Solution This solution illustrates an important technique for using connectedness. We let $x_0 \in A$ and let $B = \{y \in A \mid x_0$ and y can be joined by a continuous path$\}$. Obviously, $x_0 \in B$, so that $B \neq \varnothing$. We claim that B is both open and closed regarded as a subset of A; *i.e.*, B is both open and closed relative to A. First, B is open for the following reason. If $y \in B$, choose a disk $D(y, \varepsilon) \subset A$, which is possible since A is open. If $z \in D(y, \varepsilon)$, then $z \in B$, since we can get a continuous path from x_0 to z by concatenation of a path from x_0 to y with the straight line from y to z (the reader should prove that this produces a *continuous* path). Thus, $D(y, \varepsilon) \subset B$, so B is open. To show that B is closed, let $y_k \in B$ and $y_k \to y \in A$. Since A is open, there is an $\varepsilon > 0$ such that $D(y, \varepsilon) \subset A$. Since $y_k \to y$, there is an N such that $y_k \in D(y, \varepsilon)$ for $k \geq N$. Joining x_0 to y_N by a continuous path followed by the straight line from y_N to y, we see that $y \in B$, and so B is closed. Since $B \neq \varnothing$ and B is both open and closed in A, we get $B = A$. (Otherwise, B and $B \backslash A$ would disconnect A.) Thus, every point in A can be joined to x_0 by a continuous path, and so A is path-connected. ♦

Exercises for Chapter 3

1. Which of the following sets are compact? Which are connected?

- **a.** $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 1\}$
- **b.** $\{x \in \mathbb{R}^n \mid \|x\| \leq 10\}$
- **c.** $\{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2\}$
- **d.** $\mathbb{Z} = \{\text{integers in } \mathbb{R}\}$
- **e.** A finite set in $\mathbb{R}$
- **f.** $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (distinguish between the cases $n = 1$ and $n \geq 2$)
- **g.** Perimeter of the unit square in $\mathbb{R}^2$
- **h.** The boundary of a bounded set in $\mathbb{R}$
- **i.** The rationals in $[0, 1]$
- **j.** A closed set in $[0, 1]$

2. Prove that a set $A \subset \mathbb{R}^n$ is not connected iff we can write $A \subset F_1 \cup F_2$, where F_1, F_2 are closed, $A \cap F_1 \cap F_2 = \varnothing$, $F_1 \cap A \neq \varnothing$, $F_2 \cap A \neq \varnothing$.

3. Prove that in $\mathbb{R}^n$, a bounded infinite set A has an accumulation point.

4. Show that a set A is bounded iff there is a constant M such that $d(x, y) \leq M$ for all $x, y \in A$. Give a plausible definition of the diameter of a set and reformulate your result.

5. Show that the following sets are not compact, by exhibiting an open cover with no finite subcover.

a. $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$

b. $\mathbb{Z}$, the integers in $\mathbb{R}$

6. Suppose that F_k is a sequence of compact nonempty sets satisfying the nested set property such that diameter$(F_k) \to 0$ as $k \to \infty$. Show that there is exactly one point in $\cap\{F_k\}$. (By definition, diameter$(F_k) = \sup\{d(x,y) \mid x, y \in F_k\}$).

7. Let x_k be a sequence in $\mathbb{R}^n$ that converges to x and let $A_k = \{x_k, x_{k+1}, \ldots\}$. Show that $\{x\} = \cap_{k=1}^{\infty} \mathrm{cl}(A_k)$. Is this true in *any* metric space?

8. Let $A \subset \mathbb{R}^n$ be compact and let x_k be a Cauchy sequence in $\mathbb{R}^n$ with $x_k \in A$. Show that x_k converges to a point in A.

9. Determine (by proof or counterexample) the truth or falsity of the following statements:

a. (A is compact in $\mathbb{R}^n$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected).

b. (A is connected in $\mathbb{R}^n$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected).

c. (A is connected in $\mathbb{R}^n$) $\Rightarrow$ (A is open or closed).

d. ($A = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$) $\Rightarrow$ ($\mathbb{R}^n \backslash A$ is connected). [Hint: Check the cases $n = 1$ and $n \geq 2$.]

10. A metric space M is said to be ***locally path-connected*** if each point in M has a neighborhood U such that U is path-connected. (This terminology differs somewhat from that of some topology books.) Show that (M is connected and locally path-connected) $\Leftrightarrow$ (M is path-connected).

11. **a.** Prove that if A is connected in a metric space M and $A \subset B \subset \mathrm{cl}(A)$, then B is connected.

b. Deduce from **a** that the components of a set A are relatively closed. Give an example in which they are not relatively open. ($C \subset A$ is called ***relatively closed*** in A if C is the intersection of some closed set in M with A, *i.e.*, if C is closed in the metric space A.)

c. Show that if a family $\{B_i\}$ of connected sets is such that $B_i \cap B_j \neq \varnothing$ for all i, j, then $\cup_i B_i$ is connected.

d. Deduce from **c** that every point of a set lies in a unique component.

e. Use **c** to show that $\mathbb{R}^n$ is connected, starting with the fact that lines in $\mathbb{R}^n$ are connected.

12. Let S be a set of real numbers that is nonempty and bounded above. Let $-S = \{x \in \mathbb{R} \mid -x \in S\}$. Prove that

a. $-S$ is bounded below.

b. $\sup S = -\inf(-S)$.

13. Let M be a complete metric space and F_n a collection of closed nonempty subsets (not necessarily compact) of M such that $F_{n+1} \subset F_n$ and diameter $(F_n) \to 0$. Prove that $\cap_{n=1}^{\infty} F_n$ consists of a single point; compare Exercise 6.

14. a. A point $x \in A \subset M$ is said to be ***isolated*** in the set A if there is a neighborhood U of x such that $U \cap A = \{x\}$. Show that this is equivalent to saying that there is an $\varepsilon > 0$ such that for all $y \in A$, $y \neq x$, we have $d(x, y) > \varepsilon$.

b. A set is called ***discrete*** if all its points are isolated. Give some examples. Show that a discrete set is compact iff it is finite.

15. Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be path-connected (respectively, connected, compact). Show that $K_1 \times K_2$ is path-connected (respectively, connected, compact) in $M_1 \times M_2$.

16. If $x_k \to x$ in a normed space, prove that $||x_k|| \to ||x||$. Is the converse true? Use this to prove that $\{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is closed, using sequences.

17. Let K be a nonempty closed set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n \backslash K$. Prove that there is a $y \in K$ such that $d(x, y) = \inf\{d(x, z) \mid z \in K\}$. Is this true for open sets? Is it true in general metric spaces?

18. Let $F_n \subset \mathbb{R}$ be defined by $F_n = \{x \mid x \geq 0 \text{ and } 2 - 1/n \leq x^2 \leq 2 + 1/n\}$. Show that $\cap_{n=1}^{\infty} F_n \neq \varnothing$. Use this to show the existence of $\sqrt{2}$.

19. Let $V_n \subset M$ be open sets such that $\mathrm{cl}(V_n)$ is compact, $V_n \neq \varnothing$, and $\mathrm{cl}(V_n) \subset V_{n-1}$. Prove $\cap_{n=1}^{\infty} V_n \neq \varnothing$.

20. Prove that a compact subset of a metric space must be closed as follows: Let x be in the complement of A. For each $y \in A$, choose disjoint neighborhoods U_y of y and V_y of x. Consider the open cover $\{U_y\}_{y \in A}$ of A to show the complement of A is open.

21. a. Prove: a set $A \subset M$ is connected iff $\varnothing$ and A are the only subsets of A that are open and closed relative to A. (A set $U \subset A$ is called ***open relative to*** A if $U = V \cap A$ for some open set $V \subset M$; "closed relative to A" is defined similarly.)

b. Prove that $\varnothing$ and $\mathbb{R}^n$ are the only subsets of $\mathbb{R}^n$ that are both open and closed.

22. Find two subsets $A, B \subset \mathbb{R}^2$ and a point $x_0 \in \mathbb{R}^2$ such that $A \cup B$ is not connected but $A \cup B \cup \{x_0\}$ is connected.

23. Let $\mathbb{Q}$ denote the rationals in $\mathbb{R}$. Show that both $\mathbb{Q}$ and the irrationals $\mathbb{R} \backslash \mathbb{Q}$ are not connected.

24. Prove that a set $A \subset M$ is not connected if we can write A as the disjoint union of two sets B and C such that $B \cap A \neq \varnothing$, $C \cap A \neq \varnothing$, and neither of the sets B or C has a point of accumulation belonging to the other set.

25. Prove that there is a sequence of distinct integers $n_1, n_2, \ldots \to \infty$ such that $\lim_{k \to \infty} \sin n_k$ exists.

26. Show that the completeness property of $\mathbb{R}$ may be replaced by the *Nested Interval Property*. If $\{F_n\}_1^\infty$ is a sequence of closed bounded intervals in $\mathbb{R}$ such that $F_{n+1} \subset F_n$ for all $n = 1, 2, 3, \ldots$, then there is at least one point in $\cap_{n=1}^\infty F_n$.

27. Let $A \subset \mathbb{R}$ be a bounded set. Show that A is closed iff for every sequence $x_n \in A$, $\limsup x_n \in A$ and $\liminf x_n \in A$.

28. Let $A \subset M$ be connected and contain more than one point. Show that every point of A is an accumulation point of A.

29. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^4 = 1\}$. Show that A is compact. Is it connected?

30. Let U_k be a sequence of open bounded sets in $\mathbb{R}^n$. Prove or disprove:

a. $\bigcup_{k=1}^\infty U_k$ is open.

b. $\bigcap_{k=1}^\infty U_k$ is open.

c. $\bigcap_{k=1}^\infty (\mathbb{R}^n \backslash U_k)$ is closed.

d. $\bigcap_{k=1}^\infty (\mathbb{R}^n \backslash U_k)$ is compact.

31. Suppose $A \subset \mathbb{R}^n$ is not compact. Show that there exists a sequence $F_1 \supset F_2 \supset F_3 \cdots$ of closed sets such that $F_k \cap A \neq \varnothing$ for all k and

$$\left(\bigcap_{k=1}^\infty F_k \right) \bigcap A = \varnothing.$$

32. Let x_n be a sequence in $\mathbb{R}^3$ such that $\|x_{n+1} - x_n\| \leq 1/(n^2 + n)$, $n \geq 1$. Show that x_n converges.

33. ***Baire category theorem.*** A set S in a metric space is called ***nowhere dense*** if for each nonempty open set U, we have $\mathrm{cl}(S) \cap U \neq U$, or equivalently, $\mathrm{int}(\mathrm{cl}(S)) = \varnothing$. Show that $\mathbb{R}^n$ cannot be written as the countable union of nowhere dense sets.

34. Prove that each closed set $A \subset M$ is an intersection of a countable family of open sets.

35. Let $a \in \mathbb{R}$ and define the sequence $a_1, a_2, \ldots$ in $\mathbb{R}$ by $a_1 = a$, and $a_n = a_{n-1}^2 - a_{n-1} + 1$ if $n > 1$. For what $a \in \mathbb{R}$ is the sequence

a. Monotone?

b. Bounded?

c. Convergent?

Compute the limit in the cases of convergence.

36. Let $A \subset \mathbb{R}^n$ be uncountable. Prove that A has an accumulation point.

37. Let $A, B \subset M$ with A compact, B closed, and $A \cap B = \varnothing$.

a. Show that there is an $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for all $x \in A$ and $y \in B$.

b. Is **a** true if A, B are merely closed?

38. Show that $A \subset M$ is not connected iff there exist two *disjoint* open sets U, V such that $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$, and $A \subset U \cup V$.

39. Let $F_1 = [0, 1/3] \cup [2/3, 1]$ be obtained from $[0, 1]$ by removing the middle third. Repeat, obtaining

$$F_2 = [0, 1/9] \ \cup \ [2/9, 1/3)] \ \cup \ [2/3, 7/9] \ \cup \ [8/9, 1].$$

In general, F_n is a union of intervals and F_{n+1} is obtained by removing the middle third of these intervals. Let $C = \cap_{n=1}^{\infty} F_n$, the ***Cantor set.*** Prove:

a. C is compact.

b. C has infinitely many points. [Hint: Look at the endpoints of F_n.]

c. $\text{int}(C) = \varnothing$.

d. C is ***perfect;*** that is, it is closed with no isolated points.

e. Show that C is ***totally disconnected;*** that is, if $x, y \in C$ and $x \neq y$ then $x \in U$ and $y \in V$ where U and V are open sets that disconnect C.

40. Let F_k be a nest of compact sets (that is, $F_{k+1} \subset F_k$). Furthermore, suppose each F_k is connected. Prove that $\cap_{k=1}^{\infty}\{F_k\}$ is connected. Give an example to show that compactness is an essential condition and we cannot just assume that "F_k is a nest of closed connected sets."