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Math 136	Real Analysis II	February 22, 2018
T. Quinto & L. Tu	Exam 1 (100 points)	Noon – 1:20 p.m.

In Questions 1 and 2, write your answers on the exam. In the other questions, write your solutions in the bluebook provided. Please simplify your answers. Hand in the exam with the bluebook. Unless instructed otherwise, show your work.

The exam adds up to 102 points, which means you can get two bonus points.

1. (15 points)

   (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$. State the *definition of the first derivative* $f'(a)$.

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   (b) For $a \neq -1$ use the definition of the derivative to compute $f'(a)$ for $f(x) = \dfrac{1}{x+1}$.

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   (c) Let $\mathcal{O} \subset \mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$ be continuously differentiable. Let $\mathbf{x} \in \mathcal{O}$. State the *First Order Approximation Theorem* for f at $\mathbf{x}$ including all of its hypotheses and conclusions.

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(*Exam continues on the other side of this page.*)

2. (9 points) True or False. No justification necessary and no partial credit.

   (a) ______________ If $z = f(x, y)$ has a tangent plane at (x_0, y_0), then f has all directional derivatives at (x_0, y_0).

   (b) ______________ For any function $f\colon \mathbb{R}^2 \to \mathbb{R}$, if the mixed second partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist at (x_0, y_0), then $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

   (c) ______________ If the function $f : \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions at $(0, 0)$, then the graph of $z = f(x, y)$ has a tangent plane at $(0, 0)$.

For the rest of the problems, provide complete answers in your bluebook.

3. (10 points) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Assume $f'(x) > 0$ for all $x \in (a, b)$. Prove that $f(b) > f(a)$.

4. (12 points) Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ or prove that the limit does not exist.

5. (15 points) Consider the equations $\mathbf{F}(x, y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$.

   (a) Find the derivative matrix $\mathbf{DF}(x, y)$ for $(x, y) \in \mathbb{R}^2$.

   (b) Near which points $(x, y) \in \mathbb{R}^2$ can one use the Inverse Function Theorem to solve for (x, y) locally as a function of (u, v)? Why?

   (c) You may assume the Inverse Function theorem holds near $(1, 1)$. Use the information from part (a) to find $\frac{\partial x}{\partial u}(2, 0)$. Note that $\mathbf{F}(1, 1) = (2, 0)$

6. (17 points) Denote points in $\mathbb{R}^3$ by (x, y, z). Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^2$ be continuously differentiable (i.e., in $C^1(\mathbb{R}^3, \mathbb{R}^2)$). Assume $\mathbf{F}(1, 2, 3) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\mathbf{DF}(1, 2, 3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \end{pmatrix}$.

   (a) Explain why one can solve $\mathbf{F}(x, y, z) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ for (y, z) in terms of x for x in a neighborhood of the point $x = 1$?

   (b) Explain why $\mathbf{F}(x, y, z) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ has an infinite number of solutions.

   (c) Can one use the Implicit Function Theorem to solve $\mathbf{F}(x, y, z) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ for (x, z) in terms of y locally near $y = 2$? Why or why not?

7. (12 points) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}$. Use the definition of differentiability to prove that f is continuous at a.

8. (12 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and let $\mathbf{p}$ be a nonzero vector in $\mathbb{R}^n$. Let $\mathbf{x_0} \in \mathbb{R}^n$. Use either the Mean Value Theorem or the First Order Approximation Theorem for f to prove that the directional derivative of f at $\mathbf{x_0}$ in the direction $\mathbf{p}$ is

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x_0}) = \langle \nabla f(\mathbf{x_0}), \mathbf{p} \rangle.$$

Justify each step of your proof.

(End of Exam)