A proper subset of the problems will be selected for grading.

Here are some useful theorems and definitions. Refer to them by number if you use them.

Theorem 1 *[Lebesgue's Theorem] Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let f be a bounded function from $\mathbb{I}$ to $\mathbb{R}$. Then, f is integrable if and only if the set of discontinuities of f, $D(f, \mathbb{I}) = \left\{\mathbf{x} \in \mathbb{I} \,\middle|\, f \text{ is discontinuous at } \mathbf{x}\right\}$ has measure zero.*

Definition 2 *Let $A \subset \mathbb{R}^n$ and let f and g be functions from A to $\mathbb{R}$. Then, $f = g$* almost everywhere, *$f = g$ a.e., if $\left\{\mathbf{x} \in A \,\middle|\, f(x) \neq g(x)\right\}$ has measure zero.*

Theorem 3 *Let A be a bounded subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded integrable function. If A has measure zero then $\int_A f = 0$.*

Theorem 4 *Let A be a bounded subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded integrable function.*

(a) If $f = 0$ a.e. on A, then $\int_A f = 0$.

(b) If $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in A$, then $\int_A f = 0$ if and only if $f = 0$ a.e. on A.

Problems:

1. (20 points) Let $\mathbb{I} = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$ and let $h : \mathbb{I} \to \mathbb{R}$ be a continuous function. For each $y \in [c, d]$ define
$$g(y) = \int_a^b h(x, y)\, dx.$$
Prove that the function $g : [c, d] \to \mathbb{R}$ is continuous.

   HINT: First, note that h is uniformly continuous. You may assume that this implies that for each $\epsilon > 0$ there is a $\delta > 0$ such that if y and y_0 are points in $[c, d]$ and $|y - y_0| < \delta$, then for all $x \in [a, b]$, $|h(x, y) - h(x, y_0)| < \frac{\epsilon}{2(b-a)}$. Use this to prove the bound $-\epsilon < g(y) - g(y_0) < \epsilon$, which will prove g is (uniformly) continuous for $y \in [c, d]$.

   Solution: Let $\epsilon > 0$. We will assume the hint, so let $\delta > 0$ such that if y and y_0 are points in $[c, d]$ and $|y - y_0| < \delta$, then
$$-\frac{\epsilon}{2(b-a)} < h(x, y) - h(x, y_0) < \frac{\epsilon}{2(b-a)} \quad \forall x \in [a, b]. \tag{1}$$
Now, let $|y - y_0| < \delta$, then
$$g(y) - g(y_0) = \int_a^b \left(h(x, y) - h(x, y_0)\right) dx$$
and, by starting with (1) and integrating, we see
$$-\frac{\epsilon}{2} = \int_a^b -\frac{\epsilon}{2(b-a)}\, dx \leq \int_a^b \left(h(x, y) - h(x, y_0)\right) dx \leq \int_a^b \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{2}.$$
This shows $|g(y) - g(y_0)| < \epsilon$ and so g is (uniformly) continuous.

2. (15 points) Show that a.e. is an equivalence relation. That is, if $A \subset \mathbb{R}^n$ and f, g, and h are functions from A to $\mathbb{R}$ (or $\mathbb{C}$), then

   (a) $f = f$ a.e.; (b) if $f = g$ a.e. then $g = f$ a.e.; (c) if $f = g$ a.e. and $g = h$ a.e. then $f = h$ a.e.

Solution: To determine if $f = f$ a.e., we calculate $\{x \in [a,b] \mid f(x) \neq f(x)\}$. This is the empty set, so it has measure zero (can you prove this?). Therefore $f = f$ a.e.

Now, let $f = g$ a.e. Then, $\{x \in [a,b] \mid f(x) \neq g(x)\}$ has measure zero However this is the same set as $\{x \in [a,b] \mid g(x) \neq f(x)\}$. Therefore $g = f$ a.e.

Finally, let $f = g$ a.e., and $g = h$ a.e. I claim that $\{x \in [a,b] \mid f(x) \neq h(x)\} \subset \{x \in [a,b] \mid f(x) \neq g(x)\} \cup \{x \in [a,b] \mid g(x) \neq h(x)\}$. Since the two sets on the right hand side of this containment have measure zero, so does the set on the left, $f = h$ a.e.

Here's a proof. If $f(x) = g(x)$ and $g(x) = h(x)$, then $f(x) = h(x)$. The contrapositive of this is:

If $f(x) \neq h(x)$, then either $f(x) \neq g(x)$ or $g(x) \neq h(x)$.

This justifies the following containment (read the definition of these sets and compare with the statement just above):

$$\{x \in [a,b] \mid f(x) \neq h(x)\} \subset \{x \in [a,b] \mid f(x) \neq g(x)\} \cup \{x \in [a,b] \mid g(x) \neq h(x)\}.$$

The next problems are about the set $\mathcal{L}^2([a,b],\mathbb{R})$ which is the set of bounded integrable functions from $[a,b]$ to $\mathbb{R}$ with the inner product $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$.

This is a warm-up for $\mathcal{L}^2([a,b],\mathbb{C})$, which we will consider in class.

3. (20 points) Let f and g be functions in $\mathcal{L}^2([a,b],\mathbb{R})$. We will let fg be the product function $fg : [a,b] \to \mathbb{C}$ defined by $fg(x) = f(x)g(x)$.

   (a) Prove that $D(fg,[a,b]) \subset D(f,[a,b]) \cup D(g,[a,b])$.
   HINT: You know that if f and g are both continuous at a point $x \in [a,b]$ then fg continuous at x. What is the contrapositive of this statement?
   Solution: The contrapositive of this statement is, if fg is not continuous at x then, either f is not continuous at x or g is not continuous at x. This is equivalent to:
   $$D(fg,[a,b]) \subset D(f,[a,b]) \cup D(g,[a,b]). \tag{2}$$

   (b) Prove that the product $fg : [a,b] \to \mathbb{R}$ is integrable (thus, $fg \in \mathcal{L}^2([a,b],\mathbb{R})$).
   Solution: Since f and g are integrable, $D(f,[a,b])$ and $D(g,[a,b])$ have measure zero. Therefore by (2), $D(fg,[a,b])$ has measure zero. As fg is bounded, Lebesgue's Theorem shows fg is integrable.

   (c) Explain why the inner product $\langle f,g\rangle$ is defined on $\mathcal{L}^2([a,b],\mathbb{R})$ (i.e., why is the integral defined).
   Solution: As fg is integrable, $\langle f,g\rangle = \int_a^b fg\,dx$ is defined and so the product $\langle f,g\rangle$ is defined.

   (d) Prove that $\langle f,f\rangle = 0$ if and only if $f = 0$ a.e.
   Solution: $0 = \langle f,f\rangle = \int_a^b f^2\,dx$ if and only if the integrand, f^2, is zero almost everywhere. This is true by Theorem 4.

4. (25 points) Let f, g, and h be functions in $\mathcal{L}^2([a,b],\mathbb{R})$ and let $c \in \mathbb{R}$.

   (a) Prove that $f+g$ and cf are in $\mathcal{L}^2([a,b],\mathbb{R})$. Since the zero function is in $\mathcal{L}^2([a,b],\mathbb{R})$, this shows that $\mathcal{L}^2([a,b],\mathbb{R})$ is a vector space (as it is a subspace of the vector all functions from $[a,b]$ to $\mathbb{R}$).
   Solution: First, note that $\mathcal{L}^2([a,b],\mathbb{R})$ is the set of bounded integrable real valued functions. Therefore, if f and g are integrable, then so is $f+g$ and cf, since the integral is linear, which was one of our first theorems about integrable functions.

(b) Prove that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Solution: This follows directly from linearity of the integral, since $f+g$ is integrable.

(c) Show that $\langle f, g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$, $\langle f,g\rangle = \langle g,f\rangle$, and $\langle cf,g\rangle = c\langle f,g\rangle$.

Solution: As multiplication is commutative, $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g,f\rangle$. As the integral is linear, $\langle f,g+h\rangle = \int_a^b f(x)(g(x)+h(x))\,dx = \int_a^b f(x)g(x)\,dx = \int_a^b f(x)h(x)\,dx = \langle f,g\rangle + \langle f,h\rangle$. Similarly, $\langle cf,g\rangle = \int_a^b cf(x)g(x)\,dx = c\int_a^b f(x)g(x)\,dx = \langle f,g\rangle$.

(d) Is $\mathcal{L}^2([a,b],\mathbb{R})$ an inner product space? Why or why not?

Solution: The product we defined on $\mathcal{L}^2([a,b],\mathbb{R})$ is positive semidefinite as, for $f\in\mathcal{L}^2([a,b],\mathbb{R})$, $\langle f,f\rangle = \int_a^b f^2(x)\,dx \geq 0$ as $f^2(x)\geq 0$.

However, this product is not positive definite as $0 = \langle f,f\rangle = \int_a^b f^2(x)\,dx$ if and only if $f^2(x) = 0$ a.e. on $[a,b]$. That is $\langle f,f\rangle = 0$ iff $f = 0$ a.e. This does not imply $f = g$ in $\mathcal{L}^2([a,b],\mathbb{R})$ because $f = g$ in $\mathcal{L}^2([a,b],\mathbb{R})$ if $\forall x\in[a,b],\ f(x) = g(x)$, that is, if f and g are equal as functions. But there are functions that are equal to zero a.e. but are not the zero function.

5. (20 points) Let $L^2([a,b],\mathbb{R})$ be the set $\mathcal{L}^2([a,b],\mathbb{R})$ with equality defined as $f = g$ in $L^2([a,b],\mathbb{R})$ if $f = g$ a.e.

(a) Show that under this definition of equality, if $f = g$ in $L^2([a,b],\mathbb{R})$ then $\int_a^b f = \int_a^b g$.

Solution: Let f and g be in $L^2([a,b],\mathbb{R})$. Then, $f = g$ a.e. as functions. Therefore, $f - g = 0$ a.e. (can you prove this?). By Theorem 4 this implies $\int_a^b (f-g) = 0$ and so, as f and g are integrable, $\int_a^b f = \int_a^b g$. This says that the integral is well-defined in $L^2([a,b],\mathbb{R})$.

(b) Let f_1, f_2, g_1, and g_2 be functions in $\mathcal{L}^2([a,b],\mathbb{R})$. Assume $f_1 = f_2$ a.e. and $g_1 = g_2$ a.e. Prove $\langle f_1,g_1\rangle = \langle f_2,g_2\rangle$. This shows that $\langle f,g\rangle$ is well-defined in $L^2([a,b],\mathbb{R})$.

Solution: We use a similar argument to that in part (a). Let $f_1 = f_2$ a.e. and $g_1 = g_2$ a.e. I claim that the products $f_1g_1 = f_2g_2$ a.e. Here's the proof. If $x\in[a,b]$ and $f_1(x) = f_2(x)$ *and* $g_1(x) = g_2(x)$, then $f_1(x)g_1(x) = f_2(x)g_2(x)$. Taking the contrapositive of this, we see that if $f_1(x)g_1(x) \neq f_2(x)g_2(x)$, then either $f_1(x)\neq f_2(x)$ *or* $g_1(x)\neq g_2(x)$. Therefore $\{x\in[a,b] \mid f_1(x)g_1(x)\neq f_2(x)g_2(x)\} \subset \{x\in[a,b] \mid f_1(x)\neq f_2(x)\} \cup \{x\in[a,b] \mid g_1(x)\neq g_2(x)\}$. Since the two sets on the right side of this containment have measure zero, the set on the left has measure zero, Therefore $f_1g_1 = f_2g_2$ a.e. Then, by the result of part (a), this shows $\langle f_1,g_1\rangle = \langle f_2,g_2\rangle$.

(c) Show that under this definition, the inner product $\langle f,g\rangle = \int_a^b fg$ is positive definite (that is, $\langle f,f\rangle\geq 0$ and $\langle f,f\rangle = 0$ if and only if $f = 0$ in $L^2([a,b],\mathbb{R})$.

HINT: What does equality in $L^2([a,b],\mathbb{R})$ mean?

Solution: We already showed for $f\in\mathcal{L}^2([a,b],\mathbb{R})$ that $\langle f,f\rangle\geq 0$. Now, we show that $\langle f,f\rangle = 0$ if and only if $f = 0$ a.e. Of course, if $f = 0$ a.e., then $\langle f,f\rangle = \int_a^b f^2 = 0$ by Theorem 4.

Similarly, if $\langle f,f\rangle = 0$ then, since $f^2\geq 0$, $f = 0$ a.e. by Theorem 4. That means $f = 0$ in $L^2([a,b],\mathbb{R})$.

Here are optional extra-credit challenge problems. Todd will grade them.

1. (2 points) Use the fact that $f:\mathbb{I}\to\mathbb{R}$ is uniformly continuous to prove the hint to problem 1 of this homework.

Solution: Because h is continuous on the compact set $\mathbb{I} = [a,b]\times[c,d]$, h is uniformly continuous on $\mathbb{I}$. Let $\epsilon > 0$, then there is a $\delta > 0$ such that if $\|(x,y)-(x_0,y_0)\| < \delta$ then $|h(x,y) - h(x_0,y_0| < \epsilon$. Now, we just choose $x = x_0$, then, if $|y-y_0| < \delta$, then $\|(x,y)-(x,y_0)\| < \delta$. Therefore $|h(x,y)-h(x,y_0)| < \epsilon$.

2. (2 points–this should be worth much more) Define $f:[0,1]\to\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x\notin\mathbb{Q}\cap[0,1] \\ \frac{1}{n} & \text{if } x = \frac{m}{n}\in\mathbb{Q}\cap[0,1] \text{ and } \frac{m}{n} \text{ is in lowest form} \end{cases}.$$

So, for example, $f(0.75) = \frac{1}{4}$ and $f(\sqrt{2}/2) = 0$.

(a) Show that $D(f,[0,1]) = \mathbb{Q} \cap [0,1]$

Solution: We first show that f is discontinuous at each rational number in $[0,1]$. Then, we will show that f is continuos at each irrational number in $[0,1]$.

I should have mentioned that $f(0) = f(0/1) = 1/1 = 1$.

Let q be a rational number in $[0,1]$. Then $f(q) > 0$ since $f(q) = 1/n$ where $q = p/n$ when written in reduced form, and $n \in \mathbb{N}$.

Let $q \in \mathbb{Q} \cap [0,1]$. then, by density of the irrationals in $[0,1]$, there is a sequence of irrational numbers in $[0,1]$ $\{r_j\}$ that converges to q. By definition $f(r_j) = 0$ for all j so $f(r_j) \to 0 \neq f(q)$

Now, let r be an irrational number in $[0,1]$. Then, $f(r) = 0$. We will show that the only rational numbers "near enough" to r have large denominators, and this will allow us to make $|f(x) - f(r)|$ small for x sufficiently near r. Here goes!

For $N \in \mathbb{N}$, define

$$\mathcal{N}_N = \left\{ \frac{k}{n} \mid 0 \leq k \leq n \leq N \right\}.$$

Then,$\mathcal{N}_N$ is the set of all rational numbers in $[0,1]$ with denominator less than or equal to N.

So, for example $\mathcal{N}_4 = \{1/4, 1/3, 1/2, 2/3, 3/4, 1\}$ and any rational number in $[0,1] \setminus \mathcal{N}_N$ has denominator greater than 4, and so $f(q) < 1/4$ for q a rational number that is not in $\mathcal{N}_4$.

This is true in general, and for $N \in \mathbb{N}$, any rational number $q \in \mathcal{N}_N$ satisfies $f(q) \geq 1/N$.

The only rational numbers that have $f(q) < 1/N$ must have denominators in lowest form that are greater than N and therefore, not in $\mathcal{N}_N$.

This means that if q is a rational number in $[0,1] \setminus \mathcal{N}_N$ then $f(q) < 1/N$

Let r be an irrational number in $[0,1]$. Let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.

Let $\delta = \min \left\{ |q - r| \;\middle|\; q \in \mathcal{N}_N \right\}$. Then, $\delta > 0$ since r is irrational and $\mathcal{N}_N \subset \mathbb{Q}$.

If $|x - r| < \delta$, then $x \notin \mathcal{N}_N$ and therefore $0 \leq f(x) < 1/N < \epsilon$ (if x is irrational, then $f(x) = 0$, and if x is rational, its denominator is greater than N, so $f(x) < 1/N < \epsilon$. To summarise, if $\epsilon > 0$, there is a $\delta > 0$, such that if $x \in [0,1]$ and $|x - r| < \delta$, then $|f(x) - f(r)| < \epsilon$

This shows f is continuous at irrational numbers and discontinuous at rationals: $D(f,[0,1]) = \mathbb{Q} \cap [0,1]$.

(b) Show that f is integrable on $[0,1]$.

Solution: This part is much easier. Since f is integrable ($|f(x)| \leq 1$) and $D(f,[0,1])$ has measure zero as it is the countable set $\mathbb{Q} \cap [0,1]$, f is integrable by Lebesgue's Theorem.