Question. (a) *Write the definition of Cauchy sequence*

Cauchy sequence is $\forall \varepsilon \in \mathbb{Q}^+ \; \exists m \in \mathbb{N} \; \forall n_1, n_2$
$|a_{n_1} - a_{n_2}| < \varepsilon$

(b) *Show that* $a_n = \frac{(-1)^n}{n}$ *is a Cauchy sequence.*

$a_{n_1} = \frac{(-1)^n}{n} \quad a_{n_2} = \frac{(-1)^{n+k}}{n+k}$

$\lim_{n \to \infty} |a_{n_1} - a_{n_2}| = \lim_{n \to \infty} \frac{1}{n} - \frac{1}{n+k} = 0$

(c) *Show that* $a_n = \frac{(-1)^n(n+3)}{n+2}$ *is not a Cauchy sequence.*

$a_{n_1} = \frac{(-1)^n(n+3)}{n+2} \quad a_{n_2} = \frac{(-1)^{n+k}(n+3+k)}{n+2+k} \quad k \in \mathbb{N}$ - if n even, k odd

$\lim_{n \to \infty} |a_{n_1} - a_{n_2}| = \lim_{n \to \infty} (-1)^n - (-1)^{n+k} \neq 0 \; \forall n, k$ meaning not a Cauchy sequence as it doesn't converge to 0.

(d) *Write the condition* $\lim_{n \to \infty}(a_n) = 0$.

$\forall \varepsilon \in \mathbb{Q}^+ \; \exists m \; \forall n \geq m \; |a_n| < \varepsilon$

(e) *Let* (a_n) *be a Cauchy sequence. Write the condition that* $([a_n]) \neq 0$. *Your sentence should have a "greater than" sign* ($>$) *at the end.*

$\exists \varepsilon \in \mathbb{Q}^+ \; \forall m \; \exists n \geq m \; |a_n| > \varepsilon$

(f) *Show that if $[(a_n)] \neq 0$, then there exists some m such that either for all $n \geq m$ $a_n > 0$ or for all $n \geq m$ $a_n < 0$.*

If $[(a_n)] \neq 0$, then $\exists \epsilon \in \mathbb{Q}^+, \forall m \; \exists n \geq m$ $|a_n| > \epsilon$. This means that a_n has to be $a_n > 0$ or $a_n < 0$ as $\epsilon \in \mathbb{Q}^+$ as $|a_n| > \epsilon$. Additionally $|[(a_n)]| \neq 0$, since $a_n > 0$ a Cauchy sequence then (a_n) converges to some number (or diverges) and so $a_n > 0$ or $a_n < 0$ for all other values of $n \geq m$.

(g) *How do you reconcile your examples in (b) and (c) with your statement in (f)*

No need to reconcile with b) as $[(a_n)] = 0$

For part c, it violates this principal as the limit of the series is alternating between -2, 0, or 2, so it doesn't work, as not for all $n \geq m$ $a_n > 0$ or $a_n < 0$

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