Tufts University
Department of Mathematics

Math 135 **Homework 10**[1] **Fall 2022**

Readings for Problem Set 10

§11.3: Path Connectedness and the Intermediate Value Theorem
§11.4: Connectedness and the Intermediate Value Property

Problem Set 10
(Due Wednesday, November 30, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (10 points) (**Path-connectedness of a union**) §11.3, p. 309, #2.
   Let A and B be path-connected subsets of $\mathbb{R}^n$ whose intersection is nonempty. Prove that the union $A \cup B$ is also path-connected.

2. (10 points) (**Path-connectedness of an ellipse**) §11.3, p. 309, #3.
   Let a and b be positive real numbers. Use the path-connectedness of a graph on a path-connected domain and the previous problem to prove that the ellipse
   $$\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$$
   is path-connected. (*Hint*: Write the ellipse as the union of its closed upper half and its closed lower half. Then use #1.)

3. (10 points) (**Path-connected subsets of** $\mathbb{Q}$)
   (a) Describe all path-connected subsets of $\mathbb{Q}$.
   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and let $A \subset \mathbb{R}^n$. Assume A is path-connected *and* assume $f(A) \subset \mathbb{Q}$. What can you say about the image $f(A)$?

4. (10 points) (**Connectedness**) §11.4, p. 313, #3.
   Let A be a connected subset of $\mathbb{R}^3$. Suppose that the points $(0,0,1)$ and $(4,3,0)$ are in A.
   (a) Prove that there is a point in A whose second component is 2.
   (b) Prove that there is a point in A whose norm is 4.
   (*Hint*: Use the intermediate-value property of connectedness. In (a), which function is 2 an intermediate value of?)

5. (10 points) (**Rational and irrational numbers**) §11.3, p. 309, #6.
   Show that the set $S = \{(x,y) \in \mathbb{R}^2 \mid x \text{ or } y \text{ is rational}\}$ is path-connected. (*Hint*: You may draw a picture and use it to describe paths between any two points in S.)

6. (20 points) (**Connectedness and the intermediate-value property**)
   In this problem, you will show some useful facts about continuous functions. Let A be a subset of $\mathbb{R}^n$ and let $F : A \to \mathbb{R}^m$ be continuous.
   (a) Let B be a nonempty subset of A. Prove using the definition that the function F on the smaller domain B, $F : B \to \mathbb{R}^m$ is continuous.
   (b) Let U and V be *disjoint open* subsets of $\mathbb{R}^n$. Prove that the function $f : U \cup V \to \mathbb{R}$ defined by
   $f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$ is continuous.
   (c) Now let A be a set in $\mathbb{R}^n$ that is not connected. Find a function $f : A \to \mathbb{R}$ that is continuous and such that $f(A)$ is not an interval.
   (d) (2 points) Does a disconnected set A have the intermediate-value property? Why or why not?

7. (10 points) (**Convex sets**)
   (a) Let A and B be convex sets in $\mathbb{R}^n$. Is the intersection, $A \cap B$ convex? Either prove this or draw a counterexample and explain why it is a counterexample.
   (b) Let A and B be pathwise connected sets in $\mathbb{R}^n$. Is the intersection $A \cap B$ pathwise connected? Either prove this or draw a counterexample and explain why it is a counterexample.
   (c) (3 points) Why can't the argument from part (a) be used to prove $A \cap B$ is pathwise connected in part (b)?

8. (10 points) (**Topologist's sine curve**) §11.4, p 313, #7
   Let K be the closed interval $\{0\} \times [-1, 1]$ and $G = \{(x, \sin 1/x) \mid x \in (0, 1]\}$. The topologist's sine curve is the union $A = K \cup G$ (Example 11.38, p. 312). Show that the topologist's sine curve is not path-connected. (*Hint*: Suppose that there is a parametrized path γ: $[0, 1] \to \mathbb{R}^2$ in A joining $(0, 1)$ to $(1, \sin 1)$. Let $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big)$. Define t_* to be the supremum of the points t in $[0, 1]$ such that γ maps the interval $[0, t]$ into K. This means $\gamma_1(t) \in K$ for all $t < t_*$, so $\gamma_1(t) = 0$ for all $t < t_*$. By continuity, $\gamma_1(t_*) = 0$. Express $\gamma_2(t)$ for $t > t_*$ in terms of $\gamma_1(t)$ and then show that $\gamma_2(t)$ is not continuous at t_*.)

(End of Problem Set 10)