Wednesday, April 5

Saturday, April 1, 2023 22:32

Student Hours :

11:00 AM -12:00 noon on Mondays
2:00-3:00 PM on Thursdays
1:30-2:30+ PM on Fridays in the Math Department Lounge
+ just ask if you want to talk at other times

Note changes to the new syllabus highlighted in Saturday's email.

We don't have class on Wednesday, April 19.

There will be no group work this week.

This week we will read Section 19.1 of Fitzpatrick plus parts of Chapter 8 and the beginning of chapter 10 of Marsden-Hoffman, which is on Canvas. (Files/Marsden-Hoffman)

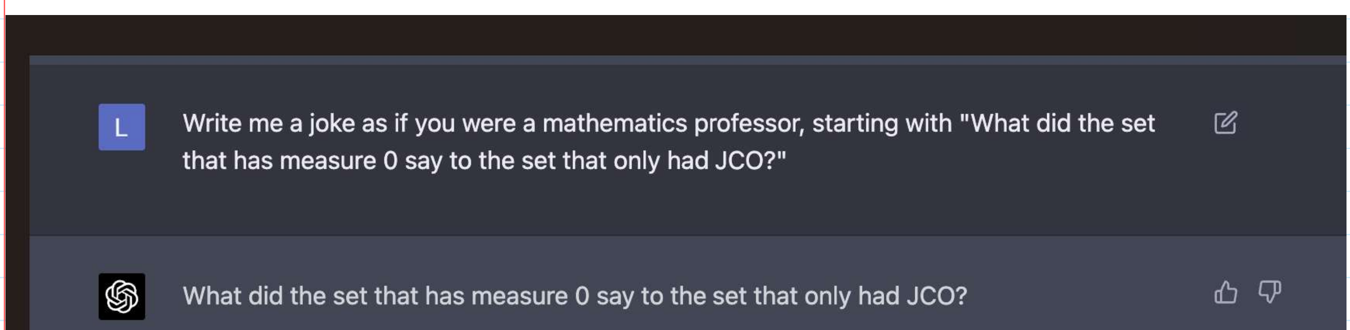
Students are finding we have a lot of new definitions, so you might want to keep an integration definition and theorem sheet if you don't already have one.

Steven Strogatz, the Jacob Gould Schurman Professor of Applied Mathematics at Cornell University, and a wonderful speaker!! will speak on "Infinite Powers: a History of Calculus" on Friday, April 7, 4 p.m. to 5 p.m. In JCC 270!

By the end, Prof. Strogatz hopes to convince you that calculus is one of the greatest triumphs of human creativity ever.

If you're interested in learning more, read his charming book Infinite Powers, How CALCULUS Reveals the Secrets of the Universe.

Luca G. asked Chat GPT:

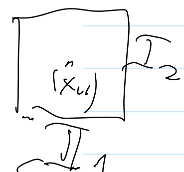


"You may have JCO, but I'm Lebesgue-ndary!"

Theorem 1 (Fubini's Theorem in $\mathbb{R}^n$)**.** *Let $\mathbb{I}_1$ be a generalized rectangle in $\mathbb{R}^n$ and let $\mathbb{I}_2$ be a generalized rectangle in $\mathbb{R}^k$. Let f be a bounded integrable function from the rectangle $\mathbb{I}_1 \times \mathbb{I}_2$ in $\mathbb{R}^{n+k}$ to $\mathbb{R}$.*

For $\mathbf{x} \in \mathbb{I}_1$ and $\mathbf{y} \in \mathbb{I}_2$, $f(\mathbf{x}, \mathbf{y}) \in \mathbb{R}$.

(i) Assume for each $\mathbf{x} \in \mathbb{I}_1$ the function $F_{\mathbf{x}}(\mathbf{y}) = f(\mathbf{x}, \mathbf{y})$ is integrable function for $\mathbf{y} \in \mathbb{I}_2$. Let $A(\mathbf{x}) = \int_{\mathbb{I}_2} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$. Then,



(1) $$\int_{\mathbb{I}_1 \times \mathbb{I}_2} f = \int_{\mathbf{x} \in \mathbb{I}_1} A(\mathbf{x})\, d\mathbf{x} \cancel{= \int_{\mathbf{y} \in \mathbb{I}_2} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}\, d\mathbf{x}} = \int_{x \in \mathbb{I}_1} \int_{y \in \mathbb{I}_2} f(x,y)\, dy\, dx$$

(ii) Assume for each $\mathbf{y} \in \mathbb{I}_2$ the function $F_{\mathbf{y}}(\mathbf{x}) f(\mathbf{x}, \mathbf{y})$ is an integrable function for $\mathbf{x} \in \mathbb{I}_1$. Let $B(\mathbf{y}) = \int_{\mathbb{I}_1} f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}$. Then,

(2) $$\int_{\mathbb{I}_1 \times \mathbb{I}_2} f = \int_{\mathbf{y} \in \mathbb{I}_2} B(\mathbf{y})\, d\mathbf{y} \cancel{= \int_{\mathbf{y} \in \mathbb{I}_2} f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y}} = \int_{y \in \mathbb{I}_2} \left(\int_{x \in \mathbb{I}_1} f(x,y)\, dx \right) dy$$

Therefore, whenever $\mathbf{y} \mapsto f(\mathbf{x}, \mathbf{y})$ is integrable for $\mathbf{y} \in \mathbb{I}_2$ and $\mathbf{x} \mapsto f(\mathbf{x}, \mathbf{y})$ is integrable for $\mathbf{x} \in \mathbb{I}_1$, the hypotheses of *(i)* and of *(ii)* are true, so one can *fubinate* the iterated integrals (switch the integrals over $\mathbb{I}_1$ and $\mathbb{I}_2$).

Note that if f is continuous on $\mathbb{I}_1 \times \mathbb{I}_2$, then both (1) and (2) hold. Therefore, one can fubinate the iterated integrals.

The proof is goes along the lines of the proof in $\mathbb{R}^2$, but it is a little more complicated to write down.

Theorem 2. *Let $\mathbb{I} = [a,b] \times [c,d]$ and let $f : \mathbb{I} \to \mathbb{R}$ be continuous on $\mathbb{I}$. Assume $\frac{\partial f}{\partial y}$ is continuous on $\operatorname{int}(\mathbb{I})$ and extends to a continuous function on $\mathbb{I}$.*

Let $F(y) = \int_{x=a}^{b} f(x,y)\, dx$. Then, F is differentiable on (c,d) and $F'(y) = \int_{x=a}^{b} \frac{\partial f}{\partial y}(x,y)\, dx$. Equivalently,

$$\frac{d}{dy} \int_{x=a}^{b} f(x,y)\, dx = \int_{x=a}^{b} \frac{\partial f}{\partial y}(x,y)\, dx.$$

Namely, one can take the derivative in y inside the integral with respect to x.

The proof will be in HW.

Complex numbers $\quad z = a + bi \in \mathbb{C}$

$a = re(z)$ The real part of z
$a \in \mathbb{R}$

$b = im(z)$ imaginary part of z
$b \in \mathbb{R}$

$z = a + bi \quad w = c + di$

$z = w$ if $a = c$ $b = d$

$(i)^2 = -1$

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$z + w = (a+c) + (b+d)i$

$zw = (a+bi)(c+di) = (ac - bd) + (ad + bc)i$

complex conjugate $\bar{z} = a - bi$

$z\bar{z} = a^2 + b^2 \geq 0$ $z\bar{z} = 0$ iff $z = 0 + 0i = 0$

define modulus of z to be $|z| = \sqrt{z\bar{z}}$
absolute value

Complex valued fns $A \subset \mathbb{R}^n$ bndd.

$f : A \to \mathbb{C}$ f can be written

$f(x) = r(x) + s(x)i$ $r(x) = re(f(x))$
$s(x) = im(f(x))$

f is integrable on A
if $r(x)$ and $s(x)$ are integrable on A

int his case $\int_A f(x) = \left(\int_A r(x)dx\right) + \left(\int_A s(x)dx\right)i$

eg $\int_0^{\frac{\pi}{2}} e^{ix} dx = \int_0^{\frac{\pi}{2}} \cos x\, dx + \left(\int_0^{\frac{\pi}{2}} \sin x\, dx\right)i$

$e^{ix} = \cos x + (\sin x)i$

$= \left(\sin\frac{\pi}{2} - \underbrace{\sin 0}_{0}\right) + \left(-\cos\frac{\pi}{2} + \cos 0\right)i$

$= 1 + i$

Defn V is a complex vector space
(VS)

if V satisfies the properties of VS
where the field of scalars is $\mathbb{C}$

$v \in V$ $w \in V$ there is a binary operator
$v + w \in V$

for $c \in \mathbb{C}$ $v \in V$ scalar multiplication

$cv \in V$

for $c \in \mathbb{C}$ $v \in V$ scalar multiplication

$cv \in V$

satisfying axioms of vs

ex $\mathbb{C}^n$ all n-tuples of complex number

$z = (z_1, z_2, \ldots z_n) \in \mathbb{C}^n$ $z_1, z_2, \ldots z_n$ all in $\mathbb{C}$

(eg $(1, 5, -i+7) \in \mathbb{C}^3$)

$z = (z_1, \ldots z_n)$ $w = (w_1, \ldots w_n)$

$z = w$ if $z_i = w_i$ for all $i = 1, \ldots n$.

$z + w = (z_1 + w_1, \ldots , z_n + w_n)$

$c \in \mathbb{C}$ $cz = (cz_1, \ldots, cz_n)$

Defn V is a cpx (complex) inner product space (i p space)

if V is a cpx vs and

there is a fn $\langle \, , \, \rangle : V \times V \to \mathbb{C}$

s.t. $\forall$ u, v, w in V $\forall c \in \mathbb{C}$

linearity in 1st argument

$\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$\langle cu, v \rangle = c \langle u, v \rangle$

$\langle v, w \rangle = \overline{\langle w, v \rangle}$

positive semidefiniteness

$\langle v, v \rangle \geq 0$ (ie $\langle v, v \rangle$ is a non negative real num)

the ip $\langle \, , \, \rangle$ is positive def

if in addition to properties above

$\langle v, v \rangle = 0$ iff $v = 0$

$\langle v, v\rangle = 0$ iff $v = 0$

eg $\mathbb{C}^n$ $z = (z_1, \dots, z_n) \in \mathbb{C}^n$

$w = (w_1, \dots, w_n) \in \mathbb{C}^n$

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$$

is cpx ip space

pf you check properties besides last 2. $z = (z_1, z_2, \dots, z_n)$

$$\langle z, z\rangle = z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}$$

$$= |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \geq 0$$

$|z| = \sqrt{z\bar{z}}$, $z \in \mathbb{C}$

and this sum $= 0$ iff $z = 0 = (0, \dots, 0)$

Defn (Norm on ip space) Let $(V, \langle\ ,\ \rangle)$ be cpx ip space

we define, for $v \in V$ $\|v\| = \sqrt{\langle v, v\rangle}$

(as $\langle v, v\rangle \geq 0$, $\sqrt{\langle v, v\rangle} \in [0, \infty)$)

Claim: $\|\ \|$ is a norm on V

ie $\forall c \in \mathbb{C}$ v, w in V

1. $\|v+w\| \leq \|v\| + \|w\|$
2. $\|cv\| = |c|\,\|v\|$
3. $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$

Cauchy Schwartz Bunyakovski ineq

$$|\langle v, w\rangle| \leq \|v\|\,\|w\|$$

pf sketch 2 follows from properties of ip pf sketch ② follows from properties of ip

$\|v\| = \sqrt{\langle v,v\rangle}$

① follows from CSB

hint $0 \leq \langle u+v, u+v\rangle = \|u+v\|^2$

pf ③ $v \in V$ $\|v\| = \sqrt{\langle v,v\rangle} \geq 0$ by def $\sqrt{}$

$0 = \|v\| = \sqrt{\langle v,v\rangle}$ iff $\langle v,v\rangle = 0$

iff $v = 0$

or ip is + definite

pf of CSB prove $|\langle v,w\rangle| \leq \|v\|\,\|w\|$

if $w=0$ all cool.

now assume $w \neq 0$. let $\lambda = \dfrac{\langle v,w\rangle}{\langle w,w\rangle}$

$0 \leq \langle v-\lambda w, v-\lambda w\rangle$

$= \langle v, v-\lambda w\rangle - \langle \lambda w, v-\lambda w\rangle$

$\langle \lambda w, v-\lambda w\rangle = \langle \lambda w, v\rangle - \langle \lambda w, \lambda w\rangle$

$= \lambda \langle w,v\rangle - \lambda\bar{\lambda}\langle w,w\rangle$

or $\langle v, \lambda w\rangle = \bar{\lambda}\langle v,w\rangle$

$= \lambda\left(\langle w,v\rangle - \dfrac{\langle w,v\rangle\langle w,w\rangle}{\langle w,w\rangle}\right)$

$= 0$

so $0 \leq \langle v, v-\lambda w\rangle = \langle v,v\rangle - \bar{\lambda}\langle v,w\rangle$

$\bar{\lambda}\langle v,w\rangle \leq \langle v,v\rangle$

$\lambda = \dfrac{\langle v,w\rangle}{\langle w,w\rangle}$ $\dfrac{\langle w,v\rangle}{\langle w,w\rangle}\langle v,w\rangle \leq \langle v,v\rangle$

$\overline{\langle v,w\rangle}\langle v,w\rangle \leq \langle v,v\rangle\langle w,w\rangle$

$|\langle v,w\rangle|^2 \leq \|v\|^2\,\|w\|^2$ ✓

Recall in normed linear space (v.s. with

Recall in normed linear space (v.s. with a norm) V, if $\{f_k\}$ is a sequence in V, $f \in V$, then $f_k \to f$ if $\|f_k - f\| \to 0$ ie

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $\forall k \geq N$ $\|f_k - f\| < \varepsilon$

$\{f_k\}$ is a Cauchy seq in V if
$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $\forall k \geq N$ $l \geq N$

$\|f_k - f_l\| < \varepsilon$.

A normed linear space V is complete if every Cauchy seq in V converges to a point in V

(complete normed linear space is called a Banach space)

An IPS pace that is complete under the norm $\|v\| = \sqrt{\langle v, v \rangle}$ is called a <u>Hilbert Space</u>

<u>eg</u> $C([a,b], \mathbb{C})$ has the sup norm.
$\|f\| = \sup\{|f(x)| \mid x \in [a,b]\}$
& $C([a,b], \mathbb{C})$ is complete under this norm [Math 135]

we define an inner product on $C([a,b], \mathbb{C}) = C$

and show it is positive definite

for f, g in C define

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$$\langle f, g\rangle = \int_a^b f(x)\,\overline{g(x)}\,dx$$

you can check linearity properties and $\langle f, g\rangle = \overline{\langle g, f\rangle}$

we show $\langle\ ,\ \rangle$ is + definite

let $f \in C$ $\quad \langle f, f\rangle = \int_a^b f(x)\,\overline{f(x)}\,dx$

$$= \int_a^b |f(x)|^2\,dx \geq 0$$

if $\int_a^b |f(x)|^2\,dx = 0$ then $|f(x)|^2 = 0$

$\forall x \in [a,b]$ by hw problem

as f is cont. $\quad \therefore \langle\ ,\ \rangle$ is ip

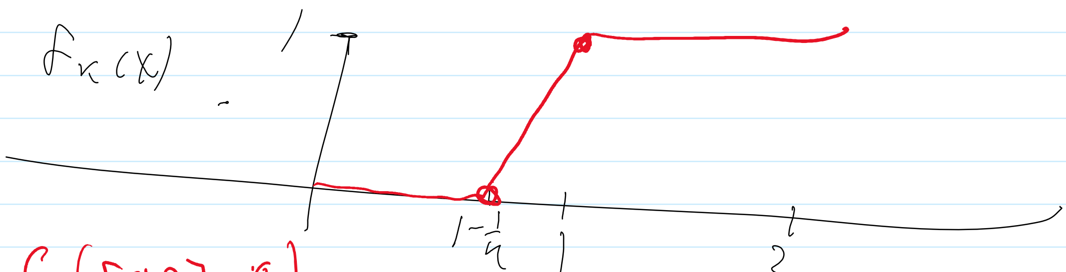
norm $\quad \|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b |f(x)|^2\,dx}$

is C a Hilbert space?

ex let $f_n : [0,2] \to \mathbb{R}$ $\quad n = 2, 3, 4 \ldots$

$$f_n(x) = \begin{cases} 0 & 0 \leq x < 1 - \frac{1}{n} \\ 1 - n(1-x) & 1 - \frac{1}{n} \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$$

graph $f_n(x)$

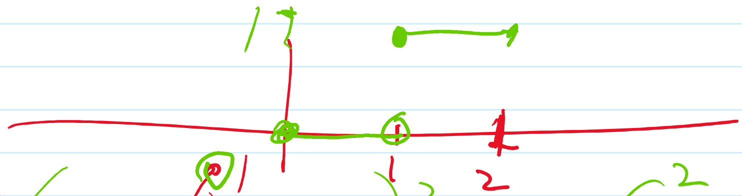


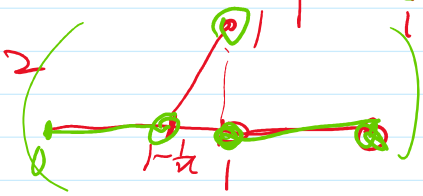
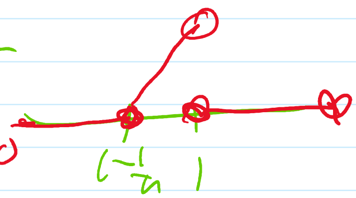
$f_n \in C([0,2], \mathbb{R})$

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$$

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x \le 2 \end{cases}$$

f is not cont



$\|f_n - f\|^2 = \int_0^2 \Big($  $\Big)^2 \le \int_0^2$ 

$= \frac{1}{n} \cdot 1 \xrightarrow[n \to \infty]{} 0$

$\therefore$ $C([0,2], \mathbb{C})$ not Hilbert space.

NOTE: That {f_k} converges to a function in the norm we defined, {f_k} must be a Cauchy sequence. Since that function is not in C (It is not even equal to a continuous function a.e., C is not complete. C is not a Hilbert space.