Math 70 Hw 3

Question 1 a) Let $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

By the rule of scalar multiplication, $0v = \begin{bmatrix} 0 \cdot v_1 \\ \vdots \\ 0 \cdot v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$0v = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$ Q.E.D

Question 1b) Let $\vec{v} = \begin{bmatrix} v_1 \\ v_n \end{bmatrix}$ By scalar multiplication, $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_n \end{bmatrix}$

Thus $\begin{bmatrix} cv_1 \\ cv_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives the system $cv_1 = 0, cv_2 = 0, \dots cv_n = 0$

Since $c \neq 0$, by rules of multiplication, $v_1 = 0, v_2 = 0, \dots v_n = 0$

This means $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$ $\boxed{\text{Q.E.D}}$

Question 2a) To show that if $\vec{w}$ is a unique linear combination then S is linearly independent, this means that $c_1\vec{v}_1 + c_2 v_2 + \dots c_n \vec{v}_n = \vec{0}$ $c_1 \dots c_n \in \mathbb{R}$ has only the trivial solution $c_1 \dots c_n = 0$

Since $\vec{w} \in \text{span}(S)$, $\vec{w} = c_1\vec{v}_1 + c_2 \vec{v}_2 + \dots c_n v_n$

where $c_1 \dots c_n \in \mathbb{R}$. This means the only way $\vec{0}$ can be written is with $c_1 \dots c_n = 0$, which shows that S is linearly independent

Question 2b) To prove $\vec{w}$ has a unique representation as a linear combination of S, let us assume there exists a second way of writing $\vec{w}$, using a as a coefficient.

$\vec{w} = c_1\vec{v}_1 + c_2 \vec{v}_2 + \dots c_n v_n$ $c_1 \dots c_n \in \mathbb{R}$

$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots a_n\vec{v}_n$ $a_1 \dots a_n \in \mathbb{R}$

$\vec{w} - \vec{w} = (c_1 - a_1)\vec{v}_1 + (c_2 - a_2)\vec{v}_2 + \dots (c_n - a_n)\vec{v}_n$

$\vec{0} = C(C_1 - a_1)\vec{v}_1 + (C_2 - a_2)\vec{v}_2 + \ldots C(C_n - a_n)\vec{v}_n$
Since $v_1 \ldots v_n$ are linearly independent, then the coefficients of $\vec{v}_1 \ldots \vec{v}_n = 0$.
Thus, $C_n - a_n = 0$, $C_n = a_n$. Since $C_n = a_n$ this means there is only one way of writing $\vec{u}$. Q.E.D

Question 3) Let $A = [\vec{a}_1\ \vec{a}_2\ \ldots\ \vec{a}_n]$ where $\vec{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

If the columns of A are linearly independent, Then $C_1\vec{a}_1 + C_2\vec{a}_2 + \ldots C_n\vec{a}_n = \vec{0}$ has only the trivial solution $C_1 \ldots C_n = 0$. In other terms, this mean the matrix A has no free variables, as a free variables means there are infinitely many solutions to $C_1\vec{a}_1 + C_2\vec{a}_2 + \ldots C_n\vec{a}_n = 0$. For A to have no free variables, A must have a pivot, in every column. If $n > m$, this is impossible, as only 3 pivot positions can exist in 4 columns. This means $\boxed{m \geq n}$ which would allow for a pivot in each column. QED

Question 4) Let $\vec{u} = e_1 + e_2 + e_3$, then, $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{u}\}$. To show S is linearly dependent, 1 vector must be written as a linear combination of the others.
$1 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2 + 1 \cdot \vec{e}_3 = \vec{u} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$ QED.

To show a set containing 3 of them is linearly independent, there are 2 cases we need to consider:
Case 1: The new set $S' = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$
Case 2: the new set S' contains $\vec{u}$ and is missing $\vec{e}_1$, $\vec{e}_2$, or e_3.

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Question 4 (cont) Case 1. $\vec{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

For $\vec{S}$ to be linearly independent, $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \vec{0}$ must only have the trivial solution $c_1, c_2, c_3 = 0$

To prove:

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$, meaning $c_1, c_2, c_3 = 0$.

QED

Case 2: WLOG, $\vec{S} = \{\vec{e}_1, \vec{e}_2, \vec{w}\}$ $\vec{w} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$

For $\vec{S}$ to be linearly independent, $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{w} = \vec{0}$, must only have $c_1 \ldots c_3 = 0$

To show:

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2 + c_3\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$, this yields the linear system.

$$\begin{cases} c_1 = 0 \\ c_2 + c_3 = 0 \\ c_3 = 0 \end{cases}$$

Since $c_3 = 0$, $c_2 = 0$

The solution to the system is $c_1, c_2, c_3 = 0$

Since $c_1, c_2, c_3 = 0$, $\vec{S}$ is linearly indepent

QED