MATH 42 HOMEWORK 3 SOLUTIONS

Topics covered: planes, surfaces, curve length from §13.5, 13.6, 14.4, and 15.1

This homework is due at 11:59 p.m. (Eastern Time) on Wednesday, September 30. Scan the completed homework and upload it **as one pdf file** to Gradescope. The Canvas module "Written Assignments" has instructions for how to upload the assignment to Gradescope.

(1) Find the length of the parameterized curve over the specified time interval:
   (a) $\vec{r}(t) = \langle \sin(t), \cos(t), t\sqrt{3} \rangle$ for $t \in [0, 5]$
   (b) $\vec{r}(t) = \langle \frac{1}{2}t^2, \frac{2\sqrt{2}}{3}\pi^{1/4}t^{3/2}, t\sqrt{\pi} \rangle$ for $t \in [0, \pi]$

Solution: The length, S, of a parameterized curve is given by

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

For the first part:

$$\begin{aligned}
S &= \int_0^5 \sqrt{\left(\frac{\mathrm{d}\sin(t)}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\cos(t)}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}t\sqrt{3}}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= \int_0^5 \sqrt{(\cos(t))^2 + (-\sin(t))^2 + \left(\sqrt{3}\right)^2}\,\mathrm{d}t \\
&= \int_0^5 \sqrt{1+3}\,\mathrm{d}t \\
&= \int_0^5 2\,\mathrm{d}t \\
&= 2t\Big|_{t=0}^{t=5} \\
&= 10,
\end{aligned}$$

where the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ was used.

For the second part:

$$
\begin{aligned}
S &= \int_0^\pi \sqrt{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}t^2\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{2\sqrt{2}}{3}\pi^{1/4}t^{3/2}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}t\sqrt{\pi}\right)^2}\mathrm{d}t \\
&= \int_0^\pi \sqrt{t^2 + \left(\sqrt{2\pi^{1/2}t}\right)^2 + \left(\sqrt{\pi}\right)^2}\mathrm{d}t \\
&= \int_0^\pi \sqrt{t^2 + 2\sqrt{\pi}t + \pi}\mathrm{d}t \\
&= \int_0^\pi \sqrt{(t+\sqrt{\pi})^2}\mathrm{d}t \\
&= \int_0^\pi t + \sqrt{\pi}\mathrm{d}t \\
&= \left(\frac{t^2}{2} + \sqrt{\pi}t\right)\Bigg|_{t=0}^{t=\pi} \\
&= \frac{\pi^2}{2} + \pi^{3/2} \approx 10.503,
\end{aligned}
$$

where a square of a binomial was recognized in an intermediate step.

(2) Consider the plane passing through $A(-2,4,3)$, $B(1,0,-3)$, and $C(3,2,-1)$.
  (a) Identify the unit normal vector of the plane.
  (b) Express the equation of the plane in the form $\alpha x + \beta y + \gamma z + \delta = 0$ where α, β, γ, and δ are integers with greatest common denominator of 1. Hint: $|\delta| = 19$ in this form.

Solution: Given three points in the plane, two vectors that lie within the plane are easily identified; for example, consider $\overrightarrow{BA} = \langle 3,-4,-6\rangle$ and $\overrightarrow{BC} = \langle -2,-2,-2\rangle$. The cross product of these vectors is normal to the plane so we have $\overrightarrow{n} = \overrightarrow{BA} \times \overrightarrow{BC} = \langle -4,18,-14\rangle$, which may be divided by its magnitude to give the unit normal vector $\widehat{n} = \frac{1}{\sqrt{134}}\langle -2,9,-7\rangle$.

To determine the expression of the plane, we use the fact that any vector that lies within the plane must be orthogonal to the unit normal vector (or any scaling of the normal vector). We may select an arbitrary vector in the plane by using point A as one terminus and an arbitrary point in the plane given by (x,y,z). Hence $\langle -2-x, 4-y, 3-z\rangle \cdot \widehat{n} = 0$ is an equation of the plane. With some work this simplifies to $2x - 9y + 7z + 19 = 0$, or the same expression with opposite algebraic signs on each term.

(3) Consider the plane P with unit normal vector $\hat{n} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$ that passes through the origin, $O(0,0,0)$.
  (a) The *dihedral angle* is the angle between two planes that intersect. Find the dihedral angle between the plane in problem (2) and the plane P.
  (b) The intersection between the plane in problem (2) and the plane P is a line. In what direction does this line point? Express the answer as a unit vector.

Hint: Use knowledge of the unit normal vector of each plane.

Solution: The dihedral angle, θ, of two planes that intersect may be found via the angle between the normal vectors of each plane. We are not restricted to using the unit-length representations of each normal vector because

$$\theta = \arccos\left(\frac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{||\overrightarrow{n_1}|| * ||\overrightarrow{n_2}||}\right),$$

where $\overrightarrow{n_1}$ is any scaling of the normal vector from problem 2 and $\overrightarrow{n_2}$ is any scaling of the normal vector given in problem 3. If we were to use the unit normal vectors, then the denominator of the expression would simply be 1. If we quickly examine $\overrightarrow{n_1} \cdot \overrightarrow{n_2} = \langle -4, 18, -14\rangle \cdot \langle 1, 1, 1\rangle = -4 + 18 - 14 = 0$, we see that irrespective of the magnitude of the vectors the argument of $\arccos(\cdot)$ is 0 so we must solve $\arccos(0) = \theta$ to find the dihedral angle. This is realized when $\theta = \frac{\pi}{2}$. We may conclude that the planes are orthogonal.

The intersection between two planes results in a line with infinite extent. This line of intersection must also be orthogonal to the planes' normal vectors. Why? Because any line segment of the intersection is a vector on each of the planes, and we know that the normal vector of a plane is orthogonal to any vector on the plane. Therefore, we seek a vector that is orthogonal to both normal vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$. The cross product is the vector operation that produces a vector orthogonal to its arguments so a vector on the line of intersection $\overrightarrow{l} = \overrightarrow{n_1} \times \overrightarrow{n_2}$ (alternatively, we could take the cross product of $\overrightarrow{n_1} \times \overrightarrow{n_2}$, which would simply produce the opposite algebraic sign on $\overrightarrow{l}$).

If we normalize $\overrightarrow{l}$, we obtain $\widehat{l} = \frac{1}{\sqrt{402}}\langle 16, -5, -11\rangle$.

(4) For any point D selected in three-dimensions, can a plane be found that passes through D and is orthogonal to the plane in problem (2)? If one such plane exists, do others and if so, how many? Explain your reasoning.

Solution: Note: In this solution, we refer to the plane in problem 2 as a general *reference plane* because there is not anything special about the plane in problem 2.

For any point D in three-dimensions, a plane can be found that passes through D and is orthogonal to any reference plane, including the plane in problem 2. This includes the case when D is on the reference plane. In fact, infinite planes may be found that satisfy the requirement that D is on the plane and that the new plane is orthogonal to the reference plane.

Consider a fixed reference plane and some arbitrary point D. We may find the shortest line segment between D and the reference plane, which is necessarily orthogonal to the plane. Hence, this shortest line segment connecting D and the reference plane is a scaling of the reference plane's normal vector, which we denote $\overrightarrow{n_r}$.

We know that a normal vector and a single point is necessary and sufficient to specify a **unique** plane; in light of this, observe that the normal vector(s) of the plane(s) that we seek are orthogonal to $\overrightarrow{n_r}$. But there are an infinite number of vectors orthogonal to $\overrightarrow{n_r}$; in fact, the set of vectors orthogonal to $\overrightarrow{n_r}$ all lie in the reference plane. Therefore, there are infinite unique planes that pass through D and are orthogonal to the reference plane. These infinite planes are rotations around the line segment between D and the reference plane. A simple drawing is relatively convincing of this fact.

(5) Sketch the traces and at least one cross section (for a non-zero value of x, y, or z) of the surfaces:
(a) $y = \frac{1}{4}x^2 - 2z^2$
(b) $0 = 2 - x^2 + 2y^2 + 4z^2$
(c) $x = y^2 + z^2$

Solution: See the final pages for plots produced in *Mathematica*. The terms trace or cross section may refer to either two-dimensional plots where the constant coordinate is labeled or three-dimensional plots where the constant coordinate is illustrated by drawing a plane intersecting the surface. In the solutions we opt for the latter by showing the intersection of the three-dimensional surface with the two-dimensional plane with a dashed, blue line. If asked to draw two-dimensional plots, then the dashed, blue line along with axes labels, a few function values, and a label indicating the constant coordinate would be sufficient.

(6) Consider the equation $f(x,y,z) = \sqrt{(x-3)^2 + (y+4)^2 + z^2}$ and its level surfaces at $f(x,y,z) = 1, 2, 3, \ldots$ Qualitatively describe the separation of these level surfaces.

Solution: The level surfaces at the values specified are spherical shells with radii equal to the function values (which were given as the natural numbers), so the separation between the level surfaces is constant.

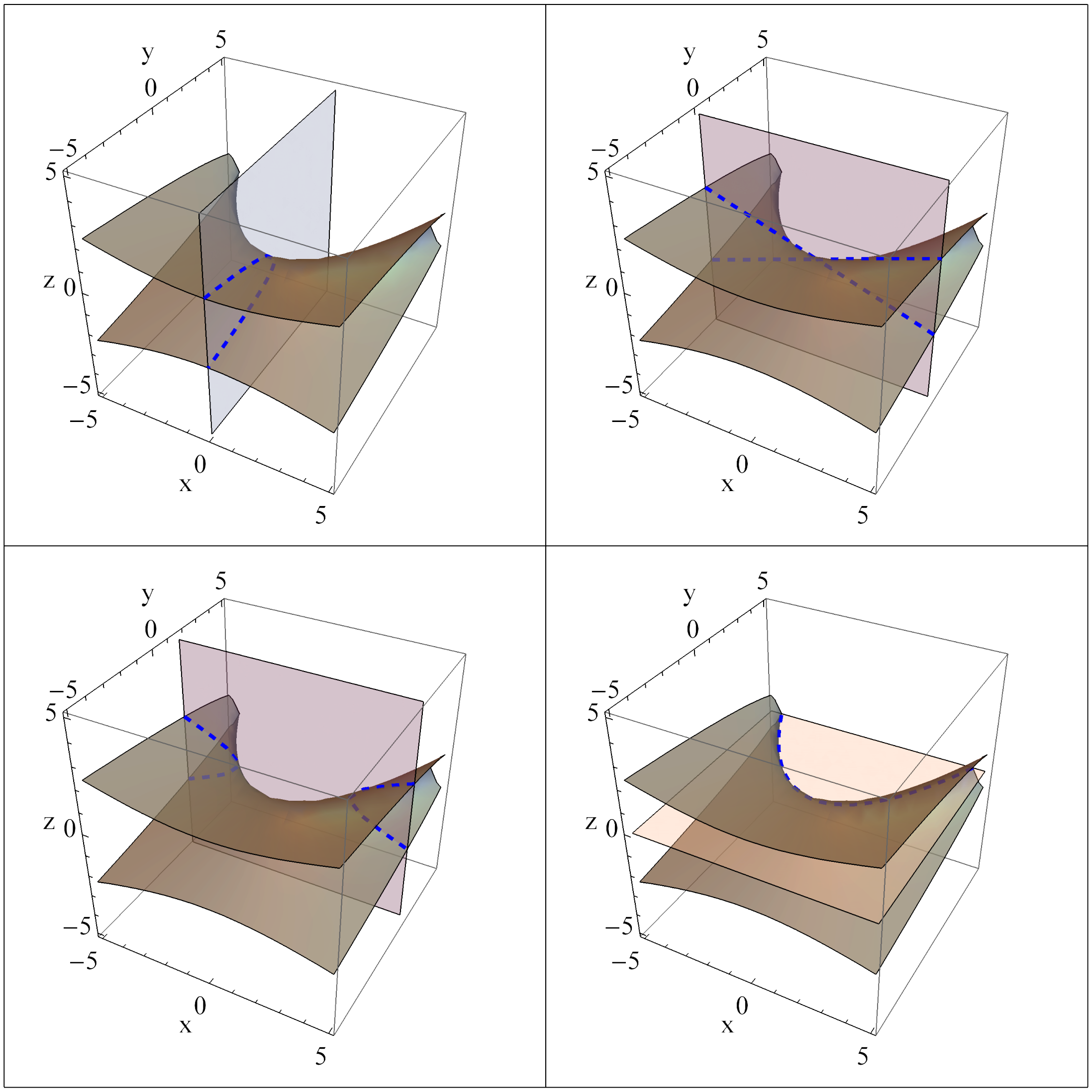

FIGURE 1. Traces and non-zero cross section for problem 5 part (a).

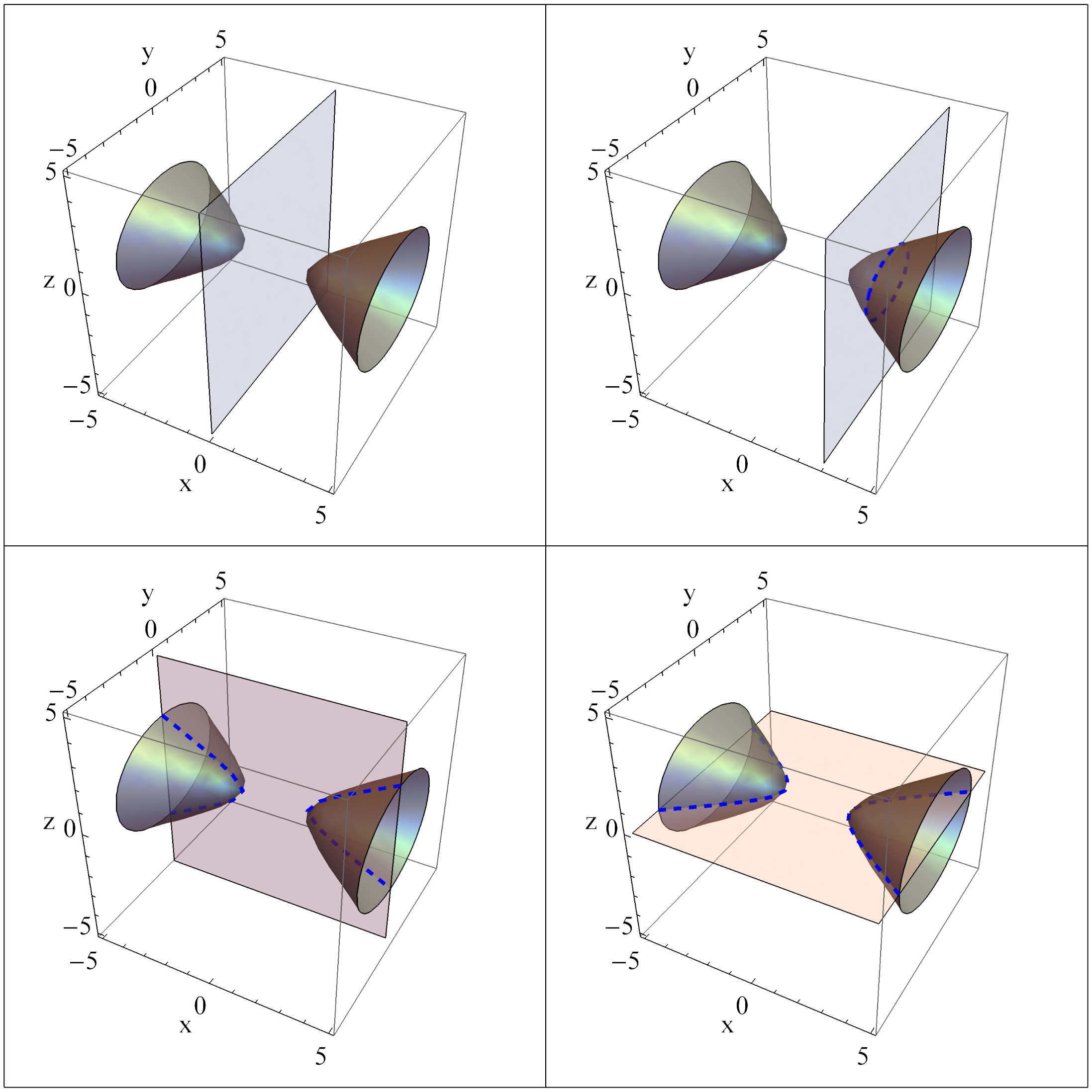

FIGURE 2. Traces and non-zero cross section for problem 5 part (b).

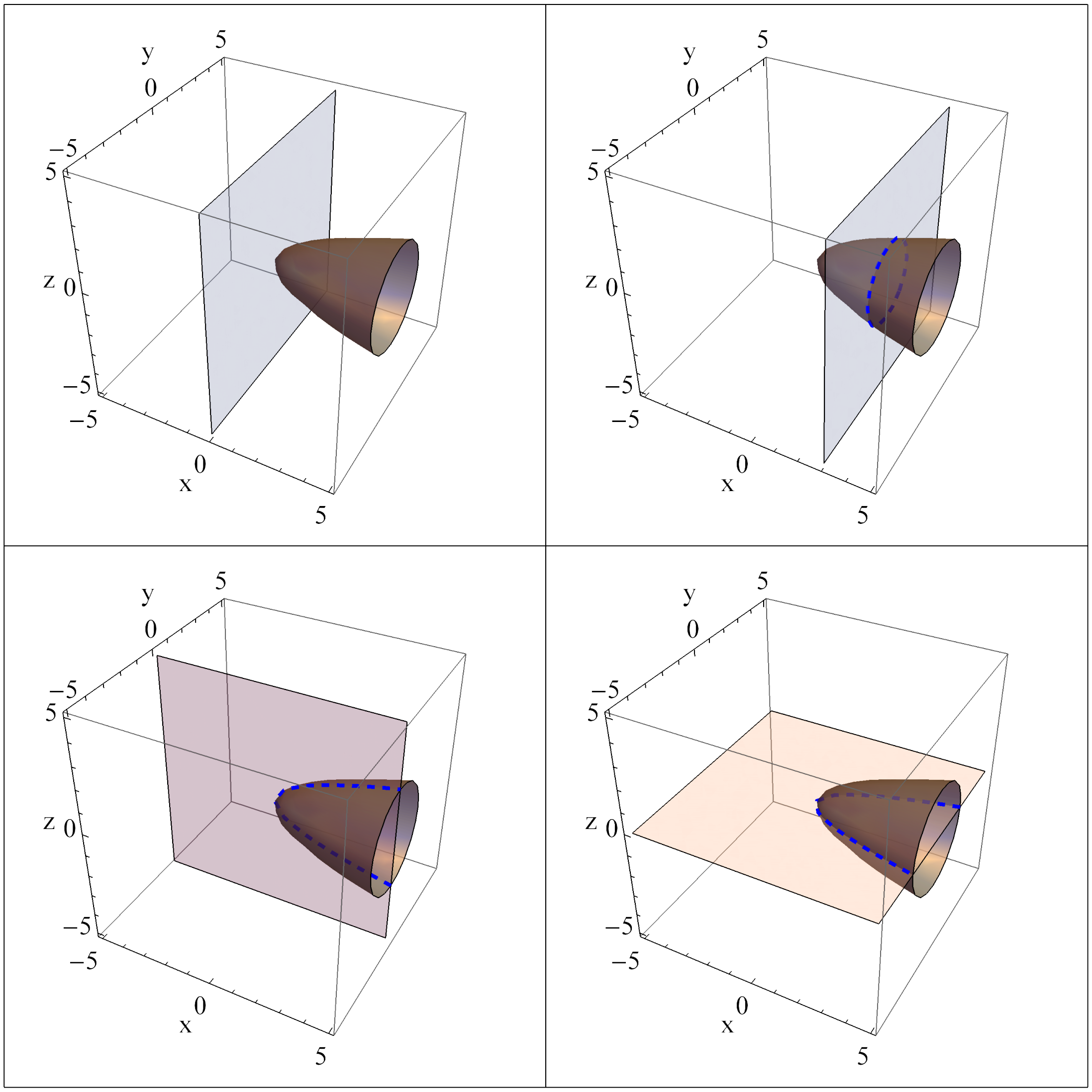

FIGURE 3. Traces and non-zero cross section for problem 5 part (c).