1. (10 points) (**Negation, inequality**)
   (a) Negate the statement:
   $\forall \varepsilon > 0,\ \exists \delta > 0$ *such that* $\forall x, y \in D$, *if* $|x-y| < \delta$, *then* $|f(x) - f(y)| < \varepsilon$.
   (b) Prove that for all $a, b \in \mathbb{R}$,
   $$|\,|a| - |b|\,| \leq |a-b|.$$
   (*Hint*: Write $a = (a-b) + b$ and apply the triangle inequality.)

Solution: (a) $\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$, $\exists\, x, y \in D$ such that $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

(To negate a quantifier statement,
$\forall$ becomes $\exists$,
$\exists$ becomes $\forall$,
comma becomes "such that",
"such that" becomes a comma,
the ending statement becomes its negation.
The negation of "if p, then q" is "p and not q.")

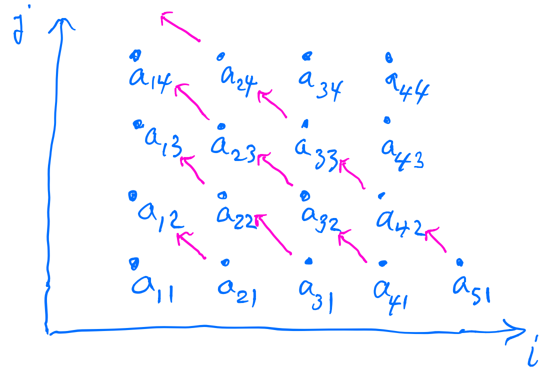
(b)
Proof. By the triangle inequality,
$$|a| = |(a-b) + b| \leq |a-b| + |b|.$$
Subtracting $|b|$ from both sides gives
$$|a| - |b| \leq |a-b|.$$
Reversing the roles of a and b, we get
$$|b| - |a| \leq |b-a| = |a-b|.$$
Therefore, $$|\,|a| - |b|\,| = \pm(|a| - |b|) \leq |a-b|. \qquad \square$$

2. Suppose that $A_1, A_2, A_3, \ldots$ are countably infinite set. Show that their union $A_1 \cup A_2 \cup A_3 \cup \cdots$ is countable.

<u>Proof</u>. We use the same argument that proves that $\mathbb{Q}^+$, a doubly indexed set, is countable. Let

$$A_1 = \{ a_{11}, a_{12}, a_{13}, \ldots \},$$
$$A_2 = \{ a_{21}, a_{22}, a_{23}, \ldots \},$$
$$A_3 = \{ a_{31}, a_{32}, a_{33}, \ldots \}, \ldots \quad \text{and so on.}$$

We can arrange all the points of the infinite union as a lattice in the first quadrant



Then enumerate the entire set by going up diagonally to the left:

$$A_1 \cup A_2 \cup A_3 \cup \cdots = \{ a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, \ldots \}.$$

If we hit a point that has appeared previously, simply delete it. This way we can ensure that the list has no repeating elements. Such a list proves that a countable union of countable sets is countable. $\square$

3. Let $S = \{ x \in [0,1] \mid x = .a_1 a_2 a_3 \ldots \}$ in base 10 decimal expansion, where $a_i \in \{0,1,2\}$. Prove that S is not countable.

Proof. Assume the contrary that S is countable. Then we can enumerate S as a list:

$$S = \{ x_1, x_2, x_3, \ldots \},$$

where

$$x_1 = .x_{11}\, x_{12}\, x_{13} \ldots$$
$$x_2 = .x_{21}\, x_{22}\, x_{23} \ldots$$
$$x_3 = .x_{31}\, x_{32}\, x_{33} \ldots$$
$$\vdots$$

This is supposed to be the complete list of elements in S. Because we are not using the digit 9, the decimal expansion is unique. Now mimicking Cantor's diagonal argument, we create a new element $b \in S$ that is different from all the x_i's. Let $b = .b_1 b_2 b_3 \cdots$, where we choose $b_i \in \{0,1,2\} - \{x_{ii}\}$, that is, b_i is 0, 1, or 2, but is different from x_{ii}.

Then $b \neq x_1$ because the first digit b_1 of b is different from the first digit x_{11} of x_1. Similarly, $b \neq x_2$ because the second digit b_2 of b is different from the second digit x_{22} of x_2. In general, $\forall\, i \in \mathbb{N}$, $b \neq x_i$ because the ith digit b_i of b is different from the ith digit x_{ii} of x_i. Therefore, b is an element of S that is not in the list, so the list is not complete, a contradiction. Hence, S is countable. $\square$

4. (10 points) (**Definition of convergence**) Using the ε-N definition of convergence, prove that the sequence $\left\{\dfrac{n^2+2n+1}{n^2+n-1}\right\}$ converges. (In this problem, you may not use limit rules or the sandwich lemma in your proof. you must start with an $\varepsilon > 0$ and try to find an $N \in \mathbb{N}$. In your scratch work, you may use anything you like. First divide both the numerator and the denominator by n^2, the highest power of n, to guess the limit.)

Solution. Scratch work:

$$\lim \frac{n^2+2n+1}{n^2+n-1} = \lim \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{1}{n}-\frac{1}{n^2}}$$ (Dividing both top and bottom by n^2)

$$= 1$$ (Using sum and quotient rules, since $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$)

Thus, the limit is 1.

Let $\varepsilon > 0$, We want to find $N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$\left| \frac{n^2+2n+1}{n^2+n-1} - 1 \right| < \varepsilon .$$

$$\left| \frac{n^2+2n+1}{n^2+n-1} - 1 \right| = \frac{n^2+2n+1-(n^2+n-1)}{n^2+n-1} = \frac{n+2}{n^2+n-1}$$

Scratch work:

We need an upper bound for $\frac{n+2}{n^2+n-1}$, hence a lower bound for n^2+n-1. Since $n \geq 1$, $n-1 \geq 0$ and $n^2+n-1 \geq n^2$.

$$\frac{n+2}{n^2+n-1} \leq \frac{n+2}{n^2} \leq \frac{3n}{n^2} = \frac{3}{n}$$ $(n+2 \leq 3n \Leftrightarrow 2 \leq 2n \Leftrightarrow 1 \leq n)$

$$\frac{3}{n} < \varepsilon \Leftrightarrow n > \frac{3}{\varepsilon}.$$

Choose $N \in \mathbb{N}$ such that $N > \frac{3}{\varepsilon}$ Then $\forall n \geq N$,

$$\left| \frac{n^2+2n+1}{n^2+n-1} - 1 \right| = \frac{n+2}{n^2+n-1} \leq \frac{n+2}{n^2} \leq \frac{3n}{n^2} = \frac{3}{n} < \varepsilon$$

because $n \geq N > \frac{3}{\varepsilon} \Rightarrow \frac{1}{n} < \frac{\varepsilon}{3} \Rightarrow \frac{3}{n} < \varepsilon$.

Therefore, $\frac{n^2+2n+1}{n^2+n-1}$ converges to 1. $\square$

5. Show that $\frac{2^k}{k!} \to 0$.

Proof. $$\frac{2^k}{k!} = \frac{2}{k} \cdot \frac{2}{k-1} \cdot \frac{2}{k-2} \cdot \dots \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{2}{1}$$

Note that for $k \geq 4$, $k-1 \geq 3$ and $\frac{2}{k-1} \leq \frac{2}{3} < 1$.
Assume $k \geq 4$. Then

$$0 \leq \frac{2^k}{k!} \leq \frac{2}{k} \frac{2}{3} \cdot \frac{2}{3} \cdots \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{2}{1}$$

$$\leq \frac{2}{k} \cdot 1 \cdot 2 = \frac{4}{k}.$$

Since $\lim \frac{4}{k} = 0$, by the sandwich lemma,

$$\lim \frac{2^k}{k!} = 0. \qquad \square$$

- This exercise shows that the exponential 2^k is much smaller than the factorial $k!$ for k large.
- We proved $\frac{2^k}{k!} \leq \frac{4}{k}$ directly, but you might also enjoy proving it by induction.

6. (10 points) (**Completeness axiom**) Does the set $\mathbb{Z}$ satisfy the completeness axiom? That is, if S is a nonempty subset of $\mathbb{Z}$ that is bounded above, is there an element of $\mathbb{Z}$ that is $\sup(S)$? Prove your result. (*Hint*: You may use the completeness axiom for $\mathbb{R}$.)

Solution. By the completeness axiom for $\mathbb{R}$, the nonempty set S has a $\sup S$ that is a real number b. Suppose b is not an integer. Let N be the largest integer $< b$. Since there is no integer between N and b, the integer N is an upper bound for S. Thus, b cannot be l.u.b.(S). This contradiction proves that the l.u.b. b must be an integer. Hence, $\mathbb{Z}$ satisfies the completeness axiom. $\square$

7. (15 points) (**Sup, inf, max, min**) Let $S = \left\{ 4 - \frac{2}{\sqrt{n}} \,\middle|\, n \in \mathbb{N} \right\}$.

(a) Find $\sup(S)$ and prove your result.
(b) Does S have a maximum (i.e., an element $s \in S$ that is an upper bound for S)?
(c) Find $\inf(S)$ and prove your result.
(d) Does S have a minimum (i.e., an element $s \in S$ that is a lower bound for S)?

Solution. (a) As $n \to \infty$, $2/\sqrt{n} \to 0$ and $4 - 2/\sqrt{n} \to 4$.

We claim that $\sup S = 4$.

Proof. Let $\varepsilon > 0$. We will find an $n \in \mathbb{N}$ such that

$$4 - \varepsilon < 4 - \frac{2}{\sqrt{n}}.$$

This is equivalent to $-\frac{2}{\sqrt{n}} > -\varepsilon$

$$\Leftrightarrow \frac{2}{\sqrt{n}} < \varepsilon \Leftrightarrow \frac{\sqrt{n}}{2} > \frac{1}{\varepsilon} \Leftrightarrow \sqrt{n} > \frac{2}{\varepsilon}$$

$$\Leftrightarrow n > \frac{4}{\varepsilon^2}.$$

Choose $n \in \mathbb{N}$ such that $n > \frac{4}{\varepsilon^2}$. As we have shown, this is equivalent to $4 - \varepsilon < 4 - \frac{2}{\sqrt{n}} \in S$. By the ε-criterion for sup, $\sup S = 4$.

(b) No. As n increases, $2/\sqrt{n}$ decreases and the sequence $4 - 2/\sqrt{n}$ monotonically increases to 4, but it never reaches the l.u.b. 4. Since $4 \notin S$, S does not have a maximum.

(c) Since $4 - 2/\sqrt{n}$ is monotonically increasing, the least element is the first, $4 - 2/\sqrt{1} = 2 \in S$. Thus, 2 is a minimum and is the infimum.

(d) Yes, $\min(S) = 2$. $\square$