Tufts University
Department of Mathematics

Math 135 | **Homework 2 v2**[1] | **Fall 2022**

(Changes in red)

Readings for Problem Set 2

Notes:: Cardinality and Cantor's diagonal argument
§**1.2:** Completeness
§**1.3:** Inequalities and identities
§**2.1:** The convergence of sequences

Readings for Problem Set 3

§**2.1:** Uniqueness of a limit, limit properties
§**2.2:** Boundedness, sequential density
§**2.3:** The monotone convergence theorem

Problem Set 2
(Due Wednesday, September 21, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

To prove that a number is the sup of a set S, You may use the following *ε-criterion for sup*.

Theorem. *Suppose $A \subset \mathbb{R}$ is a nonempty set that is bounded above. Then a real number b is $\sup A$ if and only if b is an upper bound for A and for every $\varepsilon > 0$, there is an element $a \in A$ such that $a > b - \varepsilon$.*

1. (10 points) (**Negation, inequality**)
   (a) Negate the statement:
   $$\forall \varepsilon > 0,\ \exists \delta > 0 \textit{ such that } \forall x, y \in D, \textit{ if } |x-y| < \delta, \textit{ then } |f(x) - f(y)| < \varepsilon.$$
   (b) Prove that for all $a, b \in \mathbb{R}$,
   $$|\,|a| - |b|\,| \leq |a-b|.$$
   (*Hint*: Write $a = (a-b) + b$ and apply the triangle inequality.)

2. (10 points) (**Union of countable sets**) Suppose that $A_1, A_2, A_3, \ldots$ are countably infinite sets. Show that their union $A_1 \cup A_2 \cup A_3 \cup \cdots$ is countable.

3. (10 points) (**Uncountable set**) Let
   $$S = \{x \in [0,1] \mid x = .a_1 a_2 a_3 \cdots \text{ in base 10 decimal expansion}\},$$
   where $a_i = 0, 1$, or $2\}$. Prove that S is not countable. (*Hint*: Mimic Cantor's diagonal argument.)

4. (10 points) (**Definition of convergence**) Using the ε-N definition of convergence, prove that the sequence $\left\{\dfrac{n^2+2n+1}{n^2+n-1}\right\}$ converges. (In this problem, you may not use limit rules or the sandwich lemma in your proof. you must start with an $\varepsilon > 0$ and try to find an $N \in \mathbb{N}$. In your scratch work, you may use anything you like. First divide both the numerator and the denominator by n^2, the highest power of n, to guess the limit.)

5. (10 points) (**Limit of a sequence**) Show that $\dfrac{2^k}{k!} \to 0$. (*Hint*: use the comparison lemma (sandwich lemma).)

6. (10 points) (**Completeness axiom**) Does the set $\mathbb{Z}$ satisfy the completeness axiom? That is, if S is a nonempty subset of $\mathbb{Z}$ that is bounded above, is there an element of $\mathbb{Z}$ that is $\sup(S)$? Prove your result. (*Hint*: You may use the completeness axiom for $\mathbb{R}$.)

7. (15 points) (**Sup, inf, max, min**) Let $S = \left\{4 - \dfrac{2}{\sqrt{n}} \,\middle|\, n \in \mathbb{N}\right\}$.
   (a) Find $\sup(S)$ and prove your result.
   (b) Does S have a maximum (i.e., an element $s \in S$ that is an upper bound for S)?
   (c) Find $\inf(S)$ and prove your result.
   (d) Does S have a minimum (i.e., an element $s \in S$ that is a lower bound for S)?

(End of Problem Set 2)