Math 135 HW 9

1 Let $\{a_k\} \in A$ and $\{b_k\} \in B$. Suppose $a_k \to a \in A$ and $b_k \to b \in B$. As A, B sequentially compact, $\exists \{a_{k_i}\} \in A \to a$ and $\exists \{b_{k_j}\} \in B \to b$. As a_{k_i} and b_{k_j} are bounded, $\exists \{a_{k_{i_m}}\} \to a$ and $\exists \{b_{k_{j_m}}\} \to b$
So for any $A \times B$, have the subsequence $(a_{k_{i_m}}, b_{k_{j_m}}) \to (a, b)$
$(a, b) \in A \times B$ so $A \times B$ is sequentially compact

2 a) $x \in F^{-1}(Y \setminus C)$ iff $F(x) \in Y \setminus C$ iff $F(x) \notin C$ iff $x \notin F^{-1}(C)$ so $x \in X \setminus F^{-1}(C)$
Therefore, $F^{-1}(Y \setminus C) = X \setminus F^{-1}(C)$

b) $\Rightarrow$ If continuous then $F^{-1}(\text{closed}) = \text{closed}$
Since C is closed, $\mathbb{R}^m \setminus C$ is open. From class, F is continuous iff $F^{-1}(\text{open set}) = \text{open}$.
$F^{-1}(\mathbb{R}^m \setminus C) = \mathbb{R}^n \setminus F^{-1}(C)$ and $\mathbb{R}^n \setminus F^{-1}(C)$ then is open, so $F^{-1}(C)$ is closed in $\mathbb{R}^n$.

$\Leftarrow$ $F^{-1}(\text{closed}) = \text{closed}$ then F is continuous
$V \subset \mathbb{R}^m$ be an open set. $\mathbb{R}^m \setminus V$ is closed
$f^{-1}(\mathbb{R}^m \setminus V)$ is closed, as $f^{-1}(\text{closed}) = \text{closed}$
$f^{-1}(\mathbb{R}^m \setminus V) = \mathbb{R}^n \setminus f^{-1}(V)$, $\mathbb{R}^n \setminus F^{-1}(V)$ is closed
so $F^{-1}(V)$ is open and as V is open, F is continuous. $\square$

3 a) $D(\vec{w}) = \text{dist}(\vec{w}, \vec{v})$ is continuous if: for $\vec{w}, \vec{v} \in \mathbb{R}^n$
$\forall \epsilon > 0, \exists \delta > 0$ s.t for $\vec{u} \in \mathbb{R}^n$ s.t. if $\|u - u_0\| < \delta$
then $|D(\vec{w}) - D(\vec{w}_0)| < \epsilon$
By triangle inequality: Scratchwork
$|\text{dist}(u, v) - \text{dist}(u_0, v)| \le \text{dist}(u, v) + \text{dist}(u_0, v) \le \text{dist}(u, u_0) < \epsilon$
Suppose $\delta < \epsilon$
If $\|u - u_0\| < \epsilon$ then $|\text{dist}(u, v) - \text{dist}(u_0, v)| \le \text{dist}(u, v) + \text{dist}(u_0, v) \le \text{dist}(u, u_0)$
$\|u - u_0\| = \text{dist}(u, u_0)$ so $\text{dist}(u, u_0) < \epsilon$ thus ϵ-δ criterion is satisfied

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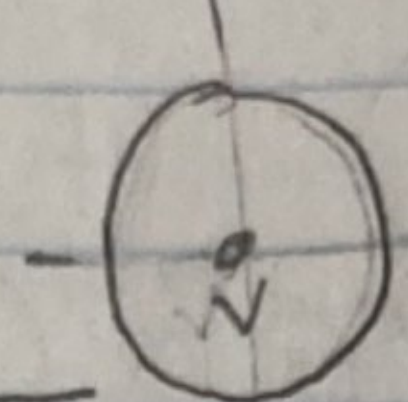
3 b) A $\subset \mathbb{R}^n$ is sequentially compact set and by part a, $D(\vec{w}) = dist(\vec{w},\vec{v})$ is continuous for $\vec{w},\vec{v} \in \mathbb{R}^n$
So, by extreme value theorem, the function realizes a minimum and maximum on A, so $\forall \vec{w} \in A$, $\exists \vec{w_0} \in A$ s.t. $dist(w_0,v) \le dist(w,v)$.
This isn't always unique, consider $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ and $\vec{w}\in A$ and $v=(0,0)$ or:



$dist(w,v)=1 \ \forall w \in A$.

closed ball

c) Yes, suppose for $\overline{B_r(\vec{v})}$ r is large enough s.t. $\overline{B_r(\vec{v})} \cap A \neq \emptyset$. $\exists w \in A$ s.t. $dist(w,v) \le r$
All points not in $\overline{B_r(\vec{v})} \cap A$. As both sets are closed $A \cap \overline{B_r(\vec{v})}$ is closed. Furthermore the set is bounded, as for $w \in A$ $dist(w,v) \le r$ but is greater than some bound on k s.t. $A \cap B_k(\vec{v}) = \emptyset$.
Therefore, this set is closed and bounded, so it is sequentially compact so by extreme value theorem, $D(\vec{w})$ realizes a minimum on $A \cap \overline{B_r(\vec{v})}$ so the conclusion holds if A is closed.

4 a) Let $\epsilon > 0$. $a \in \mathbb{R}$ is $\in bdA$ if $\forall \epsilon > 0$, $B_\epsilon(a)$ has an interior and an exterior point where $B_\epsilon(a)$ is open ball.
By definition of infimum, if $a = \inf A$, $\exists x \in A$ where $x < a + \epsilon$. So, by definition, $x \in B_\epsilon(a)$ as $dist(x,a) < \epsilon$
To show $B_\epsilon(a)$ has an exterior point, consider $a - \epsilon/2$. as $a - \epsilon/2 < a$, and $a = \inf A$, $a - \epsilon/2 \notin A$. However, as $dist(a, a-\epsilon/2) = \epsilon/2 < \epsilon$ so $B_\epsilon(a)$ has an exterior point.
Therefore $\inf A \in bdA$, as $B_\epsilon(a)$ has an interior and exterior point

b) If A is closed, $bdA \subset A$ then $\inf A \in bdA \subset A$
so $\inf A \in A$ $\square$

5a) $A=\{(x,y)\in\mathbb{R}^2 \mid x^4+y^4=1\}$ True, is sequentially compact

$F(x,y)=x^4+y^4-1=0$

$S=\{F(x,y)\in\mathbb{R} \mid F(x,y)=0\}$

$F^{-1}([0])=$ closed.

As $|x^4+y^4=1$, $x^4\le 1$, and $y^4\le 1$ so $|x|\le 1$ and $|y|\le 1$ so the set is bounded. Therefore the set is Sequentially Compact.

b) False, let $A=[1,\infty)$ which is closed.
$F(x)=\frac{1}{x}$ is continuous on A, but $F(A)=(0,1]$ which is not closed.

c) True
Consider the bounded set A, so $\forall w\in A, \|w\|<M$.
$A\subseteq B_M(0)$ where $B_M(0)$ is a closed ball of radius M.
so it is sequentially compact. Therefore, from class $F(B_M(0))$ is also compact. Therefore as $F(A)\subset F(B_M(0))$ and $F(B_M(0))$ is bounded, so and, $F(A)$ is also bounded.

d) False. Consider $f(x)=\sin x$. Let $C=[-1,1]$ which is sequentially compact in $\mathbb{R}$.
$f^{-1}([-1,1])=\mathbb{R}$ as $|\sin x|\le 1$ $\forall x\in\mathbb{R}$ so as $\mathbb{R}$ is unbounded, $f^{-1}([-1,1])$ is unbounded and not sequentially compact

6 Since $[0,1]$ is closed, $f^{-1}([0,1])$ is closed.
If $\vec{x}\in f^{-1}([0,1])$ then $0\le f(x)\le 1$ which becomes $0\le\|\vec{x}\|\le f(\vec{x})\le 1$. We can see clearly $f(\vec{x})$ is bounded, and x is bounded, so $f^{-1}([0,1])$ is sequentially compact.

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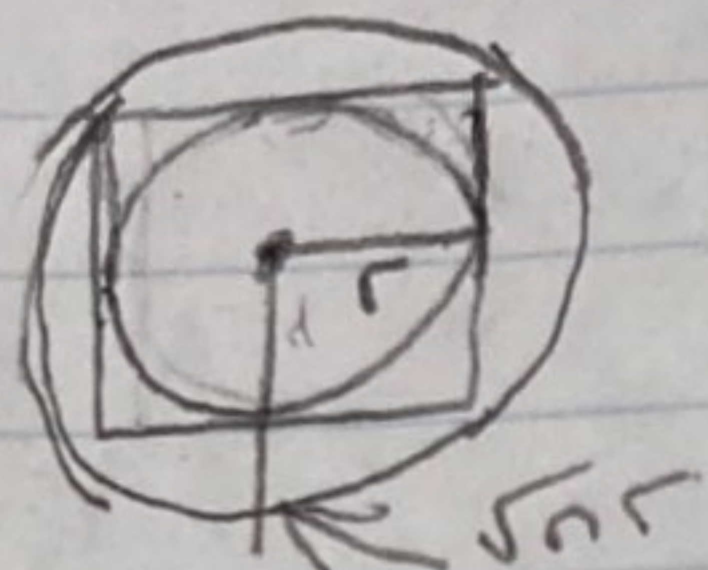

$B(\vec{0},r) \subset K(\vec{0},r) \subseteq B(\vec{0},\sqrt{n}r)$

Let $\vec{x} \in B(\vec{0},r)$ show $\vec{x} \in K(\vec{0},r)$

Since $\vec{x} \in B(\vec{0},r)$, $||x||<r$ so $\sqrt{x_1^2+x_2^2+\dots+x_n^2}<r$

WLOG, $\sqrt{x_1^2} < \sqrt{x_1^2+x_2^2+\dots+x_n^2} < r$

so $\sqrt{x_1^2}<r$, $|x_1|<r$

This holds $\forall i$, so $|x_i|<r$ meaning $x \in K(\vec{0},r)$

Next, show if $x \in K(\vec{0},r)$ then $x \in B(\vec{0},\sqrt{n}r)$

If $x \in K(\vec{0},r)$ $\forall i$, $|x_i|<r$

So $||x|| = \sqrt{x_1^2+x_2^2+\dots+x_n^2} < \sqrt{r_1^2+r_2^2+\dots+r_n^2} = \sqrt{nr^2} = \sqrt{n}r$

So $||x|| < \sqrt{n}r$ so $x \in B(\vec{0},\sqrt{n}r)$

Therefore $B(\vec{0},r) \subset K(\vec{0},r) \subseteq B(\vec{0},\sqrt{n}r)$.