(1) Let $A = \{1, 2\}$, $B = \{1, 2, 3, 5\}$, $C = \{1, 3, 4, 6\}$.

(a) Compute the intersections:

$\cap$	A	B	C
A			
B			
C			

(b) Compute the unions:

$\cup$	A	B	C
A			
B			
C			

(c) Prove that if X, Y, Z are arbitrary sets and $X \subseteq Y$, then $X \cap Z \subseteq Y \cap Z$.

(d) Is it true that if $X \cap Z \subseteq Y \cap Z$, then $X \subseteq Y$? Prove or give a counterexample.

(2) Prove or find a counterexample: If A and B are sets, then $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(3) Let $X = \{a, b, c, d\}, Y = \{1, 2, 3, 4, 5\}$, and let $f : X \to Y$ be the function defined by the table

x	a	b	c	d
$f(x)$	4	1	2	4

(a) Complete the table by computing the appropriate sets for each subset $A \subseteq X$:

A	$f(A)$	$f^{-1}(f(A))$
$\{a\}$		
$\{b\}$		
$\{a, b\}$		
$\{c\}$		
$\{a, b, c\}$		
$\{d\}$		
X		

(b) Complete the table by computing the appropriate sets for each subset $B \subseteq Y$

B	$f^{-1}(B)$	$f(f^{-1}(B))$
$\{1\}$		
$\{2,4\}$		
$\{3,5\}$		
$\{2,4,5\}$		
Y		

(4) Prove or find a counterexample:

(a) If $f : X \to Y$ is a function and $A, A' \subseteq X$ are two subsets of X, then $f(A \cup A') = f(A) \cup f(A')$.

(b) If $f : X \to Y$ is a function and $A, A' \subseteq X$ are two subsets of X, then $f(A \cap A') = f(A') \cap f(A)$.

(5) Find and fix the problem in the following proof:

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 1 - x$. Then f is a surjective function.*

Proof. In order to show that f is surjective, we need to show that any element of the codomain has a pre-image. Let $1 - y \in \mathbb{R}$ be any element. Let $x = y$. Then $f(x) = 1 - x = 1 - y$, so the element $1 - y$ is hit by x. This shows that every element of $\mathbb{R}$ has a pre-image, so f is surjective. $\square$

(6) Define an operation $+$ on sets by $A + B = (A \cup B) - (A \cap B)$.

(a) Sketch a Venn diagram of $A + B$. Conclude that $A + B = B + A$ for all sets A, B.

(b) If A is any set, what is $A + A$? What about $A + \emptyset$?

(c) Prove that for any sets A, B, C, we have $(A + B) + C = A + (B + C)$. (A Venn diagram computation is fine.)

(d) Prove that for any sets A, B, C, we have $A \cap (B + C) = (A \cap B) + (A \cap C)$.

(e) Does it make sense to define a sum of sets $\sum_{i \in I} A_i$ in this way for an arbitrary (possibly infinite) collection of sets $\{A_i\}_{i \in I}$?

(7) Given a function $f : X \to Y$, we can think of the operation of taking pre-images as defining a function between the powersets of X and Y:

$$\begin{aligned} f^{-1} : \mathcal{P}(Y) &\to \mathcal{P}(X) \\ B &\mapsto f^{-1}(B). \end{aligned}$$

(a) Show that if $\mathrm{id}_X : X \to X$ is the identity function (defined by $\mathrm{id}_X(x) = x$ for all $x \in X$), then

$$(\mathrm{id}_X)^{-1} = \mathrm{id}_{\mathcal{P}(X)} : \mathcal{P}(X) \to \mathcal{P}(X)$$

is the identity function on $\mathcal{P}(X)$.

(b) If $f : X \to Y$ and $g : Y \to Z$ are functions, we have associated functions

$$\begin{aligned} g^{-1} &: \mathcal{P}(Z) \to \mathcal{P}(Y), \\ f^{-1} &: \mathcal{P}(Y) \to \mathcal{P}(X), \\ \text{and } (g \circ f)^{-1} &: \mathcal{P}(Z) \to \mathcal{P}(X). \end{aligned}$$

Show that

$$f^{-1} \circ g^{-1} = (g \circ f)^{-1}.$$

Remark 2. The last exercise shows that taking powersets extends to what is known as a “contravariant functor." We should think this means that taking powersets is a very orderly, structure-preserving thing to do.