MATH 70 WORKSHEET 1 SOLUTIONS

Instructions: This worksheet is due on Gradescope at 11:59 p.m. Eastern Time on Sunday, September 20. You are encouraged to work with others, but the final results must be your own.[1]

Please give complete reasoning for all worksheet answers.

1. (6 points) Let A be an 4×6 matrix and assume that for some $\mathbf{b} \in \mathbb{R}^4$, $A\mathbf{x} = \mathbf{b}$ has no solution. Explain why A does not have a pivot in every row.

   Solution: Ideal answer would mention the fact that having a pivot in each row means the augmented matrix $[A|\mathbf{b}]$ can be reduced to a form where we can always pick $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ via the row-reduction process. Arguments like "if $A\mathbf{x} = \mathbf{b}$ has no solutions, then the echelon form of $[A|\mathbf{b}]$ has a row like $[0...b]$ which is inconsistent" should cite a theorem in the textbook or explain why such a row must exist.

2. (14 points) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ be vectors in $\mathbb{R}^4$.

   (a) Use the definitions of vector addition and scalar multiplication on $\mathbb{R}^4$ (defined entry-by-entry, see p. 29 of the book) to show that

   $$2(\mathbf{x} + \mathbf{y}) = 2\mathbf{y} + 2\mathbf{x}$$

   (note the change in order of $\mathbf{x}$ and $\mathbf{y}$ on the right-hand side). At each step explain the properties of vector addition and the rules of arithmetic that you are using.

Solution:

$$2(\mathbf{x}+\mathbf{y}) \overset{\text{(definition of } \mathbf{x}, \mathbf{y}\text{)}}{=} 2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}\right) \overset{\text{(definition of vector addition)}}{=} 2\left(\begin{bmatrix} x_1+y_1 \\ x_1+y_2 \\ x_3+y_3 \\ x_4+y_4 \end{bmatrix}\right) \overset{\text{(definition of scalar mult.)}}{=} \begin{bmatrix} 2(x_1+y_1) \\ 2(x_1+y_2) \\ 2(x_3+y_3) \\ 2(x_4+y_4) \end{bmatrix} \overset{\text{(distributive prop)}}{=} \begin{bmatrix} 2x_1+2y_1 \\ 2x_1+2y_2 \\ 2x_3+2y_3 \\ 2x_4+2y_4 \end{bmatrix}$$

$$\underset{\text{(commutativity of addition)}}{=} \begin{bmatrix} 2y_1+2x_1 \\ 2y_1+2x_2 \\ 2y_3+2x_3 \\ 2y_4+2x_4 \end{bmatrix} \underset{\text{(definition of vector addition)}}{=} \begin{bmatrix} 2y_1 \\ 2y_1 \\ 2y_3 \\ 2y_4 \end{bmatrix} + \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \\ 2x_4 \end{bmatrix} \underset{\text{(definition of scalar mult.)}}{=} 2\begin{bmatrix} y_1 \\ y_1 \\ y_3 \\ y_4 \end{bmatrix} + 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \underset{\text{(definition of } \mathbf{x},\mathbf{y}\text{)}}{=} 2\mathbf{y} + 2\mathbf{x}$$

(b) Use the definitions of vector addition and scalar multiplication on $\mathbb{R}^4$ (defined entry-by-entry, see p. 29 of the book) to show that

$$\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}.$$

At each step explain the properties of vector addition and the rules of arithmetic that you are using.

Solution:

$$\mathbf{x} + (-1)\mathbf{x} \overset{\text{(definition of } \mathbf{x}\text{)}}{=} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + (-1)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \underset{\text{(definition of scalar mult.)}}{=} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \\ -x_4 \end{bmatrix} \overset{\text{(definition of vector addition)}}{=} \begin{bmatrix} x_1-x_1 \\ x_2-x_2 \\ x_3-x_3 \\ x_4-x_4 \end{bmatrix} \underset{\text{(additive inverses)}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

3. **This problem is only required for section/s 1.** Use the three points $(-1, 12), (2, 18), (3, 0)$ to answer the following.

(a) The the equation of a line looks like $x_1 x + x_2 = b$ where x_1, x_2 and b are constants. Use the three given points to fill in x and b in the equation of a line to get a system of three linear equations (in 2 variables x_1 and x_2).

Solution: To find the equation of the line through the three points we need to find an x_1 and x_2 so that

$$\begin{aligned} x_1(-1) + x_2 &= 12 \\ x_1(2) + x_2 &= 18 \\ x_1(2) + x_2 &= 0 \end{aligned}$$

(b) Set up the matrix used to solve this system, and solve it using the matrix. Give an equation of a line through the three points or explain how you can tell by looking at the matrix that it doesn't have a solution. If there are many solutions explain how you can tell this from the matrix.

Solution: This is a system of equations which we can solve by row reducing the augmented matrix

$$\begin{bmatrix} -1 & 1 & 12 \\ 2 & 1 & 18 \\ 3 & 1 & 0 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The last row of the matrix requires that $0x_2 = 1$ which cant be, so there is no solution.

(c) The the equation of a quadratic looks like $x_1x^2 + x_2x + x_3 = b$ where x_1, x_2, x_3 and b are constants. Use the three given points to fill in x and b in the quadratic equation to get a system of three linear equations (in 3 variables x_1, x_2 and x_3).

Solution: To find the equation of the quadratic through the three points we need to find an x_1, x_2 and x_2 so that

$$\begin{aligned} x_1(-1)^2 + x_2(-1) + x_3 &= 12 \\ x_1 2^2 + x_2 2 + x_3 &= 18 \\ x_1 3^2 + x_2 3 + x_3 &= 0 \end{aligned}$$

(d) Set up the matrix used to solve this system and solve it using the matrix. Give the equation of a quadratic through the three points or explain how you can tell by looking at the matrix that it doesn't have a solution. If there are many solutions explain how you can tell this from the matrix.

Solution: This is a system of equations which we can solve by row reducing the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 12 \\ 4 & 2 & 1 & 18 \\ 9 & 3 & 1 & 0 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 24 \end{bmatrix}$$

This means that $x_1 = -5, x_2 = 7$ and $x_3 = 24$ so the quadratic solution is $-5x^2 + 7x + 24 = p(x)$.

(e) The the equation of a cubic looks like $x_1x^3 + x_2x^2 + x_3x + x_4 = b$ where x_1, x_2, x_3, x_4 and b are constants. Use the three given points to fill in x and b in the cubic equation to get a system of three linear equations (in 4 variables x_1, x_2, x_3 and x_4).

Solution: To find the equation of the quadratic through the three points we need to find an x_1, x_2 and x_2 so that

$$\begin{aligned} x_1(-1)^3 + x_2(-1)^2 + x_3(-1) + x_4 &= 12 \\ x_1(2)^3 + x_2 2^2 + x_3 2 + x_4 &= 18 \\ x_1(3)^3 + x_2 3^2 + x_3 3 + x_3 &= 0 \end{aligned}$$

(f) Set up the matrix used to solve this system and solve it using the matrix. Give the equation of a cubic through the three points or explain how you can tell by looking at the matrix that it doesn't have a solution. If there are many solutions explain how you can tell this from the matrix.

Solution: This is a system of equations which we can solve by row reducing the augmented matrix

$$\begin{bmatrix} -1 & 1 & -1 & 1 & 12 \\ 8 & 4 & 2 & 1 & 18 \\ 27 & 9 & 3 & 1 & 0 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{6} & -5 \\ 0 & 1 & 0 & \frac{2}{3} & 7 \\ 0 & 0 & 1 & -\frac{1}{6} & 24 \end{bmatrix}$$

Since there is a free variable (or a column with no pivot) there are many (infinte) solutions. The matrix gives us, x_4 is free, $x_3 = \frac{1}{6}x_4 + 24$, $x_2 = -\frac{2}{3}x_4 + 7$ and $x_1 = \frac{1}{6}x_4 - 5$. Choosing $x_4 = 6$ we get one solution $6x^3 + 25x^2 + 3x - 4 = p(x)$.