Tufts University
Department of Mathematics

Math 235 **Midterm Exam**[1] **Fall, 2022**

This is an open notes exam and you are allowed to consult your class notes and Heil's textbook.

Please upload the test paper with your answers to Gradescope by 1:30 p.m. on Wednesday, October 19. When you upload your answers, please scan all your answers into one PDF file starting with your signed cover page and with the answers clearly numbered and in order. Please identify the page(s) for each problem as you upload to Gradescope.

Please sign the following pledge and submit the signed pledge with your answers:

The Tufts University statement on academic integrity holds that: "Academic integrity is the joint responsibility of faculty, students, and staff. Each member of the community is responsible for integrity in their own behavior and for contributing to an overall environment of integrity at the university." I accept this responsibility, affirm that I am an honest person who can be trusted with to do the right thing, and certify that the work I will do on this exam is mine alone.

I pledge that I have used only the reference sources I have cited in my answers.

Signature____________________

The test problems start on the next page.

Problem 1. Let $E = \{(x, -2x), x \in [-1, 2]\} \subset \mathbb{R}^2$. Prove that E is Lebesgue measurable and find its measure $|E|$.

Problem 2. Prove that a subset $E \subset \mathbb{R}$ is measurable if and only if for every $\epsilon > 0$ there exist an open set $O \supset E$ and a closed set $F \subset E$ such that $|O \setminus F|_e < \epsilon$.

Problem 3. Suppose that $E = A \cup B$ where A and B are measurable subsets of $\mathbb{R}^d$. Prove that a function $f : E \to \mathbb{R}$ is measurable if and only if f is measurable on A and on B.

Problem 4. Suppose $E \subset \mathbb{R}^d$ is a measurable set of positive Lebesgue measure, and let $f : E \to \mathbb{R}$ be a measurable function.

4.1 Prove that there exists a measurable set $A \subset E$, $|A| > 0$, and such that $\sup_{x \in A} |f(x)| < \infty$.

4.2 Assume in addition that there exists a real number $\delta > 0$ such that $|\{|f| > \delta\}| < \infty$. Prove that for every $\epsilon > 0$, there a bounded measurable function g on E such that $|\{f \neq g\}| \leq \epsilon$.

Problem 5. Let $E \subset \mathbb{R}^d$ be measurable, and let $f_n, f, g : E \to [-\infty, \infty]$ be measurable functions that are finite a.e. Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous.

5.1 Suppose $f_n \xrightarrow{m} f$ and Suppose $f_n \xrightarrow{m} g$, and prove that $f = g$ a.e.

5.2 Suppose $f_n \xrightarrow{m} f$ and φ is uniformly continuous, and prove that $\varphi \circ f_n \xrightarrow{m} \varphi \circ f$.

5.3 Prove that if $f_n \xrightarrow{m} f$ and $|E| < \infty$, then $\varphi \circ f_n \xrightarrow{m} \varphi \circ f$.