Carefully PRINT your full name:

CIRCLE your section: Section 1 (Tu) Section 2 (Tu) Section 3 (Hasselblatt)

MATH 135 Exam 1 October 17, 2022

(100 points) 12 noon–1:20 p.m.

Instructions: No books, notes, calculators, or external help from any person or device are allowed. Except in the true-false questions, justify all of your steps. Write only in the space provided and do not attach any extra page.

Please sign the following pledge: *I pledge that in this exam I have neither given nor received assistance or cheated in any other way.*

Signature: ______________________________

1. (10 points) Circle either True or False. You do not need to justify your choice.

   (a) **True / False:** $\mathbb{Z}$ is dense in $\mathbb{R}$.

   (b) **True / False:** Every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.

   (c) **True / False:** The product of monotone sequences is monotone.

   (d) **True / False:** $\mathbb{Q} \cap [0, 1]$ is a closed set in $\mathbb{R}$.

   (e) **True / False:** An unbounded sequence does not converge.

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2. (30 points)

   (a) (5 pts) We learned that $\mathbb{Q}$ is countable, so there is a sequence $\{a_n\}$ in which every rational number appears as a term.
   Does this sequence necessarily have a monotone subsequence? Justify your answer by quoting the statement of a theorem.

   (b) (5 pts) Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x^8 + x^3 \sin 3 + \sqrt{5}$. Is this function uniformly continuous? Justify your answer by quoting statements of theorems.

   (c) (10 pts) Let $f\colon D \to \mathbb{R}$. State the ϵ-δ-criterion for continuity at a point $x_0 \in D$.

   (d) (10 pts) Choose the statement to negate depending on your section. The following definition is implicit in our textbook:
   (**Prof. Hasselblatt's section**) A sequence $\{a_n\}$ is said to *converge* if
   $\boxed{\exists a \in \mathbb{R}\ \forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N \quad |a_n - a| < \epsilon}$.
   State a definition of "A sequence is said to diverge if…" by negating this statement.
   (**Prof. Tu's section**) Negate $\boxed{\exists a \in \mathbb{R} \text{ such that } \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, |a_n - a| < \epsilon}$.

3. (10 points) Suppose $A \subset \mathbb{R}$ is a nonempty set which is bounded below. Prove that there is a sequence $\{a_n\}$ in A that converges to $\inf A$.

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4. (10 points) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} x & \text{if } x \notin \mathbb{Z} \\ -x & \text{if } x \in \mathbb{Z}. \end{cases}$

   Using the definition of continuity at a point, prove that f is not continuous at $x = 1$.

5. (15 points) Prove the sandwich theorem (squeeze theorem): Let $L \in \mathbb{R}$ and let $\{x_n\}$ and $\{z_n\}$ be sequences that both converge to L. Assume $\{y_n\}$ is a sequence such that $\forall n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. Prove that $y_n \to L$ as $n \to \infty$ using the ϵ-N-definition of limit.

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6. (10 points) Assuming $\sin x$ is a continuous function of x, prove that there is a solution of the equation $x^7 + \sin x + 15 = 0$.

7. (15 points) Using the ϵ-N-definition of limit, prove that $\dfrac{n^2-4}{2n^2-n+1}$ converges to $1/2$. (Hint: $2n^2-n+1 = n^2+(n^2-n)+1$.)

(End of Exam)