MATH 42 HOMEWORK 9

This homework is due at 11:59 p.m. (Eastern Time) on Tuesday, November 24. Scan the completed homework and upload it **as one pdf file** to Gradescope. The Canvas module "Written Assignments" has instructions for how to upload the assignment to Gradescope. This assignment covers §17.4−17.6.

Be sure to show work (integration by parts, substitutions, etc.) when calculating integrals. Unless stated in the problem, it is insufficient to simply respond with a numerical evaluation of definite integrals or an antiderivative of a non-standard integrand.

(1) Use Green's Theorem to:
   (a) Find the area of the region bounded by $\mathbf{r}(t) = \langle t(1-t^2), 1-t^2\rangle$ for $t \in [-1, 1]$. Plot the curve and be cognizant of the orientation.
   (b) Find the area of the astroid $x^{2/3} + y^{2/3} = 4$. Plot the curve. Hint: Show that the parameterization $x = 8\cos^3 t, y = 8\sin^3 t$ for $t \in [0, 2\pi)$ works.
   (c) Evaluate $\displaystyle\oint_C 2x\,\mathrm{d}y - 3y\,\mathrm{d}x$ where C is the square with vertices $(0, 2), (2, 0), (-2, 0)$, and $(0, -2)$, oriented counterclockwise.

   Important: If you used any curve plotting software tools to help you solve the problem above, please list them.

(2) Without finding a potential function, calculate $\displaystyle\int_C e^x \cos y\,\mathrm{d}x - e^x \sin y\,\mathrm{d}y$ where C is the curve (not closed) from $(\ln 2, 0)$ to $(0, 1)$ to $(-\ln 2, 0)$. Hint: Consider the straight-line closing of the curve and apply Green's Theorem.

(3) A three-dimensional *central force*, such as gravity or Coulomb attraction, may be written as $\mathbf{F}(r) = f(r)\mathbf{r} = f(r)\langle x, y, z\rangle$, where f is some scalar-valued function of the magnitude r. Show that the curl of any central force is irrotational, $\nabla \times \mathbf{F} = \vec{0}$.

(4) Let $f(x, y, z)$ be a scalar-valued function and let $\mathbf{A}$ and let $\mathbf{B}$ be vector fields on $\mathbb{R}^3$. Prove the following identities:
   (a) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$.
   (b) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$.
   (c) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$.

(5) Compute the surface integral $\displaystyle\iint_S xyz\,\mathrm{d}S$, where S is the portion of the plane $z + y = 6$ that lies inside the cylinder $x^2 + y^2 = 4$.

(6) Let the vector $\mathbf{r}$ denote the vector between the origin and (x, y, z). Let p be a real number and $|\mathbf{r}| = r$. Where possible, simplify work by performing the calculation for one coordinate and arguing that the other two coordinates follow a similar pattern.
   (a) Show that $\nabla(1/r^p) = \frac{-p\mathbf{r}}{r^{p+2}}$.
   (b) Show that $\nabla \cdot (\mathbf{r}/r^p) = \frac{3-p}{r^p}$.
   (c) Show that $\nabla \cdot \nabla(1/r^p) = \frac{p(p-1)}{r^{p+2}}$. Hint: Use the previous parts.
   (d) Sketch the vector field $-\nabla(1/r) = \hat{\mathbf{r}}/r^2$ and show that its divergence, $\nabla \cdot (\hat{\mathbf{r}}/r^2)$, is zero.

(e) Remark on the vector field drawn compared to the divergence calculated.
(f) Compute, $\iint_{\text{sphere}} r^{-2}\,\mathrm{d}S$, the surface integral over an origin-centered sphere of radius R. Is the value obtained the same for any $R > 0$?

See below for some remarks on the importance of this computation.[1]

[1]The calculation in part (c) tells us the divergence of a vector field given by the gradient of the scalar field $1/r^p$. For $p = 1$, the divergence of the vector field is zero almost everywhere (except at the origin) yet the surface integral over any sphere is non-zero and constant. This suggests that something exciting happens at the origin. One example of what we have encountered is a single point in space (with no dimensional extent) that is endowed with electric charge, referred to as a *point charge*.

The potential function for an electric point charge q at the origin is $\phi = (4\pi\epsilon_0)^{-1}(q/r)$ and the force $\mathbf{F} = -\nabla\phi \propto r^{-2}$. Likewise $\nabla \cdot \mathbf{F} = \rho/\epsilon_0$ where $\rho(x, y, z)$ is the electric charge density – so everywhere, except for the exact location of the point charge, the divergence is zero. This indicates that a point charge is a source/sink of electric field.

Recall that all of these computations were done assuming three-dimensional space: The answer to part (c) disappears only when $p = 1$ so there is something special about nature's choice to have many forces follow inverse-square laws. One view is that this phenomenon is a geometric manifestation of the dimensions of space in which our world resides. Perhaps some physical laws would need modification if more complicated understandings of the universe's geometry were considered…