Hypothesis testing and decision rules

The Generalized Likelihood Ratio

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Outline

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Generalized Likelihood Ratio (GLR)

- Suppose $y_1, y_2, \ldots, y_n$ is random sample from the uniform pdf on $[0, \theta]$
- The parameter θ is unknown
- We wish to conduct a hypothesis test at level of significance α between the alternatives
  - $H_0 : \theta = \theta_0$
  - $H_1 : \theta < \theta_0$

The sets ω and Ω

- Define ω to be the set of parameter values possible under the constraints of H_0
- Define Ω to be the set of all unknown parameters
- In the example of the uniform distribution on $[0, \theta]$,
  - $\omega = \{\theta \mid \theta = \theta_0\}$
  - $\Omega = \{\theta \mid 0 < \theta \leq \theta_0\}$
- Note that $\omega \subset \Omega$

Maximizing likelihood over ω and Ω

- Let L be the likelihood function which, in our example, is

$$L(\theta) = \prod_j^n f_Y(y_j;\theta) = \begin{cases} \left(\frac{1}{\theta}\right)^n & \text{if } y_{\max} \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

- Maximize likelihood over all $\theta \in \omega$, in other words $\theta = \theta_0$
  - This will be L evaluated at θ_0
  - $\max_{\theta\in\omega} L(\theta) = \left(\frac{1}{\theta_0}\right)^n$
- Maximize likelihood over all $\theta \in \Omega$.
  - This will be L evaluated at the maximum likelihood estimate
  - $\max_{\theta\in\Omega} L(\theta) = \left(\frac{1}{y_{\max}}\right)^n$

First example: GLR for uniform distribution

- *Generalized Likelihood Ratio (GLR)* is then defined to be

$$\lambda = \frac{\max_{\theta \in \omega} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$

- For our example, we have

$$\lambda = \frac{\max_{\theta \in \omega} L(\theta)}{\max_{\theta \in \Omega} L(\theta)} = \frac{\left(\frac{1}{\theta_0}\right)^n}{\left(\frac{1}{y_{\max}}\right)^n} = \left(\frac{y_{\max}}{\theta_0}\right)^n$$

- Note that λ will be positive, but always strictly ≤ 1.
- Values of λ near one are compatible with H_0
- Values of λ near zero suggest that we reject H_0

General definition of Generalized Likelihood Ratio

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- Let $y_1, y_2, \ldots, y_n$ be random sample from $f_Y(y; \theta_1, \ldots, \theta_k)$
- The GLR is defined to be

$$\lambda = \frac{\max_{\theta \in \omega} L(\theta_1, \ldots, \theta_k)}{\max_{\theta \in \Omega} L(\theta_1, \ldots, \theta_k)}$$

- Note that the generalization to k parameters is, in principle, straightforward.

Hypothesis testing with the GLR

- Recall that we have determined
  - Note that λ will be positive, but always strictly ≤ 1.
  - Values of λ near one are compatible with H_0
  - Values of λ near zero suggest that we reject H_0
- GLR Test (GLRT) rejects H_0 whenever $0 < \lambda \leq \lambda^*$
- Here λ^* is chosen to satisfy

$$P\left(0 < \Lambda \leq \lambda^* \mid H_0 \text{ is true}\right) = \alpha,$$

  where Λ is the random variable associated with λ.
- Note that if we knew $f_\Lambda(\lambda \mid H_0)$, then λ^* could be determined from

$$\alpha = \int_0^{\lambda^*} d\lambda\ f_\Lambda(\lambda \mid H_0)$$

Hypothesis testing with GLR (continued)

- We would like to choose our cutoff so that

$$\alpha = \int_0^{\lambda^*} d\lambda \; f_\Lambda(\lambda \mid H_0)$$

- Unfortunately, the pdf $f_\Lambda(\lambda \mid H_0)$ may not be so easy to determine
- For the case of $\lambda = \left(\frac{y_{\max}}{\theta_0}\right)^n$, we have

$$\begin{aligned}\alpha =& P\left(\Lambda \leq \lambda^* \mid H_0 \text{ is true}\right)\\ =& P\left[\left(\frac{Y_{\max}}{\theta_0}\right)^n \leq \lambda^* \mid H_0 \text{ is true}\right]\\ =& P\left(Y_{\max} \leq \theta_0 \sqrt[n]{\lambda^*} \mid H_0 \text{ is true}\right)\end{aligned}$$

Hypothesis testing with GLR (continued)

- From last page

$$
\begin{aligned}
\alpha =& P\left(Y_{\max} \leq \theta_0 \sqrt[n]{\lambda^*} \mid H_0 \text{ is true}\right) \\
=& F_{Y_{\max}}\left(\theta_0 \sqrt[n]{\lambda^*}\right) \\
=& \left[F_Y\left(\theta_0 \sqrt[n]{\lambda^*}\right)\right]^n \\
=& \left(\frac{\theta_0 \sqrt[n]{\lambda^*}}{\theta_0}\right)^n \\
=& \lambda^*
\end{aligned}
$$

- So reject H_0 if $\lambda \leq \lambda^* = \alpha$, or equivalently $y_{\max} \leq \theta_0 \sqrt[n]{\alpha}$

Second example: Binomial distribution

- Bernoulli trial $p_X(k;p) = p^k(1-p)^{1-k}$
- Likelihood is

$$L(p) = \prod_j^n p^{k_j}(1-p)^{1-k_j} = p^k(1-p)^{n-k}$$

where $k = \sum_j^n k_j$.

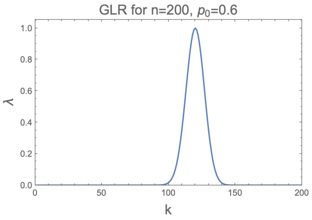
- Log likelihood $\ln L(p) = k \ln p + (n-k)\ln(1-p)$
- Max likelihood $0 = \frac{d}{dp}\ln L(p) = \frac{k}{p} - \frac{n-k}{1-p}$ so $p_e = \frac{k}{n}$
- Max likelihood estimator $\hat{p}(\vec{X}) = \frac{1}{n}\sum_j^n X_j = \overline{X}$

Binomial distribution: GLR hypothesis testing

- Test $H_0 : p = p_0$ versus $H_1 : p \neq p_0$
- $\max_{p\in\omega} L(p) = L(p_0) = p_0^k(1-p_0)^{n-k}$
- $\max_{p\in\Omega} L(p) = L(\hat{p}) = \left(\frac{k}{n}\right)^k \left(1-\frac{k}{n}\right)^{n-k}$
- GLR is then

$$\lambda = \frac{\max_{p\in\omega} L(p)}{\max_{p\in\Omega} L(p)} = \frac{p_0^k(1-p_0)^{n-k}}{\left({}^k/_n\right)^k\left(1-{}^k/_n\right)^{n-k}} = \left(\frac{p_0}{{}^k/_n}\right)^k\left(\frac{1-p_0}{1-{}^k/_n}\right)^{n-k}$$

- Example for $n = 200$ and $p_0 = 0.6$ (so $np_0 = 120$):



Binomial distribution: GLR hypothesis testing

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- Demand that

$$\alpha = \text{Prob}\left(\lambda < \lambda^*\right) = \text{Prob}\left(\left(\frac{p_0}{k/n}\right)^k \left(\frac{1-p_0}{1-k/n}\right)^{n-k} < \lambda^*\right)$$

- Given p_0, work out correspondence between λ^* and α.

- **Ex.:** For $n = 200$ and $p_0 = 0.60$, and taking $\lambda^* = 0.25$
  - Reject H_0 if $\lambda < \lambda^*$
  - Reject H_0 if $k \leq 108$ or $k \geq 132$

$$\alpha = \sum_{k=0}^{108} \binom{200}{k} (0.6)^k (1-0.6)^{200-k} + \sum_{k=132}^{200} \binom{200}{k} (0.6)^k (1-0.6)^{200-k} = 0.0972$$

  - About 90% confidence we will not reject H_0 if it is true.

Binomial distribution: GLR hypothesis testing

- **Ex.:** For $n = 200$ and $p_0 = 0.60$, and taking $\lambda^* = 0.15$
  - Reject H_0 if $\lambda < \lambda^*$
  - Reject H_0 if $k \leq 106$ or $k \geq 134$

$$\alpha = \sum_{k=0}^{106} \binom{200}{k} (0.6)^k (1 - 0.6)^{200-k} + \sum_{k=134}^{200} \binom{200}{k} (0.6)^k (1 - 0.6)^{200-k} = 0.0513$$

  - About 95% confidence we will not reject H_0 if it is true.

Binomial distribution: GLR hypothesis testing

- **Ex.:** For $n = 200$ and $p_0 = 0.60$, and taking $\lambda^* = 0.0369$
  - Reject H_0 if $\lambda < \lambda^*$
  - Reject H_0 if $k \leq 102$ or $k \geq 139$

$$\alpha = \sum_{k=0}^{102} \binom{200}{k} (0.6)^k (1-0.6)^{200-k} + \sum_{k=139}^{200} \binom{200}{k} (0.6)^k (1-0.6)^{200-k} = 0.00913$$

  - About 99% confidence we will not reject H_0 if it is true.

Plot of α versus λ^*

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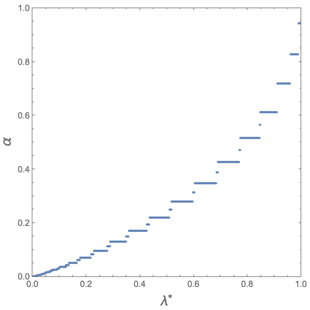
- **Ex.:** For $n = 200$ and $p_0 = 0.60$, plot α versus λ^*
- Smaller α results from smaller thresholds λ^*.
- Discrete "steps" due to discreteness of r.v. $k \in \mathbb{Z}$.



Summary

- We introduced and defined Generalized Likelihood Ratio
- We demonstrated how to use GLR for hypothesis testing
  - For uniform distribution
  - For binomial distribution
- Reject H_0 when $\lambda < \lambda^*$.
- Hence smaller λ^* mean smaller α, and greater confidence against rejecting H_0 if it is true.