1. Math 61, Fall 20, Review Sheet for the final exam

Question 1.1. *Let T be the set of all Tufts students Denote with $K(x, y)$ the statement " student x knows student y"*

(a) Use quantifiers and connections to express the statement "For any two students at Tufts, there is always a third student that knows both".

(b) Write the negation of the above statement so that any negative sign appears next to a "$K(-,-)$".

Question 1.2. *To which of the statements below is $\neg(p \wedge q)$ equivalent to? Justify your answer.*

(a) $(\neg p) \wedge (\neg q)$

(b) $(\neg p) \vee (\neg q)$

(c) $p \to \neg q$

(d) $\neg p \to q$

Question 1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\forall x, y \in \mathbb{R}\ f(xy) = xf(y) + yf(x)$*

(a) Show that $f(1) = 0$.

(b) Show that if $u \in \mathbb{R}, n \in \mathbb{N}$, then $f(u^n) = nu^{n-1}f(u)$.

Question 1.4. *A car has a license plate which has three letters (chosen from among 26) followed by three digits (each of which is one of $0, 1, 2, \dots 9$). The three letters include the ordered consecutive pair AS and the three digits include each of the digits 5 and 2 exactly once in some order. How many license plates fit this description?*

Question 1.5. *(a) Prove algebraically that if $k < n$ are natural numbers, then*

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

(b) Give a combinatorial argument to prove the equation in (a).

Question 1.6. *Let $n_1, n_2, \dots, n_k$ be natural numbers such that $n = n_1 + n_2 + \cdots + n_k$. Show that*

$$\sum_{j=1}^{k} \binom{n_j}{2} \leq \binom{n}{2}$$

Hint: think of each side of the equation as counting some edges in a graph.

Question 1.7. *A forest is a not necessarily connected graph with no simple closed loops. A certain forest has 50 vertices and 40 edges. How many connected components does it have?*

Question 1.8. *If a connected planar graph has 10 vertices of degree 4, 12 vertices of degree 3 and 3 of degree 2, how many edges does it have? In how many regions does it divide the plane?*

Question 1.9. *Let G be a connected graph. A Hamilton circuit is a circuit (closed path) that contains all vertices exactly once, except for the first and last vertices that are the same.*

(a) Show that if G has a Hamilton circuit, then $\deg(v) \geq 2$ for all vertices v.

(b) Show that if G has a Hamilton circuit but no Euler circuit then $deg(v) \geq 3$ for at least <u>two</u> vertices in G.

(c) *Show that $K_n, n \geq 3$ has $(n-1)!$ Hamilton circuits.*

Question 1.10. *Show that if a tree T has a vertex v of degree k, then T has at least k vertices of degree 1.*

Question 1.11. *Suppose that every vertex of a graph G has degree at most k. Prove that the chromatic number of G satisfies $CN(G) \leq k+1$. Show that this bound is the best possible by exhibiting (for every k) a graph with maximum degree k and chromatic number $k+1$. Hint: Use induction on the number of vertices.*

Question 1.12. *Your friend is an avid tennis player.The fraction of the games that she wins against each of her four opponents is .6, .5, .45, .4 respectively. Suppose that your friend plays 30 matches against each of the first two, and 20 matches against each of the second two.*

(a) *What is the probability that your friend wins one of her matches.*

(b) *Given that your friend won a particular match, what is the probability that she was playing the first opponent.*

(c) *Given that your friend lost a particular match, what is the probability that she was playing the fourth opponent (assume there aren't any ties).*

Question 1.13. (a) *Define what it means for a set to be countable.*

(b) *Show that $\mathbb{Z} \times \mathbb{N}$ is countable*

Question 1.14. (a) *Consider the assignment* $\begin{array}{cccc} f: & \mathbb{Q}\times\mathbb{Q} & \to & \mathbb{Q} \\ & (\frac{a}{b}, \frac{c}{d}) & \to & \frac{a+b}{c+d} \end{array}$ *Prove or disprove that this is a well defined function.*

(b) *Consider the assignment* $\begin{array}{cccc} f: & \mathbb{Q}\times\mathbb{Q} & \to & \mathbb{Q} \\ & (\frac{a}{b}, \frac{c}{d}) & \to & \frac{ad+bc}{bd} \end{array}$ *Prove or disprove that this is a well defined function.*

Question 1.15. *Let $f : A \to B$ be a function. Show that f is onto if and only if there exists a function $g : B \to A$ such that $f \circ g = I_B$.*

Question 1.16. *For a natural number a written in decimal expression, we will denote its digits with $a_n a_{n-1} \dots a_1 a_0$. For example, for the number 367, $n = 2, a_2 = 3, a_1 = 6, a_0 = 7$.*

(a) *Show that $a = a_n a_{n-1} \dots a_1 a_0$ is divisible by 3 if and only if $a_n + a_{n-1} + \cdots + a_1 + a_0$ is divisible by 3.*

(b) *Show that $a = a_n a_{n-1} \dots a_1 a_0$ is divisible by 9 if and only if $a_n + a_{n-1} + \cdots + a_1 + a_0$ is divisible by 9.*

Question 1.17. (a) *Show that the sequence $a_n = \frac{4n+5}{n+1} \in \mathbb{Q}$ satisfies $\lim_{n\to\infty} a_n = 4$.*

(b) *Show that the sequence $a_n = \frac{(-1)^n(4n+5)}{n+1} \in \mathbb{Q}$ does not have a limit.*

Question 1.18. (a) *Define subtraction of real numbers by $[(a_n)] - [(b_n)] = [(a_n - b_n)]$. Show that this gives a well defined operation.*

(b) *Prove the associative property for addition of real numbers.*

2. Solutions

Question 2.1. *Let T be the set of all Tufts students Denote with $K(x, y)$ the statement " student x knows student y"*

(a) Use quantifiers and connections to express the statement "For any two students at Tufts, there is always a third student that knows both".

(b) Write the negation of the above statement so that any negative sign appears next to a "$K(-,-)$".

Answer 1. (a) $\forall x \in T,\ \forall y \in T\ \exists z \in T\ (K(z, x) \wedge K(z, y))$
(b) $\exists\ x \in T,\ \exists\ y \in T\ \forall z \in T\ (\neg K(z, x) \vee \neg K(z, y))$

Question 2.2. *To which of the statements below is $\neg(p \wedge q)$ equivalent to? Justify your answer.*

(a) $(\neg p) \wedge (\neg q)$
(b) $(\neg p) \vee (\neg q)$
(c) $p \to \neg q$
(d) $\neg p \to q$

Answer 2. $\neg(p \wedge q) \leftrightarrow (\neg p) \vee (\neg q) \leftrightarrow (p \to \neg q)$ so (b) and (c) are both correct.

Question 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\forall x, y \in \mathbb{R}\ f(xy) = xf(y) + yf(x)$*

(a) Show that $f(1) = 0$.

(b) Show that if $u \in \mathbb{R}, n \in \mathbb{N}$, then $f(u^n) = nu^{n-1}f(u)$.

Answer 3. We assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $\forall x, y \in \mathbb{R}\ f(xy) = xf(y) + yf(x)$

(a) From the condition on f, $f(1) = f(1 \times 1) = 1f(1) + 1f(1) = 2f(1)$, that is $f(1) = 2f(1)$. Therefore, $f(1) = 0$.

(b) We use induction on n to show that if $u \in \mathbb{R}, n \in \mathbb{N}$, then $f(u^n) = nu^{n-1}f(u)$.

For $n = 1$, we need to see that $\forall u \in \mathbb{R},\ f(u^1) = u^0 f(u)$. Equivalently, $\forall u \in \mathbb{R},\ f(u) = 1 \times f(u)$ which is true.

Assume now the result true for n that is $f(u^n) = nu^{n-1}f(u)$ and prove it for $n + 1$.

$$f(u^{n+1}) = f(u(u^n)) = uf(u^n) + u^n f(u) = u(nu^{n-1}f(u)) + u^n f(u) = (nu^n + u^n)f(u) =$$
$$= (n + 1)u^n f(u) = (n + 1)u^{(n+1)-1}f(u).$$

which proves the result for $n + 1$ and therefore concludes the proof by induction.

Question 2.4. *A car has a license plate which has three letters (chosen from among 26) followed by three digits (each of which is one of $0, 1, 2, \dots 9$). The three letters include the ordered consecutive pair AS and the three digits include each of the digits 5 and 2 exactly once in some order. How many license plates fit this description?*

Answer 4. Let us assume that letters can be repeated. We must take one letter and the symbol AS either before or after it. There are 26 choices for the letters. The numbers are $5, 2$ and one more digit to be chosen from the remaining 8 digits. They can be ordered in $3!$ ways, so $8 \cdot 3!$ choices. Total $2 \cdot 26 \cdot 8 \cdot 3!$ possibilities.

Question 2.5. *(a) Prove algebraically that if $k < n$ are natural numbers, then*

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

(b) Give a combinatorial argument to prove the equation in (a).

Answer 5. (a) We can write

$$\binom{k}{2} + k(n-k) + \binom{n-k}{2} = \frac{k(k-1)}{2} + k(n-k) + \frac{(n-k)(n-k-1)}{2} =$$

$$= \frac{k(k-1) + 2k(n-k) + (n-k)(n-k-1)}{2} = \frac{k^2 + k + 2k(n-k) + (n-k)^2 - (n-k)}{2} =$$

$$= \frac{[k + (n-k)]^2 - k - (n-k)}{2} = \frac{(n)^2 - n)}{2} = \binom{n}{2}$$

(b) A set of n objects is divided into a subset with k and a subset with the remaining $n-k$. If we want to choose two objects from the4 set, we could choose both from one subset (which gives us $\binom{k}{2}$ and $\binom{n-k}{2}$ options respectively or we could choose one from each which gives $k(n-k)$ options.

Question 2.6. *Let $n_1, n_2, \ldots, n_k$ be natural numbers such that $n = n_1 + n_2 + \cdots + n_k$. Show that*

$$\sum_{j=1}^{k} \binom{n_j}{2} \leq \binom{n}{2}$$

Hint: think of each side of the equation as counting some edges in a graph.

Answer 6. Let $n_1, n_2, \ldots, n_k$ be natural numbers such that $n = n_1 + n_2 + \cdots + n_k$. Take K_n and divide its vertices into k disjoint sets with $n_1, n_2, \ldots, n_k$ vertices respectively. The complete subgraphs $K_{n_1}, \ldots K_{n_k}$ on each of these groups of vertices are disjoint (they have no common vertices and hence no common edges). They have $\binom{n_1}{2}, \binom{n_2}{2}, \ldots \binom{n_k}{2}$ edges. As they are disjoint, the sum of the number of its edges is at most the number of edges of K_n. Hence,

$$\sum_{j=1}^{k} \binom{n_j}{2} \leq \binom{n}{2}$$

Question 2.7. *A forest is a not necessarily connected graph with no simple closed loops. A certain forest has 50 vertices and 40 edges. How many connected components does it have?*

Answer 7. Each connected component of a forest is a tree. Hence, if the connected component $C_i,\ i = 1 \cdots c$ has n_i vertices and m_i edges, then $n_i = m_i + 1$. Adding over the various components, the sum of the number of vertices of each component is the total number of vertices n while the sum of the number of edges is the total number of edges m

$$n = \sum_i n_i = \sum_i m_i + \sum_i 1 = m + c$$

In our case, $50 = 40 + c$. Hence, the number of components c is 10.

Question 2.8. *If a connected planar graph has 10 vertices of degree 4, 12 vertices of degree 3 and 3 of degree 2, how many edges does it have? In how many regions does it divide the plane?*

Answer 8. A planar graph has 10 vertices of degree 4, 12 vertices of degree 3 and 3 of degree 2. By the handshake lemma, twice the number m of edges is the sum of the degrees of the vertices. Hence,

$$2m = 10 \times 4 + 12 \times 3 + 3 \times 2 = 82$$

So $m = 41$. The total number of vertices is $n = 10+12+3 = 25$. Then by Euler's formula, the number of faces satisfies the equation $n-m+f = 2$. Hence, $f = 2+m-n = 2+41-25 = 18$.

Question 2.9. *Let G be a connected graph. A Hamilton circuit is a circuit (closed path) that contains all vertices exactly once, except for the first and last vertices that are the same.*

(a) Show that if G has a Hamilton circuit, then $\deg(v) \geq 2$ for all vertices v.

(b) Show that if G has a Hamilton circuit but no Euler circuit then $deg(v) \geq 3$ for at least two vertices in G.

(c) Show that $K_n, n \geq 3$ has $(n-1)!$ Hamilton circuits.

Answer 9. (a) If G has a Hamilton circuit (simple circuit that contains all vertices), then there are at least two edges at each vertex. Therefore, the degree of every vertex is at least 2.

(b) If the degree of each vertex is even, then the graph has an Euler circuit. So if such circuit does not exist, then the degree of at least one vertex is odd. As the number of vertices of odd degree is even, there are at least two vertices of odd degree. Now the smallest odd number greater than or equal to 2 is 3. This shows

(c) To construct a Hamilton circuit for $K_n, n \geq 3$ start at any vertex. We can choose any of the $n-1$ edges through that vertex. Then we can choose to add any of the n-2 edges through the second vertex that do not end up on the first vertex.... Once we reach v_k, we can choose to add any of the edges starting at v_k and ending at one vertex other than $v_1, \cdots v_{k-1}$. There are $n-k$ such choices. Hence, the total number of possible Hamilton circuits is $(n-1)(n-2)\cdots 2.1 = (n-1)!$.

Question 2.10. *Show that if a tree T has a vertex v of degree k, then T has at least k vertices of degree 1.*

Answer 10. Assume that T is a tree and has a vertex of $\deg k$. Write α for the number of vertices of $\deg 1$. Let v be the total number of vertices. There are then $v - \alpha - 1$ vertices of degree at least two.

The sum of the degrees of the vertices is $2v-2$ and, based on our assumptions, can be bounded as

$$2v - 2 = \sum_{i=1\ldots v} d_i = k + \alpha + \sum_{i=\alpha+2\ldots v} d_i \geq k + \alpha + 2(v - \alpha - 1) = 2v - 2 + k - \alpha$$

Therefore, $\alpha \geq k$.

Question 2.11. *Suppose that every vertex of a graph G has degree at most k. Prove that the chromatic number of G satisfies $CN(G) \leq k+1$. Show that this bound is the best possible by exhibiting (for every k) a graph with maximum degree k and chromatic number $k+1$. Hint: Use induction on the number of vertices.*

Answer 11. We prove this by induction on n, the number of vertices in the graph. Since the one-vertex graph can be one-colored, the proposition holds for $n = 1$. Now assume the proposition holds for $n-1$, take a graph G_n with n vertices and maximal degree at most

k, and remove a single vertex v (along with its j connecting edges) to get a smaller graph G_{n-1}. Since removing edges cannot increase the maximal degree, we know G_{n-1} can be $k+1$-colored by the induction hypothesis. W e just need to color v in a compatible way. Since $j \leq k$, there are at most k colors represented inthe neighbors of v, so at least one of the $k+1$ colors must not be represented. By choosing this color for v we have constructed a coloring of G_n.

To show this is the best possible bound, consider the complete graphmK_{k+1}, which has chromatic number $k+1$ since its vertices are pairwise connected and thus must have pairwise distinct colors.

Question 2.12. *Your friend is an avid tennis player.The fraction of the games that she wins against each of her four opponents is .6, .5, .45, .4 respectively. Suppose that your friend plays 30 matches against each of the first two, and 20 matches against each of the second two.*

(a) What is the probability that your friend wins one of her matches.

(b) Given that your friend won a particular match, what is the probability that she was playing the first opponent.

(c) Given that your friend lost a particular match, what is the probability that she was playing the fourth opponent (assume there aren't any ties).

Answer 12. Denote by W the event " Your friend wins", by G_i the event " your friend plays against the i^{th} opponent". We know that

$$P(G_1) = P(G_2) = \frac{30}{100} = 0.3,\ P(G_3) = P(G_4) = \frac{20}{100} = 0.2,$$
$$P(W|G_1) = .6,\ P(W|G_2) = .5,\ P(W|G_1) = .45,\ P(W|G_4) = .4$$

(a) As $W = (W \cap G_1) \cup (W \cap G_2) \cup (W \cap G_3) \cup (W \cap G_4)$

$$P(W) = P(W \cap G_1) + P(W \cap G_2) + P(W \cap G_3) + P(W \cap G_4) =$$
$$= P(W|G_1)P(G_1) + P(W|G_2)P(G_2) + P(W|G_3)P(G_3) + P(W|G_4)P(G_4) =$$
$$= .6 \times .3 + .5 \times .3 + .45 \times .2 + .4 \times .2 = .5$$

(b) The question asks for the probability of playing the first opponent conditioned to winning $P(G_1|W)$. We know that

$$P(G_1|W) = \frac{P(G_1 \cap W)}{P(W)} = \frac{P(W|G_1)P(G_1)}{P(W)} = \frac{.6 \times .3}{.5} = .36$$

(c) As there are no ties, the event " your friend loses a game" is the complement of " your friend wins a game" . Therefore

$$P(\bar{W}) = .5,\ P(\bar{W}|G_4) = 1 - P(W|G_4) = 1 - .4 = .6$$
$$P(G_4|\bar{W}) = \frac{P(G_4 \cap \bar{W})}{P(\bar{W})} = \frac{P(\bar{W}|G_4)P(G_4)}{P(\bar{W})} = \frac{.6 \times .2}{.5} = .24$$

Question 2.13. *(a) Define what it means for a set to be countable.*
(b) Show that $\mathbb{Z} \times \mathbb{N}$ is countable

Answer 13. (a) A set is countable if there is a bijection between the set and $\mathbb{N}$

(b) We know that $\mathbb{Z}$ is countable. This means that there is a bijection $h : \mathbb{Z} \to \mathbb{N}$. Then, taking h in the first component and the identity in the second, we obtain a bijection

$$(h, I_{\mathbb{N}}) : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}.$$

We know that $\mathbb{N} \times \mathbb{N}$ is countable. This means that there is a bijection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Then, the composition

$$g \circ (h, I_{\mathbb{N}} : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$$

is a bijection, as it is the composition of two bijections and shows that $\mathbb{Z} \times \mathbb{N}$ is countable.

Question 2.14. *(a) Consider the assignment* $\begin{array}{cccc} f: & \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \\ & (\frac{a}{b}, \frac{c}{d}) & \to & \frac{a+b}{c+d} \end{array}$ *Prove or disprove that this is a well defined function.*

(b) Consider the assignment $\begin{array}{cccc} f: & \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \\ & (\frac{a}{b}, \frac{c}{d}) & \to & \frac{ad+bc}{bd} \end{array}$ *Prove or disprove that this is a well defined function.*

Answer 14. (a) The first assignment is not well defined as $\frac{1}{2} = \frac{2}{4}, \frac{1}{3} = \frac{5}{15}$ while

$$\frac{1+1}{2+3} = \frac{2}{5} \neq \frac{7}{19} = \frac{2+5}{4+15}.$$

(b) The second assignment is well defined: assume $\frac{a}{b} = \frac{a'}{b'}, \frac{c}{d} = \frac{c'}{d'}$. By definition of the equivalence relation giving rise to the rational numbers, this means that $ab' = a'b, cd' = c'd$. Multiplying the first equation with dd' and the second equation with bb', we obtain

$$ab'dd' = a'bdd', cd'bb' = c'dbb'.$$

Adding these two equations, we obtain

$$(ad + bc)b'd' = ab'dd' + cd'bb' = a'bdd' + c'dbb' = (a'd' + b'c')bd$$

which is equivalent to saying that $\frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'}$. Therefore, the second assignment is a well defined function.

Question 2.15. *Let $f : A \to B$ be a function. Show that f is onto if and only if there exists a function $g : B \to A$ such that $f \circ g = I_B$.*

Answer 15. If f is onto, for every $b \in B$, we can find $a \in A$ such that $f(a) = b$. This a is not unique, but we just choose one of them for each b. We then define $g(b) = a$ for that chosen a depending on b. Then, for $b \in B$, $(f \circ g)(b) = f(g(b)) = f(a) = b$ where the a is the one we chose and the last equality comes from our choice of a.

Conversely, if there exists $g : B \to A$ such that $f \circ g = I_B$, then for each $b \in B, b = I_B(b) = (f \circ g)(b) = f(g(b))$. As $g(b) \in A$, we are displaying b as the image of an element in A, namely $g(b)$. Hence, f is onto.

Question 2.16. *For a natural number a written in decimal expression, we will denote its digits with $a_n a_{n-1} \dots a_1 a_0$. For example, for the number 367, $n = 2, a_2 = 3, a_1 = 6, a_0 = 7$.*

(a) Show that $a = a_n a_{n-1} \dots a_1 a_0$ is divisible by 3 if and only if $a_n + a_{n-1} + \dots + a_1 + a_0$ is divisible by 3.

(b) Show that $a = a_n a_{n-1} \dots a_1 a_0$ is divisible by 9 if and only if $a_n + a_{n-1} + \dots + a_1 + a_0$ is divisible by 9.

Answer 16. We saw in Example 13.6 of the Notes that after fixing a natural number n, one can define an equivalence relation in the integers by

$$z_1, z_2 \in \mathbb{Z}, \text{ then } z_1 \sim z_2 \iff \exists k \in \mathbb{Z}, z_1 - z_2 = nk.$$

There are n equivalence classes $[0]_n, [1]_{,n} \dots [n-1]_n$. In example 14.4, we checked that addition and product of equivalence classes is well defined. We will use the cases $n = 3$ and $n = 9$ below.

Note also, that given an integer by giving its digits, $a = a_n a_{n-1} \dots a_1 a_0$, the integer can be written as

$$a = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + 10 a_1 + a_0$$

(a) Saying that $a = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + 10a_1 + a_0$ is divisible by 3 is the same as saying that its coset $[a]_3 = [0]_3$. As taking cosets is compatible with sums and products, we have

$$[a]_3 = [a_n 10^n + a_{n-1} 10^{n-1} + \cdots + 10a_1 + a_0]_3 = [a_n]_3 [10]_3^n + [a_{n-1}]_3 [10]_3^{n-1} + \cdots + [10]_3 [a_1]_3 + [a_0]_3$$

As $10 = 3 \times 3 + 1$,

$$[10]_3 = [3]_3 \times [3]_3 + [1]_3 = [0]_3 \times [0]_3 + [1]_3 = [1]_3.$$

Therefore, $[10^n]_3 = ([10]_3)^n = ([1]_3)^n = [1^n]_3 = [1]_3$. It follows that

$$[a]_3 = [a_n]_3 [10]_3^n + [a_{n-1}]_3 [10]_3^{n-1} + \cdots + [10]_3 [a_1]_3 + [a_0]_3 = [a_n]_3 + [a_{n-1}]_3 + \cdots + [a_1]_3 + [a_0]_3 =$$
$$= [a_n + a_{n-1} + \cdots + a_1 + a_0]_3$$

so that $a = a_n a_{n-1} \dots a_1 a_0$ is divisible by 3 if and only if $a_n + a_{n-1} + \cdots + a_1 + a_0$ is divisible by 3.

(b) The proof is identical to the previous one using that $10 = 9 + 1$ and therefore

$$[10]_9 = [9]_9 + [1]_9 = [1]_9, \quad [10^k]_9 = ([10]_9)^k = ([1]_9)^k = [1^k]_9 = [1]_9.$$

It follows that $a = a_n a_{n-1} \dots a_1 a_0$ is divisible by 9 if and only if $a_n + a_{n-1} + \cdots + a_1 + a_0$ is divisible by 9.

Question 2.17. *(a) Show that the sequence $a_n = \frac{4n+5}{n+1} \in \mathbb{Q}$ satisfies $\lim_{n\to\infty} a_n = 4$.*
(b) Show that the sequence $a_n = \frac{(-1)^n(4n+5)}{n+1} \in \mathbb{Q}$ does not have a limit.

Answer 17. (a) The condition $\lim_{n\to\infty}(a_n) = 4$ means that

$$\forall \epsilon > 0, \ \exists m \in \mathbb{N} \text{ such that } \forall n \geq m, \ |a_n - 4| < \epsilon$$

Choose $\epsilon \in \mathbb{Q}^+$ and take $m \geq \frac{1}{\epsilon} - 1$. Then, for $n \geq m$, we have $n + 1 \geq \frac{1}{\epsilon}$, so $\frac{1}{n+1} < \epsilon$

$$|a_n - 4| = |\frac{4n+5}{n+1} - 4| = \frac{1}{n+1} < \epsilon$$

(b) We showed that every convergent sequence is a Cauchy sequence (the other way around is NOT true for rational numbers). So, in order to show that this sequence does not converge, it suffices to check that it is not a Cauchy sequence The condition for Cauchy sequence is Choose $\epsilon \in \mathbb{Q}^+$

$$\exists m \in \mathbb{N}, \forall n_1, n_2 \geq m, |a_{n_1} - a_{n_2}| < \epsilon$$

The negation of the condition of Cauchy sequence is obtained by negating this statement. Recall that the negation of a quantifier such as "for all x, $P(x)$ is "there exists some x not

$P(x)$". The negation of "exists $x, P(x)$ is "for all x not $P(x)$". Therefore, the negation of (a_n) being a Cauchy sequence is

$$\exists \epsilon_0 > 0, \ \forall m \in \mathbb{N} \ \exists n_1, n_2 \geq m, \ |a_{n_1} - a_{n_2}| > \epsilon$$

Take $\epsilon_0 = 8$. Choose any $m \in \mathbb{N}$ and choose $n_1 > m$ such that n_1. even, $n_2 = n_1 + 1$ For n_1 even, $a_{n_1} = \frac{(4n_1+5)}{n_1+1} > \frac{(4(n_1+1)}{n_1+2} = 4$. Now $n_2 = n_1 + 1$ is odd and $a_{n_2} = -\frac{(4n_2+5)}{n_2+1} < -4$. Hence, if n_1 is even , $n_2 = n_1 + 1$, then $|a_{n_1} - a_{n_1+1}| = a_{n_1} - a_{n_1+1} > 4 - (-4) = 8$ contradicting the condition for Cauchy sequence.

Question 2.18. *(a) Define subtraction of real numbers by* $[(a_n)] - [(b_n)] = [(a_n - b_n)]$*. Show that this gives a well defined operation.*

(b) Prove the associative property for addition of real numbers.

Answer 18. (a) Given Cauchy sequences with $(a_n) \sim (a'_n), (b_n) \sim (b'_n)$, we need to show that $(a_n - b_n) \sim (a'_n - b'_n)$ Recall that addition and product with scalars preserve Cauchy sequences. Therefore, if $(a_n), (b_n)$ are Cauchy sequences, then $-1(b_n)$ and $(a_n) + (-1)(b_n) = (a_n - b_n)$ is a Cauchy sequence (and similarly $(a'_n - b'_n)$ is a Cauchy sequence). From the definition of equivalence, $\lim_{n\to\infty}(a_n - a'_n) = 0, \lim_{n\to\infty}(b_n - b'_n) = 0$. As the limit of the sum is sum of limits and multiplying with a constant preserves limits,

$$\lim_{n\to\infty}(a_n - b_n) - (a'_n - b'_n) = \lim_{n\to\infty}(a_n - a'_n) - (b_n - b'_n) = \lim_{n\to\infty}(a_n - a'_n) - \lim_{n\to\infty}(b_n - b'_n) = 0 - 0 = 0$$

proving that $(a_n - b_n) \sim (a'_n - b'_n)$.

(b) Using the definition of addition of cosets of sequences

$$([(a_n)] + [(b_n)]) + [(c_n)] = ([(a_n + b_n)]) + [(c_n)] = [(a_n + b_n) + c_n]]$$

Using the associative property of addition in the set of rational numbers $(a_n + b_n) + c_n = a_n + (b_n + c_n)$ then using again the definition of addition and product of cosets of sequences,

$$[(a_n + (b_n + c_n))] = [(a_n)] + ([(b_n)] + [(c_n)])$$