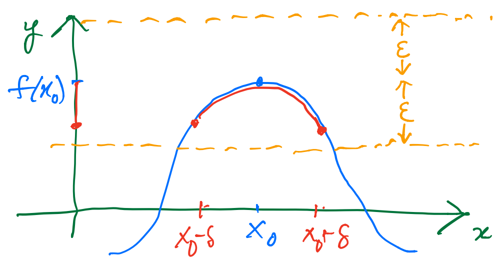
Math 135 Homework 6 Solutions

1. (Elephant Under the Rug Theorem)
Let $D \subset \mathbb{R}$. Suppose $f: D \to \mathbb{R}$ is continuous and $f(x_0) \neq 0$ at some point $x_0 \in D$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all x in the interval $(x_0 - \delta, x_0 + \delta)$.

Proof.



Let $\varepsilon = f(x_0)/2 > 0$. By the ε-δ criterion for continuity, there is a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$, or

$$x \in (x_0 - \delta, x + \delta_0) \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon).$$

In other words, the δ-interval about x_0 is mapped into the ε-interval about $f(x_0)$. But $f(x_0) - \varepsilon = f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0$,

so $\forall\, x \in (x_0 - \delta, x + \delta)$,

$$f(x) > f(x_0) - \varepsilon = \frac{f(x_0)}{2} > 0. \qquad \square$$

2. Suppose $\{x_n\}$ satisfies

$$|x_{n+1} - x_n| < \frac{1}{2^n}.$$

Show that $\{x_n\}$ converges. (Hint. Write

$x_{n+k} - x_n = (x_{n+k} - x_{n+k-1}) + (x_{n+k-1} - x_{n+k-2}) + \cdots + (x_{n+1} - x_n).$)

Proof. Let $m = n+k > n$ be two natural numbers. Then

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)|$$

(all the intermediate terms cancel out)

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

(Δ inequality)

$$< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n} = \frac{1}{2^{n+k-1}} + \cdots + \frac{1}{2^n}$$

$$= \frac{1}{2^n}\left(\frac{1}{2^{k-1}} + \cdots + 1\right) = \frac{1}{2^n}\,\frac{1-\left(\frac{1}{2}\right)^k}{1-\frac{1}{2}}$$

$$= \frac{1}{2^n}\left(2 - \left(\frac{1}{2}\right)^{k-1}\right) < \frac{1}{2^n}(2) = \frac{1}{2^{n-1}}$$

n	1	2	3	4	5	6	$\cdots$
2^{n-1}	1	2	4	8	16	32	$\cdots$

From this table, it is clear that $2^{n-1} \geq n$ for all $n \in \mathbb{N}$. (This can be easily proven by induction.)

Hence,

$$|x_m - x_n| < \frac{1}{2^{n-1}} \leq \frac{1}{n}.$$

It suffices to solve $\frac{1}{n} < \varepsilon$.

Taking reciprocals gives $n > \frac{1}{\varepsilon}$.

Choose a natural number $N > \frac{1}{\varepsilon}$. Then $\forall\ n > N$ and $k \geq 1$,

$$|x_{n+k} - x_n| < \varepsilon.$$

This proves that $\{x_n\}$ is Cauchy and therefore convergent. $\square$

An alternative approach is to note that, as above,

$$|x_m - x_n| < \frac{1}{2^m} + \frac{1}{2^{m-1}} + \cdots + \frac{1}{2^n}.$$

The right hand side is the difference $s_m - s_{n-1}$ of the partial sums of $\sum_{k=0}^{\infty} \frac{1}{2^k}$, where

$$s_n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k.$$

Since $\sum_{k=0}^{\infty} \frac{1}{2^k}$ is convergent, it is Cauchy and therefore for any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\forall\, m, n \geq N$,

$$|s_m - s_{n-1}| < \varepsilon.$$

Therefore, the same N guarantees that $\forall\, m, n \geq N$,

$$|x_m - x_n| < |s_m - s_{n-1}| < \varepsilon.$$

This proves that $\{x_n\}$ is Cauchy and therefore convergent. $\square$

3. (10 points) **Uniform convergence.**
Assume $\{a_n\}$ is a bounded sequence of numbers and M is an upper bound, $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $r > 0$. Prove that the series

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

converges uniformly on $[-r, r]$ to a continuous function.

Solution. For $|x| \leq r$,

$$\left| \frac{a_n x^n}{n!} \right| \leq \frac{|a_n|\,|x|^n}{n!} \leq \frac{M r^n}{n!} := M_n .$$

Consider the ratio:

$$\frac{M_{n+1}}{M_n} = \frac{M r^{n+1}}{(n+1)!} \frac{n!}{M r^n} = \frac{r}{n+1} \to 0 .$$

By the ratio test, $\sum M_n$ cg.

By the M-test, $\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$ converges uniformly on $[-r, r]$. $\square$

4. (15 points) **Continuity of ~~log~~ ln.**
In Calculus II, you showed, for $x \in (-1,1]$, that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

(a) Show that this series converges uniformly on the interval $[-r,r]$ for any $r \in (0,1)$. (You may use any series convergence test that is in our book.)
(b) Prove that $\ln(1+x)$ is a continuous function on $(-1,1)$.

(a) For $|x| \le r$,

$$\left|\frac{(-1)^{n+1}x^n}{n}\right| = \frac{|x|^n}{n} \le \frac{r^n}{n} := M_n.$$

Consider the ratio

$$\frac{M_{n+1}}{M_n} = \frac{r^{n+1}}{n+1}\frac{n}{r^n} = \frac{n}{n+1}r \longrightarrow r < 1.$$

By the ratio test, $\sum_{n=1}^{\infty} M_n$ converges.
By the M-test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ converges uniformly on $[-r,r] \subset (-1,1)$.

(b) Being a polynomial in x, the partial sum $s_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}x^n}{n}$ is continuous on $[-r,r]$.

As the uniform limit of continuous functions, $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ is continuous on $[-r,r]$ for all $0 < r < 1$. Any $x_0 \in (-1,1)$ is in $[-r,r]$ for some $r \in (0,1)$, for example, $r = (|x_0|+1)/2$.

Hence, $\ln(1+x)$ is continuous at any $x_0 \in (-1,1)$, i.e., continuous on $(-1,1)$. $\square$

5. (10 points) **Sum of uniformly convergent sequences.**
Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions from $\mathbb{R}$ to $\mathbb{R}$. Assume $f_n \to f$ uniformly on $\mathbb{R}$ and $g_n \to g$ uniformly on $\mathbb{R}$. Prove that $f_n + g_n$ converges to $f + g$ uniformly on $\mathbb{R}$.

Proof. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly on $\mathbb{R}$, $\exists N_1 \in \mathbb{N}$ (not depending on x) such that $\forall x \in \mathbb{R}$ and $\forall n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon/2.$$

Similarly, since $g_n \to g$ uniformly on $\mathbb{R}$, $\exists N_2 \in \mathbb{N}$ (not depending on x) such that $\forall x \in \mathbb{R}$ and $\forall n \geq N$,

$$|g_n(x) - g(x)| < \varepsilon/2.$$

Choose $N = \max(N_1, N_2)$. By the Δ inequality, $\forall x \in \mathbb{R}$ and $\forall n \geq N$,

$$\begin{aligned} |(f_n + g_n)(x) - (f+g)(x)| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, $f_n + g_n \to f + g$ uniformly on $\mathbb{R}$. $\square$

6. (15 points) **Uniform convergence.**
For each of the following sequences, find the limit as $n \to \infty$ and determine whether it converges pointwise or uniformly. Show your reasoning.

(a) $\dfrac{\sin x}{n}$ on $\mathbb{R}$ (b) $\dfrac{x}{nx+1}$ on $(0,1)$ (c) x^n on $[0, 0.999]$

(a) $\left|\frac{\sin x}{n}\right| \le \frac{1}{n} \to 0$ (since $|\sin x| \le 1$)

Since

$$\left|\frac{\sin x}{n} - 0\right| \le \frac{1}{n} \text{ and } \frac{1}{n} \to 0,$$

by the Comparison theorem for uniform convergence, $(\sin x)/n \to 0$ uniformly on $\mathbb{R}$.

(b) $\frac{x}{nx+1} = \frac{1}{n+\frac{1}{x}} \to 0$ as $n \to \infty$ for each $x \in (0,1)$.

Since

$$\left|\frac{x}{nx+1} - 0\right| = \frac{1}{n+\frac{1}{x}} \le \frac{1}{n} \text{ and } \frac{1}{n} \to 0,$$

by the comparison theorem for uniform convergence, $x/(nx+1)$ converges uniformly to 0 on $(0,1)$.

(c) Since $|x| < 1$ for $x \in [0, 0.999]$, $x^n \to 0$.

Since

$$|x^n - 0| = |x|^n \le (.999)^n \text{ and } (.999)^n \to 0,$$

by the comparison theorem for uniform convergence, $x^n \to 0$ uniformly on $[0, .999]$.