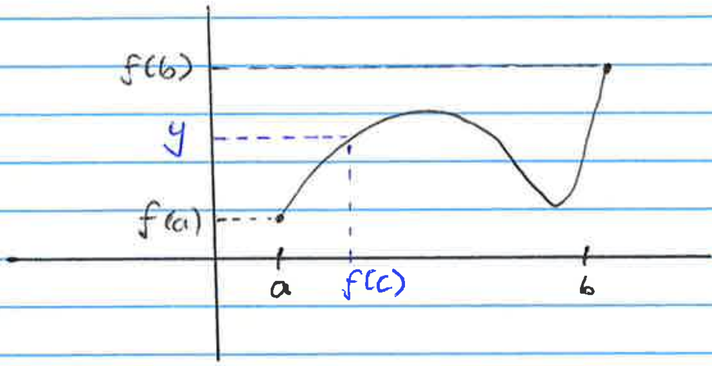
MATH 126 Lecture 4

Intermediate value theorem (IVT)

Let f be a continuous function on the interval $[a, b]$. Then f realizes every value between $f(a)$ and $f(b)$. More precisely, if y is a number between $f(a)$ and $f(b)$, then there exists a number c with $a \le c \le b$ such that $f(c) = y$.



Exercise Show that $f(x) = x^2 - 3$ on the interval $[1, 3]$ must take on the values 0 and 1.

solution $f(1) = -2$; $f(3) = 6$

Therefore f takes all values between -2 and 6.

continuous limits Let f be a function that is continuous in a neighborhood of x_0, and assume that $\lim_{n \to \infty} x_n = x_0$. Then $\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right)$

$= f(x_0)$

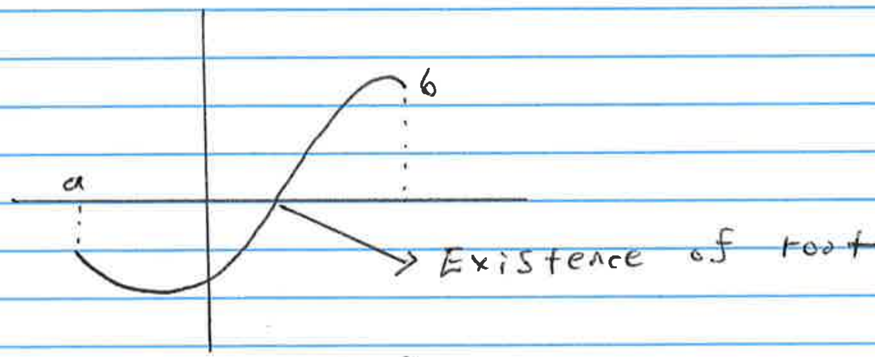
You can bring limits inside continuous functions

Definition $x = r$ is a root of the function f if $f(r) = 0$.

The first natural question, given a function $f(x)$, is to ask if it has a root i.e existence of root.

Theorem Let f be a continuous function on $[a,b]$ satisfying $f(a)\,f(b) < 0$. Then f has a root between a and b i.e there exists a number r $a < r < b$ such that $f(r) = 0$.

Proof This follows from IVT.



Bisection method

Initial estimate : (a, b) such that $f(a)$ and $f(b)$ have opposite signs

Step 1 Divide the interval into two parts (a, c) (c, b) where $c = (a+b)/2$

i) If $f(c) = 0$, c is a root.

ii) If $f(a)\,f(c) < 0$, define a new interval as (a, c)

iii) If $f(a)\,f(c) > 0$, define a new interval as (c, b)

Repeat the above procedure until a stopping criterion is met. Typically, $|f(c)| <$ tol where tol is a small parameter (e.g 10^{-5})

Pseudocode

```
Initialize: (a, b)
For k = 1, 2, 3 ...
    c = (a + b)/2
    If |f(c)| < ε1 or |a-b| < ε2
        return x* ≈ c
    else if f(a)·f(c)<0 then
        b = c
    else
        a = c
```

Analysis of bisection method

$I_0 = (a_0, b_0)$
$I_1 = (a_1, b_1)$
$I_2 = (a_2, b_2) \ \cdots \quad I_n = (a_n, b_n)$

Note that $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) \qquad n \geq 0$

$$
\begin{aligned}
b_{n+1} - a_{n+1} &= \frac{1}{2}(b_n - a_n) \\
&= \frac{1}{2} \cdot \frac{1}{2}(b_{n-1} - a_{n-1}) \\
&= \frac{1}{2^2}(b_{n-1} - a_{n-1}) = \frac{1}{2^3}(b_{n-2} - a_{n-2}) \ldots \\
&= \frac{1}{2^{n+1}}(b_0 - a_0)
\end{aligned}
$$

In general,

$$\boxed{b_n - a_n = 2^{-n}(b_0 - a_0)}$$

Let's consider the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n \geq 0}$

$\{a_n\}$ is non-decreasing and bounded above

$a_0 \leq a_1 \leq a_2 \leq \ldots \leq b_0$

$\{b_n\}$ is non-increasing and bounded below

$b_0 \geq b_1 \geq b_2 \geq \ldots \geq a_0$

Applying the monotone convergence theorem, $\{a_n\}$ and $\{b_n\}$ converge.

consider

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} 2^{-n}(b_0 - a_0) = 0$$

Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \mu$

Since $f(a_n) f(b_n) \leq 0$ Take limit on inequality

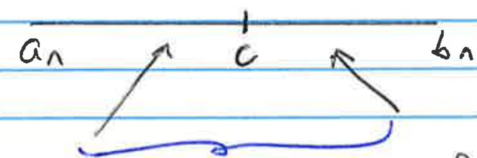
$f(\mu)^2 \leq 0 \implies (f(\mu)^2$ is non-negative$) \implies f(\mu) = 0$

Important remark: The above analysis only informs us what happens in the limit $n \to \infty$. In practice, we are interested in the error at some finite n.

$$c_n = \frac{a_n + b_n}{2} \quad \text{—— Estimate of root given interval } [a_n, b_n]$$

Therefore, it is of interest to bound $|r - c_n|$

<u>Exercise</u> Show that $|r - c_n| \le \dfrac{b_n - a_n}{2}$



$$|r - c_n| \le \frac{1}{2}(b_n - a_n) \le \frac{1}{2} \cdot 2^{-n}(b_0 - a_0) = 2^{-(n+1)}(b_0 - a_0)$$

<u>Exercise</u> What is the order of the convergence of the bisection method?

$$E_n = |r - c_n| \le 2^{-n}(b_0 - a_0)$$
$$E_{n+1} = |r - c_{n+1}| \le 2^{-(n+1)}(b_0 - a_0)$$

If E_n and E_{n+1} attain the worst bound, linear convergence.

* In general, it is <u>not</u> linear convergence as some textbooks / papers wrongly suggest

How many iterations to attain $|E_n| < \varepsilon$

$$2^{-(n+1)}(b_0 - a_0) < \varepsilon$$
$$2^{-(n+1)} < \frac{\varepsilon}{b_0 - a_0} \quad \text{(Take } \log_2 \text{ on both sides)}$$
$$-(n+1) < \log_2\left(\frac{\varepsilon}{b_0 - a_0}\right)$$
$$n+1 > \log_2\left(\frac{b_0 - a_0}{\varepsilon}\right)$$
$$\boxed{n > \log_2\left(\frac{b_0 - a_0}{\varepsilon}\right) - 1} \quad *$$

Set n according to *

Exercise How many function evaluations at the n-th iteration?

Solution In each iterations,

- $f(a_n)$ $f(c_n)$
- compute c_n

* However, we re-use either $f(a_n)$, $f(c_n)$ or both

Exercise What is a good method to use if f is a polynomial?

Bisection

Pros

i) simple

ii) Minimal assumption on f

Cons

i) slow convergence

ii) No generalization to multiple variable case

iii) Requires good initial estimate

Discussion Explain why we can find roots of positive numbers using bisection method

Next time $\Rightarrow$ use information about derivative

$f(x_0) + f'(x_0)(x - x_0)$