Math 125 **Project 2** **Due: Monday 12/12 at 11:59 p.m. EST**

Instruction: You only have to do two out of the three problems. In addition to the solution of the problems, include with your submission a brief report that summarizes in non-technical terms the problem, the method, merits of the method e.g. simplicity, scalability, convergence, stability, limitations of the method and limitations of the model if one was to use the proposed numerical methods in a practical setting. For both problems, include a printout of your code with your project submission. You should submit the project on Gradescope.

Remark: For all questions that ask you to do a numerical implementation, the implementation must be your own and not based on calling routines/functions from existing libraries in MATLAB/Python.

1 Numerical integration

In the following two problems, we consider the approximation of integrals via numerical quadrature.

1.1 Integral of curves

We consider a curve whose parametric equations are given by

$$x(t) = a\cos(t)\,, y(t) = b\sin(t) \quad t \in (-\pi, \pi)$$

The arc-length of the parametric curve is given by

$$L = \int_{-\pi}^{\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

(a) Let $a = b$. Find the arc-length of the curve. What is the curve?

(b) For a general a and b, prove that the arc-length of the curve is

$$L = 4a \int_{0}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2(t)}\, dt,$$

where $k^2 = 1 - \frac{b^2}{a^2}$.

(c) Let $a = 3$ and $b = 2$. The resulting curve is shown below. Using a numerical integration method of your choice, compute L with an error that is at most 10^{-6}.

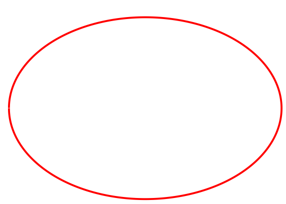


Figure 1: Curve with parametric equation $x = 3\cos(t)$ and $y = 2\sin(t)$ with $-\pi \leq t \leq \pi$.

(d) **Extra credit**: Unlike the case of a circle, finding the arc-length of an ellipse is not simple. Do research to find out what other approximations exist to estimate the perimeter of an ellipse. Compare the estimations to your result in (c).

Solution:

For details of implementation, refer to `arclength_trapz.m` in the `project1_soln` folder.

(a) The arc-length of the curve is

$$L = \int_{-\pi}^{\pi} \sqrt{a^2(\sin^2(t) + \cos^2(t))}\, dt = \int_{-\pi}^{\pi} a\, dt = a\, t|_{-\pi}^{\pi} = 2\pi a.$$

When $a = b$, the curve is a circle with circumference is $2\pi a$.

(b) We compute the arc-length as follows.

$$\begin{aligned} L &= \int_{-\pi}^{\pi} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt \\ &= \int_{-\pi}^{-\pi/2} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt + \int_{-\pi/2}^{0} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt \\ &+ \int_{0}^{\pi/2} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt + \int_{\pi/2}^{\pi} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt \end{aligned}$$

We make the following change of variables for the integrand: $\theta = t + \pi$ for the first integrand, $\theta = -t$ for the second integrand and $\theta = t - \pi/2$ for the last integrand. With this, we obtain

$$L = \int_{-\pi}^{\pi} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt = 4 \int_{0}^{\pi/2} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt$$

In what follows, we further simplify this integral.

$$\begin{aligned} L &= 4 \int_{0}^{\pi/2} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\, dt = 4 \int_{0}^{\pi/2} \sqrt{a^2 \left(1 - \cos^2(t) + \frac{b^2}{a^2} \cos^2(t)\right)}\, dt \\ &= 4a \int_{0}^{\pi/2} \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right) \cos^2(t)}\, dt = 4a \int_{0}^{\pi/2} \sqrt{1 - k^2 \cos^2(t)}\, dt, \end{aligned}$$

We now consider the change of variable $\theta = \pi/2 - t$ and obtain

$$L = 4a \int_{0}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2(t)}\, dt.$$

(c) We use the composite Trapezoid method. Note that on the interval $[x_0, x_n]$ with m sub-intervals and $h = \frac{1}{x_n - x_0}$, the error is $E = -\frac{(x_n - x_0)h^2}{12} f''(c)$ where $c \in (x_0, x_n)$. In our case, $f(t) = \sqrt{1 - k^2 \sin^2(t)}$. We first find the derivatives of f.

$$f'(t) = \frac{-k^2 \cos(t) \sin(t)}{\sqrt{1 - k^2 \sin^2(t)}} = \frac{-k^2 \sin(2t)}{2\sqrt{1 - k^2 \sin^2(t)}},$$

$$f''(t) = \frac{k^4 \sin^4(x) + 2k^2 \cos^2(x) - k^2}{\left(1 - k^2 \sin^2(t)\right)^{\frac{3}{2}}}$$

To bound $f''(t)$, we maximize the numerator and minimize the denominator for $t \in (0, \frac{\pi}{2})$.

$$|f''(t)| = \left| \frac{k^4 \sin^4(x) + 2k^2 \cos^2(x) - k^2}{\left(1 - k^2 \sin^2(t)\right)^{\frac{3}{2}}} \right| \le \frac{k^4 + 2k^2 + k^2}{(1 - k^2)^{\frac{3}{2}}}$$

With $a = 3$ and $b = 2$, $k^2 = 1 - 4/9 = 5/9$. Therefore, $|f''(t)| \leq \frac{(32 \cdot 5)/(81)}{8/27} = \frac{20}{3}$ on the interval $t \in (0, \frac{\pi}{2})$. We can now bound the error as follows

$$E = -\frac{(x_n - x_0)h^2}{12} f''(c) \leq |\frac{(x_n - x_0)h^2}{12} f''(c)| \leq \frac{\pi}{24} h^2 \cdot \frac{20}{3}$$

For the error to be at most 10^{-6}, $\frac{\pi}{24} h^2 \cdot \frac{20}{3} < 10^{-6}$. Solving for this, we get $h = 0.00107047446$. Accordingly, the number of sub-intervals is 1465. The numerical estimation of the arc-length of the curve is 15.865439589290586.

(d) The following approximation due to Ramanujan approximates the arc-length of ellipse as

$$L \approx \pi(a+b)\left(1 + \frac{3p^2}{10 + \sqrt{4 - 3p^2}}\right),$$

where $p = \frac{a-b}{a+b}$. This estimate gives 15.865439589251233 which agrees to 10-decimals with our numerical integration estimate.

1.2 Integral of $\exp\left(-\frac{x^2}{2}\right)$

In this problem, we consider the integral of $\exp\left(-\frac{x^2}{2}\right)$ using Gaussian quadrature.

(a) In brief terms, describe how Gaussian quadrature can be applied to compute the integral of a function $f(x)$ on the interval $[-1, 1]$ i.e. $\int_{-1}^{1} f(x)\, dx$.

(b) Implement Gaussian quadrature for the following integral

$$\int_{-1}^{1} \exp\left(-\frac{x^2}{2}\right) dx$$

Compute the integral using number of points $n = 2$, $n = 3$ and $n = 4$. The roots of the n-degree Legendre polynomial are noted in the table below.

n	x_i
2	$-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{3}}$
3	$-\sqrt{\frac{3}{5}}$ 0 $\sqrt{\frac{3}{5}}$
4	−0.86113631159405 −0.33998104358486 0.33998104358486 0.86113631159405

Table 1: The roots x_i of the n-degree Legendre polynomial.

(c) **Extra credit**: Relate the integral in (b) to the probability density function of the normal distribution. Using numerical values from exisiting solvers, check the error for your answer in b for all the different values of n.

Solution:

(a) Let the roots of the Legendre polynomial be denoted by $x_1, ..., x_n$. In Gaussian quadrature, the integrand $f(x)$ is approximated as $f(x) = \sum_{i=1}^{n} f(x_i)l_i(x)$ where $l_i(x)$ is the Lagrange interpolating polynomial. Integrating both sides, we obtain

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} \sum_{i=1}^{n} f(x_i)l_i(x)\,dx = \sum_{i=1}^{n} f(x_i) \int_{-1}^{1} l_i(x)\,dx = \sum_{i=1}^{n} c_i f(x_i),$$

where $c_i = \int_{-1}^{1} l_i(x)\,dx$.

(b) For $n = 2$, Gaussian quadrature approximation is

$$c_1 f(x_1) + c_2 f(x_2) = 1 f\left(-\sqrt{\frac{1}{3}}\right) + 1 f\left(\sqrt{\frac{1}{3}}\right) = 1.692963449781228$$

For $n = 2$, Gaussian quadrature approximation is

$$c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) = 1.712020245201909$$

For $n = 4$, Gaussian quadrature approximation is

$$\begin{aligned}
&c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) + c_4 f(x_4) \\
&= 0.347854845137445 f(-0.86113631159405) + 0.65214515486255 f(-0.33998104358486) \\
&+ 0.65214515486255 f(0.33998104358486) + 0.34785484513745 f(0.86113631159405) \\
&= 1.711224504599491
\end{aligned}$$

(c) Recall the probability density function of the normal distribution

$$\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

Let I denote $\int_{-1}^{1} f(x)\,dx$. Then, $I = \sqrt{2\pi}\,(\Phi(1) - \Phi(-1))$ where Φ denotes the cumulative distribution function (CDF). Using the following command in MATLAB,
`sqrt(2*pi)*(normcdf(1)-normcdf(-1))`
we obtain 1.711248783784297 which agrees with the estimate in (b) to 4-decimal digits.

2 Iterative methods for linear systems

We study the problem of solving the following linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ where $\mathbf{A} \in \mathcal{R}^{n\times n}$ is the matrix defined as:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 2 \end{pmatrix}$$

and $\boldsymbol{b} \in \mathcal{R}^n$ is the right hand size vector defined as:

$$\boldsymbol{b} = h^2 \begin{pmatrix} \sin(\pi h) \\ \sin(2\pi h) \\ \vdots \\ \vdots \\ \sin(n\pi h) \end{pmatrix}$$

(a) Show that the Jacobi iteration converges for this problem.

(b) Prove that Gauss Seidel iteration converges for this problem.

(c) Implement the Jacobi method to solve the linear system with $n = 1000$ and $h = 0.1$. Plot the number of iterations versus the residual error in a semilog y plot.

(d) Implement the Gauss Seidel method to solve the linear system with $n = 1000$ and $h = 0.1$. Plot the number of iterations versus the residual error in a semilog y plot.

(e) Using your results in (c) and (d), what can you say about the convergence rate of these methods to solve this problem?

Solution

(a) For implementation, refer to `test_iterative.m` in `project2_Soln` folder on Canvas. The iteration matrix for Jacobi is $\boldsymbol{M} = \boldsymbol{I} - \boldsymbol{D}^{-1}\boldsymbol{A}$. We consider three ways to show convergence.
(1) We numerically check if all the eigenvalues of $\boldsymbol{M}$ are strictly less than 1 in absolute value. Since this is indeed the case, the spectral radius of $\boldsymbol{M}$ is strictly less than 1. It follows that Jacobi iteration converges for this problem.
(2) Formally, one could show that the eigenvalues of $\mathbf{A}$ have the following form: $\lambda_j = \cos(\frac{j\pi}{n+1}), 1 \leq j \leq n$. This implies that the spectral radius is strictly less than 1, and hence convergence.
(3) Note that $\boldsymbol{M}$ has the following form

$$\boldsymbol{M} = \begin{pmatrix} 0 & 1/2 & & & & 0 \\ 1/2 & 0 & 1/2 & & & \\ & 1/2 & 0 & 1/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1/2 & 0 & 1/2 \\ 0 & & & & 1/2 & 0 \end{pmatrix}$$

We note that $\mathbf{e}_k = \boldsymbol{M}^k \mathbf{e}_0$ where $\mathbf{e}_k$ denotes the error at the k-th iteration and $\mathbf{e}_0$ is the initial error. Let $\mathbf{e}_0 = c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + ... + c_n\mathbf{p}_n$ where $\mathbf{p}_i$ is a vector of zeros except a 1 at the i-th position. It can be easily verified that $\lim_{k\to\infty} \boldsymbol{M}^k \mathbf{p}_i = \mathbf{0}$. For instance, if we consider $\mathbf{p}_1$, we have

$$\boldsymbol{M}^k \mathbf{p}_1 = \begin{cases} \frac{1}{2^{(k+2)/2}} (\mathbf{p}_1 + \mathbf{p}_3) & k \text{ is even} \\ -\frac{1}{2^{(k+1)/2}} \mathbf{p}_2 & k \text{ is odd} \end{cases}$$

Hence, $\lim_{k\to\infty} \boldsymbol{M}^k \mathbf{p}_1 = \mathbf{0}$. Similar arguments can be made for other $\mathbf{p}_i' s$ and we can then conclude that $\lim_{k\to\infty} \boldsymbol{M}^k \mathbf{e}_0 = \mathbf{0}$.

(b) The iteration matrix for Gauss Seidel is $\boldsymbol{M} = -(\boldsymbol{L}+\boldsymbol{D})^{-1}\boldsymbol{U}$. We compute $\boldsymbol{M}$ as follows:

$$\boldsymbol{M} = -\begin{pmatrix} 2 & 0 & & & & 0 \\ -1 & 2 & 0 & & & \\ & -1 & 2 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & 0 \\ 0 & & & & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & & & & 0 \\ 0 & 0 & -1 & & & \\ & 0 & 0 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & -1 \\ 0 & & & & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{M} = -\begin{pmatrix} \frac{1}{2} & 0 & & & & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & & & \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{2^n} & \frac{1}{2^{n-1}} & \cdots & & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 & & & & 0 \\ 0 & 0 & -1 & & & \\ & 0 & 0 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & -1 \\ 0 & & & & 0 & 0 \end{pmatrix}$$

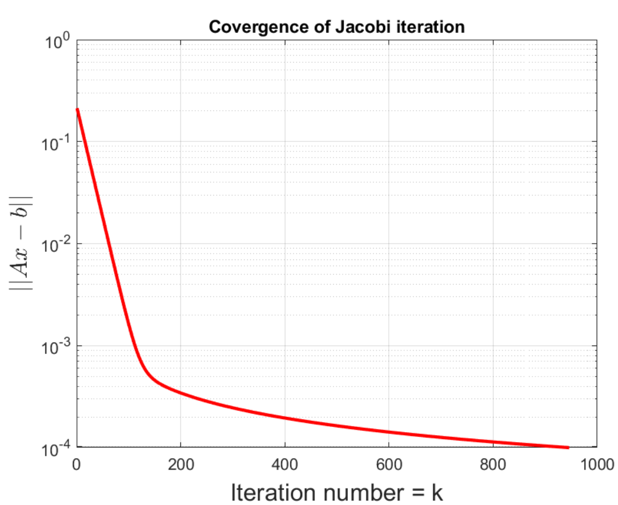
$$= \begin{pmatrix} 0 & \frac{1}{2} & & & & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & & & \\ 0 & \frac{1}{8} & \frac{1}{4} & 0 & & \\ 0 & \frac{1}{16} & \frac{1}{8} & \ddots & \ddots & \\ & & & \vdots & \vdots & \frac{1}{2} \\ 0 & \frac{1}{2^n} & \frac{1}{2^{n-1}} & \cdots & & \frac{1}{4} \end{pmatrix}$$

We now compute the infinity norm of the iteration matrix.

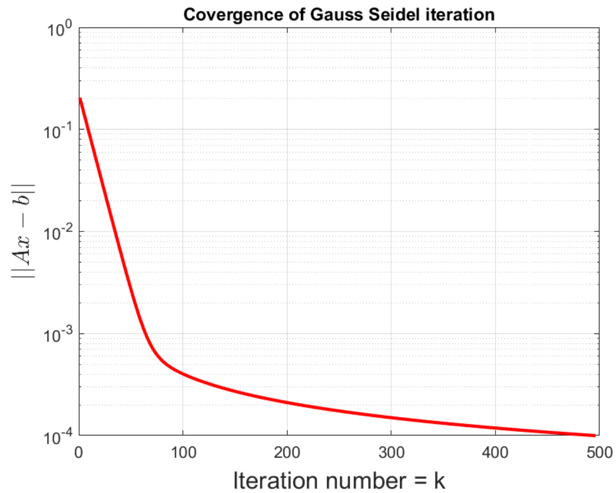
$$||\boldsymbol{M}||_\infty = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} = 1 - 2^{1-n}$$

Note that the last equality is evaluated using the formula for the sum of a geometric series. Clearly, $||\boldsymbol{M}||_\infty < 1$. Since $\rho(\boldsymbol{M}) \leq ||\boldsymbol{M}||_\infty$, it follows that the spectral radius of the iteration matrix is strictly less than 1. Gauss Seidel iteration converges for this problem.

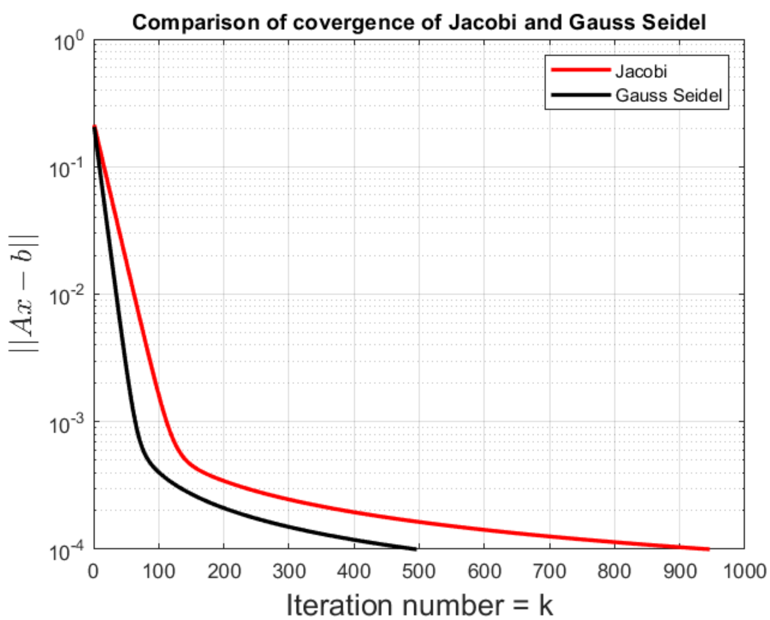
(c) For implementation, refer to `jacobi.m`. The figure below shows a plot of the residual error with respect to number of iterations.



(d) For implementation, refer to `gauss_seidel.m`. The figure below shows a plot of the residual error with respect to number of iterations.



(e) The figure below shows a comparison of Jacobi iteration and Gauss Seidel iteration methods to solve the linear system. Gauss Seidel converges twice as fast as Jacobi iteration.



3 Steady state of a dynamical system

A matrix $\mathbf{A} \in \mathcal{R}^{n\times n}$ is column-stochastic if all of its entries are non-negative and the entries of each column sum to 1. Here on, assume that $\mathbf{A}$ is a column-stochastic matrix.

(a) Prove that $\lambda = 1$ is an eigenvalue of $\mathbf{A}$.

(b) Prove that the largest eigenvalue of $\mathbf{A}$ is 1.

(c) Let $\mathbf{x}^{(0)} \in \mathcal{R}^n$ be a non-negative vector whose entries sum to 1. Consider the following iterative procedure

$$\mathbf{x}^{(k)} = A\mathbf{x}^{(k-1)} \quad \text{for } k = 1, 2, 3, \ldots$$

If $\mathbf{x}^{(k)}$ converges to $\mathbf{x}^*$, what limiting equation do we obtain?

(d) Perron–Frobenius Theorem states that the dominant eigenvector of a positive column-stochastic matrix has positive entries and its entries sum to 1. Load the matrix `stochastic_matrix.mat` or `stochastic_matrix.csv` from the `Project2` folder. Implement the power method and find the dominant eigenvector and eigenvalue of the matrix.

(e) **Extra credit**: Comment how your work in (a)-(d) connects to the page rank algorithm we discussed in class.

Solution

(a) Since $\mathbf{A}^T$ is row-stochastic, $\mathbf{A}^T\mathbf{1} = \mathbf{1}$. Therefore, $\mathbf{1}$ is an eigenvector with corresponding eigenvalue 1. We now use the fact that $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues. This follows from $\det(\mathbf{A}^T - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})$ which implies that the characteristic polynomials of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. Therefore, $\lambda = 1$ is an eigenvalue of $\mathbf{A}$.

(b) We apply the Gershgorin circle theorem to $\mathbf{A}^T$. We note that all the disks are centered at $\mathbf{A}_i i$ with radius $\sum_{j\neq i}\mathbf{A}_j j$. The maximum eigenvalue in each disk is bounded by $\mathbf{A}_i i + \sum_{j\neq i}\mathbf{A}_j j = 1$. Therefore, the largest eigenvalue of $\mathbf{A}^T$ is 1. Arguing similarly as in (a), we conclude that the largest eigenvalue of $\mathbf{A}$ is 1.

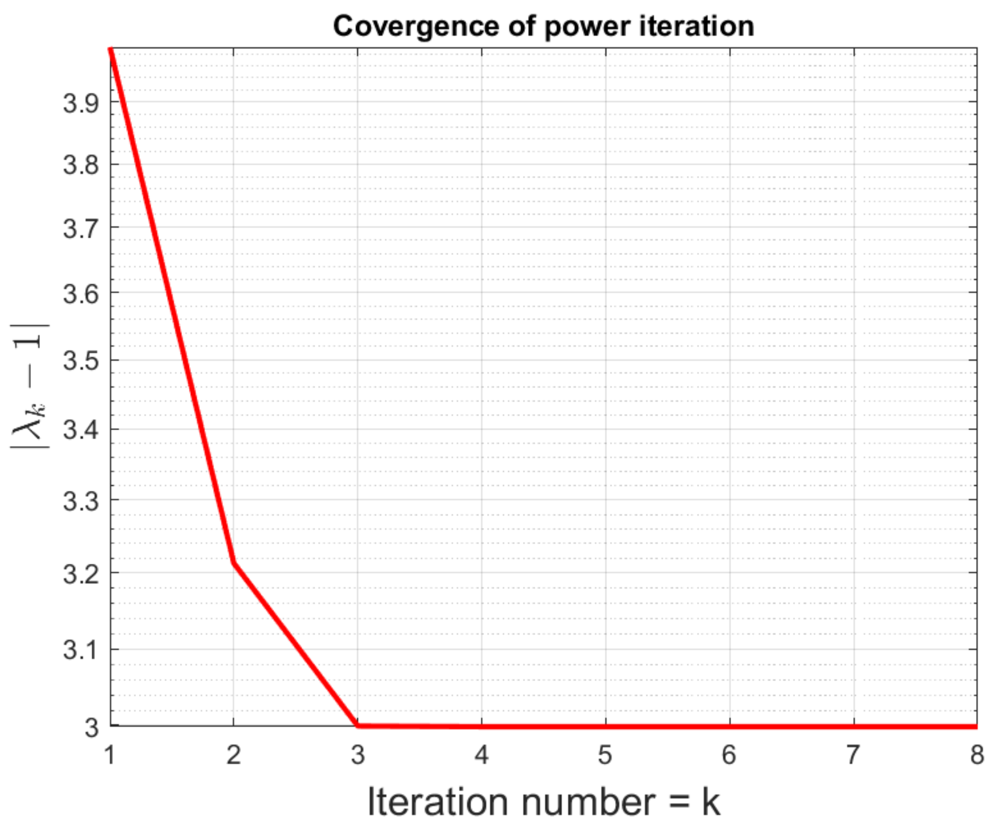
(c) Note that

$$\lim_{k\to\infty}\mathbf{x}^{(k)} = \lim_{k\to\infty} A\mathbf{x}^{(k-1)} \longrightarrow \mathbf{x}^* = A\lim_{k\to\infty}\mathbf{x}^{(k)} \longrightarrow \mathbf{x}^* = \mathbf{A}\mathbf{x}^*.$$

Therefore, the limiting equation is an eigenvalue/eigenvector problem which informs us that $\mathbf{x}^*$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue $\mathbf{A}$.

(d) For implementation details, refer to `power_method.m` in the `project2_soln` folder. Power method converges fast to the dominant eigenvector. The convergence is shown below.

(e) Note that in page rank $\mathbf{x}^0$ indicates the state vector where $(\mathbf{x}^0)_i$ is the probability that the user is in web-page i at the initial time. The matrix $\mathbf{A}$ is a transition probability matrix where thee (i, j) entry indicates the probability of transitioning to web-page j given that the user is in web-page i. At time t_1, the state vector of the user is $\mathbf{A}\mathbf{x}^0$. After k steps i.e. the statee vector of the user will be $\mathbf{A}^k\mathbf{x}^0$. The limiting equation informs us where the user will end up from any starting point $\mathbf{x}^0$. Since $\mathbf{x}^0$ is any starting point, if $\mathbf{x}^*$ does exists, it informs us which pages users will visit in the long run i.e. the importance or rank of a web-page can be inferred from entries of $\mathbf{x}^*$.


Covergence of power iteration
3.9
3.8
3.7
3.6
3.5
3.4
3.3
3.2
3.1
3
$|\lambda_k - 1|$
1
2
3
4
5
6
7
8
Iteration number = k