Abstract Algebra I, Practice exam 1

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This is longer and a bit more difficult than what the real exam will be like.

1. (a) Let G be a group and let $g \in G$ be any element. Let n be a positive integer. Show that $(g^{-1})^n = (g^n)^{-1}$.

   (b) Let G be a group and let g_1, g_2, g_3, g_4, g_5 be any elements. Show that

$$((g_1 \cdot (g_2 \cdot g_3)) \cdot (g_4 \cdot g_5)) = (g_1 \cdot g_2) \cdot ((g_3 \cdot g_4) \cdot g_5).$$

2. Let $\phi : G \to H$ be an isomorphism of groups. Suppose that g_1, g_2, g_3 generate G. Show that $\phi(g_1), \phi(g_2), \phi(g_3)$ generate H.

3. Show that the multiplicative group $((\mathbb{Z}/12\mathbb{Z})^\times, \times)$ is not cyclic.

4. Let S_n be the symmetric group on n letters. Suppose that σ is a 3-cycle and that τ is a 5-cycle. Suppose further that $\sigma\tau$ is an m-cycle, for some integer m.

   (a) Show that $m < 9$.

   (b) Show that $m \neq 1$.

   (c) Show that m is odd.

   (d) Give examples of σ, τ for which $\sigma\tau$ is an m-cycle, where $m = 3, 5, 7$.

5. Show that the groups D_{12} and S_4 are not isomorphic.

6. There are 16 subgroups of D_6. Find them all, and prove that there are no others.

   This one takes some effort; on a real exam, I probably would not ask you to write down a proof that your list is exhaustive. Nevertheless, it is a good exercise to think through.