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Math 87 HW 1
Due 2/02/22

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1. A high-end gym wishes to offer a New Year's special to all customers who sign up for a yearlong membership before February 1st. The gym typically charges \$150 a month and is trying to decide how much of a discount to offer. It is estimated that for every \$100 off the yearly price, the number of gym memberships purchased will increase by 15%.

   a) How much of a discount will maximize the gyms profits on this special? Model the question as a single-variable optimization problem.

   Let P(x) represent total profit, and x is the number of \$100 given off the yearly price, and m is a constant for number of original members. Profit is members * price, so:

$$P(x) = (m + 0.15mx)(1800 - 100x) \tag{1}$$

   We can factor out m and ignore it, as it won't alter where the maximum discount is located. m is actually ignored for most other computations.

   (a) Using Pandas yields the following data table:

	Number of price decrease	Total profit
0	0	1,800.00
1	1	1,955.00
2	2	2,080.00
3	3	2,175.00
4	4	2,240.00
5	5	2,275.00
6	6	2,280.00
7	7	2,255.00
8	8	2,200.00
9	9	2,115.00
10	10	2,000.00
11	11	1,855.00
12	12	1,680.00
13	13	1,475.00
14	14	1,240.00
15	15	975.00
16	16	680.00
17	17	355.00

   We can see that profit is maximized when $x = 6$ meaning that the gym should give a 600 dollar discount to maximize profit.

(b) Compute the sensitivity of the optimal discount and the corresponding profit to the 15% assumption.

$$S(x, P(x)) = \frac{dP}{dx} * \frac{x}{P(x)} \tag{2}$$

$$P(x) = (1 + 0.15x)(1800 - 100x) = -15x^2 + 170x + 1800 \tag{3}$$

$$\frac{dP}{dx} = -30x + 170 \tag{4}$$

$$S(x, P(x)) = \frac{-30x^2 + 170x}{-15x^2 + 170x + 1800} \tag{5}$$

The number of members does not change the sensitivity because if we multiply $P(x)$ by m then $\frac{dP}{dx}$ also increases by a factor of m, and they cancel each other out in $S(x, P(x))$
Evaluating, S(6, P(6)) = -0.026, which is our sensitivity

(c) Suppose that the special only generates a 10% increase in sales per \$100. What is the effect?
We can use the equation for P(x) derived above (ignoring m), but substitute out 0.15 with 0.1, so $P(x) = (1 + 0.1x)(1800 - 100x)$

Using Pandas and Python yields the following table:

	Number of price decrease	Total profit
0	0	1,800.00
1	1	1,870.00
2	2	1,920.00
3	3	1,950.00
4	4	1,960.00
5	5	1,950.00
6	6	1,920.00
7	7	1,870.00
8	8	1,800.00
9	9	1,710.00
10	10	1,600.00
11	11	1,470.00
12	12	1,320.00
13	13	1,150.00
14	14	960.00
15	15	750.00
16	16	520.00
17	17	270.00

The max value here is \$1960, which here is at 4 price decreases, or a total discount of \$400.

(d) Under what circumstances would an offer of a special cause a reduction in profit?
The general equation with \$100 price decrease intervals is

$$P(x) = m(1 + m_i x)(1800 - 100x) \tag{6}$$

Where m_i is the percent increase in members, m is the starting number of members, and x is the number of \$100 price decreases.
We also can ignore m and will do so, as m is a constant and doesn't effect where the maximum is, only it's value.
We want to find what values of m_i lead to $P(x)$ to be decreasing for $\forall x \ \epsilon \ [0, 18]$

$$P\prime(x) = -1 + 1800m_i - 2m_i x \tag{7}$$

We set the derivative to 0, rearrange terms and end up with:

$$900 - \frac{1}{2m_i} = x \tag{8}$$

We want to see when firm would lose money, the max has to be at 0, so $x = 0$, which makes the equation: $900 = \frac{1}{2m_i}$ and solving for m_i gives an answer of $m_i = 0.000555$
The result means that for every \$100 off price, if there is an increase in customers less than 5.5%, the gym should not offer the deal, as their profit will decrease.

2. A chemist is synthesizing a compound. In the last step, she must dissolve her reagents in a solution with a particular pH level H, for 1.2 H 2.7, and heated to a temperature T (in degrees Celsius), for 66 T 98. Her goal is to maximize her percent yield as a percentage of the initial mass of the reagents. The equation determining the percentage F (H, T) is:

$$F(H, T) = -0.038T^2 - 0.0223TH - 10.982H^2 + 7.112T + 60.912H - 328.898 \tag{9}$$

a) Find the optimal temperature and pH level in the allowed range.
We can start by finding $\frac{\partial F}{\partial H}$ and $\frac{\partial F}{\partial T}$ and we get:

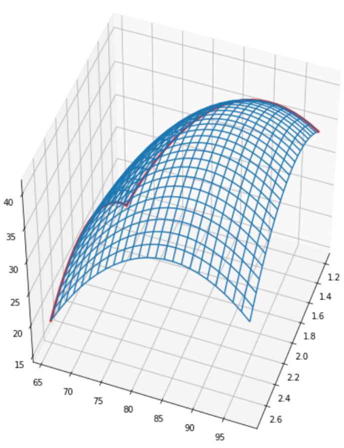
$$\frac{\partial F}{\partial H} = -0.223T - 21.964H + 60.912 \tag{10}$$

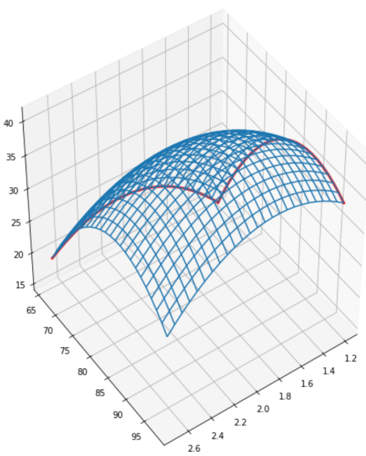
$$\frac{\partial F}{\partial T} = -0.076T - 0.223H + 7.112 \tag{11}$$

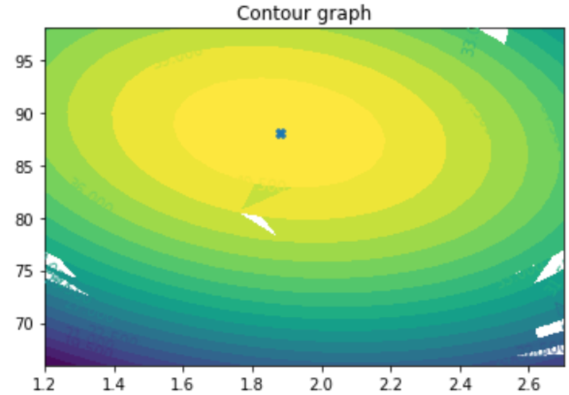
We can set both to 0, then are left with a linear system, that when solved, gives the solution $(H, T) = (1.892, 88.029)$
To verify this is a maximum, we can calculate the Determinant of the Hessian matrix of F(H, T), by taking the second partial derivatives we get the following Hessian matrix:
$\begin{bmatrix} -21.964 & -0.223 \\ -0.223 & -0.076 \end{bmatrix}$

The determinant of the above matrix is approximately 1.62 and as since $\frac{\partial^2 F}{\partial H^2} = -21.964 < 0$ we have a maximum at the critical point. The optimal pH is 1.892 and the optimal temperature is 88.029 degrees.

(b) Below, are my graphs for F(H, T) and a contour plot.







3. Human blood is generally classified in the "ABO" system, with four blood types: A, B, O, and AB. These four types reflect six gene pairs (genotypes), with blood type A corresponding to gene pairs AA and AO, blood type B corresponding to gene pairs BB and BO, blood type O corresponding to gene pair OO, and blood type AB corresponding to gene pair AB. Let p be the proportion of gene A in the population, q be the proportion of gene B in the population, and be the proportion of gene O in the population.

   a) The Hardy-Weinberg principle states that p, q, and r are fixed from generation to generation, as are the frequencies of the different genotypes. Under this assumption, what is the probability that an individual has genotype AA? BB? OO? What is the probability of an individual having two different genes?

   If we assume getting each gene is independent, and P(x) is the probability of event x happening, then $P(AA) = P(A)P(A)$, $P(BB) = P(B)P(B)$, and $P(OO) = P(O)P(O)$ Since P(A) = p, P(B) = q, and P(O) = r, then $P(AA) = p^2$, $P(BB) = q^2$, and $P(OO) = r^2$
   Since $p + q + r = 1$ then $(p + q + r)^2 = 1$

$$(p + q + r)^2 = p^2 + q^2 + r^2 + 2pq + 2pr + 2qr \tag{12}$$

   p^2, q^2, r^2 added together are the probability of someone having two identical genes, then the remaining terms, $2pq+2pr+2qr$ is the probability of someone having different genes.

(b) Find the maximum percentage of the population that can have two different genes under the Hardy-Weinberg principle in two different ways, by directly maximizing a function of only two variables and by using the method of Lagrange multipliers.

We want to maximize $f(p, q, r) = pq + 2pr + 2qr$ under the constraint $p + q + r = 1$. First, since have to use 2 variables, $r = 1 - p - q$, and our equation becomes:

$$f(p, q, r) = 2pq + 2p(1 - p - q) + 2q(1 - p - q) \tag{13}$$

$$f(p, q) = 2p - 2p^2 + 2q - 2pq - 2q^2 \tag{14}$$

Next, can take partial derivatives of $\frac{\partial f}{\partial p}$ and $\frac{\partial f}{\partial q}$ and get the system of equations:

$$\frac{\partial f}{\partial p} = 2 - 4p - 2q = 0 \tag{15}$$

$$\frac{\partial f}{\partial q} = 2 - 4q - 2p = 0 \tag{16}$$

The solution to this is (p, q) = ($\frac{1}{3}$, $\frac{1}{3}$) and then we can solve and get r = $\frac{1}{3}$, plugging this in to $f(p, q, r) = \frac{2}{3}$
which is the max percentage that could have different genes.
To solve using Lagrange Modifiers, we have the constraint $p + q + r = 1$ and want to maximize the equation $g(p, q, r) = 2pr + 2qr + 2pq$

$$L = g(p, q, r) - \lambda \cdot (1 - p - q - r) \tag{17}$$

$$L = 2pr + 2qr + 2pq - \lambda(1 - p - q - r) \tag{18}$$

Taking partial derivatives we get the following system of equations

$$\begin{aligned}\frac{\partial L}{\partial p} &= 2r + 2q + \lambda = 0\\ \frac{\partial L}{\partial q} &= 2p + 2q + \lambda = 0\\ \frac{\partial L}{\partial r} &= 2p + 2r + \lambda = 0\\ \frac{\partial L}{\partial \lambda} &= p + q + r - 1 = 0\end{aligned}$$

This system can be solved using Gaussian Elimination and get:
(p, q, r, λ) = ($\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{-4}{3}$).
As a side note, depending on how you set up your system, you get $\lambda = \pm\frac{4}{3}$ Plugging in the values of p, q, and r, into the initial function gives a max percentage that have different genes at $\frac{2}{3}$

(c) Can you say what the Lagrange multiplier represents in the above example?
λ represents by what factor the probabilities would change if another gene were to be introduced.

4. Consider the function $f(x) = e^x - xe^x$

(a) Find the roots of f

$$0 = e^x - xe^x \tag{19}$$

$$0 = e^x(1 - x) \tag{20}$$

The root of the function is $x = 1$

(b) Test Newton's method, the secant method, and bisection for finding this root. For Newton, use initial points x$_0$ = 1, 2, 10, $\frac{-1}{2}$, 5. For secant, use initial points (x_0, x_1) = (0, 2), (0, 10), (-1, 2), (-5, 5), (-10, 2). For bisection, use [xL, xR] = [0, 2], [-5, 5], [-10, 2], [-1, 2], [0, 1]. Report the iteration count and approximation to the root for each method.

N/A means it did not converge within the default max iterations, which is 50.

x_0	iters	x_*
$1/2$	7	1
2	7	1
10	17	1
$-1/2$	N/A	N/A
-5	N/A	N/A

Table 1: Newton's Method

(x_0, x_1)	iters	x_*
$(0, 2)$	13	1
$(0, 5)$	N/A	N/A
$(-1, 2)$	N/A	N/A
$(-5, 5)$	N/A	N/A
$(-10, 2)$	N/A	N/A

Table 2: Secant Method

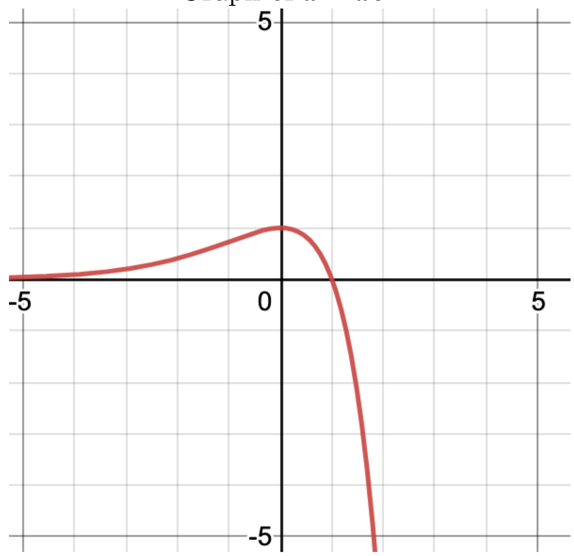
$[x_L, x_R]$	iters	x_*
$[0, 2]$	3	1
$[-5, 5]$	43	1
$[-10, 2]$	43	1
$[-1, 2]$	41	1
$[0, 1]$	32542	1

Table 3: Bisection Method

(c) How do the initial parameters impact the success in finding the root for the three methods?
Newton's method: Newton's method doesn't appear to work with negative x_0 this is explained in d) in further depth. The secant method runs into issues with most parameters, with the only one providing a correct solution being the interval (0, 2). This might be because the secant lines might tend towards $-\infty$ Though bisection is considered the least accurate, it was surprisingly able to get a solution for all. Generally, the bisection method took longer than the other two methods, but it worked for all test cases. when it worked. I am unsure why the interval [0, 1] took 32628 iterations, but that is what the function returned.

(d) Sketch or plot the function f. Use a geometric argument to explain why the negative initial parameters are not successful for some of the methods. What happens as x $\to -\infty$?

Graph of $x - xe^x$



The reason negative parameters might not be successful is that by following the graph with a negative parameter, Newton's method might tend to move towards the left, causing f(x) to get closer to zero, but x moves away from the root, and x has no root x→−∞

(e) This is an example as to why a "stopping criteria" to stop iterating at the kth step should NOT be checking $f(x_k) = 0$. How does your answer from (d) justify that? What is a better termination condition to determine convergence?
My answer from (d) justifies this idea because with implementing Newton's algorithm with some functions, $f(x_k) \not\approx 0$, meaning the stopping condition is never reached. With f(x), the the limit as x→−∞ = 0, but this is never reached. Therefore, you may get a result that is invalid. A better termination condition could be checking if $f(x_{k+1}) - f(x_k) \approx 0$, as when this condition is true, Newton's method has converged to a value.