Consider the relation on $\mathbb{R}^3 - \{(0,0,0)\}$ defined by

$(x, y, z) \sim (x', y', z') \iff$ there exists $c \in \mathbb{R} - \{\mathbf{0}\}$ such that $(cx, cy, cz) = (x', y', z')$.

(1) Verify that $\sim$ is an equivalence relation.

Definition 1. Denote the equivalence class of $(x, y, z) \in \mathbb{R}^3 - \{\mathbf{0}\}$ by $[x : y : z]$.

The set of equivalence classes of $\sim$ is called the **real projective plane** and denoted

$$\begin{aligned}\mathbb{RP}^2 &:= (\mathbb{R}^3 - \{\mathbf{0}\})/\sim \\ &= \{[x : y : z] \mid (x, y, z) \in \mathbb{R}^3 - \{\mathbf{0}\}\}.\end{aligned}$$

(In general, n-dimensional real projective space $\mathbb{RP}^n$ is $(\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$ where two $n+1$-dimensional vectors are equivalent if and only if one is a non-zero scalar multiple of the other.)

(2) What is the equivalence class of $(1, 0, 0)$ as a subset of $\mathbb{R}^3 - \{\mathbf{0}\}$? What about $(1, 1, 1)$? $(0, 1, 0)$?

We consider $\mathbb{RP}^2$ as a topological space by giving it the quotient topology. Write

$$\begin{aligned} p : \mathbb{R}^3 - \{\mathbf{0}\} &\to \mathbb{RP}^2 \\ (x, y, z) &\mapsto [x : y : z] \end{aligned}$$

for the quotient map.

(1) Denote by $D(z)$ the subset of $\mathbb{RP}^2$ defined by

$$D(z) = \{[x : y : z] \in \mathbb{RP}^2 \mid z \neq 0\}$$

(This is well-defined since the third coordinate of (x, y, z) is 0 if and only if the third coordinate of (cx, cy, cz) is 0 for a nonzero number c.)

Show that $D(z)$ is an open subset of $\mathbb{RP}^2$.

(2) Write

$$\tilde{D}(z) := p^{-1}(D(z)) = \{(x, y, z) \in \mathbb{R}^3 - \{0\} \mid z \neq 0\}.$$

Show that the restriction $p_z : \tilde{D}(z) \to D(z)$ of p to $\tilde{D}(z)$ is a quotient map.

(3) Show that

$$\begin{aligned} \varphi : \mathbb{R}^2 &\to D(z) \\ (x, y) &\mapsto [x : y : 1] \end{aligned}$$

is a homeomorphism with inverse

$$\begin{aligned} \psi : D(z) &\to \mathbb{R}^2 \\ [x : y : z] &\mapsto \left(\frac{x}{z}, \frac{y}{z}\right). \end{aligned}$$

(Hint: To show φ is continuous, write it as a composite of functions $\mathbb{R}^2 \to \tilde{D}(z) \to D(z)$. To show ψ is continuous, use the universal property of quotients.)

Remark 2. One can define similar open subsets $D(x)$ and $D(y)$ which are also homeomorphic to $\mathbb{R}^2$. Therefore $\mathbb{RP}^2$ has an open cover by open subsets homeomorphic to $\mathbb{R}^2$. We say that $\mathbb{RP}^2$ is a **topological manifold**.

(4) Any line L in $\mathbb{R}^2$ can be written in the form
$$L = \{(x, y) \in \mathbb{R}^2 \mid ax + by + c = 0\}$$
where $a, b, c \in \mathbb{R}$ and at least one of a, b is nonzero. Consider the set
$$\overline{L} = \{[x : y : z] \in \mathbb{RP}^2 \mid ax + by + cz = 0\}.$$
Show that $\overline{L}$ is well-defined and that $\varphi(L) = \overline{L} \cap D(z)$. How many points are in $\overline{L}$ but not L? (The notation comes from the fact that $\overline{L}$ is the closure of $\varphi(L)$ in $\mathbb{RP}^2$.)

(5) Let L_1 be the line in $\mathbb{R}^2$ defined by $x = 0$ and L_2 the line defined by $x = 1$. Form $\overline{L_1}$ and $\overline{L_2}$ as in the previous problem. Compute and compare $L_1 \cap L_2$ with $\overline{L_1} \cap \overline{L_2}$.

(6) In fact, any algebraic variety in $\mathbb{R}^2$ extends in a nice way to $\mathbb{RP}^2$. Let $f(x,y) = \sum_{i,j} c_{ij}x^i y^j$ be a polynomial of degree d, i.e., the largest value of $i+j$ for which $c_{ij} \neq 0$ is d. Write

$$V = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; f(x,y) = \sum_{i,j} c_{ij}x^i y^j = 0 \right\}$$

for the variety it defines. Consider the set

$$\overline{V} = \left\{ [x:y:z] \in \mathbb{RP}^2 \;\middle|\; \sum_{i,j} c_{ij}x^i y^j z^{d-i-j} = 0 \right\}.$$

Show that $\overline{V}$ is well-defined and $\varphi(V) = \overline{V} \cap D(z)$.