Bayesian estimation

Parameter estimation using Bayesian reasoning

Bruce M. Boghosian

Department of Mathematics

Tufts University

Outline

1 Bayesian estimation

2 Example of Bayesian estimation

3 Summary

Application to estimation

Bruce M. Boghosian

Bayesian estimation

Example of Bayesian estimation

Summary

- Let W be a statistic dependent on a parameter θ. Call its pdf $f_W(w \mid \theta)$.
- Suppose that θ is the value of a random variable Θ.
- Denote the prior distribution of Θ by
  - $p_\Theta(\theta)$ if Θ is discrete
  - $f_\Theta(\theta)$ if Θ is continuous
- Posterior distribution of Θ, given that $W = w$, is then

$$g_\Theta(\theta \mid W = w) = \begin{cases} \dfrac{p_W(w \mid \theta) f_\Theta(\theta)}{\int d\xi \; p_W(w \mid \xi) f_\Theta(\xi)} & \text{if } W \text{ is discrete} \\ \\ \dfrac{f_W(w \mid \theta) f_\Theta(\theta)}{\int d\xi \; f_W(w \mid \xi) f_\Theta(\xi)} & \text{if } W \text{ is continuous} \end{cases}$$

- If Θ is discrete
  - Replace the $f_\Theta(\theta)$ by $p_\Theta(\theta)$.
  - Replace the integrals over θ by sums.

Example of Bayesian estimation

- Exponentially distributed random variable W

$$f_W(w \mid \lambda) = \lambda e^{-\lambda w}$$

- Recall the mean is $E(w) = 1/\lambda$
- Suppose also that your prior for λ is the uniform distribution on $[a, b]$,

$$f_\Lambda(\lambda) = \begin{cases} \frac{1}{b-a} & \text{if } 0 < a \leq \lambda \leq b \\ 0 & \text{otherwise} \end{cases}$$

- Suppose that you sample W and find a value w_s.

Example (continued)

- Suppose that you sample W and find a value w_s.
- The posterior distribution of λ is

$$g_\Lambda(\lambda \mid W = w_s) = \frac{f_W(w_s \mid \lambda) f_\Lambda(\lambda)}{\int_{-\infty}^{+\infty} d\xi \; f_W(w_s \mid \xi) f_\Lambda(\xi)}$$

$$= \begin{cases} \frac{\lambda e^{-\lambda w_s} \frac{1}{b-a}}{\int_a^b d\xi \; \xi e^{-\xi w_s} \frac{1}{b-a}} & \text{if } a \le \lambda \le b \\ 0 & \text{otherwise} \end{cases}$$

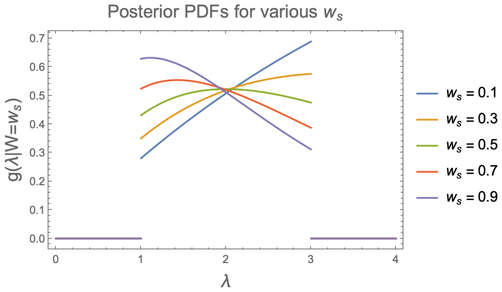
$$= \begin{cases} \frac{\lambda w^2 e^{-\lambda w_s}}{(1+aw_s)e^{-aw_s} - (1+bw_s)e^{-bw_s}} & \text{if } a \le \lambda \le b \\ 0 & \text{otherwise} \end{cases}$$

- What does this posterior distribution look like for various samples w_s?

Example (continued)

- Suppose your prior for $f_\Lambda(\lambda)$ has $a = 1$ and $b = 3$
- The posterior distribution of λ is

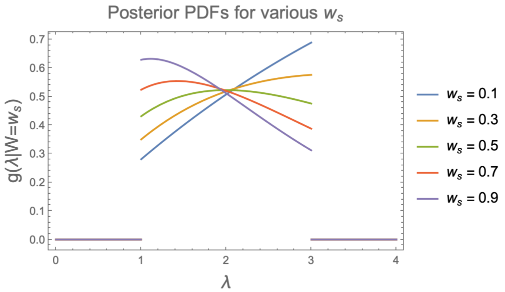
$$g_\Lambda(\lambda \mid W = w_s) = \begin{cases} \frac{\lambda w^2 e^{-\lambda w_s}}{(1+aw_s)e^{-aw_s} - (1+bw_s)e^{-bw_s}} & \text{if } a \leq \lambda \leq b \\ 0 & \text{otherwise} \end{cases}$$



A potential drawback of Bayesian estimation

- There is no way that the posterior $g_\Lambda(\lambda \mid W = w_s)$ can be nonzero anywhere outside of the region $[a, b]$, where the first prior $f_\Lambda(\lambda)$ was nonzero.
- It is clear that this is generally true from the equation

$$g_\Lambda(\lambda \mid W = w_s) = \frac{f_W(w_s \mid \lambda) f_\Lambda(\lambda)}{\int_{-\infty}^{+\infty} d\xi \; f_W(w_s \mid \xi) f_\Lambda(\xi)}$$



Summary

- We have studied Bayesian estimation of parameters
- We have provided an example of Bayesian estimation for the exponential distribution.
- We have identified a drawback of Bayesian reasoning – if a prior pdf begins at zero at some point in parameter space, it can never become nonzero at that point.