Tuesday, November 9

Let $x = x(t)$ be a time-dependent population.

1. The simplest model of population growth is

$$\frac{dx}{dt} = rx$$

   where $r > 0$ is a constant. Given that $x(0) = x_0$, what is $x(t)$? Also, explain why this equation reflects the belief that the population increase per individual per unit time is constant.

2. The second-simplest model is the *logistic equation*,

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \tag{1}$$

   where $r > 0$ and $K > 0$ are constants.

   (a) What might motivate equation (1)?

   (b) Assume $x(0) = x_0$ lies between 0 and K. Explain why $x(t))$ is increasing and converges to K. Does it have an inflection point?

   (c) Using separation of variables, and assuming $0 < x < K$, show that (1) means that the function

$$y(t) = \frac{x(t)}{K - x(t)}$$

   satisfies

$$\frac{dy}{dt} = ry.$$

   So it is not x which grows exponentially, but $x(t)/(K - x(t))$.

3. Now think about this equation:

$$\frac{dx}{dt} = rx(x-a)\left(1 - \frac{x}{K}\right). \tag{2}$$

   where $0 < a < K$. (Think $a \ll K$, although mathematically that's not necessary.)

   (a) What does the solution look like when $0 < x_0 < a$? Is there an inflection point?

   (b) What does the solution look like when $a < x_0 < K$? Is there an inflection point?

   (c) What might motivate (2)?