1. (10 points) **Pointwise vs. uniform convergence**.

Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{1-x^{4n}}{1+x^{4n}}$.

(a) Find the function f that the sequence $\{f_n\}$ converges to pointwise on $\mathbb{R}$.

(b) Does $\{f_n\}$ converge uniformly to f on $\mathbb{R}$? Why or why not?

Solution.

(a) If $|x| < 1$, $x^{4n} \to 0$, so $\lim f_n(x) = \frac{1-0}{1-0} = 1$.

If $|x| = 1$, $f_n(x) = \frac{1-1}{1+1} = 0$.

If $|x| > 1$, then $x^{4n} \to +\infty$, so

$$\lim \frac{1-x^{4n}}{1+x^{4n}} = \lim \frac{\frac{1}{x^{4n}} - 1}{\frac{1}{x^{4n}} + 1} = \frac{0-1}{0+1} = -1.$$

Thus, $f_n(x)$ converges pointwise to

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1, \\ 0 & \text{for } |x| = 1, \\ -1 & \text{for } |x| > 1. \end{cases}$$

(b) Since $f_n(x)$ is continuous for all n, if $f_n \to f$ uniformly, f would be continuous on $\mathbb{R}$. This is clearly not the case. Hence, f_n does not converge uniformly to f on $\mathbb{R}$. $\square$

2. (15 points) **Pointwise vs. uniform convergence.**

Let $f_n(x) = e^{-nx}$.

(a) Find the pointwise limit, f, of the sequence $\{f_n\}$ on $[0,\infty)$.
(*Hint*: Consider the point $x = 0$ separately.)

(b) Show that $\{f_n\}$ converges to this function f uniformly on $[1,\infty)$.

(c) (2 points) Explain why $\{f_n\}$ does not converge to f uniformly on $[0,\infty)$.

(d) (4 points) Does $\{f_n\}$ converge to f uniformly on $(0,\infty)$? Prove your answer.

Solution. (a) If $x > 0$, then $nx \to \infty$ and $f_n(x) = \frac{1}{e^{nx}} \to 0$.

If $x = 0$, then $f_n(0) = e^0 = 1 \to 1$.

Thus,
$$f(x) = \begin{cases} 0 & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

(b) For $x \geq 1$, we have

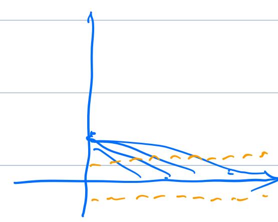
$$nx \geq n \Rightarrow e^{nx} \geq e^n \Rightarrow \frac{1}{e^{nx}} \leq \frac{1}{e^n}$$
$$\Rightarrow |f_n(x) - f(x)| = \left|\frac{1}{e^{nx}} - 0\right| = \frac{1}{e^{nx}} \leq \frac{1}{e^n}$$

Since $a_n := \frac{1}{e^n} \to 0$, by the Comparison test for uniform convergence,

$$f_n(x) = \frac{1}{e^{nx}} \to f(x) = 0 \text{ uniformly on } [1,\infty).$$

(c) $f_n \not\to f$ uniformly on $[0,\infty)$ because $f_n(x) = e^{-nx}$ is continuous on $[0,\infty)$ but $f(x)$ is not.

(d) Suppose $f_n(x) \to f$ uniformly on $(0,\infty)$.



Let $\varepsilon = 1/2$. $\exists N \in \mathbb{N}$ such that $\forall x \in (0,\infty)$ and $\forall n \geq N$,

$$\left|\frac{1}{e^{nx}} - 0\right| = \frac{1}{e^{nx}} < \frac{1}{2}.$$

This is not true because for $x = 1/n$, $\frac{1}{e^{nx}} = 1 > \frac{1}{2}$.

Hence, $f_n(x) = 1/e^{nx} \not\to f$ uniformly on $(0,\infty)$. $\square$

3. (10 points) (**Domain of convergence of a power series**) §9.5, p. 262, # 1a,b,c.
Determine the domain of convergence of each of the following power series:

(a) $\sum_{k=1}^{\infty} \frac{x^k}{k5^k}$ (b) $\sum_{k=1}^{\infty} k!x^k$ (c) $\sum_{k=\cancel{0}\,1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k+1)!}$

a. $\sum_{k=1}^{\infty} \frac{x^k}{k5^k}$

Solution. a. Let $a_k = \frac{x^k}{k5^k}$. Then

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\cancel{x^{k+1}}\,x}{(k+1)\,\cancel{5^{k+1}}\,5} \cdot \frac{k\,\cancel{5^k}}{\cancel{x^k}}\right| = \left|\frac{k\,x}{(k+1)\,5}\right| \longrightarrow \frac{|x|}{5} \quad \text{as } k \to \infty.$$

By the ratio test, the series converges for $|x| < 5$ and diverges for $|x| > 5$. At $x = 5$, the series is $\sum \frac{1}{k}$, which diverges. At $x = -5$, the series is $\sum (-1)^k \frac{1}{k}$, which converges by the alternating series test. Therefore, the domain of convergence is $\boxed{[-5, 5)}$.

b. $\sum_{k=1}^{\infty} k!\,x^k$

Solution. Let $a_k = k!\,x^k$. Then if $x \neq 0$,

$$\left|\frac{a^{k+1}}{a^k}\right| = \left|\frac{(k+1)!\;x^{k+1}}{k!\;x^k}\right| = (k+1)|x| \longrightarrow +\infty \text{ as } k \to \infty.$$

Therefore, $\sum k!\,x^k$ diverges for all $x \neq 0$, and the domain of convergence is $\boxed{\{0\}}$.

§9.5, p. 262, #1.

c. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k+1)!}$ (I think this series has to start with $k=1$, because if $k=0$, then $x^{2k-1} = x^{-1}$, which is not allowed in a power series. It's a misprint in the book.)

Let $a_k = \dfrac{(-1)^k x^{2k-1}}{(2k+1)!}$. Then for $x \neq 0$,

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{2k+1}}{(2k+3)!} \frac{(2k+1)!}{x^{2k-1}}\right| = \frac{x^2}{(2k+3)(2k+2)} \to 0 \text{ as } k \to \infty.$$

By the ratio test, the series converges for all x. $\square$

4. (10 points) (**Polarization identity**) §10.1, p. 276: # 5.
Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Prove that

$$\langle \vec{u}, \vec{v} \rangle = \frac{\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2}{4}.$$

This identity shows that the scalar product can be expressed in terms of the norm. It is called the ***polarization identity***.

Proof. $\|u+v\|^2 = \langle u+v, u+v \rangle = \langle u,u \rangle + 2\langle u,v \rangle + \langle v,v \rangle$,
$\|u-v\|^2 = \langle u-v, u-v \rangle = \langle u,u \rangle - 2\langle u,v \rangle + \langle v,v \rangle$.
Hence, subtracting one from the other,
$\|u+v\|^2 - \|u-v\|^2 = 4\langle u,v \rangle$,
so $\langle u,v \rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$. □

5. (10 points) (**Absolute value of a sum vs. Euclidean length**) §10.1, p. 276: #7.
For a natural number n and real numbers $a_1, \ldots, a_n$, prove that

$$|a_1 + \cdots + a_n| \leq \sqrt{n}\sqrt{a_1^2 + \cdots + a_n^2}.$$

(Hint: Apply the Cauchy–Schwarz inequality to two cleverly chosen vectors.)

Proof. Let $u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$.

By the Cauchy-Schwarz inequality,
$|\langle u,v \rangle| = |a_1 + \cdots + a_n| \leq \|u\| \, \|v\|$
$= \sqrt{n}\sqrt{a_1^2 + \cdots + a_n^2}$. □

6. (10 points) (**Convexity of an open ball**) §10.1, p. 277: # 10.
Let $\vec{u}$ be a point in $\mathbb{R}^n$ and let r be a positive number. Suppose that the points $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are at a distance less than r from the point $\vec{u}$. Prove that if $0 \le t \le 1$, then the point $t\vec{v} + (1-t)\vec{w}$ is also at a distance less than r from $\vec{u}$. (What this problem means geometrically is that the open ball with center $\vec{u}$ and radius r is convex. Draw a picture to convince yourself of this.)

Proof.
$$\begin{aligned}\|(tv + (1-t)w) - u\| &= \| tv - tu + (1-t)w - (1-t)u \| \\ &\le t\|v-u\| + (1-t)\|w-u\| \quad (\text{by } \Delta \text{ ineq}) \\ &< tr + (1-t)r \quad (\text{because } v, w \in B(u,r)) \\ &= r.\end{aligned}$$
□