MATH 70 WORKSHEET 5

You are encouraged to work with others, but the final results must be your own.[1]

1. (3 points) Let A, B, and C be invertible $n \times n$ matrices. Use the definition of invertibility to prove that ABC is invertible and that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Solution:

Since A, B and C are invertible, let $P = C^{-1}B^{-1}A^{-1} \in \mathbf{R}^{n\times n}$. Thus,

$$\begin{aligned}(ABC)\cdot P &= (ABC)(C^{-1}B^{-1}A^{-1})\\ &= AB(CC^{-1})B^{-1}A^{-1}\\ &= A(BB^{-1})A^{-1}\\ &= AA^{-1} = I_n\end{aligned}$$

Similarly

$$\begin{aligned}P\cdot(ABC) &= (C^{-1}B^{-1}A^{-1})(ABC)\\ &= C^{-1}B^{-1}BC\\ &= C^{-1}C = I_n\end{aligned}$$

Thus there exists a matrix P such that $P\cdot(ABC) = (ABC)\cdot P = I_n$, so (ABC) is invertible by definition and $(ABC)^{-1} = P = C^{-1}B^{-1}A^{-1}$

2. (4 points) Let A be an invertible $n \times n$ matrix, show that $\det(cA) = c^n \det(A)$ and $\det(A^{-1}) = (\det(A))^{-1}$.

Solution:

(1) By Theorem 3 in Chapter 3, multiplying one row of matrix A by constant c to produce matrix B means $\det B = c(\det A)$.
Since cA is the matrix where every row of A is multiplied by c, and $A \in \mathbf{R}^{n\times n}$ has n rows, we have $\det(cA) = c^n \det A$.
(2) Since A is invertible, we know there exists an inverse matrix A^{-1} such that

$A^{-1}A = I_n$. Thus

$$\begin{aligned}
& \det(A^{-1}A) = \det(I_n) \\
=> \quad & \det(A^{-1})\det(A) = \det(I_n) \quad \text{(by theorem 6 ch.3)} \\
=> \quad & \det(A^{-1})\det(A) = 1 \\
=> \quad & \det(A^{-1}) = \frac{1}{\det(A)} \quad \text{(since } \det(A) \neq 0 \text{ by theorem 4 ch.3).}
\end{aligned}$$

3. (5 points) Let W be the subset of $\mathbb{P}_3$ of polynomials such that $p(0) = p(1)$,

$$W = \{p \in \mathbb{P}_3 \,\big|\, p(0) = p(1)\}.$$

Prove that W is a subspace of $\mathbb{P}_3$.

Solution:

Recall that a general polynomial in $\mathbb{P}_3$ is of the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. Evaluating this polynomial at $t = 0$ and $t = 1$, we get

$$\begin{aligned}
p(0) &= a_0 \\
p(1) &= a_0 + a_1 + a_2 + a_3.
\end{aligned}$$

The polynomial $p(t)$ is in W if and only if $p(0) = p(1)$, which is equivalent to saying $a_1 + a_2 + a_3 = 0$.
First, the zero vector $z(t) = 0$ clearly satisfies $z(0) = z(1) = 0$.
Next, we check that W is closed under addition. Suppose $q(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$ is another polynomial in W (so $b_1 + b_2 + b_3 = 0$), then the sum of p and q is

$$(p+q)(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3,$$

where $c_i = a_i + b_i$ for $i = 0, 1, 2, 3$. Then

$$\begin{aligned}
c_1 + c_2 + c_3 &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) \\
&= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) \\
&= 0.
\end{aligned}$$

Hence $(p+q)(t)$ is also in W.
Finally, we check that W is closed under scalar multiplication. Let c be a scalar. Then

$$(p+q)(t) = d_0 + d_1 t + d_2 t^2 + d_3 t^3,$$

where $d_i = c a_i$ for $i = 0, 1, 2, 3$. Then

$$\begin{aligned}
d_1 + d_2 + d_3 &= c a_1 + c a_2 + c a_3 \\
&= c(a_1 + a_2 + a_3) \\
&= 0.
\end{aligned}$$

Thus $cp(t)$ is on W.

ALTERNATIVE PROOF: By definition $(p+q)(t) = p(t) + q(t)$ and $(cp)(t) = cp(t)$. You can use these facts to more concisely check that W is closed under addition and scalar multiplication.

4. (4 points) Note that $M_{2\times 2}$ is the vector space of 2×2 matrices. Decide whether the set $W = \{A \in M_{2\times 2} \mid \det(A) = 0\}$ is a subspace of $M_{2\times 2}$. If W is a subspace, prove it using the definition. If W is not a subspace, find a specific counterexample, i.e., define the matrices and scalars in your counterexample using specific numbers.

Solution:

Here, W is not a subspace, and there is a variety of counterexamples you could use to prove it.
For example, consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

A quick calculation shows that $\det(A) = \det(B) = 0$. But since $A + B = I$, the identity matrix which has determinant 1, W is *not* closed under addition.

5. (4 points) Let V be a vector space.

(a) Let $c \in \mathbb{R}$ and let $\mathbf{0}$ denote the zero vector in V. Prove that $c\mathbf{0} = \mathbf{0}$.

(b) Let $\mathbf{v} \in V$. Prove that if $c \in \mathbb{R}$ and $c \neq 0$ and $c\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.

(c) Prove that for each $\mathbf{x} \in V$, its additive inverse is unique. That is if $\mathbf{x} + \mathbf{u} = \mathbf{0}$ and $\mathbf{x} + \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{v}$.

Solution:

(a) Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$, we have that

$$c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}.$$

Subtracting $c\mathbf{0}$ from both sides, we get that $c\mathbf{0} = \mathbf{0}$.

(b) Since $c \neq 0$, the reciprocal $\frac{1}{c}$ exists. Scale both sides of $c\mathbf{v} = \mathbf{0}$ by $\frac{1}{c}$ to get $\mathbf{v} = 0$.

(c) Suppose that $\mathbf{x}+\mathbf{u}=\mathbf{u}+\mathbf{x}=\mathbf{0}$ and $\mathbf{x}+\mathbf{v}=\mathbf{v}+\mathbf{x}=\mathbf{0}$. Then using properties of vector addition, we have

$$\begin{aligned} \mathbf{u} &= \mathbf{u}+\mathbf{0} \\ &= \mathbf{u}+(\mathbf{x}+\mathbf{v}) \\ &= (\mathbf{u}+\mathbf{x})+\mathbf{v} \\ &= \mathbf{0}+\mathbf{v} \\ &= \mathbf{v}. \end{aligned}$$

Thus $\mathbf{u}=\mathbf{v}$.