Wednesday, February 1

Friday, January 27, 2023 11:22

TA Help session 10:30 Fridays, Math library, JCC 574
Student hours with Todd 1:30-3:00 my office JCC 575 (end of hall)

Student hours will start on Friday at 2:00 and we can continue to 3:30 (because of AWM panel and lunch)

Still get-to-know-you meeting slots available and when they fill up, I'll add more
https://docs.google.com/spreadsheets/d/1T8o6af3Oe3uA3aswPvv1pm0FdnmQ6oaiF5Le623wdLY/edit?usp=sharing

MATHEMATICAL CONTEST IN MODELING: February 16-20, 2023. TEAMS OF THREE UNDERGRADS
https://www.contest.comap.com/undergraduate/contests/

DIRECTED READING PROGRAM: grad student and undergrad read a math book or article and learn about it together
A list of projects and descriptions can be found
here: https://drive.google.com/file/d/1ffyVId43yPtFP-9GiODrtHf3ZlJ2Nc2S/view?usp=sharing

Application: https://forms.gle/P46BCsEKvdnzftLo9

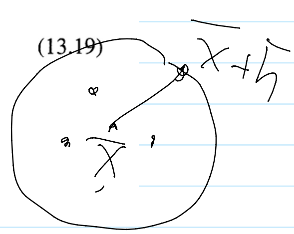
Save the date! AWM Panel & Lunch with Malena Espanol
Friday February 3rd at 1pm in JCC 501

Malena Espanol is an assistant professor in the school of Mathematical and Statistical Sciences at Arizona State University. She earned a Ph.D. in math from Tufts in 2009. The Tufts AWM chapter is excited to host Dr. Espanol for a Q&A over lunch! Everyone in the Tufts community is welcome to join.

Please RSVP at https://tufts.qualtrics.com/jfe/form/SV_0cR5K8g15jJQ7eC

Proposition 13.15 The Mean Value Proposition Let $\mathbf{x}$ be a point in $\mathbb{R}^n$ and let r be a positive number. Suppose that the function $f : \mathcal{B}_r(\mathbf{x}) \to \mathbb{R}$ has first-order partial derivatives. Then if the point $\mathbf{x}+\mathbf{h}$ belongs to $\mathcal{B}_r(\mathbf{x})$, there are points $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ in $\mathcal{B}_r(\mathbf{x})$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i), \qquad (13.19)$$



and

$$\|\mathbf{x} - \mathbf{z}_i\| < \|\mathbf{h}\| \qquad \text{for each index } i \text{ with } 1 \le i \le n.$$

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives at $\mathbf{x}$. We define the *gradient* of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$, denoted by $\nabla f(\mathbf{x})$, to be the point in $\mathbb{R}^n$ given by

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right).$$

Math 136 section 2 start reading these notes at the proof of the Directional Derivative Theorem on the next page.

Directional Deriv Thm $\ominus = \ominus$ $\theta \in \mathbb{R}$

$\ominus$ open in $\mathbb{R}^n$ $f : \ominus \to \mathbb{R}$ set $\bar{x} \in \ominus$

$\bar{h} = (h_1, h_2, \dots h_n) \in \mathbb{R}^n \setminus \{0\}$

$f \in C^1(\ominus)$ (cont. diff)

Then $\frac{\partial f}{\partial \bar{h}}(\bar{x}) = \langle \nabla f(\bar{x}), \bar{h} \rangle$

$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{x})\, h_i$

Eg $f : \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^2 y + 12$

$\bar{h} = (3,4)$ Find $\frac{\partial f}{\partial \bar{h}}(1,2)$

Soln. as $f \in C^1(\mathbb{R}^2)$

$\frac{\partial f}{\partial \bar{h}}(1,2) = \langle \nabla f(1,2), (3,4) \rangle$

$= \langle (4, 1), (3,4) \rangle$

$\frac{\partial f}{\partial x} = 2xy$ $\frac{\partial f}{\partial y} = x^2$ eval $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ @ $(1,2)$

$= 12 + 4 = 16$

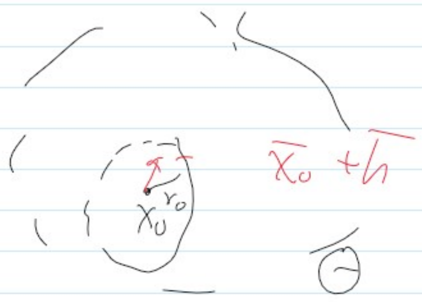
Pf of thm $\ominus$ open in $\mathbb{R}^n$ $\bar{h} = (h_1, h_2, \dots h_n) \neq \bar{0}$

$f \in C^1(\ominus)$

Let $r_0 > 0$ st $B_{r_0}(\bar{x}) \subset \ominus$

let $r_0 > 0$ st $B_{r_0}(\bar{x}) \subseteq \Theta$

let $t \in \mathbb{R}$ $\quad |t| < r_0 / \|h\|$



By Mean value prop

$$f(\bar{x} + t\bar{h}) - f(\bar{x}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\bar{z}_j)\, t h_j$$

$$\|\bar{z}_j - \bar{x}\| < \|t\bar{h}\| = |t|\,\|\bar{h}\|$$

so $\displaystyle \frac{\partial f}{\partial \bar{h}}(\bar{x}) = \lim_{t \to 0} \frac{f(\bar{x} + t\bar{h}) - f(\bar{x})}{t}$

$\displaystyle = \lim_{t \to 0} \frac{t}{t} \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\bar{z}_j)\, h_j \qquad \|\bar{z}_j - \bar{x}\| < |t|\,\|h\|$

$\bar{z}_j \to \bar{x}$ so $\displaystyle \frac{\partial f}{\partial x_j}(\bar{z}_j) \to \frac{\partial f}{\partial x_j}(\bar{x})$

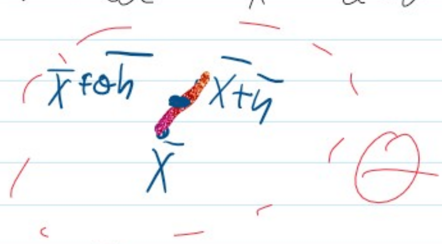
as f is C' (cont diff

$\therefore \displaystyle \frac{\partial f}{\partial \bar{h}}(\bar{x}) = \sum \frac{\partial f}{\partial x_j}(\bar{x})\, h_j = \langle \nabla f(\bar{x}), \bar{h} \rangle$

<u>Mean Value Thm</u> for Θ open in $\mathbb{R}^n$

$f \in C'(\Theta)$ assum $\bar{x} \in \Theta$ $\quad \bar{h} \in \mathbb{R}^n \setminus 0$

$\bar{h} = (h_1, \ldots h_n)$ assume the segment between $\bar{x}$ and $\bar{x} + \bar{h}$ is in Θ



then

$$f(\bar{x} + \bar{h}) - f(\bar{x}) = \langle \nabla(\bar{x} + \theta\bar{h}), \bar{h} \rangle$$

for some $\theta \in (0,1)$

pf $\quad$ define $\varphi(t) = f(\bar{x} + t\bar{h})$

$\varphi(0) = f(\bar{x}) \qquad \varphi(1) = f(\bar{x} + \bar{h})$

Pf define $\phi(t) = f(\bar{x} + t\bar{h})$

$\phi(0) = f(\bar{x})$ $\quad \phi(1) = f(\bar{x}+\bar{h})$

let I be open interval st $[0,1] \subset I$ and if $t \in I$, $\bar{x} + t\bar{h} \in \Theta$

Claim ϕ is diff on I and

$$\phi'(t) = \langle \nabla f(\bar{x} + t\bar{h}), \bar{h} \rangle$$

pf $\quad \phi(t) = f(\bar{x} + t\bar{h})$

$$\phi'(t) = \lim_{s \to 0} \frac{\phi(t+s) - \phi(t)}{s} \quad \text{if it } \exists$$

$$= \lim_{s \to 0} \frac{f(\bar{x} + t\bar{h} + s\bar{h}) - f(\bar{x} + t\bar{h})}{s}$$

$$= \frac{\partial f}{\partial \bar{h}}(\bar{x} + t\bar{h}) = \langle \nabla f(\bar{x} + t\bar{h}), \bar{h} \rangle$$

Dir deriv, $f \in C^1$

Use MVT on $\mathbb{R}$

$\phi : [0,1] \to \mathbb{R}$ cont as ϕ is diff on I

ϕ diff on $(0,1)$

$$\therefore \frac{\phi(1) - \phi(0)}{1 - 0} = \phi'(\theta) = \langle \nabla f(\bar{x} + \theta\bar{h}), \bar{h} \rangle$$

some $\theta \in (0,1)$

$$f(\bar{x} + \bar{h}) - f(\bar{x}) = \langle \nabla f(\bar{x} + \theta\bar{h}), \bar{h} \rangle$$

for some $\theta \in (0,1)$

Thm let $f \in C^1(\Theta)$

then f is continuous

pf let $\bar{x} \in \Theta$ $\quad r > 0$ st $B_r(\bar{x}) \subset \Theta$

and assume $\mathrm{cl}\, B_r(\bar{x}) \subset \Theta$

as f is C^1 on $\mathrm{cl}(B_r(\bar{x}))$

($\bar{x}$ r) Θ as ∇f is C on $cl(B_r(x))$ and $cl(B_r(x))$ is seq cpt

each $\frac{\partial f}{\partial x_j}$ is bounded on $cl(B_r(x))$ as $f \in C^1$

So $\exists M > 0$ st

$\forall \bar{y} \in cl B_r(\bar{x})$ $\|\nabla f(\bar{y})\| \leq M'$ some M'

let $\|\bar{h}\| < r$ by MVT

$$f(\bar{x}+\bar{h}) - f(\bar{x}) = \langle \nabla f(\bar{x}+\theta\bar{h}), \bar{h} \rangle$$

some $\theta \in (0,1)$

$$0 \leq |f(\bar{x}+\bar{h}) - f(\bar{x})| = |\langle \nabla f(\bar{x}+\theta\bar{h}), \bar{h} \rangle|$$

$x\theta h$, $x+\theta h$, x

$$\leq \|\nabla f(\bar{x}+\theta\bar{h})\| \, \|\bar{h}\|$$

CSB $\leq M' \|\bar{h}\|$

to check cont at $\bar{x}$ let $\bar{h} \to 0$

0 0

$\therefore$ fn cont at $\bar{x}$ ie C^1

Best affine approx. to fns.

Defn $k > 0$ (or $k \in \mathbb{N}$) let Θ open in $\mathbb{R}^n$

$\bar{x} \in \Theta$ $f : \Theta \to \mathbb{R}$ $g : \Theta \to \mathbb{R}$

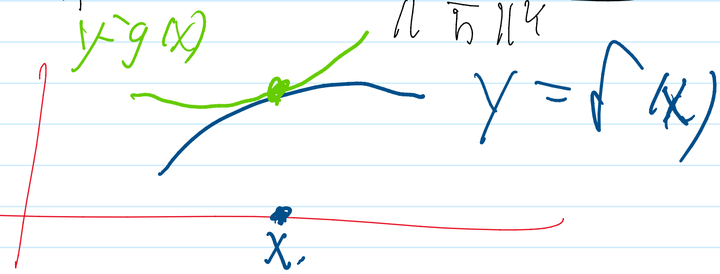
we say g approximates f to order k near $\bar{x}$ if

$$\text{if} \quad \lim_{\bar{y} \to \bar{x}} \frac{g(y) - f(y)}{\|y - x\|^k} = 0$$

equiv $\lim \quad g(\bar{x}+\bar{h}) - f(\bar{x}+\bar{h}) = 0$

equiv $$\lim_{\bar{h}\to\bar{0}} \frac{g(\bar{x}+\bar{h}) - f(\bar{x}+\bar{h})}{\|\bar{h}\|^k} = 0$$

pic



ex I, open interval $x_0 \in I$

$f: I \to \mathbb{R}$ diff at x_0

Then the tangent lin to $y = f(x)$ at x_0

approx f to order 1 at x_0

$$g(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$\lim_{h\to 0} \frac{g(x_0+h) - f(x_0+h)}{|h|}$$

$$= \lim_{h\to 0} \frac{f(x_0) + f'(x_0)(h) - f(x_0+h)}{|h|} \overset{?}{=} 0$$

$$= \lim_{h\to 0} \left| \frac{f(x_0) + f'(x_0)h - f(x_0+h)}{h} \right|$$

$$= \lim_{h\to 0} \left| \frac{f(x_0) - f(x_0+h)}{h} + f'(x_0) \right| =$$

$$= \lim_{h\to 0} \left| \underbrace{f'(x_0)}_{f'(x_0)} - \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{f'(x_0)} \right| = 0$$

Math 136-02 you can stop reading here. The other material will be covered on Monday, 2/6 in section 2.

Def let $a \in \mathbb{R}$ $\bar{h} \in \mathbb{R}^n$

Defn let $a \in \mathbb{R}$ $\bar{b} \in \mathbb{R}^n$

then the fn $g(\bar{x}) = a + \langle \bar{b}, \bar{x} \rangle$

is an affine fn $g: \mathbb{R}^n \to \mathbb{R}$

if $\bar{x}_0 \in \mathbb{R}^n$ we can rewrite g

$$g(\bar{x}_0 + \bar{h}) = \tilde{a} + \langle \bar{b}, \bar{h} \rangle$$

where $\tilde{a} = a + \langle \bar{b}, \bar{x}_0 \rangle$

Thm $\Theta \subset \mathbb{R}^n$ open $f \in C^1(\Theta)$ $\bar{x}_0 \in \Theta$

Then the affine fn

$$g(\bar{x}) = f(\bar{x}_0) + \langle \nabla f(\bar{x}_0), (\bar{x} - \bar{x}_0) \rangle$$

approx f to order 1 at $\bar{x}_0$

Pf outline use MVT in $\mathbb{R}^n$