(1) Let $X = \mathbb{R}$ and consider the collection of subsets

$$\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}.$$

(a) Check that $\mathcal{B}$ is a basis.

(b) Call the topology generated by $\mathcal{B}$ the **lower limit topology**. Intuitively, in this topology, points are "close" to the points to their right and "far" from the points to their left. Write $\mathbb{R}_\ell$ for $\mathbb{R}$ with the lower limit topology. Write $\mathbb{R}$ for $\mathbb{R}$ with the usual topology. How do the two topologies compare?

(c) Consider the function $f : \mathbb{R}_\ell \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Is f continuous?

(d) Consider the function $g : \mathbb{R}_\ell \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

Is g continuous?

(e) Do you have a guess for which functions $f : \mathbb{R}_\ell \to \mathbb{R}$ are continuous?

(2) (a) Show that any two open intervals (a, b) and (a', b') are homeomorphic.

(b) Show that any two closed intervals $[a, b]$ and $[a', b']$ are homeomorphic.

(c) Take a guess: is $(0, 1)$ homeomorphic to $[0, 1]$? (It's actually a bit difficult to answer this without some more topology.)

(3) Let τ be the usual topology on $\mathbb{R}$. One of our first examples of an unusual topology was the following topology on $\mathbb{R}$:

$$\tau' = \{U \subseteq \mathbb{R} \mid U \in \tau\} \cup \{U \cup \{0\} \mid U \in \tau\}.$$

Prove that $(\mathbb{R}, \tau')$ is homeomorphic to $(-\infty, -1) \cup \{0\} \cup (1, \infty)$ with the usual topology.

Definition 1. The n**-dimensional sphere** $\mathbb{S}^n$ is the topological space with underlying set of points

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

together with the subspace topology from $\mathbb{R}^{n+1}$.

(4) Here's a fun example which is important in complex analysis. Let $N = (0, 0, 1)$ be the "north pole" of $\mathbb{S}^2$. We will show that

$$\mathbb{S}^2 - \{N\} \cong \mathbb{R}^2$$

via **stereographic projection**. If (x, y, z) is a point of the sphere other than the north pole, we obtain its stereographic projection $\phi(x, y, z)$ by the following procedure:

(i) Construct a line L through N and (x, y, z).
(ii) Find the unique intersection point $P = (X, Y, 0)$ of L with the $z = 0$ plane.
(iii) Set $\phi(x, y, z) = (X, Y)$.

(a) Find a formula for ϕ. Conclude that ϕ is a continuous function. (Some geometry from Calc III may be handy.)

(b) Find a formula for an inverse function

$$\begin{aligned} \psi : \mathbb{R}^2 &\to \mathbb{S}^2 - \{N\} \\ (X, Y) &\mapsto ? \end{aligned}$$

of ϕ. Conclude that ϕ is a homeomorphism. (One way about it: Parametrize the line from N to $P = \langle X, Y, 0 \rangle$ by the vector formula $\ell(t) = N + t(P - N)$. What is the non-zero value of t for which $\ell(t)$ lies on the unit sphere?)