Assume local truncation error is $e_i \le c h^{k+1}$ for some c and $k \ge 0$. Then, under the assumptions we discussed, the global truncation error is

$$g_i = |w_i - y_i| \le \frac{ch^k}{L}\left(e^{L(t_i-a)} - 1\right) \quad *$$

If $*$ is satisfied for ODE scheme as $h \to 0$, the scheme has order k.

Example Euler's method : $e_i \le \frac{Mh^2}{2}$

Therefore, order of Euler's method is 1.

Explicit Trapezoid method

$$w_0 = y_0$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_i + h, \underbrace{w_i + h f(t_i, w_i)}_{w_i + h f(t_i, w_i)})\right]$$

slope from left end point

slope from right end point using Euler

* Why is this called Trapezoid?

Assume $f(t, y)$ is independent of y

$$\int_{t_i}^{t_i+h} y'(t)\, dt = y(t_i+h) - y(t_i)$$

$$\int_{t_i}^{t_i+h} f(t)\, dt \approx \frac{h}{2}\left[f(t_i) + f(t_i+h)\right]$$ using Trapezoid method

* We can show that the local truncation error is $O(h^3)$. The method is order 2

Assume $(k+1)$-times differentiability of $y(t)$

$$y(t+h) = y(t) + h y'(t) + \frac{h^2}{2} y''(t) + \dots + \frac{1}{k} h^k y^{(k)}(t) + \frac{1}{k+1} h^{k+1} y^{(k+1)}(c) \qquad c \in (t, t+h)$$

$$w_0 = y_0$$

$$w_{i+1} = w_i + h f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) + \dots + \frac{h^k}{k!} f^{(k-1)}(t_i, w_i)$$

$f' \equiv$ derivative with respect to t

Example $f' = f_t(t,y) + f_y(t,y)\, f(t,y)$

Let $w_i = y_i$

$y_{i+1} - w_{i+1} = \dfrac{h^{k+1}}{(k+1)!}\, y^{(k+1)}(c)$ order k

* Although this allows for arbitrary higher orders, it is expensive as it requires computing partial derivatives

$$y_{i+1} = y_i + h\left[a f(t_i, y_i) + b\, f(t_i + \alpha h,\ y_i + \beta h f_i)\right] \quad (A)$$

$$y(t_i + h) = y(t_i) + h y'(t_i) + \frac{1}{2} h^2 y''(t_i) + \dots$$

$$= y(t_i) + h f + \frac{1}{2} h^2 (f_t + f_y f) + \dots \quad (B1)$$

* consider RHS of (A)

$$y + h\left[a f + b\, f(t_i + \alpha h,\ y + \beta h f)\right]$$

$$= y + h a f + h b (f + \alpha h f_t + \beta h f f_y) + \dots$$

$$= y + h(a+b) f + h^2 b(\alpha f_t + \beta f f_y) + O(h^3) \quad (B2)$$

NOTE $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$+ \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

Compare (B1) and (B2)

$a + b = 1$

$\alpha b = \frac{1}{2}$

$\beta b = 1/2$

$$\left.\begin{aligned} a + b &= 1 \\ \alpha b &= 1/2 \\ \beta b &= 1/2 \end{aligned}\right\} \text{ solve this}$$

(i) $a = b = \frac{1}{2}$, $\alpha = \beta = 1$ Heun

(ii) $a = 0$, $b = 1$, $\alpha = \beta = 1/2$ Midpoint

(iii) $a = \frac{1}{4}$, $b = \frac{3}{4}$, $\alpha = \beta = 2/3$

$y_{j+1} = y_j + c_1 k_1 + c_2 k_2 + c_3 k_3 + c_4 k_4$ 12 constants

$k_1 = h f(t_j, y_j)$

$k_2 = h f(t_j + \alpha_2 h,\ y_j + \beta_{21} k_1)$ RK4

$k_3 = h f(t_j + \alpha_3 h,\ y_j + \beta_{31} k_1 + \beta_{32} k_2)$

$k_4 = h f(t_j + h,\ y_j + \beta_{41} k_1 + \beta_{42} k_2 + \beta_{43} k_3)$