Tufts University
Department of Mathematics
Math 235 **Homework #8 : Due on 12/1**[1] **Fall, 2023**
All sections referred to Heil's Introduction to real analysis.

1. **Reading assignment**
   1.1 5.1–5.5
   1.2 6.1
2. **Problems: The problems with bolded numbers will be collected for grading for both completeness and correctness. However, credit will also be given for attempts that are fully justified.**
   2.1 5.1.5, 5.1.6, **5.1.7**.
   2.2 5.2.18, **5.2.21, 5.2.22**, 5.2.23, 5.2.25.
   2.3 **5.4.5, 5.4.6**, 5.4.7.
   2.4 5.5.14, **5.5.19, 5.5.20, 5.5.21**
3. These are extra problems for practice. They will not be collected.
   3.1. Evaluate the following limits:
      3.1.1 $\lim_{n\to\infty} \int_0^1 \frac{ne^x}{1+n^2x^{1/2}}\, dx$.
      3.1.2 $\sum_{n=0}^{\infty} \frac{1}{n} \int_1^2 \ln^n x dx$.
   3.2. Let f be an integrable function on a measurable set $E \subset \mathbb{R}^d$. Show that if $\{A_n\}$ is a sequence of measurable subsets of E such that $|A_n| \to 0$, then $\int_{A_n} f \to 0$.
   3.3. Compute the following limit for each of the cases $a > 0$, $a = 0$, and $a < 0$: $\lim_{n\to\infty} \int_a^{\infty} \frac{n}{1+n^2x^2} dx$.
   3.4. Prove that if $f \in L^1(\mathbb{R})$, then $\lim_{n\to\infty} f(x) \sin nx dx = 0$.
   3.5. Let $E \subset \mathbb{R}^d$ be measurable. Assume that functions $f_n \in L^(E)$ satisfy $\|f_n\|_1 \to 0$, and there exists some $g \in L^1(E)$ such that $|f_n|^2 \le g$ a.e. for every n. Prove that $\int_E |f_n|^2 \to 0$.
   3.6. Let $f \in L^1(\mathbb{R})$ and let $g_n(x) = \int_{x-n}^{x+n} f(t)dt$.
      3.6.1 Prove that g_n is continuous.
      3.6.2 Given $x \in \mathbb{R}$, determine whether $\lim_{n\to\infty} g_n(x)$ exists, and if so find it.
      3.6.3 Given $n \in \mathbb{N}$, determine whether $\lim_{x\to\infty} g_n(x)$ exists, and if so find it.
   3.7 Compute $\iint |f|$ and the two iterated integrals of f. Do They exist? Are they equal? where $f(x, y) = \frac{xy}{(x^2+y^2)^2}$ on the domain $[-1, 1]^2$.
   3.8 Let $g : \mathbb{R} \to \mathbb{R}$ be a nonmeasurable function and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} g(x) : & y \in \mathbb{Q} \\ e^{-|x|-|y|} & y \notin \mathbb{Q} \end{cases}$.
      Is f measurable? Is f integrable?
   3.9 Let f be a bounded measurable function on a measurable set $E \subset \mathbb{R}^d$, and suppose that there exist some constants $C > 0$ and $\alpha \in (0, 1)$ such that $|\{|f| > t\}| \le Ct^{-\alpha}$ for all $t > 0$. Prove that f is integrable.
   3.10 This question has two parts.
      3.10.1 Prove that $f(x) = \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{\sin(x-t)}{1+(x-t)^2} dt$ is continuous on $\mathbb{R}$.
      3.10.2 Is $F(x, t) = \frac{1}{1+t^2} \frac{\sin(x-t)}{1+(x-t)^2}$ integrable on $\mathbb{R}^2$?