The Generalized Likelihood Ratio

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Outline

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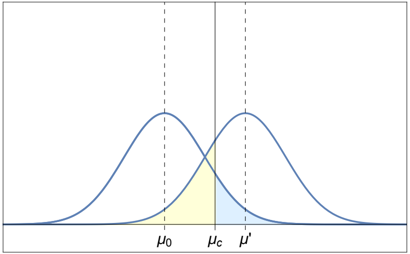
1. Type I and Type II errors

2. The Generalized Likelihood Ratio (GLR)

3. Using the GLR for hypothesis testing

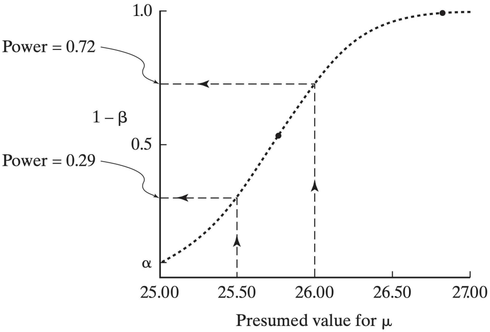
4. Summary

Graphical depiction of Type I and Type II errors



- Type I (blue): $\frac{\mu_c-\mu_0}{\sigma/\sqrt{n}} = +z_\alpha$ so $\mu_c = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$
- Type II (yellow): $\frac{\mu_c-\mu'}{\sigma/\sqrt{n}} = -z_\beta$ so $\mu_c = \mu' - z_\beta \frac{\sigma}{\sqrt{n}}$
- Eliminating μ_c yields $(z_\alpha + z_\beta)\frac{\sigma}{\sqrt{n}} = \mu' - \mu_0$
- Note that as $\mu' \to \mu_0$, we have $z_\alpha = -z_\beta = z_{1-\beta}$, so that $1-\beta = \alpha$

Power curves

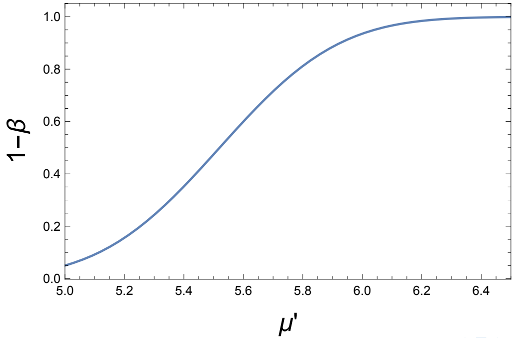


From Larsen & Marx, Fig. 6.4.4, p. 362

- Recall $\mu_0 = 25$ in this example.
- As $\mu' \to \mu_0$, the vertical axis intercept of the power curve is $1 - \beta = \alpha$.

More on power curves

- Return to $(z_\alpha + z_\beta)\frac{\sigma}{\sqrt{n}} = \mu' - \mu_0$
- Solve for $z_\beta = -z_\alpha + \frac{\sqrt{n}}{\sigma}(\mu' - \mu_0)$
- Fix $\alpha = 0.05$, $n = 10$, $\sigma = 1$, $\mu_0 = 5$, and plot $1 - \beta$ vs. μ'



Generalized Likelihood Ratio (GLR)

- Suppose $y_1, y_2, \ldots, y_n$ is random sample from the uniform pdf on $[0, \theta]$
- The parameter θ is unknown
- We wish to conduct a hypothesis test at level of significance α between the alternatives
  - $H_0 : \theta = \theta_0$
  - $H_1 : \theta < \theta_0$

The sets ω and Ω

- Define ω to be the set of parameter values possible under the constraints of H_0
- Define Ω to be the set of all unknown parameters
- In the example of the uniform distribution on $[0, \theta]$,
  - $\omega = \{\theta \mid \theta = \theta_0\}$
  - $\Omega = \{\theta \mid 0 < \theta \leq \theta_0\}$
- Note that $\omega \subset \Omega$

Maximizing likelihood over ω and Ω

- Let L be the likelihood function which, in our example, is

$$L(\theta) = \prod_{j}^{n} f_Y(y_j;\theta) = \begin{cases} \left(\frac{1}{\theta}\right)^n & \text{if } y_{\max} \le \theta \\ 0 & \text{otherwise} \end{cases}$$

- Maximize likelihood over all $\theta \in \omega$, in other words $\theta = \theta_0$
  - This will be L evaluated at θ_0
  - $\max_{\theta\in\omega} L(\theta) = \left(\frac{1}{\theta_0}\right)^n$
- Maximize likelihood over all $\theta \in \Omega$.
  - This will be L evaluated at the maximum likelihood estimate
  - $\max_{\theta\in\Omega} L(\theta) = \left(\frac{1}{y_{\max}}\right)^n$

Generalized Likelihood Ratio

- The *Generalized Likelihood Ratio (GLR)* is then defined to be

$$\lambda = \frac{\max_{\theta\in\omega} L(\theta)}{\max_{\theta\in\Omega} L(\theta)}$$

- For our example, we have

$$\lambda = \frac{\max_{\theta\in\omega} L(\theta)}{\max_{\theta\in\Omega} L(\theta)} = \frac{\left(\frac{1}{\theta_0}\right)^n}{\left(\frac{1}{y_{\max}}\right)^n} = \left(\frac{y_{\max}}{\theta_0}\right)^n$$

- Note that λ will be positive, but always strictly less than one.
- Values of λ near one are compatible with H_0
- Values of λ near zero suggest that we reject H_0

General definition of Generalized Likelihood Ratio

- Let $y_1, y_2, \ldots, y_n$ be a random sample from $f_Y(y; \theta_1, \ldots, \theta_k)$
- The GLR is defined to be

$$\lambda = \frac{\max_{\theta\in\omega} L(\theta_1, \ldots, \theta_k)}{\max_{\theta\in\Omega} L(\theta_1, \ldots, \theta_k)}$$

- Note that the generalization to k parameters is, in principle, straightforward

Hypothesis testing with the Generalized Likelihood Ratio

- Recall that we have determined
  - Note that λ will be positive, but always strictly less than one.
  - Values of λ near one are compatible with H_0
  - Values of λ near zero suggest that we reject H_0
- GLR Test (GLRT) rejects H_0 whenever $0 < \lambda \leq \lambda^*$
- Here λ^* is chosen to satisfy

$$P\left(0 < \Lambda \leq \lambda^* \mid H_0 \text{ is true}\right) = \alpha,$$

where Λ is the random variable associated with λ.

- Note that if we knew $f_\Lambda(\lambda \mid H_0)$, then λ^* could be determined from

$$\alpha = \int_0^{\lambda^*} d\lambda \; f_\Lambda(\lambda \mid H_0)$$

Hypothesis testing with the Generalized Likelihood Ratio

- We would like to choose our cutoff so that

$$\alpha = \int_0^{\lambda^*} d\lambda\, f_\Lambda(\lambda \mid H_0)$$

- Unfortunately, the pdf $f_\Lambda(\lambda \mid H_0)$ may not be so easy to determine
- For the case of $\lambda = \left(\frac{y_{\max}}{\theta_0}\right)^n$, we have

$$\begin{aligned}\alpha =& P\left(\Lambda \leq \lambda^* \mid H_0 \text{ is true}\right)\\ =& P\left[\left(\frac{Y_{\max}}{\theta_0}\right)^n \leq \lambda^* \mid H_0 \text{ is true}\right]\\ =& P\left(Y_{\max} \leq \theta_0 \sqrt[n]{\lambda^*} \mid H_0 \text{ is true}\right)\end{aligned}$$

Hypothesis testing with the Generalized Likelihood Ratio

- From last page

$$\begin{aligned}\alpha =& P\left(Y_{\max} \le \theta_0 \sqrt[n]{\lambda^*} \mid H_0 \text{ is true}\right)\\ =& F_{Y_{\max}}\left(\theta_0 \sqrt[n]{\lambda^*}\right)\\ =& \left[F_Y\left(\theta_0 \sqrt[n]{\lambda^*}\right)\right]^n\\ =& \left(\frac{\theta_0 \sqrt[n]{\lambda^*}}{\theta_0}\right)^n\\ =& \lambda^*\end{aligned}$$

- So reject H_0 if $\lambda \le \lambda^* = \alpha$, or equivalently $y_{\max} \le \theta_0 \sqrt[n]{\alpha}$

Second example: Binomial distribution

- Bernoulli trial $p_X(k;p) = p^k(1-p)^{1-k}$
- Likelihood $L(p) = \prod_j^n p^{k_j}(1-p)^{1-k_j} = p^k(1-p)^{n-k}$ where $k = \sum_j^n k_j$
- Log likelihood $\ln L(p) = k \ln p + (n-k)\ln(1-p)$
- Max likelihood $0 = \frac{d}{dp}\ln L(p) = \frac{k}{p} - \frac{n-k}{1-p}$ so $p_e = \frac{k}{n}$
- Max likelihood estimator $\hat{p}(\vec{X}) = \frac{1}{n}\sum_j^n X_j = \overline{X}$

Binomial distribution: GLR hypothesis testing

- Test $H_0 : p = p_0$ versus $H_1 : p \neq p_0$
- $\max_{p\in\omega} L(p) = L(p_0) = p_0^k(1-p_0)^{n-k}$
- $\max_{p\in\Omega} L(p) = L(\hat{p}) = \overline{X}^k(1-\overline{X})^{n-k}$
- GLR

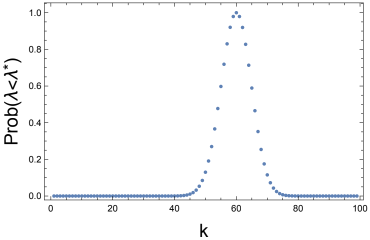
$$\lambda = \frac{\max_{p\in\omega} L(p)}{\max_{p\in\Omega} L(p)} = \frac{p_0^k(1-p_0)^{n-k}}{\overline{X}^k(1-\overline{X})^{n-k}} = \left(\frac{p_0}{k/n}\right)^k \left(\frac{1-p_0}{1-k/n}\right)^{n-k}$$

Binomial distribution: GLR hypothesis testing

- Demand that

$$\alpha = \text{Prob}\left(\lambda < \lambda^*\right) = \text{Prob}\left(\left(\frac{p_0}{k/n}\right)^k\left(\frac{1-p_0}{1-k/n}\right)^{n-k} < \lambda^*\right)$$

- Given p_0, left-hand side of inequality can be tabulated for various values of k
- For $n = 100$ and $p_0 = 0.60$,



- $\sum_{k=0}^{46} \text{Prob}(\lambda < \lambda^*) = 0.0399$
- $\sum_{k=73}^{100} \text{Prob}(\lambda < \lambda^*) = 0.0507$

- Rejecting H_0 if $k \leq 46$ or $k \geq 73$ gives $\alpha \approx 0.1$.

Summary

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- We have reviewed Type I and Type II errors and added some observations
- We have introduced and defined the Generalized Likelihood Ratio
- We have demonstrated how to use the GLR for hypothesis testing
  - For uniform distribution
  - For binomial distribution