I am going to make some suggestions of things you could with your journal. You can't follow them all, and you need not follow any of them.

Tuesday, September 16

1. Our discrete model:

$$\begin{aligned} S_{k+1} &= S_k - \alpha I_k S_k \\ I_{k+1} &= \alpha I_k S_k \\ R_{k+1} &= R_k + I_k \end{aligned}$$

   We noticed that $\alpha S_0 < 1$ and $\alpha I_0 < 1$ imply that the sequence $\{S_k\}$ decreases, the sequence $\{I_k\}$ decreases, and $S_k \geq 0$, $I_k \geq 0$ for all k. In other words, the epidemic peters out. If $\alpha I_0 > 1$, then S_1 is negative, so the model becomes nonsensical. If $\alpha S_0 > 1$, then $I_1 > I_0$, so it's an epidemic that initially gets worse.

   Notice that $\alpha S_0 < 1$ is typically a much stronger condition than $\alpha I_0 < 1$. Typically, I_0 is small and S_0 is almost the whole population size.

   This seems like a boring conclusion: $\alpha S_0 < 1$ means that the mean number of people whom one infected will infect (the "R number", typically denoted by R_0, but that means something else in our modeling, so we'll call it αS_0) is smaller than 1. Then *of course* the epidemic peters out. $\alpha S_0 > 1$ means that the mean number of people whom one infected will infect is greater than 1. Then *of course* the epidemic will grow exponentially.

   The *interesting* insight that one gets from the modeling is that the *final outcome* for αS_0 slightly greater than 1 is *dramatically* worse than for αS_0 slightly less than 1.

   Let's see if we ca see that by analysis. Prove: If $\alpha S_0 < 1$, and $R_0 = 0$, then

$$\lim_{k \to \infty} R_k < \sum_{k=0}^{\infty} (\alpha S_0)^k I_0 = \frac{I_0}{1 - \alpha S_0}$$

   (Look up "geometric series" if you don't remember to understand the equation, but the inequality is the main point to understand.) So in the sub-threshold case ($\alpha S_0 < 1$), the number of affected people is a constant — namely $1/(1 - \alpha S_0)$ — times the initial number of infected people.

2. Now the harder case. Suppose αS_0 is modestly greater than 1 — think 1.2, or 1.5, or 2. Could it still be the case that only a modest multiple of the initially infected are affected by the epidemic? Suppose $I_k \leq r I_0$ for some number $r > 1$, for all k. Suppose that r is some modest number — 5 or 10 or something. Then $S_{k+1} = (1 - \alpha I_k) S_k \geq (1 - \alpha r I_0) S_k$. So the number of infected people increases at least until $(1 - \alpha r I_0)^k \alpha S_0$ becomes smaller than 1, which means it increases at least until

$$k \geq \frac{\ln(\alpha S_0)}{|\ln(1 - \alpha r I_0)|}.$$

This implies that the number of affected people is at least

$$\frac{\ln(\alpha S_0)}{|\ln(1-\alpha r I_0)|} I_0.$$

If we assume $\alpha r I_0 \ll 1$ (this would be the case if r were of modest size and αS_0 would be modestly greater than 1, $I_0 \ll S_0$), then we could use the approximation $\ln(1-\alpha r I_0) \approx -\alpha r I_0$ (using the local linear approximation of ln at 1), therefore

$$\frac{\ln(\alpha S_0)}{|\ln(1-\alpha r I_0)|} I_0 \approx \frac{\ln(\alpha S_0)}{r(\alpha S_0)} S_0.$$

So now the total number of affected people is at least some constant (the constant being $\ln(\alpha S_0)/(r\alpha S_0)$) times S_0, not times I_0. So now not *a modest multiple of the initially infected* fall victim to the disease, but *at least a modest multiple of the initially susceptible*.

Write this argument up for yourself, verify each step. It's not a very pretty and precise argument — can you make it better?

3. We will now get into differential equations. In preparation, you could already read Chapter 2 in Strogatz's book, and there are plenty of exercises, any of them will be good to do.