You are encouraged to work on problems with other Math 136 students and to talk with Todd and our TA Wentao, but your answers should be in your own words.

A proper subset of the problems will be selected for grading.

To address some questions, here is the definition of *complex inner product space.*

Definition 1 *[Complex (Hermetian) Inner Product Space] Let V be a vector space over the complex numbers and let $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$ be a function.*

Then, $\langle\cdot,\cdot\rangle$ is an inner product and $(V,\langle\cdot,\cdot\rangle)$ is an inner product space if the following properties hold. Let $\mathbf{u},\mathbf{v},\mathbf{w}$ be arbitrary vectors in V and let c be an arbitrary complex number.

(a) Linearity in first argument: $\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{w}\rangle+\langle\mathbf{v},\mathbf{w}\rangle$ and $\langle c\mathbf{u},\mathbf{w}\rangle = c\langle\mathbf{u},\mathbf{w}\rangle$.

(b) Conjugate symmetry: $\langle\mathbf{v},\mathbf{w}\rangle = \overline{\langle\mathbf{w},\mathbf{v}\rangle}$.

(c) Positive definiteness: $\langle\mathbf{v},\mathbf{v}\rangle\geq 0$ and $\langle\mathbf{v},\mathbf{v}\rangle = 0$ if and only if $\mathbf{v}=\mathbf{0}$.

If $\langle\cdot,\cdot\rangle$ satisfies all of the properties above *except* $\langle\mathbf{v},\mathbf{v}\rangle = 0$ iff $\mathbf{v}=0$, then it is called a *positive semidefinite* inner product. For $(V,\langle\cdot,\cdot\rangle)$ to be an inner product space, the inner product must be positive definite.

The product for $\mathcal{L}^2([a,b],\mathbb{C})$ is only positive semidefinite, so $\mathcal{L}^2([a,b],\mathbb{C})$ is not an inner product space.

Problems:

1. (30 points) Let $(V,\langle\cdot,\cdot\rangle)$ be a complex inner product space.

   Let $h\in V$ and define the function $\Lambda : V\to\mathbb{C}$ by $\Lambda(f)=\langle f,h\rangle$ for all $f\in V$.

   (a) Prove that Λ is linear, that is if f and g are in V and $c\in\mathbb{C}$ then you need to prove: $\Lambda(f+g)=\Lambda(f)+\Lambda(g)$, and $\Lambda(cf)=c\Lambda(f)$.

   Solution: Since $\Lambda(f)=\langle f,h\rangle$, $\langle\cdot,\cdot\rangle$ is defined in terms of the inner product on V by Definition 1,(1),with given $h\in V$, $\langle f+g,h\rangle=\langle f,h\rangle+\langle g,h\rangle$ and $\langle cf,h\rangle = c\langle f,h\rangle$ for all $c\in\mathbb{C}$ and $\forall f,g\in$ V; this gives linearity of Λ

   (b) Show that Λ is continuous. HINT: It is easiest to prove Λ is Lipschitz, I.e., show there is a constant $c>0$ such that $|\Lambda(f-g)|\leq c\,\|f-g\|$ for f and g in V.

   Solution: For f and g in V, by Λ's linearity, $|\Lambda(f)-\Lambda(g)| = |\Lambda(f-g)| = |\langle f-g,h\rangle|\leq\|f-g\|\,\|h\|$, the last inequality is by Cauchy-Schwartz inequality in inner product space. Taking $c=\|h\|$, this proves Λ is Lipschitz. (If $\|h\|=0$, then let $c=1$.)

2. (30 points) In this problem, you will show a couple of the properties of ℓ^2, the set of all complex sequences $\mathbf{a}=(a_0,a_1,a_2,\dots)$ such that $\|\mathbf{a}\|=\sqrt{\sum_{j=0}^\infty |a_j|^2}<\infty$ (i.e., the sum converges).

   Let $\mathbf{a}=(a_0,a_1,a_2,\dots)$ and $\mathbf{b}=(b_0,b_1,b_2,\dots)$ be vectors in ℓ^2.

   (a) Prove that $\mathbf{a}+\mathbf{b}\in\ell^2$. That is, show that the $\sum_{j=0}^\infty |a_j+b_j|^2$ converges absolutely.

   HINT: Let a and b be complex numbers. Then, $|a+b|^2\leq 2(|a|^2+|b|^2)$.

   Solution: Since $\mathbf{a},\mathbf{b}\in\ell^2$, $\sum_{k=0}^\infty |a_k|^2$ and $\sum_{k=0}^\infty |b_k|^2$ converge so $\sum_{k=0}^\infty 2(|a_k|^2+|b_k|^2)$ converges. Therefore, by inequality $\forall a_k,b_k\in\mathbb{C}, |a_k+b_k|^2\leq 2(|a_k|^2+|b_k|^2)$, it yields $\sum_{k=0}^\infty |a_k+b_k|^2$ converges.

   (b) Prove that $\langle a,b\rangle=\sum_{k=0}^\infty a_k\overline{b_k}$ converges absolutely. This shows that $\langle\mathbf{a},\mathbf{b}\rangle\in\mathbb{C}$ so $\langle\cdot,\cdot\rangle : \ell^2\times\ell^2\to\mathbb{C}$. HINT: Let a and b be complex numbers. Then, $|ab|\leq\frac{1}{2}(|a|^2+|b|^2)$.

   Solution: Since $\forall a_k,\overline{b_k}\in\mathbb{C}$, $|a_k\overline{b_k}|\leq\frac{1}{2}(|a_k|^2+|\overline{b_k}|^2)=\frac{1}{2}(|a_k|^2+|b_k|^2)$ and as a and b are in ℓ^2, $\sum_{k=0}^\infty |a_k|^2$ and $\sum_{k=0}^\infty |b_k|^2$ converge, so $\sum_{k=0}^\infty |a_k\overline{b_k}|$ converges. Thus $\langle a,b\rangle=\sum_{k=0}^\infty a_k\overline{b_k}$ converges absolutely.

3. (40 points) Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous, and assume $\frac{\partial f}{\partial y}(x,y)$ is defined on $(a,b)\times(c,d)$ and extends to a continuous function on $[a,b]\times[c,d]$.

For all $y\in[c,d]$, define $F(y)=\int_{x=a}^{b} f(x,y)\,dx$. You will prove that F is differentiable on (c,d) and

$$\frac{d}{dy}\int_{x=a}^{b} f(x,y)\,dx=\frac{dF}{dy}(y)=\int_{x=a}^{b}\frac{\partial f}{\partial y}(x,y)\,dx. \tag{1}$$

That is, you can bring the derivative with respect to y inside this integral with respect to x.

First, define $g(y)=\displaystyle\int_{x=a}^{b}\frac{\partial f}{\partial y}(x,y)\,dx$ for $y\in(c,d)$.

You may assume g is continuous for $y\in[c,d]$ (the proof was problem 1 on HW 7). Now, use the following steps to show identity (1) .

(a) (short answer) Let $y\in[c,d]$. Explain why $\displaystyle\int_{t=c}^{y} g(t)\,dt=\int_{t=c}^{y}\int_{x=a}^{b}\frac{\partial f}{\partial y}(x,t)\,dx\,dt.$

Solution: g is a continuous function on $[c,d]$, so it is integrable, and $\int_{t=c}^{y} g(t)dt=\int_{t=c}^{y}\int_{x=a}^{b}\frac{\partial f}{\partial y}(x,t)\,dx\,dt$ by the definition of g.

(b) Justify why $\displaystyle\int_{t=c}^{y}\int_{x=a}^{b}\frac{\partial f}{\partial y}(x,t)\,dx\,dt=\int_{x=a}^{b}\int_{t=c}^{y}\frac{\partial f}{\partial y}(x,t)\,dt\,dx$ for each $y\in[c,d]$.

Solution: We know that $\frac{\partial f}{\partial y}$ is defined on $(a,b)\times(c,d)$ and extends to a continuous function on $[a,b]\times[c,d]$. Therefore it is integrable on this rectangle and satisfies the hypothesis of Fubini's Theorem. So, we conclude that for each $y\in[c,d]$ we have:
$\int_{t=c}^{y}\int_{x=a}^{b}\frac{\partial f}{\partial y}(x,t)\,dx\,dt=\int_{x=a}^{b}\int_{t=c}^{y}\frac{\partial f}{\partial y}(x,t)\,dt\,dx$

(c) Now, justify why $\displaystyle\int_{t=c}^{y} g(t)\,dt=\int_{x=a}^{b}[f(x,y)-f(x,c)]\,dx$ for each $y\in[c,d]$.

Solution: For each fixed $x\in[a,b]$, we can apply the FTC to the continuous function $t\to\frac{\partial f}{\partial y}(x,t)$ whose antiderivative is the function $t\to f(x,t)$. It follows that for each $y\in[c,d]$ we have:
$\int_{t=c}^{y} g(t)\,dt=\int_{x=a}^{b}\int_{t=c}^{y}\frac{\partial f}{\partial y}(x,t)\,dt\,dx=\int_{x=a}^{b}[f(x,y)-f(x,c)]\,dx.$

(d) Use the result of part (c) to prove (1).

Solution: By the FTC, it follows that:
$g(y)=\int_{x=a}^{b}\frac{\partial f}{\partial y}(x,y)\,dx=g(y)=\frac{d}{dy}\int_{t=c}^{y} g(t)\,dt=\frac{d}{dy}\int_{x=a}^{b} f(x,y)\,dx-\frac{d}{dy}\int_{x=a}^{b} f(x,c)dx=\frac{d}{dy}\int_{x=a}^{b} f(x,y)\,dx$

Here is an optional extra-credit challenge problem. Todd will grade it.

1. (2 points) Prove the inequalities in the hints for both parts of problem 2.

Solution: Let a and b be complex numbers. We now prove that $|a+b|^2\le 2(|a|^2+|b|^2)$.

As an auxiliary calculation, we first note that

$$0\le(|a|-|b|)^2=|a|^2-2|a||b|+|b|^2.$$

Rearranging, we see

$$2|a||b|\le|a|^2+|b|^2 \tag{2}$$

We now use the triangle inequality to prove the $|a+b|\le|a|+|b|$:

$$\begin{aligned}|a+b|^2&\le(|a|+|b|)^2=|a|^2+2|a||b|+|b|^2\\&\le 2(|a|^2+|b|^2)\end{aligned}$$

where we replaced $2|a||b|$ in the second line using (2).

To prove $|ab|\le\frac{1}{2}(|a|^2+|b|^2)$, we just divide inequality (2) by two.