \langle f, \phi_m \rangle \phi_m \right\rangle$$

$$= \sum_{n=0}^{N} \sum_{m=0}^{N} \langle \langle f, \phi_n \rangle \phi_n , \langle f, \phi_m \rangle \phi_m \rangle$$

$$= \sum_{n=0}^{N} \sum_{m=0}^{N} \langle f, \phi_n \rangle \langle f, \phi_m \rangle \langle \phi_n , \phi_m \rangle$$

recall $\langle \phi_n, \phi_m \rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$ Φ on

$$\|f\|_V^2 = \sum_{n=0}^{N} \langle f, \phi_n \rangle \overline{\langle f, \phi_n \rangle} = \sum_{n=0}^{N} |\langle f, \phi_n \rangle|^2 \quad \text{qed}$$

Thm let $(V, \langle , \rangle)$ be i space

$\Phi = \{\phi_0, \phi_1, \phi_2, \dots\}$ be on set (assume Φ is countably infinite set)

let $f \in V$ then

$$\|f\|^2 \geq \sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|^2$$ Bessel's ineq.

Φ is a complete on set iff

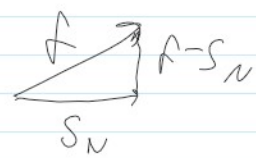
$$\forall f \in V, \quad \|f\|^2 = \sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|^2$$ Parseval's Thm.

pf $f \in V$ let $S_N = \sum_{n=0}^{N} \langle f, \phi_n \rangle \phi_n$ $N \in \mathbb{N}$

claim $\langle f - S_N , S_N \rangle = 0$ pf calculat similar to lemma

So by Pythagorean thm

$$\|f\|^2 = \|S_N\|^2 + \|f - S_N\|^2$$



as $S_N \perp f - S_N$

as $S_N \perp$ $n > N$

$$S_N = \sum_{n=0}^{N} \langle f, \phi_n \rangle \phi_n \quad \therefore \text{ by lem}$$

$$\|S_N\|^2 = \sum_{n=0}^{N} |\langle f, \phi_n \rangle|^2$$

and $$\|f\|^2 \geq \|S_N\|^2 = \sum_{n=0}^{N} |\langle f, \phi_n \rangle|^2$$

so as $N \to \infty$ $\sum_{n=0}^{N} |\langle f, \phi_n \rangle|^2$ increases

bndd above by $\|f\|^2$ $\qquad$ $S_1, S_2 \to \lim$ $\quad \|f\|^2$

ie $$\|f\|^2 \geq \sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|^2$$ qed bessel.

parseval

assume Φ is complete

so $\forall f \in V$ $$f = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n$$

$$S_N = \underbrace{\sum_{n=0}^{N} \langle f, \phi_n \rangle \phi_n}_{S_N} \longrightarrow \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n = f$$

So $$\|f - S_N\| \to 0$$

so $$\|f\|^2 = \|f - S_N\|^2 + \|S_N\|^2$$
$$= \|f - S_N\|^2 + \sum_{n=0}^{N} |\langle f, \phi_n \rangle|^2$$

let $N \to \infty$

$$\|f\|^2 = 0 + \sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|^2$$

so Φ complete $\to$ $\|f\|^2 = \sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|^2$

now assume $\|f\|^2 = \sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|^2 \quad \forall f$

$$\|f\|^2 = \|f - S_N\|^2 + \|S_N\|^2 \quad \forall n \ \forall f$$

$$\|f\|^2 = \|f - S_N\|^2 + \sum_{n=0}^{N} |\langle f, \phi_n \rangle|^2$$

$N \to \infty$

$N \to \infty \quad \|f\|^2 = 0 \qquad \sum_{n=0}^{\infty} |\langle f, \phi_n \rangle|^2 = \|f\|^2$

So $S_N \to f$ as f is arbitrary Φ complete

Fourier coefficient map.

$(V, \langle , \rangle)$ ip spa $\Phi = \{\phi_0, \phi_1, \phi_2, \dots\}$

o.n. set in V

for $f \in V$ define the Fourier coefficient map of f to be

$\mathcal{F}(f) = (\langle f, \phi_0 \rangle, \langle f, \phi_1 \rangle, \langle f, \phi_2 \rangle \dots)$

$\mathcal{F} : V \to \mathbb{C}^\infty$

ex $x - \pi = \sum_{n=1}^{\infty} \frac{-2}{n} \sin nx$ in $L^2([0, 2\pi], \mathbb{C})$

write using normalized sine & cosine set.

$$x - \pi = A_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(A_n \frac{\cos nx}{\sqrt{\pi}} + B_n \frac{\sin nx}{\sqrt{\pi}} \right)$$

$\Phi = \{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}} \dots \}$

Find $\mathcal{F}(x - \pi)$

So if $f = A_0 \frac{1}{\sqrt{2\pi}} + \sum \left(A_n \frac{\cos nx}{\sqrt{\pi}} + B_n \frac{\sin nx}{\sqrt{\pi}} \right)$

$\mathcal{F}(f) = (A_0, A_1, B_1, A_2, B_2 \dots)$

$x - \pi = \sum_{n=1}^{\infty} \frac{-2}{n} \sin nx$

$A_0 = 0 \quad A_n = 0 \qquad B_n \frac{\sin nx}{\sqrt{\pi}} = \frac{-2}{n} \sin nx$

$$B_n = \frac{-2\sqrt{\pi}}{n}$$

$$\Phi(x-\pi) = \left(0,\ 0,\ \frac{-2\sqrt{\pi}}{1},\ 0,\ \frac{-2\sqrt{\pi}}{2},\ \dots\right)$$

$\forall f \in V^2$

$$\|f\|^2 = |A_0|^2 + \sum\left(|A_n|^2 + |B_n|^2\right) \quad \text{as normalized sine/cosine set is complete}$$

when $f = A_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(A_n \frac{\cos nx}{\sqrt{\pi}} + B_n \frac{\sin nx}{\sqrt{\pi}}\right)$

$$f = x - \pi = \sum \frac{-2\sqrt{\pi}}{n} \frac{\sin nx}{\sqrt{\pi}}$$

$$\|f\|^2 = \|x-\pi\|^2 = \sum_{n=0}^{\infty}\left|\frac{-2\sqrt{\pi}}{n}\right|^2 = \sum_{n=1}^{\infty} \frac{4\pi}{n^2}$$

$$\frac{1}{4\pi}\int_0^{2\pi}(x-\pi)^2\,dx = \frac{\|x-\pi\|^2}{4\pi} = \quad = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Back to general case $\Phi = \{\varphi_0, \varphi_1, \varphi_2, \dots\}$ ON

$f \in V$ $\Phi(f) = (\langle f, \varphi_0\rangle, \langle f, \varphi_1\rangle, \dots, \langle f, \varphi_n\rangle, \dots) \in \ell^2$

and $\forall f \in V$ $\|f\|_V^2 \geq \|\Phi f\|_{\ell^2}^2$

Φ is

$$\|\Phi f\|^2 = \langle \Phi f, \Phi f\rangle$$

$$= \sum_{n=0}^{\infty} |\langle f, \varphi_n\rangle|^2 \leq \|f\|^2 \quad \text{by Bessel}$$

$\therefore\ \Phi : V \to \ell^2$