Hw 2

1) Reflexive $x=(x_0,y_0)$ then $x_0^2+y_0^2=x_0^2+y_0^2$ $x\sim x$
Symmetric $x=(x_0,y_0)$ $y=(x_1,y_1)$ $x\sim y$ so
$x_0^2+y_0^2=x_1^2+y_1^2 \rightarrow y_1^2+x_1^2=y_0^2+x_0^2$ so $y\sim x$

Transitive) $x=(x_0,y_0)$ $y=(x_1,y_1)$ $z=(x_2,y_2)$
$x\sim y$ and $y\sim z$ so:
$x_0^2+y_0^2=x_1^2+y_1^2$ $x_1^2+y_1^2=x_2^2+y_2^2$ so
$x_0^2+y_0^2=x_2^2+y_2^2$ and $x\sim z$
This equivalence class is set of circles centered at $(0,0)$ w/radius $\sqrt{x^2+y^2}$

2) There isn't neccesarly a b s.t. $a\sim b$
so it doesn't hold

3a) Reflexive: $f(x_1)=f(x_1)$ so $x_1\sim x_1$
Symmettric: let $x_1\sim x_2$ $f(x_1)=f(x_2)\rightarrow f(x_2)=f(x_1)$
so $x_2\sim x_1$
transitive. Let $x_0\sim x_1$ and $x_1\sim x_2$ so
$f(x_0)=f(x_1)$ $f(x_1)=f(x_2)$ so
$f(x_0)=f(x_2)$ and $x_0\sim x_2$
Therefore, this is an equivalence relation.

b) Let $f_2: X/\sim \rightarrow Y$
$[x]\rightarrow f_2([x])$
Since all elements in an equivalence class have the same image, it is well defined.

To show injective, let $x_1, x_2 \in X$ ~~then x_2~~
If $f_2([x_1])=f_2([x_2])$ then $f(x_1)=f(x_2)$ so $x_1\sim x_2$
meaning $[x_1]=[x_2]$ and f_2 is injective
functions.
To show surjective let $g: X\rightarrow X/\sim$
$x\rightarrow [x]$

From earlier, $f(x)=f_2(g(x))$.

By assumption, $f(x)$ is surjective, meaning that f_2 must be surjective. as shown back in HW 1s. Since f_2 is surjective and injective, a bijective correspondance exists.

4 Comparability: $x \neq y$ then $x \prec y$ or $y \prec x$
WLOG $x < y$ so $x^2 < y^2$, but if $x^2 = y^2$ then $x < y$ as $x \neq y$.
Antireflexivity: $\forall x \in \mathbb{R}$, $x \not\prec x$. $x^2 = x^2$ and $x = x$
So $x \not\prec x$.
Transitivity. $\forall x, y, z \in \mathbb{R}$ if $x \prec y$ and $y \prec z$ then $x \prec z$
Since $x \prec y$ $x^2 < y^2$ or if $x^2 = y^2$ $x < y$
Since $y \prec z$ $y^2 < z^2$ or if $y^2 = z^2$ $y < z$.
If $x^2 \neq y^2 \neq z^2$ then $x^2 < y^2 < z^2 \rightarrow x^2 < z^2$ so $x \prec z$
If $x^2 = y^2 \neq z^2$ WLOG, $x^2 = y^2$ but $x < y$ and $y^2 < z^2$
$x^2 = y^2 < z^2$ so $x^2 < z^2$ so $x \prec z$.
If $x^2 = y^2 = z^2$ then $x < y$ and $y < z$ so
$x < y < z \rightarrow x < z$ and $x \prec z$
Therefore $\prec$ is an order relation.

5) Let $L = \{ l \mid l$ is a lower bound of $A \}$
Denote $\sup L = b$ and $\forall l \in L$, $l \leq b$.
Let $U = \{ u \mid u$ is an upper bound of $L \}$
$A \subseteq U$ as $\forall a \in A$ and $\forall l \in L$, $l \leq a$ as l is a lower bound on A so A is an upper bound on L. So, $\forall u \in U$, $b \leq u$ and $b \leq a$, so $b = \max L$ meaning $b = \inf A$.

6a) Let $b \in B = (b_1, b_2 \ldots b_n)$
Since $B_1 \subseteq A_1$, $B_2 \subseteq A_2 \ldots$ then $b_1 \in B_1 \subseteq A_1$
$b_1 \in B_2 \subseteq A_2 \ldots$ and $b_1 \in A_1$, $b_2 \in A_2 \ldots$
So $b \in A_1 \times A_2 \times \ldots = A$ and $B \subseteq A$

b) Let $b = (b_1, b_2 \ldots) \in B$ and each B_i is nonempty.
Since $B \subseteq A$, then $b \in A$, so $b_1 \in A_1$, $b_2 \in A_2$ .... But $b_i \in B_i$ and so $b_i \in B_i$ is the stronger restriction. and $B_i \subseteq A_i$. otherwise, if $A_i \subseteq B_i$ then $\exists b \in B_i$ s.t. $b \notin A_i$ which doesn't work as $B \subseteq A$. Therefore, the converse holds

c) If A is nonempty then no A_i can be empty. If $\exists A_i$ s.t. $A_i = \phi$, then $A = \phi$ as for any $A \times B$, if $B = \phi$ then $A \times \phi = \phi$. Therefore no A_i can be empty if A is nonempty

The converse, if each A_i is nonempty then A is nonempty holds. The contrapositive, if A is empty then $\exists A_i$ that is empty holds, as discussed previously, so the original statement, the converse, holds.

d) $(A_1 \cup B_1) \times (A_2 \cup B_2) \times \ldots \supseteq A \cup B$
Note this as $\prod_{i=1} (A_i \cup B_i) \supseteq A \cup B$

~~Note the~~ Cartesian product is countable, so can use induction to show this

$A \cup B = (A_1 \times A_2 \times \ldots) \cup (B_1 \times B_2 \times \ldots)$

But $\prod_{i=1}^{n} (A_i \cup B_i) = (A_1 \cup B_1) \times (A_2 \cup B_2) \times \ldots$

By properties of cartesian product

for $n=2$. $= (A_1 \times A_2) \cup (A_1 \times B_2) \cup (B_1 \times A_2) \cup (B_1 \times B_2)$

Then this becomes all permutations of A_i and B_i as add more terms so $= (A_1 \times A_2 \times \ldots A) \cup (A_1 \times B_2 \times \ldots)$ and so on, but $A = A_1 \times A_2 \times \ldots$ and $B = B_1 \times B_2 \times \ldots$ are in this set so $A \cup B \subseteq \prod_{i=1} (A_i \cup B_i)$

Note this holds if A or B is empty as wlog A is empty $A \cup B = B$ but $\exists A_i$ s.t $A_i \cup B_i = B_i$, you still have the rest of the product. Note if $B_i = \emptyset$ this still holds as then $B = \emptyset$ and $\emptyset \supseteq \emptyset$

For $\prod (A_i \cap B_i) \subseteq A \cap B$

Using a similar argument as earlier, for an i

$$\prod (A_i \cap B_i) = \overbrace{(A_1 \times A_2 \times \ldots \times A_i) \cap (A_1 \times B_2 \times \ldots) \cap \ldots \cap (B_1 \times B_2 \times \ldots \times B_i)}^{A}_{B}$$

But the intersection restricts

so $\prod (A_i \cap B_i) \subseteq A \cap B$

This holds regardless if A_i or B_i is empty as if an A_i is empty then $A_i \cap B_i = \emptyset$ so $\prod (A_i \cap B_i) = \emptyset$ and $A = \emptyset$ so $A \cap B = \emptyset$.