Theorem Proofs for Chapter 3

3.1.3 Bolzano-Weierstrass Theorem *A subset of a metric space is compact iff it is sequentially compact.*

Proof We begin with two lemmas.

Lemma 1 *A compact set $A \subset M$ is closed.*

Proof We will show that $M \backslash A$ is open. Let $x \in M \backslash A$ and consider the following collection of open sets: $U_n = \{y \mid d(y, x) > 1/n\}$. Since every $y \in M$ with $y \neq x$ has $d(y, x) > 0$, y lies in some U_n. Thus, the U_n cover A, and so there must be a finite subcover. One of these has a largest index, say, U_N. If $\varepsilon = 1/N$, then, by construction, $D(x, 1/N) \subset M \backslash A$, and so $M \backslash A$ is open. ▼

Lemma 2 *If M is a compact metric space and $B \subset M$ is closed, then B is compact.*

Proof Let $\{U_i\}$ be an open covering of B and let $V = M \backslash B$, so that V is open. Thus $\{U_i, V\}$ is an open cover of M. Therefore, M has a finite cover, say, $\{U_1, \ldots, U_N, V\}$. Then $\{U_1, \ldots, U_N\}$ is a finite open cover of B. ▼

Proof of 3.1.3 Let A be compact. Assume there exists a sequence $x_k \in A$ that has *no* convergent subsequences. In particular, this means that x_k has infinitely many distinct points, say, $y_1, y_2, \ldots$. Since there are no convergent subsequences, there is some neighborhood U_k of y_k containing no other y_i. This is because if every neighborhood of y_k contained another y_j, we could, by choosing the neighborhoods $D(y_k, 1/m)$, $m = 1, 2, \ldots$, select a subsequence converging to y_k. We claim that the set $\{y_1, y_2, \ldots\}$ is closed. Indeed, it has no accumulation points, by the assumption that there are no convergent subsequences. Applying Lemma 2 to $\{y_1, y_2, \ldots\}$ as a subset of A, we find that $\{y_1, y_2, \ldots\}$ is compact. But $\{U_k\}$ is an open cover that has no finite subcover, a contradiction. Thus x_k has a convergent subsequence. The limit lies in A, since A is closed, by Lemma 1.

Conversely, assume that A is sequentially compact. To prove that A is compact, let $\{U_i\}$ be an open cover of A. We need to prove that this has a finite subcover. To show this, we proceed in several steps.

Lemma 3 *There is an* $r > 0$ *such that for each* $y \in A$, $D(y, r) \subset U_i$ *for some* U_i. *The number* r *is called a* Lebesgue number *for the covering. The infimum of all such* r *is called* the *Lebesgue number for the covering.*

Proof If not, then for every integer n, there is some y_n such that $D(y_n, 1/n)$ is not contained in any U_i. By hypothesis, y_n has a convergent subsequence, say, $z_n \to z \in A$. Since the U_i cover A, $z \in U_{i_0}$ for some U_{i_0}. Choose $\varepsilon > 0$ such that $D(z, \varepsilon) \subset U_{i_0}$, which is possible since U_{i_0} is open. Choose N large enough so that $d(z_N, z) < \varepsilon/2$ and $1/N < \varepsilon/2$. Then $D(z_N, 1/N) \subset U_{i_0}$, a contradiction. ▼

Lemma 4 *A is totally bounded (see Definition* **3.1.4**).

Proof If A is not totally bounded, then for some $\varepsilon > 0$ we cannot cover A with finitely many disks. Choose $y_1 \in A$ and $y_2 \in A \backslash D(y_1, \varepsilon)$. By assumption, we can repeat; choose $y_n \in A \backslash [D(y_1, \varepsilon) \cup \cdots \cup D(y_{n-1}, \varepsilon)]$. This is a sequence with $d(y_n, y_m) \geq \varepsilon$ for all n and m, and so y_n has no convergent subsequence, a contradiction to the assumption that A is sequentially compact. ▼

To complete our proof, let r be as in Lemma **3**. By Lemma **4** we can write $A \subset D(y_1, r) \cup \cdots \cup D(y_n, r)$ for finitely many y_j. By Lemma **3**, $D(y_j, r) \subset U_{i_j}$, $j = 1, \ldots, n$, for some index i_j. Then $U_{i_1}, \ldots, U_{i_n}$ cover A. ■

3.1.5 Theorem *A metric space is compact iff it is complete and totally bounded.*

Proof First assume that M is compact. By **3.1.3**, it is sequentially compact. Thus, if x_k is a Cauchy sequence, it has a convergent subsequence, and so, as in **1.4.7**, the whole sequence converges. Thus M is complete. It is also totally bounded, by Lemma **4**.

Conversely, assume that M is complete and totally bounded. By **3.1.3**, it is enough to show that M is sequentially compact. Let y_k be a sequence in M. We can assume that the y_k are all distinct, for if y_k has infinitely many repetitions, there is a trivially convergent subsequence, and if there are finite repetitions we may delete them. Given an integer N, cover M with finitely many balls $D(x_{L_1}, 1/N), \ldots, D(x_{L_N}, 1/N)$. An infinite number of the y_k lie in one of these balls. Start with $N = 1$. Write $M = D(x_{L_1}, 1) \cup \cdots \cup D(x_{L_N}, 1)$, and so we can select a subsequence of y_k lying entirely in one of these balls. Repeat for $N = 2$, getting a further subsequence lying in a fixed ball of radius $1/2$, and so on. Now choose the "diagonal" subsequence, the first member from the first sequence, the second from the second, and so on. This sequence is Cauchy and since M is complete, it converges. ■

3.2.1 Heine-Borel Theorem *A set $A \subset \mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof We have already proved that compact sets are closed and bounded. We must now show that a set $A \subset \mathbb{R}^n$ is compact if it is closed and bounded. We will give two proofs of this.

First Proof This proof is based on the Bolzano-Weierstrass theorem and the fact that any bounded sequence in $\mathbb{R}$ has a convergent subsequence, proved in 1.4.3. In fact, we shall prove that a closed and bounded set A is sequentially compact. Let $x_k = (x_k^1, x_k^2, \ldots, x_k^n) \in \mathbb{R}^n$ be a sequence. Since A is bounded, x_k^1 has a convergent subsequence, say, ${x^1}_{f_1(k)}$. Then ${x^2}_{f_1(k)}$ has a convergent subsequence, say ${x^2}_{f_2(k)}$. Continuing, we get a further subsequence $x_{f_n(k)} = ({x^1}_{f_n(k)}, \ldots, {x^n}_{f_n(k)})$, all of whose components converge. Thus $x_{f_n(k)}$ converges in $\mathbb{R}^n$. The limit lies in A since A is closed. Thus A is sequentially compact, and so is compact. ■

Second Proof This proof uses the definition of compactness in terms of open covers directly. We begin with a special case:

Lemma 1 *Closed intervals $[a, b]$ in $\mathbb{R}$ are compact.*

Proof Let $\mathcal{U} = \{U_i\}$ be an open covering of $[a, b]$. Define

$$C = \{x \in [a, b] \mid \text{the set } [a, x] \text{ can be covered by a finite collection of the } U_i\}.$$

We want to show that $C = [a, b]$. To this end, let $c = \sup(C)$. The sup exists because $C \neq \varnothing$ (since $a \in C$) and C is bounded above by b. Since $a \in C$ and b is an upper bound for C, $c \in [a, b]$, by definition of $\sup(C)$. Suppose $c \in U_{i_0}$; such a U_{i_0} exists, since the U_i's cover $[a, b]$. Since U_{i_0} is open, there is an $\varepsilon > 0$ such that $]c - \varepsilon, c + \varepsilon[\subset U_{i_0}$. Since $c = \sup(C)$, there exists an $x \in C$ such that $c - \varepsilon < x \leq c$ (see Proposition **1.3.2**). Because $x \in C$, $[a, x]$ has a finite subcover, say, $U_1, \ldots, U_N$; then $[a, c + \varepsilon/2]$ also has the finite subcover $U_1, \ldots, U_N, U_{i_0}$. Thus we conclude that $c \in C$ and moreover that $c = b$. Indeed, if $c < b$, we would get a member of C larger than c, since $[a, c + \varepsilon/2]$ has a finite subcover. The latter cannot happen, since $c = \sup(C)$. ▼

Note. Why does this proof fail for $]a,b]$, $[a,b[$ or $[a,\infty[$?

Lemma 2 *If $A \subset \mathbb{R}^n$ is compact and $x_0 \in \mathbb{R}^m$, then $A \times \{x_0\} \subset \mathbb{R}^n \times \mathbb{R}^m$ is compact.*

Proof Let $\mathcal{U}$ be an open cover of $A \times \{x_0\}$, and

$$\mathcal{V} = \{V \mid V = \{y \mid (y, x_0) \in U\}, \text{ for some } U \in \mathcal{U}\}.$$

Then $\mathcal{V}$ is an open cover of A in $\mathbb{R}^n$, and hence $\mathcal{V}$ has a finite subcover of A, say, $\mathcal{V}' = \{V_1, \ldots, V_k\}$. Each $V_i \in \mathcal{V}'$ corresponds to a $U_i \in \mathcal{U}$, and $\mathcal{U}' = \{U_1, \ldots, U_k\}$ is then a finite subcover in $\mathbb{R}^n \times \mathbb{R}^m$ of $A \times \{x_0\}$. ▼

The next step is an induction argument.

Lemma 3 *If $[-R,R]^{n-1} \subset \mathbb{R}^{n-1}$ is compact, then $[-R,R]^n \subset \mathbb{R}^n$ is compact, where $[-R,R]^n = [-R,R] \times \cdots \times [-R,R]$, n times.*

Proof Suppose that $[-R,R]^{n-1}$ is compact and that $\mathcal{U}$ is an open cover of $[-R,R]^n$. Define

$$S = \{x \in [-R,R] \mid [-R,R]^{n-1} \times [-R,x] \subset \mathbb{R}^n \text{ has a finite subcover in } \mathcal{U}\}.$$

Now $-R \in S$, since $[-R,R]^{n-1}$ is compact, by hypothesis, and so, by Lemma **2**, $[-R,R]^{n-1} \times \{-R\}$ has a finite subcover in $\mathcal{U}$. Since S is bounded above by R, it has a supremum, say, x_0. We will show that $x_0 = R$, which will prove the lemma.

Let $\mathcal{U}' \subset \mathcal{U}$ be a finite subcover of $[-R,R]^{n-1} \times \{x_0\}$. For each point $(y, x_0) \in [-R,R]^{n-1} \times \{x_0\}$, there exists $\varepsilon_y > 0$ such that $D((y,x_0), \sqrt{2}\varepsilon_y)$ is covered by $\mathcal{U}'$. Because

$$V_y = D(y, \varepsilon_y) \times]x_0 - \varepsilon_y, x_0 + \varepsilon_y[\subset D((y,x_0), \sqrt{2}\varepsilon_y),$$

it is covered by $\mathcal{U}'$. Consider the open cover $\mathcal{V} = \{V_y \mid y \in [-R,R]^{n-1}\}$ of $[-R,R]^{n-1} \times \{x_0\}$. By Lemma **2**, $\mathcal{V}$ has a finite subcover of $[-R,R]^{n-1} \times \{x_0\}$, say $\{V_{y_1}, \ldots, V_{y_N}\}$. Let $\varepsilon = \inf\{\varepsilon_{y_1}, \ldots, \varepsilon_{y_N}\}$. Then

$$[-R,R]^{n-1} \times]x_0 - \varepsilon, x_0 + \varepsilon[\subset \bigcup_{i=1}^{\infty} V_{y_i},$$

and so $[-R,R]^{n-1} \times]x_0 - \varepsilon, x_0 + \varepsilon[$ is covered by $\mathcal{U}'$.

With this ε, there exists $x \in S$ such that $x_0 - \varepsilon < x \leq x_0$. Since $x \in S$, there exists a finite subcover $\mathcal{U}'' \subset \mathcal{U}$ which covers $[-R,R]^{n-1} \times [-R,x]$, and $\mathcal{U}' \cup \mathcal{U}''$ is a finite cover of $[-R,R]^{n-1} \times [-R,x_0+\varepsilon[$. Thus $x_0 \in S$. Suppose $x_0 < R$; then choose δ such that $x_0+\delta < R$ and $x_0+\delta < x_0+\varepsilon$. Thus $[-R,R]^{n-1} \times [-R,x_0+\delta]$ is covered by $\mathcal{U}' \cup \mathcal{U}''$, and $x_0+\delta \in S$, a contradiction, and therefore $x_0 = R$. ▼

To conclude the proof of the Heine-Borel theorem, let $A \subset \mathbb{R}^n$ be closed and bounded. Since it is bounded, there is an $R > 0$ such that $A \subset [-R,R]^n$. Lemmas **1** and **3** show that $[-R,R]^n$ is compact. Lemma **2** in the proof of **3.1.3** shows that A is compact, since it is a closed subspace of the compact set $[-R,R]^n$. ■

3.3.1 Nested Set Property *Let F_k be a sequence of compact nonempty sets in a metric space M such that $F_{k+1} \subset F_k$ for all $k = 1,2,\ldots$. Then there is at least one point in $\bigcap_{i=1}^{\infty} F_k$.*

Proof In the compact set $A = F_1$, the sets $F_1, F_2, \ldots$ have the finite intersection property, since the intersection of any finite collection equals the F_k with the highest index. By Example **3.1.9**,

$$F_1 \bigcap \left(\bigcap_{k=1}^{\infty} F_k\right) = \bigcap_{k=1}^{\infty}\{F_k\} \neq \varnothing. \quad ■$$

3.5.2 Theorem *Path-connected sets are connected.*

We begin by first proving a special case of the theorem.

Lemma *The interval $[a,b]$ is connected.*

Proof Suppose the interval were not connected. Then there would be two open sets U and V with $U \cap [a,b] \neq \varnothing$ and $V \cap [a,b] \neq \varnothing$, $[a,b] \cap U \cap V = \varnothing$, and $[a,b] \subset U \cup V$. Further, suppose that $b \in V$. Let $c = \sup(U \cap [a,b])$, which exists, since $U \cap [a,b]$ is nonempty and is bounded above. The set $U \cap [a,b]$ is closed, since its complement is $V \cup (\mathbb{R} \backslash [a,b])$, which is open. Thus $c \in U \cap [a,b]$ (see Exercise 8, Chapter 2). Now $c \neq b$, since $c \notin V$ and $b \in V$. We claim that any neighborhood of c intersects $V \cap [a,b]$. To see this, note that $c \neq b$ and no