1. f is differentiable at $(1,1)$ as $\nabla f = <$ which is continuous. Since $\nabla f = <3x^2+y, x>$ which is continuous. Since f is differentiable, it has an affine approximation at $(1,1)$ and has a tangent plane of form

$$g(x,y) = f(1,1) + < \nabla f(1,1), (x-1_0, y-1_0) >$$
$$= 3 + <(4,1), (x-1, y-1)>$$
$$g(x,y) = 3 + 4(x-1) + y - 1$$

$g(x,y) = 4x+y-2$ is the tangent plane to f at $(1,1)$ $\square$

2a) $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$2x\sin\frac{1}{x} - x\cos\frac{1}{x}$ is defined $\forall x \neq 0$ and as $f'(x) = 0$ at 0, then f is differentiable everywhere.

b) $f \notin C^1(\mathbb{R})$. As $f'(x) \to 0$ from either direction, $2x\sin\frac{1}{x} \to 0$ but $\cos\frac{1}{x}$ oscillates back and forth from -1 to 1 and $\lim_{x\to 0} 2x\sin\frac{1}{x} - \cos\frac{1}{x} =$ DNE and as the derivative isn't continuous on $\mathbb{R}$ $f \notin C^1(\mathbb{R})$

3a) $g(x)$ is affine and to show first order approx then

$$\lim_{h\to 0} \frac{g(x+h) - f(x+h)}{\|h\|} = 0 \qquad h = x - x_0$$

$$= \lim_{h\to 0} \frac{|g(x_0+h) - f(x_0+h)|}{\|h\|} = 0$$

$$= \lim_{h\to\vec{0}} \frac{|f(x_0) + <b,h> - f(x_0+h)|}{\|h\|}$$

$$= \lim_{h \to 0} \frac{|f(x_0+h) - [f(x_0) + \langle b, h \rangle]|}{\|h\|} = 0$$

which is true as $\hat{f}$ is differentiable

So g_f is a first order affine approximation of f.

b)

Let $t_k \in \mathbb{R} \setminus \{0\}$ s.t. $t_k \to 0$ and $x_0 + t_k e_i \in \mathcal{O}$ and let $\{h_k = t_k e_i\}$

$$\lim_{k \to \infty} \frac{|f(x_0 + t_k e_i) - [f(x_0) + \langle b, t_k e_i \rangle|}{\|t_k e_i\|} = 0$$

As e_i is zeros everywhere except for i, $\langle b, t_k e_i \rangle = b_i t_{ik}$

$$= \lim_{k \to \infty} \frac{|f(x_0 + t_k e_i) - f(x_0) - b_i t_{ki}|}{\|t_k e_i\|} = 0$$

Can drop absolute value and have

$$\lim_{k \to \infty} \frac{f(x_0 + t_k e_i) - f(x_0)}{\|t_k e_i\|} = \lim_{k \to \infty} \frac{b_i t_{ki}}{\|t_k e_i\|}$$

Using same logic as earlier $\|t_k e_i\| = t_{ki}$

So have:

$$\lim_{k \to \infty} \frac{f(x_0 + t_k e_i) - f(x_0)}{t_{ki}} = \lim_{k \to \infty} \frac{b_i \cancel{t_{ki}}}{\cancel{t_{ki}}}$$

and as $t_k \to 0$ $t_{ki} \to 0$ so have

$$\lim_{t_k \to 0} \frac{f(x_0 + t_k e_i) - f(x_0)}{t_{ki}} = b_i$$

$$\|$$

from def $[D_{e_i} f(x_0) = b_i$

$$\frac{\partial f}{\partial x_i}(x_0) = b_i \quad \square$$

3c) The gradient is the vector of $\frac{\partial f}{\partial x_i}$ where x_i is the unit direction or e_i, and as $b_i = \frac{\partial f}{\partial x_i}(x_0)$ then this holds for all b and $\nabla f(x_0) = b$

d) Make a guess $b = \nabla f(x_0) + \nabla g(x_0)$

By def of differentiability, want:

$$= \lim_{h \to 0} \frac{f(x_0+h) + g(x_0+h) - [f(x_0) + g(x_0) + \langle \nabla f(x_0) + \nabla g(x_0), h\rangle}{|h|} = 0$$

Using linearity of inner product

$$= \lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - \langle \nabla f(x_0), h\rangle + g(x_0+h) - g(x_0) - \langle \nabla g(x_0), h\rangle|}{\|h\|}$$

By triangle inequality

$$\leq \lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - \langle \nabla f(x_0), h\rangle|}{\|h\|} + \frac{|g(x_0+h) - g(x_0) - \langle \nabla g(x_0), h\rangle|}{\|h\|}$$

$$= \lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - \langle \nabla f(x_0), h\rangle|}{\|h\|} + \lim_{h \to 0} \frac{|g(x_0+h) - g(x) - \langle \nabla g(x_0), h\rangle|}{\|h\|}$$

By assumption f and g are differentiable so get both limits are 0 and our original limit ≤ 0.

Since we are taking absolute value and dividing by $\|h\|$, then the original limit is nonnegative so $= 0$.

Therefore, $f+g$ is differentiable and $\nabla(f+g)(x_0) = \nabla f(x_0) + \nabla g(x_0)$

4) a) $\nabla f(x,y) = \langle y+2, x \rangle$

b) let $b = (\nabla f(x))$, and $\vec{x}_0 = (x_0, y_0)$
and $\vec{h} = (h,k)$

Have $\displaystyle\lim_{\vec{h}\to 0} \frac{|f(x+h, y+k) - [f(x,y) + \langle \nabla f(x,y), (h,k)\rangle]|}{\|\vec{h}\|}$

$$= \lim_{(h,k)\to 0} \frac{|(x+h)(y+k) + 2(x+h) - [xy + 2x + \langle (y+2, x), (h,k)\rangle]|}{\|\vec{h}\|}$$

$$= \lim_{(h,k)\to 0} \frac{|\cancel{xy} + \cancel{xk} + \cancel{hy} + hk + \cancel{2x} + \cancel{2h} - \cancel{xy} - \cancel{2x} - \cancel{hy} - \cancel{2h} - \cancel{xk}]|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}}$$

Since $\displaystyle\lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = 0$, $\displaystyle\lim_{(h,k)\to 0,0} \frac{|hk|}{\sqrt{h^2+k^2}} = 0$

Therefore, f is differentiable at (x,y) as the limit $= 0$ $\square$