4 Determine if the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1/2 \\ 0 \\ x_1 + 4x_2 \end{bmatrix}$$

is a linear transformation. If yes, prove it. If not, use explicit numbers to prove it isn't.

Two things to check:

i) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

ii) $T(c\vec{x}) = c \cdot T(\vec{x})$

Pf. of i): Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (x_1+y_1)/2 \\ 0 \\ (x_1+y_1)+4(x_2+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1/2 + y_1/2 \\ 0 \\ (x_1+4x_2) + (y_1+4y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1/2 \\ 0 \\ (x_1+4x_2) \end{bmatrix} + \begin{bmatrix} y_1/2 \\ 0 \\ (y_1+4y_2) \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

Pf of ii):

$$T(c\vec{x}) = T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} cx_1/2 \\ 0 \\ cx_1 + 4cx_2 \end{bmatrix}$$

$$= \begin{bmatrix} cx_1/2 \\ c \cdot 0 \\ c(x_1+4x_2) \end{bmatrix}$$

$$= c \cdot \begin{bmatrix} x_1/2 \\ 0 \\ x_1+4x_2 \end{bmatrix}$$

$$= c \cdot T(\vec{x}).$$

$$y = T(\vec{x}) + T(\vec{y}).$$

5 Let $\{v_1, v_2, v_3\}$ be a linearly independent set in $\mathbb{R}^3$.

(a) Prove that the set $\{v_1, v_1 + v_2, v_2 + v_3\}$ is linearly independent.

(b) Prove that the set $\{v_1 - v_2, v_2 - v_3, v_3 - v_1\}$ is linearly dependent.

a) Show $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3\}$. lin indep.

Suppose

$$c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_2 + \vec{v}_3) = \vec{0}$$

$$c_1\vec{v}_1 + c_2\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$(c_1 + c_2)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. indep., so

$$c_1 + c_2 = 0 \Rightarrow c_1 + 0 = 0 \Rightarrow c_1 = 0$$
$$c_2 + c_3 = 0 \Rightarrow c_2 + 0 = 0 \Rightarrow c_2 = 0$$
$$c_3 = 0$$

So $c_1 = 0,\ c_2 = 0,\ c_3 = 0.$

$\Rightarrow \{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3\}$ lin indep.

[illegible]

Let,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{x_1} \\ 0 \\ x_1 + 4x_2 \end{bmatrix}$$

Is this linear?

$$\sqrt{1+1} = \sqrt{2} \neq \sqrt{1} + \sqrt{1} = 1 + 1 = 2$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2} \\ 0 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{1} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \neq \begin{bmatrix} \sqrt{2} \\ 0 \\ 2 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$