Math 136 Exam 1 Solutions

1. (a)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(b) $$\frac{f(x) - f(a)}{x - a} = \frac{\frac{1}{x+1} - \frac{1}{a+1}}{x - a} = \frac{a - x}{(x+1)(a+1)(x-a)}$$

$$= -\frac{1}{(x+1)(a+1)}.$$

Thus, $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} -\frac{1}{(x+1)(a+1)}$$

$$= -\frac{1}{(a+1)^2}.$$

(c) $$\lim_{\vec{h} \to 0} \frac{f(\vec{x} + \vec{h}) - f(\vec{x}) - \langle \vec{\nabla} f(\vec{x}), \vec{h} \rangle}{\| \vec{h} \|} = 0,$$

i.e., $f(\vec{x}) + \langle \vec{\nabla} f(\vec{x}), \vec{h} \rangle$ is a first-order approximation to $f(\vec{x} + \vec{h})$ at $\vec{x}$.

2. (a) True

(b) False, $\partial^2 f / \partial x \partial y$ and $\partial^2 f / \partial y \partial x$ must be cont.

(c) False, § 14.1, #17, $f(x,y) = \begin{cases} \left(\sin \frac{y^2}{x}\right) \sqrt{x^2 + y^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$.

3. By the mean-value theorem, $\exists\ c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b-a) > 0$ since $f'(c) > 0$ and $b - a > 0$. Thus, $f(b) > f(a)$.

4. $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

First try $y = mx$. $$\lim_{x\to 0} \frac{x^2(mx)}{x^4 + m^2 x^2} = \lim_{x\to 0} \frac{mx}{x^2 + m^2} = 0.$$

Next try $y = x^2$. $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}.$$

Thus, the limit does not exist.

5. Consider $\vec{F}(x,y) = \begin{bmatrix} x^2+y^2 \\ x^2-y^2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$.

(a) $D\vec{F}(x,y) = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$

(b) $\det D\vec{F}(x,y) = -8xy \neq 0$ iff $x \neq 0$ and $y \neq 0$.

If $x \neq 0$ and $y \neq 0$, then (x,y) can be solved locally as a function of u, v.

(c) $\begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix}(2,0) = \left(D\vec{F}(1,1)\right)^{-1}$

$= \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{-8}\begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$

Here, $\partial x/\partial u\,(2,0) = \frac{1}{4}$.

6. (a) $\vec{F}(1,2,3) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, $D\vec{F}(1,2,3) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \end{bmatrix} = \dfrac{\partial(F_1, F_2)}{\partial(x, y, z)}$

Since $\dfrac{\partial(F_1, F_2)}{\partial(y, z)} = \det \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix} = 6 \neq 0$,

by the implicit function theorem, y, z can be solved in terms of x for x in a nbd of 1.

(b) Why is $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^2$ not injective?

To say that y, z can be solved in terms of x means that $y = y(x)$ and $z = z(x)$ and $F(x, y(x), z(x)) \equiv 0$ for x near 1.

Thus, there are infinitely many values of (x, y, z) for which $F(x, y, z) = 0$.

(c) No, because $\dfrac{\partial(F_1, F_2)}{\partial(x, z)} = \det \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} = 0$,

one cannot apply the Implicit Function Theorem.

7. Prove that f diff $\Rightarrow$ f cont at a.

Write
$$f(x) = \frac{f(x)-f(a)}{x-a}(x-a) + f(a).$$

Take limit as $x \to a$,
$$\lim_{x\to a} f(x) = f'(a) \cdot \lim_{x \to a} x-a + f(a)$$
$$= f(a).$$

Hence, f is cont at a.

8. Use the mean-value th to prove

$$\frac{\partial f}{\partial \vec{p}}(x_0) = \langle \vec{\nabla} f(x_0), \vec{p} \rangle.$$

Solution.

$$\frac{\partial f}{\partial \vec{p}}(x_0) = \lim_{t \to 0} \frac{f(x_0 + t\vec{p}) - f(x_0)}{t}$$ (def. of $\partial f / \partial \vec{p}(x_0)$)

$$= \lim_{t \to 0} \frac{\langle \vec{\nabla} f(x_0 + \theta \vec{p}), t\vec{p} \rangle}{t}$$

for some $0 < \theta < t$

$$= \langle \vec{\nabla} f(x_0), \vec{p} \rangle$$ because $\vec{\nabla} f$ is cont.