Tuesday, December 7

1. **The flow map.** The Lorenz equations stretch (exponentially) and fold (to keep things within a bounded region in space). But they do one more thing, they *compress*. In one sense, that's quite obvious from what we have said about the equations: I told you that *all of space* eventually ends up on the two-winged butterfly, the Lorenz attractor. So all of space is compressed onto the buttterfly.

   To make the compression quantitative, first think about one differential equation, instead of the Lorenz system of three differential equations. Let's say our equation is
$$\frac{dx}{dt} = f(x).$$
   Given $x_0 \in \mathbb{R}$, we can solve the initial value problem
$$\frac{dx}{dt} = f(x), \quad x(0) = x_0.$$
   The solution is some function $x(t)$, which of course depends on x_0, and to make that clear, we now write
$$x(t;x_0)$$
   instead of just $x(t)$. We usually fix x_0, and think of $x(t;x_0)$ as a function of t. But we can also think the other way around: Fix t, and think of $x(t;x_0)$ as a function of x_0. This is called the *flow map*:
$$x_0 \mapsto x(t;x_0).$$

   For the special case
$$f(x) = rx,$$
   where $r \in \mathbb{R}$ is a given constant parameter, work out an explicit formula for the flow map.

2. **The image of an interval under a flow map in one dimension.** The notation is as in the previous problem. Suppose that $[a,b]$ is an interval of real numbers. For a given $t > 0$, think about the set of all points
$$x(t;x_0) \quad \text{with } x_0 \in [a,b].$$
   So these are all the possible locations of the moving point at time t if the initial position is in $[a,b]$. Explain why this has to be the interval
$$[x(t;a), x(t;b)].$$
   So this is an interval that changes with time. We'll call the interval $[a(t), b(t)]$, so $a(t) = x(t;a)$ and $b(t) = x(t;b)$.

3. **The rate of change of the length of the interval.** Explain why

$$\frac{d}{dt}(b(t) - a(t)) = \int_{a(t)}^{b(t)} f'(x)dx.$$

So the rate at which the interval changes its length is the integral of $f'(x)$ over the interval. For instance, if you start with the interval $[0, 1/2]$ and let it flow under the flow map associated with the logistic equation

$$\frac{dx}{dt} = x(1-x),$$

what is the rate at which the length of the interval changes at time 0?