MATH 123   Lecture 14

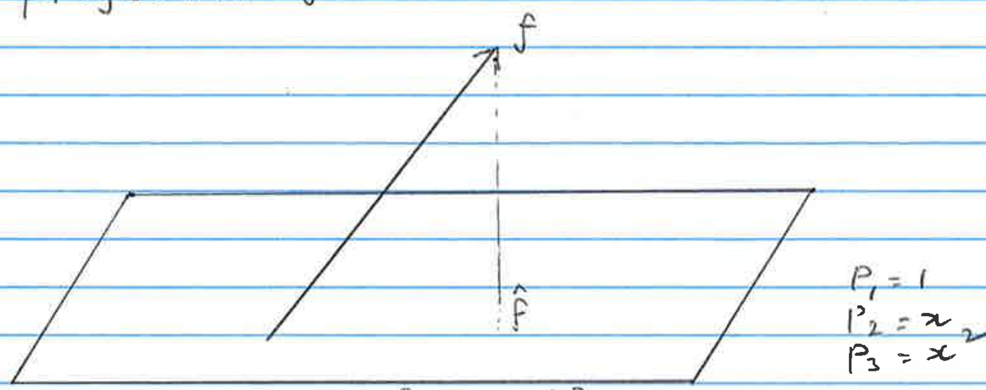
- In the last lecture, we started asking the question: "What is the "nearest" element of W closes to $f(x) = \sin(x)$? ($W \equiv \text{span}\{1, x, x^2\}$)
- We formalize this problem in this lecture.

<u>Example</u> $W \equiv \text{span}\{1, x, x^2\}$

Is W a linear space?

Let $f(x) = \sin(x)$

By best approximation theorem, the closest element to f in W is its orthogonal projection $\hat{f}$.



$W \equiv \text{span}\{1, x, x^2\}$

Note $f - \hat{f} \perp z$ $z \in W$ (For any $z \in W$)

Therefore

$$f - \hat{f} \perp P_1 = 0 \implies \langle f - \hat{f}, P_1 \rangle = 0 \quad (1)$$
$$f - \hat{f} \perp P_2 = 0 \implies \langle f - \hat{f}, P_2 \rangle = 0 \quad (2)$$
$$f - \hat{f} \perp P_3 = 0 \implies \langle f - \hat{f}, P_3 \rangle = 0 \quad (3)$$

Since $\hat{f} \in W$, $\hat{f} = c_1 P_1 + c_2 P_2 + c_3 P_3$ for some $c_1, c_2, c_3 \in \mathbb{R}$

using (1), we have

$$\langle \hat{f}, P_1 \rangle = c_1 \langle P_1, P_1 \rangle + c_2 \langle P_2, P_2 \rangle + c_3 \langle P_3, P_1 \rangle = \langle f, P_1 \rangle$$

<u>orthogonal polynomials</u>: Assume $\{P_1, P_2, P_3\}$ is an orthogonal set i.e. $\langle P_i, P_j \rangle = 0$ if $i \neq j$

$$\langle \hat{f}, P_1 \rangle = c_1 \langle P_1, P_1 \rangle + 0 + 0 = \langle f, P_1 \rangle$$

$$\therefore \quad c_1 = \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle}$$

Similarly $c_2 = \dfrac{\langle f, P_2 \rangle}{\langle P_2, P_2 \rangle}$ and $c_3 = \dfrac{\langle f, P_3 \rangle}{\langle P_3, P_3 \rangle}$

Let's now write a more general statement

Theorem Let $W \equiv \text{span}\{p_1, p_2, \ldots p_k\}$ where the set $\{p_1 \ldots p_k\}$ is orthogonal. Given a function f, the closest element to f in W is

$$\hat{f} = \frac{\langle f, p_1\rangle}{\langle p_1, p_1\rangle} p_1 + \frac{\langle f, p_2\rangle}{\langle p_2, p_2\rangle} p_2 + \ldots + \frac{\langle f, p_k\rangle}{\langle p_k, p_k\rangle} p_k$$

$$= \sum_{i=1}^{k} \frac{\langle f, p_k\rangle}{\langle p_k, p_k\rangle}$$

Exercise Find least square approximation of $\sqrt{x}$ on $[0, 1]$ using $\phi_0(x) = 1$ and $\phi_1(x) = (x - 1/2)$
$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$

Solution

First, we note that

$$\langle \phi_0, \phi_1\rangle = \int_0^1 (x - 1/2)\, dx = \frac{x^2}{2} - \frac{1}{2}x \Big|_0^1 = 0$$

Therefore, ϕ_0 and ϕ_1 are orthogonal

Let $f(x) = \sqrt{x}$

$$\hat{f} = \underbrace{\frac{\langle f, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle}}_{c_0} \phi_0 + \underbrace{\frac{\langle f, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle}}_{c_1} \phi_1$$

c_0: $\langle f, \phi_0\rangle = \int_0^1 \sqrt{x} \cdot 1\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = 2/3$

$\langle \phi_0, \phi_0\rangle = \int_0^1 1\, dx = x \Big|_0^1 = 1$

Therefore, $c_0 = \frac{2}{3}/1 = 2/3$

c_1: $\langle f, \phi_1\rangle = \int_0^1 x^{1/2}(x - 1/2)\, dx = \frac{2}{5} x^{5/2} - \frac{1}{3} x^{3/2} \Big|_0^1 = \frac{2}{5} - \frac{1}{3} = 1/15$

$\langle \phi_1, \phi_1\rangle = \int_0^1 (x - 1/2)^2\, dx = \int_0^1 \left(x^2 + \frac{1}{4} - x\right) dx$

$$= \frac{x^3}{3} + \frac{1}{4}x - \frac{x^2}{2} \Big|_0^1 = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$$

Therefore, $c_1 = \dfrac{\frac{1}{15}}{\frac{1}{12}} = 4/5$

$$\therefore \hat{f} = \frac{2}{3} + \frac{4}{5}(x - 1/2)$$

What if $\{p_1, \ldots p_k\}$ is not orthogonal set?
In that case, we have

$$\langle f, p_1 \rangle = c_1 \langle p_1, p_1 \rangle + c_2 \langle p_2, p_1 \rangle + \cdots + c_k \langle p_k, p_1 \rangle$$
$$\vdots$$
$$\langle f, p_k \rangle = c_1 \langle p_1, p_k \rangle + c_2 \langle p_2, p_k \rangle + \cdots + c_k \langle p_k, p_k \rangle$$

Define $G \in \mathbb{R}^{k \times k}$ as follows

$$G = \underbrace{\begin{pmatrix} \langle p_1, p_1 \rangle & \langle p_2, p_1 \rangle & \cdots & \langle p_k, p_1 \rangle \\ \vdots & \vdots & & \vdots \\ \langle p_1, p_k \rangle & \langle p_2, p_k \rangle & \cdots & \langle p_k, p_k \rangle \end{pmatrix}}_{\text{Gram matrix}} \qquad G_{ij} = \langle p_i, p_j \rangle$$

Therefore,

$$\begin{pmatrix} \langle f, p_1 \rangle \\ \langle f, p_2 \rangle \\ \vdots \\ \langle f, p_k \rangle \end{pmatrix} = G \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \Rightarrow$$ To find $c_1, c_2, \ldots c_k$ solve the linear system

Exercise Is G always invertible?

Uniqueness When does least square solution give us a unique answer?
Recall that we solve $A^T A x = A^T y$ where A is $m \times n$

claim $\text{Null}(A) = \text{Null}(A^T A)$

proof $\Rightarrow$ $x \in \text{Null}(A)$

$A x = \vec{0}$

$A^T A x = \vec{0} \Rightarrow x \in \text{Null}(A^T A)$

$\Leftarrow$ $x \in \text{Null}(A^T A)$

$A^T A \vec{x} = \vec{0}$

$x^T A^T A x = 0$

$(Ax)^T A x = 0 \Rightarrow \|Ax\|_2^2 = 0 \Rightarrow Ax = 0 \Rightarrow x \in \text{Null}(A)$

$\text{rank}(A) + \text{nullity}(A) = n$
$\text{rank}(A^T A) + \text{nullity}(A^T A) = n$ $\longrightarrow$ $\text{rank}(A) = \text{rank}(A^T A)$

dimension theorem

$A^T A x = A^T y$ has a unique solution if $A^T A$ is invertible

$\Rightarrow A^T A$ has rank n $\Rightarrow$ A has rank n

$\therefore$ The columns of A must be linearly independent to obtain unique solutions

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Example $\quad p(t) = a + bt$

$$A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_n \end{bmatrix}$$

columns of A are linearly independent unless $t_1 = t_2 = \dots = t_n$

Example $\quad p(t) = a + bt + ct^2$

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 \end{bmatrix}$$

consider $Az = 0$

$$\left.\begin{array}{l} z_1 + t_1 z_2 + t_1^2 z_3 = 0 \\ \vdots \\ z_1 + t_n z_2 + t_n^2 z_3 = 0 \end{array}\right] *$$

t_1 is a root of the polynomial $z_1 + z_2 t + z_3 t^2$

If we have $n \geq 3$ distinct points, the only way $*$ could hold is if $z_1 = z_2 = z_3 = 0$ (otherwise it would mean that a quadratic equation has n zeros).

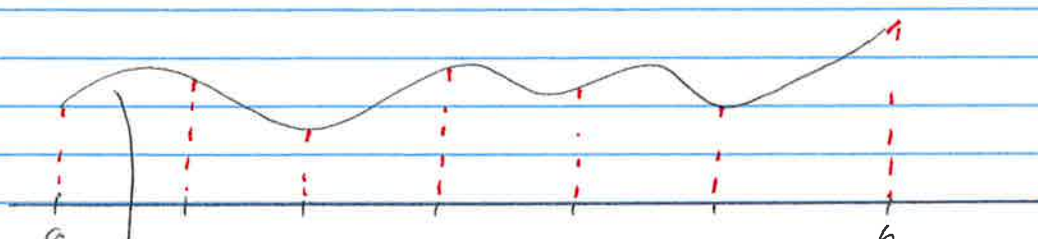
$\therefore$ A also has linearly independent columns

Numerical integration

$$I = \int_a^b f(x)\,dx$$

$f(x)$ might be difficult or expensive to integrate exactly

Idea



- Interpolate by polynomial
- Integrate the polynomial

(Different choices lead to different methods)