

1. MATH 65, REVIEW SHEET FOR EXAM 2 FALL 2021.

This review sheet is provided as a help tool when reviewing. It is a complement to and not a substitute for reviewing notes, homework problems and quizzes. Try to solve the problems without looking at the answers and ask for help only after you have devoted some time to think about it yourself. Please keep in mind that in the test, your explanations will be given at least as much credit as a numerical answer.

Question 1. The bookstore has 3 copies of the Algebra textbook, 10 copies of the Calculus book and 5 copies of the Discrete Math textbook.

- (a) In how many ways can these books be placed in a shelf if they are all used copies, very different from each other?
- (b) In how many ways can these books be placed in a shelf if they are all used copies, very different from each other and you want to keep the books for the same course together?
- (c) In how many ways can these books be placed in a shelf if they are all new copies and so the books for each given course are indistinguishable?
- (d) In how many ways can these books be placed in a shelf if they are all new copies and so the books for each given course are indistinguishable and you want to keep the books for the same course together?

Question 2. How many (meaningless) words can you form with the letters of "committee on committees" (form just a single word, no spaces).

Question 3. Write the coefficient of x^9 in $(2x - 3)^{20}$.

Question 4. Give a combinatorial proof of the following summation identities

- (a) $\sum_{i=1}^n (i - 1) = \binom{n}{2}$. Hint: when choosing two elements in $\{1, 2, 3, \dots, n\}$, distinguish according to the value of the largest of the two.
- (b) $\sum_{i=1}^n (i - 1)(n - i) = \binom{n}{3}$. Hint: when choosing three elements in $\{1, 2, 3, \dots, n\}$, distinguish according to the value of the middle one..

Question 5. A store carries 5 different types of cookie mix (sugar, chocolate, chocolate chunk, peanut, and cheese swirl). A couple sends their children to the store. What is the number of packages the parents should ask their kids to bring back if they want to make sure that at least 3 packages are of the same kind (and therefore can be mixed together)?

Question 6. Let n be a positive integer and x_0, \dots, x_n be integers. Show that there exists at least one pair x_i, x_j such that $x_i - x_j$ is divisible by n . Hint: Think about the possible remainders of x_i for the division by n .

Question 7. At 5 p.m., there are only 4 kinds of donuts left in the shop: jelly, coconut, plain and chocolate.

- (a) In how many ways can you choose a dozen donuts so that at least 4 are chocolate AND at most 3 are coconut?
- (b) In how many ways can you choose 12 donuts so that at least 4 are chocolate, at most 3 are coconut AND at most one is jelly?

Question 8. In how many ways can you place 50 books on 3 shelves if the first shelf can hold at most 10

- (a) if all 50 books are identical.

(b) If all 50 books are different and you care about the order of the books on the shelves.

Question 9. Consider the following relation R on the set $A = \{a, b, c, d\}$.

$$R = \{(a, a), (a, b), (a, c), (c, a), (c, b), (c, c), (b, c), (d, d)\}.$$

Determine whether the relation is reflexive, symmetric, antisymmetric, and/or transitive. Give short explanations with your answers.

Question 10. Let $A = \{1, 2, 3\}$. Give the partition $A = \{1\} \cup \{2, 3\}$, determine the equivalence relation associated to this partition.

Question 11. Prove or disprove each of the following statements

- (a) Consider the assignment $f : \mathbb{Q} \rightarrow \mathbb{Z}$ given by $f(\frac{a}{b}) = a+b$. Is this a well defined function?
- (b) Consider the assignment $g : \mathbb{Q} \rightarrow \mathbb{Z}$ given by $g(\frac{a}{b}) = \frac{3a}{5b}$. Is this a well defined function?
- (c) Consider the assignment $g : \mathbb{Q} \rightarrow \mathbb{Q}$ given by $g(\frac{a}{b}) = \frac{3a}{5b}$. Is this a well defined function?

Question 12. Prove or disprove each of the following statements

- (a) If a poset has a maximal element, then it has to be unique.
- (b) If a poset has only one maximal element, then it is a maximum.
- (c) It is possible for a poset to have an element that is both maximal and minimal but it is not maximum or minimum.
- (d) If x and y are incomparable then neither is a maximum.
- (e) If x and y are both maximal and different, then they are incomparable.

Question 13. A bit string is a row of digits each being 0 or 1.

- (a) When choosing a bit string of length 6 at random, what is the probability that it contains exactly three zeroes?
- (b) When choosing a bit string of length 6 at random, what is the probability that it contains exactly three zeroes and either contain the string 000 or 111(or both)?

Question 14. Three dice are thrown simultaneously.

- (a) Compute the probability that at least one of the dice shows a 3.
- (b) Compute the probability that the sum of the numbers that show on the three dice is 5.
- (c) Are the events in (a) and (b) independent?

Question 15. Three identical coins are thrown simultaneously. Consider the random variable that assigns to each possible output the number of heads H .

- (a) Write the outputs of the random variable and the probability of each of them.
- (b) Compute the expected value.
- (c) Compute the variance.