MATH 70 ANSWERS TO WORKSHEET 9, Spring 2021

You are encouraged to work with others, but the final results must be your own. Please give complete reasoning for all worksheet answers.[1]

1. (4 points) Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^n$ and let $V = \text{Span}\{\mathbf{v}\} = \{c\mathbf{v} c \in \mathbb{R}\}$. Show that $V^\perp$ has dimension $n-1$. HINT: note that $V^\perp = \{\mathbf{x} \in \mathbb{R}^n \mathbf{v}\cdot\mathbf{x} = 0\}$. What sort of equation is $\mathbf{v}\cdot\mathbf{x} = \mathbf{v}^T\mathbf{x} = 0$?

Solution:

Using the hint, consider the equation $\mathbf{v}\cdot\mathbf{x} = \mathbf{v}^T\mathbf{x} = 0$. This is equivalent to $x_1v_1 + x_2v_2 + ... + x_nv_n = 0$. Without loss of generality, assume that $v_1 \neq 0$. The solution to the homogeneous equation is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_2 \begin{pmatrix} -\frac{v_2}{v_1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{v_3}{v_1} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ldots + x_n \begin{pmatrix} -\frac{v_n}{v_1} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Therefore, the subspace $V^\perp$ is spanned by $n-1$ basis vectors. It follows that $V^\perp$ has dimension $n-1$. $\mathbf{v}\cdot\mathbf{x} = \mathbf{v}^T\mathbf{x} = 0$ is the equation of a hyperplane with normal vector $\mathbf{v}$.

2. (6 points) Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be an orthogonal set in $\mathbb{R}^n$ and let $\alpha_1, \alpha_2, \cdots, \alpha_p$ are scalars, show that

$$\|\sum_{i=1}^{p} \alpha_i\, \mathbf{v}_i\|^2 = \sum_{i=1}^{p} |\alpha_i|^2 \|\mathbf{v}_i\|^2.$$

Solution:

$$\begin{aligned}
||\sum_{i=1}^{p} \alpha_i\, \mathbf{v}_i||^2 &= \left(\sum_{i=1}^{p} \alpha_i\, \mathbf{v}_i\right) \cdot \left(\sum_{i=1}^{p} \alpha_i\, \mathbf{v}_i\right) \\
&= (\alpha_1\mathbf{v}_1 + \alpha_1\mathbf{v}_2 + ... + \alpha_p\mathbf{v}_p) \cdot (\alpha_1\mathbf{v}_1 + \alpha_1\mathbf{v}_2 + ... + \alpha_p\mathbf{v}_p) \\
&= \left(\alpha_1^2\mathbf{v}_1\cdot\mathbf{v}_1 + \alpha_2^2\mathbf{v}_2\cdot\mathbf{v}_2 + ... + \alpha_p^2\mathbf{v}_p\cdot\mathbf{v}_p\right) + \sum_{i=1}^{p}\sum_{j=1, j\neq i}^{p} \alpha_i\alpha_j\mathbf{v}_i\cdot\mathbf{v}_j \\
&= \left(\alpha_1^2||\mathbf{v}_1||^2 + \alpha_2^2||\mathbf{v}_2||^2 + ... + \alpha_p^2||\mathbf{v}_p||^2\right) + 0 \\
&= \sum_{i=1}^{p} |\alpha_i|^2 \|\mathbf{v}_i\|^2
\end{aligned}$$

Above, the fourth equality follows from the orthogonality of the given set i.e. $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$.

3. (10 points) Consider two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Let $\mathbf{w} = \mathbf{u} - \text{Proj}_{\mathbf{v}}\,\mathbf{u}$. Recall that $\text{Proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$

(a) (1 point) Show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

(b) (1 point) Show that $\operatorname{Proj}_{\mathbf{v}} \mathbf{u}$ and $\mathbf{w}$ are orthogonal.

(c) (1 point) Apply Pythagorean theorem to $\mathbf{u} = \operatorname{Proj}_{\mathbf{v}} \mathbf{u} + \mathbf{w}$ and show that

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

(d) (2 points)Use the above inequality to show that

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Solution:

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot (\mathbf{u} - \operatorname{Proj}_{\mathbf{v}} \mathbf{u}) = \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \operatorname{Proj}_{\mathbf{v}} \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right) \mathbf{v} = \mathbf{v} \cdot \mathbf{u} - \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\right) \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} = 0$. Therefore, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

(b) We compute the inner product of $\operatorname{Proj}_{\mathbf{v}} \mathbf{u}$ and $\mathbf{w}$.

$$\begin{aligned}
\operatorname{Proj}_{\mathbf{v}} \mathbf{u} \cdot \mathbf{w} &= \operatorname{Proj}_{\mathbf{v}} \mathbf{u} \cdot (\mathbf{u} - \operatorname{Proj}_{\mathbf{v}} \mathbf{u}) \\
&= \operatorname{Proj}_{\mathbf{v}} \mathbf{u} \cdot \mathbf{u} - \operatorname{Proj}_{\mathbf{v}} \mathbf{u} \cdot \operatorname{Proj}_{\mathbf{v}} \mathbf{u} \\
&= \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right) \cdot \mathbf{u} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right) \cdot \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right) \\
&= \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} \cdot \mathbf{u} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right)^2 \mathbf{v} \cdot \mathbf{v} \\
&= \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} \cdot \mathbf{u} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} \cdot \mathbf{u} = 0
\end{aligned}$$

Therefore, $\operatorname{Proj}_{\mathbf{v}} \mathbf{u}$ and $\mathbf{w}$ are orthogonal.

(c) Applying the Pythagorean theorem to $\mathbf{u} = \operatorname{Proj}_{\mathbf{v}} \mathbf{u} + \mathbf{w}$, we have

$$\begin{aligned}
||\mathbf{u}||^2 &= ||\operatorname{Proj}_{\mathbf{v}} \mathbf{u}||^2 + ||\mathbf{w}||^2 \\
&= \operatorname{Proj}_{\mathbf{v}} \mathbf{u} \cdot \operatorname{Proj}_{\mathbf{v}} \mathbf{u} + ||\mathbf{w}||^2 \\
&= \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right) \cdot \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right) + ||\mathbf{w}||^2 \\
&= \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\right)^2 \mathbf{v} \cdot \mathbf{v} + ||\mathbf{w}||^2 \\
&= \left(\frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{v}||^2}\right)^2 ||\mathbf{v}||^2 + ||\mathbf{w}||^2
\end{aligned}$$

From the last equation, $\left(\frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{v}||^2}\right)^2 ||\mathbf{v}||^2 = ||\mathbf{u}||^2 - ||\mathbf{w}||^2$. Therefore, $\left(\frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{v}||^2}\right)^2 ||\mathbf{v}||^2 \leq ||\mathbf{u}||^2$. This is equivalent to $\frac{(\mathbf{v} \cdot \mathbf{u})^2}{||\mathbf{v}||^2} \leq ||\mathbf{u}||^2$. Multiplying both sides of the inequality by $||\mathbf{v}||^2$, we obtain $(\mathbf{v} \cdot \mathbf{u})^2 \leq ||\mathbf{u}||^2||\mathbf{v}||^2$. Taking square root on both sides, we arrive at the desired inequality

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

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(d) Consider $||\mathbf{u} + \mathbf{v}||^2$.

$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = ||\mathbf{u}||^2 + 2\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||^2$

Above, the third equality uses the commutativity of the inner product, $\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}$. We will now make use of the result in (c) to derive the inequality.

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}||^2 &\leq ||\mathbf{u}||^2 + 2\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||^2 \\ &\leq ||\mathbf{u}||^2 + 2|\mathbf{u} \cdot \mathbf{v}| + ||\mathbf{v}||^2 \\ &\leq ||\mathbf{u}||^2 + 2||\mathbf{u}|| \, ||\mathbf{v}|| + ||\mathbf{v}||^2 \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$

Taking square roots on both sides, we obtain the desired inequality

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$