Solutions

(1) Consider the following subset of the real line:

$$S = \{0\} \cup (1,2) \cup (2,3) \cup (\mathbb{Q} \cap (4,5)).$$

Compute the following sets: (You may take for granted that $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}^\circ = \emptyset$.)

(a) $\overline{S}$

$\{0\} \cup [1,2] \cup [2,3] \cup [4,5]$

(b) $\left(\overline{S}\right)^\circ$

$(1,3) \cup (4,5)$

(c) $\overline{\left(\overline{S}\right)^\circ}$

$[1,3] \cup [4,5]$

(d) S°

$(1,2) \cup (2,3)$

(e) $\overline{S^\circ}$

$[1,2] \cup [2,3] = [1,3]$

(f) $\left(\overline{S^\circ}\right)^\circ$

$(1,3)$

Remark 1. S is an example of a **Kuratowski 14 set**.

(2) Let A and B be two subsets of a topological space X. Prove the following statements or find a counterexample:

(a) If $A \subseteq B$, then $\operatorname{Int} A \subseteq \operatorname{Int} B$

Int A is the largest open subset of A, so it is also an open subset of B.

By definition of Int B, $\operatorname{Int} A \subseteq \operatorname{Int} B$.

(b) $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$

$\operatorname{Int} A \subseteq A$, $\operatorname{Int} B \subseteq B$, both are open

$\Rightarrow \operatorname{Int} A \cap \operatorname{Int} B \subseteq A \cap B$ and $\operatorname{Int} A \cap \operatorname{Int} B$ is open

$\Rightarrow \operatorname{Int} A \cap \operatorname{Int} B \subseteq \operatorname{Int}(A \cap B)$.

Conversely, by (a), $\operatorname{Int}(A \cap B) \subseteq A \cap B$, $A \cap B \subseteq A$ $\Rightarrow \operatorname{Int}(A \cap B) \subseteq \operatorname{Int}(A)$

and similarly $\operatorname{Int}(A \cap B) \subseteq \operatorname{Int}(B)$

$\Rightarrow \operatorname{Int}(A \cap B) \subseteq \operatorname{Int}(A) \cap \operatorname{Int}(B)$.

Altogether, $\operatorname{Int}(A \cap B) = \operatorname{Int}(A) \cap \operatorname{Int}(B)$.

(c) $\operatorname{Int}(A \cup B) = \operatorname{Int} A \cup \operatorname{Int} B$

False $A = [1,2]$ $B = [2,3]$

$\operatorname{Int}(A \cup B) = \operatorname{Int}([1,3]) = (1,3)$

$\operatorname{Int}[1,2] \cup \operatorname{Int}[2,3] = (1,2) \cup (2,3)$ ≠

(d) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

~~False~~ False $A = (1,2)$, $B = (2,3)$

$A \cap B = \emptyset \Rightarrow \overline{A \cap B} = \emptyset$

$\overline{A} = [1,2]$, $\overline{B} = [2,3] \Rightarrow \overline{A} \cap \overline{B} = \{2\} \neq \emptyset$.

(3) Let X and Y be topological spaces. We've seen that a function $f : X \to Y$ is continuous if and only if it is locally continuous. That is f is continuous if and only if there exists an open cover $\{U_i\}_{i\in I}$ such that $f|_{U_i} : U_i \to Y$ is continuous for all $i \in I$.

More generally a property $\mathcal{P}$ of functions is said to be **local** if for any open cover $\{U_i\}_{i\in I}$ of X, f has property $\mathcal{P}$ if and only if $f|_{U_i} : U_i \to Y$ has property $\mathcal{P}$ for all $i \in I$.

(a) Consider the property of being a constant function. Let $\{U_i\}_{i\in I}$ be an open cover of X and let $f : X \to Y$ be a function. Is it true that if $f : X \to Y$ is constant, then ~~is~~ $f|_{U_i} : U_i \to Y$ is constant for all $i \in I$?

yes, the restriction of a constant function is constant

(b) In the same situation, if $f|_{U_i} : U_i \to Y$ is constant for all $i \in I$, is it necessarily true that $f : X \to Y$ is constant? Prove or give a counterexample.

no : Let $X = (1,2) \cup (3,4)$.

Then $U_1 = (1,2)$, $U_2 = (3,4)$ is open cover

$$f : x \longmapsto \begin{cases} 1 & \text{if } x \in (1,2) \\ 2 & \text{if } x \in (3,4) \end{cases}$$

has $f|_{U_1} = 1$, $f|_{U_2} = 2$, but f is not constant.

(4) Let X and Y be topological spaces. Recall that a function $f : X \to Y$ is said to be **open** if for all open subsets $V \subseteq X$, the image $f(V) \subseteq Y$ is an open subset of Y. Let's show that being an open function is a local property of functions.

(a) Let $\{U_i\}_{i\in I}$ be an open cover of X. Let $f : X \to Y$ be an open function. Show that for each $i \in I$, $f|_{U_i} : U_i \to Y$ is also an open function.

if $V \subseteq U_i$ is open, then $V \subseteq X$ is open, so

$f|_{U_i}(V) = f(V)$ is open by hypothesis.

$\therefore f|_{U_i} : U_i \to Y$ is open.

(b) Let $f : X \to Y$ be a function so that $f|_{U_i} : U_i \to Y$ is an open function for each $i \in I$. Show that f is also an open function.

Let $V \subseteq X$ be an open set.

We know $f(V \cap U_i) = f|_{U_i}(V \cap U_i)$ is open for all i

$\Rightarrow \bigcup_{i\in I} f(V \cup U_i) = f(\bigcup_{i\in I} (V \cup U_i))$

$= f(V \cup (\bigcup_{i\in I} U_i))$

$= f(V)$ is open.

$\therefore f$ is an open function.