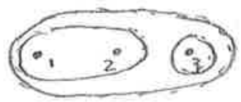
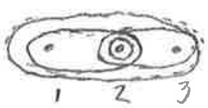
Solutions

(1) Let $X = \{1,2,3\}$. Work with your group to find an example of a topology on X other than the discrete topology containing at least 4 open sets. Prove that your example is a topology.



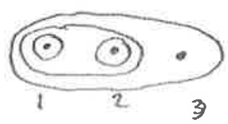
$\tau = \{ \{\}, \{1,2\}, \{3\}, \{1,2,3\}\}$

or



$\tau = \{ \emptyset, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

or



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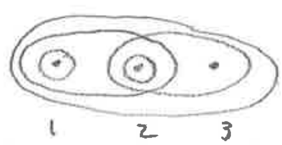

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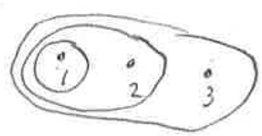


up to relabeling

(2) Let $X = \{1, 2\}$. There are four functions $X \to X$, namely

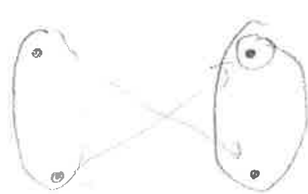
x	$f_{11}(x)$	$f_{12}(x)$	$f_{21}(x)$	$f_{22}(x)$
1	1	1	2	2
2	1	2	1	2

We have been thinking about three topologies on X:

$$\begin{aligned}\tau_d &= \{\emptyset, \{1\}, \{2\}, \{1,2\}\}\\ \tau_s &= \{\emptyset, \{1\}, \{1,2\}\}\\ \tau_i &= \{\emptyset, \{1,2\}\}.\end{aligned}$$

Given the following topologies on the domain and codomain, decide whether each function $X \to X$ is continuous. Use any shortcuts you can. Do you see any patterns?

Domain topology	Codomain topology	f_{11}	f_{12}	f_{21}	f_{22}
τ_d	τ_d	Y	Y	Y	Y
τ_d	τ_s	Y	Y	Y	Y
τ_d	τ_i	Y	Y	Y	Y
τ_s	τ_d	Y	N	N	Y
τ_s	τ_s	Y	Y	N	Y
τ_s	τ_i	Y	Y	Y	Y
τ_i	τ_d	Y	N	N	Y
τ_i	τ_s	Y	N	N	Y
τ_i	τ_i	Y	Y	Y	Y



(3) (The finite complement topology) Let X be any set. The collection of subsets

$$\tau = \{U \subseteq X \mid U^c \text{ is finite}\} \cup \{\varnothing\}$$

is called the **finite complement topology**.

(a) Show that the finite complement topology is a topology.

$\varnothing \in \tau$

X has finite complement $\in \tau$.

$U_i \in \tau \Rightarrow \left(\bigcup_i U_i\right)^c = \underbrace{\bigcap_i \underset{\text{finite}}{U_i^c}}_{\therefore \text{ finite}} \therefore \in \tau.$

$U_1, \dots, U_n \in \tau \Rightarrow \left(\bigcap_{i=1}^n U_i\right)^c = \underbrace{\bigcup_{i=1}^n \underbrace{U_i^c}_{\text{finite}}}_{\text{finite union of finite} \therefore \text{finite}} \therefore \in \tau.$

(b) How does the finite complement topology on $\mathbb{R}$ compare with the usual topology on $\mathbb{R}$? Justify your answer.

It is coarser. Each set open in the finite complement topology is open in the usual topology on $\mathbb{R}$:

s'ps U w/ finite complement.

Then if $x \in U$, we can say $\{z_1, \dots, z_n\} = U^c$,

set $\varepsilon = \min_i \{|x - z_i|\}$

Then $B(x, \varepsilon) \subseteq U$,

so U is open in the usual topology

(4) Let Z be a finite subset of $\mathbb{R}$. What is the subspace topology on Z?

It's discrete.

Let $Z = \{z_1, \dots, z_n\}$ $z_i \neq z_j$ for $i \neq j$.

Set $\varepsilon = \min_{1 \leq i \neq j \leq n} |z_i - z_j|$. Then $B(z_i, \varepsilon) \cap Z = \{z_i\}$

is open in the subspace topology on Z for each i.

This implies Z has the discrete topology, since every subset of Z is a union of $\{z_i\}$'s.

(5) Let $Z = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. Give Z the subspace topology it inherits from $\mathbb{R}$. Does Z have the discrete topology?

Nope! Let U be any open subset of ~~$\mathbb{R}$~~ Z containing 0.

Then $\exists\, V$ open in $\mathbb{R}$ s.t. $U = V \cap \mathbb{Z}$.

Since V is open, $\exists\, \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq V$.

This implies all but finitely many elements of Z belong to U, contradicting that U is an arbitrary subset containing 0.

(6) (Open subsets in the finite complement topology are big.) Show that any two nonempty subsets of $\mathbb{R}$ that are open in the finite complement topology have a non-empty intersection.

Let $U = \mathbb{R} - \{z_1, \dots, z_k\}$

$V = \mathbb{R} - \{w_1, \dots, w_\ell\}$

be nonempty open subsets in the finite complement topology.

Then $U \cap V = \mathbb{R} - (\{z_1, \dots, z_k\} \cup \{w_1, \dots, w_k\})$

is the complement of finitely many points, so is non-empty.

(7) Let τ be the finite complement topology on $\mathbb{R}$. What are the continuous functions $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$? Give an answer in terms of the preimages of points.

f is continuous iff the preimages of closed sets are closed.

The closed subsets of $\mathbb{R}$ in finite complement topology are the finite sets and $\mathbb{R}$.

I claim f is continuous iff the preimage of each point of $\mathbb{R}$ is finite or, all of $\mathbb{R}$.

Suppose f is continuous. Then $\{x\}$ is closed in $(\mathbb{R}, \tau)$, so $f^{-1}(\{x\})$ is closed, i.e, it's a finite set or all of $\mathbb{R}$.

Conversely, if the preimage of each point of $\mathbb{R}$ is either finite or all of $\mathbb{R}$, let's check f is continuous, $f^{-1}(\mathbb{R}) = \mathbb{R}$ automatically. The only remaining closed sets are the finite sets, $Z = \{z_1, \dots, z_n\}$. For these $f^{-1}(\{z_1, \dots, z_n\}) = \bigcup_{i=1}^{n} f^{-1}(\{z_i\})$, which is either $\mathbb{R}$, or a finite union of finite sets, hence finite. ∎