MATH 70 WORKSHEET 8, Spring 2021

In problems 1 and 2 of this worksheet you will investigate the eigenvectors and eigenvalues of *linear transformations*. First, here is the definition.

Let V be a vector space and let $T : V \to V$ be a linear transformation.

The number $\lambda \in \mathbb{R}$ is an *eigenvalue* of T if for some *nonzero vector* $\mathbf{v} \in V$, $T(\mathbf{v}) = \lambda\mathbf{v}$. In this case, $\mathbf{v}$ is called an *eigenvector of T associated to λ*.

1. (5 points) Let V be a vector space of dimension n and let $T : V \to V$ be a linear transformation. Assume T has eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\lambda_n$. For each $j = 1, 2, \ldots, n$, let $\mathbf{b}_j$is an eigenvector of T associated to λ_j. Assume $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ is a basis of V (one calls $\mathcal{B}$ an *eigenvector basis* for T). Therefore,

$$T(\mathbf{b}_j) = \lambda_j \mathbf{b}_j \text{ for } 1 \leq j \leq n$$

Now, assume $\lambda_j \neq 0$ for all j.

(a) Find the dimension of the kernel of T. Justify your answer. HINT: $\mathcal{B}$ is a basis of V, and $\forall j,\ \lambda_j \neq 0$.

(b) Find the dimension of the range of T. Justify your answer. HINT: $\mathcal{B}$ is a basis of V, and $\forall j,\ \lambda_j \neq 0$.

Solution:

(a) I claim $\ker T = \{\mathbf{0}\}$, so the dimension of the kernel is zero. Suppose $\mathbf{v} \in \ker T$. Since $\mathcal{B}$ is a basis, there are unique weights $c_1, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{b_1} + \ldots + c_n\mathbf{b_n}.$$

We will show that in fact all these weights must be zero, so that $\mathbf{v}$ must be the zero vector. Applying T to both sides, we have

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{b_1} + \ldots + c_n\mathbf{b_n} \\ &= c_1T(\mathbf{b_1}) + \ldots + c_nT(\mathbf{b_n}) \\ &= c_1\lambda_1\mathbf{b_1} + \ldots + c_n\lambda_n\mathbf{b_n}, \end{aligned}$$

where we've used the linearity of T and the fact that the $\mathbf{b_j}$ are eigenvectors. On the other hand, since $\mathbf{v} \in \ker T$, we must have $T(\mathbf{v}) = \mathbf{0}$, so

$$c_1\lambda_1\mathbf{b_1} + \ldots + c_n\lambda_n\mathbf{b_n} = \mathbf{0}.$$

But $\mathcal{B}$ is a basis, so by linear independence the only way this equation can hold is if $c_1\lambda_1 = \ldots = c_n\lambda_n = 0$. By assumption, for all j, $\lambda_j \neq 0$, so the only way $\lambda_j c_j = 0$ is if $c_j = 0$. Thus we must have $c_1 = \ldots = c_n = 0$, so that $\mathbf{v} = \mathbf{0}$, as claimed.

(b) By the rank-nullity theorem for linear transformations,

$$\dim(\ker T) + \dim(T) = \dim(V).$$

Since we've shown $\dim(\ker T) = 0$, we must have

$$\dim(T) = \dim(V) = n.$$

(Alternatively, we could check explicitly that $\{T(\mathbf{b_1}), \ldots, T(\mathbf{b_n})\}$ is a basis for V.)

2. (9 points) Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ by $T(a_0 + a_1t + a_2t^2) = (a_0 + a_1 + a_2) + \frac{1}{2}a_1t + \frac{1}{2}a_2t^2$ and $\mathcal{B} = \{1, t, t^2\}$ be the standard basis for $\mathcal{P}_2$.

(a) (1 points) Compute the matrix for T relative to $\mathcal{B}$, $[T]_{\mathcal{B}}$. Call this matrix A.

(b) (2 points) Find the eigenvalues for A and basis for each of their respective eigenspaces.

(c) (1 point) Find a basis for $\mathbb{R}^3$ of eigenvectors of A. Call this basis $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(d) (2 points) For each of the vectors $\mathbf{v}_j$, $j = 1, 2, 3$, in $\mathcal{B}'$, compute the vector $\mathbf{w}_j \in V$ such that $[\mathbf{w}_j]_{\mathcal{B}} = \mathbf{v}_j$ and show that $\mathbf{w}_j$ is an eigenvector for T.

(e) (1 point) Express A as PDP^{-1} where D is diagonal matrix. In your answer, you need to explicitly find P and D.

(f) (2 points) The notation $T^k(\mathbf{v})$ means $\underbrace{(T \circ \ldots \circ T)}_{k-times}(\mathbf{v})$, that is k compositions of T with itself evaluated at $\mathbf{v}$. Compute $T^4(3 + 2t - 4t^2)$. Hint: first compute $A^4[3 + 2t - 4t^2]_{\mathcal{B}}$. You may use that $A^4 = \left[T^4\right]_{\mathcal{B}}$.

* *Optional–not required:* If your interested in a challenge, try to estimate $T^k(3 + 2t - 4t^2)$ for large values of k. For an even more interesting challenge, try to estimate $T^k(a_0 + 2a_1t - 4a_2t^2)$ for large k!!

Solution:

(a) $\mathrm{A} = \begin{pmatrix} [T(1)]_{\mathcal{B}} & [T(t)]_{\mathcal{B}} & [T(t^2)]_{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

(b) The characteristic polynomial is

$$\det(A - \lambda I) = (1 - \lambda)(1/2 - \lambda)(1/2 - \lambda),$$

so the eigenvalues are $\lambda_1 = 1$, with multiplicity 1, and $\lambda_2 = 1/2$, with multiplicity two.

The eigenvectors corresponding to $\lambda_1 = 1$ are solutions to the homogeneous system

$$(A - I)\mathbf{x} = \mathbf{0}$$

with coefficient matrix

$$A - I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

This is row-equivalent to the reduced echelon matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so solutions are of the form

$$\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$$

with x_1 free. A possible basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

The eigenvectors corresponding to $\lambda_2 = 1/2$ are solutions to the homogeneous system

$$(A - (1/2)I)\mathbf{x} = \mathbf{0}$$

which has coefficient matrix

$$\begin{pmatrix} 1/2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is row equivalent to

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solutions must satisfy $x_1 = -2x_2 - 2x_3$, with x_2, x_3 free. Thus the solutions are of the form

$$x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

so a basis is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(c) We can obtain an eigenvector basis for $\mathbb{R}^3$ by combining the bases of the two eigenspaces:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(d)

$$\begin{aligned} \mathbf{w_1} &= 1 \\ \mathbf{w_2} &= -2 + t \\ \mathbf{w_3} &= -2 + t^2 \end{aligned}$$

Verifying these are eigenvectors:

$$\begin{aligned} &T(\mathbf{w_1}) = 1 = 1 \cdot \mathbf{w_1} \\ &T(\mathbf{w_2}) = (-2+1) + (1/2)1t = -1 + t/2 = (1/2)(-2+t) = (1/2)\mathbf{w_2} \\ &T(\mathbf{w_3}) = (-2+1) + (1/2)t^2 = (1/2)(-2+t^2) = (1/2)\mathbf{w_3}, \end{aligned}$$

so these are indeed eigenvectors of T.

(e) D will have the same eigenvalues (with the same multiplicities) as A, so we can take

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

The matrix P has columns corresponding to the eigenvector basis computed above:

$$P = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that here we write the eigenvectors in the same order as the eigenvalues were listed in D. Thus

$$\begin{aligned} A &= \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(f) Using the diagonalization, we have

$$\begin{aligned} A^4 &= (PDP^{-1})^4 \\ &= PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1} \\ &= PD^4P^{-1} \\ &= \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}^4 \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{16} \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{15}{8} & \frac{15}{8} \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{16} \end{pmatrix} \end{aligned}$$

Thus

$$A^4[3+2t+4t^2]_{\mathcal{B}} = \begin{pmatrix} 1 & \frac{15}{8} & \frac{15}{8} \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{1}{16} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1/8 \\ -1/4 \end{pmatrix}.$$

Now since $A^4 = [T^4]_{\mathcal{B}}$, we have

$$[T^4(3+2t-4t^2)]_{\mathcal{B}} = A^4[3+2t-4t^2]_{\mathcal{B}},$$

so to find $T^4(3+2t-4t^2)$ we have to find the polynomial whose coordinate vector relative to $\mathcal{B}$ is $(-3/4, 1/8, -1/4)$. It follows that

$$T^4(3+2t-4t^2) = -3/4 + t/8 - t^2/4.$$

3. (3 points) Consider the matrix $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{pmatrix}$. Find necessary and sufficient conditions on the real numbers a, b, c such that A is diagonalizable.

Solution:

We first compute the characteristic polynomial of A:

$$\det(A - \lambda I) = (\lambda - 1)(\lambda - 1)(\lambda - 2).$$

So the eigenvalues of A are $\lambda_1 = 1$, with multiplicity 2, and $\lambda_2 = 2$, with multiplicity 2. A will be diagonalizable exactly when the dimension of each eigenspace equals the multiplicity of the corresponding eigenvalue.
Since the dimension of the eigenspace is never greater than the multiplicity of the eigenvalue, and any eigenspace has dimension at least one, the λ_2 eigenspace must have dimension 1. So A is diagonalizable if and only if the dimension of the λ_1 eigenspace is 2.
For $\lambda_1 = 1$, the eigenvectors are solutions to the equation

$$(A - I)\mathbf{x} = \mathbf{0}.$$

This has coefficient matrix

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 2 \end{pmatrix}$$

There are two cases: $a = 0$ or $a \neq 0$. If a is nonzero, the echelon form of this matrix will have two pivots, hence two basic variables and one free variable, so the eigenspace will be one dimensional. If a is zero, there will be only one pivot, so one basic variable and two variables, so the eigenspace will be two dimensional. Thus A will be diagonalizable if and only if $a = 0$.

4. (3 points) Let A and B be n by n matrices. If A is invertible and AB is diagonalizable, prove that BA is also diagonalizable.

Solution:

Since A is invertible, A^{-1} exists. Multiplying BA by $I = A^{-1}A$, we have

$$BA = A^{-1}ABA = A^{-1}(AB)A,$$

which shows that BA is similar to AB. Since AB is diagonalizable, it is similar to some diagonal matrix D, and since similarity is transitive, BA must be similar to D as well. Hence BA is diagonalizable.