Math 125 **Problem Set 4** **Due: Thursday 11/10 at 11:59 p.m. EST**

Instruction: Read the assignment policy. For problem 3, include a printout your code with your homework submission. You should submit your assignment on Gradescope.

1. Let $C[0,1]$ be the vector space of real-valued functions that are continuous on $[0,1]$. Prove that the following definition gives a valid inner product

$$\langle f\,, g\rangle = \int_0^1 f(x)\, g(x)\, dx,$$

where f and g are elements of $C[0,1]$.

Remark: An inner product $\langle\cdot,\cdot\rangle$ satisfies the following four properties. Let f, g, and h be vectors and c be a scalar.

1. $\langle f+g,h\rangle = \langle f,h\rangle + \langle g,h\rangle$.
2. $\langle cg,h\rangle = c\langle g,h\rangle$.
3. $\langle f,g\rangle = \langle g,f\rangle$.
4. $\langle f,f\rangle \geq 0$ and equal to 0 if and only if $f = 0$.

2. Let W be a subspace of functions on the interval $[0,1]$ with basis $\{f_1, f_2, f_3\}$. We define an inner product on this space as follows

$$\langle f\,, g\rangle = \int_0^1 f(x)\, g(x)\, dx,$$

where f and g are elements of W.

(a) Define the following matrix $\mathbf{G} = \begin{bmatrix} \langle f_1\,, f_1\rangle & \langle f_1\,, f_2\rangle & \langle f_1\,, f_3\rangle \\ \langle f_2\,, f_1\rangle & \langle f_2\,, f_2\rangle & \langle f_2\,, f_3\rangle \\ \langle f_3\,, f_1\rangle & \langle f_3\,, f_2\rangle & \langle f_3\,, f_3\rangle \end{bmatrix}$. Prove that G is symmetric and positive definite. [Hint: Show that $\mathbf{z}^T\mathbf{G}\mathbf{z} > 0$ for any nonzero vector $\mathbf{z}$].

(b) Any function g that is in W can be written as a linear combination of f_1, f_2 and f_3 i.e. $g = c_1 f_1 + c_2 f_2 + c_3 f_3$. Let $\mathbf{c}$ denote the vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Prove that $\mathbf{c}$ is a solution to the linear system $\mathbf{G}\mathbf{c} = \mathbf{y}$ where $\mathbf{y} = \begin{bmatrix} \langle g\,, f_1\rangle \\ \langle g\,, f_2\rangle \\ \langle g\,, f_3\rangle \end{bmatrix}$. Is there a unique solution?

(c) Let h be a function that is not in W. From the best approximation theorem, we know that the closest element in W to h is the orthogonal projection of h onto W denoted by $\hat{h}$. Find an expression for $\hat{h}$. [Hint: Use orthogonality relations and the results in (b)].

Remark: The set $\{f_1, f_2, f_3\}$ is a basis but not necessarily **orthogonal**. Note that $\mathbf{z}^T\mathbf{G}\mathbf{z}$ can be written as

$$\mathbf{z}^T\mathbf{G}\mathbf{z} = \sum_{i=1}^{3}\sum_{j=1}^{3} G_{i,j} z_i z_j$$

3. Download the dataset `least_square_data.mat` from HW3 folder. If you use Python for programming, download `least_square_data.csv`.

(a) Find the best linear squares fit to the data. Plot the linear fit.

(b) Find the best quadratic squares fit to the data. Plot the quadratic fit.

[Remark: Note that the best least squares fit is determined from the normal equations $(\mathbf{A}^T\mathbf{A})\mathbf{c} = \mathbf{A}^T\mathbf{y}$. For the above problems, it suffices to define $\mathbf{A}$ and set up a linear system. You can use any numerical solver to find the coefficients $\mathbf{c}$ that obtain the least squares fit].

4. We study the least problem for the case where all observations might not be equally reliable. The standard least square formulation assumes that each data point is equally reliable as the other. In certain applications, due to the nature of acquisition of data, this assumption is violated. In this setting, the observations can be assigned different weights. A larger weight indicate that a data point in consideration is more reliable. To integrate the weight in the least square formulation, define the following inner product: $\langle \boldsymbol{x}\,, \boldsymbol{y}\rangle = \boldsymbol{x}^T\boldsymbol{W}^T\boldsymbol{W}\boldsymbol{y}$ where $\boldsymbol{W}$ is a non-singular matrix of weights.

(a) Following the derivation of the normal equations for the least squares problem , show that the weighted least square solution is equivalent to solving the the weighted normal equation: $\boldsymbol{A}^T\boldsymbol{W}^T\boldsymbol{W}\boldsymbol{A}\mathbf{x} = \boldsymbol{A}^T\boldsymbol{W}^T\boldsymbol{W}\boldsymbol{b}$?

(b) Find the weighted least-squares solution to $\mathbf{A}\mathbf{x} = \boldsymbol{b}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{W} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Plot the data and the least-squares fit on a graph.

(c) **Extra credit**: Why is it necessary to require invertibility of $\mathbf{W}$? [**Hint**: Check when the inner product is well-defined].

5. In this problem, we study the approximation of a function using Chebyshev polynomials. Recall the definition of a Chebyshev polynomial $T_n(x) = \cos(n \arccos(x))$ where $x \in [-1, 1]$. In lecture 8, we noted that $T_0(x) = 1, T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

(a) Prove that the set $\{T_0(x), T_1(x), T_2(x), ..., T_n(x)\}$ is orthogonal with the following inner product

$$\langle T_i(x)\,, T_j(x)\rangle = \int_{-1}^{1} \left(T_i(x)T_j(x)\frac{1}{\sqrt{1-x^2}} \right) dx.$$

(b) **Extra credit**: Find the least-squares approximation of the function $f(x) = \sin(x)$ on the interval $[-1, 1]$ using the Chebyshev basis $\{T_0(x), T_1(x), T_2(x)\}$. Plot the approximation and the function $f(x) = \sin(x)$ on the same plot in the interval $[-1, 1]$. Define the error of the approximation to be $E = \max\limits_{-1\leq x\leq 1} |f(x) - \hat{f}(x)|$ where $\hat{f}$ denotes the least squares approximation. What is E?