Hw 4

2) So define $h: \mathbb{R}^n \to \mathbb{R}^2$
$x \mapsto (f(x), g(x))$

By question 1, this is continuous as $f(x)$ and $g(x)$ are continuous.

So we know addition is continuous ~~(x)~~ $\circ h(x)$

$k: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to x+y$

and $k \circ h(x) = f(x) + g(x)$ is continuous as composition of functions is continuous.

For multiplication, define h the same way and $k': \mathbb{R}^2 \longrightarrow \mathbb{R}$ and from definitions k'
$(x,y) \to xy$ is continuous

So $k' \circ h(x) = f(x)g(x)$ and again composition of continuous functions is continuous so $f(x)g(x)$ is continuous

3) $\Rightarrow$ Suppose $F: \mathbb{R}^n \to \mathbb{R}^m$ is continuous and K is closed in $\mathbb{R}^m$. Then $O = \mathbb{R}^m \setminus K$ is open and $F^{-1}(O) = F^{-1}(\mathbb{R}^m \setminus K) = \mathbb{R}^n \setminus F^{-1}(K)$

By def. $F^{-1}(O)$ is open so $\mathbb{R}^n \setminus F^{-1}(K)$ is open and $F^{-1}(K)$ is closed as complement of an open set is closed

$\Leftarrow$ Suppose $F^{-1}(K)$ is closed in $\mathbb{R}^n$.

Let O be open in $\mathbb{R}^m$ then $\mathbb{R}^m \setminus O$ is closed

$F^{-1}(\mathbb{R}^m \setminus O) = \mathbb{R}^n \setminus F^{-1}(O)$ which is closed so $F^{-1}(O)$ is open and therefore F is continuous as $F: \mathbb{R}^n \to \mathbb{R}^m$ is continuous iff $F^{-1}(\text{open}) = \text{open}$

4) Let $v \in B(x, \epsilon)$. If $B(x, \epsilon)$ is open, then $\exists r > 0$ s.t. $B(v, r) \subset B(x, \epsilon)$

Let $r = \epsilon - |x - v|$ and $\vec{y} \in B(v, r)$.

By triangle inequality:

$$|\vec{y} - \vec{x}| \leq |\vec{y} - \vec{v}| + |\vec{x} - \vec{v}|$$

$$|\vec{y} - \vec{x}| < r + |\vec{x} - \vec{v}| \quad \text{as } B(v, r)$$

$$|\vec{y} - \vec{x}| < \epsilon - |\vec{x} - \vec{v}| + |\vec{x} - \vec{v}|$$

$|\vec{y} - \vec{x}| < \epsilon$ so $\vec{y} \in B(x, \epsilon)$ and $B(x, \epsilon)$ is an open subset of $\mathbb{R}^n$. $\square$

5 $Z = \{(x,y) \in \mathbb{R}^2 \mid f(x) = g(y)\}$

$Z = \{(x,y) \in \mathbb{R}^2 \mid f(x) - g(y) = 0\}$

Define $h(x,y) = f(x) - g(y)$ which is continuous through similar reasoning of question 2

$$Z = \{(x,y) \in \mathbb{R}^2 \mid h(x,y) = 0\}$$

So $h^{-1}(\{0\}) =$ closed as h is continuous $\{0\}$ is closed, and $h^{-1}(\{0\}) = Z$ so Z is a closed subset of $\mathbb{R}^2$

We can see that $U = Z^c$ and since Z is closed in $\mathbb{R}^2$, then $Z^c = U$ is open, so U is open in $\mathbb{R}^2$.