This is a double homework because of spring break.

A proper subset of the problems will be selected for grading.

Here are some useful theorems. Theorems 1 and 2 are from Math 135 and Theorem 5 will be proven after break. Refer to them by number if you use them.

Theorem 1 *Let A and B be sets in $\mathbb{R}^n$. Then,* $\mathrm{bd}(A \cup B) \subset \mathrm{bd}(A) \cup \mathrm{bd}(B)$.

Theorem 2 *Let $A \subset \mathbb{R}^n$, then the closure of A, $\mathrm{cl}(A)$, is the smallest closed set containing A.*

Theorem 3 *If R is a generalized rectangle in $\mathbb{R}^n$, then R is a Jordan domain.*

Theorem 4 *The finite union of Jordan domains is a Jordan domain. That is, if $A_1, A_2, \dots, A_m$ are Jordan domains then so is $\bigcup_{j=1}^m A_j$.*

Theorem 5 *Let A be a compact set (equiv. sequentially compact set) with measure zero. Then A has volume zero.*

Problems: NOTE: you may not use the results in Marsden-Hoffman that aren't in Fitzpatrick to prove problems 1-6 (and you don't need to).

1. (10 points) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by $f(x,y) = xy^2$. Use the Archimedes-Riemann Theorem and the sequence of $\{\mathbb{P}_n\}$ of regular partitions $\mathbb{P}_n = (\{0, 1/n, 2/n, \dots, 1\}, \{0, 1/n, 2/n, \dots, 1\})$ of $[0,1] \times [0,1]$ to show f is integrable and find $\int_{\mathbb{I}} f$.

   HINT: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

   Solution: When you don't see clever things to do, trying a regular partition is good. So, let P_n be the regular partition of $[0,1]$ with n subintervals, each of length $\Delta x = 1/n$. Let $x_i = i/n$ for $i = 0, \dots, n$ be the endpoints of the subintervals. Let $\mathbb{P}_n = (P_n, P_n)$ and let $\mathbb{I}_{ij} = [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ for i and j in $\{1, \dots, n\}$.

   First we show $\{\mathbb{P}_n\}$ is an Archimedean Sequence for f. Because $f(x,y) = xy^2$ increases whenever x and y increase, the minimum on $\mathbb{I}_{ij}$ is $f((i-1)/n, (j-1)/n) = (i-1)(j-1)^2/n^3$ and the maximum on $\mathbb{I}_{ij}$ is $f(i/n, j/n) = i(j^2)/n^3$. After a bit of calculation and using the hint, one sees

   $$U(f, \mathbb{P}_n) - L(f, \mathbb{P}_n) = \sum_{i=1}^n \sum_{j=1}^n \frac{i(2j-1) + (j-1)^2}{n^5} = \frac{n^3(n+1)}{2n^5} + \frac{(n-1)n(2n-1)}{6n^5}$$

   and the right hand side converges to zero as $n \to \infty$ because the numerator in each term has lower degree than the denominator.

   Now that we know $\{\mathbb{P}_n\}$ is an Archimedean sequence, we note $U(f, \mathbb{P}_n) = \sum_{i=1}^n \sum_{j=1}^n \frac{ij^2}{n^5} = \frac{n^2(n+1)^2(2n+1)}{12n^5}$

   $$\lim_{n\to\infty} U(f, \mathbb{P}_n) = \lim_{n\to\infty} \frac{n^2(n+1)^2(2n+1)}{12n^5} = \frac{1}{6}.$$

   Now, that's more fun than doing the integral using Fubini's theorem and the FTC, right!

2. (10 points) Let S_1 and S_2 be sets in $\mathbb{R}^n$ with Jordan content zero. Use the definition of Jordan content zero to prove that $S_1 \cup S_2$ has Jordan Content zero.

   Solution: We use that the union of a cover of S_1 and a cover of S_2 to cover their union. Let $\epsilon > 0$. Since S_1 has JC0, there are a finite collection of rectangles, $R_1, R_2, \dots, R_N$ that cover S_1 and $\sum_{i=1}^N \mathrm{Vol}(R_i) < \epsilon/2$. Similarly, since S_2 has JC0, there are a finite collection of rectangles, $J_1, J_2, \dots, J_M$ that cover S_2 and $\sum_{k=1}^M \mathrm{Vol}(J_k) < \epsilon/2$. So, the collection of rectangles $\{R_1, R_2, \dots, R_N, J_1, J_2, \dots, J_M\}$ cover $S_1 \cup S_2$ and the sum of their volumes, $\sum_{i=1}^N \mathrm{Vol}(R_i) + \sum_{k=1}^N \mathrm{Vol}(J_k) < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore, $S_1 \cup S_2$ had JC0.

3. (10 points) Let A_1 and A_2 be Jordan domains. Prove that $A_1 \cup A_2$ also is also a Jordan domain.

NOTE: you may not use Theorem 4 at the start of the test to prove this result. This is the first step in an induction proof of Theorem 4.

Solution: Note that $\text{bd}(A_1)$ has JC0 because A_1 is a Jordan domain. Similarly, $\text{bd}(A_2)$ has JC0. Therefore, $\text{bd}(A_1) \cup \text{bd}(A_2)$ has JC0 by problem 2 of this assignment. By Theorem 1 on this homework assignment, $\text{bd}(S_1 \cup S_2)$ is a subset of $\text{bd}(S_1) \cup \text{bd}(S_2)$ which has JC0. Since the union $\text{bd}(S_1) \cup \text{bd}(S_2)$ has JC0, $S_1 \cup S_2$ is a Jordan domain.

4. (15 points) Let $f : [0,1]^2 \to \mathbb{R}$ be a bounded function that is continuous on the interior, $(0,1)^2$. Prove that f is integrable on $[0,1]^2$. The proof outlined below uses a couple of arguments similar to those in the proof of the Jordan Integrability Theorem.

Assume $M > 0$ such that $-M \le f(\mathbf{x}) \le M$, $\forall \mathbf{x} \in [0,1]^2$. Recall the partition from group work 5: Let $\delta \in (0, 1/2)$, then $\mathbb{P}_\delta = (\{0, \delta, 1-\delta, 1\}, \{0, \delta, 1-\delta, 1\})$.

(a) Why is there a partition $\mathbb{P}'$ of $[\delta, 1-\delta] \times [\delta, 1-\delta]$ such that $U(f, \mathbb{P}') - L(f, \mathbb{P}') < \epsilon/2$?

Note, I should have chosen $\epsilon > 0$ in the problem. Let's do it now! Let $\epsilon > 0$!

Solution: Because f is continuous on $\text{int}([0,1]^2)$, f is continuous on the rectangle $[\delta, 1-\delta]^2$. Because of this, f is integrable on $[\delta, 1-\delta]^2$. By Riemann's condition, there is a partition $\mathbb{P}'$ of $[\delta, 1-\delta]^2$ such that $U(f, \mathbb{P}') - L(f, \mathbb{P}') < \epsilon/2$.

(b) Let $\mathbb{P}^* = \mathbb{P}_\delta \overline{\cup} \mathbb{P}'$. How can you choose δ such that $U(f, \mathbb{P}^*) - L(f, \mathbb{P}^*) < \epsilon$?

HINT: divide the rectangles in $\mathbb{P}^*$ into those contained in $[\delta, 1-\delta]^2$ and those that meet this rectangle only on their boundary.

Solution: There was some question in the TA's office hours about how to interpret $\mathbb{P}_\delta \overline{\cup} \mathbb{P}'$ and here is how. You take the union of the points in the respective partitions that make up $\mathbb{P}_\delta$ and $\mathbb{P}'$. that is:

$$\mathbb{P}_\delta = (\{0, \delta, 1-\delta, 1\}, \{0, \delta, 1-\delta, 1\}$$
$$\mathbb{P}' = (\{\delta, x_1, x_2, \ldots, x_{n-1}, 1-\delta\}, \{\delta, y_1, y_2, \ldots, y_{m-1}, 1-\delta\})$$

where $\delta = x_0 < x_1 < x_2 < \ldots, x_{n-1} < x_n = 1-\delta$ and $\delta = y_0 < y_1 < y_2 < \ldots, y_{m-1} < y_n = 1-\delta$. So, the union is

$$\mathbb{P}_\delta \overline{\cup} \mathbb{P}' = (\{0, \delta, x_1, x_2, \ldots, x_{n-1}, 1-\delta, 1\}, \{0, \delta, y_1, y_2, \ldots, y_{n-1}, 1-\delta, 1\}).$$

Now to the proof. We will calculate δ at the end, and for now, let $\delta \in (0, 1/2)$.

The rectangles in $\mathbb{P}^*$ that are in $[\delta, 1-\delta]^2$ are those from $\mathbb{P}'$ since $\mathbb{P}_\delta$ has no partition points in $(\delta, 1-\delta)$. Therefore,

$$\begin{aligned} U(f, \mathbb{P}^*) - L(f, \mathbb{P}^*) &= \sum_{J \text{ in } \mathbb{P}^*} (M(f,J) - m(f,J)) \operatorname{Vol}(J) \\ &= \sum_{J \text{ in } \mathbb{P}^*, J \subset [\delta, 1-\delta]^2} (M(f,J) - m(f,J)) \operatorname{Vol}(J) \qquad (1) \\ &\quad + \sum_{J \text{ in } \mathbb{P}^*, J \not\subset [\delta, 1-\delta]^2} (M(f,J) - m(f,J)) \operatorname{Vol}(J) \qquad (2) \end{aligned}$$

The sum (1) is less than equal to $\epsilon/2$ since those rectangles are those from $\mathbb{P}'$.

We now estimate the sum (2) and then choose δ so this sum is bounded above by $\epsilon/2$:

$$\begin{aligned} \sum_{J \text{ in } \mathbb{P}^*, J \not\subset [\delta, 1-\delta]^2} (M(f,J) - m(f,J)) \operatorname{Vol}(J) &\le \sum_{J \text{ in } \mathbb{P}^*, J \not\subset [\delta, 1-\delta]^2} (M - (-M)) \operatorname{Vol}(J) \\ &\le 2M(1 - (1-2\delta)^2) = 2M(4\delta - 4\delta^2) < 8M\delta \end{aligned}$$

This tells us that we need to choose $0 < \delta < \epsilon/(16M)$ and then the sum (2) is less than $\epsilon/2$ so $U(f, \mathbb{P}^*) - L(f, \mathbb{P}^*) < \epsilon$. This shows f is integrable on $[0,1]^2$!

5. (15 points) Let A be a set of Jordan content zero in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded function. Prove that f is integrable and that $\int_A f = 0$. You may use the following steps or develop your own proof.

Let $\mathbb{I}$ be a generalized rectangle such that $A \subset \mathbb{I}$, and let $\hat{f}$ be the zero extension of f to $\mathbb{I}$.

Let $M > 0$ such that $-M \le \hat{f}(\mathbf{x}) \le M$ for all $\mathbf{x} \in \mathbb{I}$. Let $\epsilon > 0$.

You may use any part of the problem, in a subsequent part, even if you are not sure how to prove it.

(a) Explain why there is a finite set of generalized rectangles $\{R_1, R_2, \dots, R_N\}$ contained in $\mathbb{I}$ such that $A \subset \bigcup_{i=1}^N R_i$ and $\sum_{i=1}^N \mathrm{Vol}(R_i) < \frac{\epsilon}{2M}$.

Solution: As A has Jordan content zero, there are a finite number of generalized rectangles in $\mathbb{R}^n$, $S_1, \dots, S_N$ that cover A and $\sum_{i=1}^N \mathrm{Vol}(S_i) < \frac{\epsilon}{2M}$.

For each $i = 1, \dots, N$, let $R_i = S_i \cap \mathbb{I}$. Then, R_i is contained in $\mathbb{I}$. Furthermore, each R_i is a generalized rectangle since it is the intersection of two generalized rectangles. Therefore, the rectangles $R_1, \dots, R_N$ are all rectangles in $\mathbb{I}$ that cover A (since A is a subset of $\mathbb{I}$), and such that $\sum_{i=1}^N \mathrm{Vol}(R_i) \le \sum_{i=1}^N \mathrm{Vol}(S_i) < \frac{\epsilon}{2M}$.

Now, let $F = \bigcup_{i=1}^N R_i$

(b) Explain why the characteristic function $\mathbb{1}_F$ is integrable.

Solution: This follows from the conclusion Theorem 4 on this homework assignment since F is the finite union of generalized rectangles which are Jordan domains (by Theorem 3) Therefore, F is a Jordan domain. As we showed in class, the characteristic function of any Jordan domain is integrable as Jordan domains have volume.

(c) Show that $\int_{\mathbb{I}} M \mathbb{1}_F \le \frac{\epsilon}{2}$.

Solution: I claim $\mathbb{1}_F(\mathbf{x}) \le \sum_{i=1}^N \mathbb{1}_{R_i}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

By the definition of F, if $\mathbb{1}_F(\mathbf{x}) = 1$, then $\mathbf{x} \in R_{i_0}$ for at least one $i_0 \in \{1, 2, \dots, N\}$. Therefore, $\mathbb{1}_{R_{i_0}}(\mathbf{x}) = 1$. This proves the claim for $\mathbf{x} \in F$. Now, if $\mathbf{x} \notin F$, then $\mathbf{x}$ is in no R_i and so $\sum_{i=1}^N \mathbb{1}_{R_i}(\mathbf{x}) = 0 = \mathbb{1}_F(\mathbf{x})$. This shows the claim.

Therefore,

$$\int M \mathbb{1}_F \le \int M \left(\sum_{i=1}^N \mathbb{1}_{R_i} \right) = \sum_{i=1}^N M \,\mathrm{Vol}(R_i) < M \frac{\epsilon}{2M} = \frac{\epsilon}{2}.$$

(d) Explain why $-M\mathbb{1}_F(\mathbf{x}) \le \hat{f}(\mathbf{x}) \le M\mathbb{1}_F(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{I}$.

Solution: For $\mathbf{x} \in F$, $-M = -M\mathbb{1}_F(\mathbf{x}) \le \hat{f}(\mathbf{x}) = f(\mathbf{x}) \le M\mathbb{1}_F(\mathbf{x}) = M$ since M is an upper bound for $|f|$. For $\mathbf{x} \notin M$, all terms are equal to zero, so the assertion holds.

(e) Show that $-\frac{\epsilon}{2} \le \underline{\int}_{\mathbb{I}} \hat{f} \le \overline{\int}_{\mathbb{I}} \hat{f} \le \frac{\epsilon}{2}$. HINT: you may assume that if g and h are bounded functions on $\mathbb{I}$ and $g \le h$ then $\underline{\int} g \le \underline{\int} h$ and $\overline{\int} g \le \overline{\int} h$.

Solution: We use the hint for this part of the problem

$$
\begin{aligned}
-\frac{\epsilon}{2} < \int -M\mathbb{1}_F = \underline{\int} -M\mathbb{1}_F \quad & \text{by part (c) and that } \mathbb{1}_F \text{ is integrable} \\
\le \underline{\int}_{\mathbb{I}} \hat{f} \le \overline{\int}_{\mathbb{I}} \hat{f} \quad & \text{by the hint to this part of the problem and then because lower integrals} \\
& \quad \text{are less than or equal to upper integrals} \\
\le \overline{\int}_{\mathbb{I}} M\mathbb{1}_F < \frac{\epsilon}{2} \quad & \text{where we use the monotonicity of upper integrals,} \\
& \quad \text{that } \mathbb{1}_F \text{ is integrable, and part (c).}
\end{aligned}
$$

(f) Now explain why $\hat{f}$ is integrable on $\mathbb{I}$ and find $\int_{\mathbb{I}} \hat{f}$.

Solution: We have that for every $\epsilon > 0$ the *numbers* $\underline{\int}_{\mathbb{I}} \hat{f}$ and $\overline{\int}_{\mathbb{I}} f$ are in the interval $(-\epsilon/2, \epsilon/2)$, and so they must both be equal to zero. Therefore, $\hat{f}$ is integrable on $\mathbb{I}$ with integral zero.

(g) (1 point) Briefly explain why f is integrable on A and why $\int_A f = 0$.

Solution: In the last part of the problem, we showed $\hat{f}$ is integrable on $\mathbb{I}$ with integral zero. Since $\hat{f}$ is integrable on $\mathbb{I}$, f is integrable on A by definition. Furthermore, $\int_A f = \int_{\mathbb{I}} \hat{f}$ by definition, so $\int_A f = 0$.

6. (10 points) Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let $f : \mathbb{I} \to \mathbb{R}$ be integrable. Assume $\int f \neq 0$. Prove there is a nontrivial generalized rectangle $R \subset \mathbb{I}$ such that $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in R$.
SMALL HINT: WLOG, you may assume $\int f > 0$.

Solution: As the hint suggests, WLOG assume $\int f > 0$.

By Theorem 18.7 Archimedean-Riemann Theorem, there is an Archimedean sequence of partitions $\{\mathbb{P}_k\}$ for $f : \mathbb{I} \to \mathbb{R}$ s.t., $\lim_{k\to\infty} L(f, \mathbb{P}_k) = \int f := M > 0$ Hence there is some k_0 such that the partition $\mathbb{P}' = \mathbb{P}_{k_0}$ has $L(f, \mathbb{P}') = \sum_{J \text{ in } \mathbb{P}'} \inf\{f(x)|x \in J\} > M/2 > 0$. For this to be true, at least one of the terms, $\inf\{f(x)|x \in J\}$ must be positive, thus $f(x) > 0$ on this rectangle, J. Thus $f \neq 0$ for all $\mathbf{x}$ in one of the rectangles in $\mathbb{P}'$. This proves the claim.

7. (10 points)

(a) Let S be a bounded set with Jordan content zero. Prove that $\mathrm{cl}(S)$ has Jordan content zero.
Solution: We use the fact that $\mathrm{cl}(S)$ is the smallest closed set containing S by Theorem 2. So, let $\epsilon > 0$ and let $\{S_1, S_2, \ldots, S_N\}$ be rectangles that cover S and for which $\Sigma_{i=1}^N Vol(S_i) < \epsilon$. Since $S_1 \cup S_2 \cup \ldots \cup S_N$ is closed and contains S, $S_1 \cup S_2 \cup \ldots \cup S_N$ must contain cl(S) by the fact we just gave. Since $\Sigma_{i=1}^N Vol(S_i) < \epsilon$ and $S_1 \cup S_2 \cup \ldots \cup S_N$ covers $\mathrm{cl}(S)$, $\mathrm{cl}(S)$ has Jordan Content zero.

(b) If S has measure zero, does $\mathrm{cl}(S)$ have measure zero? Why or why not?

Solution: No, the rational number set $\mathbb{Q} \cap [0, 1]$ is a set with measure zero but $\mathrm{cl}(\mathbb{Q} \cap [0, 1]) = [0, 1]$ does not have Jordan content zero (since $[0, 1]$ is a set with volume and that volume is positive, not zero volume=JC0).

8. (10 points) Let A be a bounded subset of $\mathbb{R}^n$. Prove that A has volume if and only if $\mathrm{bd}(A)$ has Jordan content zero.

Solution: By Marsden-Ch8 Lebesgue's Theorem 8.3.2 Corollary, bounded set $A \subseteq \mathbb{R}^n$ has volume iff the $\mathrm{bd}(A)$ has measure zero. This follows from Lebesgue's Theorem. As noted in class, Jordan Content zero implies measure zero. So $\mathrm{bd}(A)$ has Jordan content 0 implies it has measure 0.

Since A is bounded, thus $\mathrm{bd}(A)$ is also bounded. Also $\mathrm{bd}(A) = (\mathrm{int}(A) \cup \mathrm{ext}(A))^c$ thus it is closed. By the Heine-Borel Theorem, a bounded closed subset of $\mathbb{R}^n$ is compact. By Theorem 5 in the beginning of homework, a compact set with measure 0 also has volume 0 thus Jordan content 0 by class proof.

For bounded A, $\mathrm{bd}(A)$ is measure 0 iff it is Jordan content 0.

A has volume iff bd(A) is Jordan content zero.

9. (10 points) Let $\mathbb{I}$ be a generalized rectangle and let f be an integrable function from $\mathbb{I}$ to $\mathbb{R}$. Recall that $D(f, \mathbb{I}) = \{\mathbf{x} \in \mathbb{I} \mid f \text{ is not continuous at } \mathbf{x}\}$ is the set of discontinuities of f on $\mathbb{I}$

(a) Let $g(\mathbf{x})$ be the composite function $g(\mathbf{x}) = \sin(f(\mathbf{x}))$. Prove that $D(g, \mathbb{I}) \subset D(f, \mathbb{I})$. HINT: This is equivalent to showing that if f is continuous at $\mathbf{x}_0 \in \mathbb{I}$ then $g = \sin \circ f$ is continuous at $\mathbf{x}_0$.
Solution: As the hint suggests it suffices to show if f is continuous at $\mathbf{x}_0 \in \mathbb{I}$ then $g = \sin \circ f$ is continuous at $\mathbf{x}_0$. For any sequence $\{x_n\} \to \mathbf{x}_0$, by the continuity of f, $\lim\limits_{n\to\infty} f(x_n) = f(\mathbf{x}_0)$, then by the continuity of sin, $\lim\limits_{n\to\infty} \sin \circ f(x_n) = sin\left(\lim\limits_{n\to\infty} f(x_n)\right) = \sin(f(\mathbf{x}_0))$, hence $\sin \circ f$ is continuous at $\mathbf{x}_0$.
To finish this off, we note that the contrapositive of "if f is continuous at $\mathbf{x}_0 \in \mathbb{I}$ then $\sin \circ f$ is also continuous is: if $\sin \circ f$ is not continuous at $\mathbf{x}_0$, then f is not continuous; that is $D(\sin \circ f, \mathbb{I}) \subset D(f, \mathbb{I})$.

(b) Prove that g is integrable on $\mathbb{I}$.

Solution: By Lebesgue's Theorem and integrability of f, $D(\sin\circ f, \mathbb{I})$ has measure zero, being a subset of the set $D(f, \mathbb{I})$, which has measure zero by assumption.

Here are optional challenge problems that will give you extra points if you successfully do them. Todd will grade them.

You may do up to three *of them for credit.*

1. (1 point) Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let A be a subset of $\mathbb{I}$ with Jordan content zero. Assume $f : \mathbb{I} \to \mathbb{R}$ is integrable and assume $g : \mathbb{I} \to \mathbb{R}$ is bounded and

$$g(\mathbf{x}) = f(\mathbf{x}), \ \forall \mathbf{x} \in \mathbb{I} \smallsetminus A.$$

Prove that g is integrable on $\mathbb{I}$ and that $\int_{\mathbb{I}} g = \int_{\mathbb{I}} f$. You may use any other homework problem on this assignment.

Solution: Let $h = g - f$, then h is zero except on the Jordan Content zero set A. By problem 5 on this HW, h is integrable on $\mathbb{I}$ and $\int_{\mathbb{I}} h = 0$.

Since both h and f are integrable on $\mathbb{I}$, $g = h + f$ is integrable on $\mathbb{I}$ and $\int_{\mathbb{I}} g = \int_{\mathbb{I}} h + \int_{\mathbb{I}} f = 0 + \int_{\mathbb{I}} f = \int_{\mathbb{I}} f$.

2. (1 points) Let $S_1, S_2, \dots, S_m$ be a finite number of sets of Jordan content zero. Prove that $\bigcup_{j=1}^m S_j$ also has Jordan content zero.

Solution: We use induction. The induction hypothesis for n sets is:

hyp(n) = *The union of n sets of Jordan content zero has JC0.*

hyp(1) is trivial: the union of one set, S_1 of JC0 is S_1, and S_1 has JC0 by assumption.

Now, assume hyp(n). namely, if $S_1, S_2, \dots, S_n$ are n sets of JC0, then their union has JC0.

Let $S_1, S_2, \dots, S_n, S_{n+1}$ be $n+1$ sets of JC0. Then,

$$\bigcup_{j=1}^{n+1} S_j = \left(\bigcup_{j=1}^{n} S_j\right) \bigcup S_{n+1}.$$

By the induction hypothesis $\bigcup_{j=1}^n S_j$ has JC0, and by problem 2 on this HW the union of two sets of JC0 has JC0, so $\left(\bigcup_{j=1}^n S_j\right) \bigcup S_{n+1}$ has JC0.

3. (1 point) Let S be a subset of $\mathbb{R}^n$ with measure zero. Prove that $\operatorname{int}(S) = \varnothing$.

Solution: Assume $\operatorname{int}(S) \neq \varnothing$. As $\operatorname{int}(S)$ is open and nonempty, $\operatorname{int}(S)$ contains an open ball $B_r(\mathbf{x})$ for some $\mathbf{x} \in \operatorname{int}(S)$ and $r > 0$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ then you can check the rectangle $J = [x_1, x_1 + \delta] \times [x_2, x_2 + \delta] \times \cdots \times [x_n, x_n + \delta]$ where $\delta = \dfrac{r}{2\sqrt{n}}$ is contained in $B_r(\mathbf{x})$. If S has measure zero, then J also has measure zero being a subset of S. However, J is compact and compact sets with measure zero have volume zero = JC0. However, J has positive volume equal to δ^n! This finishes problem since we proved contrapositive of the statement in the problem.

4. (1 points) Prove that if R is a generalized rectangle in $\mathbb{R}^n$, then R is a Jordan domain (that is, prove Theorem 3).

Solution: Because this is worth one point and it's late, I'll be brief (and I will expect you to be brief, too). Let J be a generalized rectangle in $\mathbb{R}^n$. We will show each face of J has JC0 so $\operatorname{bd}(J)$ has JC0. Let $J = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$. We consider the face $F = \{a_1\} \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Let $V = \operatorname{Vol}([a_2, b_2] \times \cdots \times [a_n, b_n])$ in $\mathbb{R}^{n-1}$ and let $\epsilon > 0$. Let $R = [a_1, a_1 + (\epsilon/(2V))] \times [a_2, b_2] \times \cdots [a_n, b_n]$. Then, R covers F and $\operatorname{Vol}(R) = V(\epsilon/(2V)) = \epsilon/2 < \epsilon$. This shows F has JC0. You can do the sam arguments for all the other faces and so $\operatorname{bd}(J)$ has JC0 being the finite union of JC0 sets.