Homework 10

Early problem due on Gradescope at 11:59 pm on Wednesday, April 12th.
Due on Gradescope at 11:59 pm on Friday, April 14th.

(1) (Early problem) The n-sphere is the subspace

$$\mathbb{S}^n := \{\vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1\}$$

of $\mathbb{R}^{n+1}$ with the subspace topology.

Prove that $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is homeomorphic to $[0,1]/\sim$ where $\sim$ is the equivalence relation where $0 \sim 1$, $1 \sim 0$, and $x \sim x$ for all $x \in [0,1]$.

(2) Let $U_1 = \{1,2\} \times (0,3)$, $U_2 = \{3,4\} \times (0,3)$. Let U_{12} be the open subset $\{1,2\} \times ((0,1) \cup (2,3))$ of U_1 and let U_{21} be the open subset $\{3,4\} \times ((0,1) \cup (2,3))$ of U_2.

(a) Find a homeomorphism $\phi_{12} : U_{12} \to U_{21}$ so that the space X obtained by gluing U_1 to U_2 with ϕ_{12} is homeomorphic to a disjoint union of two copies of $\mathbb{S}^1$. You do not need to prove that X is homeomorphic to two copies of $\mathbb{S}^1$.

(b) Find a homeomorphism $\psi_{12} : U_{12} \to U_{21}$ so that the space Y obtained by gluing U_1 to U_2 with ψ_{12} is homeomorphic to $\mathbb{S}^1$. You do not need to prove that Y is homeomorphic to $\mathbb{S}^1$.

(3) Consider the equivalence relation $\sim$ on $\mathbb{S}^1$ defined by $\vec{x} \sim \vec{y}$ if and only if $\vec{x} = \pm\vec{y}$. Find a homeomorphism of $\mathbb{S}^1/\sim$ with a familiar space and prove that your map is a homeomorphism.

(4) Let τ and τ' be two topologies on X. If τ' is finer than τ, what does connectedness of X in one topology imply about connectedness in the other?

(5) Let X be a topological space. For each integer $n \geq 1$, let A_n be a connected subspace of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup_{n \in \mathbb{Z}_{>0}} A_n$ is connected.

(6) Recall that the lower limit topology is the topology on $\mathbb{R}$ generated by the basis

$$\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}\}.$$

(See Recitation 6 or Section 13 of the text.) Denote $\mathbb{R}$ with the lower limit topology by $\mathbb{R}_\ell$. Is $\mathbb{R}_\ell$ connected or disconnected? Justify your answer.