Homework 7

Due on Gradescope at 11:59 pm on Friday, March 17th.

(1) Suppose X and Y are topological spaces and $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for their respective topologies. Prove that

$$\mathcal{B} = \{B_X \times B_Y \mid B_X \in \mathcal{B}_X, B_Y \in \mathcal{B}_Y\}$$

is a basis for the product topology on $X \times Y$.

Definition 1. A topological space X is said to be T_0 or *Kolmogorov* if, for each pair of distinct points $x_1, x_2 \in X$, either x_1 has an open neighborhood not containing x_2 or x_2 has an open neighborhood not containing x_1.

Remark. We've seen this property before: it's the hypothesis we needed to add to Problem 2 of Homework 5.

Recall from recitation:

Definition 2. A topological space X is said to be *Hausdorff* if for each pair of distinct points $x_1, x_2 \in X$, there are open neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$.

(2) (a) Prove that if a topological space is Hausdorff, then it is T_0.
(b) If a topological space is T_0, must it be Hausdorff? Prove or give a counterexample.

(3) Prove that if X is homeomorphic to Y and X is T_0, then Y is T_0.

(4) Suppose that X and Y are T_0 topological spaces. Prove that $X \times Y$ is a T_0 topological space.

(5) Prove that X is a Hausdorff topological space if and only if

$$\Delta = \{(x, x) \in X \times X\}$$

is a closed subset of $X \times X$ with the product topology.

Definition 3. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *open* or an *open map* if for each open subset U of X, $f(U)$ is open in Y

(6) Let X be a topological space and let $Z \subseteq X$. Give Z the subspace topology. Prove that the inclusion function

$$\begin{aligned} i : Z &\to X \\ z &\mapsto z \end{aligned}$$

is an open map if and only if Z is an open subset of X.

(7) Let $\pi_1 : X \times Y \to X$ be the projection map $(x, y) \mapsto x$. Prove that π_1 is an open map.