Math 42 Worksheet 1 Solutions

September 23, 2020

1. **Using the Pythagorean Theorem, we showed in class that the distance between two points $(a_1, a_2), (b_1, b_2)$ in the plane is given by**

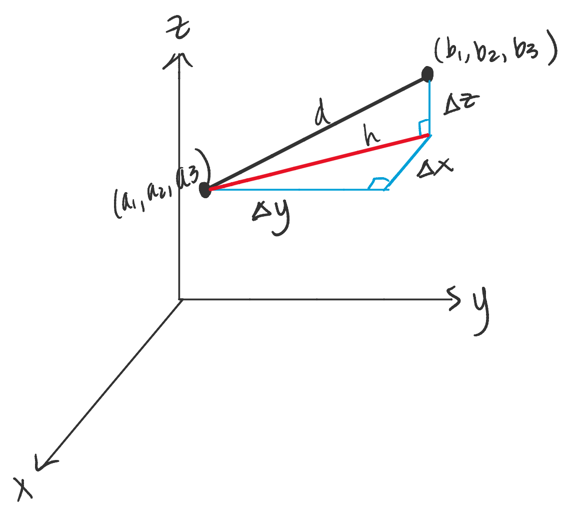
$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

Show that the distance between two points $(a_1, a_2, a_3), (b_1, b_2, b_3)$ in 3-space is given by

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

Hint: use the Pythagorean Theorem twice.

Solution: First, draw a picture!!



We can compute the changes in the x, y, and z directions:

$$\Delta x = b_1 - a_1$$
$$\Delta y = b_2 - a_2$$
$$\Delta z = b_3 - a_3$$

Using this information, we want to compute d.

(i) Find the hypotenuse of the triangle in the x-y plane:

$$\begin{aligned} h &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} \end{aligned}$$

(ii) Now, find the hypotenuse, d, of the triangle made up from sides $h, \Delta z, d$.

$$\begin{aligned} d &= \sqrt{h^2 + (\Delta z)^2} \\ &= \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2} \end{aligned}$$

2. **Look at the equation $\boldsymbol{x^2 + y^2 + z^2 = a}$. If $\boldsymbol{a}$ is negative, the equation has no solutions. If $\boldsymbol{a = 0}$ the equation has a unique point as solution namely $\boldsymbol{(0,0,0)}$. If $\boldsymbol{a > 0}$, the set of solutions form a sphere. Look now at the equation $\boldsymbol{x^2 + y^2 + z^2 - 4x + 8y - 10z = a}$. Classify the set of solutions depending on the value of $\boldsymbol{a}$.**

   Solution: It might be helpful to consider the original equation as

$$(x-0)^2 + (y-0)^2 + (z-0)^2 = a$$

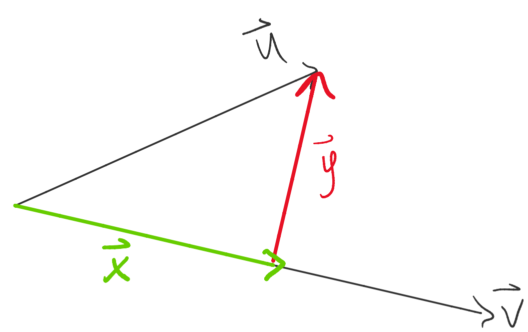
   and consider the cases when the right-hand side, a, is positive, negative, and zero. Now, we want to re-write the second equation in this form. So, we complete the square.

$$\begin{aligned} x^2 + y^2 + z^2 - 4x + 8y - 10z &= a \\ x^2 - 4x + \mathbf{4} + y^2 + 8y + \mathbf{16} + z^2 - 10z + \mathbf{25} &= a + \mathbf{4} + \mathbf{16} + \mathbf{25} \\ (x-2)^2 + (y+4)^2 + (z-5)^2 &= a + 45 \end{aligned}$$

   Using the reasoning above, if the right-hand-side (RHS) is negative, we have no solutions, so if $a < -45$, no solutions exist. If the RHS is zero, we have a single point solution. Therefore, $a = -45$ has a point solution $(2, -4, 5)$. If the RHS is positive, $a > -45$, the solution is a sphere centered at $(2, -4, 5)$ with radius $\sqrt{a + 45}$.

3. **Given the vectors $\boldsymbol{\vec{u} = \langle 3, 3, 4 \rangle}$ and $\boldsymbol{\vec{v} = \langle 2, 0, 1 \rangle}$ find vectors $\boldsymbol{\vec{x}, \vec{y}}$ such that $\boldsymbol{\vec{x}}$ is parallel to $\boldsymbol{\vec{v}}$, $\boldsymbol{\vec{y}}$ is perpendicular to $\boldsymbol{\vec{v}}$, and $\boldsymbol{\vec{u} = \vec{x} + \vec{y}}$.**

   Solution: We start by drawing a picture to set up the system.



Since $\vec{x} \,||\, \vec{v}$ we want to find the component of $\vec{u}$ that is in the direction of $\vec{v}$, which tells us we want $\vec{x} = \text{proj}_{\vec{v}}\vec{u}$.

$$\begin{aligned}\vec{x} &= \text{proj}_{\vec{v}}\vec{u}\\ &= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}\\ &= \langle 4, 0, 2\rangle\end{aligned}$$

Now, we find $\vec{y}$.

$$\vec{y} = \vec{u} - \vec{x} = \langle -1, 3, 2\rangle$$

4. **For the given vectors u and v, calculate the** $proj_v u$ **and** $scal_v u$**:**

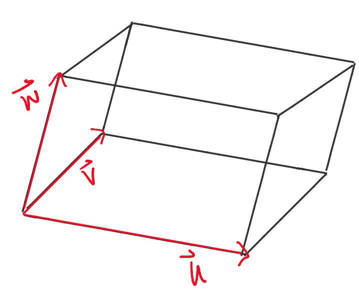
   (a) $\vec{u} =< -1, 4 >$ **and** $\vec{v} =< -4, 2 >$
   Solution: We follow the definition on Page 786.
   Step 1:
   $scal_v u = \frac{u\cdot v}{|v|} = \frac{<-1,4>\cdot<-4,2>}{\sqrt{(-4)^2+(2)^2}} = \frac{4+8}{\sqrt{20}} = \frac{12}{2\sqrt{5}} = \frac{6}{\sqrt{5}}$
   Step 2:
   $proj_v u = (scal_v u)(\frac{v}{|v|}) = (\frac{6}{\sqrt{5}})(\frac{<-4,2>}{\sqrt{20}}) = (\frac{6}{\sqrt{5}})(\frac{2<-2,1>}{2\sqrt{5}}) = \frac{6}{5} < -2, 1 >$

   (b) $\vec{u} =< 3, 3, -3 >$ **and** $\vec{v} =< 1, -1, 2 >$
   Solution: We follow the definition on Page 786.
   Step 1:
   $scal_v u = \frac{u\cdot v}{|v|} = \frac{<3,3,-3>\cdot<1,-1,2>}{\sqrt{(1)^2+(-1)^2+(2)^2}} = \frac{3-3-6}{\sqrt{6}} = \frac{-6}{\sqrt{6}} = \frac{-6\sqrt{6}}{6} = -\sqrt{6}$
   Step 2:
   $proj_v u = (scal_v u)(\frac{v}{|v|}) = (-\sqrt{6})(\frac{<1,-1,2>}{\sqrt{6}}) = - < 1, -1, 2 >=< -1, 1, -2 >$

   (c) $\vec{u} = 5i + j - 5k$ **and** $\vec{v} = -i + j - 2k$
   Solution: We follow the definition on Page 786.
   Note: these vectors are using the coordinate unit vector form so $\mathbf{i} =< 1, 0, 0 >, \mathbf{j} =< 0, 1, 0 >, \mathbf{k} =< 0, 0, 1 >$
   Step 1:
   $scal_v u = \frac{u\cdot v}{|v|} = \frac{<5,1,-5>\cdot<-1,1,-2>}{\sqrt{(-1)^2+(1)^2+(-2)^2}} = \frac{-5+1+10}{\sqrt{6}} = \frac{6}{\sqrt{6}} = \frac{6\sqrt{6}}{6} = \sqrt{6}$
   Step 2:
   $proj_v u = (scal_v u)(\frac{v}{|v|}) = (\sqrt{6})(\frac{<-1,1,-2>}{\sqrt{6}}) =< -1, 1, -2 >$

5. **Find the volume of the parallelepiped (slanted box) such that its edges are parallel to the vectors** $\boldsymbol{\vec{u} = \langle 1, 0, 3\rangle}$, $\boldsymbol{\vec{v} = \langle 0, 2, 1\rangle}$, $\boldsymbol{\vec{w} = \langle 1, 1, 1\rangle}$.

   Solution: As always, we draw a picture.



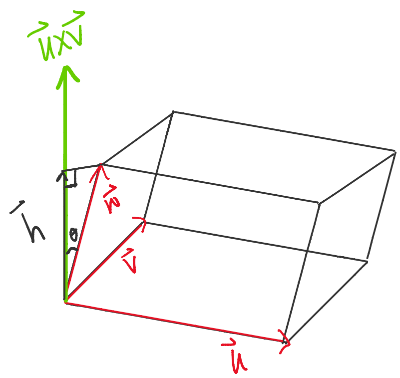
We know that volume is given by

$$V = A_{base} \cdot h$$

Let's define the base as the parallelogram defined by $\vec{u}$ and $\vec{v}$. From a property of the cross product, we have immediately that

$$A_{base} = |\vec{u} \times \vec{v}|$$

Now, we need to determine the "height". The height is defined in the direction that is perpendicular to the base. Again, from the definition of the cross product and the right hand rule, we know that this is the direction of $\vec{u} \times \vec{v}$.



This means that the height is the component of $\vec{w}$ that is in the $\vec{u} \times \vec{v}$ direction.

$$\begin{aligned} |\vec{h}| &= |\text{proj}_{(\vec{u}\times\vec{v})}\vec{w}| \\ &= \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|} \end{aligned}$$

Plugging this into our volume formula, we see that

$$\begin{aligned} V &= A_{base} \cdot h \\ &= |\vec{u} \times \vec{v}| \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|} \\ &= |\vec{w} \cdot (\vec{u} \times \vec{v})| \\ &= 5 \end{aligned}$$

Note that the final term $|\vec{w} \cdot (\vec{u} \times \vec{v})|$ is the *triple scalar* derived in class.