

Tufts University  
Department of Mathematics  
Final Exam<sup>1</sup>

Math 235

Fall, 2022

*This is an open book and open notes exam but you may not consult anyone other than Shuang Guan or myself, and you may not ask for help online (e.g., message boards, forums). You are allowed to consult your class notes and Heil's textbook.*

*The test will be available on Gradescope starting 8:30 a.m. Eastern Time on Thursday December 22nd. You will have 2 hours from the time you download the test to finish it. You will have an extra 15 minutes to upload it to Gradescope. Since it can take time to upload, do not wait until this 15 minutes is almost up. You must finish uploading your answers by 11:59 p.m. on Thursday December 22nd. When you upload your answers to Gradescope, please scan this signed signature page first and then scan all your answers into one PDF file. Please number the problems clearly and in order.*

*Please sign the following pledge and submit the signed pledge with your answers:*

The Tufts University statement on academic integrity holds that: "Academic integrity is the joint responsibility of faculty, students, and staff. Each member of the community is responsible for integrity in their own behavior and for contributing to an overall environment of integrity at the university." I accept this responsibility, affirm that I am an honest person who can be trusted with to do the right thing, and certify that the work I will do on this exam is mine alone.

I pledge that I have used only the reference sources I have cited in my answers.

Signature \_\_\_\_\_

*The test problems start on the next page.*

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Problem 1. Let  $E = \{(x, x^3), x \in \mathbb{R}\} \subset \mathbb{R}^2$ . Using only the definition, prove that  $E$  is Lebesgue measurable and find its measure  $|E|$ .

Problem 2. Let  $T = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x < \infty\}$ .

Evaluate  $\lim_{n \rightarrow \infty} \iint_T e^{-(x + \frac{1}{n} + \frac{n+1}{n}y)}(dxdy)$  justifying your answer.

Problem 3. Assume that  $f \in AC[0, 1]$  and there is a function  $g$  continuous on  $[0, 1]$  such that  $f' = g$  a.e. Show that  $f$  is differentiable everywhere on  $[0, 1]$ , and that  $f'(x) = g(x)$  for all  $x \in [0, 1]$ . Show by an example that the hypothesis of absolute continuity is necessary.

Problem 4. Let  $f \in L^1(\mathbb{R})$  and define  $F(x) = \int_{\mathbb{R}} f(t) \frac{\sin tx}{t} dt$ .

- (a) Prove that  $F$  is differentiable on  $\mathbb{R}$  and find  $F'$
- (b) Is  $F$  absolutely continuous on each compact subinterval of  $\mathbb{R}$ ? Justify your answer.

Problem 5. Let  $f \in L^p(\mathbb{R})$ , where  $1 \leq p < \infty$ . For  $\alpha > 0$ , define

$$E_\alpha(f) = \{x \in \mathbb{R} : |f(x)| > \alpha\}.$$

- (a) Show that  $E_\alpha$  has finite Lebesgue measure.
- (b) Use (a) to show that every  $f \in L^p(\mathbb{R})$ ,  $1 \leq p \leq 2$ , can be decomposed as  $f_1 + f_2$  where  $f_1 \in L^1(\mathbb{R})$  and  $f_2 \in L^2(\mathbb{R})$ .