Theorems:

Theorem 1 ch.1: page 13 (4th ed.)

Uniqueness of reduced echelon form. Each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem 2 ch.1 page 21 (4th ed.)

Existence and Uniqueness theorem. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. That is if and only if the augmented matrix has no row of the form

$$[\,0 \quad \dots \quad 0 \quad b\,] \qquad \text{with } b \neq 0.$$

If a linear system is consistent, then the solution set contains either

(i) a unique solution, when there are no free variables, or

(ii) infinitely many solutions, if there are free variables.

Theorem 3 ch.1: page 36 (4th ed.)

If $A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_3 \end{bmatrix}$ is an $m \times n$ matrix and $\vec{\mathbf{b}}$ is a vector in $\mathbb{R}^m$, the matrix equation

$$A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

has the same solution set as the vector equation

$$x_1\,\vec{\mathbf{a}}_1 + x_2\,\vec{\mathbf{a}}_2 + \dots + x_n\,\vec{\mathbf{a}}_n = \vec{\mathbf{b}}$$

which in turn has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_n & \vec{\mathbf{b}} \end{bmatrix}$$

Theorem 4 ch.1 (important): page 37 (4th ed.)

Let A be an $m \times n$ matrix. Then the following statements are equivalent (they are either all true of all false).

(a) For each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$, the equation $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution.

(b) Each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$; that is $\mathbf{Span}\left\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n\right\} = \mathbb{R}^m$.

(d) A has a pivot position in every row.

Theorem 5 ch.1: page 39 (4th ed.)

If A is an $m \times n$ matrix, $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ vectors in $\mathbb{R}^n$ and c a scalar, then:

a) $A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = A\,\vec{\mathbf{u}} + A\,\vec{\mathbf{v}}$;

b) $A(c\,\vec{\mathbf{u}}) = c(A\,\vec{\mathbf{u}})$.

Theorem 6 ch.1: page 46 (4th ed.)

Suppose $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ is consistent for some $\vec{\mathbf{b}}$, and let $\vec{\mathbf{p}}$ be a solution. Then the solution set of $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ is the set of vectors of the form $\vec{\mathbf{w}}=\vec{\mathbf{p}}+\vec{\mathbf{v}}_h$, where $\vec{\mathbf{v}}_h$ is any solution to $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$.

Theorem 7 ch.1: page 58 (4th ed.)

Characterization of Linearly Independent Sets An indexed set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the rest.

Theorem 8 ch.1: page 59 (4th ed.)

If a set contains more vectors than there are entries in each vector then the set is linearly dependent, that is, any set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ of vectors from $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem 9 ch.1: page 59 (4th ed.)

If a set of vectors contains the zero vector then it is a linearly dependent set.

Theorem 10 ch.1: page 71 (4th ed.)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}} \qquad \text{for all } \vec{\mathbf{x}} \text{ in } \mathbb{R}^n.$$

Furthermore the matrix A is:

$$A = \begin{bmatrix} T(\vec{\mathbf{e}}_1) & T(\vec{\mathbf{e}}_2) & \ldots & T(\vec{\mathbf{e}}_n) \end{bmatrix}$$

This matrix is called **the standard matrix for the linear transformation** T.

Theorem 11 ch.1: page 76 (4th ed.)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\vec{\mathbf{x}})=\vec{\mathbf{0}}$ has only the trivial solution.

Theorem 12 ch.1: page 77 (4th ed.)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A. Then

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

Theorem 1 ch.2 (matrix operations): page 93 (4th ed.)

Let A, B and C be matrices of the same size, 0 will be the matrix of all zeros with the same size as A and C, and let r and s be scalars. Then:

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + 0 = A$

(iv) $r(A + B) = rA + rB$

(v) $(r + s)A = rA + sA$

(vi) $r(sA) = (rs)A$

Theorem 2 ch.2 (matrix multiplication properties): page 97 (4th ed.)

Let A, B and C be matrices of the appropriate sizes and let r be a scalar. Then:

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(A + B)C = AC + BC$

(iv) $r(AB) = (rA)B = A(rB)$

(v) $I_mA = A = AI_n$

Theorem 3 ch.2: page 99 (4th ed.)

Let A and B denote matrices whose sizes are appropriate for the following and r a scalar, then:

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^TA^T$

Theorem 4 ch.2: page 103 (4th ed.)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. id $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc = 0$ the matrix is not invertible.

Theorem 5 ch.2: page 104 (4th ed.)

If A is an invertible $n \times n$ matrix then, for each $\vec{\mathbf{b}}$ in $\mathbb{R}^n$, the equation $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$ the unique solution $\vec{\mathbf{x}} = A^{-1}\,\vec{\mathbf{b}}$.

Theorem 6 ch.2: page 105 (4th ed.)

a. If A is an invertible matrix then A^{-1} is invertible and $(A^{-1})^{-1} = A$

b. If A and B are $n \times n$ invertible matrices then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

c. If A is an invertible matrix then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 7 ch.2: page 107 (4th ed.)

An $n \times n$ matrix is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduce A to I_n also transform I_n into A^{-1}

Theorem 8 ch.2 page 112 (4th ed.)

Let A be an $n \times n$ matrix, then the following are equivalent.

a) A has a right inverse, that is, there is a matrix D such that $AD = I_n$.

a$_1$) For any $n \times p$ matrix B, $AX = B$ has at least one solution. For $p = 1$ this is matrix vector multiplication.

a$_1'$) The columns of A span $\mathbb{R}^n$.

a$_2$) A has n pivots (one for each row.)

a$_2'$) $A \sim I_n$

a$_3$) A is the product of elementary matrices.

b) A has a left inverse. That is, there is a matrix C such that $CA = I_n$.

b$_1$) For B an $n \times p$ matrix, solutions of $AX = B$ are unique when they exist. For $p = 1$ this is matrix vector multiplication.

b$_1'$) The only solution to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

b$_1''$) The columns of A are linearly independent.

b$_2$) A has n pivots, (1 for each column).

b$_2'$) $A \sim I_n$.

b$_3$) A is the product of elementary matrices.

Theorem 9 ch.2 page 114 (4th ed.)

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation with standard matrix A, then T is invertible if and only if A is an invertible matrix. Furthermore $T^{-1}(\vec{\mathbf{x}}) = A^{-1}\vec{\mathbf{x}}$.

Theorem 1 ch.3: page 166 (4th ed.)

The determinate of A can be computed by cofactor expansion across any row or column. That is the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is, using expansion along row i

$$\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

or using expansion along column j is

$$\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Theorem 2 ch.3: page 167 (4th ed.)

If A is a triangular matrix, then $\det(A)$ is the product of the entries along the diagonal.

Theorem 3 ch.3: page 169 (4th ed.)

Row operations: Let A be a square matrix.

a) If a multiple of one row a A is added to another row to produce a matrix B then $\det(B) = \det A$.
b) If two rows of A are interchanged to produce a matrix B then $\det(B) = -\det(A)$.
c) If one row of A is multiplied by k to produce a matrix B, then $\det(B) = k \cdot \det(B)$.

Theorem 4 ch.3: page 171 (4th ed.)

A square matrix A is invertible if and only if $\det(A) \neq 0$

Theorem 5 ch.3: page 172 (4th ed.)

If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

Theorem 6 ch.3: page 173 (4th ed.)

If A and B are matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

Theorem 1 ch.4: page 195 (4th ed.)

If $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ are vectors in$V$, then Span$\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$ is a subspace of V.

Theorem 2 ch.4: page 199 (4th ed.)

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Theorem 3 ch.4: page 201 (4th ed.)

If A is an $m \times n$ matrix, then $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Theorem 4 ch.4: page 208 (4th ed.)

An indexed set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ of two or more vectors with $\vec{\mathbf{v}}_1 \neq \vec{\mathbf{0}}$ is linearly dependent if and only if some $\vec{\mathbf{v}}_j$ with $j > 1$ is a linear combination of the preceding vectors, $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$.

Theorem 5 ch.4: page 210 (4th ed.)

The spanning set theorem: Let $S = \left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ be a set in V, and let $H = \text{Span}\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$.

a. If one of the vectors in S, say $\vec{\mathbf{v}}_k$ is a linear combination of the remaining vectors in S, then the set formed by removing $\vec{\mathbf{v}}_k$ from S still spans S.
b. If $H \neq \left\{\vec{\mathbf{0}}\right\}$, some subset of S is a basis for H.

Theorem 6: page 212 (4th ed.)

The pivot columns of a matrix A form a basis for $\mathrm{Col}(A)$.

Theorem 7 ch.4: page 216 (4th ed.)

The Unique Representation theorem Let $\mathcal{B} = \left\{ \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n \right\}$ be a basis for the vector space V. Then for each $\vec{\mathbf{x}}$ in V, there is a unique set of scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{\mathbf{x}} = c_1 \vec{\mathbf{b}}_1 + c_2 \vec{\mathbf{b}}_2 + \ldots + c_n \vec{\mathbf{b}}_n$$

That is $\vec{\mathbf{x}}$ has a unique representation as a linear combination as basis vectors from $\mathcal{B}$.

Theorem 8 ch.4: page 219 (4th ed.)

Let $\mathcal{B} = \left\{ \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_2 \ldots, \vec{\mathbf{b}}_n \right\}$ be a basis for a vector space V. Then the coordinate mapping $x \to [x]_{\mathcal{B}}$ is a one-to-one, onto linear transformation from V onto $\mathbb{R}^n$

Theorem 9 ch.4: page 225 (4th ed.)

If a vector space has a basis $\mathcal{B} = \left\{ \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n \right\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10 ch.4: page 226 (4th ed.)

If a vector space V has a basis of n vectors, then every basis must consist of V must consist on n vectors.

Theorem 11 ch.4: page 227 (4th ed.)

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded to be a basis for H. Also, H is finite-dimensional and

$$\mathrm{Dim}(H) \leq \mathrm{Dim}(V)$$

Theorem 12 ch.4: page 227 (4th ed.)

The Basis Theorem: Let V be a p-dimensional vector space, with $p \geq 1$. Any linearly independent set of exactly p vectors in V is automatically a basis for V. Any set of p elements that span V is automatically a basis for V.

Theorem 13 ch.4: page 231 (4th ed.)

If two matrices are row equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for the row space of A as well as B

Theorem 14 ch.4: page 233 (4th ed.)

The rank theorem: Let A be an $m \times n$ matrix, then

1. $\mathrm{Rank}(A) = \mathrm{Dim}(\mathrm{Col}(A)) =$ the number of pivots of $A = \mathrm{Dim}(\mathrm{Row}(A))$
2. $$\begin{aligned} \mathrm{Rank}(A) + \mathrm{Dim}(\mathrm{Nul}(A)) &= \mathrm{Dim}(\mathrm{Col}(A)) + \mathrm{Dim}(\mathrm{Nul}(A)) \\ &= \mathrm{Dim}(\mathrm{Row}(A)) + \mathrm{Dim}(\mathrm{Nul}(A)) = n \end{aligned}$$

Invertible Matrix Theorem (continued): page 235 (4th ed.)

Let A be an $n \times n$ matrix, then the following are equivalent.

a) A is an invertible matrix.
b) $A \sim I_n$.
c) A has n pivots.
d) The only solution to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
e) The columns of A are linearly independent.
f) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is one-to-one.
g) The equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at least one solution for each $\vec{\mathbf{b}}$ in $\mathbb{R}^n$.
h) The columns of A span $\mathbb{R}^n$.
i) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is onto.
j) A has a left inverse. That is, there is a matrix C such that $CA = I_n$.
k) A has a right inverse, that is, there is a matrix D such that $AD = I_n$.
l) A^T is an invertible matrix.
m) The columns of A form a basis for $\mathbb{R}^n$.
n) $\mathrm{Col}(A) = \mathbb{R}^n$.
o) $\mathrm{Dim}(\mathrm{Col}(A)) = n$.
p) $\mathrm{Rank}(A) = n$.
q) $\mathrm{Nul}(A) = \left\{\vec{\mathbf{0}}\right\}$.
r) $\mathrm{Dim}(\mathrm{Nul}(A)) = 0$.
∗) A is the product of elementary matrices.

Theorem 15 ch.4: page 240 (4th ed.)

Let $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n\right\}$ and $\mathcal{C} = \left\{\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2, \ldots, \vec{\mathbf{c}}_n\right\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ called the change of basis matrix (from $\mathcal{B}$ to $\mathcal{C}$) which converts a vector $[\vec{\mathbf{x}}]_{\mathcal{B}}$ to a vector $[\vec{\mathbf{x}}]_{\mathcal{C}}$ where $\vec{\mathbf{x}}$ is a vector in V. That is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{\mathbf{x}}]_{\mathcal{B}} = [\vec{\mathbf{x}}]_{\mathcal{C}}$$

Theorem 1 ch.5: page 269 (4th ed.)

The eigenvalues of a triangular matrix are the entries on it's main diagonal.

Theorem 2 ch.5: page 270 (4th ed.)

If $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_r\right\}$ is linearly independent

Inverse Matrix theorem continued: page 275 (4th ed.)

Let A be an $n \times n$ matrix, then the following are equivalent.

a) A is an invertible matrix.
b) $A \sim I_n$.
c) A has n pivots.
d) The only solution to $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$ is $\vec{\mathbf{x}}=\vec{\mathbf{0}}$.
e) The columns of A are linearly independent.
f) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is one-to-one.
g) The equation $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ has at least one solution for each $\vec{\mathbf{b}}$ in $\mathbb{R}^n$.
h) The columns of A span $\mathbb{R}^n$.
i) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is onto.
j) A has a left inverse. That is, there is a matrix C such that $CA = I_n$.
k) A has a right inverse, that is, there is a matrix D such that $AD = I_n$.
l) A^T is an invertible matrix.
m) The columns of A form a basis for $\mathbb{R}^n$.
n) $\text{Col}(A) = \mathbb{R}^n$.
o) $\text{Dim}(\text{Col}(A)) = n$.
p) $\text{Rank}(A) = n$.
q) $\text{Nul}(A) = \left\{\vec{\mathbf{0}}\right\}$.
r) $\text{Dim}(\text{Nul}(A)) = 0$.
s) The number zero is not an eigenvalue for A.
t) The $\det(A) \neq 0$.
$*$) A is the product of elementary matrices.

Theorem 4 ch.5: page 277 (4th ed.)

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomials, and hence eigenvalues.

Theorem 5 ch.5: page 282 (4th ed.)

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PAP^{-1}$ with D a diagonal matrix if and only if the columns of P are n linearly independent eigenvectors of A, and in this case, the diagonal entries of D are the corresponding eigenvectors.

Theorem 6 ch.5: page 284 (4th ed.)

An $n \times n$ matrix with distinct eigenvalues is diagonalizable.

Theorem 7 ch.5: page 285 (4th ed.)

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of λ_k.

b) The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n. This happens if and only if

   (i) The Characteristic polynomial of A factors completely into linear factors and

   (ii) The dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace of λ_k, then the total collection of vectors in the sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ forms a basis for $\mathbb{R}^n$.

Theorem 8 ch.5: page 291 (4th ed.)

Diagonal Matrix Representation: Suppose $A = PDP^{-1}$, where D is diagonal $n \times n$ matrix. If $\mathcal{B}$ is a basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\vec{\mathbf{x}} \rightarrow A\,\vec{\mathbf{x}}$.

Theorem 1 ch.6: page 331 (4th ed.)

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then,

a) $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$

b) $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{w}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$

c) $(c\,\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\,\vec{\mathbf{v}})$

d) $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \geq 0$ and $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = 0$ if and only if $\vec{\mathbf{u}} = \vec{\mathbf{0}}$

Theorem 2 ch.6: page 334 (4th ed.)

The Pythagorean Theorem: Two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal if and only if

$$|| \vec{\mathbf{u}} + \vec{\mathbf{v}} ||^2 = || \vec{\mathbf{u}} ||^2 + || \vec{\mathbf{v}} ||^2$$

Theorem 3 ch.6: page 335 (4th ed.)

Let A be an $m \times n$ matrix, then,

$$(\text{Row}(A))^\perp = \text{Nul}(A) \quad \text{and} \quad (\text{Col}(A))^\perp = \text{Nul}(A^T)$$

Theorem 3 ch.6: page 335 (4th ed.)

Let A be an $m \times n$ matrix, then,

$$(\text{Row}(A))^\perp = \text{Nul}(A) \quad \text{and} \quad (\text{Col}(A))^\perp = \text{Nul}(A^T)$$

Theorem 4 ch.6: page 238 (4th ed.)

If a set $S = \left\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_p\right\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence a basis for the subspace spanned by S.

Theorem 5 ch.6 : page 339 (4th ed.)

Let $\left\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_p\right\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\vec{\mathbf{y}}$ in W, the weights in the linear combination

$$\vec{\mathbf{y}} = c_1 \vec{\mathbf{u}}_1 + \ldots + c_p \vec{\mathbf{u}}_p \quad \text{are given by} \quad c_j = \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_i}$$

that is

$$\vec{\mathbf{y}} = \sum_{i=1}^{p} \left(\frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_i} \right) \vec{\mathbf{u}}_i$$

Theorem 6 ch.6: page 334 (4th ed.)

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Theorem 7 ch.6: page 334 (4th ed.)

Let U be a matrix with orthonormal columns, and let $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ be vectors in $\mathbb{R}^n$. Then,

1. $||U \vec{\mathbf{x}}|| = || \vec{\mathbf{x}} ||$
2. $(U \vec{\mathbf{x}}) \cdot (U \vec{\mathbf{y}}) = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$
3. $(U \vec{\mathbf{x}}) \cdot (U \vec{\mathbf{y}}) = 0$ if and only if $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = 0$

Theorem 8 ch.6: page 348 (4th ed.)

The Orthogonal Decomposition Theorem: Let W be a subspace of $\mathbb{R}^n$. then each $\vec{\mathbf{y}}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$y = \hat{\mathbf{y}} + \vec{\mathbf{z}}$$

where $\hat{\mathbf{y}} \in W$ and $\vec{\mathbf{z}} \in W^{\perp}$. Furthermore if $\left\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_p\right\}$ is an orthogonal basis of W then

$$\hat{\mathbf{y}} = \text{proj}_W \vec{\mathbf{y}} = \sum_{i=1}^{p} \left(\frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_i} \right) \vec{\mathbf{u}}_i$$

and $\vec{\mathbf{z}} = \vec{\mathbf{y}} - \hat{\mathbf{y}}$.

Note, if $\vec{\mathbf{y}}$ is in W then $\vec{\mathbf{y}} = \text{proj}_W \vec{\mathbf{y}}$.

Theorem 9 ch.6: page 350 (4th ed.)

The Best Approximation Theorem Let W be a subspace of $\mathbb{R}^n$, let $\vec{\mathbf{y}}$ be any vector in $\mathbb{R}^n$, then $\hat{\mathbf{y}}$ is the closest point in W to $\vec{\mathbf{y}}$, that is

$$|| \vec{\mathbf{y}} - \hat{\mathbf{y}} || \leq || \vec{\mathbf{y}} - \vec{\mathbf{v}} || \quad \text{for all } \vec{\mathbf{v}} \text{ in } W$$

$\boxed{\textbf{Theorem 10 ch.6:}}$ page (4th ed.)

If $\left\{\vec{\mathbf{u}}_1, \ldots \vec{\mathbf{u}}_p\right\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\mathrm{proj}_W \vec{\mathbf{y}} = \sum_{i=1}^{p} \left(\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i\right) \vec{\mathbf{u}}_i$$

If $U = \left[\vec{\mathbf{u}}_1, \ldots \vec{\mathbf{u}}_p\right]$ then,

$$\mathrm{proj}_W \vec{\mathbf{y}} = UU^T \vec{\mathbf{y}} \qquad \text{for all } \vec{\mathbf{y}} \text{ in } \mathbb{R}^n.$$

$\boxed{\textbf{Theorem 11 ch.6:}}$ page 355 (4th ed.)

The Gram-Schmit Process: Given a basis $\left\{\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_2\right\}$ for a non zero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\vec{\mathbf{v}}_1 &= \vec{\mathbf{x}}_1 \\
\vec{\mathbf{v}}_2 &= \vec{\mathbf{x}}_2 - \left(\tfrac{\vec{\mathbf{x}}_2 \cdot \vec{\mathbf{v}}_1}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1}\right) \vec{\mathbf{v}}_1 \\
\vec{\mathbf{v}}_3 &= \vec{\mathbf{x}}_3 - \left(\tfrac{\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{v}}_1}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1}\right) - \left(\tfrac{\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{v}}_2}{\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2}\right) \vec{\mathbf{v}}_2 \\
&\vdots \\
\vec{\mathbf{v}}_p &= \vec{\mathbf{x}}_p - \left(\frac{\vec{\mathbf{x}}_p \cdot \vec{\mathbf{v}}_1}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1}\right) \vec{\mathbf{v}}_1 - \left(\frac{\vec{\mathbf{x}}_p \cdot \vec{\mathbf{v}}_2}{\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2}\right) \vec{\mathbf{v}}_2 - \ldots - \left(\frac{\vec{\mathbf{x}}_p \cdot \vec{\mathbf{v}}_{p-1}}{\vec{\mathbf{v}}_{p-1} \cdot \vec{\mathbf{v}}_{p-1}}\right) \vec{\mathbf{v}}_{p-1}
\end{aligned}$$

$\boxed{\textbf{Theorem 13 ch.6}}$ page 361 (4th ed.)

The set of least squares solutions to $A \vec{\mathbf{x}} = \vec{\mathbf{b}}$ concides with the nonempty set of solutions to the normal equations $A^T A \vec{\mathbf{x}} = A^T \vec{\mathbf{b}}$.

Section 1.1, Systems of Linear Equations:

Linear equations, coefficients: page 2 (4th ed.)

A **linear equation** in the variables $x_1, x_2, ...x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where b and the **coefficients** $a_1, a_2, ...a_n$ are real or complex numbers.

System of Linear Equations (Linear System): page 2 (4th ed.)

A **system of linear equations** or **linear system** is a collection of one or more linear equations.

Solution: page 2 (4th ed.)

A **solution** of a linear system is a list $(s_1, s_2, ..., s_n)$ of numbers that makes each equation a true statement.

Solution set: page 3 (4th ed.)

The set of all possible solutions of a linear system is called a **solution set**.

Equivalent: page 3 (4th ed.)

Two linear systems are **equivalent** if they have the same solution set.

Fact: page 4 (4th ed.)

A linear system has

1. no solutions, or
2. exactly one solution, or
3. infinitely many solutions

Consistent, Inconsistent: page 4 (4th ed.)

A linear system is said to be **consistent** if it has a solution (either one or infinity many) and **inconsistent** if it has no solution

Matrix, Coefficient Matrix (Matrix of Coefficients), Augmented Matrix: page 4 (4th ed.)

The essential information from a given a linear system,

$$\begin{array}{ccccccccc} a_{11}\, x_1 & + & a_{12}\, x_2 & + & \dots & + & a_{1n}\, x_n & = & b_1 \\ a_{21}\, x_1 & + & a_{22}\, x_2 & + & \dots & + & a_{2n}\, x_n & = & b_2 \\ \vdots & & & & & & & & \\ a_{m1}\, x_1 & + & a_{m2}\, x_2 & + & \dots & + & a_{mn}\, x_n & = & b_m \end{array}$$

can be recorded in a rectangular array called a **matrix**. The **coefficient matrix** (or **matrix of coefficients**) is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and the **augmented matrix** of the system is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

.

Size of a Matrix: page 4 (4th ed.)

The **size** of a matrix tells how many rows and columns it has, written $m \times n$ if it has m rows and n columns. It is said the matrix is an m by n matrix

Elementary Row Operations: page 6 (4th ed.)

Given a matrix M, there are three elementary row operations:

1. (Replacement) Replace one row by the sum of itself and multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row of a by a nonzero constant.

Row Equivalent: page 6 (4th ed.)

Two matrices are called **row equivalent** if there is a series or elementary row operations that transforms one matrix into the other.

Fact: page 7 (4th ed.)

If the augmented matrices of two linear systems are row equivalent then the two systems have the same solution set.

Section 1.2, Row Reduction and Echelon Form:

Nonzero Row, Leading Entry: page 12 (4th ed.)

A **nonzero row** of a matrix is one where at least one entry is not zero. A **leading entry** of a row refers to the left most nonzero entry (in a nonzero row).

Pivot Position page 14 (4th ed.)

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Echelon Form (Row Echelon Form), Reduced Echelon Form (Reduced Row Echelon Form), Echelon Matrix, Reduced Echelon Matrix:
page 13 (4th ed.)

A matrix is in **echelon form** (or **row echelon form**) if it has the following properties:

1. All nonzero rows appear above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix is in echelon form and satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**:

1. The leading entry in each row is a 1.
2. Each leading 1 is the only nonzero entry in the column.

A matrix in echelon form is called an **echelon matrix** and one in reduced echelon form is called a **reduced echelon matrix**.

Row Reduced: page 13 (4th ed.)

Any nonzero matrix may be **row reduced**. Meaning it may be changed through elementary row operations into a (nonunique) matrix that is in echelon form, and into a unique matrix in reduced echelon form.

Theorem 1 (of chapter 1): page 13 (4th ed.)

Uniqueness of reduced echelon form. Each matrix is row equivalent to one and only one reduced echelon matrix.

An Echelon form, The Reduced Echelon Form: page 14 (4th ed.)

If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form of A**; if U is in reduced echelon form we call U **the reduced echelon form of A**.

Pivot Position: page 14 (4th ed.)

A **pivot position** in a matrix A is a location in A that coresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column in A that contains a pivot position.

Row Reduction Algorithm, Forward Phase, Backward Phase: page 15-17 (4th ed.)

Read text.

Basic Variables, Free Variables: page 18 (4th ed.)

The vaiables in a linear system that corespond to pivot columns in coefficient matrix are called **basic variables**, the others are called **free variables**.

Theorem 2 (of chapter 1) page 21 (4th ed.)

Existence and Uniqueness theorem. A linear system is consistant if and only if the rightmost column of the augmented matrix is not a pivot column. That is if and only if the augmented matrix has no row of the form

$$[\,0 \quad \dots \quad 0 \quad b\,] \qquad \text{with } b \neq 0.$$

If a linear system is consistent, then the solution set contains either

(i) a unique solution, when there are no free variables, or

(ii) infinitely many solutions, if there are free variables.

Using Row Reduction: page 21 (4th ed.)

Read text.

Section 1.3, Vector Equations:

Column vector (vector): page 24 (4th ed.)

A matrix with only one column is a column **vector** or just **vector** Notations eused for column vectors are: $\vec{\mathbf{x}}$, $\mathbf{x}$, $\vec{x}$. Two vectors are equal if and only if their corresponding entries are the same The zero vector is a vector with zeros in all positions, written $\vec{\mathbf{0}}$.

Vector Addition: page 24 (4th ed.)

Given two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ with dimension $1 \times n$, their sum is computes as follows:

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Scalar Multiplication: page 24 (4th ed.)

Given a vectors $\vec{\mathbf{u}}$ with dimension $1 \times n$ and a scalar (constant real number) c, the **scalar multiplication** of $\vec{\mathbf{u}}$ by c is computed as follows:

$$c\,\vec{\mathbf{u}} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

Parallelogram Rule for Vector addition in $\mathbb{R}^2$: page 26 (4th ed.)

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are vectors in $\mathbb{R}^2$ the vector $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ corresponds to the fourth vertex of the parallelogram whose other vectors are $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{0}}$.

Algebraic Properties of $\mathbb{R}^n$: page 27 (4th ed.)

For all vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ in $\mathbb{R}^n$ and scalars c and d:

(i) $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$

(ii) $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}})$

(iii) $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{u}} = \vec{\mathbf{u}}$

(iv) $\vec{\mathbf{u}} + (-1)\,\vec{\mathbf{u}} = (-1)\,\vec{\mathbf{u}} + \vec{\mathbf{u}} = \vec{\mathbf{0}}$

(v) $c(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = c\,\vec{\mathbf{u}} + c\,\vec{\mathbf{v}}$

(vi) $(c+d)\,\vec{\mathbf{u}} = c\,\vec{\mathbf{u}} + d\,\vec{\mathbf{u}}$

(vii) $c(d\,\vec{\mathbf{u}}) = (cd)\,\vec{\mathbf{u}}$

(viii) $1\,\vec{\mathbf{u}} = \vec{\mathbf{u}}$

Linear Combination and Weights: page 27 (4th ed.)

Let $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_p$ be vectors in $\mathbb{R}^n$ and $c_1, c_2, \ldots, c_p$ be scalars, the vector $\vec{\mathbf{y}}$ defined by

$$\vec{\mathbf{y}} = \sum_{i=1}^{p} c_i \vec{\mathbf{x}}_i = c_1 \vec{\mathbf{x}}_1 + c_2 \vec{\mathbf{x}}_2 + \ldots + c_p \vec{\mathbf{x}}_p$$

Is called a **linear combination** of the vectors $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_p$ with **weights** $c_1, c_2, \ldots, c_p$.

Vector Equation: page 29 (4th ed.)

A **vector equation** has the form, $x_1 \vec{\mathbf{a}}_1 + x_2 \vec{\mathbf{a}}_2 + \ldots + x_n \vec{\mathbf{a}}_n = \vec{\mathbf{b}}$, where $x_1, x_2, \ldots, x_n$ can be variables. The above vector equation has the same solution set as the augmented matrix

$$\begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \ldots & \vec{\mathbf{a}}_n & \vec{\mathbf{b}} \end{bmatrix}$$

Spanned, Generated: page 30 (4th ed.)

If $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_p$ are vectors in $\mathbb{R}^n$, then the set of linear combinations of $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_p$ is denoted **Span** $\left\{\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_p\right\}$ and is called the subset of $\mathbb{R}^n$ **spanned** by or **generated** by $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_p$. That is **Span** $\left\{\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \ldots, \vec{\mathbf{x}}_p\right\}$ is all the vectors of the form

$$c_1 \vec{\mathbf{x}}_1 + c_2 \vec{\mathbf{x}}_2 + \ldots + c_p \vec{\mathbf{x}}_p$$

where $c_1, c_2, \ldots, c_p$ are scalars.

Section 1.4, The Matrix Equation $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$:

The Product of a Matrix and a Vector: page 35 (4th ed.)

If $A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_n \end{bmatrix}$ is an $m \times n$ matrix and $\vec{\mathbf{x}}$ a vector in $\mathbb{R}^n$, then the **product of** A **and** $\vec{\mathbf{x}}$, denoted by $A\vec{\mathbf{x}}$, is the linear combination of the columns of A using the corresponding entries of $\vec{\mathbf{x}}$ as weights; that is

$$A\vec{\mathbf{x}} = x_1\vec{\mathbf{a}}_1 + x_2\vec{\mathbf{a}}_2 + \ldots + x_n\vec{\mathbf{a}}_n$$

Matrix Equation: page 35 (4th ed.)

Let A be an $m \times n$ matrix $\vec{\mathbf{x}}$ a variable vector in $\mathbb{R}^n$ and $\vec{\mathbf{b}}$ a vector in $\mathbb{R}^m$. Then $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ is called a matrix equation; that is

$$A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{\mathbf{b}}$$

Theorem 3 ch.1: page 36 (4th ed.)

If $A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_3 \end{bmatrix}$ is an $m \times n$ matrix and $\vec{\mathbf{b}}$ is a vector in $\mathbb{R}^m$, the matrix equation

$$A\vec{\mathbf{x}}=\vec{\mathbf{b}}$$

has the same solution set as the vector equation

$$x_1\vec{\mathbf{a}}_1 + x_2\vec{\mathbf{a}}_2 + \ldots + x_n\vec{\mathbf{a}}_n = \vec{\mathbf{b}}$$

which in turn has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_n & \vec{\mathbf{b}} \end{bmatrix}$$

Fact: page 36 (4th ed.)

The equation $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ has a solution if and only if $\vec{\mathbf{b}}$ is a linear combination of the columns of A.

Theorem 4 ch.1 (important): page 37 (4th ed.)

Let A be an $m \times n$ matrix. Then the following statements are equivalent (they are either all true of all false).

(a) For each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$, the equation $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ has a solution.

(b) Each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$; that is $\textbf{Span}\left\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n\right\} = \mathbb{R}^m$.

(d) A has a pivot position in every row.

Theorem 5 ch.1: page 39 (4th ed.)

If A is an $m \times n$ matrix, $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ vectors in $\mathbb{R}^n$ and c a scalar, then:

a) $A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = A\,\vec{\mathbf{u}} + A\,\vec{\mathbf{v}}$;

b) $A(c\,\vec{\mathbf{u}}) = c(A\,\vec{\mathbf{u}})$.

Section 1.5, Solution Set of Linear Systems:

Homogenous System: page 43 (4th ed.)

A system of equations is called **homogenous** if it can be written in the form $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$ for some matrix A. Such a system always has $\vec{\mathbf{x}}=\vec{\mathbf{0}}$ as a solution (called the **trivial solution**, other solutions are called **nontrivial** solutions.

Note that a homogenous system of the form $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$ has a solution if and only if the equation has a free variable.

Parametric vector form: page 44 (4th ed.)

Read text.

Translated: page 45 (4th ed.)

Given $\vec{\mathbf{v}}$ and $\vec{\mathbf{p}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ the effect of adding $\vec{\mathbf{p}}$ to $\vec{\mathbf{v}}$ is to move $\vec{\mathbf{v}}$ in a direction parallel to the line through $\vec{\mathbf{p}}$ and $\vec{\mathbf{0}}$. We say v is **translated** by $\vec{\mathbf{p}}$ to $\vec{\mathbf{p}}+\vec{\mathbf{v}}$.

We call $\vec{\mathbf{x}}=\vec{\mathbf{p}}+t\vec{\mathbf{v}}$ **the equation of the line through $\vec{\mathbf{p}}$ parallel to $\vec{\mathbf{v}}$** so the solution set to $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ is the line through $\vec{\mathbf{p}}$ parallel to the solution set of $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$.

Theorem 6 ch.1: page 46 (4th ed.)

Suppose $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ is consistent for some $\vec{\mathbf{b}}$, and let $\vec{\mathbf{p}}$ be a solution. Then the solution set of $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ is the set of vectors of the form $\vec{\mathbf{w}}=\vec{\mathbf{p}}+\vec{\mathbf{v}}_h$, where $\vec{\mathbf{v}}_h$ is any solution to $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$.

Section 1.7, Linear independence:

Linearly independent/dependent: page 56 (4th ed.)

An indexed set of vecors $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1 \vec{\mathbf{v}}_1 + \quad x_2 \vec{\mathbf{v}}_2 + \quad \ldots \quad + x_p \vec{\mathbf{v}}_p \quad = \vec{\mathbf{0}}$$

has only the trivial solution. the set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ is said to be **linearly dependent** if there are $x_1, x_2, \ldots, x_p$ which are not all zero so that,

$$x_1 \vec{\mathbf{v}}_1 + \quad x_2 \vec{\mathbf{v}}_2 + \quad \ldots \quad + x_p \vec{\mathbf{v}}_p \quad = \vec{\mathbf{0}}$$

The above equation is called a **linear dependence relation**

Theorem 7 ch.1: page 58 (4th ed.)

Characterization of Linearly Independent Sets An indexed set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the rest.

Theorem 8 ch.1: page 59 (4th ed.)

If a set contains more vectors than there are entries in each vector then the set is linearly dependent, that is, any set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ of vectors from $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem 9 ch.1: page 59 (4th ed.)

If a set of vectors contains the zero vector then it is a linearly dependent set.

Section 1.8, Introduction to Linear Transformations:

Transformation (function, mapping): page 63 (4th ed.)

A **transformation (or function or mapping)** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector in $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ a vector $T(\vec{\mathbf{x}})$ in $\mathbb{R}^m$ (note, each $\vec{\mathbf{x}}$ is assigned one and only one vector in $\mathbb{R}^m$).

The set $\mathbb{R}^n$ is called the **domain** of T and the set $\mathbb{R}^m$ the **codomain** of T. The notation

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Indicates that T is a transformation with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$.

For $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ the vector $T(\vec{\mathbf{x}})$ is called the image of $\vec{\mathbf{x}}$ under the action of T. The set of all images are called the **range** of T.

Notation that I will use that is not in the book.

1. dom(T) will mean the domain of T,
2. ran(T) will mean the range of T.

Linear Transformation: page 65 (4th ed.)

A transformation T is **linear** if:

(i) $T(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = T(\vec{\mathbf{u}}) + T(\vec{\mathbf{v}})$ for all $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in the domain of T

(ii) $T(c\,\vec{\mathbf{u}}) = cT(\vec{\mathbf{u}})$ for all $\vec{\mathbf{u}}$in the domain of T and scalars c.

Shear Transformation : page 65 (4th ed.)

Read text.

Section 1.8, Introduction to Linear Transformations:

Theorem 10 ch.1: page 71 (4th ed.)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\vec{\mathbf{x}}) = A\,\vec{\mathbf{x}} \qquad \text{for all } \vec{\mathbf{x}} \text{ in } \mathbb{R}^n.$$

Furthermore the matrix A is:

$$A = \begin{bmatrix} T(\vec{\mathbf{e}}_1) & T(\vec{\mathbf{e}}_2) & \dots & T(\vec{\mathbf{e}}_n) \end{bmatrix}$$

This matrix is called **the standard matrix for the linear transformation** T.

Transformations in $\mathbb{R}^2$: page 73-75 (4th ed.)

Read text for descriptions, the matrices given are the standard matrix for each transformation:

1. Reflection through x_1-axis, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. Reflection through x_2-axis, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
3. Reflection through the line $x_2 = x_1$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
4. Reflection through the line $x_2 = -x_1$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.
5. Reflection through the origin $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
6. Horizontal contraction (for $0 < k < 1$) and expansion (for $1 < k$), $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
7. Vertical contraction (for $0 < k < 1$) and expansion (for $1 < k$), $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$.
8. Horizontal shear left (for $k < 0$) and right (for $0 < k$), $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.
9. Vertical shear down (for $k < 0$) and up (for $0 < k$), $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.
10. Projection onto x_1-axis, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
11. Projection onto x_2-axis, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
12. Rotation conterclockwise by θ radians $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Onto (surjective): page 75 (4th ed.)

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** if each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ is the image of at least one $\vec{\mathbf{x}}$ in $\mathbb{R}^n$. This is equivalent to saying, the range of T equals the codomain of T, or the range of T spans $\mathbb{R}^m$.

One-to-one (injective): page 75 (4th ed.)

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one-to-one** if each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ is the image of at most one $\vec{\mathbf{x}}$ in $\mathbb{R}^n$. This is equivalent to saying, if $T(\vec{\mathbf{x}}) = T(\vec{\mathbf{y}})$ then $\vec{\mathbf{x}} = \vec{\mathbf{y}}$.

Theorem 11 ch.1: page 76 (4th ed.)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$ has only the trivial solution.

Theorem 12 ch.1: page 77 (4th ed.)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A. Then

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

Section 2.1, Matrix Operations:

Diagonal entries: page 92 (4th ed.)

The **diagonal entries** of an $m \times n$ matrix $A = [a_{ij}]$ are the entries a_{ii} together they form the **main diagonal** of A. A Diagonal matrix is an $n \times n$ matrix with nondiagonal entries that are zeros.

Matrix sum: page 92 (4th ed.)

If $A = [a_{ij}] =$ and $B = [b_{ij}]$ are matrices of the same size, then their **sum (matrix sum)** is the matrix

$$A + B = [a_{ij} + b_{ij}]$$

or the sum of the matrices A and B is the matrix whith positions equal to the sum of the coresponding positions of A nad B.

Matrix scalar multiplication: page 92 (4th ed.)

If $A = [a_{ij}]$ is a matrix and r a scalar, then the **scalar multiple** is

$$rA = [ra_{ij}]$$

or rA is the matrix with positions equal to the coresponding position of A multipiled by r.

Theorem 1 ch.2 (matrix operations): page 93 (4th ed.)

Let A, B and C be matrices of the same size, 0 will be the matrix of all zeros with the same size as A and C, and let r and s be scalars. Then:

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + 0 = A$

(iv) $r(A + B) = rA + rB$

(v) $(r + s)A = rA + sA$

(vi) $r(sA) = (rs)A$

Matrix multiplication: page 95 (4th ed.)

If B is an $m \times n$ matrix and $A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \ldots & \vec{\mathbf{a}}_p \end{bmatrix}$ is an $n \times p$ matrix, then their **product** is the $m \times p$ matrix

$$BA = \begin{bmatrix} B\,\vec{\mathbf{a}}_1 & B\,\vec{\mathbf{a}}_2 & \ldots & B\,\vec{\mathbf{a}}_p \end{bmatrix}$$

To compute the i, j-th position of BA, use the formula

$$\sum_{k=1}^{n} b_{ik} a_{kj}$$

Where b_{ij} is the i, j position of B and a_{ij} is the i, j-th position of A.

Define $A^k = \underbrace{A \ldots A}_{k}$

Theorem 2 ch.2 (matrix multiplication properties): page 97 (4th ed.)

Let A, B and C be matrices of the appropriate sizes and let r be a scalar. Then:

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(A + B)C = AC + BC$

(iv) $r(AB) = (rA)B = A(rB)$

(v) $I_m A = A = AI_n$

Transpose: page 99 (4th ed.)

If $A = [a_{ij}]$ is a matrix, then the the **transpose** of A is the matrix $A^T = [a_{ji}]$.

Theorem 3 ch.2: page 99 (4th ed.)

Let A and B denote matrices whose sizes are appropriate for the following and r a scalar, then:

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

Section 2.2, Matrix Inverse:

Invertible: page 103 (4th ed.)

An $n \times n$ matrix is said to be **invertible** if and only if there is an $n \times n$ matrix C such that,

$$CA = I_n \qquad \text{and} \qquad AC = I_n$$

where I_n is the diagonal matrix with all 1's on the diagonal. If there is such a C for A we call C **the inverse** of A and write A^{-1}. It turns out that if C exists, it will be unique so we can write *the inverse.*

We call invertible matrices **non-singular** and noninvertable matricies **singular**

Theorem 4 ch.2: page 103 (4th ed.)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. id $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc = 0$ the matrix is not invertible.

Determinant: page 103 (4th ed.)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $ad - bc$ is called the **determinant** of A.

Theorem 5 ch.2: page 104 (4th ed.)

If A is an invertible $n \times n$ matrix then, for each $\vec{\mathbf{b}}$ in $\mathbb{R}^n$, the equation $A\ \vec{\mathbf{x}} = \vec{\mathbf{b}}$ the unique solution $\vec{\mathbf{x}} = A^{-1}\ \vec{\mathbf{b}}$.

Theorem 6 ch.2: page 105 (4th ed.)

a. If A is an invertible matrix then A^{-1} is invertible and $(A^{-1})^{-1} = A$
b. If A and B are $n \times n$ invertible matrices then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
c. If A is an invertible matrix then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

Elementary matrices: page 106 (4th ed.)

There are three types of elementary matrices. Each type corresponds to a row operation.

1. Let A be an $m \times n$ matrix and B be the matrix obtained by using the replacement row operation, that is replacing row i of A by row i plus k row j, that is

$$A \quad \mathrm{R}_i + k\mathrm{R}_j \;=\; B$$

The elementary matrix for replacement is the $m \times m$ diagonal matrix with 1's on the diagonal except there is a k in the i, j-position. For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ ka_{11}+a_{31} & ka_{12}+a_{32} & ka_{13}+a_{33} & ka_{14}+a_{34} \end{pmatrix}$$

is the same as,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \quad \mathrm{R}_3 = k\mathrm{R}_1 + \mathrm{R}_3 \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ ka_{11}+a_{31} & ka_{12}+a_{32} & ka_{13}+a_{33} & ka_{14}+a_{34} \end{pmatrix}$$

2. Let A be an $m \times n$ matrix and B be the matrix obtained by using the interchange row operation, that is interchanging row i of A by row j of A, that is

$$A \quad \mathrm{R}_i = \mathrm{R}_j \;=\; B$$

The elementary matrix for interchange is the $m \times m$ diagonal matrix with 1's on the diagonal except there is a 0 in position ii a 0 in positionjj, a 1 in position ij and a 1 in position ji. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

is the same as,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \quad \mathrm{R}_2 = \mathrm{R}_3 \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

3. Let A be an $m \times n$ matrix and B be the matrix obtained by using the scaling row operation, that is scaling row i of A by a multiple k, that is

$$A \quad \mathrm{R}_i = k\mathrm{R}_i \;=\; B$$

The elementary matrix for interchange is the $m \times m$ diagonal matrix with 1's on the diagonal except there is a k in position ii. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ ka_{31} & ka_{32} & ka_{33} & ka_{34} \end{pmatrix}$$

is the same as,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \quad \mathrm{R}_3 = k\mathrm{R}_3 \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ ka_{31} & ka_{32} & ka_{33} & ka_{34} \end{pmatrix}$$

Theorem 7 ch.2: page 107 (4th ed.)

An $n \times n$ matrix is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduce A to I_n also transform I_n into A^{-1}

Section 2.3, Characterization of Invertible matrices:

Theorem 8 ch.2 page 112 (4th ed.)

Let A be an $n \times n$ matrix, then the following are equivalent.

- a) A has a right inverse, that is, there is a matrix D such that $AD = I_n$.
- a_1) For any $n \times p$ matrix B, $AX = B$ has at least one solution. For $p = 1$ this is matrix vector multiplication.
- a_1') The columns of A span $\mathbb{R}^n$.
- a_2) A has n pivots (one for each row.)
- a_2') $A \sim I_n$
- a_3) A is the product of elementary matrices.
- b) A has a left inverse. That is, there is a matrix C such that $CA = I_n$.
- b_1) For B an $n \times p$ matrix, solutions of $AX = B$ are unique when they exist. For $p = 1$ this is matrix vector multiplication.
- b_1') The only solution to $A \vec{\mathbf{x}} = \vec{\mathbf{0}}$ is $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
- b_1'') The columns of A are linearly independent.
- b_2) A has n pivots, (1 for each column).
- b_2') $A \sim I_n$.
- b_3) A is the product of elementary matrices.

Theorem 9 ch.2 page 114 (4th ed.)

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation with standard matrix A, then T is invertible if and only if A is an invertible matrix. Furthermore $T^{-1}(\vec{\mathbf{x}}) = A^{-1} \vec{\mathbf{x}}$.

Section 3.1, Introducing Determinants:

Determinant: page 165 (4th ed.)

For $n \geq 2$ the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is,

$$\sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

Here A_{ij} is the $n-1 \times n-1$ matrix defined by removing row i and column j from A.

(i,j)-Cofactor: page 165 (4th ed.)

Given an $n \times n$ matrix $A = [a_{ij}]$ for $n \geq 2$ the (i,j)**-cofactor** is written C_{ij} and is given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Theorem 1 ch.3: page 166 (4th ed.)

The determinate of A can be computed by cofactor expansion across any row or column. That is the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is, using expansion along row i

$$\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

or using expansion along column j is

$$\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Theorem 2 ch.3: page 167 (4th ed.)

If A is a triangular matrix, then $\det(A)$ is the product of the entries along the diagonal.

Section 3.2, Properties of Determinants:

Theorem 3 ch.3: page 169 (4th ed.)

Row operations: Let A be a square matrix.

a) If a multiple of one row a A is added to another row to produce a matrix B then $\det(B) = \det A$.

b) If two rows of A are interchanged to produce a matrix B then $\det(B) = -\det(A)$.

c) If one row of A is multiplied by k to produce a matrix B, then $\det(B) = k \cdot \det(B)$.

Fact: page 171 (4th ed.)

Let A be matrices and U be an echelon form for A where no row multiplication is used in the row reduction. Then

$$\det(A) = \begin{cases} (-1)^r \cdot (\text{product of pivots of } U) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible.} \end{cases}$$

Theorem 4 ch.3: page 171 (4th ed.)

A square matrix A is invertible if and only if $\det(A) \neq 0$

Theorem 5 ch.3: page 172 (4th ed.)

If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

Theorem 6 ch.3: page 173 (4th ed.)

If A and B are matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

Section 4.1 Vector spaces:

Vector Space: page 190 (4th ed.)

A **vector space** is a nonempty set V of objects called *vectors*, on which are defined two operations, called *addition* and *scalar multiplication* (scalars will be real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ and for all scalars c and d.

1. V is closed under addition, that is, $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is also in V.
2. Addition in V is commutative, that is, $\vec{\mathbf{u}} + \vec{\mathbf{v}}=\vec{\mathbf{v}} + \vec{\mathbf{u}}$.
3. Addition in V is associative, that is, $(\vec{\mathbf{u}} + \vec{\mathbf{v}})+ \vec{\mathbf{w}}=\vec{\mathbf{u}} +(\vec{\mathbf{v}} + \vec{\mathbf{w}})$.
4. V has a zero vector, that is there is a vector $\vec{\mathbf{0}}$ in V such that $\vec{\mathbf{u}} + \vec{\mathbf{0}}=\vec{\mathbf{u}}$.
5. Additive inverses. For each $\vec{\mathbf{u}}$ in V there is another vector in V which we call $-\vec{\mathbf{u}}$ such that $\vec{\mathbf{u}} +(-\vec{\mathbf{u}}) =\vec{\mathbf{0}}$. Note, it turns out that $(-1)\vec{\mathbf{u}}$ ($\vec{\mathbf{u}}$ multiplied by the scaler -1 is $\vec{\mathbf{u}}$ inverse, so the notation $-\vec{\mathbf{u}}$ is unambiguous.
6. V is closed under scalar multiplication, that is, $c\vec{\mathbf{u}}$ is in V.
7. Distributivity (scalars distribute over vectors. $c(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = c\vec{\mathbf{u}} +c\vec{\mathbf{v}}$.
8. Distributivity (vectors distribute over scalars). $(c + d)\vec{\mathbf{u}}= c\vec{\mathbf{u}} +c\vec{\mathbf{v}}$
9. Associativity of scalar multiplication. $c(d\vec{\mathbf{v}}) = (cd)\vec{\mathbf{v}}$.
10. One is the scalar identity, that is $1\vec{\mathbf{u}}=\vec{\mathbf{u}}$

Subspace: page 193 (4th ed.)

A **subspace** of a vector space V is a subset H of V that has three properties:

a) The zero vector of V is in H.
b) H is closed under vector addition.
c) H is closed under scalar multiplication

Theorem 1 ch.4: page 195 (4th ed.)

If $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ are vectors in$V$, then Span $\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$ is a subspace of V.

The subspace spanned by: page 194 (4th ed.)

We call Span$\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$ **the subspace spanned by** $\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$ or generated by $\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$. Given any subspace H of V, a **spanning** (or **generating**) **set** for H, is a set $\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$ such that $H = \text{Span}\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$.

Section 4.2 Null Spaces, Column spaces and Linear Transformations:

Null Space: page 199 (4th ed.)

The **null space** of an $m \times n$ matrix A, written Nul(A) is the set of solutions to the homogenious equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$. In set notation,

$$\text{Nul}(A) = \left\{ \vec{\mathbf{x}} : \vec{\mathbf{x}} \in \mathbb{R}^n \text{ and } A\vec{\mathbf{x}} = \vec{\mathbf{0}} \right\}$$

Theorem 2 ch.4: page 199 (4th ed.)

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Column Space: page 201 (4th ed.)

The **column space** of an $m \times n$ matrix A written Col(A), is the set of linear combinations of the columns of A. If $A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_n \end{bmatrix}$, then

$$\text{Col}(A) = \text{Span}\left\{ \vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n \right\}$$

Theorem 3 ch.4: page 201 (4th ed.)

If A is an $m \times n$ matrix, then Col(A) is a subspace of $\mathbb{R}^m$.

Linear transformation (on vector spaces): page 204 (4th ed.)

A **Linear transformation** from a vector space V to a vector space W is a rule which assigns to each vector $\vec{\mathbf{x}}$ in V a unique vector $T(\vec{\mathbf{x}})$ in W, such that

i) $T(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = T(\vec{\mathbf{u}}) + T(\vec{\mathbf{v}})$ for all $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in V.

ii) $T(c\vec{\mathbf{u}}) = cT(\vec{\mathbf{u}})$ for all scalars c and $\vec{\mathbf{u}}$ in V.

Kernel: page 204 (4th ed.)

The **Kernel** or **null space** of a linear transformation is the set of vectors $\vec{\mathbf{x}}$ in V such that $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$.

Section 4.3 Linear Independent sets; Basis:

Linearly Independent: page 208 (4th ed.)

A set of vectors $\left\{ \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_p \right\}$ in a vector space is said to be **linearly independent** if the vector equation,

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \dots + c_p\vec{\mathbf{v}}_p = \vec{\mathbf{0}}$$

has only the trivial solution. They are said to be **linearly dependent** if not. If they are linearly dependent then (1) is called a **dependence relation**.

Theorem 4 ch.4: page 208 (4th ed.)

An indexed set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ of two or more vectors with $\vec{\mathbf{v}}_1 \neq \vec{\mathbf{0}}$ is linearly dependent if and only if some $\vec{\mathbf{v}}_j$ with $j > 1$ is a linear combination of the preceding vectors, $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$.

Basis: page pg.209 (4th ed.)

Let H be a subspave of a vector space V. An indexed set $B = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_p\right\}$ in V is a **basis** for H if

(i) B is a linearly independent set, and
(ii) $H = \mathrm{Span}(B)$.

Standard basis for: page 209 (4th ed.)

The set $\left\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_p\right\}$ is called the **standard basis for** $\mathbb{R}^n$. The **standard basis** for $\mathbb{P}_n$ is $\left\{1, t, t^2, \ldots, t^n\right\}$.

Theorem 5 ch.4: page 210 (4th ed.)

The spanning set theorem: Let $S = \left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$ be a set in V, and let $H = \mathrm{Span}\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_p\right\}$.

a. If one of the vectors in S, say $\vec{\mathbf{v}}_k$ is a linear combination of the remaining vectors in S, then the set formed by removing $\vec{\mathbf{v}}_k$ from S still spans S.
b. If $H \neq \left\{\vec{\mathbf{0}}\right\}$, some subset of S is a basis for H.

Theorem 6: page 212 (4th ed.)

The pivot columns of a matrix A form a basis for $\mathrm{Col}(A)$.

Section 4.4 Coordinate systems:

Theorem 7 ch.4: page 216 (4th ed.)

The Unique Representation theorem Let $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n\right\}$ be a basis for the vector space V. Then for each $\vec{\mathbf{x}}$ in V, there is a unique set of scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{\mathbf{x}} = c_1\, \vec{\mathbf{b}}_1 + c_2\, \vec{\mathbf{b}}_2 + \ldots + c_n\, \vec{\mathbf{b}}_n$$

That is $\vec{\mathbf{x}}$ has a unique representation as a linear combination as basis vectors from $\mathcal{B}$.

$\mathcal{B}$-coordinates: page 216 (4th ed.)

Suppose $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n\right\}$ is a basis for V and $\vec{\mathbf{x}}$ is in V. the **coordinates of** x **relative to the basis** $\mathcal{B}$ (or the $\mathcal{B}$-coordinates) are the weights $c_1, \ldots c_n$ such that

$$\vec{\mathbf{x}} = c_1\, \vec{\mathbf{b}}_1 + c_2\, \vec{\mathbf{b}}_2 + \ldots + c_n\, \vec{\mathbf{b}}_n$$

.

$$[\vec{\mathbf{x}}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of** $\vec{\mathbf{x}}$ **relative to** $\mathcal{B}$ or the **$\mathcal{B}$-coordinate vector of** $\vec{\mathbf{x}}$

The map $\vec{\mathbf{x}} \to [\vec{\mathbf{x}}]_{\mathcal{B}} : V \longrightarrow \mathbb{R}^n$ is the **coordinate mapping determined by** $\mathcal{B}$.

$P_{\mathcal{B}}$: page 219 (4th ed.)

The matrix $P_{\mathcal{B}}$ changes the $\mathcal{B}$-coordinates of a vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ to the standard basis. That is if $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n\right\}$ is a basis for $\mathbb{R}^n$ then

$$P_{\mathcal{B}}[\vec{\mathbf{x}}]_{\mathcal{B}} = \begin{bmatrix} \vec{\mathbf{b}}_1 & \vec{\mathbf{b}}_2 & \ldots & \vec{\mathbf{b}}_n \end{bmatrix} [\vec{\mathbf{x}}]_{\mathcal{B}} = \vec{\mathbf{x}}$$

Theorem 8 ch.4: page 219 (4th ed.)

Let $\mathcal{B} = \left\{\vec{\mathbf{b}}_2, \vec{\mathbf{b}}_2 \ldots, \vec{\mathbf{b}}_n\right\}$ be a basis for a vector space V. Then the coordinate mapping $x \to [x]_{\mathcal{B}}$ is a one-to-one, onto linear transformation from V onto $\mathbb{R}^n$

Isomorphism: page 220 (4th ed.)

In the context of linear algebra, a one-to-one, onto, linear map between vector spaces is called an **isomorphism**

Section 4.5 Dimension:

Theorem 9 ch.4: page 225 (4th ed.)

If a vector space has a basis $\mathcal{B} = \left\{ \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n \right\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10 ch.4: page 226 (4th ed.)

If a vector space V has a basis of n vectors, then every basis must consist of V must consist on n vectors.

Dimension: page 226 (4th ed.)

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written $\text{Dim}(V)$, is the number of vectors in a basis for V. The dimension of the vector space $\left\{ \vec{\mathbf{0}} \right\}$ is zero. If V is a vector space not spanned by a finite set, then V is said to be **infinite-dimensional**.

Theorem 11 ch.4: page 227 (4th ed.)

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded to be a basis for H. Also, H is finite-dimensional and

$$\text{Dim}(H) \leq \text{Dim}(V)$$

Theorem 12 ch.4: page 227 (4th ed.)

The Basis Theorem: Let V be a p-dimensional vector space, with $p \geq 1$. Any linearly independent set of exactly p vectors in V is automatically a basis for V. Any set of p elements that span V is automatically a basis for V.

Section 4.6 Rank:

Row space: page 231 (4th ed.)

Let A be an $m \times n$ matrix, each row of A can be identified with a vector in $\mathbb{R}^n$, these vectors are called the **row vectors of** A. The set of all linear combinations of the row vectors is called the **row space** of A, denote this by $\text{Row}(A)$. Not that the $\text{Row}(A)$ is a subspace of $\mathbb{R}^n$. Also, $\text{Row}(A)$ is the span of A^T (and thus a vector subspace).

Theorem 13 ch.4: page 231 (4th ed.)

If two matrices are row equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for the row space of A as well as B

Rank: page 233 (4th ed.)

The **rank** of a matrix A is the dimension of the column space of A, denoted $\text{Rank}(A)$

Spaces of a matrix: page no page (4th ed.)

Let A be an $m \times n$ matrix then,

1. The null space of A written $\text{Nul}(A)$ is the set of vectors $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ such that $A\,\vec{\mathbf{x}}=\vec{\mathbf{0}}$, or
$$\text{Nul}(A) = \left\{ \vec{\mathbf{x}} : \vec{\mathbf{x}} \in \mathbb{R}^n \text{ and } A\,\vec{\mathbf{x}}=\vec{\mathbf{0}} \right\}$$
The dimension of the null space can be called the nullity.
2. The column space of A written $\text{Col}(A)$ is the set of vectors $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ in the span of the columns of A, or the set of vectors $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ such that there is a vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ and $A\,\vec{\mathbf{x}}=\vec{\mathbf{b}}$, or
$$\text{Col}(A) = \left\{ \vec{\mathbf{b}} : \text{ there is a vector } \vec{\mathbf{x}} \in \mathbb{R}^n \text{ and } A\,\vec{\mathbf{x}}=\vec{\mathbf{b}} \right\}$$
The dimension of the column space is called the rank of A. Note $\text{Col}(A) = \text{ran}(A\,\vec{\mathbf{x}})$.
3. The Row space of A written $\text{Row}(A)$ is the set of vectors $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ such that $\vec{\mathbf{x}}$ is in the span of the row vectors of A or the set of vectors $\vec{\mathbf{x}}$ such $\vec{\mathbf{x}} \in \text{Col}(A^T)$ Note $\text{Row}(A) = \text{ran}(A^T\,\vec{\mathbf{x}})$.

   The dimension of the row space is by a theorem the rank of A.

Theorem 14 ch.4: page 233 (4th ed.)

The rank theorem: Let A be an $m \times n$ matrix, then

1. $\text{Rank}(A) = \text{Dim}(\text{Col}(A)) = \text{the number of pivots of } A = \text{Dim}(\text{Row}(A))$
2. $$\begin{aligned} \text{Rank}(A) + \text{Dim}(\text{Nul}(A)) &= \text{Dim}(\text{Col}(A)) + \text{Dim}(\text{Nul}(A)) \\ &= \text{Dim}(\text{Row}(A)) + \text{Dim}(\text{Nul}(A)) = n \end{aligned}$$

Invertible Matrix Theorem (continued): page 235 (4th ed.)

Let A be an $n \times n$ matrix, then the following are equivalent.

a) A is an invertible matrix.
b) $A \sim I_n$.
c) A has n pivots.
d) The only solution to $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$ is $\vec{\mathbf{x}}=\vec{\mathbf{0}}$.
e) The columns of A are linearly independent.
f) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is one-to-one.
g) The equation $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ has at least one solution for each $\vec{\mathbf{b}}$ in $\mathbb{R}^n$.
h) The columns of A span $\mathbb{R}^n$.
i) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is onto.
j) A has a left inverse. That is, there is a matrix C such that $CA = I_n$.
k) A has a right inverse, that is, there is a matrix D such that $AD = I_n$.
l) A^T is an invertible matrix.
m) The columns of A form a basis for $\mathbb{R}^n$.
n) $\mathrm{Col}(A) = \mathbb{R}^n$.
o) $\mathrm{Dim}(\mathrm{Col}(A)) = n$.
p) $\mathrm{Rank}(A) = n$.
q) $\mathrm{Nul}(A) = \left\{\vec{\mathbf{0}}\right\}$.
r) $\mathrm{Dim}(\mathrm{Nul}(A)) = 0$.
∗) A is the product of elementary matrices.

Section 4.7 Change of Basis:

Theorem 15 ch.4: page 240 (4th ed.)

Let $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n\right\}$ and $\mathcal{C} = \left\{\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2, \ldots, \vec{\mathbf{c}}_n\right\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ called the change of basis matrix (from $\mathcal{B}$ to $\mathcal{C}$) which converts a vector $[\vec{\mathbf{x}}]_{\mathcal{B}}$ to a vector $[\vec{\mathbf{x}}]_{\mathcal{C}}$ where $\vec{\mathbf{x}}$ is a vector in V. That is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{\mathbf{x}}]_{\mathcal{B}} = [\vec{\mathbf{x}}]_{\mathcal{C}}$$

Section 5.1 Eigenvectors and Eigenvalues:

Eigenvector/Eigenvalue: page 267 (4th ed.)

An **eigenvector** of an $n \times n$ matrix is A is a nonzero vector $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\vec{\mathbf{x}}$ of $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$; such a vector is called an eigenvector corresponding to λ

Eigenspace: page 268 (4th ed.)

For a given λ, the set of all solution to $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is called the eigenspace of A corresponding to λ

Theorem 1 ch.5: page 269 (4th ed.)

The eigenvalues of a triangular matrix are the entries on it's main diagonal.

Theorem 2 ch.5: page 270 (4th ed.)

If $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_r\right\}$ is linearly independent

Section 5.2 Characteristic equation:

Characteristic Equation: page 276 (4th ed.)

For an $n \times n$ matrix A, the polynomial $\det(A - \lambda I) = 0$ is the characteristic equation for A. Solutions to this equation are exactly the eigenvalues of A.

Inverse Matrix theorem continued: page 275 (4th ed.)

Let A be an $n \times n$ matrix, then the following are equivalent.

a) A is an invertible matrix.
b) $A \sim I_n$.
c) A has n pivots.
d) The only solution to $A\vec{\mathbf{x}}=\vec{\mathbf{0}}$ is $\vec{\mathbf{x}}=\vec{\mathbf{0}}$.
e) The columns of A are linearly independent.
f) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is one-to-one.
g) The equation $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$ has at least one solution for each $\vec{\mathbf{b}}$ in $\mathbb{R}^n$.
h) The columns of A span $\mathbb{R}^n$.
i) The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is onto.
j) A has a left inverse. That is, there is a matrix C such that $CA = I_n$.
k) A has a right inverse, that is, there is a matrix D such that $AD = I_n$.
l) A^T is an invertible matrix.
m) The columns of A form a basis for $\mathbb{R}^n$.
n) $\text{Col}(A) = \mathbb{R}^n$.
o) $\text{Dim}(\text{Col}(A)) = n$.
p) $\text{Rank}(A) = n$.
q) $\text{Nul}(A) = \left\{\vec{\mathbf{0}}\right\}$.
r) $\text{Dim}(\text{Nul}(A)) = 0$.
s) The number zero is not an eigenvalue for A.
t) The $\det(A) \neq 0$.
$*$) A is the product of elementary matrices.

Properties of determinants: page 275 (4th ed.)

Let A and B be $n \times n$ matrices.

1. A is invertible if and only if $\det(A) \neq 0$.
2. $\det(AB) = \det(A)\det(B)$.
3. $\det(A^T) = \det(A)$.
4. If A is triangular then $\det(A)$ is the product of the diagonals.
5. Row replacement does not change determinants, row swapping changes the sign of determinants. A row scaling also scales the determinant by the same amount.

Algebraic multiplicity: page 276 (4th ed.)

Let $p(\lambda)$ be a polynomial and a a root of the polynomial. The **algebraic multiplicity** of a is the number of factors or $(\lambda - a)$ the polynomial has.

Matrix similarity: page 277 (4th ed.)

A matrix A is said to be **similar** to a matrix B, it there is an invertible matrix P such that $A = PAP^{-1}$. Changing A into PAP^{-1} is called the **similarity transform**.

Theorem 4 ch.5: page 277 (4th ed.)

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomials, and hence eigenvalues.

Section 5.3 Diagonalization:

Diagonalizable: page 282 (4th ed.)

A square matrix A is said to be **diagonalizable** is a is similar to a diagonal matrix.

Theorem 5 ch.5: page 282 (4th ed.)

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PAP^{-1}$ with D a diagonal matrix if and only if the columns of P are n linearly independent eigenvectors of A, and in this case, the diagonal entries of D are the corresponding eigenvectors.

Theorem 6 ch.5: page 284 (4th ed.)

An $n \times n$ matrix with distinct eigenvalues is diagonalizable.

Theorem 7 ch.5: page 285 (4th ed.)

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of λ_k.

b) The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n. This happens if and only if

   (i) The Characteristic polynomial of A factors completely into linear factors and

   (ii) The dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace of λ_k, then the total collection of vectors in the sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ forms a basis for $\mathbb{R}^n$.

Section 5.4 Eigenvectors and linear transformations:

Matrix for T relative to the basis $\mathcal{B}$ and $\mathcal{C}$: page 289 (4th ed.)

Given a vector space V with basis $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, ..., \vec{\mathbf{b}}_n\right\}$, a vector space W with basis $\mathcal{C}$ and a linear transformation $T : V \longrightarrow W$, the matrix

$$\left[[T(\vec{\mathbf{b}}_1)]_{\mathcal{C}} \quad [T(\vec{\mathbf{b}}_2)]_{\mathcal{C}} \quad \ldots \quad [T(\vec{\mathbf{b}}_n)]_{\mathcal{C}} \right]$$

is called **the matrix for T relative to the basis $\mathcal{B}$ and $\mathcal{C}$**. If $W = V$ and $\mathcal{B} = \mathcal{C}$, then the matrix is called **the matrix for T relative to $\mathcal{B}$** or **the $\mathcal{B}$-matrix for T**. In this case the matrix is denoted $[T]_{\mathcal{B}}$, that is

$$[T]_{\mathcal{B}} = \left[[T(\vec{\mathbf{b}}_1)]_{\mathcal{B}} \ \ldots \ [T(\vec{\mathbf{b}}_n)]_{\mathcal{B}} \right]$$

Theorem 8 ch.5: page 291 (4th ed.)

Diagonal Matrix Representation: Suppose $A = PDP^{-1}$, where D is diagonal $n \times n$ matrix. If $\mathcal{B}$ is a basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\vec{\mathbf{x}} \rightarrow A\,\vec{\mathbf{x}}$.

Section 6.1 Inner Product, Length and Orthogonality:

Inner product: page 330 (4th ed.)

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are vectors in $\mathbb{R}^n$ then $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}}^T \vec{\mathbf{v}}$ is called the **inner product** of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, or the **dot product** of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Note:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = v_1u_1 + v_2u_2 + \ldots + u_nv_n$$

Theorem 1 ch.6: page 331 (4th ed.)

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then,

a) $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$

b) $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{w}} + \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$

c) $(c\,\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\,\vec{\mathbf{v}})$

d) $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} \geq 0$ and $\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = 0$ if and only if $\vec{\mathbf{u}} = \vec{\mathbf{0}}$

Length (norm) of a vector: page 331 (4th ed.)

The **length** or **norm** of a vector $\vec{\mathbf{v}}$ is the non negative scalar $||\,\vec{\mathbf{v}}\,||$ defined by,

$$||\,\vec{\mathbf{v}}\,|| = \sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}} = \sqrt{v_1^2 + v_2^2 + \ldots v_n^2} \quad \text{and} \quad ||\,\vec{\mathbf{v}}\,||^2 = \vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$$

Note that $||c\,\vec{\mathbf{v}}\,|| = |c|\;||\,\vec{\mathbf{v}}\,||$ for scalars c.

Unit vector and normalizing: page 332 (4th ed.)

A vector with norm 1 is called a **unit vector**. If $\vec{\mathbf{v}}$ is a nonzero vector then the vector $\dfrac{\vec{\mathbf{v}}}{||\,\vec{\mathbf{v}}\,||}$ is a unit vector which has the same "direction as $\vec{\mathbf{v}}$, the process of multiplying the vector $\vec{\mathbf{v}}$ by $\frac{1}{||\vec{\mathbf{v}}||}$ is called **normalizing** $\vec{\mathbf{v}}$.

Distance: page 333 (4th ed.)

The **distance between two vectors** $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$, written $\text{dist}(\vec{\mathbf{v}}, \vec{\mathbf{u}})$ is the length of the vectors $\vec{\mathbf{u}} - \vec{\mathbf{v}}$. That is

$$\text{dist}(\vec{\mathbf{v}}, \vec{\mathbf{u}}) = ||\,\vec{\mathbf{v}} - \vec{\mathbf{u}}\,||$$

Orthogonal vectors: page 334 (4th ed.)

Two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbb{R}^n$ are orthogonal to each other if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$

Theorem 2 ch.6: page 334 (4th ed.)

The Pythagorean Theorem: Two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal if and only if

$$||\,\vec{\mathbf{u}} + \vec{\mathbf{v}}\,||^2 = ||\,\vec{\mathbf{u}}\,||^2 + ||\,\vec{\mathbf{v}}\,||^2$$

Orthogonal Complement: page 334 (4th ed.)

If W is a vector subspace of $\mathbb{R}^n$, then a vector $\vec{\mathbf{u}}$ is said to be orthogonal to W if $\vec{\mathbf{u}}$ is orthogonal to every vector in W, I will denote this by $\vec{\mathbf{u}} \perp W$ (this is not notation in the book). Then

$$W^{\perp} = \left\{ \vec{\mathbf{u}} \;\mid\; \vec{\mathbf{u}} \perp W \right\}$$

is called the **orthogonal complement** of W and is a vector subspace of $\mathbb{R}^n$.

Theorem 3 ch.6: page 335 (4th ed.)

Let A be an $m \times n$ matrix, then,

$$(\text{Row}(A))^{\perp} = \text{Nul}(A) \quad \text{and} \quad (\text{Col}(A))^{\perp} = \text{Nul}(A^T)$$

Fact: page 335 (4th ed.)

For vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ (in $\mathbb{R}^2$ or $\mathbb{R}^3$), if θ is the vector between $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = ||\,\vec{\mathbf{u}}\,||\;||\,\vec{\mathbf{v}}\,||\cos(\theta)$$

Section 6.2 Orthogonal sets:

Orthogonal set: page 238 (4th ed.)

A set of vectors $\left\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_p\right\}$ is said to be an **orthogonal set**, if each pair of vectors is orthogonal if the vectors are also unit vectors (lenght 1) then the set is called an orthonormal set.

Theorem 4 ch.6: page 238 (4th ed.)

If a set $S = \left\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_p\right\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence a basis for the subspace spanned by S.

Orthogonal basis: page 339 (4th ed.)

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set. An **orthonormal basis** is an orthogonal basis where each of the basis elements are length 1.

Theorem 5 ch.6 : page 339 (4th ed.)

Let $\left\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_p\right\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\vec{\mathbf{y}}$ in W, the weights in the linear combination

$$\vec{\mathbf{y}} = c_1 \vec{\mathbf{u}}_1 + \ldots + c_p \vec{\mathbf{u}}_p \quad \text{are given by} \quad c_j = \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_i}$$

that is

$$\vec{\mathbf{y}} = \sum_{i=1}^{p} \left(\frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_i} \right) \vec{\mathbf{u}}_i$$

Orthogonal Projection: page 340 (4th ed.)

Given a vector $\vec{\mathbf{y}}$ and a line L defined by the vector $\vec{\mathbf{u}}$ (all multiples of the vector),

$$\hat{\mathbf{y}} = \mathrm{proj}_L \vec{\mathbf{y}} = \left(\frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_i} \right) \vec{\mathbf{u}}_i$$

Is called the **orthogonal projection of $\vec{\mathbf{y}}$ onto** L (or $\vec{\mathbf{u}}$). Read the text for a more intuitive explanation.

Theorem 6 ch.6: page 334 (4th ed.)

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Theorem 7 ch.6: page 334 (4th ed.)

Let U be a matrix with orthonormal columns, and let $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ be vectors in $\mathbb{R}^n$. Then,

1. $||U \vec{\mathbf{x}}\, || = || \vec{\mathbf{x}}\, ||$
2. $(U \vec{\mathbf{x}}) \cdot (U \vec{\mathbf{y}}) = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$
3. $(U \vec{\mathbf{x}}) \cdot (U \vec{\mathbf{y}}) = 0$ if and only if $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = 0$

Section 6.2 Orthogonal Projections:

Theorem 8 ch.6: page 348 (4th ed.)

The Orthogonal Decomposition Theorem: Let W be a subspace of $\mathbb{R}^n$. then each $\vec{\mathbf{y}}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$y = \widehat{\mathbf{y}} + \vec{\mathbf{z}}$$

where $\widehat{\mathbf{y}} \in W$ and $\vec{\mathbf{z}} \in W^{\perp}$. Furthermore if $\left\{\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_p\right\}$ is an orthogonal basis of W then

$$\widehat{\mathbf{y}} = \text{proj}_W \ \vec{\mathbf{y}} = \sum_{i=1}^{p} \left(\frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \cdot \vec{\mathbf{u}}_i} \right) \vec{\mathbf{u}}_i$$

and $\vec{\mathbf{z}} = \vec{\mathbf{y}} - \widehat{\mathbf{y}}$.

Note, if $\vec{\mathbf{y}}$ is in W then $\vec{\mathbf{y}} = \text{proj}_W \ \vec{\mathbf{y}}$.

Theorem 9 ch.6: page 350 (4th ed.)

The Best Approximation Theorem Let W be a subspace of $\mathbb{R}^n$, let $\vec{\mathbf{y}}$ be any vector in $\mathbb{R}^n$, then $\widehat{\mathbf{y}}$ is the closest point in W to $\vec{\mathbf{y}}$, that is

$$|| \vec{\mathbf{y}} - \widehat{\mathbf{y}} || \leq || \vec{\mathbf{y}} - \vec{\mathbf{v}} || \quad \text{for all } \vec{\mathbf{v}} \text{ in } W$$

Theorem 10 ch.6: page 351 (4th ed.)

If $\left\{\vec{\mathbf{u}}_1, \ldots \vec{\mathbf{u}}_p\right\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W \ \vec{\mathbf{y}} = \sum_{i=1}^{p} \left(\vec{\mathbf{y}} \cdot \vec{\mathbf{u}}_i \right) \vec{\mathbf{u}}_i$$

If $U = \left[\vec{\mathbf{u}}_1, \ldots \vec{\mathbf{u}}_p\right]$ then,

$$\text{proj}_W \ \vec{\mathbf{y}} = UU^T \ \vec{\mathbf{y}} \qquad \text{for all } \vec{\mathbf{y}} \text{ in } \mathbb{R}^n.$$

Section 6.4 The Gram-Schmidt Process:

Theorem 11 ch.6: page 355 (4th ed.)

The Gram-Schmit Process: Given a basis $\left\{\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_2\right\}$ for a non zero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\vec{\mathbf{v}}_1 &= \vec{\mathbf{x}}_1 \\
\vec{\mathbf{v}}_2 &= \vec{\mathbf{x}}_2 - \left(\frac{\vec{\mathbf{x}}_2 \cdot \vec{\mathbf{v}}_1}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1}\right) \vec{\mathbf{v}}_1 \\
\vec{\mathbf{v}}_3 &= \vec{\mathbf{x}}_3 - \left(\frac{\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{v}}_1}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1}\right) - \left(\frac{\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{v}}_2}{\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2}\right) \vec{\mathbf{v}}_2 \\
&\vdots \\
\vec{\mathbf{v}}_p &= \vec{\mathbf{x}}_p - \left(\frac{\vec{\mathbf{x}}_p \cdot \vec{\mathbf{v}}_1}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1}\right) \vec{\mathbf{v}}_p - \left(\frac{\vec{\mathbf{x}}_p \cdot \vec{\mathbf{v}}_2}{\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2}\right) \vec{\mathbf{v}}_2 - \ldots - \left(\frac{\vec{\mathbf{x}}_p \cdot \vec{\mathbf{v}}_{p-1}}{\vec{\mathbf{v}}_{p-1} \cdot \vec{\mathbf{v}}_{p-1}}\right) \vec{\mathbf{v}}_{p-1}
\end{aligned}$$

Then $\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\right\}$ is an orthogonal basis for W, In addition ,

$$\operatorname{Span}\left(\left\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k\right\}\right) = \operatorname{Span}\left(\left\{\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k\right\}\right) \quad \text{for } 1 \leq k \leq p$$

Section 6.5 The Least-squares problem:

Least squares solution: page 360 (4th ed.)

If A is an $m \times n$ matrix and $\vec{\mathbf{b}}$ id in $\mathbb{R}^n$, **a least-squares solution** of $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is an $\widehat{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$\|\,\vec{\mathbf{b}} - A\widehat{\mathbf{x}}\| \leq \|\,\vec{\mathbf{b}} - A\,\vec{\mathbf{x}}\,\|$$

for all $\vec{\mathbf{x}}$ in $\mathbb{R}^n$.

Normal equations: page 361 (4th ed.)

The matrix equation $A^T A\,\vec{\mathbf{x}} = A^T\,\vec{\mathbf{b}}$ represents a system of linear equations called the **normal equations** for $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$. A solution to these equations is often called $\widehat{\mathbf{x}}$.

Theorem 13 ch.6 page 361 (4th ed.)

The set of least squares solutions to $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$ coincides with the nonempty set of solutions to the normal equations $A^T A\,\vec{\mathbf{x}} = A^T\,\vec{\mathbf{b}}$.