Math 65 HW3

1 Smallest n is $n=1$, so for $n=1$, have $\frac{(1)^2(1+1)^2}{4} = 1 = 1^3$

Since it works for $n=1$, by induction, we can show it holds true up to $n-1$.

For $n-1$, $= \frac{(n-1)^2 n^2}{4}$ for n terms. $\frac{(n-1)^2 n^2}{4} + n^3$

$$\rightarrow \frac{(n-1)^2 n^2 + 4n^3}{4} = \frac{n^2((n-1)^2 + 4n)}{4} = \frac{n^2(n^2+2n+1)}{4} = \frac{n^2(n+1)^2}{4}$$ QED

2 $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = (1-\tfrac{1}{2}) + (\tfrac{1}{2}-\tfrac{1}{3}) + (\tfrac{1}{3}-\tfrac{1}{4}) + \dots (+\tfrac{1}{n} - \tfrac{1}{n+1})$$

$$S_n = 1 - \frac{1}{n+1}$$

To prove $S_n = 1 - \frac{1}{n+1}$ w/ induction, lets start from $n=1$

$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2}$, $S_n = 1 - \frac{1}{2} = \frac{1}{2}$

For $n-1$, $S_{n-1} = 1 - \frac{1}{(n-1)+1} = 1 - \frac{1}{n}$ $S_n = 1 - \frac{1}{n} + \frac{1}{n(n+1)}$

$$S_n = 1 - \frac{(n+1)}{n(n+1)} + \frac{1}{n(n+1)} = 1 - \frac{n}{n(n+1)} = 1 - \frac{1}{n+1}$$

3 $n=0$, $3>0$ $n=1$, $9>1$, $n=2$, $27>16$, $n=3$, $81=81$,

$n=4$, $243 \not> 256$, for $n=5$, $729>625$, Works for $n \geq 5$

$n=5$ is the base case, and $3^6 > 5^4$; $729 > 625$

We assume true for n, so let's try $n+1$.

$3^{n+1} > n^4 \rightarrow 3 \cdot 3^{n+1} \geq 3n^4 \geq (n+1)^4$

$3n^4 \geq n^4 + 4n^3 + 6n^2 + 4n + 1$

$\frac{2n^4 \geq 4n^3 + 6n^2 + 4n + 1}{2} \rightarrow n^4 \geq 2n^3 + 3n^2 + 2n + \frac{1}{2}$

Our assumption is for $n \geq 5$, so $n^4 \geq 5n^3 \geq 2n^3 + 3n^3$

$2n^3 + 3n^3 \geq 2n^3 + 5 \cdot 3n^2$, $2n^3 + 3n^3 \geq 2n^3 + 15n^2$

$2n^3 + 15n^2 \geq 2n^3 + 3n^2 + 12n^2$, $2n^3 + 15n^2 \geq 2n^3 + 3n^2 + 60n$

$2n^3 + 3n^2 + 60n \geq 2n^3 + 3n^2 + 2n + \frac{1}{2}$, for $n \geq 5$, since this inequality is true, then $3^{n+1} > n^4$ for $n \geq 5$

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4 a) p is prime if p's only factors are 1 and itself.

b) smallest case is $n=2$, which only factors are 1, 2 making it prime. For all $k \in \mathbb{Z} \leq n$, k is prime or multiplication of prime factors. Let's assume k has two factors, $a \geq 2$ and $b \leq n$ and $ab = n+1$. $n+1$ is not prime, by the definition, meaning a, b are prime factors or prime. If a, b don't exist, then k is prime.

5 For $n \geq 2$, there are k RedSox fans in line, and $n-k$ yankees fans, where $n-k \neq 0$. Since a Yankee is last, at some point in the line before the final person, there are no more Red sox fans, with only Yankees fans remaining. Therefore, for this to happen, the last Red Sox fan must be next to a Yankee, as after the last Red Sox fan, only Yankees remain

6 a) $b^n = 6b^{n-1} - 9b^{n-2}$

$b^n - 6b^{n-1} + 9b^{n-2} = 0$

$b^{n-2}(b^2 - 6b + 9) = 0$

$b^{n-2}(b-3)^2 = 0$, $\boxed{b=3}$

b) $n \cdot 3^n = 6(n-1)3^{n-1} - 9(n-2)3^{n-2}$

$n \cdot 3^n = 2(n-1) \cdot 3^n - (n-2)3^n$

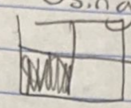
$n \cdot 3^n = 2n \cdot 3^n - 2 \cdot 3^n - (n3^n + 2 \cdot 3^n)$

$n \cdot 3^n = n \cdot 3^n$

6c) $C_0 = -2$, $C_1 = 6$. 3^n and $n \cdot 3^n$ are solutions
so want to find:
$C_n = d \cdot 3^k + k \cdot n 3^n$ $\quad k, d \in \mathbb{Z}$
$C_0 = d$, $-2 = d$

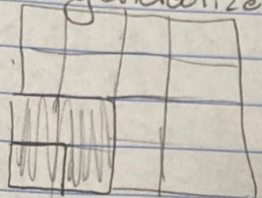
$C_1 = 3d + 3k$, $\frac{6 = 3d + 3k}{3}$, $\boxed{C_n = (-2) \cdot 3^n + 4n \cdot 3^n}$
$2 = k + d$, $d = -2$, so $k = 4$

7 1) Using induction, base case of $k=1$, gives 2×2.



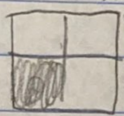
→ this is fillable w/ 1 basic L shape

To generalize to $k-1$, by adding to k our shape becomes:



We know the square in the lower left is fillable, and the remaining square forms a L which we proved was able to be filled with basic L-shapes in quiz on 9/22.

2) Using induction, have base case $k=1$.



← No matter what square removed, it can still be filled by a basic L-shape

For $2^{k-1} \cdot 2^{k-1}$ square, there exists a solution.
If we take a board of $2^k \cdot 2^k$, it consists of 4 squares of size $2^{k-1} \cdot 2^{k-1}$. To show, $4(2^{k-1} \cdot 2^{k-1}) = 2^2 \cdot 2^{k-1} \cdot 2^{k-1} = 2^{2k} = 2^k \cdot 2^k$. If we pick one of these quadrants and remove a square, we already know a solution exists for that quadrant. The remaining 3 quadrants form an L-shape, which we proved in 9/22 quiz that a solution exists for an L shape where it can be covered by basic L-shapes