Math 135 Solutions to HW 4 L. Tu

1. (10 points) Prove that if a set S in $\mathbb{R}$ is sequentially compact, then it is bounded.

Proof. Suppose S is unbounded. Then for every $n \in \mathbb{N}$, there is an element $s_n \in S$ such that $|s_n| > n$. As $n \to \infty$, $\{s_n\}$ diverges (though not necessarily to $+\infty$, for example, $\{1, -1, 2, -2, 3, -3, \dots\}$ diverges but not to $+\infty$). If $\{s_{n_k}\}$ is any subsequence of $\{s_n\}$, then

$$|s_{n_k}| > n_k \geq k,$$

so $\{s_{n_k}\}$ also diverges. This proves that $\{s_n\}$ is a sequence in S that has no convergent subsequence. Hence, S is not sequentially compact. $\square$

2. (10 points) Prove that if a set S is sequentially compact, then it is closed.

Proof. Suppose S is not closed. Then there is a convergent sequence $s_n \in S$ such that s_n converges to a number $a \notin S$. Since every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ converges to the same limit $a \notin S$ (Proposition 2.30), S cannot be sequentially compact. (The definition of S being sequentially compact is that every sequence $s_n \in S$ has a convergent subsequence s_{n_k} that converges to a point of S.) $\square$

3. (15 points) **Square root rule**. Using the ε-N definition of the limit, prove that If $a_n \to a$, then $\lim \sqrt{a_n} = \sqrt{a}$.

Solution.

Case 1: $a \neq 0$. For $\sqrt{a_n}$ to be defined, we must assume $a_n \geq 0$.

Suppose $a_n \to a$. To prove $\lim \sqrt{a_n} = \sqrt{a}$, let $\varepsilon > 0$.

We need to find $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|\sqrt{a_n} - \sqrt{a}| < \varepsilon.$$

Scratch work:

$$|\sqrt{a_n} - \sqrt{a}| = \left|\frac{(\sqrt{a_n} - \sqrt{a})(\sqrt{a_n} + \sqrt{a})}{\sqrt{a_n} + \sqrt{a}}\right|$$

$$= \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} \quad \text{because } \sqrt{a_n} + \sqrt{a} \geq \sqrt{a}.$$

$$< \varepsilon \quad \text{iff} \quad |a_n - a| < \sqrt{a}\,\varepsilon := \varepsilon_1.$$

Let $\varepsilon_1 = \sqrt{a}\,\varepsilon > 0$. Since $a_n \to a$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|a_n - a| < \varepsilon_1 = \sqrt{a}\,\varepsilon.$$

Then

$$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} < \frac{\sqrt{a}\,\varepsilon}{\sqrt{a}} = \varepsilon.$$

This proves that $\sqrt{a_n} \to \sqrt{a}$.

Case 2: $a = 0$. Assume $a_n \to 0$, $a_n \geq 0$. To prove $\sqrt{a_n} \to 0$, let $\varepsilon > 0$.

We need to find $N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|\sqrt{a_n} - 0| = \sqrt{a_n} < \varepsilon,$$

or

$$a_n < \varepsilon^2.$$

Use $\varepsilon_1 = \varepsilon^2 > 0$ as our new ε. Since $a_n \to 0$, $\exists N \in \mathbb{N}$ s.t.

$\forall n \geq N$, $|a_n - 0| = a_n < \varepsilon_1 = \varepsilon^2$. Then $|\sqrt{a_n} - 0| < \varepsilon$.

This proves that $\sqrt{a_n} \to 0$. $\square$

4. §3.1, p.57, #6.

Define $g:\mathbb{R}\to\mathbb{R}$ by
$$g(x)=\begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational.}\end{cases}$$

At what points is the function g continuous? Justify your answer.

Let x_0 be a nonzero rational number, and $\{x_n\}$ a sequence of irrational numbers that converges to x_0, e.g. $x_n = x_0 + \frac{\sqrt{2}}{n}$. Then $g(x_n) = -x_n^2$ converges to $-x_0^2 \neq g(x_0)$. Hence, g is not continuous at any nonzero rational number.

Similarly, let x_0 be a nonzero irrational number. and $\{x_n\}$ a seq of rational numbers that converges to x_0. Then $g(x_n) = x_n^2$ converges $x_0^2 \neq g(x_0)$. Hence, g is not continuous at any nonzero irrational number.

We claim that g is continuous at $x=0$.

Suppose $x_n \to 0$. We will prove that $g(x_n)\to 0$.

Let $\varepsilon>0$. We need to find an $N\in\mathbb{N}$ s.t. $\forall\, n\geq N$,
$$|g(x_n)-g(0)| = |\pm x_n^2 - 0| = x_n^2 < \varepsilon.$$
i.e.,
$$|x_n| < \sqrt{\varepsilon}.$$
Since $x_n\to 0$, $\exists\, N$ s.t. $\forall\, n\geq N$, $|x_n - 0| < \sqrt{\varepsilon}$.

For this N, if $n\geq N$, then
$$|g(x_n)-g(0)| = |\pm x_n^2 - 0| = x_n^2 < \varepsilon.$$
Hence, $\lim_{n\to\infty} g(x_n) = g(0)$. Thus, g is continuous at $x=0$ and not continuous at all other points in $\mathbb{R}$.

#5.

(Sequential compactness) Which of the following are sequentially compact? Justify.

(a) $\mathbb{Q} \cap [0,1)$ No, because not closed.

(b) $\mathbb{Q} \cap [0,1]$ No, not closed. A sequence of rational numbers in $[0,1]$ can converge to an irrational number.

(c) $[0,1] \cup [2,3]$ Yes, closed and bounded.

(d) $\mathbb{Z} \cap [1,100]$ Yes, because closed and bounded.

(e) $[0,\infty)$ No, because not bounded.

#6. §3.1, p.58, #11.

Suppose $g: \mathbb{R} \to \mathbb{R}$ is continuous and $g(x) = 0$ if x is rational. Prove that $g(x) = 0$ for all x in $\mathbb{R}$.

Let x be any real number. By the density of the rational numbers, for every $n \in \mathbb{N}$, there is a rational number r_n in $(x, x + \frac{1}{n})$. By the sandwich lemma, $\lim_{n \to \infty} r_n = x$. So there is a sequence of rational numbers converging to x. Since g is continuous at x,

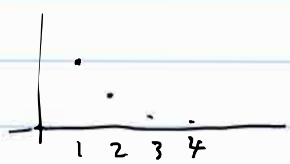
$$g(x) = \lim_{n \to \infty} g(r_n) = \lim_{n \to \infty} 0 = 0.$$

Hence, $g(x) = 0$ for all x in $\mathbb{R}$.

#7. §2.4, p.47, #4.

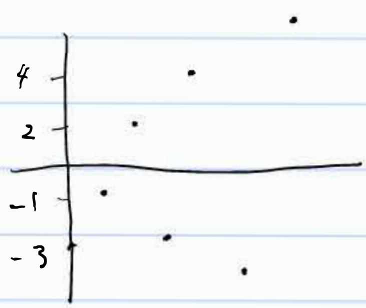
Find the peak indices.

(a) $\{\frac{1}{n}\}$



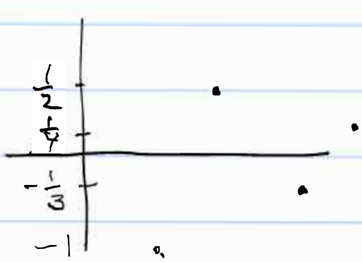
$\{1, 2, 3, 4, \dots\}$

(c) $\{(-1)^n n\}$



There are no peak indices, because there is a strictly increasing subsequence.

(d) $\left\{\frac{(-1)^n}{n}\right\}$



$\{2, 4, 6, 8, \dots\}$