The Method of Moments:

The Poisson distribution and the normal distribution

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Outline

The method of moments

- Make n measurements of Y, $Y_j = y_j$ for $j = 1, \ldots, n$.
- Posited distribution has s parameters, $f_Y(y; \theta_1, \ldots, \theta_s)$
- Set s *moments*, equal to corresponding *sample moments*

$$E(Y) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y = \frac{1}{n} \sum_{j=1}^{n} y_j$$

$$E(Y^2) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y^2 = \frac{1}{n} \sum_{j=1}^{n} y_j^2$$

$$\ldots$$

$$E(Y^s) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y^s = \frac{1}{n} \sum_{j=1}^{n} y_j^s$$

- Yields s simultaneous equations for the s parameters.

The Poisson distribution and its moments

- The *Poisson distribution* is

$$\boxed{p_X(k) = \text{Prob}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

- *Moments* of the Poisson distribution

Moment	Expression	Result
Normalization	$E(1) = \sum_{k=0}^{\infty} p_X(k)$	$= 1$
Mean	$E(X) = \sum_{k=0}^{\infty} p_X(k)k$	$= \lambda$

Method of moments for the Poisson distribution

- The *Poisson distribution* is

$$\boxed{p_X(k) = \mathrm{Prob}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

- The theoretical mean is $E(X) = \lambda$
- One parameter, so estimate λ by the mean of data

$$\boxed{\lambda_e = \frac{1}{n}\sum_{j=1}^{n} k_j.}$$

- This is identical to our result from MLE.

MM estimators for the Poisson distribution

- Since we have

$$\lambda_e = \frac{1}{n}\sum_{j=1}^{n} k_j$$

- We define the estimator function

$$\boxed{\hat{\lambda}(\vec{x}) = \frac{1}{n}\sum_{j=1}^{n} x_j}$$

- Again, this is identical to our result from MLE.

The normal distribution and its moments

- The *normal distribution pdf* is

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(x-\mu)^2}{2v}\right],}$$

- *Moments* of the normal distribution

Moment	Expression	Result
Normalization	$E(1) = \int_{\mathbb{R}} dx\ f_X(x)$	$= 1$
Mean	$E(X) = \int_{\mathbb{R}} dx\ f_X(x) x$	$= \mu$
Mean square	$E(X^2) = \int_{\mathbb{R}} dx\ f_X(x) x^2$	$= \mu^2 + v$
Variance	$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$	$= v$

Method of moments for the normal distribution

- The *normal distribution pdf* is

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(x-\mu)^2}{2v}\right],}$$

- Theoretical mean is $E(X) = \mu$
- Theoretical mean square is $E(X^2) = \mu^2 + v$
- PDF has two parameters, so set

$$\mu_e = \frac{1}{n}\sum_{j=1}^{n} x_j \quad \text{and} \quad \mu_e^2 + v_e = \frac{1}{n}\sum_{j=1}^{n} x_j^2$$

- Solve for μ_e and v_e
- MM estimates identical to MLE estimates for μ_e and v_e.

Normal distribution (continued)

- To verify the last point for v_e, note

$$\begin{aligned}
v_e =& \frac{1}{n}\sum_{j=1}^{n} x_j^2 - \mu_e^2 \\
=& \frac{1}{n}\sum_{j=1}^{n} x_j^2 - 2\mu_e^2 + \mu_e^2 \\
=& \frac{1}{n}\sum_{j=1}^{n} x_j^2 - 2\mu_e \frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} \mu_e^2 \\
=& \frac{1}{n}\sum_{j=1}^{n} \left(x_j^2 - 2\mu_e x_j + \mu_e^2\right) \\
=& \frac{1}{n}\sum_{j=1}^{n} \left(x_j - \mu_e\right)^2 \\
=& \frac{1}{n}\sum_{j=1}^{n} \left[x_j - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]^2
\end{aligned}$$

MM estimators for the normal distribution

- Since we have

$$\mu_e = \frac{1}{n}\sum_{j=1}^{n} x_j$$

$$v_e = \frac{1}{n}\sum_{j=1}^{n}\left[x_j - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]^2$$

- We define the estimator functions

$$\boxed{\hat{\mu}(\vec{x}) = \frac{1}{n}\sum_{j=1}^{n} x_j}$$

$$\boxed{\hat{v}(\vec{x}) = \frac{1}{n}\sum_{j=1}^{n}\left[x_j - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]^2}$$

- Again, this is identical to our results from MLE.

Summary

- We reviewed the *method of moments* (MM).
- We applied MM to the *Poisson distribution*.
- We applied MM to the *normal distribution*.
- Our results, for estimates and estimators, were identical to those obtained from MLE.