Math 65 HW 7

1 a) $\sim$ is reflexive, for $x \in \mathbb{R}$ $x \sim x$, as $|x-x| = 0 \le 2$

b) $\sim$ Symmetric as if $x \sim y$, then $|x-y| \le 2$ and $y \sim x$ is $|y-x| \le 2 \to |-(x-y)| \le 2 \to |x-y| \le 2$ meaning $\sim$ it is symmetric.

c) It is not antisymmetric, as if $x=2, y=1$ $x \sim y$ and $y \sim x$ as $|x-y| = 1 \le 2$ and $|y-x| = 1 \le 2$, but $x \neq y$.

d) ~~Have~~ $x=6$ $y=4$ $z=2$ $|6-4| \le 2$, $|4-2| \le 2$, but $|6-2| \not\le 2$ therefore it is not transitive.

$-2 \le y-z \le 2$

$-2 \le x-y \le 2$

$+ \; -2 \le y-z \le 2$

$-4 \le x-z \le 4, \to |x-z| \le 4$

$|\frac{x}{2} - \frac{z}{2}| \le 2,$

Therefore, $x \sim z$ and $\sim$ is transitive.

2 ~~Take~~ Consider $A = \{1, 2, 3, 4\}$ and the relation $R = \{(1,2), (2,1), (2,2), (1,1)\}$

The relation isn't reflexive as $3 \not\sim 3$.

The relation is symmetric as for every a in each (a,b) pair where $a, b \in \mathbb{N}$, if $a \sim b$ then $b \sim a$ is included.

The relation is also transitive, if we consider $1 \sim 2$ and $2 \sim 1$, by transitive property $1 \sim 1$, which is true. The same can be repeated for all transitive relations.

3 Conditions for equivalence relation:
Reflexive: xRx, any number has same amount of digits as itself.
Symmetric: xRy means x and y have same number of digits so they yRx as they have same number of digits
Transitive: Have $k \in \mathbb{Z}$ if xRy and yRk then xRk as x and y have same amount of digits, but so do y and k, therefore x and k have same amount of digits and the relation is transitive

The relation is an equivalence relation and it creates a partition for each number with a different amount of digits.

4 # equivalence relations = # partitions
partitions
- 1 way to split it into a group of 3: $\{1,2,3\}$
- $\binom{3}{2} = 3$ ways to split into a group of 2 and a group of 1.
- 1 way to put each element ~~into it's own group~~

There are 5 partitions, so there are 5 equivalence relations.

5a) It is well defined each $z \in \mathbb{Z}$ is only defined in one equivalence class making it well defined

b) f_2 is well defined. To prove, need to show for if $z_1, z_2 \in \mathbb{Z}$ and if $[z_1]_{10} = [z_2]_{10}$ then $[z_1]_5 = [z_2]_5$
If condition is true, $\exists k \in \mathbb{Z}$ $z_1 - z_2 = 10k$ and want $\exists k' \in \mathbb{Z}$ such that $z_1 - z_2 = 5k'$
$z_1 - z_2 = 5(2k)$ showing it is equivalent to $z_1 - z_2 = 5k'$ and f_2 is well defined

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5c) f_3 is not well defined. Consider the case where $z_1, z_2 \in \mathbb{Z}$ and $z_1 = 3$, $z_2 = 8$

$$f_3([z_1]_5) = f_3([z_2]_{10})$$

$$f_3([3]_5) = f_3([8]_5)$$

However the outputs, $[3]_{10} \neq [8]_{10}$ meaning f_3 isn't well defined.

6a) Consider that have an $a \in \mathbb{Z}$ our solutions are those that solve $[a+2]_{10} = [8]_{10}$ and let $k \in \mathbb{Z}$

$$a+2 = 10k+8$$

$$a = 10k+6$$

Meaning that the solutions to the equation are $[10k+6]_{10}$ where $k \in \mathbb{Z}$

b) Have $[2a]_{10} = [8]_{10}$

$$\frac{2a}{2} = \frac{10k+8}{2}$$

a is substitute for y meaning

$a = 5k+4$, so $\boxed{y = [5k+4]_{10} \text{ where } k \in \mathbb{Z}}$

c) Have $[2a]_{10} = [3]_{10}$

$$2a = 10k+3 \quad \text{where } k \in \mathbb{Z}.$$

$$2a = 2(5k+1)+1$$

this means the left side is even and the right side is odd so they never equal and there are no solutions.

7a) Conditions for equivalence relation

Reflexive: $(a,b) \sim (a,b)$ true as $a+b = a+b$.

Symmetric (if $(a,b) \sim (c,d)$ then $(c,d) \sim (a,b)$)

if $(c,d) \sim (a,b)$ then $c+b = a+d$ which is flipped equation of $a+d = b+c$, meaning it is symmetric

Reflexive: Have $(e,f) \in (\mathbb{N}-\{0\} \times \{\mathbb{N}-\{0\}\})$

If $(a,b) \sim (c,d)$ $a+d = c+b$ if $(c,d) \sim (e,f)$,

$c+f = d+e$, $c = d+e-f$.

Rewrite $a+d = b+c$ as $a+d = b+d+e-f \rightarrow a+f = b+e$,

$(a,b) \sim (e,f)$

Therefore, $\sim$ is an equivalence relation.

b) To show well defined, let us have

$[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$.

If well defined both sums give same solution.

Let $a', b', c', d' \in \mathbb{N}-\{0\}$

If $[(a,b)] = [(a',b')]$ then $a+b' = b+a'$

and for $[(c,d)] = [(c',d')]$ then $c+d' = d+c'$

Next, taking the sums

$[(a,b)] + [(c,d)] = [(a+c, b+d)]$

$[(a',b')] + [(c',d')] = [(a'+c', b'+d')]$

Lets add the equations from earlier giving us

$a+c+b'+d' = b+a'+d+c'$ which is our sum when added together meaning it is well defined on the cosets.

7c) $\tilde{f}([(a,b)]) = a-b$ on $\sim$, $a+d = b+c$, or $a-b = c-d$.

meaning that each difference for $a-b$ maps to a coset and since we know the difference between 2 numbers can't have 2 values, it is well defined

7 c (cont) We know $\bar{f}$ is one-to-one as

$\bar{f}([a_1, b_1]) = \bar{f}([a_2, b_2]) \Rightarrow a_1 - b_1 = a_2 - b_2$

$a_1 - b_1 + b_1 + b_2 = a_2 - b_2 + b_1 + b_2$

$a_1 + b_2 = a_2 + b_1 \longrightarrow [(a_1, b_1)] = [(a_2, b_2)]$

To prove onto, show $\forall k \in \mathbb{Z}\ \exists\, a, b \in \mathbb{N} - \{0\}$ that $a - b = k$

3 cases to consider:

If $k > 0$:

Set $b = 1$, so $a - 1 = k$, $a = k + 1$, a, b still $\in \mathbb{N} - \{0\}$

if $k = 0$:

$b - a = 0$, $b = a$

if $k < 0$

Set $a = 1$, $1 - b = k$ $1 - k = b$, but since $k < 0$, b is positive and still a natural number.

$\bar{f}$ is one-to-one and onto, meaning it is a bijection, and as earlier is well defined.

d) $\bar{f}([a, b]) + \bar{f}([c, d]) = \bar{f}([a + c, b + d])$

$(a - b) + (c - d) = (a + c) - (b + d)$

$(a - b) + (c - d) = (a - b) + (c - d)$