Tufts University
Department of Mathematics

Math 135 **Homework 11**[1] **Fall 2022**

Readings for Problem Set 11

Note that most of the proofs for a metric space are exactly the same as for $\mathbb{R}^n$.

§12.1: Metric Spaces: Open and Closed Sets, Convergence
§12.2: Completeness and the Contraction Mapping Principle

Problem Set 11
(Due Wednesday, December 7, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (15 points) (**A function space**) Let $a, b \in \mathbb{R}$. Recall that we define for f and g in $C([a,b],\mathbb{R})$,
$$d(f,g) = \mathbf{max}\left\{|f(x) - g(x)| \,\middle|\, x \in [a,b]\right\}. \tag{1}$$
   (a) Let $\mathscr{C}_b((a,b),\mathbb{R})$ be the set of continuous **bounded** functions on the **open** interval (a,b). Why can one *not* define a distance on $\mathscr{C}_b((a,b),\mathbb{R})$ using (1)?
   With this in mind, for f and g in $\mathscr{C}_b((a,b),\mathbb{R})$ define $d(f,g)$ by
$$d(f,g) = \mathbf{sup}\left\{|f(x) - g(x)| \,\middle|\, x \in (a,b)\right\}. \tag{2}$$
   (b) Prove for f and g in $\mathscr{C}_b((a,b),\mathbb{R})$ that $d(f,g) = d(g,f)$.
   (c) Prove for f and g in $\mathscr{C}_b((a,b),\mathbb{R})$ that $d(f,g) \geq 0$ and $d(f,g) = 0$ if and only if $f = g$ (i.e., $f(x) = g(x)$ for all $x \in (a,b)$).
   (d) Let f, g, and h be in $\mathscr{C}_b((a,b),\mathbb{R})$. Prove that for each $x_0 \in (a,b)$,
$$|f(x_0) - g(x_0)| \leq |f(x_0) - h(x_0)| + |h(x_0) - g(x_0)|.$$
   (e) Use this to prove that for each $x_0 \in (a,b)$ $\quad |f(x_0) - g(x_0)| \leq d(f,h) + d(h,g)$.
   (f) Now use this to prove the triangle inequality for the distance function d in (2).

   You have now proven that the set $\left(\mathscr{C}_b((a,b),\mathbb{R}), d\right)$ of continuous bounded functions on (a,b) with the sup metric is a metric space.

2. (20 points) (**Topology of a metric space**) Let (X,d) be a metric space and let $A \subset X$. We have already defined $\text{int}(A)$ as the set of $x \in A$ such that for some $\varepsilon > 0$, $B_\varepsilon(x) \subset A$.
   We now introduce the following definitions.

   Definition 1. *Let (X,d) be a metric space and let $A \subset X$. The* boundary of A, $\text{bd}(A)$ *is the set of points $x \in X$ such that*
$$\textit{for all } \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset, \textit{ and } B_\varepsilon(x) \cap (X \setminus A) \neq \emptyset.$$
   The exterior of A *is the set of all $x \in X$ such that for some $\varepsilon > 0$, $B_\varepsilon(x) \subset A^c$, where $A^c = X \setminus A$.*
   The closure of A *is the set* $\text{cl}(A) = \text{int}(A) \cup \text{bd}(A)$.

   (a) Prove that $\text{int}(A)$ is open.
   (b) Prove that $\text{ext}(A)$ is open.

(c) Prove that $\mathrm{bd}(A)$ is closed.
(d) Prove that $X = \mathrm{int}(A) \cup \mathrm{bd}(A) \cup \mathrm{ext}(A)$.
(e) Prove that $\mathrm{int}(A)$, $\mathrm{bd}(A)$, and $\mathrm{ext}(A)$ are pairwise disjoint.

3. (10 points) (**Closure**) You will use the definition above in this problem. Let (X,d) be a metric space and let $A \subset X$
 (a) Prove that $\mathrm{cl}(A)$ is closed.
 (b) Prove that if B is a closed subset of X and $A \subset B$, then $\mathrm{cl}(A) \subset B$. This shows that $\mathrm{cl}(A)$ is the smallest closed subset of X containing A.

4. (5 points) (**Pointwise convergence**) §12.1, p. 321, # 3.
Let $\{f_k\}$ be the sequence in $C\big([0,1],\mathbb{R}\big)$ defined by
$$f_k(x) = (1-x)x^k \quad \text{for } x \in [0,1], k \in \mathbb{N}.$$
Prove that the sequence converges pointwise to the constant function $f = 0$. Is the sequence $\{f_k\}$ a convergent sequence in the metric space $C\big([0,1],\mathbb{R}\big)$? (*Hint*: Find the maximum of f_k on $[0,1]$ using Calc I methods.)

5. (15 points) (**Topology of a function space**) Define $A = \big\{ f \in C([0,1],\mathbb{R}) \,\big|\, |f(x)| \le x^2+1,\ \forall x \in [0,1] \big\}$.
 (a) (5 points) Prove A is closed in $C([0,1])$.
 (b) Prove that $\mathrm{int}(A) = \big\{ f \in C([0,1],\mathbb{R}) \,\big|\, |f(x)| < x^2+1,\ \forall x \in [0,1] \big\}$.
 (c) Write A^c in set notation, $A^c = \big\{ f \in C([0,1],\mathbb{R}) \,\big|\ ????\ \big\}$.
 (d) Find $\mathrm{bd}(A)$ and prove your result. See Definition 1 for the definition of $\mathrm{bd}(A)$.
 (e) Find $\mathrm{ext}(A)$ and prove your result. See Definition 1 for the definition of $\mathrm{ext}(A)$.

6. (5 points) (**Cauchy sequence in the contraction mapping principle**) §11.2, p. 328, #12.
Verify that the inequality (12.1),
$$d(p_m, p_k) \le \frac{c^k}{1-c} d\big(T(p_0), p_0\big) \quad \text{if } m > k,$$
implies that the sequence $\{p_k\}$ is Cauchy.

7. (10 points) (**Contraction mapping principle**) §11.2, p. 327, #1.
Show that none of the following mappings $f\colon X \to X$ have a fixed point and explain why the contraction mapping principle is not contradicted.
 (a) $X = (0,1) \subset \mathbb{R}$ and $f(x) = x/2$ for $x \in X$.
 (b) $X = \mathbb{R}$ and $f(x) = x+1$ for $x \in X$.
 (c) $X = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ and $f(x,y) = (-y,x)$ for $(x,y) \in X$.

8. (10 points) (**Cauchy sequence in a function space**) §11.2, p. 328, #10.
For each natural number k, define $f_k(x) = e^{x/k}$ for $0 \le x \le 1$. Is the sequence $\{f_k\colon [0,1] \to \mathbb{R}\}$ Cauchy in the metric space $C\big([0,1],\mathbb{R}\big)$? (*Hint*: Uniformly convergent $\Rightarrow$ uniformly Cauchy. Use the comparison test to prove uniform convergence.)

9. (10 points) (**Lipschitz mapping on a bounded set**) Let X be a subset of $\mathbb{R}^n$ and suppose that the mapping $T\colon X \to \mathbb{R}^m$ is Lipschitz. Prove that $T(X)$ is bounded if X is bounded. Is this result still true if the mapping T is only assumed to be continuous on X? (*Note*: X is bounded if there is an $x_0 \in X$ and $M > 0$ such that $d(x,x_0) \le M$ for all $x \in X$.)

(End of Problem Set 11)