Small-Sample Statistics

The statistics of the sample variance S^2

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Outline

Orthogonal matrices

- Column vector $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$
- Transpose is row vector $v^T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$
- Square of length of vector is given by

$$\|v\|^2 = v_1^2 + \cdots + v_n^2 = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v^T v$$

- Hence length of vector is given by

$$\|v\| = \sqrt{v^T v}$$

Orthogonal matrices (continued)

- Consider a linear transformation of an n-vector, $u = Av$, where A is an $n \times n$ matrix
- Demand that the transformation preserve length

$$0 = u^T u - v^T v = (Av)^T (Av) - v^T v = v^T A^T A v - v^T v = v^T (A^T A - I) v.$$

- Require above to be true for all vectors v, so we have

$$\boxed{A^T A = I}$$

- A matrix with this property is called an *orthogonal matrix*.
- If A is nonsingular, postmultiplication by A^{-1} yields

$$\boxed{A^T = A^{-1},}$$

and premultiplying both sides of this by A yields

$$\boxed{A A^T = I}$$

Orthogonal matrices (continued)

- Let a_j denote the jth column of A
- The equation $A^T A = I$ indicates that

$$a_j^T a_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

- Hence the rows and columns of an orthogonal matrix are unit vectors.

Orthogonal matrices (continued)

- Given an orthogonal matrix A, we have $A^T A = I$
- Take determinant of both sides and use the following determinant theorems
  - $\det(AB) = \det(A)\det(B)$
  - $\det(A^T) = \det(A)$
- The result is

$$\det(A^T A) = \det(A^T)\det(A) = [\det(A)]^2 = \det(I) = 1$$

and hence

$$\boxed{\det(A) = \pm 1.}$$

- Orthogonal matrix A with
  - $\det(A) = +1$ is *proper orthogonal transformation*
  - $\det(A) = -1$ is *improper orthogonal transformation*

Orthogonal matrices (continued)

- Transformation $u = Av$ can be written $u_i = \sum_{j=1}^{n} A_{ij} v_j$
- The (i, j)th element of Jacobian matrix is

$$\frac{\partial u_i}{\partial v_j} = A_{ij}$$

- Jacobian factor for transforming n-dimensional integral over the v is

$$J = |\det(A)| = |\pm 1| = 1.$$

- Hence if we write $du = du_1 \cdots du_n$ and $dv = dv_1 \cdots dv_n$,

$$\int du\ f(u) = \int dv\ f(Av).$$

Transformation from X to Z

- We have data $\vec{Y} = \langle Y_1, \ldots, Y_n \rangle$ with
  - mean μ,
  - variance σ^2,
  - sample variance S^2.
- Let standardized version be $X_j = \frac{Y_j - \mu}{\sigma}$ for $j = 1, \ldots, n$
- We know the X_j are $N(0, 1)$ (standard normal)
- Let A be an orthogonal matrix whose last row is

$$\begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}$$

- Then the transformation $\vec{Z} = A\vec{X}$ results in new coordinates $\vec{Z}$, the nth one of which is

$$Z_n = \frac{X_1}{\sqrt{n}} + \cdots + \frac{X_n}{\sqrt{n}} = \sqrt{n}\,\overline{X}$$

The transformation from X to Z (continued)

- Also, for variables z and x with $z = Ax$, we have

$$\|z\|^2 = z_1^2 + \cdots + z_n^2 = x_1^2 + \cdots + x_n^2 = \|x\|^2$$

- Since $\|x\|^2 = \|z\|^2$ and the Jacobian is one, the multivariate pdfs transform as

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) &= (2\pi)^{-n/2} \exp\left[-\frac{1}{2}(x_1^2 + \cdots + x_n^2)\right] \\ &= (2\pi)^{-n/2} \exp\left[-\frac{1}{2}(z_1^2 + \cdots + z_n^2)\right] = f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n) \end{aligned}$$

- Hence the Z_j are also iid $N(0,1)$ (standard normal) r.v.s

The distribution of S^2

- Finally we note that

$$\sum_{j=1}^{n} Z_j^2 = \sum_{j=1}^{n-1} Z_j^2 + n\overline{X}^2 = \sum_{j=1}^{n} X_j^2 = \sum_{j=1}^{n} (X_j - \overline{X})^2 + n\overline{X}^2$$

- Hence, using $X_j = \frac{Y_j - \mu}{\sigma}$, we have

$$\sum_{j=1}^{n-1} Z_j^2 = \sum_{j=1}^{n} (X_j - \overline{X})^2 = \frac{1}{\sigma^2} \sum_{j=1}^{n} (Y_j - \overline{Y})^2 = \frac{(n-1)S^2}{\sigma^2}$$

- Hence S^2 is independent of $Z_n^2 = n\,\overline{X}^2$, and hence of $\overline{X}$.
- Since $Y_j = \mu + \sigma X_j$, S^2 is also independent of $\overline{Y}$.
- The quantity $\frac{(n-1)S^2}{\sigma^2}$ is χ^2 distributed with $n-1$ df. $\square$

Summary

- We reviewed the linear algebra of orthogonal matrices.
- We used an orthogonal transformation to prove
  - the independence of S^2 with any of $\{Z_n^2, \overline{X}, \overline{Y}\}$.
  - the fact that $\frac{(n-1)S^2}{\sigma^2}$ is χ^2 distributed with $n-1$ df.