Math 136 **Real Analysis II**[1] **Spring 2023**

A roadmap for the Jordan integrability Theorem

Theorem 1 (Jordan Integrability Theorem)**.** *Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let f be a bounded function from $\mathbb{I}$ to $\mathbb{R}$. Assume*

(1) $$D(f,\mathbb{I}) = \left\{\mathbf{x}\in\mathbb{I}\,\middle|\, f:\mathbb{I}\to\mathbb{R} \text{ is discontinuous at } \mathbf{x}\right\}$$

has Jordan content zero (JC 0).

Then, f is integrable on $\mathbb{I}$!

We list here some of the important notation and ideas in the proof for you to use while learning the proof.

We let $\epsilon > 0$ and $\mathbb{I} = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$

We let $M > 0$ such that for all $\mathbf{x}\in\mathbb{I}$, $-M \le f(\mathbf{x}) \le M$

Let $\widetilde{J}_1, \widetilde{J}_2, \ldots, \widetilde{J}_p$ be generalized rectangles in $\mathbb{I}$ such that

(a) $D(f,\mathbb{I}) \subset \widetilde{J}_1 \cup \widetilde{J}_2 \cup\cdots\cup \widetilde{J}_p$ and

(b) $\displaystyle\sum_{i=1}^{p} \mathrm{Vol}(\widetilde{J}_i) < \frac{\epsilon}{4M}$

Let $F = \cup_{i=1}^{p}\widetilde{J}_i$. Then $D(f,\mathbb{I})\subset F$.

For $j = 1,\ldots,n$ let P_j be the partition of $[a_j,b_j]$ (the j^{th} edge of $\mathbb{I}$) that contains the j^{th} coordinates of all vertices of all the $\widetilde{J}_i$.

Let $\mathbb{P} = (P_1,P_2,\ldots,P_n)$ be the resulting partition of $\mathbb{I}$. By construction each rectangle J in $\mathbb{P}$ either

- is contained in F or
- meets F at most on its boundary.

Let $J'_1, J'_2, \ldots, J'_\ell$ be the rectangles in $\mathbb{P}$ that are contained in F.

Since the J'_k are contained in F,

(2) $$\sum_{k=1}^{\ell}\mathrm{Vol}(J'_k) < \frac{\epsilon}{4M}$$

Let $J_1, J_2, \ldots, J_m$ be those rectangles in $\mathbb{P}$ that meet F at most on a boundary. Therefore,

(3) $$f \text{ is continuous on } \mathrm{int}(J_k) \text{ and bounded for all } k = 1,\ldots,m.$$

Let $f_1:\mathbb{I}\to\mathbb{R}$ be defined by $f_1(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \mathbf{x}\in\cup_{j=1}^{m}J_j\\ 0 & \mathbf{x}\in\mathrm{int}(F)\cup\mathrm{bd}(\mathbb{I})\end{cases}$

By (3), f_1 is integrable on each J_i, $i = 1,\ldots,m$ since $f_1 = f$ there.

f_1 is integrable on each J'_k because $f_1 = 0$ in $\mathrm{int}(J'_k)$ for each k. By the Additivity over Partitions Theorem, f_1 is integrable on $\mathbb{I}$.

By Riemann's Condition, there is a partition $\mathbb{P}_1$ of $\mathbb{I}$ such that $U(f_1, \mathbb{P}_1) - L(f_1, \mathbb{P}_1) < \frac{\epsilon}{2}$.

Let $\mathbb{P}^* = \mathbb{P} \bigcup \mathbb{P}_1$. Then $U(f_1, \mathbb{P}^*) - L(f_1, \mathbb{P}^*) < \frac{\epsilon}{2}$.

Now, we divide the sum for $U(f, \mathbb{P}^*) - L(f, \mathbb{P}^*)$ into two sums

$$
\begin{aligned}
U(f, \mathbb{P}^*) - L(f, \mathbb{P}^*) &= \sum_{J \text{ in } \mathbb{P}^*,\, J \subset F} (M(f,J) - m(f,J)) \operatorname{Vol}(J) + \sum_{J \text{ in } \mathbb{P}^*,\, J \subset \cup J_k} (M(f,J) - m(f,J)) \operatorname{Vol}(J) \\
&\leq \underbrace{\sum_{J \text{ in } \mathbb{P}^*,\, J \subset F} (M - (-M)) \operatorname{Vol}(J)}_{\text{as } -M \leq f(\mathbf{x}) \leq M \text{ on } \mathbb{I}} + \underbrace{\sum_{J \text{ in } \mathbb{P}^*,\, J \subset \cup J_k} (M(f_1,J) - m(f_1,J)) \operatorname{Vol}(J)}_{\text{as } f = f_1 \text{ here}} \\
&< 2M \frac{\epsilon}{4M} + \sum_{J \text{ in } \mathbb{P}^*} (M(f_1,J) - m(f_1,J)) \operatorname{Vol}(J) \\
&\qquad \text{the sum over } J \text{ in } \mathbb{P}^* \textit{ and } J \subset \cup J_k \text{ is a part of the sum over all } J \text{ in } \mathbb{P}^*, \\
&\qquad \text{and the terms are nonnegative} \\
&= \frac{\epsilon}{2} + U(f_1, \mathbb{P}^*) - L(f_1, \mathbb{P}^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

YEA!!!!!