MATH 125 Lecture 10

1. For each of the following computations, state whether or not the computation is backwards stable.

(a) Evaluating $\tilde{p}(x) = a \otimes (x \otimes x) \oplus b \otimes x \oplus c = \tilde{p}(a,b,c;x)$

(b) $\tilde{f}(a,b,c) = \begin{pmatrix} a \otimes b \\ b \otimes c \\ a \otimes c \end{pmatrix}$

(c) Solving a linear system with an invertible matrix $D \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^3$: $\tilde{f}(D, b) = x$ such that $Dx = b$.

Solution

(a)
$$\tilde{p}(x) = a \otimes x^2(1+\varepsilon_1) \oplus bx(1+\varepsilon_2) \oplus c$$
$$= ax^2(1+\varepsilon_1)(1+\varepsilon_3) \oplus [bx(1+\varepsilon_2)+c](1+\varepsilon_4)$$
$$= [ax^2(1+\varepsilon_1)(1+\varepsilon_3) + bx(1+\varepsilon_2)(1+\varepsilon_4) + c(1+\varepsilon_4)](1+\varepsilon_5)$$
$$= a(1+\varepsilon_1)(1+\varepsilon_3)(1+\varepsilon_5)x^2 + b(1+\varepsilon_2)(1+\varepsilon_4)(1+\varepsilon_5)x + c(1+\varepsilon_4) =$$

$\left.\begin{array}{l}|\varepsilon_1| \\ |\varepsilon_2| \\ |\varepsilon_3| \\ |\varepsilon_4| \\ |\varepsilon_5|\end{array}\right\} \le \varepsilon_{mach}$

Define
$$\tilde{a} = (1+\varepsilon_1)(1+\varepsilon_3)(1+\varepsilon_5)\,a$$
$$\tilde{b} = (1+\varepsilon_2)(1+\varepsilon_4)(1+\varepsilon_5)\,b$$
$$\tilde{c} = (1+\varepsilon_4)\,c$$

$$\frac{|\tilde{a}-a|}{|a|} = \frac{|\varepsilon_1+\varepsilon_3+\varepsilon_5+\varepsilon_3\varepsilon_5+\varepsilon_1\varepsilon_3+\varepsilon_1\varepsilon_5+\varepsilon_1\varepsilon_3\varepsilon_5|\,|a|}{|a|}$$
$$\le 3\varepsilon_{mach} + 3\varepsilon_{mach}^2 + \varepsilon_{mach}^3 = O(\varepsilon_{mach})$$

Similarly $\dfrac{|\tilde{b}-b|}{|b|} = O(\varepsilon_{mach})$ and $\dfrac{|\tilde{c}-c|}{|c|} = |\varepsilon_4| \le \varepsilon_{mach}$

Therefore it is backwards stable.

(b) $\tilde{f}(a,b,c) = \begin{pmatrix} a\cdot b\,(1+\varepsilon_1) \\ b\cdot c\,(1+\varepsilon_2) \\ a\cdot c\,(1+\varepsilon_3) \end{pmatrix}$ $\quad |\varepsilon_1| \le \varepsilon_{mach}$, $|\varepsilon_2| \le \varepsilon_{mach}$, $|\varepsilon_3| \le \varepsilon_{mach}$

No way to define $\tilde{a}, \tilde{b}, \tilde{c}$ such that $\tilde{f}(a,b,c) = f(\tilde{a}, \tilde{b}, \tilde{c})$ Not backward stable

(c)

$$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\tilde{x}_1 = \frac{b_1}{d_{11}}(1+\varepsilon_1) \qquad |\varepsilon_1| \le \varepsilon_{mach}$$

$$\tilde{x}_2 = \frac{b_2}{d_{22}}(1+\varepsilon_2) \qquad |\varepsilon_2| \le \varepsilon_{mach}$$

$$\tilde{x}_3 = \frac{b_3}{d_{33}}(1+\varepsilon_3) \qquad |\varepsilon_3| \le \varepsilon_{mach}$$

$$\tilde{D} = \begin{pmatrix} d_{11} & & \\ & d_{22} & \\ & & d_{33} \end{pmatrix} \qquad \tilde{b} = \begin{pmatrix} b_1(1+\varepsilon_1) \\ b_2(1+\varepsilon_2) \\ b_3(1+\varepsilon_3) \end{pmatrix} \qquad \tilde{D}\tilde{x} = \tilde{b}$$

$$\frac{\|b-\tilde{b}\|_2^2}{\|b\|_2^2} = \frac{b_1^2\varepsilon_1^2 + b_2^2\varepsilon_2^2 + b_3^2\varepsilon_3^2}{b_1^2+b_2^2+b_3^2}$$

$$\le \frac{\varepsilon_{mach}^2 (b_1^2+b_2^2+b_3^2)}{b_1^2+b_2^2+b_3^2} = \varepsilon_{mach}^2$$

$$\therefore \frac{\|b-\tilde{b}\|_2}{\|b\|_2} = O(\varepsilon_{mach})$$

(2) i) Find the interval in which the smallest possible root of $x^3 - x - 4 = 0$ lies.

Solution

$f(0) = -4$

$f(1) = -4$ Root lies in the interval $(1,2)$

$f(2) = 2$ Note $f'(x) = 3x^2 - 1 \Rightarrow$ minimum at $x = +\sqrt{1/3}$

$\therefore$ No root between 0 and 1.

ii) Carry out three iterations of bisection method on the interval you found in (i)

a	b
1	2
1.5	2
1.75	2
1.75	1.875

~~$f(1)\ f(1.5) > 0$~~

$f(1.75)\ f(1.5) > 0$

$f(1.875)\ f(1.75) > 0$

(3) Explain why the fixed point iteration $g(x) = \cos(x)$ converges. (Assume fixed point exists)

Solution $g'(x) = [\cos(x)]' = -\sin(x)$

At any x, $|-\sin(x)| \le 1$

Note $\sin(x) = 1$ at $x = \pi/2$] $\cos(x) = 0$
$\sin(x) = -1$ at $x = 3\pi/2$] $\cos(x) = 0$

(4) Let $\phi(x)$ be a continuous function in $[a, b]$ such that $\phi(x) \in [a, b]$. Prove that there exists at least a fixed point $\alpha \in [a, b]$.

solution Note $g(x) = \phi(x) - x$ which is continuous in $[a, b]$

$g(a) = \phi(a) - a \ge 0$
$g(b) = \phi(b) - b \le 0$
Apply IVT, g has atleast one zero in $[a, b]$.
This implies that ϕ has a fixed point

(5) Find each fixed point and decide whether fixed point iteration is locally convergent to it

(a) $g(x) = \frac{1}{2}x^2 + \frac{1}{2}x$ (b) $g(x) = x^2 - \frac{1}{4}x + \frac{3}{8}$

solution (a) $g(x) = x$

$\frac{1}{2}x^2 + \frac{1}{2}x = x$

$\frac{1}{2}x^2 = \frac{1}{2}x \Rightarrow x = 1$
$\Rightarrow x = 0$

$g'(x) = 2x \cdot \frac{1}{2} + \frac{1}{2}$
$= x + 1/2$

locally convergent to 0
divergent to 1

(b) $x^2 - \frac{1}{4}x + \frac{3}{8} = x$

$x^2 - \frac{1}{4}x - \cancel{x} + \frac{3}{8} = 0$

$x^2 - \frac{5}{4}x + \frac{3}{8} = 0$

$8x^2 - 10x + 3 = 0$

$\frac{10 \pm \sqrt{100 - 96}}{16}$

$\frac{10 \pm 2}{16}$ $\Rightarrow x = 1/2$
$\Rightarrow x = 3/4$

$g'(x) = 2x - \frac{1}{4}$

locally convergent at 1/2
locally divergent at 3/4

(6) Find the Lagrange form of the interpolation polynomial that interpolates (x_0, y_0) (x_1, y_1) and (x_2, y_2)

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$P(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$