Math 135 HW10

1) Case 1: $u, v \in A$. As A is path-connected there is a path from $u \to v$
Case 2. $u, v \in B$. As B is path-connected, there is a path from $u \to v$
Case 3: WLOG $u \in A$, $v \in B$. As $A \cap B \neq \emptyset$ an $\exists c \in A \cap B$. As $c \in A$, there is a path from $u \to c$ as A is path connected. As $c \in B$ and B is path connected, there is a path from $c \to v$.
Therefore there is a path: $u \to c \to v$ so $A \cup B$ is path-connected.

2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

$A = \{(x,y) \in \mathbb{R}^2 \mid y = b\sqrt{1-\frac{x^2}{a^2}}\}$
$B = \{(x,y) \in \mathbb{R}^2 \mid y = -b\sqrt{1-x^2/a^2}\}$

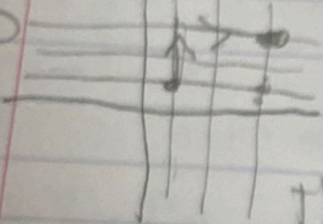
A and B are path connected as A and B are graphs on $[-a, a]$. and A and B are continuous ~~and path connected~~ and the $A \cap B = \{(-a, 0), (a, 0)\} \neq \emptyset$ so by 1) an ellipse is path connected $\square$

3 a) Let $u, v \in \mathbb{Q}$ and $u \neq v$. If a continuous path $u \to v$ exists, then by denseness of irrationals in $\mathbb{R}$ there must $\exists p \in \mathbb{Q}^c$, on the path, which on $\mathbb{Q}$ is impossible. If $u = v$ then the path maps to itself, so every rational number is it's own path connected subset of $\mathbb{Q}$.

b) $f(A)$ is path connected, but by a) then $f(A)$ has only one value, so the image is some constant value.

4 a) Let $f(x,y,z)=y$ be defined on $f:A\to\mathbb{R}$. As A is connected it has the intermediate value property. So as $f(0,0,1)=0$ and $f(4,3,0)=3$ by IVP, as $0<2<3$, $\exists (x,y,z)\in A$ s.t $f(x,y,z)=2$. So $y=2$.

b) let $\vec{w}\in\mathbb{R}^3$ and $f(\vec{w})=\|w\|$ be defind on $f:A\to\mathbb{R}^3$. As A is connected it has IVP $f((0,0,1))=1$, $f((4,3,0))=5$ as $1<4<5$ $\exists\, \vec{w}\in A$ s.t $f(\vec{w})=4$.

5 

Let $u=(u_1,u_2)$ have rational u_1 and $v=(v_1,v_2)$ have rational v_2. WLOG our assumptions. The line in $\mathbb{R}^2$, $x=u_1$ has (no holes), as u_1 is rational, so $\exists$ some z on $x=u_1$ w/ $z=(u_1,v_2)$. As v_2 is rational we can get to any x value on the line $y=v_2$ from (u_1,v_2) so there is a path $(u_1,v_2)\to(v_1,v_2)$. This can all be generalized to whatever you pick is rational. As C $u\to z\to v$, S is path connected.

6 a) $F:B\to\mathbb{R}^m$ is continuous at $b\in B$ if $u_k\to b$ then $F(u_k)\to F(b)$. As $B\subset A$ $u_k\in A$ and $b\in A$. As A is continuous, if $u_k\to b$ then $F(u_k)\to F(b)$ by definition of continuity. So, this holds and B is also continuous.

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6 b) ~~f is~~ is continuous at z_0 if $z_k \to z_0$ then $F(z_k) \to F(z_0)$
We can look at continuity on U and V.
$f: U \cup V \to \mathbb{R}$ is continuous at $u \in U$ if $u_k \to u$ then $F(u_k) \to F(u)$.
If $u_k \to u$ then $u_k \in U$, $F(u_k) = 0$ as for $x \in U$, $f(x) = 0$, and $F(u) = 0$ so $F(u_k) \to 0$ so f continuous for $x \in U$
$f: U \cup V \to \mathbb{R}$ is continuous at $v \in V$ if $v_k \to v$ then $F(v_k) \to F(v)$. By definition, $v_k \in V$, if $v_k \to v$. as $F(v) = 1$, and $F(v_k) = 1$ for $\forall k$, so continuous at $v \in V$.
So, f is continuous on $U \cup V$ as it is continuous on U and continuous on V

c) ~~The function~~ ~~in~~ part b is an example of this, as f is continuous but $f(A) = \{0, 1\}$

d) No, set is connected iff it has IVP. So, no IVP $\longleftrightarrow$ disconnected, so no disconnected set has the IVP.

7 a) Let $u, v \in A \cap B$. By def, $u, v \in A \cap B \subset A$ so $u, v \in A$. As A is convex, for any $a_1, a_2 \in A$ there is a unique line segment from $a_1 \to a_2$, so there is a segment $u \to v$ and $A \cap B$ is convex. Can be repeated w/ B.

b) Let $u, v \in A \cap B$. By def $u, v \in A \cap B \subset A$ so $u, v \in A$ As A is pathwise-connected, there is a path from $u \to v$, ~~so~~ $A \cap B$ is pathwise connected. Can do same w/ B.

c) Convex implies unique line segment, while pathwise connected this isn't the case.

8 To prove if path connected, $\exists$ a path $\gamma:[0,1]\to A$
joining $w=(0,1)$ and $v=(1,\sin 1)$
Let $t^*=\sup\{t\in[0,1]\}$ s.t. $\gamma([0,t])\subset K$
So, $\gamma(t)\in K$ for $t<t^*$
If this is true, then $\gamma(t)=(\gamma_1(t),\gamma_2(t))\Rightarrow(0,\gamma_2(t))\ \forall t\le t^*$
By definition of continuity $\gamma(t^*)\overset{?}{=}0$
Consider the sequence $t_n=t^*+\frac{1}{n}$, then
$\gamma([0,t^*+\frac{1}{n}])\not\subset K$.
$\exists t_n\in[t^*,t^*+\frac{1}{n}]$ s.t. $\gamma(t_n)\notin K$ and $\gamma(t_n)\in A$.
So $\gamma(t_n)\in G=\{(x,\sin\frac{1}{x})\mid x\in(0,1]\}$
By Sandwich theorem $t_n\to t^*$. Using the def. of continuity, then $\gamma(t_n)\to\gamma(t^*)$
$\gamma(t_n)=(\gamma_1(t_n),\gamma_2(t_n))$ $\gamma_2(t_n)=\sin\left(\frac{1}{\gamma_1(t_n)}\right)$

So $\gamma_1(t_n)\to\gamma_1(t^*)=0$ from earlier. But as
$\gamma_2(t_n)=\sin\left(\frac{1}{\gamma_1(t_n)}\right)$ which is undefined at the limit
of $\gamma_1(t_n)\to 0$, so the path is not continuous and
$A=K\cup G$ isn't pathwise connected.