Math 135 HW2

1 a) $\exists \varepsilon > 0$, such that $\forall \delta > 0$, $\exists x, y \in D$, $|x-y| < \delta$ and $|f(x)-f(y)| \geq \varepsilon$.

b) $a = (a-b)+b$

$|a| \leq |a-b| + |b| \rightarrow |a|-|b| \leq |a-b|$

Let $b = (b-a)+a$

$|b| \leq |b-a| + |a| \rightarrow |b|-|a| \leq |b-a|$

$|b|-|a| \leq |a-b|$

$|a-b| \geq \max\{|a|-|b|, |b|-|a|\} = ||a|-|b||$ so

$|a-b| \geq ||a|-|b||$

2 $A_1 \cup A_2 \cup A_3 \cup \ldots$

W.L.O.G, we can write out $A_1 \cup A_2 \cup A_3 \cup$ as

$a_{11}, a_{12}, a_{13} \ldots a_{1K}$ $\forall a_1 \in A_1$

$a_{21}, a_{22}, a_{23} \ldots a_{2K}$ $\forall a_2 \in A_2$

$a_{31}, a_{32}, a_{33} \ldots a_{3K}$ $\forall a_3 \in A_3$

and so on.

Map diagonally, so $1 \rightarrow a_{11}$, $2 \rightarrow a_{12}$, $3 \rightarrow a_{21}$, $4 \rightarrow a_{13}$, etc.

The n^{th} diagonal requires n elements, and n elements to cross it out.

Since $\mathbb{N}$ is countably infinite, we never run out and have constructed a map between $\mathbb{N} \rightarrow A_1 \cup A_2 \cup A_3 \ldots$

So $A_1 \cup A_2 \cup A_3 \cup \ldots$ is countable.

3 Assume S is countable. Then $[0,1] = \{x_1, x_2, x_3 \ldots x_n\}$

Let $x_1 = .a_{11}a_{12}a_{13} \ldots$

$x_2 = .a_{21}a_{22}a_{23} \ldots$

$x_3 = .a_{31}a_{32}a_{33}$

etc.

Let $C = .C_1C_2C_3 \ldots$ where $C_i \neq a_{ii}$. Therefore C differs from $x_1, x_2, x_3 \ldots$ by at least 1 digit.

3) As each digit $= 0, 1,$ or 2 and then $c \in [0,1]$
However, $c \notin \{x_1, x_2 \ldots x_n\}$, as $c_i \neq x_{ii}$. This contradicts our countability assumption, so S is not countable

4) $\lim\limits_{n\to\infty} \frac{n^2+2n+1/n^2}{n^2+n-1/n^2} = \lim\limits_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{1}{n}-\frac{1}{n^2}} = 1$ is a guess

Let $\varepsilon > 0$, want to find $N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$\left|\frac{n^2+2n+1}{n^2+n-1} - 1\right| < \varepsilon$$

$$\left|\frac{n^2+2n+1}{n^2+n-1} - \frac{(n^2+n-1)}{n^2+n-1}\right| = \left|\frac{n}{n^2+n-1}\right| < \varepsilon$$

$\left|\frac{n}{n^2+n-1}\right| < \left|\frac{n}{n^2}\right|$ as $n^2+n-1 > n^2$ so then get $\rightarrow$

$\left|\frac{n}{n^2+n-1}\right| < \left|\frac{1}{n}\right| < \varepsilon, \rightarrow \frac{1}{n} < \varepsilon, \; n > \frac{1}{\varepsilon}$

Choose $N > \frac{1}{\varepsilon}$, $\forall n \geq N$, $\left|\frac{n^2+2n+1}{n^2+n-1} - 1\right| = \left|\frac{n}{n^2+n-1}\right| \rightarrow$ 1

1 $\rightarrow < \frac{1}{n} < \varepsilon$ This proves $\lim\limits_{n\to\infty} \frac{n^2+2n+1}{n^2+n-1} = 1$ $\boxed{\text{W}}$

5) From class. $\{0\}_{n=1}^{\infty} = 0$ and $\{\frac{1}{n}\}_{n=1}^{\infty} = 0$

Let's prove $\frac{2^k}{k!} < \frac{1}{k}$ as $k \to \infty$ via induction

First, $2^k < (k-1)!$ This is true for $k=6$ as $2^6 < 5! \to 64 < 120$.

Let's say $2^k < (k-1)!$ up to $k=n$. Now to prove for $k=n+1$.

$$2^{n+1} < (n+1-1)! \to 2 \cdot 2^n < n! \to 2 \cdot 2^n < n(n-1)!$$

From our assumption, $2^n < (n-1)!$, since $n > 2$, then $(n-1)!$ increases by a bigger factor than 2^n, meaning the inequality holds.

Since for $k \geq 6$, $\frac{2^k}{k!} < \frac{1}{k}$ we can prove the limit

Also $0 \leq \frac{2^k}{k!}$ $\forall k$, as get $0 \leq 2$ which is true $\forall k \in \mathbb{N}$.

By Sandwich theorem

So $\lim_{k\to\infty} 0 \leq \lim_{k\to\infty} \frac{2^k}{k!} \leq \lim_{k\to\infty} \frac{1}{k}$

$$0 \leq \lim_{k\to\infty} \frac{2^k}{k!} \leq 0$$

Therefore $\lim_{k\to\infty} \frac{2^k}{k!} = 0$ $\boxed{W}$

6) Let $b = \sup S$ and $b \in \mathbb{R}$.
Suppose $b \notin \mathbb{Z}$, then $b = K.x_1x_2x_3\ldots$. Since $S \subset \mathbb{Z}$, then $\nexists$ some $s \in S$ on $(K, K.x_1x_2x_3\ldots)$.

This violates the ε-criterion of supremum, as for $\varepsilon < .x_1x_2x_3\ldots$, $\forall s \in S$, $s < b - \varepsilon$, as if $K.x_1x_2x_3\ldots$ is an least upper bound, than $\max(S) = K$. Also This is a contradiction, so $b \in \mathbb{Z}$, meaning the integers satisfy the completeness axiom.

7)a) $b = \sup(S)$ iff b is an upper bound and $\forall \varepsilon > 0$, $\exists s \in S$ s.t. $s > b - \varepsilon$.

Guess $\sup(S) = 4$. To show upper bound:

$\cancel{4} - \frac{2}{\sqrt{n}} \leq \cancel{4}$, $\frac{2}{\sqrt{n}} \geq 0$ which is true for $\forall n \in \mathbb{N}$

To show $\forall \varepsilon > 0$, $\exists s \in S$ s.t. $s > b - \varepsilon$:

Let $\varepsilon > 0$, $s > 4 - \varepsilon$

$\cancel{4} - \frac{2}{\sqrt{n}} > \cancel{4} - \varepsilon$

$-\frac{2}{\sqrt{n}} > -\varepsilon \rightarrow \frac{2}{\sqrt{n}} < \varepsilon \Rightarrow \left(\frac{2}{\varepsilon} < \sqrt{n}\right)^2$

$\rightarrow \frac{4}{\varepsilon^2} < n$. By Archimedean property, $\exists n \in \mathbb{N}$ s.t. $n > \frac{4}{\varepsilon^2}$

Both conditions are satisfied so $\sup(S) = 4$.

b) No. Let $\cancel{4} - \frac{2}{\sqrt{n}} = \cancel{4}$. $-\frac{2}{\sqrt{n}} = 0$, which is never true. Since $\sup(S) = 4$, $\forall \varepsilon > 0$, $\exists$ some $s \in S$ s.t. $s > 4 - \varepsilon$, so since it never reaches $\sup(S)$, it never has a maximum, as there is always a slightly bigger value

c) $L = \inf S$ iff L is a lower bound of S and $\forall \epsilon > 0$, then $\exists s \in S$ s.t. $s < L + \epsilon$

Guess $\inf(S) = 2$.

To show 2 is a lower bound.

$$4 - \frac{2}{\sqrt{n}} \geq 2 \rightarrow \frac{-2}{\sqrt{n}} \geq -2 \rightarrow \frac{1}{\sqrt{n}} \leq 1,\ 1 \leq \sqrt{n} \rightarrow 1 \leq n.$$

$$-4 \qquad -4 \qquad \qquad -2$$

So $\forall n \in \mathbb{N}$, $4 - \frac{2}{\sqrt{n}} \geq 2$ meaning 2 is a lower bound.

To prove 2nd condition Let $\epsilon > 0$, $s < 2 + \epsilon$.

$$4 - \frac{2}{\sqrt{n}} < 2 + \epsilon$$

$$2 - \frac{2}{\sqrt{n}} < \epsilon$$

$$\left(\frac{-2}{\sqrt{n}} < (\epsilon - 2)\right)\sqrt{n} \rightarrow \left(\frac{-2}{\epsilon - 2} < \sqrt{n}\right)^2 \quad \frac{4}{(\epsilon-2)^2} < n.$$

By Archimedian property, $\exists n \in \mathbb{N}$ s.t. $\frac{4}{(\epsilon-2)^2} < n$ $\forall \epsilon > 0$

d) min of $S = 2$. To prove, let's show $s \in S$ increases as n increases

$$\cancel{4} - \frac{2}{\sqrt{n}} < \cancel{4} - \frac{2}{\sqrt{n+1}}$$

$$\frac{-2}{\sqrt{n}} > \frac{2}{\sqrt{n+1}} \rightarrow \frac{\sqrt{n+1}}{\sqrt{n}} > 1 \rightarrow \sqrt{n+1} > \sqrt{n} \text{ since } n \in \mathbb{N}, \text{ this is true}$$

As n increases, $4 - \frac{2}{\sqrt{n}}$ increases $\forall n$, so S is at a minimum at the smallest n, which is $n = 1$, and $4 - \frac{2}{\sqrt{1}} = 2$. So minimum of S is 2 ∎

Scanned with CamScanner