Mean value theorem for integrals

Let f be a continuous function on $[a, b]$, and let g be an integrable function that does not change sign on $[a, b]$. Then there exists a number c between a and b such that

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

Proof f is continuous on $[a, b]$

Using extreme value theorem, $f(x)$ has minimum and maximum on $[a, b]$

Let $m \le f(x) \le M$

Assume g is non-negative

$$m\, g(x) \le g(x)\, f(x) \le M\, g(x)$$

$$\int_a^b m\, g(x)\, dx \le \int_a^b g(x)\, f(x)\, dx \le \int_a^b M\, g(x)\, dx$$

$$m \int_a^b g(x)\, dx \le \int_a^b g(x)\, f(x)\, dx \le M \int_a^b g(x)\, dx$$

Divide both sides $\int_a^b g(x)\, dx$

$$m \le \underbrace{\frac{\int_a^b g(x)\, f(x)\, dx}{\int_a^b g(x)\, dx}}_{\text{This is between } m \text{ and } M} \le M$$

Applying intermediate value theorem, there is a $c \in [a, b]$ such that

$$f(c) = \frac{\int_a^b g(x)\, f(x)\, dx}{\int_a^b g(x)\, dx}$$

Therefore, $\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$

Exercise i) Extend proof when g is negative

ii) What happens when $g(x) = 1$?

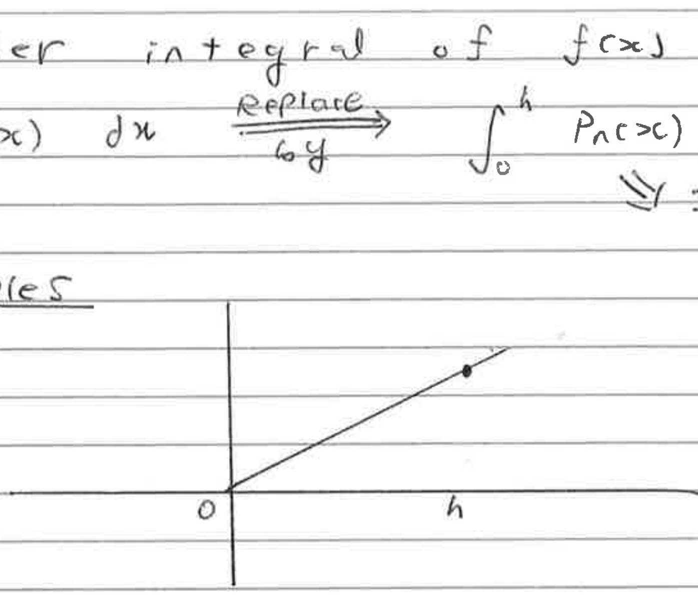
We will use this theorem for numerical integration

Consider integral of $f(x)$ in the interval $[0, h]$

$$\int_0^h f(x)\, dx \xrightarrow[\text{by}]{\text{Replace}} \int_0^h P_n(x)\, dx$$

Interpolating polynomial

Examples



Replace by linear interpolation

$f(x) \equiv$ continuous with second derivative on $[x_0, x_1]$

Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$

Degree 1 interpolating polynomial

$$f(x) = \underbrace{y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}}_{P(x)} + \underbrace{\frac{(x - x_0)(x - x_1)}{2!} f''(C_x)}_{E(x)}$$

[C_x depends continuously on x]

$$\int_{x_0}^{x_1} f(x)\, dx = \int_{x_0}^{x_1} P(x)\, dx + \int_{x_0}^{x_1} E(x)\, dx$$

Exercise: Prove $\int_{x_0}^{x_1} P(x)\, dx = h \left(\frac{y_0 + y_1}{2} \right)$ where $h = x_1 - x_0$

proof Let $w = -x + x_1$

$$\int_{x_0}^{x_1} -\frac{w}{h} (-dx) = \int_h^0 \frac{w}{h}\, dw = h/2$$

Let $w = x - x_0$

$$\int_{x_0}^{x_1} \frac{x_1 - x_0}{h}\, dx = \int_0^h \frac{w}{h}\, dw = h/2$$

We now compute the error term

$$\int_{x_0}^{x_1} E(x)\,dx = \frac{1}{2!}\int_{x_0}^{x_1} (x-x_0)(x-x_1)\, f''(C_x)\,dx$$

$$= \frac{f''(c)}{2}\int_{x_0}^{x_1} (x-x_0)(x-x_1)\,dx$$

$x - x_0 = u$

$u - h = x - x_0 - h = x - (x_0 + h) = x - x_1$

$$= \frac{f''(c)}{2}\int_0^h u(u-h)\,du = -\frac{h^3}{12} f''(c)$$

Trapezoid rule $\displaystyle\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{12} f''(c)$

$h = x_1 - x_0$ and c is between x_0 and x_1

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* Interpolation by Parabola

$$f(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f'''(C_x)$$

$$= P(x) + E(x)$$

Let $h = x_2 - x_1 = x_1 - x_0$

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} P(x)\,dx + \int_{x_0}^{x_2} E(x)\,dx$$

- $\displaystyle\int_{x_0}^{x_2} P(x)\,dx = y_0 \frac{h}{3} + y_1 \frac{4h}{3} + y_2 \frac{h}{3}$

- $\displaystyle\int_{x_0}^{x_2} E(x)\,dx = \frac{-h^5}{90} f^4(c)$

Simpson Rule : $\displaystyle\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} f^4(c)$

c is between x_0 and x_2

Example Apply the Trapezoid and Simpson's rule to approximate $\int_1^2 \ln(x)\,dx$

Solution **Trapezoid**

$$\int_1^2 \ln(x)\,dx \approx \frac{h}{2}(y_0 + y_1) = \frac{1}{2}(\ln(1) + \ln(2)) = \frac{\ln(2)}{2} \approx 0.3466$$

Note we used $x_0 = 1$ and $x_1 = 2$

Simpson

$$\int_1^2 \ln(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$= \frac{0.5}{3}\left(\ln(1) + 4\ln\frac{3}{2} + \ln(2)\right) \approx 0.3858$$

Note we used $x_0 = 1$ $x_1 = 1.5$ and $x_2 = 2$

Error

Trapezoid error : $-\frac{h^3}{12} f''(c)$

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$ **Error** : $\left\|-\frac{h^3}{12} f''(c)\right\|$ $1 < c < 2$

$f''(x) = -\frac{1}{x^2}$ $= \left|-\frac{1}{12} \cdot -\frac{1}{c^2}\right| \leq 1/12 \approx 0.0834$

Simpson error : $-h^5 f^4(c)$ $1 < c < 2$

Error : $\left|\frac{-h^5}{90} f^4(c)\right| = \left|\frac{(-1/2)^5}{90} \cdot -\frac{6}{c^4}\right|$

$f'''(x) = \frac{2}{x^3}$

$f^4(x) = -\frac{6}{x^4}$ $= \frac{6(1/2)^5}{90} = \frac{1}{480} \approx 0.0021$

Midpoint: $\int_{x_0}^{x_1} f(x)\,dx = h f(\text{midpoint}) + \frac{h^3}{24} f''(c)$

$h = x_1 - x_0$

$w = x_0 + \frac{h}{2}$

c is between x_0 and x_1