Thursday October 7

1. (If you haven't done this yet.) Think about the following equation:

$$\frac{dx}{dt} = x(x-\gamma). \tag{1}$$

   Here γ is a parameter, either positive or negative.

   (a) What are the steady states, the fixed points, the equilibria, the constant solutions? (All of these expressions denote the same thing: Solutions where $x(t)$ does not in reality depend on t.)

   (b) For each equilibrium, decide whether it is stable (when you start with an x near the equilibrium, $x(t)$ converges to the equilibrium) or unstable.

   (c) Make a plot in which the horizontal axis is γ (say between -1 and $+1$), and the vertical axis denotes equilibria of equation (2). Indicate *stable* equilibria with solid lines, and *unstable* ones with dashed lines.

   What you see here is a *transcritical bifurcation* as γ crosses zero.

2. Think about the following equation:

$$\frac{dx}{dt} = -x^2 + \gamma. \tag{2}$$

   Here γ is a parameter, either positive or negative.

   (a) What are the steady states, the fixed points, the equilibria, the constant solutions? (All of these expressions denote the same thing: Solutions where $x(t)$ does not in reality depend on t.)

   (b) For each equilibrium, decide whether it is stable (when you start with an x near the equilibrium, $x(t)$ converges to the equilibrium) or unstable.

   (c) Make a plot in which the horizontal axis is γ (say between -1 and $+1$), and the vertical axis denotes equilibria of equation (2). Indicate *stable* equilibria with solid lines, and *unstable* ones with dashed lines.

   What you see here is a *blue sky bifurcation* as γ crosses zero. The name is "blue sky" because two fixed points appear "out of the blue sky".

3. Think about the following equation:

$$\frac{dx}{dt} = x^2 + \gamma. \tag{3}$$

   Here γ is a parameter, either positive or negative.

(a) What are the steady states, the fixed points, the equilibria, the constant solutions? (All of these expressions denote the same thing: Solutions where $x(t)$ does not in reality depend on t.)

(b) For each equilibrium, decide whether it is stable (when you start with an x near the equilibrium, $x(t)$ converges to the equilibrium) or unstable.

(c) Make a plot in which the horizontal axis is γ (say between -1 and $+1$), and the vertical axis denotes equilibria of equation (2). Indicate *stable* equilibria with solid lines, and *unstable* ones with dashed lines.

What you see here is a *saddle-node bifurcation*. It's a blue sky bifurcation in reverrse. Two fixed points come together and disappear. Why is it called "saddle-node"? We can't *quite* explain that yet.

4. Think about the following equation:

$$\frac{dx}{dt} = \gamma x - x^3. \tag{4}$$

Here γ is a parameter, either positive or negative.

(a) What are the steady states, the fixed points, the equilibria, the constant solutions? (All of these expressions denote the same thing: Solutions where $x(t)$ does not in reality depend on t.)

(b) For each equilibrium, decide whether it is stable (when you start with an x near the equilibrium, $x(t)$ converges to the equilibrium) or unstable.

(c) Make a plot in which the horizontal axis is γ (say between -1 and $+1$), and the vertical axis denotes equilibria of equation (2). Indicate *stable* equilibria with solid lines, and *unstable* ones with dashed lines.

What you see here is a *pitchfork bifurcation*. Explain what it has to do with a pitchfork.

5. Think about the following equation:

$$\frac{dx}{dt} = \gamma x + x^3. \tag{5}$$

Here γ is a parameter, either positive or negative.

(a) What are the steady states, the fixed points, the equilibria, the constant solutions? (All of these expressions denote the same thing: Solutions where $x(t)$ does not in reality depend on t.)

(b) For each equilibrium, decide whether it is stable (when you start with an x near the equilibrium, $x(t)$ converges to the equilibrium) or unstable.

(c) Make a plot in which the horizontal axis is γ (say between -1 and $+1$), and the vertical axis denotes equilibria of equation (2). Indicate *stable* equilibria with solid lines, and *unstable* ones with dashed lines.

What you see here is another *pitchfork bifurcation*.

6. The pitchfork bifurcation in problem 4 is called *supercritical*, and that in problem 5 is called *subcritical*. Explain the terminology. Hint: In both cases, $x = 0$ is a stable fixed point when γ is below the critical value of 0, and an unstable fixed point above the critical value. In problem 4, the fork is above the critical value, on the side where 0 is unstable. In problem 5, the fork is below the critical value, on the side where 0 is stable. (super means above, sub means below.)

7. The supercritical pitchfork bifurcation is also called the *soft pitchfork bifurcation*. The subcritical pitchfork bifurcation is also called the *hard pitchfork bifurcation*. Explain this terminology, by picturing what a solution $x(t)$ looks like if γ is a function of t and slowly rises from negative to positive values.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and

$$\frac{dx}{dt} = f(x).$$

Show that solutions $x(t)$ cannot oscillate. They have to be monotonic. $n = 1$ is not enough to model oscillations.