MATH 70 WORKSHEET 7

Instructions: This worksheet is due on Gradescope at 11:59 p.m. Eastern Time on Monday, November 16.

1. (4 points) Let V be a vector space. Assume $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an independent set in V. Decide whether the following sets are also independent and justify your answer.

   (a) $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3\}$.

   (b) $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_3\}$.

Solution:

We will use two different possible strategies, one for each part.

(a) Since S is linearly independent, it is a basis for it's span. We can, therefore, use a coordinate map to represent

$$[\mathbf{v}_1 + \mathbf{v}_2]_S = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; [\mathbf{v}_1 - \mathbf{v}_3]_S = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \; [\mathbf{v}_2 + \mathbf{v}_3]_S = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

To check for linear independence, we set up the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is a free variable, the set is linearly dependent.

(b) Similarly, suppose there are weights c_1, c_2, c_3 such that

$$c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2\mathbf{v}_2 + c_3(\mathbf{v}_1 - \mathbf{v}_3) = \mathbf{0}.$$

Distributing the weights and collecting like terms, we have

$$(c_1 + c_3)\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 - c_3\mathbf{v}_3 = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, we know $c_1 + c_3 = 0$, $c_1 + c_2 = 0$, and $-c_3 = 0$. Since $c_3 = 0$, we can then conclude that $c_1 = 0$, and hence $c_2 = 0$. Since only the trivial solution exists, the set is linearly independent.

2. (6 points) Let V and W be finite dimensional vector spaces and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of V. Let T be a linear transformation, $T : V \to W$. Assume T maps onto W (T is surjective).

   (a) Prove that $\mathcal{B}' = \{T(\mathbf{b}_1), T(\mathbf{b}_2), \ldots, T(\mathbf{b}_n)\}$ spans W.

   (b) Prove that $\dim(W) \leq \dim(V)$.

Solution:

(a) To show that $\mathcal{B}'$ spans W, we want to take an arbitrary vector $\mathbf{w}$ in W and write it as a linear combination of the vectors in $\mathcal{B}'$. Since T is onto, there exists a vector $\mathbf{v}$ in V satisfying $T(\mathbf{v}) = \mathbf{w}$. Since $\mathcal{B}$ is a basis for V, there exists weights $c_1, \ldots, c_n$ such that
$$v = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$
Apply the transformation T to both sides to get
$$\begin{aligned} w = T(v) &= T(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) \\ &= c_1 T(\mathbf{b}_1) + \cdots + c_n T(\mathbf{b}_n). \end{aligned}$$
This proves that $\mathcal{B}'$ spans W.

(b) If the set $\mathcal{B}'$ is linearly independent, then $\mathcal{B}'$ is a basis, and hence $\dim(W) = \dim(V)$. If $\mathcal{B}'$ is not linearly independent, then at least one of the vectors is redundant. The Spanning Set Theorem tells us that we can remove redundant vectors without affecting the span of $\mathcal{B}'$ until we have a linearly independent set, and hence a basis. In this case, $\dim(W) < \dim(V)$.

3. (7 points) Let $h_1(t) = \frac{1}{2}(t-1)(t-2)$, $h_2(t) = -t(t-2)$ and $h_3(t) = \frac{1}{2}t(t-1)$.

(a) For $j = 1, 2, 3$ calculate $h_j(0)$, $h_j(1)$, and $h_j(2)$.

(b) Show that $\mathcal{B} = \{h_1(t), h_2(t), h_3(t)\}$ is basis for $\mathbb{P}_2$. HINT: Because of the result of part (a), if you take a linear combination $c_1h_1(t) + c_2h_2(t) + c_3h_3(t) = 0$ and plug in $t = 0, 1, 2$ you can easily solve for c_1, c_2, and c_3.

(c) Find the coordinate of the polynomial $p(t) = t^2 - t + 1$ relative to $\mathcal{B}$. HINT: Use the same trick as for part (b), plugging in $t = 0$, $t = 1$, and $t = 2$, into the linear combination gives the coordinates: $c_1h_1(t) + c_2h_2(t) + c_3h_3(t) = p(t)$.

The polynomials h_1, h_2, and h_3 are special (and cool!) and they are called Bernstein Polynomials.

Solution:

(a) Calculate solutions directly:
$$\begin{aligned} h_1(0) &= \frac{1}{2}(0-1)(0-2) = 1 & h_2(0) &= -0\cdot(0-2) = 0 & h_3(0) &= \frac{1}{2}0(0-1) = 0 \\ h_1(1) &= \frac{1}{2}(1-1)(1-2) = 0 & h_2(1) &= -1\cdot(1-2) = 1 & h_3(1) &= \frac{1}{2}1(1-1) = 0 \\ h_1(2) &= \frac{1}{2}(2-1)(2-2) = 0 & h_2(2) &= -2\cdot(2-2) = 0 & h_3(2) &= \frac{1}{2}2(2-1) = 1 \end{aligned}$$

(b) Let $\mathcal{B} = \{h_1(t), h_2(t), h_3(t)\}$. To prove the vectors in $\mathcal{B}$ are linearly independent, let $c_1, c_2, c_3 \in \mathbf{R}$ be any constants such that
$$c_1h_1(t) + c_2h_2(t) + c_3h_3(t) = 0$$

for **all** values of $t \in \mathbf{R}$. Thus, using the results from part a) we have

$$\begin{aligned} t = 0 \quad &=> \quad c_1h_1(0) + c_2h_2(0) + c_3h_3(0) = c_1 + 0 + 0 = 0 \\ t = 1 \quad &=> \quad c_1h_1(1) + c_2h_2(1) + c_3h_3(1) = 0 + c_2 + 0 = 0 \\ t = 2 \quad &=> \quad c_1h_1(2) + c_2h_2(2) + c_3h_3(2) = 0 + 0 + c_3 = 0 \end{aligned}$$

Thus $c_1 = c_2 = c_3 = 0$. Hence $\mathcal{B}$ is linearly independent by definition.
Since $\dim(\mathbb{P}_2) = 3$ and $h_1(t), h_2(t), h_3(t) \in P_2$ are linearly independent, $\mathcal{B}$ forms a basis of $\mathbb{P}_2$ by Ch.4 Thm.13 (the Basis Theorem).

(c) Since $\{h_1(t), h_2(t), h_3(t)\}$ is a basis for $\mathbb{P}_2$, we can write

$$p(t) = t^2 - t + 1 = c_1h_1(t) + c_2h_2(t) + c_3h_3(t)$$

for some constants $c_!, c_2, c_3 \in \mathbf{R}$. Thus, from part a) we have

$$\begin{aligned} p(0) &= 0^2 - 0 + 1 = c_1h_1(0) + c_2h_2(0) + c_3h_3(0) = c_1 + 0 + 0 \\ &=> 1 = c_1 \\ p(1) &= 1^2 - 1 + 1 = c_1h_1(1) + c_2h_2(1) + c_3h_3(1) = 0 + c_2 + 0 \\ &=> 1 = c_2 \\ p(2) &= 2^2 - 2 + 1 = c_1h_1(2) + c_2h_2(2) + c_3h_3(2) = 0 + 0 + c_3 \\ &=> 3 = c_3 \end{aligned}$$

Thus the coordinates of $p(t)$ relative to $\mathcal{B}$ are $\big[p(t)\big]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

4. (3 points) Let A be an $n \times n$ matrix. Assume A has rank n. Explain why the product A^TA is invertible.

Solution:

By the Invertible Matrix Theorem, if A has rank n, then $\det(A) \neq 0$. We also know that $\det(A^T) = \det(A)$. Finally,

$$\det(A^TA) = \det(A^T)\det(A) = \det(A)^2 \neq 0$$

since the square of a nonzero number cannot be zero. Again by the Invertible Matrix Theorem, A^TA is invertible.