MATH 42 HOMEWORK 4

This assignment covers §15.3 − 5.

(1) **Consider the surface given by $z = x^2 + 3y^2$. Find parametric equations for the tangent line to the curve of intersection of the surface and the plane $y = 1$ at the point $(1, 1, 4)$. (Hint: First, find the slope of that line.)**

To find the slope of the line on the plane $y = 1$, we need to compute $\frac{\partial z}{\partial x}$. This is because the rate of change of the function on that plane depends on z and x (i.e., y is fixed). First, we define a function implicitly $F(x, y, z) = 0$ as follows:

$$F(x, y, z) = z - x^2 - 3y^2.$$

Then, we compute the following partial derivatives:

$$F_x = -2x$$
$$F_z = 1$$

Therefore, the rate of change of z with respect to x is the following:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = 2x.$$

This makes sense intuitively if we think of $y = 1$ as a cross-section of the original function. This cross-section is $z = x^2 + 3$ which is an upward-facing parabola. If we take the derivative of both sides with respect to x, we get the same formula as above.

At the point $(1, 1, 4)$, the rate of change is $\frac{\partial z}{\partial x}|_{(1,1,4)} = 2(1) = 2$.

We have found the slope of the tangent line, now we define the line as follows:

$$\boxed{\vec{r}(t) = \langle 1, 1, 4\rangle + t\langle 1, 0, 2\rangle.}$$

(2) **Assume that the equation $e^{xyz} = \sin(x^2 + y^2 + z^2)$ implicitly defines z as a function of x, y. We will find $\frac{\partial z}{\partial x}$ using two methods:**

(a) **Go through and implicitly take partial derivatives with respect to x on both sides of the given equation, treating z as a function of x and treating y as constant, and remembering to use the chain rule when necessary. Then, solve for $\frac{\partial z}{\partial x}$. This is the method you learned in Calc I.**

We take the derivative of both sides of the equation with respect to x as follows:

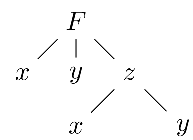
$$\frac{\partial}{\partial x}(e^{xyz}) = \frac{\partial}{\partial x}(\sin(x^2 + y^2 + z^2))$$
$$e^{xyz}(yz + xy\,\frac{\partial z}{\partial x}) = \cos(x^2 + y^2 + z^2)\,(2x + 2z\,\frac{\partial z}{\partial x})$$

We solve for $\frac{\partial z}{\partial x}$ as follows:

$$\begin{aligned}
e^{xyz}(yz + xy\,\frac{\partial z}{\partial x}) &= \cos(x^2+y^2+z^2)\,(2x + 2z\,\frac{\partial z}{\partial x}) \\
xy\,e^{xyz}\,\frac{\partial z}{\partial x} - 2z\cos(x^2+y^2+z^2)\,\frac{\partial z}{\partial x} &= 2x\cos(x^2+y^2+z^2) - yz\,e^{xyz} \\
(xy\,e^{xyz} - 2z\cos(x^2+y^2+z^2))\,\frac{\partial z}{\partial x} &= 2x\cos(x^2+y^2+z^2) - yz\,e^{xyz} \\
\frac{\partial z}{\partial x} &= \boxed{\frac{2x\cos(x^2+y^2+z^2) - yz\,e^{xyz}}{xy\,e^{xyz} - 2z\cos(x^2+y^2+z^2)}}
\end{aligned}$$

(b) (i) **Ignore the above equation for a moment, and consider the general case where some equation $\boldsymbol{F(x,y,z)=0}$ implicitly defines z as a function of $\boldsymbol{x,y}$. Use the multivariable chain rule to find a general formula for $\boldsymbol{\frac{\partial z}{\partial x}}$. Hint: draw a tree.**

Suppose we have a function defined implicitly as $F(x,y,z)=0$ where z is a function of x. We now use the following tree so we can apply the chain rule and derive a formula for $\frac{\partial z}{\partial x}$:



We now derive an expression for the partial derivative of F with respect to x following the paths in the tree as follows:

$$\begin{aligned}
\frac{\partial}{\partial x}(F(x,y,z)) &= 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\,\frac{\partial z}{\partial x} &= 0 \\
F_x + F_z\,\frac{\partial z}{\partial x} &= 0
\end{aligned}$$

Solving for $\frac{\partial z}{\partial x}$, we get the following formula:

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.}$$

(ii) **Now, use the formula you found to find $\boldsymbol{\frac{\partial z}{\partial x}}$ for the above example.**

We start by defining the original equation implicitly as follows:

$$F(x,y,z) = e^{xyz} - \sin(x^2+y^2+z^2) = 0.$$

To use the formula from part (i), we first derive the partial derivatives of F as follows:

$$\begin{aligned}
F_x &= yz\,e^{xyz} - 2x\cos(x^2+y^2+z^2) \\
F_z &= xy\,e^{xyz} - 2z\cos(x^2+y^2+z^2)
\end{aligned}$$

Then, using the formula from part (i), we get the following:

$$\boxed{\frac{\partial z}{\partial x} = -\frac{yz\,e^{xyz} - 2x\cos(x^2+y^2+z^2)}{xy\,e^{xyz} - 2z\cos(x^2+y^2+z^2)}}$$

This is exactly the same as the formula we derived in part (a), just with a negative out front.

(c) **Make sure your two answers match (otherwise, find your mistake). Which method do you prefer? Use your preferred method to find $\frac{\partial z}{\partial y}$ for the above example.**

Clearly, using the formula was much easier than deriving the derivative from the equation directly. We use the following implicit differentiation formula for this problem:

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

We have already derived F_z, and F_y is the following:

$$F_y = xz\, e^{xyz} - 2y\cos(x^2 + y^2 + z^2).$$

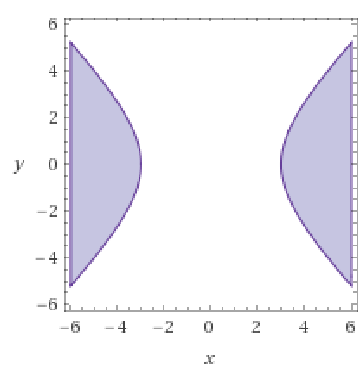
Now, we have the following for $\frac{\partial z}{\partial y}$:

$$\boxed{\frac{\partial z}{\partial y} = -\frac{xz\, e^{xyz} - 2y\cos(x^2+y^2+z^2)}{xy\, e^{xyz} - 2z\cos(x^2+y^2+z^2)}}$$

(3) **Find the domain of the function $f(x,y) = \frac{\sqrt{x^2-y^2-9}}{y-x}$. Describe the domain of f algebraically. Sketch a graph of this domain in the plane labeling any curves involved and indicating which curves are included or excluded.**

We begin by noticing that our domain cannot contain any points where $x = y$ because that would result in a division by zero. Also, we only want real solutions which restricts us so that $x^2 - y^2 - 9 \geq 0$. Since any point where $x = y$ does not satisfy $x^2 - y^2 - 9 \geq 0$, we can completely describe our domain as:

$$\boxed{x^2 - y^2 \geq 9.}$$



(4) **Compare the level curves for the three functions:**
$f(x,y) = x - y^2$, $g(x,y) = (x-y^2)^2$, $h(x,y) = (x-y^2)^3$.
Draw some sketches in the plane. In what ways are they similar and in what ways do they differ? Additionally, give the equation for the level curve of f intersecting the point $x = 2, y = 1$.

Solution at the end.

(5) **Assume $g(x, y, z) = 0$ is a smooth surface and $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a smooth curve on that surface. Use the multivariable chain rule to prove that the vector $\langle g_x, g_y, g_z\rangle$ is orthogonal to the curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ at each point of the curve. (Note: being orthogonal to a curve means being orthogonal to its tangent vector / tangent line.)**

To prove this, we start by looking at the rate of change of g along the curve $\vec{r}(t)$; that is, we find $g'(t)$. We need to use the following tree to compute the chain rule correctly:

$$\begin{array}{ccc} & g & \\ / & | & \backslash \\ x & y & z \\ | & | & | \\ t & t & t \end{array}$$

Then, we compute $g'(t)$ by differentiating $g(x, y, z) = 0$ by t on both sides; this means, $g'(t) = 0$.

$$\begin{aligned} g'(t) = \frac{\mathrm{d}g}{\mathrm{d}t} &= \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= g_x x'(t) + g_y y'(t) + g_z z'(t) \\ &= \langle g_x, g_y, g_z\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle \\ &= \langle g_x, g_y, g_z\rangle \cdot \vec{r}\,'(t) \end{aligned}$$

Recall, $\vec{r}\,'(t)$ is the vector tangent to the curve $\vec{r}(t)$ at each time t. Because $g'(t) = 0$, $\langle g_x, g_y, g_z\rangle \cdot \vec{r}\,'(t)$, and thus $\langle g_x, g_y, g_z\rangle$ is orthogonal to the curve $\vec{r}(t)$.

(6) **Let $f(x, y) = 1 - x^2/4 - y^2/16$. The point $(1, 2)$ lies in the level curve $f(x, y) = 1/2$, which is an ellipse.**

(a) **Find the gradient $\nabla f(1, 2)$.**

The gradient of f is given by

$$\nabla f = \langle f_x, f_y\rangle = \langle -\tfrac{x}{2}, -\tfrac{y}{8}\rangle.$$

Evaluating at the point (1,2) yields $\nabla f(1, 2) = \langle -\frac{1}{2}, -\frac{1}{4}\rangle$.

(b) **Find an equation of the tangent line to the ellipse $f(x, y) = 1/2$ at the point $(1, 2)$.**

We need to find the slope of the tangent line. We may compute $\partial y/\partial x$ by the same method as used in problem 1, but here we will use a separate method simply to show that there is more than one way to solve this problem. We know that the gradient of f is orthogonal to the tangent line, therefore we can set up an equation to determine a vector $\langle a, b\rangle$ pointing in the same direction as the tangent line. In general, the equation is

$$\langle f_x, f_y\rangle \cdot \langle a, b\rangle = 0$$

but at the point $(1, 2)$, using part (a), we have that

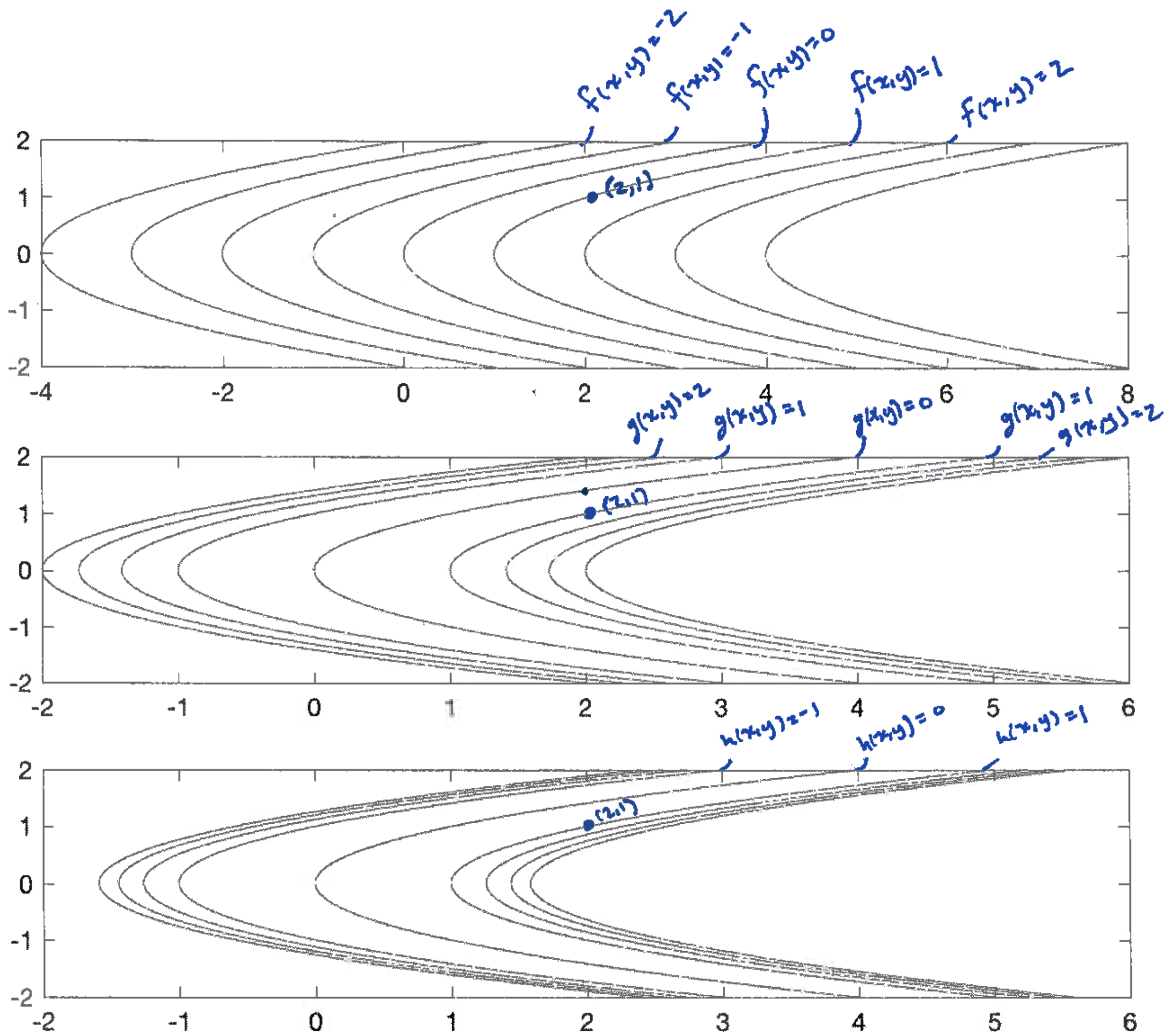
$$-\tfrac{1}{2}a - \tfrac{1}{4}b = 0 \quad \Rightarrow \quad b = -2a.$$

Note that we have some wiggle room, namely we get to pick a and b is determined – this is because there are an infinite number of vectors satisfying the equation above, up to some scalar multiplication. In other words, a suitable vector may be written

$$\langle a, -2a\rangle = a\langle 1, -2\rangle$$

where a is some nonzero scalar. At last, we have a tangent vector and therefore the equation for the tangent line may be written

$$\vec{r}(t) = \langle 1, 2\rangle + t\langle 1, -2\rangle = \langle 1 + t, 2 - 2t\rangle.$$



Point (2,1) corresponds with the level curve with equation

$$1 = x - y^2$$