Wednesday, April 12

Friday, April 7, 2023 22:16

We don't have class on Wednesday, April 19, but we do on *Friday, April 21.*

Student Hours :
11:00 AM -12:00 noon on Mondays
2:00-3:00 PM on Thursdays
1:30-2:30+ PM on Fridays in the Math Department Lounge
+ just ask if you want to talk at other times

NOTE: Thursday, 4/20: office hours end at 2:45 (I give a talk in the analysis seminar at 3:00)

Student Hours for final: 24 26
Monday, April ~~23~~, 11:00-12:00 noo***n***
Wednesday, April ~~25~~ 11:00-12:00 27
Thursday, April ~~26~~, 1:30-3:00+
Thursday, April ~~26~~, 4;30-5:45, Review Session, : Nelson Auditorium, Anderson 27

Homework 7 is due Friday, April 14, at 11:59 p.m.

Homework 8 is posted. It has three problems and will be due Friday, April 21. You should be able to do all problems now. It will be our last homework assignment.

There will be no group work this week.

This week we will cover the material in sections 10.1 and 10.2 of Marsden-Hoffman.

On Friday, April 21, we might touch on pointwise convergence of Fourier Series, section 10.3. We *will* start learning about the heat equation (starting on p. 598 of the book)

Marsden and Hoffman is on Canvas. (Files/Marsden-Hoffman)

Students are finding we have a lot of new definitions, so you might want to keep an integration definition and theorem sheet if you don't already have one.

NOTE: an Inner Product Space is a vector space with an inner product

$\langle f, g\rangle$ that is linear in f and satisfies $\langle f,g\rangle = \overline{\langle g,f\rangle}$ and is positive definite, i. e., $\langle f,f\rangle \geq 0$ *and* $\langle f,f\rangle = 0$ *if and only if* $f = 0_V$

$A \subset (B \cup C)$ *if and only if* $B^c \cap C^c \subset A^c$ *where* A^c *etc. are the completements of the sets* A, B, C.

$\mathbb{N}_0 = \{0, 1, 2 \ldots\} = \{0\} \cup \mathbb{N}$

Def V ip space $v \in V$ then v is a unit vector if $\|v\| = 1$

if $w \neq 0$, $w \in V$, $\frac{w}{\|w\|}$ is a unit vector in direction of w.

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Defn Let V be ip space, $\Phi = \{\phi_0, \phi_1, \phi_2, \ldots\}$ (finite or countably inf # vectors in V)

Then Φ is an orthogonal set (og)

if ① $\phi_j \neq 0 \ \forall j$

and ② $\langle \phi_j, \phi_k \rangle = 0$, $j \neq k$

Φ is an orthonormal set (on)

if in addition $\|\phi_j\| = 1$

(equiv. $\langle \phi_j, \phi_k \rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$

eg $\mathbb{C}^n$

$e_1 = (1, 0, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots)$

$\vdots$

$e_n = (0, 0, \ldots, 0, 1)$

$\Phi = \{e_1, e_2, \ldots, e_n\}$ is an orthonormal set

eg ℓ^2 "std basis of ℓ^2"

$e_0 = (1, 0, 0, \ldots 0 \ldots)$

$e_1 = (0, 1, 0, \ldots)$

$e_j = (\underset{0}{0}, \underset{1}{0}, \ldots, \underset{j-1}{0}, \underset{j}{1}, 0, \ldots)$

j^{th} place starting at 0^{th} place

$\Phi = \{e_0, e_1, e_2, \ldots\}$ is an an o.n set.

Given V and Φ orthogonal set in V.

if take $f \in V$ when can we write

$$f = \sum_{j=0}^{\infty} c_j \phi_j \quad (\Phi \text{ is cntbly inf})$$

for some $c_j \in \mathbb{C}$?

for some $c_j \in \mathbb{C}$

ex let V be ip span $\Phi = \{\phi_0, \phi_1, \phi_2, \cdots \phi_N\}$

finite og set in V

assume $f \in V$ and $f = \sum_{j=0}^{N} c_j \phi_j$ find c_j!

let $k \in \{0, \cdots N\}$

calculate $\langle f, \phi_k \rangle = \left\langle \sum_{j=0}^{N} c_j \phi_j , \phi_k \right\rangle$

$= \sum_{j=0}^{N} c_j \langle \phi_j, \phi_k \rangle = c_k \langle \phi_k, \phi_k \rangle$

$\langle f, \phi_k \rangle = c_k \langle \phi_k, \phi_k \rangle$

$\dfrac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} = c_k$

$\therefore \quad f = \sum_{k=0}^{N} \dfrac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k$ if $f \in \operatorname{span} \Phi$

Thm let V be ip space $\Phi = \{\phi_0, \phi_1, \phi_2 \cdots\}$

infinite o system (set) in V

assume $f \in V$ is given

and assume $f = \sum_{j=0}^{\infty} c_j \phi_j$ for some constants $c_j \in \mathbb{C}$

then $c_j = \dfrac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}$

ie $f = \sum_{j=0}^{\infty} \dfrac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} \phi_j$

ex ℓ^2 $a = (a_0, a_1, a_2 \cdots) \in \ell^2$

$\Phi = \{ e_0, e_1, e_2, \cdots \}$ o.n. set.

can I write $a = \sum_{j=0}^{\infty} c_j e_j$?

Can I write $a = \sum_{j=0}^{\infty} c_j e_j$?

$a = (a_0, a_1, a_2, \ldots)$

$= \_\_ e_0 + \_\_ e_1 + \_\_ 2 + \ldots$

$= \underbrace{a_0} (1, 0, 0, \ldots)$

$+ \underbrace{a_1} (0, 1, 0 \ldots)$

$+ \underbrace{a_2} (0, 0, 1, 0 \ldots)$

$\langle a, e_0 \rangle = \langle (a_0, a_1, a_2, \ldots), (1, 0, 0 \ldots) \rangle = a_0$

Thm pf $f \in V$ assume $f = \sum_{j=0}^{\infty} c_j \varphi_j$

goal find c_j

Lemma V ip space $g \in V$

$\Lambda : V \to \mathbb{C}$ defined by $\Lambda(f) = \langle f, g \rangle$

Then Λ is linear & cont.

now let $k \in \mathbb{N}_0$

define $\Lambda(f) = \langle f, \varphi_k \rangle =$

$= \langle \sum_{j=0}^{\infty} c_j \varphi_j , \varphi_k \rangle$

$= \langle \lim_{N \to \infty} \sum_{j=0}^{N} c_j \varphi_j , \varphi_k \rangle$

$= \Lambda \left(\lim_{N \to \infty} \sum_{j=0}^{N} c_j \varphi_j \right)$

$= \lim_{N \to \infty} \Lambda \left(\sum_{j=0}^{N} c_j \varphi_j \right)$

$= \lim_{N \to \infty} \langle \sum_{j=0}^{N} c_j \varphi_j , \varphi_k \rangle$

$= \lim_{N \to \infty} \left(\sum_{j=0}^{N} c_j \langle \varphi_j , \varphi_k \rangle \right) = \lim_{N \to \infty} c_k \langle \varphi_k, \varphi_k \rangle$

$\langle f, \varphi_k \rangle = c_k \langle \varphi_k, \varphi_k \rangle$

$\dfrac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle} = c_k$ ✓.

$\langle \varphi_k, \varphi_k \rangle$

Defn V ip spa Φ og set in V

Φ is a $\underbrace{\text{complete o.g sets}}_{\text{complete}}$

if $\forall f \in V$ $f = \sum_{j=0}^{\infty} \frac{\langle f, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} \varphi_j$

Φ Fourier series of f

if Φ is in addition an o.n. set as well as being complete

if $f \in V$ $f = \sum_{j=0}^{\infty} \langle f, \varphi_j \rangle \varphi_j$

ex in ℓ^2 $\Phi = \{e_0, e_1, e_2, \ldots\}$ is a complete on system in ℓ^2 as we showed!

Not Φ complete on system if above defn holds

V is a Hilbert space if V is a complete ip spa

Factoids about $e^{ix} = \cos x + i \sin x$

$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$ $\overline{e^{i\alpha}} = e^{-i\alpha}$

Thm the On system (exponential Fourier system)

$$\Phi_E = \left\{ \frac{e^{inx}}{\sqrt{2\pi}} \,\middle|\, n \in \mathbb{Z} \right\}$$

is a complete o.n. system in $L^2([0, 2\pi], \mathbb{C})$

Note you can chk Φ_E is o.n. system using Factoids.

[using Factoid ...
[the pf Φ_E is complete is looooong.

$\therefore$ if $f \in L^2([0,2\pi],\mathbb{C})$

then $f = \sum_{n=-\infty}^{\infty} \tilde{A}_n e^{inx}$ in L^2

where the sum is understood as

$$\sum_{n=-\infty}^{\infty} \tilde{A}_n e^{inx} = \lim_{N\to\infty} \sum_{n=0}^{N} \tilde{A}_n e^{inx}$$

$$+ \lim_{\tilde{N}\to-\infty} \sum_{n=-1}^{\tilde{N}} \tilde{A}_n e^{inx}$$

and $\tilde{A}_n = \langle f, \frac{e^{inx}}{\sqrt{2\pi}} \rangle$ (by last th)

$$\tilde{A}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, e^{-inx}\, dx \quad \text{as } e^{-inx} = \overline{e^{inx}}$$

Normalized trig Fourier series

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i\sin x + \cos x - i\sin x}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

from this and completeness of Φ_E

one can show the following th

Thm (normalized trig Fourier series is complete)

$$\Phi_N = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots \right\}$$

is a complete on system in $L^2([0,2\pi],\mathbb{C})$

so if $f \in L^2([0,2\pi],\mathbb{C})$

So if $f \in L^2([0,2\pi], \mathbb{C})$

$$f = A_0 \frac{1}{\sqrt{2\pi}} + A_1 \frac{\cos x}{\sqrt{\pi}} + B_1 \frac{\sin x}{\sqrt{\pi}}$$

$$+ A_2 \frac{\cos 2x}{\sqrt{\pi}} + B_2 \frac{\sin 2x}{\sqrt{\pi}}$$

$$+ A_3 \frac{\cos 3x}{\sqrt{\pi}} + B_3 \frac{\sin 3x}{\sqrt{\pi}} + \cdots$$

$$A_0 = \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\,dx$$

$$A_n = \left\langle f, \frac{\cos nx}{\sqrt{\pi}} \right\rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$B_n = \left\langle f, \frac{\sin(nx)}{\sqrt{\pi}} \right\rangle = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \sin nx\,dx$$

eg $$7.5 = \underbrace{A_0 \frac{1}{\sqrt{2\pi}}}_{= 7.5} + \underbrace{\cancel{A_1}}_{0} \frac{\cos x}{\sqrt{\pi}} + \underbrace{\cancel{B_1}}_{0} \frac{\sin x}{\sqrt{\pi}} + \underbrace{\cancel{A_2}}_{0} \frac{\cos 2x}{\sqrt{\pi}}$$

$$+ \underbrace{\cancel{B_2}}_{0} \frac{\sin 2x}{\sqrt{\pi}} + \cdots + \underbrace{\cancel{A_n}}_{0} \frac{\cos nx}{\sqrt{\pi}}$$

$$+ \underbrace{\cancel{B_n}}_{0} \frac{\sin nx}{\sqrt{\pi}} + \cdots$$

$$7.5 = A_0 \frac{1}{\sqrt{2\pi}}$$

$$7.5\sqrt{2\pi} = A_0 \qquad A_n = B_n = 0$$

$$n = 1, 2, \ldots$$

$$\left\langle \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}} \right\rangle = \int_0^{2\pi} \frac{\sin^2 2x}{\pi}\,dx = 1$$