For this, note that

$$2 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = \sin \frac{n\pi(x - ct)}{l} + \sin \frac{n\pi(x + ct)}{l},$$

so that

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin \frac{n\pi(x - ct)}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi(x + ct)}{l}$$

$$= \frac{1}{2}[f(x - ct) + f(x + ct)].$$

Now we verify that

$$y(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

satisfies all the conditions. First,

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}c^2[f''(x - ct + f''(x + ct)] = c^2\frac{\partial^2 y}{\partial x^2}.$$

Second, at $t = 0$, $y(x, 0) = f(x)$ and

$$\frac{\partial y}{\partial t}(x, 0) = \frac{1}{2}c[-f'(x) + f'(x) + f'(x)] = 0.$$

Third, $y(0, t) = \frac{1}{2}[f(-ct) + f(ct)] = 0$, because f is odd (when extended); and

$$y(l, t) = \frac{1}{2}[f(l - ct) + f(l + ct)] = 0,$$

because $f(l - ct) = -f(ct - l) = -f(ct + l)$, since $f(x) = f(x + 2l)$ by periodicity. ■

10.7.2 Theorem *If f is square integrable, then, for each $t > 0$,*

$$T(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/l^2} \cos \frac{n\pi x}{l}$$

converges uniformly, is differentiable, and satisfies the heat equation and boundary conditions. At $t = 0$, it equals f in the sense of convergence in the mean, and pointwise if f is of class C^1. As usual,

$$a_n = \frac{2}{l}\int_0^1 f(x) \cos \frac{n\pi x}{l}\, dx.$$

Proof To show that $T(x,t)$ satisfies the heat equation, what we must do is justify term-by-term differentiation in both x and t. For this we use Theorem 5.4.3. What we must show is that the series of derivatives

$$-\sum_{n=1}^{\infty} \frac{a_n \pi^2 n^2}{l^2} e^{-n^2\pi^2 t/l^2} \cos \frac{n\pi x}{l}$$

(which represents both $\partial T/\partial t$ and $\partial^2 T/\partial x^2$) converges uniformly in t and in x, which we do by the Weierstrass M test. Since $|a_n|$ is bounded ($a_n \to 0$, in fact), we can omit the terms $a_n \pi^2/l^2$. Now in x, let $M_n = n^2 e^{-n^2\pi^2 t/l^2}$. By the ratio test, $\sum M_n < \infty$, and so the series converges uniformly in x.

Uniformly in t means uniformly for all $t \geq \varepsilon$, where $\varepsilon > 0$ is arbitrary but fixed. In this case we let $M_n = n^2 e^{-n^2\pi^2 \varepsilon/l^2}$ and note that $\sum M_n$ converges. (We cannot allow $t = 0$.) The rest of the theorem is obvious. ■

10.7.3 Theorem *In Theorem* **10.7.2,**

$$\lim_{t \to 0, t>0} T(x,t) = f(x)$$

in the sense of convergence in mean, and, converges uniformly (and pointwise) if f is continuous, with f' sectionally continuous. More generally, for any f, if the Fourier series of f converges at x to $f(x)$, then $T(x,t) \to f(x)$ as $t \to 0$.

Proof For the first part, it will suffice to show the following.

Lemma 14 *For each $t > 0$, suppose $f_t \in \mathcal{V}$, an inner product space, and $\varphi_0, \varphi_1, \ldots$ is a complete orthonormal basis. Let*

$$f_t = \sum_{n=1}^{\infty} c_n(t)\varphi_n, \quad f = \sum_{n=1}^{\infty} c_n \varphi_n.$$

If

$$\lim_{t \to 0} \sum_{n=1}^{\infty} |c_n(t) - c_n|^2 = 0,$$

then $f_t \to f$ (in mean).

Proof The result follows from Parseval's relation $\|f_t - f\|^2 = \sum_{n=1}^{\infty} |c_n(t) - c_n|^2$. ■

In the case of Theorem **10.7.3**, we must show that

$$\lim_{t\to 0}\sum_{n=1}^{\infty}|a_n|^2(1-e^{-n^2\pi^2t/l^2})^2=0.$$

To do this, it is enough to show that the function $g(t)=\sum_{n=1}^{\infty}|a_n|^2(1-e^{-n^2\pi^2t/l^2})^2$ is continuous in t, since $g(0)=0$. To show that $g(t)$ is continuous, we shall show that the series converges uniformly in t. To do this, Abel's test will be used. The form we need is the following:

Lemma 15 *Let $\sum_{n=1}^{\infty}c_n$ be a convergent series and $\varphi_n(t)$ a uniformly bounded, decreasing (respectively, increasing) sequence defined for $t\geq 0$. Then $g(t)=\sum_{n=1}^{\infty}c_n\varphi_n(t)$ converges uniformly in t. In particular, g is continuous and $g(0)=\lim_{t\to 0}g(t)$.*

See Theorem **5.9.1** for the proof. One deduces the increasing case from the decreasing case by considering $-g(t)$, instead of $g(t)$. In our case $c_n=|a_n|^2$ and $\varphi_n(t)=(1-e^{-n^2\pi^2t/l^2})^2$. Now $\varphi_n\leq\varphi_m$ if $n\leq m$, and $|\varphi_n(t)|\leq 1$. Thus, from the lemma and the fact that $\sum c_n$ converges, we have our result.

Now suppose f' is sectionally continuous. From the proof of Theorem **10.6.1**, $\sum_{n=1}^{\infty}|a_n|<\infty$. Thus, for a given x,

$$|f(x)-T(x,t)|\leq\sum_{n=1}^{\infty}|a_n|(1-e^{-n^2\pi^2t/l^2}).$$

By an argument like the preceding, the series on the right converges uniformly, and so we can let $t\to 0$ in each term to conclude that $T(x,t)\to f(x)$ as $t\to 0$. Indeed, note that the convergence is uniform in x because we have the bound $\sum_{n=1}^{\infty}|a_n|(1-e^{-n^2\pi^2t/l^2})$, which approaches 0 as $t\to 0$ and is independent of x.

Finally, suppose $\sum_{n=1}^{\infty}a_n\cos(n\pi x/l)$ converges for some fixed x. Then we wish to show that (for this x fixed)

$$\lim_{t\to 0}g(t)=\lim_{t\to 0}\sum_{n=1}^{\infty}a_ne^{-n^2\pi^2t/l}\cos\frac{n\pi x}{l}=0.$$

Here we cannot make the same estimate, because the factor $\cos(n\pi x/l)$ is needed for $\sum a_n\cos(n\pi x/l)$ to converge. However, Lemma 15 can be applied with $c_n=a_n\cos(n\pi x/l)$ and $\varphi_n(t)=e^{-n^2\pi^2t/l^2}$ to yield the conclusion, since the φ_n are decreasing and are bounded by 1. ■

From this proof we also conclude that

$$\lim_{t\to t_0}T(x,t)=T(x,t_0);$$

that is, T is continuous in t, in each of the three cases of Theorem **10.7.3**. Indeed, we already know that for $t > 0$, $T(x,t)$ is differentiable and hence continuous. However, $T(x,t)$ may not be differentiable at $t = 0$, but the proof just given does show that we have continuity at $t = 0$.

These methods using Abel's and Dirichlet's tests are important for establishing convergence in other problems (such as Laplace's equation), as we shall see in the next proof.

10.7.4 Theorem

i. *Given g_1, let $\varphi(x,y)$ be defined by*

$$\varphi(x,y) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi(b-y)}{a} \frac{\sin(n\pi x/a)}{\sinh(n\pi b/a)}, \tag{12}$$

Suppose g_1 is of class C^2 and $g_1(0) = g_1(a) = 0$. Then φ converges uniformly, and is the solution to the Dirichlet problem with $f_1 = f_2 = g_2 = 0$, and is continuous on the whole square, and $\nabla^2\phi = 0$ on the interior.

ii. *If each of f_1, f_2, g_1, g_2 is of class C^2 and vanishes at the corners of the rectangle, then the solution $\varphi(x,y)$ is the sum of four series like Equation (12), $\nabla^2\varphi = 0$ on the interior, and ∇ is continuous on the whole rectangle and assumes the given boundary values. Furthermore, φ is C^∞ on the interior.*

iii. *If f_1, f_2, g_1, g_2 are only square integrable, then the series for φ converges on the interior, $\nabla^2\varphi = 0$, and φ is C^∞. Also, φ takes on the boundary values in the sense of convergence in mean. This means, for example, that $\lim_{y\to 0}\varphi(x,y) = \varphi(x,0) = g_1(x)$ with convergence in mean.*

Proof For simplicity, let us take the case $a = b = \pi$, the general case being obtained by a change of coordinates. To prove parts **i** and **ii** of the theorem, we show that $\varphi(x,y)$ converges uniformly in x and y and that we can differentiate twice, term by term, on the interior. In view of the preceding remarks, this suffices to prove the theorem. Part **ii** is a consequence of **i** and linearity; the boundary values are assumed because g_1 is represented by its Fourier series.

By Theorems **10.6.1** and **10.6.2**,

$$g_1(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad g_1'(x) = \sum_{n=1}^{\infty} nb_n \cos nx,$$