MATH 135 | Final Exam | December 15, 2022

Prof. Hasselblatt and Tu | (100 points) | 8:30–10:30 a.m..

• **Carefully PRINT your full name:** SOLUTIONS

• **CIRCLE your section:** Section 1 (Tu) Section 2 (Tu) Section 3 (Hasselblatt)

Instructions: No books, notes, calculators, internet, or external help from any person or device are allowed. Except in the true-false questions or when instructed otherwise, justify all of your steps.
Write only in the space provided and do not attach any extra page.

• **Please sign the following pledge:**
I pledge that in this exam I have neither given nor received assistance or cheated in any other way.

Signature: ______________________________

1. (10 points) Circle either True or False. You do not need to justify your choice.

   (a) **True** / **False:** A subset of a metric space is sequentially compact if and only if it is closed and bounded. SEE 5(a) BELOW

   (b) **True** / **False:** A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfies the intermediate-value theorem. BECAUSE $\mathbb{R}^n$ IS CONNECTED

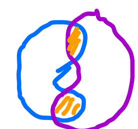
   (c) **True** / **False:** If $f\colon D \to \mathbb{R}$ is continuous, where $D \subset \mathbb{R}^2$ is connected, then its graph

   $$\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D \text{ and } z = f(x,y)\}$$

   is connected. IF $F(x,y,z) = (x,y,f(z))$ ON $\mathbb{R}^2$, THEN F IS CONTINUOUS, AND THE GRAPH IS $F(D)$.

   (d) **True** / **False:** Every connected set is pathwise connected. EXAMPLE: THE TOPOLOGIST'S SINE CURVE

   (e) **True** / **False:** The intersection of two connected sets in $\mathbb{R}^n$ is connected. EXAMPLE:

2. (10 points) Let A be a subset of $\mathbb{R}^n$ and $F\colon A \to \mathbb{R}^m$ a continuous mapping. For each of the following properties P, decide whether it is true that "if A is P, then $F(A)$ is P." Write Yes or No; no justification is necessary.

   (a) NO open CONSTANT F

   (b) NO closed $n=1$, $A=\mathbb{R}$, $F(x)=e^x$ OR $\frac{1}{x^2+1}$

   (c) YES sequentially compact

   (d) YES pathwise-connected

   (e) YES connected

   THEOREMS

Carefully PRINT your full name:

3. (15 points) No justification is needed in this question.

(a) (3 pts) Let $\{p_k\}$ be a sequence in a metric space X with metric d.

(i) Define what it means for the sequence $\{p_k\}$ to be a *Cauchy sequence* in X: $\{p_k\}$ is a *Cauchy sequence* if

$\forall \varepsilon > 0 \ \exists K \in \mathbb{N} \ \forall k, l \geq K \quad d(p_k, p_l) < \varepsilon$

ALTERNATE NOTATION:

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ SUCH THAT $\forall k, l \geq K$, $d(p_k, p_l) < \varepsilon$

(ii) Define what it means for the metric space X to be *complete*: X is *complete* if

EVERY CAUCHY SEQUENCE IN X CONVERGES (TO A POINT IN X)

(b) (4 pts) State the definition of a separation:
Let X be a metric space. Two open subsets U, V of X **separate** X if

$U \neq \emptyset$

$V \neq \emptyset$

$U \cap V = \emptyset$

$U \cup V = X$

(c) (2 pts) Negate the statement: For all $x \neq 3$, if $|x-3| < \delta$, then $|f(x) - f(3)| < \epsilon$.

THERE EXISTS AN $x \neq 3$ SUCH THAT $|x-3| < \delta$ AND $|f(x)-f(3)| \geq \epsilon$.

(d) (3 pts) Give an example which shows that the union of infinitely many closed sets need not be closed.

$$\bigcup_{x \in (0,1)} \{x\} = (0,1)$$

(IN $\mathbb{R}$ WITH THE USUAL METRIC)

[INDEED, ANY SET IS THE UNION OF ITS ELEMENTS.]

(e) (3 pts) Let X and Y be metric spaces. Define a *contraction* $T\colon X \to Y$ and state the *contraction mapping principle* including all of its hypotheses and conclusions.

(i) A map $T\colon X \to Y$ is a contraction provided that

$\exists c < 1 \ \forall x, y \in X: \quad d(f(x), f(y)) \leq c\, d(x,y)$

ALTERNATE NOTATION:

$\exists c < 1$ SUCH THAT $\forall x, y \in X$, $d(f(x), f(y)) \leq c\, d(x,y)$

(ii) The contraction mapping principle:

IF (X,d) IS A COMPLETE METRIC SPACE AND $T: X \to X$ IS A CONTRACTION, THEN T HAS A UNIQUE FIXED POINT x_T.

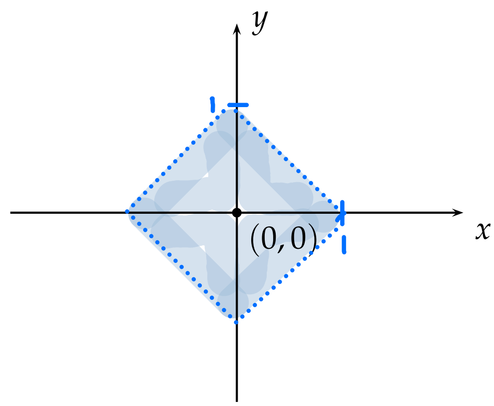
[AND $T^n(x) \longrightarrow x_T$ EXPONENTIALLY FOR ALL $x \in X$.]

Carefully PRINT your full name:

4. (11 points) No justification is needed in this question.

   (a) (3 pts) Sketch the open unit ball $B_1\big((0,0)\big)$ in $\mathbb{R}^2$ with the taxicab metric defined by

$$d\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| + |y_1 - y_2|.$$



   (b) (8 pts) Let $X = \{p_1, p_2, p_3\}$ be a set of three points with the discrete metric. Determine whether each of the following sets is open, closed, sequentially compact, or connected. Check the appropriate boxes.

      i. $A = \{p_1\}$.

      ii. $B = \{p_1, p_2\}$.

	Open	Closed	Sequentially Compact	Connected
A	✓	✓	✓	✓
B	✓	✓	✓	(NO)

ALL SETS ARE OPEN, SO ALL SETS ARE CLOSED

FINITE SETS ALWAYS ARE

$(\{p_1\}\ \{p_2\})$ IS A SEPARATION

MANY REASONS: • INTERMEDIATE-VALUE PROPERTY

• NO SEPARATION • PATH CONNECTED (BY CONSTANT PATH)

5. (9 points) For each phenomenon below, either give an example with a brief explanation or explain why none exists.

(a) A metric space with a closed and bounded set that is not sequentially compact.

EXAMPLES: ① THE CLOSED BOUNDED SET $\underbrace{\overline{B}_1(0)}_{=\{f\in C([0,1]) \mid \forall x\in[0,1],\ |f(x)|\le 1\}} \subset C([0,1])$.

, WHICH IS CLOSED (PREIMAGE OF $[0,1]$ UNDER $\psi(f)=d(0,f)$) AND BOUNDED (CONTAINED IN $B_2(0)$).

IT CONTAINS THE SEQUENCE $\{f_k\}$ FROM PART c), WHICH HAS NO CONVERGENT SUBSEQUENCE.

② AN INFINITE SET WITH THE DISCRETE METRIC: AN INJECTIVE SEQUENCE HAS NO CONVERGENT SUBSEQUENCE.

(b) A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ which is not identically zero such that $f(x) = 0$ for all $x \in \mathbb{Q}$. (A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *identically zero* if $f(x) = 0$ for all $x \in \mathbb{R}$.)

IMPOSSIBLE: BY ASSUMPTION, $\mathbb{Q} \subset f^{-1}(\{0\})$, WHICH IS CLOSED BY CONTINUITY, SO

$\mathbb{R} = \mathrm{cl}\,\mathbb{Q} \subset \mathrm{cl}\, f^{-1}(\{0\}) = f^{-1}(\{0\}) \subset \mathbb{R}$,

I.E., $f^{-1}(\{0\}) = \mathbb{R}$, SO f IS IDENTICALLY ZERO.

OR: FOR $x\in\mathbb{R}$ TAKE $x_n \in \mathbb{Q}$ WITH $\underbrace{x_n \to x}_{\text{SEQUENTIAL DENSITY OF } \mathbb{Q}}$ TO GET $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$.

(c) A sequence $\{f_k\}$ of continuous functions on $[0,1]$ which converges pointwise, but not uniformly, to a function f on $[0,1]$.

EXAMPLE: $f_k(x) = x^k \longrightarrow \begin{cases} 0 & x<1 \\ 1 & x=1 \end{cases}$ POINTWISE.

NOT UNIFORMLY, SINCE THE LIMIT IS DISCONTINUOUS

[NOTE: THE SAME HOLDS FOR ANY SUBSEQUENCE, SO NO SUBSEQUENCE CONVERGES UNIFORMLY.]

Carefully PRINT your full name:

6. (10 points) In this question you may use general theorems about uniform convergence, but you may not quote theorems about power series.

    (a) Prove that the function $f\colon (-1,1) \to \mathbb{R}$ defined by
    $$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$
    converges uniformly on the closed interval $[-r, r]$, where r is any positive number less than 1.

    (b) Prove that f defines a continuous function on $(-1,1)$.

(a) WHEN $|x| < r$, $\sum_{k=0}^{K} \frac{|x|^{2k+1}}{2k+1} \leq \sum_{k=0}^{K} r^{2k+1} \leq \frac{r}{1-r}$ FOR ANY $K \in \mathbb{N}$, (GEOMETRIC SERIES)

SO THE WEIERSTRASS M-TEST APPLIES.

(b) [NOTE THAT WE DO NOT KNOW UNIFORM CONVERGENCE ON $(-1,1)$.]
WE USE THAT f IS CONTINUOUS ON $[-r,r]$ FOR ANY $r<1$ [THE UNIFORM LIMIT OF CONTINUOUS FUNCTIONS IS CONTINUOUS.]
TO SEE THAT f IS CONTINUOUS ON $(-1,1)$ EXPLANATION

FOR $x \in (-1,1)$, TAKE $r = \frac{|x|+1}{2} < 1$ AND NOTE THAT

$x \in [-r, r]$ — WHERE f IS CONTINUOUS BY (a).

SO f IS CONTINUOUS AT x. PROOF

7. (15 points) Let $C([0,1],\mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$ with the uniform metric. For each positive integer k, define $f_k(x) = x - x^k$ and $g_k(x) = \sin(x/k)$ for $0 \le x \le 1$.

(a) Define what it means for a sequence $\{f_k\}$ of functions from $A \subset \mathbb{R}$ to $\mathbb{R}$ to *converge uniformly* to a function $f\colon A \to \mathbb{R}$ on A.

(b) Find the pointwise limit of the sequence $\{f_k\}$ on $[0,1]$.

(c) Find the pointwise limit of the sequence $\{g_k\}$ on $[0,1]$.

(d) Is the sequence $\{f_k\}$ a Cauchy sequence in the metric space $C([0,1],\mathbb{R})$?

(e) Is the sequence $\{g_k\}$ a Cauchy sequence in the metric space $C([0,1],\mathbb{R})$?

(a) $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall k \geq K, x \in A\quad |f_k(x) - f(x)| < \varepsilon.$

OR: $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ SUCH THAT $\forall k \geq K, x \in A, \quad |f_k(x) - f(x)| < \varepsilon.$

(b) $\underbrace{x - x^k}_{\to 0 \text{ WHEN } |x|<1} \begin{cases} = 0 & \text{WHEN } x = 1 \\ \xrightarrow[k\to\infty]{} x & \text{WHEN } x < 1 \end{cases}$ [DISCONTINUOUS!]

(c) $x/k \xrightarrow[k\to\infty]{} 0$ SO $\sin(x/k) \xrightarrow[k\to\infty]{} 0$ BY CONTINUITY.

(d) NO, BECAUSE THAT WOULD IMPLY UNIFORM CONVERGENCE AND HENCE CONTINUITY OF THE LIMIT.

(e) YES, BECAUSE CONVERGENCE IS UNIFORM ON $[0,1]$:

$$\left|\sin\tfrac{x}{k} - 0\right| = \sin\tfrac{x}{k} \leq \tfrac{x}{k} \leq \underbrace{\tfrac{1}{k}}_{\text{INDEPENDENT OF } x} \longrightarrow 0.$$

[MAKE SURE YOU CAN JUSTIFY EACH OF THESE!]

THIS IMPLIES "UNIFORMLY CAUCHY", WHICH IN TURN IS EQUIVALENT TO "CAUCHY IN $C([0,1],\mathbb{R})$" BY DEFINITION OF THE LATTER.

Carefully PRINT your full name:

8. (10 points)

(a) Let f and g be functions in $C([0,1],\mathbb{R})$. Define the distance $d(f,g)$ in this metric space.

(b) Let $A = \{f \in C([0,1],\mathbb{R}) \mid \forall x \in [0,1],\ f(x) \leq 3\}$. Prove that A is closed in $C([0,1],\mathbb{R})$.

(a) $d(f,g) = \max\limits_{x \in [0,1]} |f(x) - g(x)|$

(b) SUPPOSE A SEQUENCE $\{f_k\}$ IN A CONVERGES (UNIFORMLY) TO $f \in C([0,1],\mathbb{R})$.

THEN FOR ALL $x \in [0,1]$

- $f_k(x) \leq 3$, $[f_k \in A]$
- $f_k(x) \longrightarrow f(x)$,

HENCE • $f(x) \leq 3$.

THEREFORE $f \in A$.

ALTERNATIVE PROOF: PROVE (CAREFULLY!) THAT $\varphi: C([0,1],\mathbb{R}) \to \mathbb{R}$, $\varphi(f) = \max f$ IS (LIPSCHITZ-) CONTINUOUS, THEN USE THAT

[$\max g = g(x_{max}) \leq f(x_{max}) + d(g,f) \leq \max f + d(g,f)$ AND VICE VERSA.]

$A = \varphi^{-1}(\underbrace{(-\infty, 3]}_{\text{CLOSED}})$.

ALTERNATIVE PROOF: PROVE THAT THE COMPLEMENT $\{f \in C([0,1],\mathbb{R}) \mid \exists x \in [0,1] \text{ SUCH THAT } f(x) > 3\}$ IS OPEN.

[IF $f(x) > 3$ AND $d(g,f) < \delta := f(x) - 3 > 0$, THEN $g(x) > f(x) - \delta = 3$.]

9. (10 points) Let X be a sequentially compact metric space. Prove that a closed subset K of X is also sequentially compact.

IF $\{x_n\}$ IS A SEQUENCE IN K

THEN $\{x_n\}$ IS A SEQUENCE IN X, AND BY SEQUENTIAL COMPACTNESS OF X

HENCE HAS A CONVERGENT SUBSEQUENCE

$$x_{k_i} \xrightarrow[i \to \infty]{} x \in X.$$

SINCE K IS CLOSED (AND x_{k_i} IS A CONVERGENT SEQUENCE IN K),

$x \in K$. ∎

(End of Final Exam)