MATH 125 Lecture 8

Typically we choose equispaced points for interpolation. Could we do better in a way that takes into account interpolation error?

Let's consider part of the interpolation error that depends on the nodes

$$\frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} \quad *$$

Let's set the interval to be $[-1,1]$

Idea: $*$ is itself a degree $(n+1)$ polynomial
It has some maximum value in $[-1,1]$
Could we find a particular $x_0 \dots x_n$ such that the maximum value is as small as possible

\\ Minimax problem of interpolation //

Chebyshev polynomial

$T_0(x) = 1$

$T_1(x) = x$

$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

Exercise: What is $T_2(x)$?

$$T_2(x) = 2x\,T_1(x) - T_0(x)$$
$$= 2x\cdot x - 1 = 2x^2 - 1$$

Facts
i) T_n's are polynomials
ii) $\deg(T_n) = n$ and leading coefficient is 2^{n-1}
iii) $T_n(1) = 1$ $\quad T_n(-1) = (-1)^n$
iv) The maximum absolute value of $T_n(x)$ in $-1 \le x \le 1$ is 1
v) All zeros of $T_n(x)$ are located between -1 and 1
vi) $T_n(x)$ alternates between -1 and 1 $n+1$ times
~~vii)~~

$*$ $\quad T_n(x) = \cos(n \arccos(x)) \qquad -1 \le x \le 1$

$n=0 \quad T_n(x) = \cos(n \arccos(0)) = \cos(0\cdot\pi) = \cos(0) = 1$

$n=1 \quad T_n(x) = \cos(\arccos(x)) = x$

$n=2 \quad T_n(x) = \cos(2\arccos(x))$

Recall $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$T_2(x) = \cos(2z)$ where $z = \arccos(x) \Rightarrow \cos z = x$

$T_2(x) = \cos(2z) = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 2x^2 - 1$

In general

$$T_{n+1}(x) = \cos((n+1)z)$$
$$= \cos(nz + z)$$
$$= \cos(nz)\cos(z) - \sin(nz)\sin(z) \quad (1)$$

$$T_{n-1}(x) = \cos((n-1)z)$$
$$= \cos((\cancel{n-1})_{n} z + z)$$
$$= \cos(nz - z) = \cos(nz)\cos(z) - \sin(nz)\sin(-z) \quad (2)$$

Add (1) and (2) to get

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos(nz)\cos(z)$$
$$= 2x\,T_n(x)$$

Therefore,

$$\boxed{T_{n+1} = 2x\,T_n(x) - T_{n-1}(x)}$$

<u>Exercise</u> Prove Facts 4 and 5

Proof : • Note that $T_n(x) = \cos(n \arccos(x))$

$|T_n(x)| = |\cos(n\arccos(x))| \le 1$

• $T_n(x) = 0 \Rightarrow \cos(n \arccos(x)) = 0$

$n \arccos(x) = k \cdot \frac{\pi}{2}$ k is odd integer

$x = \cos\left(\frac{k \cdot \pi}{2n}\right)$

<u>Theorem</u> The choice of real numbers $-1 \le x_0 \le \dots x_n \le 1$ that makes $\max\limits_{-1 \le x \le 1} |(x - x_0)(x - x_1) \cdots (x - x_n)|$ smallest is $x_i = \cos\left(\frac{(2i+1)}{(n+1)} \frac{\pi}{2}\right)$ $i = 0, \dots, n$. Minimum value $= \frac{1}{2^n}$

<u>Exercise</u> Find a worst case error bound for the difference $[-1, 1]$ between $f(x) = e^x$ and the degree 4 chebyshev interpolating polynomial

$$f(x) - P_4(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{5!} f^{(5)}(c)$$

$x_0 = \cos\left(\frac{\pi}{10}\right)$ $x_1 = \cos\left(\frac{3\pi}{10}\right)$ $x_2 = \cos\left(\frac{5\pi}{10}\right)$ $x_3 = \cos\left(\frac{7\pi}{10}\right)$

$x_4 = \cos\left(\frac{9\pi}{10}\right)$ $(-1 < c < 1)$

Using Chebyshev theorem

$$|(x-x_0)(x-x_1)\cdots(x-x_4)| \le \frac{1}{2^4}$$

In addition, $|f^5(c)| \le e^1$ on $[-1, 1]$

$$|e^x - P_4(x)| \le \frac{e}{2^4\, 5!} \approx 0.00142$$

For arbitrary interval (a, b)

$$x_i = \frac{a+b}{2} - \frac{b-a}{2} \cos\left(\frac{2i+1}{n+1}\,\frac{\pi}{2}\right) \quad i = 0, \ldots n$$

$*$ Note $\frac{T_n(x)}{2^{n-1}}$ is monic.