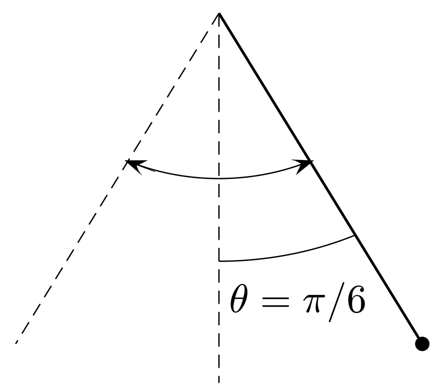
Tuesday, October 5

1. Here is another simple example of non-dimensionalization. Think of an idealized pendulum: A massless rod with a weight attached to the bottom:



The angle θ is measured from the position hanging straight down, in the clockwise direction. The length of the rod is L, and the weight has mass m. From Newton's force law, $F = ma$, one concludes:

$$-mg\sin\theta = mL\frac{d^2\theta}{dt^2}$$

where g is the gravitational acceleration of the earth (approximately 32 ft/sec^2). The left-hand side represents the component of the gravitational force in the tangential direction (tangential to the path of motion of the weight), and on the right-hand side you see mass m, times the tangential acceleration $L\frac{d^2\theta}{dt^2}$. This is explained in greater detail on Wikipedia, you can read it there.

Interestingly, m plays no role: It cancels out. So

$$-g\sin\theta = L\frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta. \tag{1}$$

Verify that g/L is the $1/\text{time}^2$, and introduce a time unit that makes the parameter g/L disappear.

2. Equation (1) is *second order*, and inn class, our main example, the Kermack-McKendrick model, has been *first order*. However, you can turn the equation into a first order *system* by artificially introducing the extra unknown function

$$\eta = \frac{d\theta}{dt}.$$

So

$$\begin{aligned}\frac{d\theta}{dt} &= \eta \\ \frac{d\eta}{dt} &= -\frac{g}{L}\sin\theta\end{aligned}$$

This is a first order system, and our theorem about existence and uniqueness of solutions applies. What does that imply about (1)? Which kind of initial conditions do you need for (1) to make the solution unique? Explain why your conclusion makes physical sense.

3. Think about the following equation:

$$\frac{dx}{dt} = x(x-\gamma). \tag{2}$$

Here γ is a parameter, either positive or negative.

(a) What are the steady states, the fixed points, the equilibria, the constant solutions? (All of these expressions denote the same thing: Solutions where $x(t)$ does not in reality depend on t.)

(b) For each equilibrium, decide whether it is stable (when you start with an x near the equilibrium, $x(t)$ converges to the equilibrium) or unstable.

(c) Make a plot in which the horizontal axis is γ (say between -1 and $+1$), and the vertical axis denotes equilibria of equation (2). Indicate *stable* equilibria with solid lines, and *unstable* ones with dashed lines.