MATH 42 HOMEWORK 6

This homework is due at 11:59 p.m. (Eastern Time) on Wednesday, October 28. Scan the completed homework and upload it **as one pdf file** to Gradescope. The Canvas module "Written Assignments" has instructions for how to upload the assignment to Gradescope. This assignment covers §16.2−16.4.

Be sure to show work (integration by parts, substitutions, etc.) when calculating integrals. Unless stated in the problem, it is insufficient to simply respond with a numerical evaluation of definite integrals or an antiderivative of a non-standard integrand.

(1) In engineering and other physical disciplines, an object's *center of mass* is an important quantity. Let $\sigma(x, y)$ denote the mass density per area at a given point within a region, $\mathcal{D}$. The center of mass has coordinates $(\bar{x}, \bar{y})$ defined as:

$$\bar{x}M = \int_{\mathcal{D}} x\,\sigma(x,y)\mathrm{d}A \text{ and } \bar{y}M = \int_{\mathcal{D}} y\ \sigma(x,y)\mathrm{d}A,$$

where $\sigma(x, y)$ is the mass density per area and $M = \int_{\mathcal{D}} \sigma(x, y)\mathrm{d}A$ is the total mass of the object. Calculate M, $\bar{x}M$, and $\bar{y}M$ for the following objects given by a region $\mathcal{D}$ and a mass density per area $\sigma(x, y)$:

(a) $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, x^5 \le y \le \sqrt{x}\}$ and $\sigma(x, y) = xy$.

(b) $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 1 \le x \le 2, \sqrt{x} \le y \le x + 2\}$ and $\sigma(x, y) = x/y$.

Solution: For the first part: the mass is given by

$$\begin{aligned} M &= \int_{x=0}^{1}\int_{y=x^5}^{\sqrt{x}} xy\mathrm{d}y\mathrm{d}x \\ &= \int_{x=0}^{1} \frac{x}{2}y^2\Big|_{y=x^5}^{\sqrt{x}}\mathrm{d}x \\ &= 2^{-1}\int_{x=0}^{1}(x^2 - x^{11})\mathrm{d}x \\ &= 2^{-1}\left(\frac{1}{3} - \frac{1}{12}\right) = \frac{1}{8} = 0.125, \end{aligned}$$

the mass-scaled $\bar{x}$ is

$$\begin{aligned} \bar{x}M &= \int_{x=0}^{1}\int_{y=x^5}^{\sqrt{x}} x^2y\mathrm{d}y\mathrm{d}x \\ &= \int_{x=0}^{1} \frac{x^2}{2}y^2\Big|_{y=x^5}^{\sqrt{x}}\mathrm{d}x \\ &= 2^{-1}\int_{x=0}^{1}(x^3 - x^{12})\mathrm{d}x \\ &= 2^{-1}\left(\frac{1}{4} - \frac{1}{13}\right) = \frac{9}{104} \approx 0.087, \end{aligned}$$

and the mass-scaled $\bar{y}$ is

$$\begin{aligned}
\bar{y}M &= \int_{x=0}^{1}\int_{y=x^5}^{\sqrt{x}} xy^2 \mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^{1} \frac{x}{3}y^3\Big|_{y=x^5}^{\sqrt{x}} \mathrm{d}x \\
&= 3^{-1}\int_{x=0}^{1} (x^{5/2} - x^{16})\mathrm{d}x \\
&= 3^{-1}\left(\frac{2}{7} - \frac{1}{17}\right) = \frac{9}{119} \approx 0.076.
\end{aligned}$$

For the second part: recall that $\int x^n \log x \mathrm{d}x$ can easily be solved using integration by parts:

$$\int x^n \log x \mathrm{d}x = \frac{x^{n+1}\log x}{n+1} - \int \frac{x^{n+1}}{x(n+1)}\mathrm{d}x + C = \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\log x - 1\right] + C,$$

and that integrals of the form $\int x^n \log(x+a)\mathrm{d}x = \int (u-a)^n \log u \mathrm{d}u$ using the substitution $u = x + a$.

The mass is given by

$$\begin{aligned}
M &= \int_{x=1}^{2}\int_{y=\sqrt{x}}^{x+2} \frac{x}{y}\mathrm{d}y\mathrm{d}x \\
&= \int_{x=1}^{2} x\log y\Big|_{y=\sqrt{x}}^{x+2} \mathrm{d}x \\
&= \int_{x=1}^{2} x\log(x+2)\mathrm{d}x - \frac{1}{2}\int_{x=1}^{2} x\log x\mathrm{d}x \\
&= \frac{5}{8} + \log\left(\frac{3\sqrt{3}}{2}\right) \approx 1.580,
\end{aligned}$$

the mass-scaled $\bar{x}$ is

$$\begin{aligned}
\bar{x}M &= \int_{x=1}^{2}\int_{y=\sqrt{x}}^{x+2} \frac{x^2}{y}\mathrm{d}y\mathrm{d}x \\
&= \int_{x=1}^{2} x^2\log y\Big|_{y=\sqrt{x}}^{x+2} \mathrm{d}x \\
&= \int_{x=1}^{2} x^2\log(x+2)\mathrm{d}x - \frac{1}{2}\int_{x=1}^{2} x^2\log x\mathrm{d}x \\
&= \frac{28\log 2}{3} - \frac{13}{18} - 3\log 3 \approx 2.451,
\end{aligned}$$

and the mass-scaled $\bar{y}$ is

$$\begin{aligned}
\bar{y}M &= \int_{x=1}^{2} \int_{y=\sqrt{x}}^{x+2} x \mathrm{d}y \mathrm{d}x \\
&= \int_{x=1}^{2} xy\Big|_{y=\sqrt{x}}^{x+2} \mathrm{d}x \\
&= \int_{x=1}^{2} x^2 + 2x - x^{3/2} \mathrm{d}x \\
&= \frac{86}{15} - \frac{8\sqrt{2}}{5} \approx 3.471,
\end{aligned}$$

(2) Consider the region of space $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : y \geq 0, \sqrt{x^2 + y^2} \leq 2\}$.
   (a) Write this region using polar coordinates, r and θ.
   (b) Endow this region with mass density $\sigma(r, \theta) = \sqrt{r} \sin\theta$. Integrate using polar coordinates to find M, $\bar{x}M$, and $\bar{y}M$
   (c) Provide an intuitive reason justifying your answer for $\bar{x}$.

Solution: The region $\{(r, \theta) \in \mathbb{R}^2 : r \leq 2, \theta \in [0, \pi]\}$ is semi-circular above the abscissa and has a reflection symmetry across the ordinate.

$$\begin{aligned}
M &= \int_{\mathcal{S}} \sigma(r, \theta) \mathrm{d}A \\
&= \int_0^{\pi} \int_0^2 r^{1/2} \sin\theta r \mathrm{d}r \mathrm{d}\theta \\
&= -(\cos\pi - \cos 0) \frac{2}{5} r^{5/2}\Big|_{r=0}^{r=2} \\
&= \frac{2^2}{5} 2^{5/2} = \frac{2^4\sqrt{2}}{5}.
\end{aligned}$$

$$\begin{aligned}
\bar{x}M &= \int_{\mathcal{S}} r\cos\theta \sigma(r, \theta) \mathrm{d}A \\
&= \int_0^{\pi} \int_0^2 r^{3/2} \cos\theta \sin\theta r \mathrm{d}r \mathrm{d}\theta \\
&= \int_0^{\pi} \cos\theta \sin\theta \mathrm{d}\theta \int_0^2 r^{5/2} \mathrm{d}r \\
&= 0,
\end{aligned}$$

where we recognize that the angular integral can be transformed via a simple substitution, $\alpha = \theta - \pi/2$ to the integral of an odd function over an origin-centered interval

$$\int_0^{\pi} \cos\theta \sin\theta \mathrm{d}\theta = \int_{\alpha=-\pi/2}^{\pi/2} \cos(\alpha + \pi/2) \sin(\alpha + \pi/2) \mathrm{d}\alpha = -\int_{\alpha=-\pi/2}^{\pi/2} \sin(\alpha) \cos(\alpha) \mathrm{d}\alpha = 0.$$

$$
\begin{aligned}
\bar{y}M &= \int_{\mathcal{S}} r\sin\theta\sigma(r,\theta)\mathrm{d}A \\
&= \int_0^{\pi}\int_0^2 r^{5/2}\sin^2\theta\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{\pi}\sin^2\theta\mathrm{d}\theta\int_0^2 r^{5/2}\mathrm{d}r \\
&= \frac{2}{7}2^{7/2}\int_0^{\pi}\frac{1}{2}-\frac{\cos 2\theta}{2}\mathrm{d}\theta \\
&= \frac{2^4\sqrt{2}}{7}\left(\frac{\pi}{2}-\frac{\sin\alpha}{4}\Bigg|_{\alpha=0}^{2\pi}\right) = \frac{8\sqrt{2}\pi}{7},
\end{aligned}
$$

where the substitution $\alpha = 2\theta$ was used to evaluate the final angular integral.

The result $\bar{x} = 0$ is intuitive because the region has reflection symmetry across the ordinate as does the mass density. Note that just one of those symmetries would be insufficient to establish that $\bar{x} = 0$; both the region and the mass density must exhibit the reflection symmetry.

(3) Find the volume of the space bounded between the surfaces:
  (a) The cone $z = 2-\sqrt{x^2+y^2}$ and the top portion of the hyperboloid $z = \sqrt{1+x^2+y^2}$.
  (b) The cylinder $(x-1)^2+y^2 = 1$, the plane $z = 0$, and the cone $z = \sqrt{x^2+y^2}$.

Solution: The cone lies above the top portion of the hyperboloid (see this by checking the heights at $(0,0)$). Both quadric surfaces are centered on the origin so there is rotational symmetry indicating the use of polar coordinates. First identify the radius at which the surfaces intersect by setting the heights, z, equal and solving for the radius, r

$$
2-r = \sqrt{1+r^2} \to 4-4r+r^2 = 1+r^2 \to 4r = 3 \to r = 3/4.
$$

By the rotational symmetry, the angular integral has limits $\theta \in [0,2\pi)$. Hence, the volume is

$$
\begin{aligned}
V &= \int_{\theta=0}^{2\pi}\int_{r=0}^{3/4}\left((2-r)-(\sqrt{1+r^2})\right)r\mathrm{d}r\mathrm{d}\theta \\
&= 2\pi\left(\int_{r=0}^{3/4}(2-r)r\mathrm{d}r-\int_{r=0}^{3/4}\sqrt{1+r^2}r\mathrm{d}r\right) \\
&= 2\pi\left[r^2-\frac{r^3}{3}\right]_{r=0}^{3/4}-2\pi\frac{1}{2}\int_{u=1}^{\frac{9}{16}+1}\sqrt{u}\mathrm{d}u \\
&= 2\pi\left(\frac{9}{16}-\frac{9}{4^3}\right)-\pi\frac{2}{3}u^{3/2}\Bigg|_{u=1}^{\frac{9}{16}+1} \\
&= \frac{5\pi}{24},
\end{aligned}
$$

where the substitution $u = 1+r^2$ was used.

For the second problem of the cylinder, plane, and cone, first manipulate the equation of the cylinder

$$\begin{aligned}1 &= (x-1)^2 + y^2\\ &= (r\cos\theta - 1)^2 + r^2\sin^2\theta\\ &= r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta\\ &= r^2 - 2r\cos\theta + 1,\end{aligned}$$

so $r^2 = 2r\cos\theta$ implying that $r = 2\cos\theta$, which is a circle in the plane offset from the origin (an alternate characterization of this origin-offset cylinder). Since r is a positive quantity, to generate this offset circle the angular variable must yield a positive value of cos so $\theta \in [-\pi/2, \pi/2]$. With the limits of integration identified, the volume may be calculated by integrating over $z = \sqrt{x^2+y^2} = r$, the height of the cone (minus the plane $z = 0$)

$$\begin{aligned}V &= \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{2\cos\theta} rr\mathrm{d}r\mathrm{d}\theta\\ &= \int_{\theta=-\pi/2}^{\pi/2} \frac{r^3}{3}\bigg|_{r=0}^{2\cos\theta}\mathrm{d}\theta\\ &= \frac{8}{3}\int_{\theta=-\pi/2}^{\pi/2}\cos^3\theta\mathrm{d}\theta\\ &= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]\bigg|_{\theta=-\pi/2}^{\pi/2}\\ &= \frac{32}{9},\end{aligned}$$

where we recall that

$$\begin{aligned}\int\cos^3\theta\mathrm{d}\theta &= \int\cos\theta(1-\sin^2\theta)\mathrm{d}\theta\\ &= \int\cos\theta\mathrm{d}\theta - \int\cos\theta\sin^2\theta\mathrm{d}\theta\\ &= \sin\theta - \int\sin^2\theta\mathrm{d}(\sin\theta)\\ &= \sin\theta - \frac{\sin^3\theta}{3} + C.\end{aligned}$$

(4) Use polar coordinates to evaluate the improper integral

$$\int_0^1\int_{\sqrt{1-x^2}}^{\infty}\frac{x}{(x^2+y^2)^2}\mathrm{d}y\mathrm{d}x + \int_1^{\infty}\int_0^{\infty}\frac{x}{(x^2+y^2)^2}\mathrm{d}y\mathrm{d}x.$$

Hint: Make a drawing of the region under consideration.

Solution: Since the integrand is the same in both integrals, first identify the region of integration as the positive quadrant with the origin-centered unit ball removed so $\{(r,\theta) \in \mathbb{R}^2 : r \geq 1, \theta \in [0, \pi/2]\}$. Second, rewrite the expression as a double integral in polar coordinates:

$$\begin{aligned}\int_0^{\pi/2}\int_1^{\infty}\frac{r\cos\theta}{r^4}r\mathrm{d}r\mathrm{d}\theta &= \int_0^{\pi/2}\cos\theta\mathrm{d}\theta\int_1^{\infty}r^{-2}\mathrm{d}r\\ &= (\sin\frac{\pi}{2}-\sin 0)(-1)\lim_{r\to\infty}(r^{-1}-1)\\ &= 1.\end{aligned}$$

(5) An integral that arises frequently in probability, statistics, and physics is $I = \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x$. In lecture, the value $I^2 = \pi$ was derived by writing $I = \int_{-\infty}^{\infty} e^{-y^2}\mathrm{d}y$ and then evaluating I^2 using polar coordinates. Give a brief justification for why $I \neq -\sqrt{\pi}$. *Hint:* What is one interpretation of the integral of a curve? What class of real values does the integrand, e^{-x^2}, take for all $x \in \mathbb{R}$?

Solution: Observe that e^{-x^2} is always positive and therefore the integral of an always-positive function can be thought of as the area bounded between the curve and the abscissa; hence, the integral of e^{-x^2} over the real line must be positive and $-\sqrt{\pi}$ is not a valid solution despite $I^2 = \pi$.

(6) Consider the *beta function*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x,$$

where $p > 0$ and $q > 0$. The beta function appears often in probability and statistics; it also appears occasionally in mechanics problems in physics and advanced theories of matter.[1]

(a) Use a simple substitution to show that $B(p,q) = B(q,p)$.

(b) Use the substitution $x = \sin^2\theta$ to write the beta function in its so-called trigonometric form

$$B(p,q) = 2\int_0^{\pi/2}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,\mathrm{d}\theta.$$

(c) Consider the *gamma function*

$$\Gamma(p) = \int_0^{\infty} t^{p-1}e^{-t}\mathrm{d}t = 2\int_0^{\infty} y^{2p-1}e^{-y^2}\mathrm{d}y.$$

Show that the final equality was obtained using the substitution $t = y^2$.

Perhaps surprisingly, it turns out that $\Gamma(n+1) = n! = n(n-1)(n-2)\cdots 1$ if $n \in \mathbb{N}$ (that is, if n is a natural number like $1, 2, 3, \ldots$). This implies that at a natural number, n, the gamma function is equal to the factorial function evaluated at the natural number one smaller, $n-1$. However, the gamma function is not restricted to being evaluated on the natural numbers; for the purposes of this problem, note that the gamma function is defined for all $p > 0$.

(d) Show that $\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q)$ for $p, q > 0$. *Hint:* Start from the left-hand side and try to recognize forms of $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ developed in the earlier parts of this problem. As we did for the Gaussian integral, begin the integration in Cartesian coordinates but then promptly switch to polar coordinates (but be careful because the limits of integration are different in this problem).

[1] In fact, the genesis of the string theory of matter involved beta functions.

(e) Using the expression of the beta function in terms of gamma functions

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

show $B(p,q) = B(q,p)$; this equivalency is almost immediate. Thus, we have shown in two ways that the beta function is symmetric in its arguments.

(f) Using the identities given and developed in this problem, calculate $B(2,3)$.

Solution: First, use the substitution $y = 1 - x$ to obtain

$$\begin{aligned} B(p,q) &= \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x \\ &= \int_{y=1}^0 (1-y)^{p-1}y^{q-1}(-\mathrm{d}y) \\ &= \int_{y=0}^1 y^{q-1}(1-y)^{p-1}\mathrm{d}y \\ &= B(q,p). \end{aligned}$$

Next, use the substitution $x = \sin^2\theta$ to determine the trigonometric form

$$\begin{aligned} B(p,q) &= \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x \\ &= \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{p-1}(1-\sin^2\theta)^{q-1}2\sin\theta\cos\theta\mathrm{d}\theta \\ &= 2\int_{\theta=0}^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta\mathrm{d}\theta, \end{aligned}$$

where the Pythagorean trigonometric identity was used. Pivoting to the gamma function, applying the substitution $t = y^2$

$$\begin{aligned} \Gamma(p) &= \int_0^\infty t^{p-1}e^{-t}\mathrm{d}t \\ &= \int_0^\infty (y^2)^{p-1}e^{-y^2}2y\mathrm{d}y \\ &= 2\int_0^\infty y^{2p-1}e^{-y^2}\mathrm{d}y. \end{aligned}$$

The simple rearranging of the gamma function is meant to rekindle memories of the integration technique for the Gaussian; in particular, the e^{-y^2} term looks promising.

Now turn to the main calculation of the problem:

$$\begin{aligned} \Gamma(p)\Gamma(q) &= \left(2\int_0^\infty x^{2p-1}e^{-x^2}\mathrm{d}x\right)\left(2\int_0^\infty y^{2q-1}e^{-y^2}\mathrm{d}y\right) \\ &= 4\int_{x=0}^\infty\int_{y=0}^\infty x^{2p-1}y^{2q-1}e^{-(x^2+y^2)}\mathrm{d}y\mathrm{d}x \\ &= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^\infty (r\cos\theta)^{2p-1}(r\sin\theta)^{2q-1}e^{-r^2}r\mathrm{d}r\mathrm{d}\theta \\ &= \left(2\int_{\theta=0}^{\pi/2}\cos^{2p-1}\theta\sin^{2q-1}\theta\mathrm{d}\theta\right)\left(2\int_{r=0}^\infty r^{2p-1+2q-1+1}e^{-r^2}\mathrm{d}r\right) \\ &= B(p,q)\Gamma(p+q) \end{aligned}$$

where we have made use of the forms developed in prior parts. Thus, it is valid to conclude that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(q,p)$$

because addition and multiplication are commutative.

Finally, making use of the gamma function's connection to the factorial function, $B(2,3) = \frac{1!2!}{4!} = \frac{1}{12}$.