Hence $D_E(x)$ does not exist. Consider a set where $D_E(x)$ attains any $\alpha \in (0,1)$. Define $\theta = 2\pi\alpha$, consider E to be a slice of circle in $\mathbb{R}^2$ with θ radians. □

- 6.1.9

Proof. The forward direction follows from Definition 6.1.1. For another direction, let $\{[a_j, b_j]\}$ be any countable collection of nonoverlapping subintervals of $[a,b]$. Assume $\sum_j^\infty |b_j - a_j| < \delta$. Then we know for all $N \in \mathbb{N}$,

$$\sum_j^N |b_j - a_j| < \delta; \qquad \sum_j^N |f(b_j) - f(a_j)| < \epsilon$$

Taking limits on the second equation above, we obtain

$$\sum_j^\infty |f(b_j) - f(a_j)| = \lim_{N\to\infty} \sum_j^N |f(b_j) - f(a_j)| < \lim_{n\to\infty} \epsilon = \epsilon$$

Therefore f is absolutely continuous. □

- 6.1.10

Proof. (a). Let $\epsilon > 0$. Since f_n converges to f in bounded variation norm there exists some n_0 such that $||f_{n_0} - f||_{\text{BV}} < \frac{\epsilon}{3}$. Also, $\exists \delta > 0$ such that any nonoverlapping subintervals $\{[a_j, b_j]\}$ of $[a,b]$ with $\sum_j (b_j - a_j) < \delta$ implies $\sum_j |f_{N_0}(b_j) - f_{N_0}(a_j)| < \frac{\epsilon}{3}$. Then

$$\begin{aligned}\sum_j |f(b_j) - f(a_j)| &\le \sum_j |(f - f_{N_0})(b_j) - (f - f_{N_0})(a_j)| + \sum_j |f_{N_0}(b_j) - f_{N_0}(a_j)| \\ &\le V[f - f_{N_0}; a, b] + \frac{\epsilon}{3} \le ||f - f_{n_0}||_{\text{BV}} < \epsilon\end{aligned}$$

Therefore f is also absolutely continuous.

(b). Consider $f_n = 0$ for $0 \le x \le \frac{1}{n\pi}$ and $f_n = x \sin \frac{1}{x}$ for $\frac{1}{n\pi} \le x \le 1$. Then f_n is absolutely continuous for each n but f is not of bounded variation hence also not absolutely continuous. □

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- 6.3.6

Proof. (1) As f is Lipschitz, $\forall\{[a_j, b_j]\}$ nonoverlapping intervals of $[a, b]$:

$$\exists K > 0 : \forall x, y \in [c, d] : |f(x) - f(y)| \leq K|x - y|$$

As $g \in [a, b]$,

$$\forall \epsilon > 0 : \exists \delta > 0 : \sum_j (b_j - a_j) < \delta \implies \sum_j |g(b_j) - g(a_j)| < \epsilon$$

We aim to show that $f \circ g \in [a, b]$, So choose δ such that $\sum_j |g(b_j) - g(a_j)| < \epsilon/K$; then by the Lipschitz property of f, $|f(g(b_j)) - f(g(a_j))| < \epsilon$.

(2) As $f \in [c, d]$, $\forall\{[a_j, b_j]\}$ nonoverlapping intervals of $[a, b]$:

$$\forall \epsilon > 0 : \exists \delta_1 > 0 : \sum_j (d_j - c_j) < \delta \implies \sum_j |f(d_j) - f(c_j)| < \epsilon$$

As $g \in [a, b]$, $\forall\{[a_j, b_j]\}$ nonoverlapping intervals of $[a, b]$:

$$\forall \delta_1 > 0 : \exists \delta_2 > 0 : \sum_j (b_j - a_j) < \delta_2 \implies \sum_j |g(b_j) - g(a_j)| < \delta_1$$

As g is monotonically increasing, $\{[g(a_j), g(b_j)\}$ is an n.o.i. in $[c, d]$. So choose δ_1 according to $\epsilon > 0$ and δ_2 according to δ_1. Then $\forall\{[a_j, b_j]\}$ nonoverlapping intervals of $[a, b]$:

$$\forall \epsilon > 0 : \exists \delta_2 > 0 : \sum_j (b_j - a_j) < \delta_2 \implies \sum_j |f(g(b_j)) - g(a_j)| < \epsilon$$

(3) If $f \circ g \in [a, b]$, it is trivial from the complex version of Theorem 6.3.1 that $f \circ g \in$ BV$[a, b]$. Now suppose $f \circ g \in$ BV$[a, b]$, $f \in [c, d]$, and $g \in [a, b]$. So by Theorem 6.3.1, both f and g map sets of measure zero to measure zero; this property is clearly preserved under composition. Furthermore, as the composition of continuous functions is continuous, $f \circ g$ is continuous. So condition (b) of Theorem 6.3.1 is satisfied, and therefore $f \circ inAC[a, b]$.

□

- 6.3.10

Proof. (1) If $f \in [a, b]$, it is trivial from Theorem 6.3.1 that $f \in BV[a, b]$. Now suppose that $f \in$ BV$[a, b]$. By Growth Lemma I (Lemma 6.2.1), f maps sets of measure zero in $[a, b]$ to sets of measure zero, as f has bounded variation on $[a, b]$ and is therefore bounded on $[a, b]$. As f is differentiable on $[a, b]$, it is continuous on $[a, b]$. So condition (b) of Theorem 6.3.1 is satisfied, and therefore $f \in AC[a, b]$.

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(2) Suppose f is constant on $[a,b]$. Then for $x \in [a,b]$,
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$$
Recalling results from chapter 4, this implies that f is constant on $[a,b]$. Now suppose $f' = 0$ a.e. on $[a,b]$. By Corollary 6.2.2, $|f([a,b])| = 0$, so by monotonicity, f maps sets of measure zero in $[a,b]$ to sets of measure zero. As $f' = 0$ a.e., $f' \in L^1[a,b]$. As f is differentiable everywhere on $[a,b]$, it is continuous on $[a,b]$. So condition (c) of Theorem 6.3.1 is satisfied, and therefore $f \in AC[a,b]$. So by Theorem 6.4.2, for $[c,d] \subseteq [a,b]$,
$$f(d) - f(c) = \int_c^d f' = 0$$
and therefore f is constant on $[a,b]$.

□

- 6.4.10

Proof. $(\Leftarrow)$. Suppose $f \in [a,b]$ and $f' \in L^\infty[a,b]$. So for arbitrary $[c,d] \subseteq [a,b]$, $f \in [c,d]$. By Theorem 6.4.2(c),
$$f(d) - f(c) = \int_c^d f'(t)dt$$
But as $f' \in L^\infty$, $f'(x) \leq K$ for some $K > 0$ for all $x \in [c,d]$ except possibly on a subset of measure zero. So by monotonicity of the integral,
$$f(d) - f(c) \leq K(d-c)$$
As $[c,d]$ is arbitrary, f is Lipschitz on $[a,b]$, as desired. □

$(\Rightarrow)$. Suppose f is Lipschitz on $[a,b]$. Then $f \in AC[a,b]$ by Lemma 6.1.3(a). Furthermore, for $x \in [a,b]$,
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \leq \lim_{h\to 0} \frac{Kh}{h} = K$$
for some constant $K > 0$ by monotonicity of the limit and the definition of a Lipschitz function. Thus, $\sup_{x\in[a,b]} f'(x) \leq K$, so $f' \in L^\infty[a,b]$ by definition. □

- 6.4.13

Proof. Suppose, to the contrary that
$$\limsup_{|x|\to\infty} |f(x)| = \alpha > 0,$$
then there exists a sequence $\{x_n\}$, s.t. $|x_n| \to \infty$ and
$$|f(x_n)| \geq \frac{\alpha}{2}. \qquad (*)$$
We may assume that $|x_n - x_{n+1}| > 1$. Since f is AC, there is a $\delta \in (0, 1/2)$, s.t.
$$|f(y) - f(x_n)| \leq \frac{\alpha}{4} \qquad (**)$$
whenever $|y - x_n| < \delta$. Observe that $(*)$ and $(**)$ imply that
$$|f(y)| \geq \frac{\alpha}{4}, \quad \forall\, y \in (x_n - \delta, x_n + \delta).$$