Scott Fullenbaum Linear Algebra Exam 1

Pledge: I pledge that I have neither given nor received assistance on this exam and have used only the reference sources cited above

- Scott F ______ → Signature

Scott Fullenbaum

1 a) $\boxed{\text{True}}$

1 b) $\boxed{\text{False}}$

1 c) $\boxed{\text{False}}$ $\quad x_1\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\\1\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

1 d) $\boxed{\text{True}}$

2 a) $x_1\begin{bmatrix}1\\-1\\2\end{bmatrix}+x_2\begin{bmatrix}2\\-4\\-1\end{bmatrix}+x_3\begin{bmatrix}-1\\1\\1\end{bmatrix}=\begin{bmatrix}-3\\5\\2\end{bmatrix}$

2 b)
$$\underbrace{\begin{bmatrix}1 & 2 & -1\\-1 & -4 & 1\\2 & -1 & -1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\vec{x}}\underbrace{=}_{=}\underbrace{\begin{bmatrix}-3\\5\\2\end{bmatrix}}_{\vec{b}}$$

2 c)
$$\begin{bmatrix}1 & 2 & -1\\-1 & -4 & 1\\2 & -1 & -1\end{bmatrix}\begin{bmatrix}2\\-1\\3\end{bmatrix}=\begin{bmatrix}-3\\5\\2\end{bmatrix}$$

2 d) When $A\vec{x}=0$,
$$\begin{bmatrix}1 & 2 & -1\\-1 & -4 & 1\\2 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\vec{0}\sim\left[\begin{array}{ccc:c}1 & 2 & -1 & 0\\-1 & -4 & 1 & 0\\2 & -1 & -1 & 0\end{array}\right]\ r_2=r_2+r_1$$

$$\sim\left[\begin{array}{ccc:c}1 & 2 & -1 & 0\\0 & -2 & 0 & 0\\2 & -1 & -1 & 0\end{array}\right]\ \overset{r_3=r_3-2r_1}{\sim}\left[\begin{array}{ccc:c}1 & 2 & -1 & 0\\0 & -2 & 0 & 0\\0 & -5 & 1 & 0\end{array}\right]\ r_3=r_3-\frac{5}{2}r_2$$

$$-5-\tfrac{5}{2}(-2)$$

$$\sim\left[\begin{array}{ccc:c}1 & 2 & -1 & 0\\0 & -2 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]\ \overset{r_1=r_1+r_2+r_3}{\sim}\left[\begin{array}{ccc:c}1 & 0 & 0 & 0\\0 & -2 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

Meaning $x_1=0$, $x_2=0$, and $x_3=0$, and since the only solution is the trivial solution, the columns of A are linearly independent.

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2 e)
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ -1 & -4 & 1 & 5 \\ 2 & -1 & -1 & 2 \end{array}\right] \overset{r_2=r_2+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -2 & 0 & 2 \\ 2 & -1 & -1 & 2 \end{array}\right]$$

$$\overset{r_3=r_3-2r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -2 & 0 & 2 \\ 0 & -5 & 1 & 8 \end{array}\right] \overset{r_3=r_3-\frac{5}{2}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\overset{r_1=r_1+r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \rightarrow \begin{cases} x_1 = 2 \\ -2x_2 = 2 \\ x_3 = -3 \end{cases} \rightarrow \boxed{\begin{matrix} x_1 = 2 \\ x_2 = -1 \\ x_3 = -3 \end{matrix}}$$

~~Solution is:~~

3 For $\vec{b}$ to be in the span of $\{v_1, v_2, v_3\}$ b, must be a linear combination of v_1, v_2, v_3 meaning:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}, \text{ where } \vec{b} \in \mathbb{R}^3, c_1, c_2, c_3 \in \mathbb{R}$$

$$c_1\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ where } c_1, c_2, c_3 \in \mathbb{R}$$

which is equivalent to the system:

$$\begin{cases} c_1 + 2c_2 + c_3 = b_1 \\ c_1 - c_3 = b_2 \\ -2c_1 + c_2 + 3c_3 = b_3 \end{cases}$$

meaning b_1, b_2, and b_3 must be solutions to the system on the left for $\vec{b}$ to be in the span of $\{v_1, v_2, v_3\}$

4 By Theorem 7, a set of at least 2 vectors is linearly dependent if one vector is a linear combination of the others. $2\vec{v}+3\vec{w} = 2(\vec{v})+3\vec{w}$, meaning $\{v, w, 2v+3w\}$ is linearly dependent.

Any pair from the set $\{v, w, 2v+3w\}$ is linearly independent because each of the sets, $\{v, w\}$, $\{v, 2v+3w\}$, and $\{w, 2v+3w\}$, cannot have one of their vectors written as a linear combination of the other, as $\{v, w\}$ is linearly independent. Thus by Theorem 7, all pairs from the set are linearly independent

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5 a) If $T(x)$ is linear, then $T(x+y) = T(x) + T(y)$,
and $T(c\vec{u}) = cT(\vec{u})$

Let $\vec{x}' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$, $\vec{y}' = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}$

$$T(\vec{x}' + \vec{y}') = T\left(\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x'_1 + y'_1 \\ x'_2 + y'_2 \end{bmatrix}\right) =$$

$$(x'_1 + y'_1)\vec{v} + (x'_2 + y'_2)\vec{u} = x'_1\vec{v} + y'_1\vec{v} + x'_2\vec{u} + y'_2\vec{u}$$

$$T(x') + T(y') = x'_1\vec{v} + y'_1\vec{v} + x'_2\vec{w} + y'_2\vec{w}$$

$$T(x+y) = T(x) + T(y)$$

Let $\vec{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $c \in \mathbb{R}$

$$T(c\vec{x}) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = cx_1\vec{v} + cx_2\vec{w}$$

$$cT(\vec{x}) = c(x_1\vec{v} + x_2\vec{w}), \quad cT(\vec{x}) = T(c\vec{x})$$

Both conditions, $T(x+y) = T(x) + T(y)$, and $T(c\vec{x}) = cT(\vec{x})$ are true, meaning $T(x)$ is a linear transformation. $\boxtimes$

5 b) By theorem 12, T is one-to-one if and only if columns of A, where A is the standard matrix, are linearly independent.
By theorem 10, $T(\vec{x}) = A\vec{x}$ as $T(x)$ is a linear transformation, meaning $T(x)$ can be rewritten as:

$$\underbrace{\begin{bmatrix} \vec{v} & \vec{u} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = T(\vec{x}),$$

For T to be one to one, the columns of A, v and u, must be linearly independent, meaning that v and u are linearly independent if and only if T is one-to-one.

$\boxtimes$

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6 a) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\text{Standard matrix } A: \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

6 b) For $T(x)$ to be one to one, by Theorem 12, the columns of A must be linearly independent. To solve for this, find solutions to

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{cases} x_2 = 0, \\ x_1 = 0 \\ x_1 + x_2 = 0 \\ x_3 = 0 \end{cases}$$

The only solution is $x_1 = 0, x_2 = 0, x_3 = 0$, which is the trivial solution meaning the columns of A are linearly independent and $T(x)$ is one-to-one.

6 c) T is on-to if and only if columns of A span $\mathbb{R}^m$, meaning there is a pivot in every row of A, which is proven by Theorem 12

Looking at A

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since A has more rows than columns, it cannot have a pivot in every row.

Since A doesn't have a pivot in every row, A does not span $\mathbb{R}^m$ or $\mathbb{R}^4$, and T is not on-to

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7a) $\boxed{\begin{array}{l} 20x_1 + 30x_2 + 0x_3 = y_1 \\ 15x_1 + 5x_2 + 40x_3 = y_2 \end{array}}$ number corresponds to each plant

7b) Since $T(x)$ is a matrix transformation,
$T(x) = A\vec{x}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ that $\overset{2\times3}{2}$ $\overset{3\times1}{\text{fulfills}}$ the system in 7a)

$$\boxed{A = \begin{bmatrix} 20 & 30 & 0 \\ 15 & 5 & 40 \end{bmatrix}}$$

7c) If $\vec{x} = \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}$ to find milk and yogurt produced,

$$\underbrace{\begin{bmatrix} 20 & 30 & 0 \\ 15 & 5 & 40 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 1200 \\ 950 \end{bmatrix},$$

$\boxed{\text{The plants produced 1200 milk and 950 yogurt}}$