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2 For $f:(X,\tau_i) \to (Y,\tau_y)$, in X the only open sets are $\emptyset$ and X. By our assumption about τ_y, $\{y\} \in Y$ is an open set in τ_y.
As f is continuous, $f^{-1}(\text{open}) = \text{open}$.
$f^{-1}(\{y\}) = X$ makes f continuous and you can only have X and preimage of nonempty set can't be $\emptyset$.
Since $f^{-1}(\{y\}) = X$, then $f: X \to Y$ is a constant mapping that maps all elements in X to a single element in Y.
Therefore the continuous functions $f:(X,\tau_i) \to (Y,\tau_y)$ are the constant mappings from (X,τ_i) to (Y,τ_y).

3 Let $U \subseteq X$ be an open subset of X. We can define $f: X \to S$ as $f(x) \begin{cases} 1 & x \in U \\ 2 & x \notin U \end{cases}$
f is the characteristic function
We can show $f: X \to S$ is continuous. As $\{1\} \in \tau_s$ then $f^{-1}(\{1\}) = U$ or $f^{-1}(\text{open}) = \text{open}$.
This follows from the function defined as $f(x) = 1$ for $x \in U$
This shows a bijection between $\{U \subseteq X \mid U \text{ is open}\}$ and $\{f: X \to S \mid f \text{ is continuous}\}$ which proves the question.

4 The subspace topology on $\mathbb{Q}$ isn't the discrete topology. To show:
If the subspace topology on $\mathbb{Q}$ is the discrete topology, then $\{q\}$ where $q \in \mathbb{Q}$ is in the subspace topology.
So $\exists U \subseteq \mathbb{R}$ where U is an open set in $\mathbb{R}$ s.t. $\{q\} = \mathbb{Q} \cap U$.
As $\{q\} \in U$, $\exists \epsilon > 0$ such that $(q-\epsilon, q+\epsilon) \subseteq U$

However, $\forall \varepsilon > 0 \; \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon < \varepsilon$ actual

~~Sum~~ So $q < q + \frac{1}{n} < q + \varepsilon$

$q + \frac{1}{n}$ is rational, as q and $\frac{1}{n}$ are rational

So $q + \frac{1}{n} \in (q - \varepsilon, q + \varepsilon)$. Since $q + \frac{1}{n} \in U$

so $\mathbb{Q} \cap U \neq \{q\}$ results in a contradiction and means no singleton set is open in the subspace topology on $\mathbb{Q}$ so the subspace topology on $\mathbb{Q}$ isn't the discrete topology.

5 a) Let $(x_0, y_0) \in \mathbb{R}^2$ and radius of circle is $r \in \mathbb{R}_{>0}$

Then $(x - x_0)^2 + (y - y_0)^2 = r^2$

$\Rightarrow (x - x_0)^2 + (y - y_0)^2 - r^2 = 0$

So $f(x,y) = (x - x_0)^2 + (y - y_0)^2 - r^2 = 0 = Z(\{f\})$

and circles are algebraic varieties

b i) let $f(x,y) = 2$

Then $Z(\{f\}) = \emptyset$ as $f(x,y) \neq 0 \; \forall (x,y) \in \mathbb{R}^2$

so $\emptyset$ is an algebraic variety

Let $g(x,y) = 0$

Then $Z(\{g\}) = \mathbb{R}^2$ as $f(x,y) = 0 \; \forall (x,y) \in \mathbb{R}^2$

So $\mathbb{R}^2$ is an algebraic variety.

ii) $\bigcap_{i \in I} V_i = \bigcap_{i \in I} Z(\{f_i\})$

$= \{(x,y) \in \mathbb{R}^2 \mid f_i(x,y) = 0 \; \forall i \in I\}$

As intersection means it holds for every V_i. and all polynomials in each V_i. This statement is the definition of an algebraic variety, so $\bigcap_{i \in I} V_i$ is an algebraic variety.

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b iii) Proof by induction. to show for $u_1 \dots u_n$ $\bigcup_{i=1}^{n} u_i$ is an algebraic variety when $u_1 \dots u_n$ are varieties

Base case $k=2$

$$U_1 \cup U_2 = \{(x,y) \in \mathbb{R}^2 \mid \forall i \in I, \forall j \in J, f_i(x,y)=0 \text{ or } f_j(x,y)=0\}$$

Note $f_i(x,y)=0$ or $f_j(x,y)=0$ iff $f_i(x,y) f_j(x,y)=0$

So $U_1 \cup U_2 = \{(x,y) \in \mathbb{R}^2 \mid \forall i \in I, j \in J \; f_i(x,y) f_j(x,y)=0\}$

This defines an algebraic variety so $U_1 U_2$ is a variety

Assume this holds for $k=n$, show this holds for $n+1$.

$$\bigcup_{i=1}^{n+1} u_i = \underbrace{\left(\bigcup_{i=1}^{n} u_i\right)}_{\text{variety be assumption.}} \cup U_{n+1}$$

u_{n+1} is an algebraic variety by assumption

As just shown, union of 2 algebraic varieties is an algebraic variety so $\bigcup_{i=1}^{n+1} u_i$ is an algebraic variety

- Can show finite $u_1 \dots u_n$ is also variety

c) $\tau_{zariski} = \tau_z$ (notationally)

i) $\emptyset \in \tau_z$ as $\emptyset^c = \mathbb{R}^2$ which is a variety

$\mathbb{R}^2 \in \tau_z$ as $\mathbb{R}^{2^c} = \emptyset$ which is an algebraic variety.

ii) let $\forall i \in I$ $u_i \in \tau_z$ show $U = \bigcup_{i \in I} u_i \in \tau_z$

$$U = \bigcup_{i \in I} u_i = \left(\bigcup_{i \in I} u_i\right)^{cc} = \Big(\underbrace{\bigcap_{i \in I} u_i^{\,c}}_{\text{algebraic Variety by b ii)}}\Big)^c$$

So it is apparent that U^c is an algebraic variety as $U^c = \bigcap_{i \in I} u_i^{\,c}$. Therefore $\bigcup_{i \in I} u_i \in \tau_z$.

c iii) let $u_1 \dots u_n \in \tau_2$ and $V = \bigcap_{i=1}^{n} u_i$

$$V = \bigcap_{i=1}^{n} u_i = \left(\bigcap_{i=1}^{n} u_i\right)^{c^c} = \left(\underbrace{\bigcup_{i=1}^{n} u_i^c}\right)^c$$

So $V = \left(\bigcup_{i=1}^{n} u_i^c\right)^c$ algebraic variety by b iii) and it is apparent that V^c is an algebraic variety so $\bigcap_{i=1}^{n} u_i \in \tau_2$ $\square$

d) $Z(\{f_i\}) = \{(x,y) \in \mathbb{R}^2 \mid f_i(x,y) = 0\}$

Since polynomials are continuous, then

$f^{-1}(\text{closed}) = \text{closed}$

$f^{-1}(\{0\}) = Z(\{f\})$

So $Z(\{f_i\})$ is ~~a~~ closed subset in the usual topology on $\mathbb{R}^2$ and algebraic varieties are closed in the usual topology on $\mathbb{R}^2$

e) In part d, we just showed algebraic varieties are closed in the usual topology. So if $u \in \tau_2$ then u^c is a variety. Since u^c is closed in the usual topology, u is open in the usual topology by the complementing characterization

We just showed for $u \in \tau_2$, $u \in$ usual topology which means $\tau_2 \subseteq$ usual topology and thus the usual topology on $\mathbb{R}^2$ is finer than the Zariski topology