1 Suppose C is an $n \times n$ matrix and there is some $\mathbf{b} \in \mathbb{R}^n$ such that $C\mathbf{x} = \mathbf{b}$ has infinitely many solutions. Do the columns of C span all of $\mathbb{R}^n$? Justify your answer.

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\vec{a}_1 \ \ \vec{a}_2]$$

$$\left[\begin{array}{cc|c} a & b & e_1 \\ c & d & e_2 \end{array}\right]$$

$$\vec{a}_1 = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Infinitely many sol'ns

$\Rightarrow$ free variable.

$\text{Span}\left\{ \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right\} = \mathbb{R}^2$? No, by Thm 4.

A linear combination of vectors $\vec{a}_1, \ldots, \vec{a}_n$ is a vector of the form

$$c_1\vec{a}_1 + \cdots + c_n\vec{a}_n$$

where $c_1, \ldots, c_n$ are real numbers.

$\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$ is the set of all linear combos of $\vec{a}_1, \ldots, \vec{a}_n$.

Thm 4: A an $m \times n$.

TFAE:

a) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$

b) Every $\vec{b}$ in $\mathbb{R}^n$ is a lin. comb. of cols of A

b) Each $\vec{b}$ in $\mathbb{R}^m$ is a lin. comb of cols of A.
c) Cols of A span $\mathbb{R}^n$.
d) A has a pivot pos'n in every row.

2

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ and let $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n} \end{bmatrix}$ be an $m \times n$ matrix. Using the definition of matrix-vector multiplication, prove that

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}.$$

(b) Suppose $\mathbf{u}$ and $\mathbf{v}$ are both solutions to the matrix-vector equation $A\mathbf{x} = \mathbf{0}$. Show that $\mathbf{u}+\mathbf{v}$ is also a solution.

a)

$$A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$$

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

$$A(\vec{u}+\vec{v}) = [\vec{a}_1 \ \cdots \ \vec{a}_n]\underbrace{\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right)}_{\vec{x}}$$

$$= [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

$$= (u_1+v_1)\vec{a}_1 + (u_2+v_2)\vec{a}_2 + \dots + (u_n+v_n)\vec{a}_n$$

$$= u_1\vec{a}_1 + v_1\vec{a}_1 + u_2\vec{a}_2 + v_2\vec{a}_2 + \dots + u_n\vec{a}_n + v_n\vec{a}_n$$

$$= (u_1\vec{a}_1 + u_2\vec{a}_2 + \dots + u_n\vec{a}_n) + (v_1\vec{a}_1 + \dots + v_n\vec{a}_n)$$

$$= [\vec{a}_1 \ \cdots \ \vec{a}_n]\begin{bmatrix} u_1 \\ \vdots \end{bmatrix} + [\vec{a}_1 \ \cdots \ \vec{a}_n]\begin{bmatrix} v_1 \\ \vdots \end{bmatrix}$$

$$= [u_1 \cdots u_n] \begin{bmatrix} \vdots \\ u_n \end{bmatrix} + [u_1 \cdots u_n] \begin{bmatrix} \vdots \\ v_n \end{bmatrix}$$

$$= A\vec{u} + A\vec{v}.$$

3 Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and let $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(a) Check that $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

(b) Check that $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is also a solution to $A\mathbf{x} = \mathbf{b}$.

(c) Is $\mathbf{u} + \mathbf{v}$ a solution to $A\mathbf{x} = \mathbf{b}$?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

a) $$A\vec{u} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{1}_{} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underbrace{1}_{} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{b} \quad \checkmark$$

b) $$A\vec{v} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{b} \quad \checkmark$$

c) $A(\vec{u} + \vec{v}) = \Big[1 \quad 1 \Big] \Big(\Big[1 \Big] + \Big[3 \Big] \Big)$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \neq \vec{b}.$$

With the least amount of work possible, decide which of the following sets of vectors are linearly independent, and give a reason for each answer.

(a)

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$$

(b) The columns of

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 7 & 8 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$$

5 cols, 4 rows $\Rightarrow$ lin dep by Thm 8.

(c)

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

lin. dep. by Thm 9.

$$0\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

(d)

$$\left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$$

$$x_1 \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_1 = 0$$

We say $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent if the equation

$$x_1\vec{v}_1 + \ldots + x_n\vec{v}_n = \vec{0}$$

$$8x_1 = 0$$
$$2x_1 = 0$$
$$x_1 = 0$$
$$4x_1 = 0$$

has only the trivial sol'n $[x_1 = 0, x_2 = 0, \ldots, x_n = 0]$.

otherwise, linearly dependent, and call

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0}$$

a linear dependence relation.

Thm 8:

If $p > n$, $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent.

Thm 9:

Any set of vectors in $\mathbb{R}^n$ that contains $\vec{0}$ is linearly dependent.

(a)

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$$

$$\vec{u}, \vec{v}$$

$$c_1 \vec{u} + c_2 \vec{v} = \vec{0}$$

If $c_1 \neq 0$,

$$\vec{u} + \left(\frac{c_2}{c_1}\right) \vec{v} = \vec{0}$$

$$u = \left(-\frac{c_2}{c_1}\right) v$$

If $c_2 \neq 0$,

$$\vec{v} = \left(-\frac{c_1}{c_2}\right)\vec{u}.$$

So $\{\vec{u}, \vec{v}\}$ is linearly dependent if and only if one vector is a multiple of the other.

<u>Q</u>: is there $c \in \mathbb{R}$ s.t.

$$\vec{u} = c\vec{v}$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = c \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} ?$$

$c = 1/3$?

$$\frac{1}{3}\begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4/3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

so there is no c that works

$\Rightarrow$ linearly independent.