1 First, need to show that π respects the equivalence relation. To show, let $\bar{f}: Q \to Q$ or $\bar{f} = id_Q$. By the UPQ $\bar{f} \circ \pi = id_Q \circ \pi = \pi$ so π satisfies the equivalence relation. The same can be said for π' and Q'.

$\pi: X \to Q$ and $\pi': X \to Q'$ let $f: X \to Q'$ as in UPQ, B is arbitrary. So, by UPQ $\exists \bar{f}$ s.t. $\pi' = \bar{f} \circ \pi$ like

$$\begin{tikzcd} X \arrow[r, "\pi'"] \arrow[d, "\pi"'] & Q' \\ Q \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

This diagram commutes so use.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\pi'"'] & Q \\ Q' \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

By UPQ, $\exists \bar{g}$ s.t. $\pi = \bar{g} \circ \pi'$

Since $\pi' = \bar{f} \circ \pi = \bar{f} \circ \bar{g} \circ \pi'$ so $\bar{f} \circ \bar{g} = id_{Q'}$

Can show the same way $\bar{g} \circ \bar{f} = id_Q$. So $\bar{f}$ and $\bar{g}$ are inverses and by UPQ is unique, so get $\bar{f}$ is the unique bijection $Q \to Q'$ where $f \circ \pi = \pi'$.

2 From the diagram $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$ Under the rule $x \mapsto x$, and as $X_1 \cap X_2 \subseteq X_1$ and $X_1 \cap X_2 \subseteq X_2$ then j_1 and J_2 have the same image and the same can be said for i_1, i_2, So $i_1 \circ j_1 = i_2 \circ J_2$. To show uniqueness let $g: X_1 \cup X_2 \to B$ so

$x \mapsto x$

$f_2 = g \circ i_2$ but but as $x \mapsto x$ and $f_2 = f \circ i_2$ $f \circ i_2 = g \circ i_2$, so $f = g$ and f is unique.

3a) $\frac{x}{1} = x\left(\frac{1}{1}\right) = x \cdot 1 = x$ so $\frac{x}{1} = x$

b) Contrapositive: If $xy = 0$ then x or $y = 0$

Case 1)

$x \neq 0$ $\frac{1}{x}(xy = 0) \Rightarrow \frac{1}{x}(xy) = \frac{1}{x} \cdot 0 \rightarrow \left(\frac{1}{x} \cdot x\right)y = 0$

so as $\frac{1}{x} \cdot x$ $1 \cdot y = 0$ and $y = 0$ so if $x \neq 0$ $y = 0$

Case 2)

$y \neq 0$ $\frac{1}{y}(xy) = 0 \rightarrow \frac{1}{y}(yx) = \frac{1}{y} \cdot 0 \rightarrow \left(\frac{1}{y} \cdot y\right)x = 0$

So $1 \cdot x = 0$ and $x = 0$ so if $y \neq 0$ $x = 0$

(The 3) other case is $x = 0, y = 0$.

Therefore as contrapositive holds, the statement $x \neq 0$ and $y \neq 0 \rightarrow xy \neq 0$ holds

c) $1 = 1$ As $yz, y, z \neq 0$ have inverses.

so $yz \cdot \frac{1}{yz} = \frac{1}{y} \cdot y \cdot \frac{1}{z} \cdot z$

$\rightarrow yz \cdot \frac{1}{yz} = y \cdot z \cdot \frac{1}{y} \cdot \frac{1}{z}$ by commutativity and as $yz \neq 0$

$= \frac{1}{yz} = \frac{1}{y} \cdot \frac{1}{z}$ $\square$

d) $w\left(\left(\frac{1}{y}\right)\left(\frac{1}{z}\right) = \left(\frac{1}{y}\right)\left(\frac{1}{z}\right)\right)$ Assume $y, z \neq 0$

$\rightarrow w\left(\frac{1}{y}\right)\left(\frac{1}{z}\right) = w\left(\frac{1}{y}\right)\left(\frac{1}{z}\right)$ by def of division

$\rightarrow x\left[\left(\frac{1}{y}\right)\left(\frac{w}{z}\right) = \left(\frac{w}{y}\right)\left(\frac{1}{z}\right)\right]$

$\Rightarrow x\left(\frac{1}{y}\right)\left(\frac{w}{z}\right) = x\left(\frac{w}{y}\right)\left(\frac{1}{z}\right)$

$\rightarrow x\left(\frac{1}{y}\right)\left(\frac{w}{z}\right) = \left(\frac{w}{y}\right)x\left(\frac{1}{z}\right)$

so $\left(\frac{x}{y}\right)\left(\frac{w}{z}\right) = \left(\frac{w}{y}\right)\left(\frac{x}{z}\right)$ $\square$

e) Since $x \neq 0$, it has an inverse $\frac{1}{x}$. If $\frac{1}{x} = 0$ then $x\left(\frac{1}{x}\right) = x \cdot 0$, $1 = 0$ which isn't true so $\frac{1}{x} \neq 0$

3 f) $w, z \neq 0$ $\frac{1}{(\frac{w}{z})}$ has an inverse $\frac{w}{z}$ so

$$\left(\frac{1}{(\frac{w}{z})} \cdot \left(\frac{w}{z}\right) = 1\right) \frac{z}{w}$$

$$\frac{1}{(\frac{w}{z})} \cdot \frac{w}{z} \cdot \frac{z}{w} = \frac{z}{w}$$

$$\frac{1}{\frac{w}{z}} \cdot \frac{w}{w} \cdot \frac{z}{z} = \frac{z}{w} \rightarrow \frac{1}{\frac{w}{z}} \cdot 1 = \frac{z}{w}$$

so $\frac{1}{\frac{w}{z}} = \frac{z}{w}$

g) $xy \left(\frac{1}{z} = \frac{1}{z}\right)$

$xy\left(\frac{1}{z}\right) = xy\left(\frac{1}{z}\right)$ by def. of division

$\frac{xy}{z} = xy\left(\frac{1}{z}\right)$ by associativity

$\frac{xy}{z} = x\left(y \cdot \frac{1}{z}\right) \rightarrow \frac{(xy)}{z} = x\left(\frac{y}{z}\right)$ $\square$

4 a) $0 < 1$, so as shown in class $-1 < 0$

So $-1 < 0 < 1$ $\square$

b) $\Rightarrow$ If $xy > 0$ then x and y are both positive/negative.

Suppose, $xy > 0$ is but ~~x~~ x and y have different signs. So, WLOG, $x > 0$ and $y < 0$.

Since $y < 0$ let $y = -z$ where $z > 0$, then

$x(-z) > 0$, but $\rightarrow -xz > 0$ and $x \cdot z > 0$ as $x, z > 0$

so contradiction as from class if $xz > 0$ then $-xz < 0$

So, x and y must both ~~pe~~ positive/negative

$\Leftarrow$ The positive case is shown from recitation.

If $x, y < 0$ then $x = -w$ where $w > 0$ and $y = -z$ where $z > 0$

So $xy = (-w)(-z) = (-1)w(-1)z = (-1)(-1)wz$
$= (-1)^2 wz$
$xy = wz$

Therefore $xy = wz$ and as $w, z > 0$ $wz > 0$
So $xy > 0$ $\square$

c) $x > 0$ so as $x(\frac{1}{x}) = 1 > 0$ and as shown in part b, this means x and $\frac{1}{x}$ have the same sign. So, since $x > 0$, $\frac{1}{x} > 0$ $\square$

d) $0 < y < x$ as $x, y > 0$ they have multiplicative inverses

So $\frac{1}{y}(0 < y < x) \to 0 < 1 < \frac{x}{y}$

$\frac{1}{x}\left(0 < 1 < x \cdot \frac{1}{y}\right) \to 0 < \frac{1}{x} < \frac{1}{x} \cdot x \cdot \frac{1}{y}$
$\to 0 < \frac{1}{x} < \frac{1}{y}$ $\square$

a) Induction on m so n is fixed
Base case. $m = 1$
$a^n \cdot a^1 = a^{n+1}$, which is how we define exponents here

Assume true up to m, so $a^n a^m = a^{n+m}$
Show for $m+1$
$a^n a^{m+1} = a^{n+m+1}$
$\to a^n a^m a = a^{n+m+1}$
$a^{n+m} a = a^{n+m+1}$ by definition get
$a^{n+m+1} = a^{n+m+1}$ $\square$

5 b) Induction on m so n is fixed

Base case $m=1$

$(a^n)^1 = a^n$ as define $a^1 = a$

$a^n = a^n$

Assume it holds up to m, so $(a^n)^m = a^{nm}$

Show for $m+1$ it holds.

$(a^n)^{m+1} = a^{n(m+1)}$

$\rightarrow (a^n)^m a^n = a^{mn+n}$ by part a)

$(a^n)^m a^n = a^{mn} a^n$ as $a^n \neq 0$.

$\rightarrow a^{nm} a^n = a^{mn} a^n$ as true to m

So $a^{nm} a^n = a^{nm} a^n$ and proves

$(a^n)^m = a^{nm}$

c) Base case for induction on m

$m=1$,

$a^1 b^1 = a \cdot b = (ab)^1$ So base case holds

Assume true up to m, show it holds for $m+1$

So need to show $a^{m+1} b^{m+1} = (ab)^{m+1}$

$a^{m+1} b^{m+1} = a^m a b^m b = a^m b^m ab$ as true up to m, $a^m b^m = (ab)^m$ and $ab = (ab)^1$ so

$(ab)^m (ab) = (ab)^{m+1}$ and by def

$(ab)^m (ab) = (ab)^m ab$ so this holds

and is true $\square$