Tuesday, November 23

1. The Poincaré-Bendixson theorem, as I stated it, goes like this:

> **Theorem.** *Let $D \subseteq \mathbb{R}^2$ be a closed region, and let $f : D \to \mathbb{R}^2$ be continuously differentiable with $f(x) \neq 0$ for all $x \in D$. Then any solution of*
> $$\frac{dx}{dt} = f(x)$$
> *with $x(t) \in D$ for all $t \geq 0$ converges to a periodic solution.*

Give an example that shows that the solution it converges to need not be attracting. Hint: Try to find an example of the form

$$\frac{dr}{dt} = g(r), \qquad \frac{d\theta}{dt} = 1.$$

2. The system

$$\frac{dr}{dt} = r - r^3, \qquad \frac{d\theta}{dt} = 1$$

has, as you know, a periodic solution, corresponding to $r = 1$:

$$x = \cos t, \quad y = \sin t.$$

Suppose we perturb the system a little:

$$\frac{dr}{dt} = r - r^3 + \frac{r\cos\theta}{10}, \qquad \frac{d\theta}{dt} = 1.$$

Is there a periodic solution? Hint: Prove that $dr/dt > 0$ for small enough positive r, and $dr/dt < 0$ for large enough positive r. So there is an annulus of the form $a \leq r \leq A$, wich $0 < a < A$, which no solution can escape. Explain why this annulus also contains no fixed point. Then apply the Poincaré-Bendixson theorem.