MATH-235 HOMEWORK 1 SOLUTION

- 1.1.20.

  Proof. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to some x. Then we have $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall k \geq N$, $d(x, x_{n_k}) < \epsilon/2$. By definition of Cauchy sequence, $\forall\epsilon > 0$, $\exists M \in \mathbb{N}$ such that $\forall n, m \geq M$, $d(x_n, x_m) < \epsilon/2$. Pick the greater one between N and M, say K, we have $\forall\epsilon > 0$, such that $\forall k \geq K$, $d(x, x_q) < d(x, x_{n_k}) + (x_k, x_{n_k}) < \epsilon$. Therefore the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x. □

- 1.4.4

  Proof. (a). Consider any $\epsilon > 0$ and $\delta = (\epsilon/k)^{1/\alpha}$. Then for all $x, y \in I$ such that $|x-y| < \delta$, we have $|f(x)-f(y)| \leq k|x-y|^\alpha$ by Holder continuity. By our assumption, $k|x-y|^\alpha < \epsilon$. Hence f is uniformly continuous.

  (b). Consider $\alpha = 1+\epsilon$ and then by Holder continuity we have
  $$|f(x) - f(y)| \leq K|x-y||x-y|^\epsilon$$
  divide both sides by $|x-y|$ and take the limit as $x \to y$, we have
  $$|f'(y)| \leq \lim_{x\to y} K|x-y|^\epsilon \to 0$$
  Therefore f is a constant on I.
  (c). Straight computation.

  (d). First, we show g is continuous at 0. Consider the limit $\lim_{x\to 0^+} |-1/\ln(x)-0| = 0$. Then we show g is continuous on $(0, 1/2)$. Consider
  $$\lim_{x\to y} |-1/\ln(x) - (-1/\ln(y))| = \lim_{x\to y} \left|\frac{-\ln y + \ln x}{\ln y \ln x}\right| = \lim_{x\to y} \left|\frac{\ln(x/y)}{\ln y \ln x}\right| = 0$$
  Therefore $g(x)$ is continuous on a compact set $[0, 1/2]$, hence it's also uniformly continuous. To show Holder continuity doesn't hold, it suffices to show $|g(x) - g(0)| > K|x|^\alpha$ where $x \to 0^+$. We prove this by contradiction, which is $|-1/\ln x| \leq K|x|^\alpha$. Divide both sides by $K|x|^\alpha$ and apply L'Hopital's rule as $x \to 0^+$, we have
  $$\lim_{x\to 0^+} K|\alpha x^{-\alpha}| \leq 1$$
  which is not possible for any $\alpha > 0$. □

- 2.1.29

  Proof. By countable subadditivity of countable sets in $\mathbb{R}^d$, we have
  $$|\cup Z_k|_e \leq \sum |Z_k|_e = 0.$$
  By non-negativity of exterior measure, we have $|\cup Z_k|_e = 0$. □

- 2.1.32. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then its graph
$$\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$
has measure zero.

  Proof. Notice that $\Gamma_f = \cup_{n=1}^{\infty} \Gamma_f^n$, where $\Gamma_f^n = \{(x, f(x)) : x \in [-n, n]\}$. Then $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x, y \in [-n, n]$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Divide $[-n, n]$ into $K_{n,\delta}$ with equal length δ intervals $[a_l, b_l]$ and consider
$$\cup_{l=1}[a_l, b_l] \times [M_n - \frac{\epsilon}{2^n}, M_n + \frac{\epsilon}{2^n}]$$
where $M_n = \sup_{x\in[-n,n]} |f(x)|$. We have $\Gamma_f^n \subseteq \cup_{l=1}[a_l, b_l] \times [M_n - \frac{\epsilon}{2^n}, M_n + \frac{\epsilon}{2^n}]$ and by monotonicity,
$$|\Gamma_f^n|_e \leq \sum_l (b_l - a_l)\frac{\epsilon}{4} < \epsilon$$
Hence we have $|\Gamma_f^n|_e = 0$ which implies $|\Gamma_f|_e = 0$. □

- 2.1.39. Given a set $E \subseteq \mathbb{R}^d$, show that $|E|_e = 0$ if and only if there exists countably many boxes Q_k such that $\sum |Q_k| < \infty$ and each point $x \in E$ belongs to infinitely many Q_k.

  Proof. Suppose $|E|_e = 0$, then $\forall n \geq 1$, there exists $\{Q_k^n\}$ boxes such that $E \in \cup_{k=1}^{\infty} Q_k^n$ and $\sum_{k=1}^{\infty} |Q_k^n| < \frac{1}{2^n}$. Notice that $E \in \cup_{k=1}^{\infty} \cup_{n=1}^{\infty} Q_k^n$ and $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |Q_k^n| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$. For all n, let $x \in E$, then there exists some k_n such that $x \in Q_{k_n}^n$, hence x is an element of infinitely many Q_k^n.

  Suppose $\exists\{Q_n\}$ such that $\sum |Q_k| < \infty$ and each point $x \in E$ belongs to infinitely many Q_k, then we have $E \subseteq \limsup Q_k$ and since $\sum |Q_k| < \infty$ we have $|\limsup Q_n| = 0$, therefore $|E|_e = 0$. □

- 2.2.34. Let $S_r = \{x \in \mathbb{R}^d : ||x|| = r\}$ be the sphere of radius r in $\mathbb{R}^d$ centered at the origin. Prove that $|S_r| = 0$.

  Proof. $\forall k \geq 1$, divide $\mathbb{R}^d$ into sides of length $\frac{1}{k}$, then there exists finitely many such sides, say $\{Q_l^k\}_{l=1}^{N_k}$. Notice that $S_r \subseteq \cup_{l=1}^{N_k} Q_l^k$ and $N_k = Ck^{d-1}$ for some constant C. Then we have
$$|S_r| \leq Ck^{d-1} \cdot \frac{1}{k^d} = \frac{C}{k}$$
which implies $|S_r| = 0$. □

- 2.2.37

  Proof. a implies b: Suppose E is Lebesgue measurable, then there exists an open set $U \supset E$ such that $|U \setminus E|_e \leq \epsilon/2$ for any $\epsilon > 0$. Also, then there exists an closed set $F \supset E$ such that $|U \setminus E|_e \leq \epsilon/2$. Therefore, $F \subset E \subset F$ and $|U \setminus F| = \epsilon$.

  b implies c: By assumption, we can construct a sequence of closed sets $F_k \subset E$ and a sequence of open sets $U_k \supset E$ and $F_k \subset E \subset U_k$ for all k such that $|U_k \setminus F_k| < 1/k$. Now consider G_δ set $G = \cap U_k$ and F_σ set $H = \cup F_k$. We have $|G \setminus H| \leq |U_k \setminus F_k| < \frac{1}{k} \to 0$ as k grows.

c implies a: Notice that $H \subset E$, which implies $G \setminus E \subset G \setminus H$ and by monotonicity we have $|G \setminus E| = 0$. Therefore E differs from a G_δ set with a set of measure 0, hence E is measurable. □

- 2.2.44

Proof. Assume A and B are measurable, then by assumption $A \cap B = \emptyset$ and $E = A \cup B$ implies $|E| = |A| + |B| = |A|_e + |B|_e$.

Assume $|E| = |A|_e + |B|_e$, then there exist G_δ sets G and H such that $A \subset G$, $B \subset H$ and $|A|_e = |G|, |B|_e = |H|$. Also, $\exists K \supset E$ such that K is a G_δ set and $|K \setminus E| = 0$. Consider $G_1 = G \cap K$ and $H_1 = H \cap K$, notice that they are both G_δ sets and

$$G_1 = G \cap K \supset A \cap E = A$$

Therefore,

$$G_1 \setminus A \subset (G_1 \setminus A) \cup (H_1 \setminus B) = (G_1 \cup H_1) \setminus E \subset K \setminus E$$

Hence we have

$$|G_1 \setminus A|_e \leq |K \setminus E| = 0$$

We conclude that A is measurable. Following a similar argument we can conclude B is also measurable. □

- 2.2.50

Proof.

(1) Since $\emptyset$ is countable trivially, we have $\emptyset \in \Sigma$ and $X \in \Sigma$.
(2) Suppose $E \in \Sigma$, then either E or E^c is countable, therefore $E^c \in \Sigma$.
(3) Suppose $\{E_k\}_{k\in} \in \Sigma$, if $\forall k \geq 1$, E_k is countable, then $\cup_{k=1}^{\infty} E_k$ is also countable, hence $\cup_{k=1}^{\infty} E_k \in \Sigma$. If $\exists k_0$ such that $E_{k_0}^c$ is countable, then $(\cup_{k=1}^{\infty} E_k)^c = \cap_{k=1}^{\infty} E_k^c \subset E_{k_0}^c$ is countable.

Hence, Σ is a σ-algebra. □