Tufts University
Department of Mathematics

Due date: 11:59 pm, Sunday, March 12, 2023 on Gradescope.

You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

A proper subset of the problems will be selected for grading.

Reading assignment: For this week, please read sections 18.1 and 18.2, integration in $\mathbb{R}^n$ and the statement that continuous functions are integrable on rectangles. This homework focusses on integration in $\mathbb{R}$

Problems:

1. (20 points) Let $a < b$ and $c \in (a, b)$. Assume $f : [a, b] \to \mathbb{R}$ is bounded and integrable on both $[a, c]$ and on $[c, b]$. Prove that f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$. Note that this is the converse of the theorem on additivity of the integral over intervals (Theorem 6.12) from the book.

2. (20 points) Let $f : [a, b] \to \mathbb{R}$ be bounded.

   (a) Assume f is continuous on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ *and* assume $f(x_0) > 0$ for some $x_0 \in [a, b]$. Prove that $\int_a^b f > 0$.

   (b) Is the conclusion in part (a) true if one assumes f is integrable on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ *and* assume $f(x_0) > 0$ for some $x_0 \in [a, b]$? If so, prove it, and if not, provide a counterexample

3. (20 points)Let f and g be bounded functions from $[a, b]$ to $\mathbb{R}$. Assume g is integrable and $f = g$ except at a finite number of points in $[a, b]$. In this problem, you will prove that f is integrable and $\int_a^b f = \int_a^b g$.

   (a) First prove that a function that is zero except at a has integral zero: Let $r \in \mathbb{R}$ constants. Define $f : [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} r & x = a \\ 0 & x \in (a, b] \end{cases}$. Prove f is integrable and $\int_a^b f = 0$. A similar proof can be used to prove $\int_a^b f = 0$ if f is equal to zero except at b.

   (b) Now, assume f is zero except for a finite number of points in $[a, b]$. Explain why $\int_a^b f = 0$. HINT: Problem 1 and part (a) could be helpful in one proof.

   (c) Let f and g be functions from $[a, b]$ to $\mathbb{R}$. Assume g is integrable, and assume $f = g$ except at a finite number of points. Prove that f is integrable on $[a, b]$ and $\int_a^b f = \int_a^b g$.

4. (20 points) Let $f : [0, 2] \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$ for $x \in [0, 2]$. Prove that f is integrable on $[0, 2]$ and find the integral using the definition of the integral or by using the Archimedean Riemann Theorem.
   HINT: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

The last problem and two extra credit problems are on the next page.

5. (20 points) Let f be an integrable function from $[a,b]$ to $\mathbb{R}$. In this problem, you will show that the absolute value of f, $|f|$, is integrable and $\left|\int_a^b f\right| \le \int_a^b |f|$.

   (a) Show $|f|$ is integrable
   You may assume that for every partition P of $[a,b]$,

$$0 \le U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P). \tag{1}$$

   (b) Now show $\left|\int_a^b f\right| \le \int_a^b |f|$.

Here are optional challenge problems that will give you extra points if you successfully do them. Todd will grade them.

1. (3 points extra credit) Prove the inequality (1) for arbitrary $f : [a,b] \to \mathbb{R}$.
   In one proof, it helps to first prove for any set $I \subset [a,b]$ and all $x \in I$, $y \in I$ that $f(x) - f(y) \le \sup_I f - \inf_I f$ and that $f(y) - f(x) \le \sup_I f - \inf_I f$.
2. (2 points extra credit) We know if f and g are integrable that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. Prove or find counterexamples to the following.
   (a) If f and g are arbitrary bounded functions is it true that $\overline{\int}_a^b (f+g) = \overline{\int}_a^b f + \overline{\int}_a^b g$?
   (b) If $f$and g are arbitrary bounded functions is it true that $\underline{\int}_a^b (f+g) = \underline{\int}_a^b f + \underline{\int}_a^b g$?