Abstract Algebra I, Practice exam 1 solutions

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1. (a) To say that $(g^{-1})^n$ is the inverse of g^n is to say that $(g^{-1})^n \cdot g^n = g^n \cdot (g^{-1})^n = e$. These can both be seen by writing out $(g^{-1})^n = g^{-1} \cdots g^{-1}$ and $g^n = g \cdots g$, and then multiplying from the inside out.

   (b) By repeated application of associativity,

$$\begin{aligned}(g_1 \cdot (g_2 \cdot g_3)) \cdot (g_4 \cdot g_5) &= ((g_1 \cdot g_2) \cdot g_3) \cdot (g_4 \cdot g_5) \\ &= (g_1 \cdot g_2) \cdot (g_3 \cdot (g_4 \cdot g_5)) \\ &= (g_1 \cdot g_2) \cdot ((g_3 \cdot g_4) \cdot g_5).\end{aligned}$$

2. We need to show that any element of h is a product of copies of $\phi(g_1), \phi(g_2), \phi(g_3), \phi(g_1)^{-1}, \phi(g_2)^{-1}, \phi(g_3)^{-1}$ in some order. Let $g = \phi^{-1}(h)$. Then, because g_1, g_2, g_3 generate G, we may write g as a product $g_1' \cdots g_n'$, where each g_i' is equal to one of $g_1, g_2, g_3, g_1^{-1}, g_2^{-1}, g_3^{-1}$. Applying ϕ, we have $h = \phi(g) = \phi(g_1' \cdots g_n') = \phi(g_1') \cdots \phi(g_n')$. This expresses h as a product in the desired way.

3. We have $((\mathbb{Z}/12\mathbb{Z})^\times, \times) = \{1, 5, 7, 11\}$, but every element squares to 1. In particular, no element generates the whole group.

4. (a) σ fixes all but $n-3$ elements, and τ fixes all but $n-5$ elements. Thus, there are at least $n-8$ elements fixed by $\tau\sigma$, so this product cannot be an m-cycle for $m > 8$.

   (b) If $m = 1$, this would mean that $\sigma = \tau^{-1}$, but the inverse of a k-cycle is a k-cycle for any k.

   (c) σ, τ are both even permutations, so their product must also be even. This means that m must be odd.

   (d) Here is one solution:

$$\begin{aligned}&(123)(15432) = (154) \\ &(354)(15432) = (14532) \\ &(167)(12345) = (1234567)\end{aligned}$$

5. There are many ways to do this, but one is to observe that D_{12} has several elements of order 12, but S_4 has none. On the other hand, an isomorphism $D_{12} \to S_4$ would send an element of order 12 to an element of order 12.

6. Let $Rot(P_6) \subset D_6$ be the cyclic subgroup of rotations in the plane. Let H be a subgroup of D_6. We will describe all possible H based on what the subgroup $H \cap Rot(P_6)$ looks like. Remember that $Rot(P_6)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$, and thus has four subgroups, generated by either r^0 (the trivial subgroup), r^1 (everything), r^2, and r^3.

   <u>Case 1:</u> $H \cap Rot(P_6) = \langle r \rangle = Rot(P_6)$. In this case, either either H is already equal to $Rot(P_6)$, or we argue that if we add any reflection $r^a s$ to H, then closure under multiplication forces $H = D_6$. We get 2 subgroups in this case.

   <u>Case 2:</u> $H \cap Rot(P_6) = \{e, r^2, r^4\}$. We may have $H = \{e, r^2, r^4\}$. If we add a reflection of the form $r^a s$ to H, then we must also add the reflection $r^{a+2}s, r^{a+4}s$ to H by closure. This gives the subgroups

$\{e, r^2, r^4, s, r^2s, r^4s\}$ and $\{e, r^2, r^4, rs, r^3s, r^5s\}$. Adding any more to these would give all of D_6, in which case we'd be back in ase 1. Therefore, we find 3 more subgroups in this case.

Case 3: $H \cap Rot(P_6) = \{e, r^3\}$. In this case, we may have $H = \{e, r^3\}$. If we add a reflection r^as to H, then we must also add the reflection $r^{a+3}s$ to H by closure. This gives the subgroups $\{e, r^3, r^as, r^{a+3}s\}$ for $a = 0, 1, 2$. On the other hand, if we add any more reflections r^bs, we will get more powers of r, which is not allowed. We get 4 subgroups in total here.

Case 4: $H \cap Rot(P_6) = \{e\}$. In this case, we must have either $H = \{e\}$, or H has order 2 and contains only a single reflection r^as in addition to the identity. Indeed, if there are two distinct reflections $r^as, r^bs \in H$, then their composition is a non-trivial power of r, which is not allowed to be in H. Because there are 6 possible axes of reflection, we get 7 subgroups in this case.