1. First Quiz, Sept 8

Show all your work and carefully explain your reasoning.

Question 1.1. (a) *Define* $A = \{n \in \mathbb{Z} |\ exists\ \ k \in \mathbb{Z}, n = 2k+1\}$, $B = \{n \in \mathbb{Z} |\ exists\ \ k \in \mathbb{Z}, n = 2k-1\}$. *Show that* $A = B$.

We can rewrite $n=2k-1$ as $n=2(k-1)+1$, if $k-1=k'$, then this is $n=2k'+1$, meaning B is shifted over 1 element from A, but since $n \in \mathbb{Z}$, both sets extend infinitely in both directions, meaning the shift is irrelevant. To show the other way, we can write $2k+1$ as $2(k+1)-1$, where $k+1=k''$, so $n=2k''+1$, this shows that the set is shifted 1 element, but using similar logic from earlier, they match, $A \subseteq B$ and $B \subseteq A$

(b) *Define* $C = \{n \in \mathbb{Z} \mid exists\ k \in \mathbb{N}, n = 2k+1\}$, $D = \{n \in \mathbb{Z} \mid exists\ k \in \mathbb{N}, n = 2k-1\}$. *Is* $C = D$?

Let's consider the 1^{st} element in each set, assuming that it is written increasing with $k=0$. as $k \in \mathbb{N}$

This means that $C = \{1, 3, 5, \ldots\ 2k+1\}$ and that $D = \{-1, 1, 3, \ldots\ 2k+1\}$. For $C = D$, then $C \subseteq D$ and $D \subseteq C$, however, $-1 \notin C$, meaning that $C \neq D$.

(c) *Consider a collection of identical coins distributed in piles. We take one coin from each pile and put them together to form a new pile. We can represent this situation by a set whose elements are the number of coins on each pile. For example, if we have two piles with one coin and one with 5 coins, the original set is $\{1,1,5\}$, the new set will be $\{4,3\} = \{3,4\}$ (the order in which we list the elements of a set does not matter). Find the collections that are invariant under this operation (for instance $\{1,1,5\}$ is not but $\{1,2\}$ is. Once you come up with a good guess, make sure that you justify that these are the only choices that will work.*

First, the only solution is where coins are distributed in piles that forms a set, S, where $S \in \mathbb{N}$ and $S = \{1, 2, 3 \ldots n\}$ $n \in \mathbb{N}$, meaning the difference between 2 consecutive elements of S is 1. Performing the operation $S = \{1, 2, 3 \ldots n\} \rightarrow S' = \{1-1, 2-1, 3-1 \ldots n-1, n\}$
$S' = \{0, 1, 2, 3 \ldots n\} = \{1, 2, 3 \ldots n\} = S$.

To show this is the only solution, let us create a set that is not continuous, $D = \{1, 2, 3 \ldots n, n+k, n+k+1, k+1, n+k+j\}$ $D, n \in \mathbb{N}$, $k \in \mathbb{N}$ but not 1, and $j \in \mathbb{N}$. Let us perform the same operation as we did for S, creating a new set D', $D' \in \mathbb{N}$, and $D' = \{1-1, 2-1, 3-1, \ldots n-1, \ldots n+k-1 \ldots n+k+j-1, n+k+j\}$
$D' = \{0, 1, 2 \ldots n-1 \ldots n+k-1 \ldots n+j+1\}$. Since $k \neq 1$ and if $k \neq 0$, then $n+k-1$ will not have existed in D as we went up by more than k. Meaning D' has at least 1 element not in D and $D' \neq D$.

For $k = 0$, the last element of D' is $n+j+1$, but since the highest element in D is $n+j$, as $k = 0$, $n+j+1$ is not in D, meaning D' has at least 1 unique element from D and $D \neq D'$. WLOG, so the any set with a discontinuity or at least 2 repeating elements is not invariant.