MATH 42 HOMEWORK 2

Topics covered: Lines & Curves; Differentiation & Integration in $\mathbb{R}^3$; § 13.4-13.5, 14.1 - 14.3

This homework is due at 11:59 pm (Eastern Time) on Wednesday, September 23. You will need to scan your completed homework and upload it **as one pdf file** to Gradescope. Please see the Canvas module "Written Assignments" for instructions on how to upload your assignment to Gradescope.

(1) Let $\vec{a} = \langle 3, 2, 1\rangle$ $\vec{b} = \langle 2, 2, 2\rangle$ $\vec{c} = \langle 4, 0, 1\rangle$. Compute the following:
  (a) $(\vec{c} \cdot \vec{b})\,(\vec{a} + \vec{b} + \vec{c})$
  (b) $(\vec{b} \times \vec{c}) \cdot \vec{b} = 0$
  (c) $\vec{a} + (\vec{b} \times \vec{a})$
  (d) $\vec{b} \cdot (\vec{a} \times \vec{c})$
  (e) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{b})$

Solution. Firstly we note the following

$$\vec{c} \cdot \vec{b} = 10$$
$$\vec{b} \times \vec{c} = -(\vec{c} \times \vec{b}) = \langle -2, 4, 2\rangle$$
$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}) = \langle 2, 1, -8\rangle$$

From these we find
  (a) $\langle 90, 40, 40\rangle$
  (b) 0
  (c) $\langle 1, 6, -1\rangle$
  (d) -10
  (e) 36

(2) For each of the following parameterizations, describe and sketch the curve it represents. Assume for all of the below that $-\infty < t < \infty$. You can check your answers on a computer graphing utility - playing around with a 3D graphing utility is a great way to build some intuition.
  (a) $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 5\rangle$
  (b) $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$
  (c) $\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle$
  (d) $\vec{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$
  (e) $\vec{r}(t) = \langle 3t^2, 4t^2, 5t^2\rangle$

Solution.

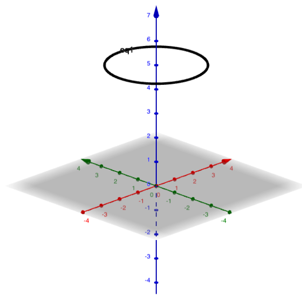

FIGURE 1. A circle, parallel to the xy-plane at height $z = 5$

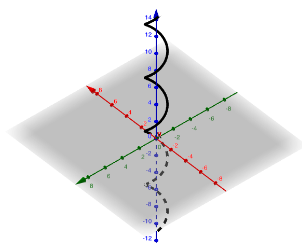

FIGURE 2. A spiral coming out of the xy-plane, wrapping around the z-axis

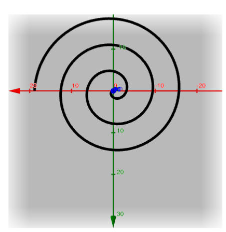

FIGURE 3. A logarithmic spiral in the plane.

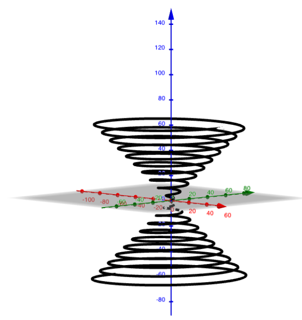

FIGURE 4. A spiral wrapping around tangent cones of infinite height, with vertex at the origin.

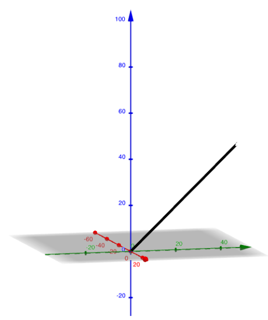

FIGURE 5. A straight line.

(3) Let $\vec{r}(t)$ be the position vector of a moving point P. Find its velocity, acceleration, and speed at the given time t.
 (a) $\vec{r}(t) = \langle \frac{2}{t}, \frac{3}{t+1} \rangle$; $t = 2$
 (b) $\vec{r}(t) = \langle e^{2t}, e^{-t} \rangle$; $t = 4$
 (c) $\vec{r}(t) = e^t \langle \cos t, \sin t, 1 \rangle$; $t = \pi/2$
 (d) $\vec{r}(t) = \langle 1 + t, 2t, 2 + 3t \rangle$; $t = 2$
 (e) $\vec{r}(t) = \langle 2t, e^{-t^2} \rangle$; $t = 1$

Solution.

(a)

$$\vec{v}(t) = \langle -2/t^2, -3/(t+1)^2 \rangle, \quad \vec{v}(2) = \langle -1/2, -1/3 \rangle$$
$$\vec{a}(t) = \langle 4/t^3, 6/(t+1)^3 \rangle, \quad \vec{a}(2) = \langle 1/2, 2/9 \rangle$$

(b)

$$\vec{v}(t) = \langle 2e^{2t}, e^{-t}\rangle, \quad \vec{v}(4) = \langle 2e^{8}, -e^{-4}\rangle$$
$$\vec{a}(t) = \langle 4e^{2t}, e^{-t}\rangle, \quad \vec{a}(4) = \langle 4e^{8}, e^{-4}\rangle$$

(c)

$$\vec{v}(t) = \langle e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, e^t\rangle, \quad \vec{v}(\pi/2) = \langle -e^{\pi/2}, e^{\pi/2}, e^{\pi/2}\rangle$$
$$\vec{a}(t) = \langle -2e^t\sin t, 2e^t\cos t, e^t\rangle, \quad \vec{a}(\pi/2) = \langle -2e^{\pi/2}, 0, e^{\pi/2}\rangle$$

(d)

$$\vec{v}(t) = \langle 1, 2, 3\rangle, \quad \vec{v}(2) = \langle 1, 2, 3\rangle$$
$$\vec{a}(t) = \langle 0, 0, 0\rangle, \quad \vec{a}(2) = \langle 0, 0, 0\rangle$$

(e)

$$\vec{v}(t) = \langle 2, -2te^{-t^2}\rangle, \quad \vec{v}(1) = \langle 2, -2/e\rangle$$
$$\vec{a}(t) = \langle 0, 4t^2e^{-t^2}\rangle, \quad \vec{a}(1) = \langle 0, 4/e\rangle$$

(4) Evaluate the following integrals.

(a) $\displaystyle\int_0^2 6t^2\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}\,dt$

(b) $\displaystyle\int_{-1}^1 -5t\mathbf{i} + 8t^3\mathbf{j} + -3t^2\mathbf{k}\,dt$

(c) $\displaystyle\int \sin t\mathbf{i} + \cos t\mathbf{j} + \tan t\mathbf{k}\,dt$

(d) $\displaystyle\int_0^1 te^{t^2}\mathbf{i} + \sqrt{t}\mathbf{j} + (t^2+1)^{-1}\mathbf{k}\,dt$

Solution.

(a) $\displaystyle\int_0^2 6t^2\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}\,dt = 2t^3|_0^2\mathbf{i} + 2t^2|_0^2 = 2t|_0^2 = \langle 16, -8, 6\rangle$

(b) $\displaystyle\int_{-1}^1 -5t\mathbf{i} + 8t^3\mathbf{j} + -3t^2\mathbf{k}\,dt = \frac{-5}{2}t^2|_{-1}^1\mathbf{i} + 2t^4|_{-1}^1\mathbf{j} - 6^3|_{-1}^1\mathbf{k} = \langle 0, 0, -2\rangle$

(c) $\displaystyle\int \sin t\mathbf{i} + \cos t\mathbf{j} + \tan t\mathbf{k}\,dt = (-\cos t + c_1)\mathbf{i} + (\sin t + c_2)\mathbf{j} - (\ln|\cos t| + c_3)k$

(d) $\displaystyle\int_0^1 te^{t^2}\mathbf{i} + \sqrt{t}\mathbf{j} + (t^2+1)^{-1}\mathbf{k}\,dt = \frac{1}{2}e^{t^2}|_0^1\mathbf{i} + \frac{2}{3}t^{3/2}|_0^1\mathbf{j} + \arctan t|_0^1 = \langle (e-1)/2, 2/3, \pi/2\rangle$

(5) Suppose that a particle has acceleration vector at time t given by

$$\vec{a}(t) = \langle e^t, t^2 + t, \sin t\cos t\rangle$$

If the particle has initial velocity $\vec{v}(0) = \langle 1, 0, 2\rangle$ and initial position, $\vec{r}(0) = \langle 0, 0, 0\rangle$, find the expressions for $\vec{v}(t)$ and $\vec{r}(t)$.

Solution.

To get the velocity, we integrate component wise and obtain

$$\int \vec{a}(t)\,dt = \vec{v}(t) = \langle e^t + c_1, \frac{1}{3}t^3 + \frac{1}{2}t^2 + c_2, \frac{1}{2}\sin^2(t) + c_3\rangle$$

From the initial condition $\vec{v}(0) = \langle 1, 0, 2 \rangle$, we see that

$$\vec{v}(t) = \langle e^t, \frac{1}{3}t^3 + \frac{1}{2}t^2, \frac{1}{2}\sin^2(t) + 2 \rangle$$

Repeating the process gives the position vector

$$\int \vec{v}(t)\, dt = \vec{s}(t) = \langle e^t + d_1, \frac{1}{12}t^4 + \frac{1}{6}t^3 + d_2, \frac{1}{4}t\frac{1}{8}\sin 2t + 2t + d_3 \rangle$$

$$\vec{v}(t) = \langle e^t - 1, \frac{1}{12}t^4 + \frac{1}{6}t^3, \frac{7}{4}t + \frac{1}{8}\sin 2(t) \rangle$$