Small-Sample Statistics

The Chi-Squared (χ^2) Distribution

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Outline

The chi squared (χ^2) distribution

- **Thm.:** If $U = \sum_{j=1}^{n} Z_j^2$ where Z_j are iid standard normal,
- Then U is gamma-distributed with $r = n/2$ df and $\lambda = 1/2$,

$$f_U(u) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} u^{(n/2)-1} e^{-u/2} \qquad \text{where } u > 0$$

- **Pf.:** First take $n = 1$. For all $u \geq 0$,

$$F_{Z^2}(u) = P(Z^2 \leq u) = P\left(-\sqrt{u} \leq Z \leq +\sqrt{u}\right) = 2P(0 \leq Z \leq \sqrt{u}).$$

or

$$F_{Z^2}(u) = \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{u}} dz\ e^{-z^2/2}$$

The chi squared (χ^2) distribution (continued)

- **Pf. (continued):** We have

$$F_{Z^2}(u) = \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{u}} dz\ e^{-z^2/2}$$

- Differentiate to find that Z is gamma-distributed with parameters $r = {}^1\!/_2$ and $\lambda = {}^1\!/_2$,

$$f_{Z^2}(u) = \frac{2}{\sqrt{2\pi}} e^{-u/2} \frac{1}{2\sqrt{u}} = \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} u^{(1/2)-1} e^{-u/2}$$

- Hence, if $U = \sum_{j=1}^n Z_j^2$, it must be that U is gamma-distributed with parameters $r = {}^n\!/_2$ and $\lambda = {}^1\!/_2$.

The chi squared (χ^2) distribution (continued)

- **Def.:** The pdf of $U = \sum_{j=1}^{n} Z_j^2$, where Z_j are iid standard normal, is called the *chi squared (χ^2) distribution with n degrees of freedom*.

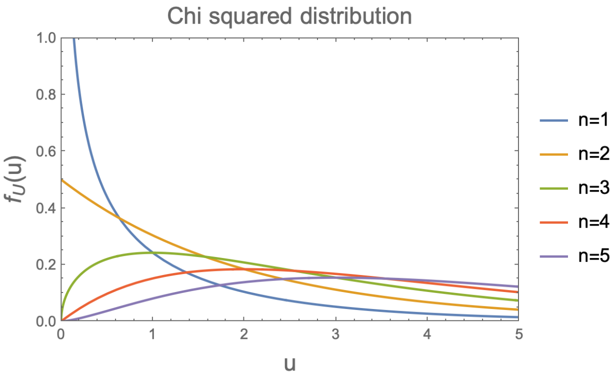
$$\boxed{f_{Z^2}(u) = \frac{\left(\frac{n}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} u^{(n/2)-1} e^{-u/2}}$$

- The χ^2 distribution is a special case of the gamma distribution with parameters $n/2$ and $1/2$.
- As such, note that the sum of a χ^2 random variable with m df and another with n df will also be χ^2 with $m+n$ df.

Tables in the appendices

- You are now in a position to understand another table in the back of the book.
- Table A.1 tabulates Z distributions for various α.
- Table A.3 tabulates χ^2 distributions for various α and n df.

Plots and properties of the χ^2 distribution



- Mean: $E(Z) = n$
- Mean square: $E(Z^2) = n(n+2)$
- Variance: $\mathrm{Var}(Z) = 2n$

Summary

- We learned about the χ^2 distribution.
- We noted that the sum of two χ^2 r.v.s is likewise χ^2.
- We learned about tables and plots for χ^2 pdfs.