① ⓐ An algorithm $\tilde{f}$ is backwards stable if $\tilde{f}(x) = f(\tilde{x})$ for every x and with $\left|\frac{x-\tilde{x}}{x}\right| = O(\varepsilon_{mach})$

ⓑ $\tilde{f}(a,b) = a^2(1+\varepsilon_1) \oplus b^2(1+\varepsilon_2)$ $|\varepsilon_1| \le \varepsilon_{mach}$; $|\varepsilon_3| \le \varepsilon_{mach}$

$= [a^2(1+\varepsilon_1) + b^2(1+\varepsilon_2)](1+\varepsilon_3)$ $|\varepsilon_2| \le \varepsilon_{mach}$

$= \underline{a^2(1+\varepsilon_1)(1+\varepsilon_3)} + \underline{b^2(1+\varepsilon_2)(1+\varepsilon_3)}$

$= \tilde{a}^2 + \tilde{b}^2$

where $\tilde{a} = a\sqrt{(1+\varepsilon_1)(1+\varepsilon_3)}$

$\tilde{b} = b\sqrt{(1+\varepsilon_2)(1+\varepsilon_3)}$

consider $\dfrac{|\tilde{a}-a|}{|a|} = \dfrac{|a\sqrt{(1+\varepsilon_1)(1+\varepsilon_3)} - a|}{|a|}$

Note $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \dots$

$a\sqrt{(1+\varepsilon_1)(1+\varepsilon_3)} = a\sqrt{1+\varepsilon_1+\varepsilon_3+\varepsilon_1\varepsilon_3}$

$= a\left(1 + \frac{\varepsilon_1}{2} + \frac{\varepsilon_3}{2} + \frac{\varepsilon_1\varepsilon_3}{2} - \frac{1}{8}(\varepsilon_1+\varepsilon_3+\varepsilon_1\varepsilon_3)^2 + O(\varepsilon_{mach}^3)\right)$

$\dfrac{|\tilde{a}-a|}{|a|} \le \left|\frac{\varepsilon_1}{2} + \frac{\varepsilon_3}{2} + \frac{\varepsilon_1\varepsilon_3}{2} + O(\varepsilon_{mach}^2)\right|$

$\le O(\varepsilon_{mach})$

Similarly, $\dfrac{|\tilde{b}-b|}{|b|} \le O(\varepsilon_{mach})$

$\tilde{f}$ is backwards stable.

ⓒ $\tilde{f}(x) = 1 \oplus x$

$= (1+x)(1+\varepsilon)$ $|\varepsilon| \le \varepsilon_{mach}$

$= 1 + \varepsilon + x + \varepsilon x$

$= 1 + \tilde{x}$ where $\tilde{x} = \varepsilon + x + \varepsilon x$

$\dfrac{|\tilde{x}-x|}{|x|} = \dfrac{|\varepsilon + x + \varepsilon x - x|}{|x|} = \dfrac{|x(\frac{\varepsilon}{x} + \varepsilon)|}{|x|} = \left|\varepsilon + \frac{\varepsilon}{x}\right|$

Near x to zero, $x \approx 0$, $\frac{\varepsilon}{x}$ could be large

$x \approx 0$, $\left|\frac{\tilde{x}-x}{x}\right| \ne O(\varepsilon_{mach})$

② ⓐ $\lim\limits_{\varepsilon \to 0} \sup\limits_{|x-\tilde{x}| \le \varepsilon} \dfrac{\frac{|f(x)-f(\tilde{x})|}{|f(x)|}}{\frac{|x-\tilde{x}|}{|x|}}$

For differentiable function f,

$$K_f(x) = \frac{|f'(x)|\,|x|}{|f(x)|}$$

(b) $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$$K_f(x) = \frac{\left|\frac{1}{2} x^{-1/2}\right| |x|}{|x^{1/2}|} = 1/2$$

Problem is well-conditioned for any $x > 0$

(c) $$K_h(x) = \frac{|h'(x)|\,|x|}{|h(x)|}$$

using chain rule, $h'(x) = f'(g(x))\, g'(x)$

$$K_h(x) = \frac{|f'(g(x))\, g'(x)|\,|x|}{|f(g(x))|} \qquad \text{Let } y = g(x)$$

$$= \frac{|f'(y)\, g'(x)|\,|x|}{|f(y)|}$$

$$= \underbrace{\frac{|y|\,|f'(y)|}{|f(y)|}}\,\underbrace{\frac{|g'(x)|\,|x|}{|g(x)|}}$$

$$= K_f \cdot K_g$$

3 (a) (i) Bisection

a	b	c	f(a) f(c)
1	2	1.5	+
1.5	2	1.75	+
1.75	2	1.875	−
1.75	1.875	1.8125	

(ii) Secant

$x_0 = 1; \quad x_1 = 2$

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{(f_k - f_{k-1})} f_k \qquad k = 1, 2, \ldots$$

$$X_2 = X_1 - \frac{(X_1 - X_0)}{(f_1 - f_0)} f_1 = 2 - \left(\frac{2-1}{4-(-10)}\right) 4 = 2 - \frac{2}{7} = 1.7143$$

$$x_3 = x_2 - \left(\frac{x_2 - x_1}{f_2 - f_1}\right) f_2$$

$$= 1.7143 - \left(\frac{1.7143 - 2}{-3.0776 - 4}\right)(-3.0776) = 1.8385$$

$$x_4 = x_3 - \left(\frac{x_3 - x_2}{f_3 - f_2}\right) f_3$$

$$= 1.8385 - \left(\frac{1.8385 - 1.7143}{-0.4135 - (-3.0776)}\right) -0.4135 = 1.8578$$

iii) Newton

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{(x_k^4 - x_k - 10)}{(4x_k^3 - 1)}$$

$$x_0 = 2$$

$$x_1 = 2 - \left(\frac{16 - 2 - 10}{32 - 1}\right) = 1.8710$$

$$x_2 = 1.8710 - \left(\frac{1.8710^4 - 1.8710 - 10}{4(1.8710)^3 - 1}\right) = 1.8710 - \frac{0.3835}{25.1988} = 1.8558$$

$$x_3 = 1.8558 - \left(\frac{1.8558^4 - 1.8558 - 10}{4(1.8558)^3 - 1}\right) = 1.8558 - \frac{0.0053}{24.5655}$$

$$= 1.8556$$

(6) Bisection - simple algorithm
- Inexpensive
- converges always
- speed of converge slow (at worst linear)

Secant - useful when derivative is not available or expensive
- More expensive than bisection
- local convergence
- superlinear convergence rate

Newton - Expensive than bisection or secant especially when derivative is complicated
- local quadratic convergence

4a) a) Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{L}$ such that $|x-y| < \delta$

$$|f(x) - f(y)| \leq L|x-y| \leq L \cdot \frac{\varepsilon}{L} \leq \varepsilon$$

Therefore, f is continuous on C

(b) For contradiction, assume two fixed points r_1 and r_2

$|r_1 - r_2| = |f(r_1) - f(r_2)|$
$\le L|r_1 - r_2|$
$< |r_1 - r_2|$

$|r_1 - r_2| < |r_1 - r_2|$ contradiction

$\therefore$ f has a unique fixed point

(c) $|x_1 - r| = |f(x_0) - r|$
$= |f(x_0) - f(r)|$
$\le L|x_0 - r|$

$|x_2 - r| = |f(x_1) - r|$
$= |f(x_1) - f(r)|$
$\le L|x_1 - r| \le L^2 |x_0 - r|$

Arguing similarly, $|x_n - r| \le L^n |x_0 - r|$

$\lim_{n \to \infty} L^n |x_0 - r| = 0$ since $L < 1$

Therefore, $\lim_{n \to \infty} x_n = r$ as desired.

4b (a) $g(x) = \cos(x) + \pi + 1$
$g'(x) = -\sin(x)$
$g'(\pi) = -\sin(\pi) = 0$ locally convergent

* The original exam had a typo, the fixed point is $r = \pi$

(b) $g(x) = e^{2x-1}$
$g'(x) = 2e^{2x}$
$g'(0) = 2$ divergent

5. (a) $(1,0)$ $(2, \ln 2)$ $(4, \ln 4)$

$P_2(x) = y_2 l_2(x) + y_3 l_3(x)$

$$= \ln(2) \frac{(x-1)(x-4)}{(2-1)(2-4)} + \ln(4) \frac{(x-1)(x-2)}{(4-1)(4-2)}$$

$$= \ln(2) \left[\frac{x^2 - 5x + 4}{-2}\right] + \frac{\ln(4)}{6} \left[x^2 - 3x + 2\right]$$

(b) $\ln 3 = 0.6931471 + 0.46209812037$

$= 1.15524$

(c) $\text{Error} \equiv \dfrac{f^{(3)}(\xi)}{3!} \prod_{i=0}^{2} 3 - x_i \qquad \xi \in [1,4]$

$= \dfrac{2}{\xi^3 \, 3!} \left(2 \cdot 1 \cdot -1 \right)$

$= \dfrac{4}{\xi^3 \, 3!}$

$|\text{Error}| = \dfrac{2}{3} \left| \dfrac{1}{\xi^3} \right| \leq 2/3$