Math 135 | **Real Analysis I** | **Fall 2022**
Prof. Hasselblatt & Tu | **Exam 1 Review**

We will have a review session on Sunday, October 16, from 1:30 to 2:45 p.m. in Bromfield-Pearson 002.

Exam 1 will be held during the Open Block from noon to 1:20 p.m. on Monday, October 17, in **JCC 270**. *There will be no Math 135 classes or office hours that day*. The exam will cover the first five problem sets and material up to and including §3.5 plus countability, Cantor's diagonal argument, basic logic (including quantifiers and negations), and mathematical induction.

The exam will include definitions and a logic problem or two, and proofs and examples in which you justify your work. The proofs will be from homework or from this review sheet, or they will be similar to them. Justify every statement you make by either referring to theorems or definitions. It can be helpful to draw pictures in order to understand problems and sets, although a picture is not a proof. On the test, you will get some credit in problems that you cannot finish if you quote relevant definitions and theorems.

What You Should Know for the Exam

In the following, by "notes" we mean the notes posted under "Readings" on the Canvas course site.

I. **Definitions**: countable set (in notes), uncountable set (in notes), supremum (p. 8), infimum (p. 10), dense (p. 15), ϵ-N definition of convergence (p. 26), bounded set (p. 35), bounded sequence (p. 35), closed set (p. 37), monotonically increasing sequence (p. 38), subsequence (p. 43), sequentially compact (p. 46), continuous at a point (p. 53), continuous function (p. 53), ϵ-δ criterion for *continuity at a point* x_0 (p. 70), uniform continuity (p. 66) and the ϵ-δ criterion for *uniform continuity on a set* D (p. 73).

II. **Statements of theorems**: the completeness axiom (p. 8), the difference of powers formula (p. 19), the geometric sum formula (p. 19), monotone convergence theorem (p. 38), every sequence has a monotone subsequence (Th. 2.32), intermediate value theorem (Th. 3.11), any continuous function on a closed bounded interval $[a, b]$ is uniformly continuous (Th. 3.17).

III. **Proofs**: Learn the following proofs: density of rationals (Lemma 14), density of irrationals (Cor. 1.10), limit rule for the sum and product of sequences (Th. 2.10), convergent $\Rightarrow$ bounded (Th. 2.18), sequential denseness theorem (Prop. 2.19), the closed interval is closed (Th. 2.22), every subsequence of a convergent sequence converges to the same limit (Prop. 2.30), every bounded sequence has a convergent subsequence (Th. 2.33—you may assume every sequence has a monotone subsequence), sequential compactness theorem (Th. 2.36) for $[a, b]$ *and* for arbitrary closed and bounded sets, closed and bounded implies sequentially compact (Th. 2.37), the composition of continuous functions is continuous (Th. 3.6), continuity in terms of ϵ-δ criterion (Th. 3.20).

IV. Review the homework problems (Problem Sets 1 – 5).

Turn over for review problems.

Math 135 **Real Analysis I** **Fall 2022**
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Solutions to the review problems will be posted on Canvas Friday evening, **October 14, 2022**. Try to do the problems on your own before then.

1. Prove that every rational number is the limit of a sequence of irrational numbers.

2. Prove the following classic theorems.

   (a) The set $\mathbb{Q}$ of rational numbers is countable.

   (b) The interval $[0, 1)$ is uncountable (Cantor's diagonal argument), and the set $\mathbb{R}$ of real numbers is uncountable.

   (c) The number $\sqrt{3}$ is irrational.

3. Show that the set T of all natural numbers divisible by 3 is countable.

4. By the monotone convergence theorem, every increasing bounded sequence of real numbers converges to a real number. Find an increasing, bounded sequence of *rational* numbers that does **not** converge to a *rational* number.

5. Prove that for any two real numbers $x < y$, there is a rational number r with $x < r < y$.

6. Prove that there is an *irrational* number between any two nonequal real numbers, i.e., $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}$, if $x < y$ then $\exists z \notin \mathbb{Q}$ such that $x < z < y$.

7. Using any method, find the following limits. Show your work.

   (a) $\displaystyle\lim_{n\to\infty} \frac{2n^2 + 3n + 5}{2n^2 - 1}$ (b) $\displaystyle\lim_{n\to\infty} (\sqrt{n} - \sqrt{n-2})\sqrt{n}$

8. Prove the sandwich theorem (squeeze theorem): Let $L \in \mathbb{R}$ and let $\{x_n\}$ and $\{z_n\}$ be sequences that both converge to L. Assume $\{y_n\}$ is a sequence such that $\forall n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. Prove $y_n \to L$ as $n \to \infty$ using the ϵ-N definition of limit.

9. Let x and y be elements of $\mathbb{R}$. Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences in $\mathbb{R}$, $x_n \to 3$ and $y_n \to 5$. Use the ϵ-N definition of the limit, to prove the following:

   (a) $\lim x_n + y_n = 8$.

   (b) $\lim x_n y_n = 15$.

   (c) $\lim 1/x_n = 1/3$.

10. Let S be the half-open interval $[1, 2)$.

    (a) Using the definition of sequential compactness, show that S is not sequentially compact.

    (b) Using the definition of a closed set, show that S is **not** closed.

(More problems on reverse side)

11. Let S be the set of rational numbers in the closed interval $[1, 2]$.

    (a) Using the definition of sequential compactness, show that S is not sequentially compact.

    (b) Using the definition of a closed set, show that S is **not** closed.

12. True or False.

    (a) The set of positive real numbers is dense in $\mathbb{R}$.

    (b) The set of rational numbers that are not integers is dense in $\mathbb{R}$.

    (c) If the sequence $\{a_n^2\}$ converges, then the sequence $\{a_n\}$ also converges.

    (d) The limit of a convergent sequence in the interval (a, b) also belongs to (a, b).

    (e) The sum of monotone sequences is monotone.

    (f) The product of monotone sequences is monotone.

    (g) The set of rational numbers in the interval $[0, 1]$ is sequentially compact.

    (h) Every function $f\colon \mathbb{N} \to \mathbb{R}$ is continuous, where $\mathbb{N}$ denotes the set of natural numbers.

13. (a) Prove that every sequentially compact set in $\mathbb{R}$ is closed and bounded.

    (b) Prove that every closed and bounded subset of $\mathbb{R}$ is sequentially compact.

14. Find the limit $\lim_{n\to\infty} \dfrac{3n^3+2}{3n^3-n+1}$ and prove your result *using the ϵ-N definition of limit.*

15. Negate the statement: For any $\epsilon > 0$, there is a finite set $x_1, \ldots, x_k$ of points in A such that $A \subset \bigcup_{i=1}^{k} B_\epsilon(x_i)$. (Although you don't need to know this for the problem $B_\epsilon(x) = \{y \in \mathbb{R} \mid |y - x| < \epsilon\} = (x - \epsilon, x + \epsilon)$.)

16. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and assume f takes on only integer values (i.e., for each $x \in \mathbb{R}$, $f(x) \in \mathbb{Z}$. Prove f is a constant function). HINT: Prove this by contradiction, and start the proof as follows. Let x_1 and x_2 be points in $\mathbb{R}$ with $f(x_1) \neq f(x_2)$. You may assume $x_1 < x_2$ without loss of generality.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$. Determine whether f is continuous at $x = 0$ and at $x = 1$ and prove your results.

18. Find an open interval of length 1 that contains a root of $f(x) = x^3 - x - 1$.

19. Let $\{a_n\}$ be a real sequence that converges to $a \in \mathbb{R}$. Let $c \in \mathbb{R}$ and assume $a < c$. Prove that there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < c$.

20. Is $f(x) = x^2$ uniformly continuous on $(0, \infty)$? Is it uniformly continuous on $[1, 5]$? Justify your answers.