HW 4

3.2.19) So first can extend $f(x)$ to $\mathbb{R}$ by saying $f(x)=0$ for $x \notin I$.

Now, define $f_n(x) = \dfrac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}$

Clearly each f_n is measurable, as f is measurable, and difference of 2 measurable functions is measurable.
Also measurable over E, as E is measurable.
It follows $\lim_{n\to\infty} f_n(x) = f'(x)$ so by 3.2.10 d), $f'(x)$ is measurable on E

3.2.20) So, for $a \in \mathbb{R}$ we have
$\{f \circ \varphi > a\} = (f \circ \varphi)^{-1}(a,\infty] = \varphi^{-1}(f^{-1}(a,\infty])$
Now, $f^{-1}(a,\infty]$ is measurable as f is a measurable function.
Furthermore, as φ^{-1} is Lipschitz, by 2.3.13, it maps measurable sets to measurable sets.
Therefore, φ^{-1} is going to be measurable on $f^{-1}(a,\infty]$
So, $\forall a \in \mathbb{R}$ $\{f \circ \varphi > a\}$ is measurable, so it follows $f \circ \varphi$ is measurable function.

3.2.21)
a) Let $A_n = \{|f| > n\}$ so $A = \{|f| = \infty\}$
A_n is measurable as f is measurable
As $|E| < \infty$ then $|A| = 0$
Also, $A_1 \supseteq A_2 \supseteq A_3 \dots$ So define $\bigcap A_n = A$
By continuity from above we have
$\left|\bigcap_{n=1}^{\infty} A_n\right| = \lim_{n\to\infty} |A_n| = 0$ so $|A_n| \to 0$.

Therefore $\exists n \in \mathbb{N}$ s.t. $\forall \varepsilon > 0$ $|A_n| < \varepsilon/2$

Con on next pg.

Now, by construction on $E \setminus A_n = \{|f|<n\}$ is bounded and measurable, so it contains a closed set. F s.t.

$$|F| > |E \setminus A_n| - \epsilon/2 = |E| - |A_n| - \epsilon/2 = |E| - \epsilon$$

So $|F| > |E| - \epsilon \Rightarrow |E| - |F| < \epsilon \rightarrow |E \setminus F| < \epsilon$
where $F \subseteq E$ closed and by construction f is bdded on F.

b) $f(x) = \sup_{x \in \mathbb{N}} |f_n(x)|$

$E_n = \{f(x) > k\}$ and so f is measurable
as each f_n is measurable by 3.2.7

Now $f(x) = \sup_{n \in \mathbb{N}} |f_n(x)| = M_x < \infty$

So f is finite everywhere so bded. Therefore, by a), we can pick an closed $F \subseteq E$ s.t. $|E \setminus F| < \epsilon$ and f is bded on F. So can say $f \leq M$ which implies $|f_n(x)| \leq f(x) \leq M \quad \forall n \in \mathbb{N}$ and $x \in F$.

3.3.9) As index by $\mathbb{R}$, $\{f_a\}_{a\in\mathbb{R}}$ is uncountable.

There are two cases. First, where WLOG $b > a+1$ so $[a, a+1] \cap [b, b+1] = \emptyset$

Now, that means $\forall x \in \mathbb{R}$ if $f_a(x) = 1$ $f_b(x) = 0$ and vice versa.

So $\|f_a - f_b\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f_a(x) - f_b(x)|$

For $x \in [a, a+1]$ $|f_a(x) - f_b(x)| \le |1-0| = 1$

For $x \in [b, b+1]$ $|f_a(x) - f_b(x)| \le |0-1| = 1$

for otw. x $|f_a(x) - f_b(x)| \le |0-0| = 0$

As $|[a, a+1]|$ and $|[b, b+1]| = 1$ then it follows $|f_a(x) - f_b(x)| \le 1$ for $\mathbb{R}$ a.e $\in \mathbb{R}$

So $\|f_a - f_b\|_\infty = 1$

Now, assume WLOG $b > a$ but s.t $[a, a+1] \cap [b, b+1] \neq \emptyset$

Now, $\|f_a - f_b\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f_a(x) - f_b(x)|$

Now, for $x \in [a, b]$ $|f_a(x) - f_b(x)| \le |1-0| = 1$

For $x \in (b, a+1)$ $|f_a(x) - f_b(x)| \le |0-0| = 0$

For $x \in [a+1, b+1]$ $|f_a(x) - f_b(x)| \le |0-1| = 1$

For else: $|f_a(x) - f_b(x)| \le |0-0| = 0$

Now, as $a \neq b$, and WLOG $b > a$, $\exists r \in \mathbb{R}^+$ s.t $b - a = r > 0$ and so $|[a, b]| = r$ and $|[a+1, b+1]| = r$. Therefore, as these sets are positive measure, then $|f_a(x) - f_b(x)| \le 1$ x a.e in $\mathbb{R}$

So $\operatorname*{ess\,sup}_{x\in\mathbb{R}} |f_a - f_b| = 1$ and $\|f_a - f_b\|_\infty = 1$

3.4.6

a) the shrinking triangle example talked abt in class as it is shown to converge almost uniformly as remove $[0, \delta]$ for some $\delta > 0$ and then it converges uniformly on $[\delta, 1]$ so converges almost everywhere

Book states this doesn't converge in L^∞

b) So from class, $f_n(x) = \chi_{[n,\infty)}$ is a sequence that converges pointwise a.e to $f(x) = 0$.

However, doesn't converge almost uniformly to 0, as can not remove a finite subinterval that makes each $f_n \to 0$ a.e. as interval is going from $n \to \infty$