Carefully PRINT your full name: SOLUTIONS

CIRCLE your section: Section 1 (Tu) Section 2 (Tu) Section 3 (Hasselblatt)

MATH 135 Exam 1 October 17, 2022

(100 points) 12 noon–1:20 p.m.

Instructions: No books, notes, calculators, or external help from any person or device are allowed. Except in the true-false questions, justify all of your steps. Write only in the space provided and do not attach any extra page.

Please sign the following pledge: *I pledge that in this exam I have neither given nor received assistance or cheated in any other way.*

Signature: ______________________________

1. (10 points) Circle either True or False. You do not need to justify your choice.

(a) **True** / **False:** $\mathbb{Z}$ is dense in $\mathbb{R}$.

(b) **True** / **False:** Every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.

(c) **True** / **False:** The product of monotone sequences is monotone.

(d) **True** / **False:** $\mathbb{Q} \cap [0, 1]$ is a closed set in $\mathbb{R}$.

(e) **True** / **False:** An unbounded sequence does not converge.

(a) $[1/3, 2/3]$ CONTAINS NO INTEGER. OR:
NO INTEGER SEQUENCE CONVERGES TO $1/2$.

(b) USE THE ε-δ CRITERION WITH $\delta < 1$. OR:
LEMMA: A CONVERGENT INTEGER SEQUENCE IS EVENTUALLY CONSTANT.
PROOF: ε-N-DEFINITION WITH $\varepsilon \leq 1$.

(c) $a_n = b_n = n - 10$

(d) BY (SEQUENTIAL) DENSITY OF $\mathbb{Q}$ THERE IS A SEQUENCE x_n IN $\mathbb{Q} \cap [0,1]$ WHICH CONVERGES TO $\frac{1}{\sqrt{2}} \notin \mathbb{Q} \cap [0,1]$.

(e) THEOREM: CONVERGENT SEQUENCES ARE BOUNDED

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2. (30 points)

   (a) (5 pts) We learned that $\mathbb{Q}$ is countable, so there is a sequence $\{a_n\}$ in which every rational number appears as a term.
   Does this sequence necessarily have a monotone subsequence? Justify your answer by quoting the statement of a theorem.

   YES. THEOREM: EVERY SEQUENCE HAS A MONOTONE SUBSEQUENCE.

   (b) (5 pts) Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x^8 + x^3 \sin 3 + \sqrt{5}$. Is this function uniformly continuous? Justify your answer by quoting statements of theorems.

   YES. ① f IS A POLYNOMIAL, HENCE CONTINUOUS
   [THEOREM: POLYNOMIALS ARE CONTINUOUS]

   ② THE DOMAIN IS A CLOSED BOUNDED INTERVAL, SO f IS UNIFORMLY CONTINUOUS.
   [THEOREM: A CONTINUOUS FUNCTION ON A CLOSED BOUNDED INTERVAL IS UNIFORMLY CONTINUOUS.]

   (c) (10 pts) Let $f\colon D \to \mathbb{R}$. State the ϵ-δ-criterion for continuity at a point $x_0 \in D$.

   $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

   OR:

   $\forall \varepsilon > 0$, $\exists \delta > 0$ SUCH THAT $\forall x \in D$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

   (d) (10 pts) Choose the statement to negate depending on your section. The following definition is implicit in our textbook:
   (**Prof. Hasselblatt's section**) A sequence $\{a_n\}$ is said to *converge* if
   $\boxed{\exists a \in \mathbb{R}\ \forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N \quad |a_n - a| < \epsilon}$.
   State a definition of "A sequence is said to diverge if…" by negating this statement.
   (**Prof. Tu's section**) Negate $\boxed{\exists a \in \mathbb{R} \text{ such that } \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, |a_n - a| < \epsilon}$.

   $\forall a \in \mathbb{R}\ \exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geq N : \quad |a_n - a| \geq \varepsilon$

   OR

   $\forall a \in \mathbb{R}$, $\exists \varepsilon > 0$ SUCH THAT $\forall N \in \mathbb{N}$, $\exists n \geq N$ SUCH THAT $|a_n - a| \geq \varepsilon$.

3. (10 points) Suppose $A \subset \mathbb{R}$ is a nonempty set which is bounded below. Prove that there is a sequence $\{a_n\}$ in A that converges to $\inf A$.

[THIS IS PROBLEM 9b FROM HOMEWORK 3.]

FOR $n \in \mathbb{N}$ THERE IS AN $a_n \in A$ SUCH THAT

$$\inf A \leq a_n < \inf A + \frac{1}{n}$$

BECAUSE $\inf A + \frac{1}{n}$ IS NOT A LOWER BOUND

[IT IS GREATER THAN $\inf A$, THE GREATEST LOWER BOUND].

EQUIVALENTLY, $0 \leq a_n - \inf A < \frac{1}{n}$.

THIS IMPLIES $a_n \to \inf A$ BY THE COMPARISON LEMMA,

OR BY THE SANDWICH LEMMA, OR BY THE DEFINITION:

LET $\varepsilon > 0$ AND $N > 1/\varepsilon$. IF $n \geq N$, THEN

$$|a_n - \inf A| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

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4. (10 points) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} x & \text{if } x \notin \mathbb{Z} \\ -x & \text{if } x \in \mathbb{Z}. \end{cases}$

Using the definition of continuity at a point, prove that f is not continuous at $x = 1$.

$x_n := 1 + \frac{1}{n} \xrightarrow[n\to\infty]{} 1$ WHILE

$f(x_n) = f(1+\frac{1}{n}) = 1 + \frac{1}{n} \xrightarrow[n\to\infty]{} 1 \neq -1 = f(1).$

5. (15 points) Prove the sandwich theorem (squeeze theorem): Let $L \in \mathbb{R}$ and let $\{x_n\}$ and $\{z_n\}$ be sequences that both converge to L. Assume $\{y_n\}$ is a sequence such that $\forall n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. Prove that $y_n \to L$ as $n \to \infty$ using the ϵ-N-definition of limit.

LET $\varepsilon > 0$.

$x_n \to L \Rightarrow \exists N_x \in \mathbb{N} \; \forall n \geq N_x \; -\varepsilon < x_n - L < \varepsilon$.

$z_n \to L \Rightarrow \exists N_z \in \mathbb{N} \; \forall n \geq N_z \; -\varepsilon < z_n - L < \varepsilon$.

LET $N = \max(N_x, N_z)$. THEN $\forall n \geq N$

$$-\varepsilon \overset{\text{BECAUSE } n \geq N_x}{<} x_n - L \leq y_n - L \leq z_n - L \overset{\text{BECAUSE } n \geq N_z}{<} \varepsilon.$$ ∎

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6. (10 points) Assuming $\sin x$ is a continuous function of x, prove that there is a solution of the equation $x^7 + \sin x + 15 = 0$.

$f(x) = x^7 + \sin x + 15$ IS CONTINUOUS BECAUSE IT IS A SUM OF CONTINUOUS FUNCTIONS ($\sin x$ AND A POLYNOMIAL).

$f(0) = 15 > 0$

$f(-2) = \underbrace{(-2)^7}_{=-2^7 < -2^4 = -16} + \underbrace{\sin(-2)}_{< 1} + 15 < 0.$

BY THE INTERMEDIATE-VALUE THEOREM THERE IS AN $x \in (-2, 0)$ FOR WHICH $f(x) = 0$. ∎

7. (15 points) Using the ϵ-N-definition of limit, prove that $\frac{n^2-4}{2n^2-n+1}$ converges to $1/2$. (Hint: $2n^2 - n + 1 = n^2 + (n^2 - n) + 1$.)

NOTE FIRST THAT

$$\frac{n^2-4}{2n^2-n+1} - \frac{1}{2} = \frac{1}{2}\frac{(2n^2-8)-(2n^2-n+1)}{2n^2-n+1}$$

$$= \frac{1}{2}\frac{n-9}{2n^2-n+1}.$$

SCRATCH WORK: WHEN $n \geq 9$ WE HAVE $0 \leq n-9 < n$.

FOR ANY n WE HAVE $2n^2-n+1 = n^2 + \underbrace{(n^2-n)}_{\geq 0} + 1 > n^2$.

COMBINED, THESE GIVE $0 \leq \frac{1}{2}\frac{n-9}{2n^2-n+1} < \frac{n}{2n^2} = \frac{1}{2n}$.

PROOF.

LET $\varepsilon > 0$ AND $N \geq 9$ SUCH THAT $N > \frac{1}{2\varepsilon}$. THEN $\forall n \geq N$

$$\left|\frac{n^2-4}{2n^2-n+1} - \frac{1}{2}\right| = \frac{1}{2}\frac{n-9}{2n^2-n+1} \overset{\text{SEE SCRATCH WORK}}{<} \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon. \quad \blacksquare$$

(End of Exam)