Part II, assignment 2

● Graded

Student

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Total Points

9.9 / 10 pts

Question 1

10.2

2 / 2 pts

✔ **– 0 pts** Correct

Question 2

10.5

2 / 2 pts

✔ **– 0 pts** Correct

– 0 pts When showing a subset of a group forms a subgroup you do not need to show associativity.

– 1 pt You need to show that the diagonal is a subgroup of $G \times G$

Question 3

10.7

2 / 2 pts

✔ **– 0 pts** Correct

– 0.5 pts Good idea but issues with execution

– 1.5 pts Incorrectly claimed that some of the groups are isomorphic

– 1 pt Insufficient explanation

Question 4

11.2

2 / 2 pts

✔ **– 0 pts** Correct

– 0.5 pts Issues with reasoning

Question 5

11.9

1.9 / 2 pts

✔ **– 0 pts** Correct

– 1 pt You did not show that there is an element with order the lcm of **all** elements.

– 0.5 pts Made a logical jump

– 1 pt Significant logic issues

– 0.1 pts You should describe how the inductive step would work even if you do not repeat every technical step.

Question 6

11.12 - Don't do this question now

0 / 0 pts

– 0 pts Submitted

Questions assigned to the following page: 1 and 2

10.2) $\mathbb{Z}$ is cyclic but $\mathbb{Z}\times\mathbb{Z}$ isn't cyclic

To see $\mathbb{Z}\times\mathbb{Z}$ isn't cyclic, suppose it is then $\exists (m,n)$ s.t $\mathbb{Z}\times\mathbb{Z} = \{k(m,n) \mid k\in\mathbb{Z}\}$

Now, to get $\mathbb{Z}$, that means $m,n = \pm 1$

So, an element like $(0,1)$ isn't in this set, so $\mathbb{Z}\times\mathbb{Z}$ isn't cyclic so $\mathbb{Z}$ not isomorphic to $\mathbb{Z}\times\mathbb{Z}$

10.5)

Closed: Let $D = \{(x,x) \mid x\in G\}$ then for $(x,x), (y,y) \in D$ $(x,x)(y,y) = (xy,xy)$

so D is closed

Associative: Clearly associative

Identity: Let $(x,x)\in D$ clearly $(e,e)\in G\times G$

So $(e,e)(x,x) = (ex, ex) = (x,x)$

$(x,x)(e,e) = (xe, xe) = (x,x)$

so $(e,e)\in G\times G$

Inverse: Let $(x,x)\in D$ then $(x,x)^{-1} = (x^{-1},x^{-1})$

To show: $(x,x)(x^{-1},x^{-1}) = (xx^{-1}, xx^{-1}) = (e,e)$

$(x^{-1},x^{-1})(x,x) = (x^{-1}x, x^{-1}x) = (e,e)$

So D forms a subgroup of $G\times G$

To show D isomorphic to G define

$\phi: D \to G$ Show this is isomorphism

$(x,x) \to x$

Injective: $\phi(x,x) = \phi(y,y) \Rightarrow x=y$ so ϕ is injective

Questions assigned to the following page: 2 and 3

Surjective: Let $y \in G$ show $\exists (x,x) \in G$ s.t. $\phi(x,x) = y$ clearly pick $(x,x) = (y,y)$ and $\phi(y,y) = y$. and $(y,y) \in D$
So surjective.

Respects group property) Let $(x,x)\ (y,y) \in D$
$\phi((x,x)(y,y)) = \phi((xy, xy)) = xy$

$\phi((x,x))\phi((y,y)) = xy$ so
this respects group relation.
Therefore, ϕ is an isomorphism, and D is isomorphic to G.

10.7) None of the groups are isomorphic
The only Abelian groups are $\mathbb{Z}_{24}$ and $\mathbb{Z}_{12} \times \mathbb{Z}_2$. $\mathbb{Z}_{24}$ is cyclic, and $\mathbb{Z}_{12} \times \mathbb{Z}_2$ isn't as $\gcd(2, 12) = 2 > 1$

So not isomorphic, and $\mathbb{Z}_{24}$ and $\mathbb{Z}_{12} \times \mathbb{Z}_2$ are in own isomorphism classes.
To show rest aren't isomorphic can check # of elements w/ order 2 as if isomorphic must have same amount.

From earlier HW, we know D_n for even n has $n+1$ elements of order 2.

So D_{12} has 13 elements of order 2

D_4 has 5 elems of order 2, $\mathbb{Z}_3$ has 0 elements of order 2, so $D_4 \times \mathbb{Z}_3$ has 5 elements of order 2.

Questions assigned to the following page: 3 and 4

D_6 has 7 elements of order 2, and so has $\mathbb{Z}_2 \times D_6$ has 15 elements of order 2 as take: $(0, d)$ $(1, d)$ $(1, 0)$ where $d \in D_6$ w/ order 2.

A_4 has 3 elems w/ order 2 $(12)(34)$ $(13)(24)$ $(14)(23)$ and so $A_4 \times \mathbb{Z}_2$ has 7 elements of order 2 as take: $(0, a)$ $(1, a)$ $(1, 0)$ where $a \in A_4$ w/ order 2.

S_4 has $\binom{4}{2}$ elements of order 2 + the 3 transpositions listed in A_4, and $\binom{4}{2} + 3 = 9$.

As they all have different numbers of elements of order 2 then $D_4 \times \mathbb{Z}_3$, D_{12}, $A_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times D_6$, and S_4 aren't isomorphic to each other.

Therefore, in conclusion none of the groups listed are isomorphic to each other $\square$

11.2)

$\Rightarrow$ $g_1 H = g_2 H$ and as $g_2 \cdot e \in g_2 H$ so $\exists h \in H$ s.t. $g_2 = g_1 h$ so $g_1^{-1} g_2 = h$ and $g_1^{-1} g_2 \in H$

$\Leftarrow$ If $g_1^{-1} g_2 \in H$, $\exists h \in H$ s.t. $g_1^{-1} g_2 = h$ So $g_2 = g_1 h \in g_1 H$. This implies that $g_1 H$ and $g_2 H$ have a common element. So, that means $g_1 H = g_2 H$ $\square$

11. a)

Let $x, y \in G$ assume order x, y are m, n respectively and that m, n are coprime. So $\text{lcm}(m,n) = mn$

$$(xy)^{mn} = x^{mn} y^{mn} = (x^m)^n (y^n)^m = e$$

So we can show $|xy| = mn$. If $|xy| < mn$, $\exists r \in \mathbb{N}$ s.t. $0 < r \leq mn$ and $(xy)^r = 1$.

However, if an r exists then it must divide mn and have m, n as multiples. However, as mn coprime, the smallest number that does this is mn so $|xy| = mn$

So, now in general given m, n not coprime so they have prime factorizations

p_i prime, $m = \prod_{i=1} p_i^{\alpha_i}$ $n = \prod p_i^{\beta_i}$

Now, define $m' = \prod_{\beta_i \leq \alpha_i} p_i^{\alpha_i}$, $n' = \prod_{\alpha_i \geq \beta_\alpha} p_i^{\beta_i}$

So m', n' are coprime and define

$$x' = x^{m/m'} \quad y' = y^{n/n'}$$ then it follows

$|x'| = m'$ and $|y'| = n'$ as suppose $|x'| = k$ then $(x')^k = e$

$$x^{mk/km'} = (x^m)^{k/m'} = e$$

So $|x'| = m'$ and by symmetry $|y'| = n'$

As m', n' are coprime, then by above $|x'y'| = m'n'$ as $\text{lcm}(m', n') = m'n'$

Now, this shows only for 2 elements

Question assigned to the following page: 5

However, this construction is generalizable to n elements through an inductive argument, as n is finite and no part of the proof relies on having only 2 elements.
However, I exclude as just gets very messy with many variables