Solutions

Recall that the field axioms for $\mathbb{R}$ are as follows:

(1) (Associativity) For all $x, y, z \in \mathbb{R}$,
$$x + (y + z) = (x + y) + z \quad \text{and } x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$
(2) (Commutativity) For all $x, y \in \mathbb{R}$,
$$x + y = y + x \quad \text{and } x \cdot y = y \cdot x.$$
(3) (Identity elements) There exists a unique element of $\mathbb{R}$ called **zero**, denoted by 0, such that for all $x \in \mathbb{R}$ we have $x + 0 = x$.
There exists a unique element of $\mathbb{R}$ called **one**, different from 0, denoted by 1, such that for all $x \in \mathbb{R}$, $x \cdot 1 = x$.
(4) (Inverses) For each element $x \in \mathbb{R}$, there exists a unique element y (called the **negative** of x and usually denoted by $-x$) such that $x + y = 0$.
Similarly, for each element $x \in \mathbb{R} - \{0\}$, there exists a unique element y (called the **reciprocal** of x and usually denoted by $1/x$ or x^{-1}) such that $x \cdot y = 1$.
(5) (Distributivity) For all $x, y, z \in \mathbb{R}$,
$$x \cdot (y + z) = x \cdot y + x \cdot z \quad \text{and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

We take $x - y$ to be an abbreviation for $x + (-y)$ and x/y to be an abbreviation for $x \cdot (1/y)$.
We have shown in class that

(P1) If $x + y = x$, then $y = 0$
(P2) $0 \cdot x = 0$
(P3) $-0 = 0$
(P4) $-(-x) = x$

(1) Using only the axioms (1)–(5) and properties proven in class, prove the following statements for all $x, y, z \in \mathbb{R}$:
(a) (-1)x = -x

axiom 4: $1 + (-1) = 0$

multiplying by x on both sides:
$$(1 + (-1)) \cdot x = 0 \cdot x$$
axiom 5: $1 \cdot x + (-1) \cdot x = 0 \cdot x$

axiom 3: $x + (-1) \cdot x = 0 \cdot x$

P2: $x + (-1) \cdot x = 0$

by the uniqueness in axiom 4,
$$(-1) \cdot x = -x.$$

(b) x(-y) = -(xy) = (-x)y

By (a), $-1(xy) = -(xy)$.

By axiom 1, $((-1)x)y = -(xy)$

By (a), $(-x)y = -(xy)$ ★

By axiom 2, $y(-x) = -(yx)$

Renaming x,y to y,x, $x(-y) = -(xy)$ ◊

★ and ◊ are the result

(c) x(y - z) = xy - xz

By definition of subtraction,

$$x(y-z) = x(y+(-z))$$

By axiom 5, $x(y-z) = x\cdot y + x\cdot(-z)$

By (b), $x(y-z) = x\cdot y + -(x\cdot z)$

By definition of subtraction,

$$x(y-z) = xy - xz.$$

(d) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$

By axiom 4, there is $1/x \in \mathbb{R}$ s.t. $x\cdot(1/x) = 1$.

Suppose $x\cdot y = x$.

Multiplying on both sides by $1/x$,

$$1/x \cdot (x\cdot y) = 1/x \cdot x$$

By axiom 1, $((1/x)\cdot x)\cdot y = (1/x)\cdot x$

By axiom 2, $(x\cdot(1/x))\cdot y = x\cdot(1/x)$

By axiom 4, $1\cdot y = 1$

By axiom 3, $y = 1$.

(e) If $x \neq 0$, then $x/x = 1$

By definition of division, $x/x = x \cdot (1/x)$

By axiom 4, $x/x = 1$.

Recall that $\mathbb{R}$ also satisfies the following axioms related to ordering:

(6) For all $x, y, z \in \mathbb{R}$, if $x > y$, then $x + z > y + z$.
For all $x, y, z \in \mathbb{R}$, if $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$.
(7) The order relation $<$ has the least upper bound property.
(8) If $x < y$, there exists an element z such that $x < z$ and $z < y$.

We have shown in class that

(P5) $x > y$ and $w > z$ implies $x + w > y + z$;
(P6) $x > 0$ and $y > 0$ implies $x + y > 0$ and $x \cdot y > 0$;
(P7) $x > 0 \iff -x < 0$

(2) Prove the following "Laws of inequalities"

(a) $x > y \iff -x < -y$

$\Rightarrow$ Suppose $x > y$. By axiom 6,

$x + (-x + -y) > y + (-x + -y)$

By axiom 1, $(x + -x) + -y > y + (-x + -y)$

By axiom 2, $(x + -x) + -y > y + (-y + -x)$

By axiom 1, $(x + -x) + -y > (y + -y) + -x$

By axiom 4, $0 + -y > 0 + -x$

By axiom 3, $-y > -x$

$\Leftarrow$ For converse, suppose $-y > -x$.

By the forward direction, $-(-y) < -(-x)$

By (P4), $y < x$.

(b) $x > y$ and $z < 0$ implies $xz < yz$

$-z > 0$, so by axiom 6, if $x > y$, then

$$x \cdot (-z) > y \cdot (-z).$$

By problem 1(b),

$$-(xz) > -(yz).$$

By 2(a),

$$xz < yz.$$

(c) $x \neq 0$ implies $x^2 > 0$, where $x^2 = x \cdot x$

If $x > 0$, by axiom 6,

$$x \cdot x > 0 \cdot x.$$

By (P2), $x^2 > 0$.

If $x < 0$, by 2(b),

$$x \cdot x > 0 \cdot x.$$

By (P2) again, $x^2 > 0$.

Altogether, if $x \neq 0$, then $x^2 > 0$.

(3) Prove that every positive number has a square root as follows.

(a) Show that if $x > 0$ and $0 < h < 1$, then

$$(x+h)^2 < x^2 + h(2x+1)$$
$$(x-h)^2 > x^2 - 2xh.$$

We'll relax with the axioms now.

$(x+h)^2 = x^2 + 2xh + h^2$; $(x-h)^2 = x^2 - 2xh + h^2$

Since $0 < h < 1$, $0 < h^2 < h$.

Therefore

$$\begin{aligned}(x+h)^2 &= x^2 + 2xh + h^2 \\ &< x^2 + 2xh + h \\ &= x^2 + h(2x+1)\end{aligned} \qquad \begin{aligned}(x-h)^2 &= x^2 - 2xh + h^2 \\ &> x^2 - 2xh\end{aligned}$$

(b) Let $x > 0$. Show that if $x^2 < a$, then $(x+h)^2 < a$ for some $h > 0$. Similarly, show that if $a < x^2$, then $a < (x-h)^2$ for some $h > 0$.

Suppose $x > 0$, $x^2 < a$. ~~Let h =~~ $\cancel{\frac{a-x^2}{2(2x+1)}}$.

If $(x+1)^2 < a$, we can take $h = 1$, and we are done.

Otherwise $x^2 < a \le (x+1)^2$

$\Rightarrow 0 < a - x^2 \le (x+1)^2 - x^2$

$\Rightarrow 0 < a - x^2 \le 2x+1$

$\Rightarrow 0 < \frac{a-x^2}{2x+1} \le 1.$

Set $h = \frac{1}{2}\left(\frac{a-x^2}{2x+1}\right)$.

By part a, $(x+h)^2 < x^2 + \frac{1}{2}\left(\frac{a-x^2}{2x+1}\right)(2x+1)$

$= x^2 + \frac{1}{2}(a - x^2) < a,$

as desired.

For the second part, if $a < (x-1)^2$, we can take $h=1$ and we are done.

Otherwise $(x-1)^2 \le a < x^2 \Rightarrow 0 < x^2 - a \le 2x - 1 < 2x \Rightarrow 0 < \frac{x^2-a}{2x} < 1.$

$$\boxed{\begin{aligned} &(x-1)^2 \le a < x^2 \\ \Rightarrow\ & x^2 - 2x + 1 \le a < x^2 \\ \Rightarrow\ & 1 - 2x \le a - x^2 < 0 \\ \Rightarrow\ & 0 < x^2 - a \le 1 - 2x \end{aligned}}$$

Set $h = \frac{x^2-a}{2x}$. By part a, $(x-h)^2 > x^2 - 2x\left(\frac{x^2-a}{2x}\right)$

$= x^2 - (x^2 - a)$

$= a.$

(c) Given $a > 0$, let B be the set of all real numbers x such that $x^2 < a$. Show that B is bounded above and contains at least one positive number. (Hint: it may help to consider the case that $a \ge 1$ separately from that case that $0 < a < 1$.)

If $a > 1$, then $1^2 < a$, so $1 \in B$.

If $0 < a < 1$, then ~~$\frac{a}{2} < 1$, so $\left(\frac{a}{2}\right)^2 <$~~ $a^2 < a$, so $a \in B$.

In either case B contains at least one positive real number.

If 1 is an upper bound on B, then B has an upper bound.

If 1 is not an upper bound on B, then $\exists x \in B$ s.t. $1 < x$.

Since $x \in B$, and $1 < x$, $1 < x < x^2 < a$. Then for all $y \in B$, either $y \le 1 < a$ or $1 < y$ and $y < y^2 < a$. Since y is arbitrary, a is an upper bound on B.

(d) Let $b = \sup B$. Show that $b^2 = a$. (Hint: Suppose $b^2 < a$, then derive a contradiction. Then do the same when $b^2 > a$.)

Suppose $b^2 < a$. Then by part b $\exists h > 0$ s.t. $(b+h)^2 < a$.
But then $b < b+h \in B$, contradicting that b is an upper bound on B. Therefore $b^2 \not< a$.

Suppose $b^2 > a$. Then by part b $\exists h > 0$ s.t. $(b-h)^2 > a$.
For all $x \in B$, $x < 0$ or $x > 0$ and $x^2 < a < (b-h)^2$.
For positive numbers c, d, $c < d \iff c^2 < d^2$, so $x < b-h$.
But then $b-h < b$ is an upper bound on B, contradicting that b is the least upper bound on B. Therefore $b^2 \not> a$.

Therefore $b^2 = a$.

(e) Show that if b and c are positive and $b^2 = c^2$, then $b = c$.

$$b^2 = c^2 \Rightarrow b^2 - c^2 = 0$$
$$\Rightarrow (b+c)(b-c) = 0$$

Since $b > 0$, $c > 0$, $b + c > 0$.
Dividing by $b+c$,
$$b - c = 0$$
$$\Rightarrow b = c.$$