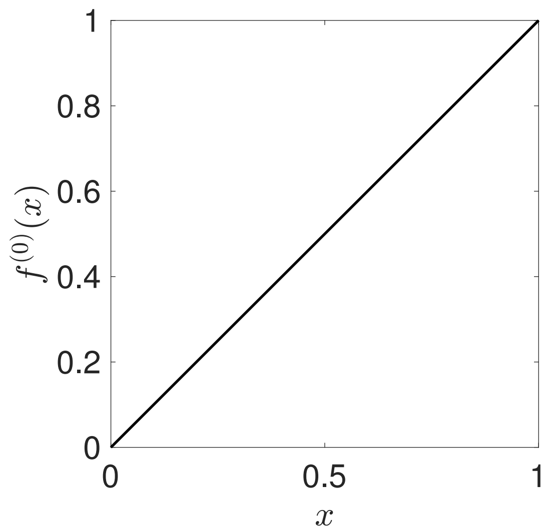
Tuesday, November 30

1. **Sets in $\mathbb{R}^N$ of similarity dimension $d < N$ are of measure zero.** (Thanks to Ethan's question.) Suppose that $S \in \mathbb{R}^N$ is a set that can be decomposed into n^d pieces that look exactly like S, scaled down (in all directions) by a factor of $1/n$. Let V be the N-dimensional volume of S. (Read "Lebesgue measure" if you know what that means, but "N-dimensional volume" will do here.) Explain:
$$V = n^d \frac{1}{n^N} V.$$
Conclude that $V = 0$ if $d < N$. So a set in $\mathbb{R}^N$ of similarity dimension $d < N$ is automatically a set of measure (N-dimensional volume) zero. Ethan, the answer to your question is "yes", and it does not "go beyond the scope of this course" at all.

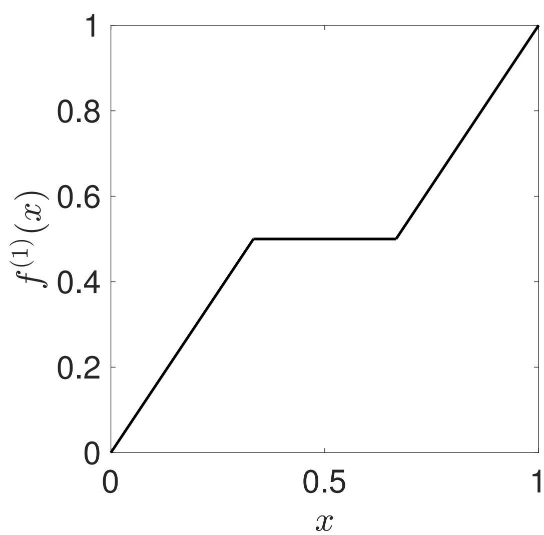
2. **Sets in $\mathbb{R}^N$ of measure zero need not have similarity dimension less than N..** Let S be the set of all *rational* numbers in $(0,1]$. (Rational numbers are ones that can be written as p/q where p and q are integers.)
   (a) Explain why this set has similarity dimension 1.
   (b) The elements of S are:
   $$\frac{1}{2}, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \cancel{\frac{2}{4}}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \cancel{\frac{2}{6}}, \cancel{\frac{3}{6}}, \cancel{\frac{4}{6}}, \frac{5}{6}, \ldots$$
   The point is, we can list them in a single list. (This is what "countably infinite" means. S is a countably infinite set.) Let's call the list
   $$S = \{s_1, s_2, s_3, s_4, s_5, \ldots\}.$$
   Let $\varepsilon > 0$. Notice that the intervals
   $$\left[s_k - \frac{\varepsilon}{2^{k+1}}, s_k + \frac{\varepsilon}{2^{k+1}} \right]$$
   cover S. What is the total length of those intervals? Why is S called a "set of measure zero"?

   So this example shows that a set of measure zero in $\mathbb{R}^N$ can have similarity dimension equal to N.

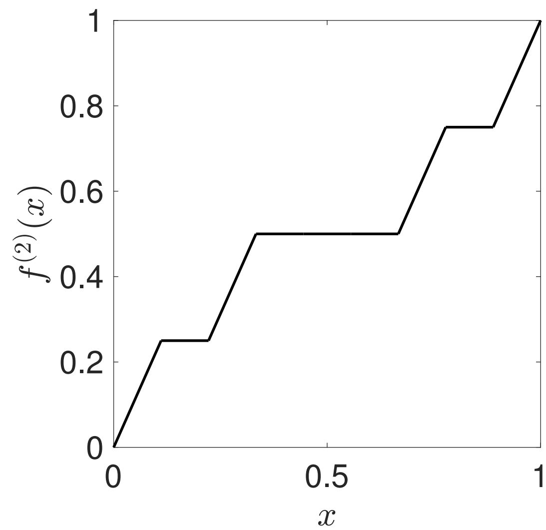
3. **The devil's staircase.** We will construct a continuous function
$$f : [0,1] \Rightarrow [0,1]$$
as follows. Begin with
$$f^{(0)}(x) = x.$$



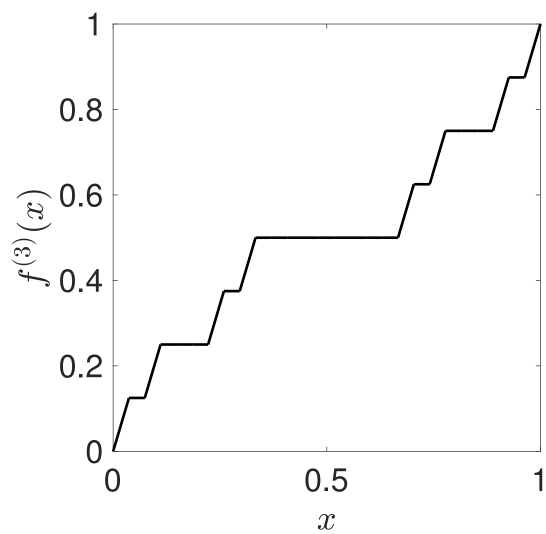
Then modify $f^{(0)}$ so that it becomes constant on the middle third $(1/3, 2/3)$, but still rises from 0 to 1. Call the result $f^{(1)}$.



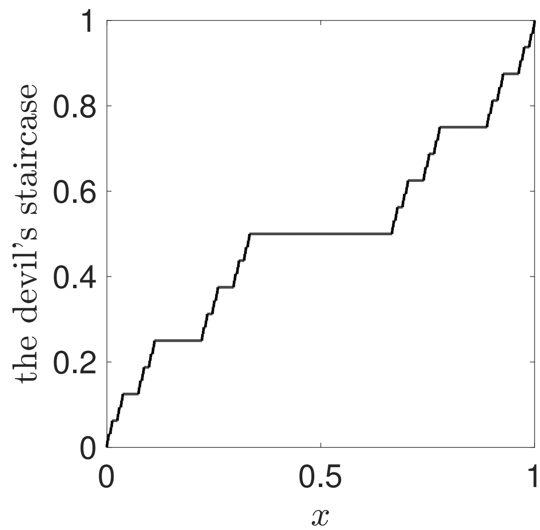
Then modify $f^{(1)}$ analogously to obtain $f^{(2)}$:



And do it again to get $f^{(3)}$:



Keep doing this. The $f^{(k)}$ converge to a continuous function called *the devil's staircase*:



Explain: $f'(x)) = 0$ for all x in the complement of the Cantor set. Nonetheless, f rises from 0 to 1 as x moves from 0 to 1. So the formula

$$\int_0^1 f'(x)dx = f(1) - f(0)$$

is invalid here, assuming that we consider the left integral to be zero, since $f'(x) = 0$ everywhere except in the Cantor set, which is a set of measure zero.