1. Math 65, Review Sheet for Exam I, Fall 2021.

Question 1. Denote by $\mathbb{R}$ the set of real numbers. Define the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $F(a, b) = (a + b, a - b)$.

(a) Define what it means for a function $f : A \to B$ to be one to one.
(b) Define what it means for a function $f : A \to B$ to be onto.
(c) Define what it means for a function $f : A \to B$ to be a bijection.
(d) For the function F defined above, prove or disprove that F is one to one.
(e) For the function F defined above, prove or disprove that F is onto.
(f) For the function F defined above, prove or disprove that F is a bijection.

Question 2. (a) Find a recurrence relation a_n for the number of strings of length n using the letters from the set $\{a, b, c, d, e\}$ that contain 2 consecutive e's.
(b) What are the initial conditions, that is, the values of the first terms that together with the recurrence would allow you to find every other term
(c) What is a_4?

Question 3. (1) Assume that $(a_n), (b_n)$ are sequences satisfying that

$$a_n = 6a_{n-1} - 8a_{n-2} \quad b_n = 6b_{n-1} - 8b_{n-2}$$

Give conditions on a_0, a_1, b_0, b_1 that guarantee that for each sequence c_n satisfying $c_n = 6c_{n-1} - 8c_{n-2}$, one can find constants A, B such that $c_n = Aa_n + Bb_n$.
(2) Find two sequences of the form $c_n = x^n$ satisfying the recurrence.
(3) Find a non recursive expression for the sequence or sequences satisfying

$$c_n = 6c_{n-1} - 8c_{n-2} \qquad c_0 = 4, c_1 = 10$$

or show that no such sequence exists.

Question 4. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

Question 5. Let $p(x)$ be the proposition: "I will go to the concert on day x."
Let $q(x)$ be the proposition: "I have an exam on day x." Using p and q and logical connectives, write the propositions that follow. Write the negation of each of these statements both in mathematical symbols and in English.

(a) "I will not go to the concert today if I have an exam tomorrow."
(b) "If I do not have an exam tomorrow, I will go to the concert today."
(c) "If I do not go to the concert today, I will not have an exam tomorrow."
(d) "I will have the exam some day"
(e) "I will never go to the concert"

Question 6. Determine the truth value of each of these statements where $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of natural numbers. If the statement is true, give a proof, if false give a counterexample. Write the negation of the statements and again give a proof or counterexample.

(a) $\exists x \in \mathbb{N}\ (-x < -3)$
(b) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}\ \exists z \in \mathbb{N}\ (x = 2y + 4z)$
(c) $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ (y \geq x)$
(d) $\forall y \in \mathbb{N}\ \exists x \in \mathbb{N}\ (x < y)$
(e) $\forall x \in \mathbb{N}\ (-2x < -x)$

Question 7. Assume that all the sets we are working with are contained in a set U. Then we will indicate with $\overline{A}$ the complement of A, that is the elements in U that are not in A, $\overline{A} = U - A$. Prove that $\overline{(A \cap B \cap C \cap D)} = \bar{A} \cup \bar{B} \cup \bar{C} \cup \bar{D}$ by showing each side is a subset of the other side.

Question 8. Let $A = \{x \in \mathbb{R} | x^2 \leq 8x\}, B = \{x \in \mathbb{R} | x^2 \leq 1\}$
(a) Find $A \cap B$.
(b) Find $A \cup B$.

Question 9. Assume that x is an integer.
(a) Prove that if $x + 5$ is odd, then $3x + 2$ is even.
(b) Prove that if $x + 5$ is even, then $3x + 2$ is odd.
(c) Prove that (b) is equivalent to proving that if $3x + 2$ is even, then $x + 5$ is odd.

Question 10. Use mathematical induction to prove that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

for all $n \geq 1$.

Question 11. Show that any amount of postage of 12 cents or more can be made with 4 cents and 5 cent stamps.

Question 12. Assume that A, B are finite sets of cardinality m and n respectively. Show that $A \times B$ is finite of cardinality mn.

Question 13. Assume that $A \neq \emptyset$ is a set $f : A \to B$ a function. Show that f is one to one if and only if there exists a function $g : B \to A$ such that $g \circ f = I_A$.

Question 14. Let $f : A \to B$ be a function. We are NOT assuming that f is a bijection, so f^{-1} is not defined as a function.
(a) If $X \subseteq A$, write the definition of $f(X)$.
(b) If $Y \subseteq B$, write the definition of $f^{-1}(Y)$.
(c) If $X \subseteq A$, show that $X \subseteq f^{-1}(f(X))$.
(d) Give an example of an $f : A \to B$ and an $X \subseteq A$ such that $X \neq f^{-1}(f(X))$.

Question 15. (a) Show that the set of natural numbers divisible by 5 is countable.
(b) Show that the real plane $\mathbb{R} \times \mathbb{R}$ is not countable.

Question 16. How many four digit numbers contain the string 123 or the string 234 (or both)?