Recall the following facts

Fact 0.1. In this course, we will freely use the basic properties of addition and product of numbers in the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ satisfy all the usual properties

(1) Associative, that is for all a, b, c in one of these sets $(a+b)+c = a+(b+c)$, $a(bc) = (ab)c$.
(2) Commutative, that is for all a, b in one of these sets $a + b = b + a$, $ab = ba$.
(3) Existence of a 0 and a 1 with $a + 0 = a, a \times 1 = a$ for all a in these sets.
(4) Distributive property: for all a, b, c in any of these sets $a(b + c) = ab + ac$.
(5) Existence of inverses for addition in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, that is for all a on each of these sets, there is another element that we call $-a$ on the same set such that $a + (-a) = 0$.
(6) Existence of inverses for product in $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, that is for all $a \neq 0$ on each of these sets, there is another element that we call $\frac{1}{a}$ on the same set such that $a \times \frac{1}{a} = 1$.
(7) Order in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ but not in $\mathbb{C}$: for any two different a, b, either $a < b$ or $b < a$ (and conversely, if $a < b$, then $a \neq b$). Moreover, if $a < b$ and $b < c$, then $a < c$.
(8) If $a < b$ and c is in the same set, then $a + c < b + c$. If $a < b$ and $c > 0$, then $ac < bc$.

Using these facts and only these facts, see how many of the following you can prove:

Question. (a) For all $a \in \mathbb{R}$, if $a < 0$, then $-a > 0$. Hint: add $-a$ to the given inequality.

(5) $a < 0$ ... $0 + (-a)$ → (1) $(a-a) < (0-a)$ → $0 < -a$

(4) $(a + (-a)) < (0 + (-a))$ (5) → 0

→ $(a-a) < (0-a)$ (3)

$0 < -a$

(b) For all $a \in \mathbb{R}$, $a \times 0 = 0$. Hint: start with $0 + 0 = 0$

(4) $a(0+0) = a0$,

5 $-(a\times 0) + a\times 0 + a\times 0 = a\times 0) + (-a\times 0))$

5 $a\times 0 - a\times 0 + a\times 0 = a\times 0 - a\times 0$

$(a\times 0) = 0$

(c) For all $a \in \mathbb{R}$, if $-(a \times b) = a \times (-b) = (-a) \times b$..

First to show $-1\times a = (-a)$

(2) $a\times 0 = 0$ from b $\rightarrow a(1+-1) = 0$

1) $a\times 1 + a\times -1 = 0$, 2) $\rightarrow$ inverse of $a = a\times 1$ is $-a$, meaning

$a\times -1 = -1\times a = -a$.

1) $-(a\times b) = a\times(-1\times(b)) \overset{2)}{=} (a\times(b\times -1)) \overset{1)}{=} (a\times b)\times -1 \overset{2)}{=} -(a\times b)$

$1 \rightarrow -(a\times b) = (-1\times a)\times b \overset{1)}{\rightarrow} (-1\times(a\times b)) \overset{1)}{} -1\times(a\times b) = -(a\times b)$

$-a$

(d) For all $a, b \in \mathbb{R}$, if $a \times b = 0$, then either $a = 0$ or $b = 0$.

$a\times b = 0$, assuming $a \neq 0$

6) $\frac{1}{a}(a\times b) = \frac{b}{a}\times 0$

1) $(\frac{1}{a}\times a)\times b = \frac{0}{a}$, 6) $1\times b = 0$ $\quad \frac{0}{a} = 0$, $\frac{1}{a}\times 0 = 0$; proven in part b

By 3); $b = 0$

$a\times b = 0$, assuming $b \neq 0$

6) $\frac{1}{b}(a\times b) = \frac{1}{b}\times 0$

1) $a\times(\frac{1}{b}\times b) = \frac{1}{b}\times 0 = \frac{0}{b}$, $\frac{0}{b} = 0$, $\frac{1}{b}\times 0$, shown in part b

3) $a\times 1 = 0$, $a = 0$.

Therefore, if $a\times b = 0$, then a or b equals 0.

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