Homework 1

Early problem due on Gradescope at 8 pm on Tuesday, January 24th.
Due on Gradescope at 8 pm on Friday, January 27th.

(1) (Early problem) Let A and B be sets of real numbers. Write the negation of each of the following statements:
   (a) For every $a \in A$, it is true that $a^2 \in B$.
   (b) For at least one $a \in A$, it is true that $a^2 \in B$.
   (c) For every $a \in A$, it is true that $a^2 \notin B$.
   (d) For at least one $a \notin A$, it is true that $a^2 \in B$.

(2) Determine which of the following statements are true for all sets A, B, C, D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether the statement becomes true if the "equals" symbol is replaced by one or the other of the inclusion symbols $\subseteq$ or $\supseteq$.
   (a) $A \subseteq B$ and $A \subseteq C \iff A \subseteq (B \cup C)$.
   (b) $A \subseteq B$ or $A \subseteq C \iff A \subseteq (B \cup C)$.
   (c) $A \subseteq B$ and $A \subseteq C \iff A \subseteq (B \cap C)$.
   (d) $A \subseteq B$ or $A \subseteq C \iff A \subseteq (B \cap C)$.
   (e) $A - (A - B) = B$.
   (f) $A - (B - A) = A - B$.
   (g) $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$

(3) Prove that De Morgan's laws hold for arbitrary unions and intersections. That is, show that if A is a set and $\{B_i\}_{i \in I}$ is an arbitrary collection of sets, then

$$A - \left(\bigcup_{i \in I} B_i \right) = \bigcap_{i \in I} (A - B_i)$$

and

$$A - \left(\bigcap_{i \in I} B_i \right) = \bigcup_{i \in I} (A - B_i).$$

(4) Let $f : X \to Y$ be a function and let $\{B_i\}_{i \in I}$ be a collection of subsets of Y. Prove that
   (a)

$$\bigcup_{i \in I} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i \in I} B_i \right).$$

   (b)

$$\bigcap_{i \in I} f^{-1}(B_i) = f^{-1}\left(\bigcap_{i \in I} B_i \right).$$

(5) Suppose $S = \{1, 2, 3\}$ and $T = \{4, 5\}$.
   (a) How many functions are there from S to T? How many from T to S?
   (b) How many of the functions from S to T are injective? How many are surjective?
   (c) How many of the functions from T to S are injective? How many are surjective?

(6) Let $f : X \to Y$ and $g : Y \to Z$ be functions.
   (a) Prove that if $g \circ f$ is injective, then f is injective.
   (b) Prove that if $g \circ f$ is surjective, then g is surjective.

(7) Prove that if $f : X \to Y$ is surjective, then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is injective.

(8) (Optional, no credit) Let $f : X \to Y$ be a function. Let $A \subseteq X$ and $B \subseteq Y$. Prove that $A \subseteq f^{-1}(B)$ if and only if $f(A) \subseteq B$. (Remark: this shows that image and preimage form what is called a *Galois connection*. It is a corollary that taking images preserves unions and taking preimages preserves intersections. Curious students are encouraged to consult the Wikipedia page.)