breakable

MATH 70 WORKSHEET 3 SOLUTIONS

Instructions: This worksheet is due on Gradescope at 11:59 p.m. Eastern Time on Sunday, October 4. You are encouraged to work with others, but the final results must be your own.[1]

Please give complete reasoning for all worksheet answers.

1. (4 points) Let $\mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

   (a) Prove that $0\mathbf{v} = \mathbf{0}$. In a similar way one can prove that $c\mathbf{0} = \mathbf{0}$ for any c.

   Solution:

   Suppose $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. Then $0\mathbf{v} = 0 \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0v_1 \\ \vdots \\ 0v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$.

   (b) Assume $c\mathbf{v} = \mathbf{0}$. Prove that if $c \neq 0$ then $\mathbf{v} = \mathbf{0}$.

   Solution:

   We assume that $c\mathbf{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$. That is, $cv_i = 0$ for all i in $\{1, \ldots, n\}$. If $c \neq 0$, we know from arithmetic of real numbers that $v_i = 0$. This is true for all i, and so $\mathbf{v} = \mathbf{0}$.

   enumerate This proves that $c\mathbf{v} = \mathbf{0}$ if and only if $c = 0$ or $\mathbf{v} = \mathbf{0}$.
   (**Note:** you can use the algebraic properties of $\mathbb{R}^n$ in your proofs for this problem.)

   (c) (8 points) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathbf{u} \in \text{Span}(S)$. Show $\mathbf{u}$ has a unique representation as a linear combination of vectors from S if and only if S is a linearly independent set.

   Note: This statement is a bi-conditional (an if and only if) so requires the following two statements to be proven:

   (i) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathbf{u}$ be in $\text{Span}(S)$. Show if $\mathbf{u}$ has a unique representation as a linear combination of vectors from S then S is a linearly independent set.

   (ii) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathbf{u}$ be in $\text{Span}(S)$. If S is a linearly independent set then $\mathbf{u}$ has a unique representation as a linear combination of vectors from S.

Solution:

($\Rightarrow$) (By contrapositive.) Assume that S is linearly dependent and let

$$x_1\mathbf{v}_1 + \quad x_2\mathbf{v}_2 + \quad \ldots \quad + x_n\mathbf{v}_n \quad = \vec{\mathbf{0}}$$

be a linear combination of vectors from S with $x_i \neq 0$ for at least one i. Since $\mathbf{u}$ is in the Span(S) we also have

$$y_1\mathbf{v}_1 + \quad y_2\mathbf{v}_2 + \quad \ldots \quad + y_n\mathbf{v}_n \quad = \mathbf{u}.$$

Examine the following computation:

$$\begin{aligned}
\mathbf{u} &= \mathbf{u} + \vec{\mathbf{0}} \\
&= (\ y_1\mathbf{v}_1 + \quad y_2\mathbf{v}_2 + \quad \ldots \quad + y_n\mathbf{v}_n\ ) + (\ x_1\mathbf{v}_1 + \quad x_2\mathbf{v}_2 + \quad \ldots \quad + x_n\mathbf{v}_n\ ) \\
&= (y_1 + x_1)\mathbf{v}_1 + (y_2 + x_2)\mathbf{v}s_2 + \ldots + (y_n + x_n)\mathbf{v}_n
\end{aligned}$$

Since there is at least one $x_i \neq 0$ then for at least one i we have $y_i \neq y_i + x_i$. This means that $y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \ldots + y_n\mathbf{v}_n$ and $(y_1+x_1)\mathbf{v}_1 + (y_2+x_2)\mathbf{v}_2 + \ldots + (y_n+x_n)\mathbf{v}_n$ are different linear combinations. Since both equal $\mathbf{u}$ we have shown there are at least two different representations for $\mathbf{u}$ meaning $\mathbf{u}$ does not have a unique representation.

($\Leftarrow$) (by contrapositive) Assume that $\mathbf{u}$ has at least two different linear representations say,

$$\begin{aligned}
\mathbf{u} &= \sum_{i=1}^{n} x_i\mathbf{v}_i \\
\mathbf{u} &= \sum_{i=1}^{n} y_i\mathbf{v}_i
\end{aligned} \qquad \text{for } \mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n \text{ in } S \text{ and for at least one } i,\ x_i \neq y_i.$$

Then we have

$$\vec{\mathbf{0}} = \mathbf{u} - \mathbf{u} = \left(\sum_{i=1}^{n} x_i\mathbf{e}_i\right) - \left(\sum_{i=1}^{n} y_i\mathbf{v}_i\right) = \sum_{i=1}^{n}(x_i - y_i)\mathbf{v}_i$$

and since for at least one i, $x_i \neq y_i$ we have $x_i - y_i \neq 0$. This gives us a linear combinations of vectors from S equal to $\vec{\mathbf{0}}$ with at least one nonzero scalar. So, S is linearly dependent.

($\Leftarrow$) ALTERNATIVE: since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set, we know that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n = \vec{\mathbf{0}}$ iff $c_1 = c_2 = \ldots = c_n = 0$.
Thus, if matrix A has columns $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$, then $A\mathbf{x} = \vec{\mathbf{0}}$ only has the trivial solution.
Thus if $\mathbf{u} \in \text{Span}(S)$, **the equation $A\mathbf{x} = \mathbf{u}$ has a unique solution**.
Thus $\mathbf{u} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n$ is the unique representation of $\mathbf{u}$ as the vectors in S.

(d) (4 points) If the columns of the $m \times n$ matrix A span $\mathbb{R}^m$ and are also linearly independent, what can you conclude about the relation between the dimensions, m and n, of A?

Solution:

Since the columns of A are linearly independent, we know that the matrix A has a pivot position in every column. Since the columns of A span all of $\mathbb{R}^m$, we know that A has a pivot position in every row. Therefore, the number of rows must equal the number of columns, and $m = n$.

(e) (4 points) Show that the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ in $\mathbb{R}^3$ is linearly dependent, but that any set containing 3 of them is linearly independent. The vector $\mathbf{e}_i$ represents the vectors with all zeros except in position i where it has a 1.

Solution:

(i) We have that $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3 = 1\mathbf{e}_1+1\mathbf{e}_2+1\mathbf{e}_3$ so $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ is linearly dependent.

(ii) We have by above proof that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent.

(iii) Suppose we have the set $S = \{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k\}$ Where i, j, k are either 1,2,3 and not equal. Then,

$$\begin{aligned} a\mathbf{e}_i + b\mathbf{e}_j + c(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k) = \vec{\mathbf{0}} \quad &\Rightarrow \\ (a-c)\mathbf{e}_i + (b-c)\mathbf{e}_j - c\mathbf{e}_k = \vec{\mathbf{0}} \quad &\Rightarrow \\ a-c=0, \quad b-c=0, \quad c=0 \quad &\Rightarrow \\ a=0, \quad b=0, \quad c=0 \end{aligned}$$

So S must be linearly independent.