Math 135 **Real Analysis I** **Fall 2022**

Exam 2 Review[1]

Exam 2 will take place during the Open Block from noon to 1:20 p.m. on Monday, November 21 in **JCC 270**. There will be no classes or office hours the day of the exam.

There will be a *review session on Sunday, November 20 from 1:30 p.m. to 2:45 p.m. in **JCC 265***.

The test will cover Sections 9.1–9.4, 10.1–10.3, 11.1–11.2, tests for uniform convergence (Notes), and power series (Notes). This corresponds to the material in Problem Sets 6 through 9. The exam will include definitions, statements of theorems, proofs and examples in which you justify your work. The proofs will be from those discussed in class, from the homework or this review sheet, or they will be similar to one of these. Justify every statement you make by either referring to theorems or definitions. On the test, you will get some credit in problems that you cannot finish if you quote relevant definitions and theorems.

What You Should Know for the Exam

I. **Definitions.** Cauchy sequence (p. 228), Pointwise and uniform convergence of a sequence $\{f_k\}$ of functions (pp. 241, 245). The scalar product and the norm in $\mathbb{R}^n$ (p. 271–272), the ε-K definition of convergence of a sequence in $\mathbb{R}^n$ (p. 278), Cauchy sequences in $\mathbb{R}^n$ (Exercise 8, Section 10.2), open balls in $\mathbb{R}^n$ (p. 282), the interior of a set (p. 282), open sets and closed sets (p. 283–284), the boundary and exterior of a set (p. 287), the closure of a set (Problem 10, p. 289), relatively open and relatively closed subsets of a set A (Problem Set 8).

Continuity of a mapping $F : A \to \mathbb{R}^m$ at a point $\mathbf{u} \in A$ and over the whole domain A (p. 290), sequentially compact set (p. 299), bounded set and sequence (p. 299), uniformly continuous map (p. 66 and p. 302), the ε-δ condition for uniform continuity (see also Theorem 3.22, p. 73), ~~the line segment $\{t\mathbf{u}+(1-t)\mathbf{v} \mid 0 \le t \le 1\}$ joining points $\mathbf{u}$ and $\mathbf{v}$, convex set (p. 305), parametrized path $\gamma : [a,b] \to \mathbb{R}^n$ and path joining points $\mathbf{u}$ and $\mathbf{v}$ (p. 305), pathwise connected set (p. 305), the intermediate-value property (p. 309), separation of a set A by open sets $\mathscr{U}$ and $\mathscr{V}$ and disconnected sets (p. 310), connected sets (p. 310).~~

II. **Statements of Theorems** absolute convergence test (Th. 9.18—if a series converges absolutely, it converges), ratio test (Cor. 9.21), Weierstrass uniform convergence criterion.
The theorem that every continuous function on a sequentially compact set is uniformly continuous (Theorem 11.25).
The ε-δ criterion for uniform continuity of a map $F : A \to \mathbb{R}^m$ (Theorem 11.27).
Comparison test for uniform convergence (Notes).
Weierstrass M-test (Notes).

III. **Statements of Theorems and Their Proofs:**
The Cauchy convergence criterion for sequences (Theorem 9.4), the Cauchy convergence criterion for series (Theorem 9.17), uniform limit of continuous functions (Theorem 9.31)

The Cauchy-Schwartz inequality (Theorem 10.6), the Triangle Inequality for norms (Theorem 10.7) and for distances (Corollary 10.8), the componentwise convergence criterion (Theorem 10.9), algebraic rules for limits of sequences (Theorem 10.10).

Open balls are open sets (Proposition 10.13), the complementing characterization for open and closed sets (Theorem 10.16), Propositions 10.17 and 10.18 on unions and intersections of open sets and of closed sets, Proposition 10.19 on characterizations of open and closed sets via their boundaries. The fact that, if $A \subset \mathbb{R}^n$, then $\mathbb{R}^n$ is the disjoint union of $\mathrm{int}A$, $\mathrm{bd}A$, and $\mathrm{ext}A$.

Theorems 11.3 and 11.5 on the continuity of sums, products, quotients, and compositions of continuous maps, the componentwise continuity criterion (Theorem 11.9), the ε-δ Criterion for continuous maps (Theorem 11.11), the criterion for continuity via inverse images (Theorem 11.12).

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence (Theorem 11.17) The sequential compactness theorem (Theorem 11.18), the image of a sequentially compact set under a continuous map is sequentially compact (Theorem 11.20), the extreme-value theorem (Theorem 11.22), continuous functions on sequentially compact sets are uniformly continuous (Theorem 11.25).

A power series $\sum a_k x^k$ has a radius of convergence (Notes).

Exam 2 Review Problems

1. (**Topology of** $\mathbb{R}^n$) For each of the sets A below, determine whether A is (i) open, (ii) closed, or (iii) sequentially compact. Give reasons. You may use the following fact: the only subsets of $\mathbb{R}^n$ that are simultaneously open and closed are the empty set $\varnothing$ and $\mathbb{R}^n$.
   (a) $A = \{\mathbf{u} \in \mathbb{R}^n \mid \|\mathbf{u}\| \leq 5\}$.
   (b) $A = \{(x,y) \in \mathbb{R}^2 \mid 1 < |x| < 2\}$.
   (c) A is any finite set in $\mathbb{R}^n$.
   (d) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^4 + y^4 + z^4 \leq 1\}$.
   (e) $A = \{z \in \mathbb{R} \mid z = x^7 + 5y^9 \text{ with } x^2 + y^2 \leq 4\}$.
   (f) $A = \{(x,y) \in \mathbb{R}^2 \mid xy > 1\}$.
   (g) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^3 + y^3 + z^3 = 1 \text{ and } xyz = 5\}$. For this A, answer only (i) and (ii).
   (h) $A = \mathbb{Q} \cap [0,1]$. In addition, what is the boundary of this set A?

2. (**Closure**) Let A be a subset of $\mathbb{R}^n$. The *closure*[2] of A, denoted $\mathrm{cl}\,A$, is the set $\mathrm{int}\,A \cup \mathrm{bd}\,A$.
   (a) Prove that $\mathrm{cl}\,A = A \cup \mathrm{bd}\,A$.
   (b) Prove that $(\mathrm{cl}\,A)^c = \mathrm{ext}\,A$, and that $\mathrm{cl}\,A$ is a closed subset of $\mathbb{R}^n$.
   (c) Prove that if B is any closed subset of $\mathbb{R}^n$ containing A, then B contains $\mathrm{cl}\,A$. Thus, $\mathrm{cl}\,A$ is the *smallest* closed subset of $\mathbb{R}^n$ containing A.
   (d) Prove that A is closed if and only if $A = \mathrm{cl}\,A$.
   (e) Prove that a point $\mathbf{u}$ belongs to $\mathrm{cl}\,A$ if and only if every open ball $B_\varepsilon(\mathbf{u})$ contains a point $\mathbf{v}$ of A.
   (f) Prove that if a set A is bounded, then its closure $\mathrm{cl}\,A$ is bounded. (*Hint*: Use (c).)
   (g) Prove that if a set A is bounded, then its boundary $\mathrm{bd}\,A$ is bounded. (*Hint*: Use (f).)

3. (**Diameter**) Let A be a nonempty subset of $\mathbb{R}^n$. The *diameter* of A, denoted $\mathrm{diam}(A)$, is the number $\sup\{d(\mathbf{u},\mathbf{v}) \mid \mathbf{u},\mathbf{v} \in A\}$, if A is bounded; in case A is unbounded, we put $\mathrm{diam}(A) = \infty$. Suppose that A is a sequentially compact subset of $\mathbb{R}^n$. Prove that A has finite diameter and that there exist points $\mathbf{u}$ and $\mathbf{v}$ in A such that $d(\mathbf{u},\mathbf{v}) = \mathrm{diam}(A)$.

4. (**The distance function is Lipschitz**) Let $A \subset \mathbb{R}^n$. A mapping $F : A \to \mathbf{R}^m$ is said to be *Lipschitz* if there is a constant M such that
$$\|F(\mathbf{u}) - F(\mathbf{v})\| \leq M\,\|\mathbf{u} - \mathbf{v}\|$$
   for all $\mathbf{u}$ and $\mathbf{v}$ in A. M is called a *Lipschitz constant* for F.
   (a) Using the ε-δ criterion, prove that a Lipschitz mapping is uniformly continuous.
   (b) For a fixed $\mathbf{v} \in \mathbb{R}^n$, define the function $F : \mathbb{R}^n \to \mathbb{R}$ by $F(\mathbf{u}) = \mathbf{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$. Prove that F is Lipschitz, and hence is uniformly continuous.

5. (**Distance from a point to a set**) Let S be a nonempty subset of $\mathbf{R}^n$. If $\mathbf{u} \in \mathbb{R}^n$, the *distance* from $\mathbf{u}$ to S is the number
$$d(\mathbf{u},S) = \inf\{d(\mathbf{u},\mathbf{v}) \mid \mathbf{v} \in \mathbf{S}\}.$$
   (a) Prove that the function $F(\mathbf{u}) = \mathbf{d}(\mathbf{u},\mathbf{S})$ is Lipschitz, and hence is a uniformly continuous function on $\mathbb{R}^n$.
   (b) Prove that $\mathrm{cl}\,S = \{\mathbf{u} \in \mathbb{R}^n \mid d(\mathbf{u},S) = 0\}$.

[2] cf. Problem 12, Section 10.3

(c) If S is closed, prove that for any point $\mathbf{u} \in \mathbb{R}^n$, there is a point $\mathbf{v} \in S$ such that $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{u}, S)$. Is the point $\mathbf{v}$ unique? What if S is not closed?

6. (**Interior, closure, boundary, and exterior**) For the following sets in $\mathbb{R}^2$, find $\text{int}A$, $\text{cl}A$, $\text{bd}A$, and $\text{ext}A$.
   (a) $A = \{(x, y) \,|\, x \text{ and } y \text{ are both rational}\}$.
   (b) $A = \{(x, y) \,|\, xy > 1\}$.
   (c) $A = \{(\frac{1}{m}, \frac{1}{n}) \,|\, m, n \in \mathbb{N}\}$.

7. (**Interior of a subset**) Let A and B be subsets of $\mathbb{R}^n$, with $A \subset B$. Is $\text{int}\, A \subset \text{int}\, B$? Prove your result or give a counterexample.

8. (**Interior of an intersection**) Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$. Is it true that $\text{int}A \cap \text{int}B = \text{int}(A \cap B)$? Either prove this or give a counterexample.

9. (**Open or closed**)
   (a) Prove using the definition of open set that $A = \{(x, y) \in \mathbb{R}^2 \,\big|\, x > 0\}$ is open.
   (b) Prove that the set $\{(x, y, z) \,|\, xyz > 1\}$ is open in $\mathbb{R}^3$.
   (c) Is the set $\{(x, y) \,|\, x^3 + x^2 y - 2y^3 = 0\}$ in $\mathbb{R}^2$ open, closed, or neither? Justify your answer.

10. (**Uniform convergence**) Discuss the convergence and uniform convergence of

    (a) $f_n(x) = \cos(x/n)$ on $\mathbb{R}$

    (b) $f_n(x) = \dfrac{x^n}{1 + x^n}$ on $[0, 2]$

    (c) $f_n(x) = x - x^n$ on $[0, 1]$

    (d) $f_n(x) = x - x^n$ on $[0, .999]$

    (e) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ on $[0, 1]$

11. (**Uniform convergence**) Prove that $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is a continuous function on $[0, 1]$.

12. (**Radius of convergence**) Compute the radius of convergence of the following power series:

    (a) $\sum (n^4/n!) x^n$

    (b) $\sum \sqrt{n} x^n$

13. (**Relatively closed sets**) Let $A \subset \mathbb{R}^n$. Prove that a subset S of A is relatively closed in A if and only if its complement $A \setminus S$ in A is relatively open in A.

14. (**Convergent subsequences**)
    (a) Let $\{\mathbf{u}_n\}$ be a bounded sequence in $\mathbb{R}^2$. Prove this sequence has a convergent subsequence without quoting Theorem 11.17.

(b) Let $A \subset \mathbb{R}^n$ be closed and bounded. Prove that A is sequentially compact without quoting Theorem 11.18.

15. (**Extremal values**) Let $A \subset \mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be continuous. Suppose that $\mathbf{u}, \mathbf{v} \in A$ and $\gamma : [0,1] \to A$ is a path in A joining $\mathbf{u}$ to $\mathbf{v}$. Show that along this path, f assumes its maximum and minimum values.

(*End of Review Problems*)