Thursday October 28

1. **The time it takes to move past the ghost left behind by a saddle-node collision.** Think about
$$\frac{dx}{dt} = x^2 + \mu.$$
As μ passes from negative to positive, there is a saddle-node bifurcation. (Make sure you know what that means, explain it.) Suppose $\mu > 0$ but $\mu \approx 0$. Let $A > 0$ and $B > 0$. (A and B not necessarily the same, thanks to Max's question in class.)

   (a) Show that the time it takes for x to move from $-A$ to B is equal to
   $$T(\mu) = \frac{\arctan(A/\sqrt{\mu}) + \arctan(B/\sqrt{\mu})}{\sqrt{\mu}}.$$
   Hint: Notice that you can think of t as a function of x, rather than x as a function of t, and
   $$\frac{dt}{dx} = \frac{1}{x^2 + \mu}.$$

   (b) Show that
   $$T(\mu) \sim \frac{\pi}{\sqrt{\mu}} \quad \text{as } \mu \to 0+$$
   no matter what A and B are. By this, we mean precisely:
   $$\lim_{\mu \to 0+} \sqrt{\mu}\, T(\mu) = \pi.$$

2. **Another differential equation exhibiting excitability.** Think about
$$\frac{dv}{dt} = v(v - \theta)(1 - v)$$
where $0 < \theta < 1$ is a fixed parameter. (I now call the unknown function $v = v(t)$ instead of $x = x(t)$ because I am thinking of the voltage inside a neuron.) Find the fixed points. Which are stable, and which are unstable? Why is this another example of "excitabiliy"? Explain: If v is raised from rest past the threshold (what do "rest" and "threshold" mean here?), then v rises further to a high value (which value?) and stays there.

3. **The FitzHugh-Nagumo model of a nerve cell.** A neuron has the ability to raise its voltage v to high values, but v quickly comes back to low values. In the differential equation of problem 2, there is no mechanism to accomplish that. But it's easy to introduce one:
$$\frac{dv}{dt} = v(v - \theta)(1 - v) - w, \tag{1}$$
$$\frac{dw}{dt} = \varepsilon(\alpha v - w), \tag{2}$$

where $\varepsilon > 0$ and $\alpha > 0$ are parameters. You should think of ε as small. The variable w models some slower process (because ε is small and appears on the right-hand side of (2), dw/dt is small) which brings v back (because w appears with a minus sign in (1), it tends to bring v down when it is positive). In a nerve cell, that might be the opening of potassium channels, which let potassium out of the nerve cell, thereby lowering the voltage inside because potassium ions carry positive charge.

We'll add one more thing: A "driving term", that is, a term that drives v towards higher values. Think of it as a current that is pumped into a nerve cell — either by an experimenter, or a current carried by positive ions flowing from the extracellular fluid into the nerve cell, as happens in normal brain function all the time. We call that "driving term" I, since I is a letter often used for electrical currents. So now the system is

$$\frac{dv}{dt} = v(v-\theta)(1-v) - w + I, \tag{3}$$

$$\frac{dw}{dt} = \varepsilon(\alpha v - w), \tag{4}$$

where $\theta \in (0,1)$, $\alpha > 0$, $\varepsilon > 0$, and $I > 0$ are parameters.

Equations (3) and (4) are known as the *FitzHugh-Nagumo equations.*[1] They represent the simplest model of a nerve cell. The first step in understanding such a system is to think about fixed points.

(a) Explain: (v_*, w_*) is a fixed point of (3), (4) (this means, a solution that is independent of t) if and only if

$$v_*(v_* - \theta)(1 - v_*) - \alpha v_* + I = 0 \quad \text{and} \quad w_* = \alpha v_*.$$

(b) Show: There is at least one fixed point, and there could be up to three, depending on the parameters (but no more than three).

(c) Show: If v were constant, then any solution of (4) would converge to αv as $t \to \infty$. So (3) drives w towards αv.

To be continued next week!

[1] Richard FitzHugh suggested the model in 1961 to capture the essence of a nerve cell. Independently, Jin-Ichi Nagumo and collaborators proposed the same model in 1962, and built an electrical circuit described by the system.