**Math 125** **Problem Set 3** **Due: Tuesday 10/25 at 11:59 p.m. EST**

**Instruction**: Read the assignment policy. For problems 2 and 3, include a printout your code with your homework submission. You should submit your assignment on Gradescope.

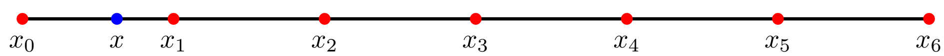
**1**. Consider a uniform distribution of $n+1$ nodes for interpolation. In particular, given $x_0$ and $h > 0$, $x_i = x_{i-1} + h$ for $i = 1, ..., n$.

(a) Prove the following inequality

$$\left| \prod_{i=0}^{n} (x - x_i) \right| \leq \frac{n! h^{n+1}}{4},$$

where $x \in (x_0, x_n)$.

[**Hint**: Without loss of generality, you can assume that $x \in (x_0, x_1)$. An example of such a grid with $n = 6$ is shown below].




(b) Define the interval $I$ as $I = (x_0, x_n)$. Let $f$ be continuously differentiable up to $n+1$ times and $P_n$ be the Lagrange interpolant polynomial of order $n$ using the data points $(x_0, f(x_0)), (x_1, f(x_2)), ..., (x_n, f(x_n))$. Show that

$$E(x) = |P_n(x) - f(x)| \leq \max_{x \in I} \frac{|f^{n+1}(x)|}{4(n+1)} h^{n+1},$$

where $x$ is any point in $I$.

(c) Is it always true that $E(x) \to 0$ as $n \to \infty$?

**2**. We explore the Runge phenomenon by considering the function $f(x) = \dfrac{1}{1+25x^2}$. We approximate $f(x)$ using $n+1$ equally spaced points $x_0, x_1, ..., x_n$ with $x_i = (2i/n) - 1$.

(a) Plot the Lagrange interpolant of degree $n$ for $n = 4, 8, 16, 32$.

(b) For each interpolant in (a), plot the error $E(x) = f(x) - P_n(x)$.

(c) Does increasing $n$ improve the quality of the approximation? Discuss the implication of this result.

(d) Repeat the exercises in (a)-(c) by using Chebyshev interpolation.

(e) If we do not have the freedom to choose the nodes, what alternatives ways are to fix the problem observed in (a)-(c)?

**3**. Assume that you have collected data $(t_1, y_1), (t_2, y_2), ..., t_{n+1}, y_n)$ and want to interpolate the data using cubic splines. Let $p_i$ denote the cubic polynomial between $t_i$ and $t_{i+1}$.

$$p_i(t) = a_i + b_i \frac{t - t_i}{t_{i+1} - t_i} + c_i \frac{(t - t_i)^2}{(t_{i+1} - t_i)^2} + d_i \frac{(t - t_i)^3}{(t_{i+1} - t_i)^3} \qquad t \in [t_i, t_{i+1}]$$

(a) Use a change of variable and argue that $p_i(u) = a_i + b_i u + c_i u^2 + d_i u^3$ for $u \in [0, 1]$.

(b) **Extra credit**: Let $D_i$, $1 \le i \le n+1$, denote the value of the first derivative of $p_i$ at the nodes. In lecture, we discussed that the coefficients $a_i, b_i, c_i, d_i$ can be determined from $\{y_i\}_{i=1}^{n+1}$ and $\{D_i\}_{i=1}^{n+1}$. In particular, we have

$$\begin{aligned} a_i &= y_i \\ b_i &= D_i \\ c_i &= 3(y_{i+1} - y_i) - 2D_i - D_{i+1} \\ d_i &= 2(y_i - y_{i+1}) + D_i + D_{i+1} \end{aligned}$$

Following the derivation in class and using natural boundary conditions, prove that the $\{D_i\}_{i=1}^{n+1}$ can be obtained from solving the following linear system

$$\begin{pmatrix} 2 & 1 & & & & 0 \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ 0 & & & & 1 & 2 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \\ D_{n+1} \end{pmatrix} = \begin{pmatrix} 3(y_2 - y_1) \\ 3(y_3 - y_1) \\ 3(y_4 - y_2) \\ \vdots \\ 3(y_{n+1} - y_{n-1}) \\ 3(y_{n+1} - y_n) \end{pmatrix}$$

(c) Prove that the matrix in (c) is invertible.

(d) Download the dataset `spline_data.mat` from HW3 folder. If you use Python for programming, download `spline_data.csv`. Use the linear system in (c) to find the cubic spline that interpolates the data points. Plot the cubic spline. When you plot the cubic between two knots $t_i$ and $t_{i+1}$, use a uniform grid with spacing $10^{-4}$.

[**Remark**: To solve the linear system, you can use `np.linalg.solve` in Python or the command `A\z` (to solve $\mathbf{Ax} = \mathbf{z}$) in MATLAB.]