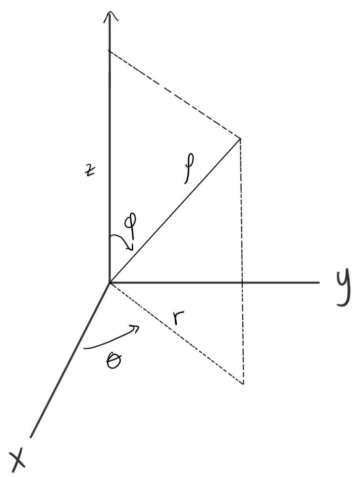
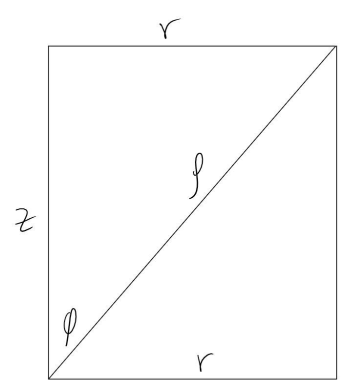
MATH 42 HOMEWORK 7 SOLUTION

(1) **Find the volume of the solid bounded by the cylinders $r = 1$ and $r = 2$ and the cones $\phi = \pi/6$ and $\phi = \pi/3$.**

Let's set this up in spherical coordinates to start. We know $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$, and the solid is rotated around the z-axis, so $0 \leq \theta \leq 2\pi$. We need to change $1 \leq r \leq 2$ into limits of ρ. Let's look at the relationship between the two coordinate systems:



Looking closer, we see



and we have that $\sin(\phi) = r/\rho$. We rearrange to solve for ρ, and plug in our values for r to see that

$$\rho = \frac{r}{\sin(\phi)} = r\csc(\phi) \quad \text{and} \quad \csc(\phi) \leq \rho \leq 2\csc(\phi)$$

Combining all of this, we set up and evaluate the integral.

$$\begin{aligned}
\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta &= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \tfrac{7}{3}\csc^3\varphi\sin\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \tfrac{7}{3}\csc^2\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi} -\frac{7}{3}\cot\varphi|_{\pi/6}^{\pi/3}\,d\theta \\
&= \int_0^{2\pi} \frac{14}{9}\sqrt{3}\,d\theta \\
&= \boxed{\frac{28\pi\sqrt{3}}{9}.}
\end{aligned}$$

Alternative setup: cylindrical coordinates.
To set up in cylindrical coordinates, we know $0 \le \theta \le 2\pi$ and $1 \le r \le 2$, so we need to convert the ϕ bounds to z bounds. Using the same triangle system from above, we have that $\tan(\phi) = r/z$ and $z = r/\tan(\phi)$. We plug in $\phi = \pi/6$ and $\phi = \pi/3$ to get that $r/\sqrt{3} \le z \le r\sqrt{3}$, and we get

$$\text{Cylindrical coordinates: } \int_0^{2\pi}\int_1^2\int_{r\sqrt{3}/3}^{r\sqrt{3}} dz\,r\,dr\,d\theta$$

(2) **Suppose the temperature inside a three dimensional ball is proportional to the square root of the distance from the center. Find a formula for the average temperature in the ball. What is the limit of the average temperature as the ball gets larger?**

Let T denote the temperature. Then $T(x, y, x) = k\sqrt{a}$ where a is the distance from the center and $k > 0$ is some proportionality constant. If we integrate T over our entire region (the ball of radius a), we will have in some sense the "total temperature" (more appropriately called "internal energy"). To find the *average* temperature, we need to divide by the volume of the sphere. Let $T(\bar{a}, k)$ denote the average temperature, as a function of k and a, and let V be the volume of the sphere. Then

$$\begin{aligned}
\bar{T}(a,k) &= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi}\int_0^a k\sqrt{\rho}\,\rho^2\sin\phi\,d\rho d\phi d\theta \\
&= \frac{2\pi k}{V}\int_0^{\pi}\int_0^a \rho^{5/2}\sin\phi\,d\rho d\phi \\
&= \frac{2\pi k}{V}\int_0^{\pi} \frac{2}{7}a^{7/2}\sin\phi\,d\phi \\
&= \frac{8\pi k}{7V}a^{7/2}
\end{aligned}$$

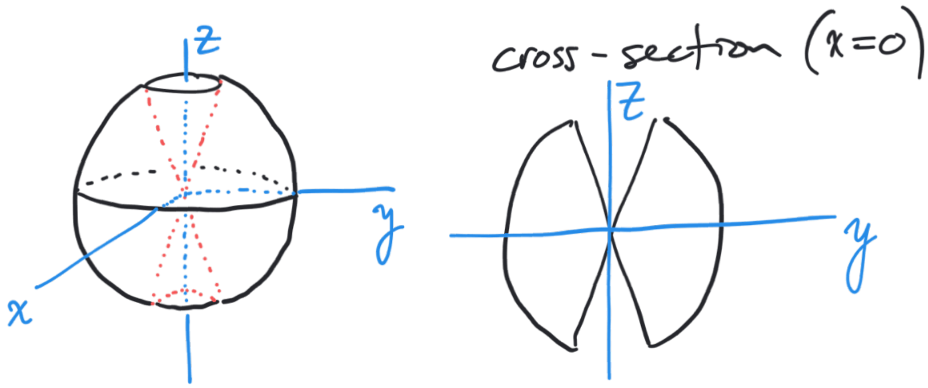
Now, we could compute the volume of a sphere with radius a, or we could remember it from high school: $\frac{4}{3}\pi a^3$. This gives us

$$\bar{T}(a,k) = \frac{8\pi k}{7V}a^{7/2} = \frac{8\pi k}{7}a^{7/2} * \frac{3}{4\pi a^3} = \frac{6k}{7}\sqrt{a}.$$

Since k is positive, in the limit as $a \to \infty$, the average temperature increases without bound.

(3) **Let $f(x, y, z) = \frac{1}{(x^2+y^2+z^2)}$ be the charge density (Coulomb per unit volume) for the solid inside the sphere $x^2+y^2+z^2 = 40$ and outside the cone $z^2 = 9(x^2+y^2)$. Set up the integral (including bounds) in both cylindrical and spherical coordinates, but compute only one of them.**

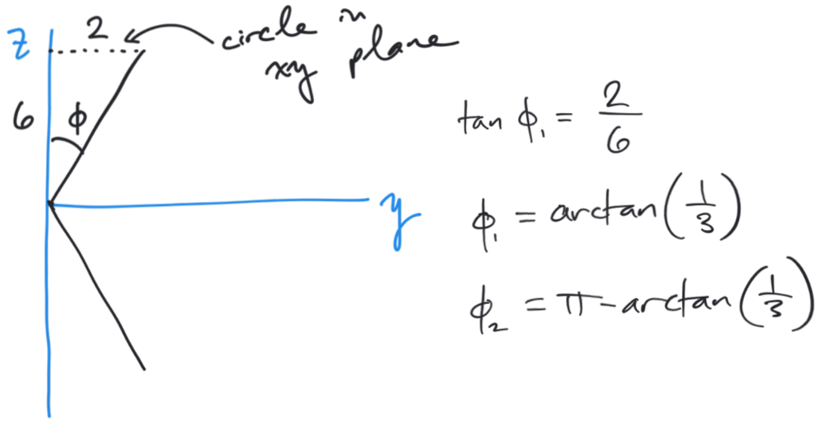
The first thing to determine is what our region looks like. Our region looks like:



Let's start with spherical coordinates. Clearly, θ ranges from 0 to 2π. The polar angle ϕ will vary between between the upper cone and the lower cone, so we need to determine these angles, while the radius will simply vary from 0 to the radius of the sphere, $\sqrt{40}$. To determine the bounds for ϕ, we will need the largest value of z that our solid takes. Substituting the cone equation in the equation for the sphere we have

$$x^2 + y^2 + 9(x^2 + y^2) = 40 \quad \Rightarrow \quad x^2 + y^2 = 4$$

so the intersection of the two surfaces is a circle of radius 2. Substituting this back into the equation of the sphere, we see that this occurs at $z = 6$. Therefore, to determine the bounds for ϕ we have the following picture:



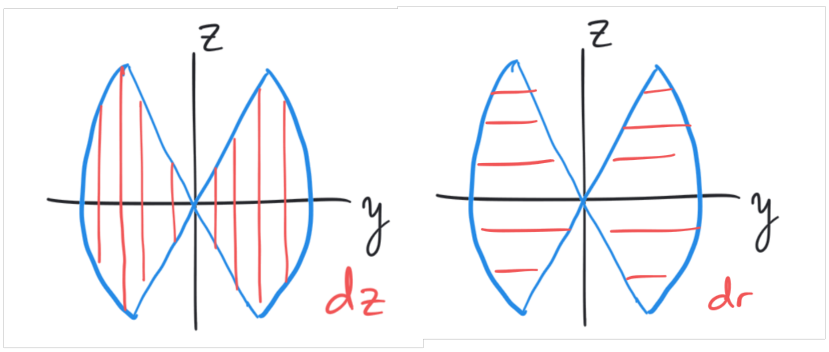
We are now ready to set up the integral. Note that f may be rewritten in spherical coordinates as ρ^{-2}. Thus, the total charge is given by

$$\int_0^{2\pi} \int_0^{\sqrt{40}} \int_{\pi-\arctan(1/3)}^{\arctan(1/3)} \sin\phi \, d\phi d\rho d\theta.$$

Let's go ahead and solve this.

$$\begin{aligned}\int_0^{2\pi}\int_0^{\sqrt{40}}\int_{\pi-\arctan(1/3)}^{\arctan(1/3)} \sin\phi\, d\phi d\rho d\theta &= -2\pi\sqrt{40}\left(\cos(\pi-\arctan(1/3)) - \cos\arctan(1/3)\right)\\ &= -2\pi\sqrt{40}\left(-\cos(-\arctan(1/3)) - \cos\arctan(1/3)\right)\\ &= -2\pi\sqrt{40}\left(-\cos(\arctan(1/3)) - \cos\arctan(1/3)\right)\\ &= 4\pi\sqrt{40}\cos\arctan 1/3\\ &= 4\pi\sqrt{40}\frac{3}{\sqrt{10}}\\ &= 24\pi\end{aligned}$$

Now let's try cylindrical. We often integrate with respect to z first, however in this case notice that if we were to do that, we would have to separate into two regions, (a) where the upper cone is above the lower cone and (b) the outermost region between the upper and lower hemispheres.



Instead, if we integrate first with respect to r, we can integrate between two surfaces and z will be bounded by constants. The bounds for r must then be functions in terms of z, but this is easy enough since the cone equation gives us one bound and the sphere gives us the other using $r^2 = x^2 + y^2$. Lastly, f may now be written as $1/(r^2+z^2)$ so that we have

$$\int_0^2 \pi \int_{-6}^{6}\int_{z/3}^{\sqrt{40-z^2}} \frac{r}{r^2+z^2}\, drdzd\theta.$$

(4) **Find the mass of the parallelepiped given by $0 \le x+y+z \le 10, 0 \le y+z \le 5$, and $0 \le z \le 2$, with density $f(x,y,z) = yz + z^2$, using a change of variables. (Note that the density is nonnegative since $yz+z^2 = z(y+z)$, and neither component may be less than 0.)**

In general, we use change of variables to make either the integrand more manageable or to make the region of integration more manageable. When choosing a change of variables, you want to try to simplify both if possible, so we look for similarities between the integrand and the bounds. Note that $f(x,y,z) = yz + z^2 = z(y+z)$ and the factors z and $y+z$ are both constraints defining the region. This suggests the transformation

$$u = x+y+z, \quad v = y+z, \quad w = z$$

where we chose u to simplify the remaining bound. The new integrand will be given by $f(u,v,w) = vw$ which is somewhat simpler than the previous case. Lastly, we need to compute the Jacobian. This is straightforward enough. Writing x, y and z in terms of u, v

and w we have $x = u - v$, $y = v - w$ and $z = w$. Then to find how our transformation scales volume (the Jacobian) we just need to compute as follows:

$$\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

The resulting integral is then

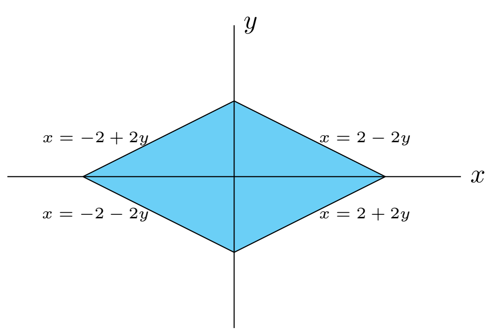
$$\int_0^2 \int_0^5 \int_0^{10} vw \, dudvdw = 250.$$

For each of the following,

- Sketch the original region of integration.
- Choose a useful change of variables based on the region of integration and the integrand (more than one choice might work here). Remember, your goal is to simplify both the region (i.e. endpoints on the integrals) and the integrand, so you get something easier to evaluate.
- Sketch the new region based on your change of variables.
- Transform the integrand to the new variables.
- Compute the Jacobian of your transformation.
- Write the integral in the new coordinate system.
- Evaluate the integral.

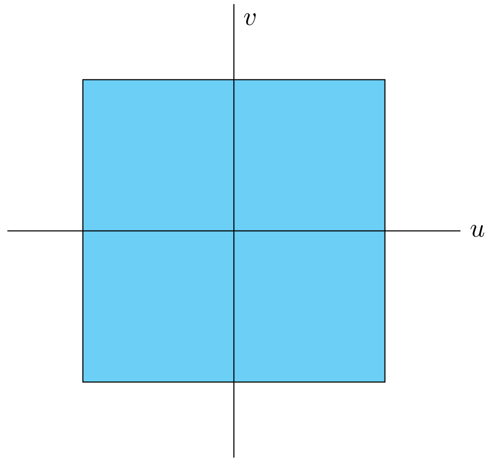
(5) $\iint_R (3x + 6y)^2 dA$, where R is the diamond bounded by $x = 2 - 2y, x = -2 + 2y, x = -2 - 2y, x = 2 + 2y$. (Hint: factor the integrand)

The region of integration is the follow:



We can try to integrate this directly by splitting into two integrals, but let's be more clever about it. Suppose we transform our variables $u = x + 2y$ and $v = x - 2y$. Then, the equations become $u = 2$, $v = -2$, $u = -2$, and $v = -2$.

Here is the new region of integration on the uv-plane:



Remember, we need to compute the Jacobian to compute this integral in the uv-plane. We can easily see that $x = \frac{u+v}{2}$ and $y = \frac{u-v}{4}$. Then, the Jacobian is the following;

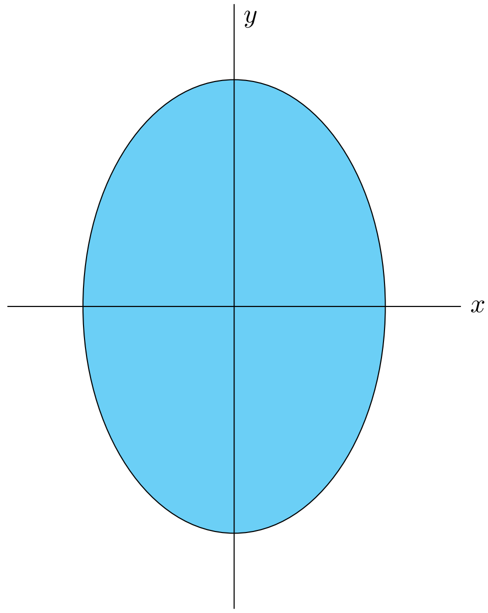
$$J(u,v) = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}.$$

The integral is the following:

$$\begin{aligned} \iint_R (3x+6y)^2 dxdy &= \int_{-2}^{2}\int_{-2}^{2} (3u)^2 |J(u,v)|\, du\, dv \\ &= \int_{-2}^{2}\int_{-2}^{2} \tfrac{9}{4}u^2\, du\, dv \\ &= \int_{-2}^{2} \tfrac{3}{4}u^3|_{-2}^{2}\, du\, dv \\ &= \int_{-2}^{2} 12\, dv \\ &= 12v|_{-2}^{2} \\ &= \boxed{48.} \end{aligned}$$

(6) $\iint_R xydA$ **where** R **is the interior of the ellipse** $9x^2+4y^2=36$**.**

The region of integration is the following:



Let's use the following transformation: $x = 2r\cos\theta$ and $y = 3r\sin\theta$ for $0 \le r \le 1$ and $0 \le \theta < 2\pi$. The Jacobian is 6. Our integral is the following now:

$$\begin{aligned}
\iint_R xy\,dx\,dy &= \int_0^{2\pi}\int_0^1 6r^2\sin\theta\cos\theta\, r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 6r^3\sin\theta\cos\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \tfrac{3}{2}r^4|_0^1 \sin\theta\cos\theta\,d\theta \\
&= \int_0^{2\pi} \tfrac{3}{2}\sin\theta\cos\theta\,d\theta
\end{aligned}$$

Use a u-substitution with $u = \sin\theta$ and $du = \cos\theta$ with new bounds $u(0) = \sin(0) = 0$ and $u(2\pi) = \sin(2\pi) = 0$:

$$\int_0^{2\pi} \tfrac{3}{2}\sin\theta\cos\theta\,d\theta = \int_0^0 \tfrac{3}{2}u\,du = \boxed{0.}$$