MATH 125 Lecture 12

Splines

- The origin of the word is devices used by shipbuilders to trace smooth shapes
- used in computer graphics

Setup collection of data

(t_1, y_1) (t_2, y_2) ... (t_{n+1}, y_{n+1})

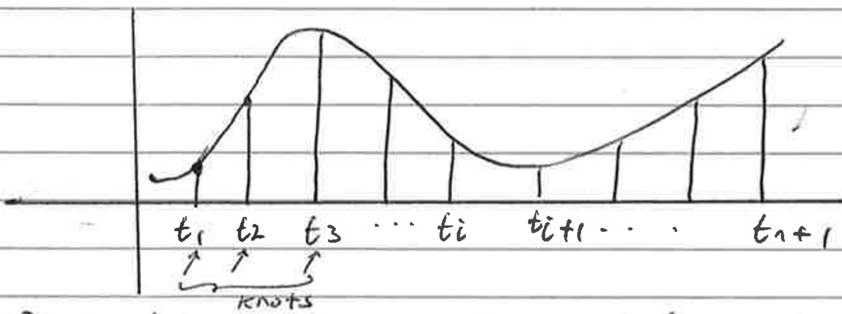

Figure 1: Illustration of cubic spline

$P_i \equiv$ denotes polynomial between t_i and t_{i+1}

$$P_i(t) = a_i + b_i \frac{(t-t_i)}{(t_{i+1}-t_i)} + c_i \frac{(t-t_i)^2}{(t_{i+1}-t_i)^2} + d_i \frac{(t-t_i)^3}{(t_{i+1}-t_i)^3} \; ; t_i \le t \le t_{i+1}$$

Change of variable: $u = \dfrac{t-t_i}{t_{i+1}-t_i}$

Then, $t_i \le t \le t_{i+1}$ becomes $0 \le u \le 1$

$$P_i(u) = a_i + b_i u + c_i u^2 + d_i u^3 \qquad 0 \le u \le 1$$

(This is simpler to work with)

Remark Although Fig. 1 assumes $t_1 \le t_2 \le \dots < t_{n+1}$, it is not a requirement to derive cubic splines.

Constraint 1 Polynomials must agree at points t_i, $2 \le i \le n$

$P_i(0) = a_i = y_i$ ①

$P_i(1) = a_i + b_i + c_i + d_i = y_{i+1}$ ②

* This gives us $2(n-1) = 2n-2$ constraints

Constraint 2 Smoothness

$D_i \equiv (1 \le i \le n+1)$: value of the first derivative at the knots

Essentially, constraint 2 says the derivatives at interior points must agree

$$③\ P_i'(0) = b_i = D_i$$
$$④\ P_i'(1) = b_i + 2c_i + 3d_i = D_{i+1}$$
$\Big] \ 2 \le i \le n$

* $(n-1)$ constraint

Let's solve for a_i, b_i, c_i and d_i using ① - ④

$$a_i = y_i$$
$$b_i = D_i$$

using ② $\quad c_i + d_i = y_{i+1} - y_i - D_i$ ⑤

using ④ $\quad 2c_i + 3d_i = D_{i+1} - D_i$ ⑥

solve for c_i and d_i

$$2\,Eq.5 - Eq.6 \Rightarrow -d_i = 2(y_{i+1} - y_i - D_i) - D_{i+1} + D_i$$
$$\therefore d_i = 2(y_i - y_{i+1}) + D_i + D_{i+1} \quad ⑦$$

using ⑤, solve for c_i.

$$c_i = y_{i+1} - y_i - D_i - 2(y_i - y_{i+1}) - D_i - D_{i+1}$$
$$c_i = 3y_{i+1} - 3y_i - 2D_i - D_{i+1}$$
$$c_i = 3(y_{i+1} - y_i) - 2D_i - D_{i+1}$$

1.Ⓐ	$a_i = y_i$
1.Ⓑ	$b_i = D_i$
1.Ⓒ	$c_i = 3(y_{i+1} - y_i) - 2D_i - D_{i+1}$
1.Ⓓ	$d_i = 2(y_i - y_{i+1}) + D_i + D_{i+1}$

* Main idea:
All the coefficients a_i, b_i, c_i, d_i follow from knowing $\{y_i\}_{i=1}^{n+1}$ and $\{D_i\}_{i=1}^{n+1}$

Constraint 3 "More smoothness"

Second derivatives match at interior points $(t_2 \ldots t_n)$

$$P_{i-1}''(1) = P_i''(0)$$
$$2c_{i-1} + 6d_{i-1} = 2c_i \qquad 2 \le i \le n$$

Also $P_1(0) = y_1$ and $P_n(1) = y_{n+1}$

$$\Rightarrow a_1 = y_1$$
$$\Rightarrow a_n + b_n + c_n + d_n = y_{n+1}$$

constraint 4

Unknowns $4n$ unknowns (n polynomials, 4 coefficients each)

# of Equations	
constraint 1	$2n-2$
constraint 2	$n-1$
constraint 3	$n-1$
constraint 4	2

Total: $4n-2$

Need 2 more equations

Natural boundary conditions $\quad P_1''(0) = 0 \quad ; P_n''(1) = 0$

$\Longrightarrow \quad c_1 = 0 \quad$ and $\quad 2c_n + 6d_n = 0$

using these equations $\left[\begin{array}{l} C_i = 3(y_{i+1} - y_i) - 2D_i - D_{i+1} \\ 2C_{i-1} + 6d_{i-1} = 2C_i \end{array} \right. \longrightarrow \quad 2 \le i \le n$

$$\Downarrow$$

$$2C_{i-1} + 6d_{i-1} = 2\left[3(y_{i+1} - y_i) - 2D_i - D_{i+1}\right]$$

Using 1.ⓒ and 1.ⓓ

$$C_{i-1} = 3(y_i - y_{i-1}) - 2D_{i-1} - D_i$$
$$d_{i-1} = 2(y_{i-1} - y_i) + D_{i-1} + D_i$$

$$2\left[3(y_i - y_{i-1}) - 2D_{i-1} - D_i\right] + 6\left[2(y_{i-1} - y_i) + D_{i-1} + D_i\right] = 2\left[3(y_{i+1} - y_i) - 2D_i - D_{i+1}\right]$$

$$\Downarrow$$

$$D_{i+1} + 4D_i + D_{i-1} = 3(y_{i+1} - y_{i-1}) \qquad 2 \le i \le n$$

we also know $\quad c_1 = 3(y_2 - y_1) - 2D_1 - D_2$

$\quad 2D_1 + D_2 = 3(y_2 - y_1)$

we also know $\quad c_n = -3d_n$

$\quad \Longrightarrow a_n + b_n - 2d_n = y_{n+1}$

Therefore, $D_n + 2D_{n+1} = 3(y_{n+1} - y_n)$

We now have the following linear system

$$\begin{pmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ \vdots \\ D_{n+1} \end{pmatrix} = \begin{pmatrix} 3(y_2 - y_1) \\ 3(y_3 - y_1) \\ 3(y_4 - y_2) \\ \vdots \\ 3(y_{n+1} - y_{n-1}) \\ 3(y_{n+1} - y_n) \end{pmatrix}$$

$$\Downarrow$$

Tridiagonal, invertible

solve for $\{D_i\}_{i=1}^{n+1}$ and use (1a) – (1d) to get the polynomials

Exercise Obtain a cubic spline fit to the data $(0,1)$, $(1,4)$, $(2,10)$, $(3,8)$ under the conditions $f''(0) = 0 = f''(3)$ and valid in the interval $[1,2]$