MATH 42 HOMEWORK 6

This homework is due at 11:59 p.m. (Eastern Time) on Wednesday, October 28. Scan the completed homework and upload it **as one pdf file** to Gradescope. The Canvas module "Written Assignments" has instructions for how to upload the assignment to Gradescope. This assignment covers §16.2−16.4.

Be sure to show work (integration by parts, substitutions, etc.) when calculating integrals. Unless stated in the problem, it is insufficient to simply respond with a numerical evaluation of definite integrals or an antiderivative of a non-standard integrand.

(1) In engineering and other physical disciplines, an object's *center of mass* is an important quantity. Let $\sigma(x,y)$ denote the mass density per area at a given point within a region, $\mathcal{D}$. The center of mass has coordinates $(\bar{x},\bar{y})$ defined as:

$$\bar{x}M = \int_{\mathcal{D}} x\,\sigma(x,y)\mathrm{d}A \text{ and } \bar{y}M = \int_{\mathcal{D}} y\ \sigma(x,y)\mathrm{d}A,$$

where $\sigma(x,y)$ is the mass density per area and $M = \int_{\mathcal{D}} \sigma(x,y)\mathrm{d}A$ is the total mass of the object. Calculate M, $\bar{x}M$, and $\bar{y}M$ for the following objects given by a region $\mathcal{D}$ and a mass density per area $\sigma(x,y)$:
  (a) $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, x^5 \le y \le \sqrt{x}\}$ and $\sigma(x,y) = xy$.
  (b) $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 2, \sqrt{x} \le y \le x+2\}$ and $\sigma(x,y) = x/y$.

(2) Consider the region of space $\mathcal{S} = \{(x,y) \in \mathbb{R}^2 : y \ge 0, \sqrt{x^2+y^2} \le 2\}$.
  (a) Write this region using polar coordinates, r and θ.
  (b) Endow this region with mass density $\sigma(r,\theta) = \sqrt{r}\sin\theta$. Integrate using polar coordinates to find M, $\bar{x}M$, and $\bar{y}M$
  (c) Provide an intuitive reason justifying your answer for $\bar{x}$.

(3) Find the volume of the space bounded between the surfaces:
  (a) The cone $z = 2 - \sqrt{x^2+y^2}$ and the top portion of the hyperboloid $z = \sqrt{1+x^2+y^2}$.
  (b) The cylinder $(x-1)^2 + y^2 = 1$, the plane $z = 0$, and the cone $z = \sqrt{x^2+y^2}$.

(4) Use polar coordinates to evaluate the improper integral

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\infty} \frac{x}{(x^2+y^2)^2}\mathrm{d}y\mathrm{d}x + \int_1^{\infty}\int_0^{\infty} \frac{x}{(x^2+y^2)^2}\mathrm{d}y\mathrm{d}x.$$

Hint: Make a drawing of the region under consideration.

(5) An integral that arises frequently in probability, statistics, and physics is $I = \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x$. In lecture, the value $I^2 = \pi$ was derived by writing $I = \int_{-\infty}^{\infty} e^{-y^2}\mathrm{d}y$ and then evaluating I^2 using polar coordinates. Give a brief justification for why $I \ne -\sqrt{\pi}$. *Hint:* What is one interpretation of the integral of a curve? What class of real values does the integrand, e^{-x^2}, take for all $x \in \mathbb{R}$?

(6) Consider the *beta function*

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x,$$

where $p>0$ and $q>0$. The beta function appears often in probability and statistics; it also appears occasionally in mechanics problems in physics and advanced theories of matter.[1]

(a) Use a simple substitution to show that $B(p,q)=B(q,p)$.

(b) Use the substitution $x=\sin^2\theta$ to write the beta function in its so-called trigonometric form

$$B(p,q)=2\int_0^{\pi/2}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,\mathrm{d}\theta.$$

(c) Consider the *gamma function*

$$\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}\mathrm{d}t=2\int_0^\infty y^{2p-1}e^{-y^2}\mathrm{d}y.$$

Show that the final equality was obtained using the substitution $t=y^2$.

Perhaps surprisingly, it turns out that $\Gamma(n+1)=n!=n(n-1)(n-2)\cdots1$ if $n\in\mathbb{N}$ (that is, if n is a natural number like $1,2,3,\ldots$). This implies that at a natural number, n, the gamma function is equal to the factorial function evaluated at the natural number one smaller, $n-1$. However, the gamma function is not restricted to being evaluated on the natural numbers; for the purposes of this problem, note that the gamma function is defined for all $p>0$.

(d) Show that $\Gamma(p)\Gamma(q)=\Gamma(p+q)B(p,q)$ for $p,q>0$. *Hint:* Start from the left-hand side and try to recognize forms of $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ developed in the earlier parts of this problem. As we did for the Gaussian integral, begin the integration in Cartesian coordinates but then promptly switch to polar coordinates (but be careful because the limits of integration are different in this problem).

(e) Using the expression of the beta function in terms of gamma functions

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

show $B(p,q)=B(q,p)$; this equivalency is almost immediate. Thus, we have shown in two ways that the beta function is symmetric in its arguments.

(f) Using the identities given and developed in this problem, calculate $B(2,3)$.

[1] In fact, the genesis of the string theory of matter involved beta functions.