Problem 1. A streaming service wants to model the annual growth rate of its subscribers. The data shows that they can roughly predict the behavior of customers based on how long they have been users of the service: Let p_a be the current number of subscribers who have had the service for less than a year,and p_b be the current number of subscribers who have had the service for more than a year (so in total there are $p_a + p_b$ subscribers). For convenience we will refer to the two types of subscriber as "new" and "old". Let p= $\begin{bmatrix} p_n \\ p_o \end{bmatrix}$ be a vector recording the current subscriber population, and let p^* be the subscriber population in a year's time. Then the data shows that p^* = Ap, where: A = $\begin{bmatrix} 1 & 0 \\ 0.8 & 0.5 \end{bmatrix}$

1. What percentage of current new users will still be subscribed by the start of the next year? How does this retention rate differ for old users?

   To solve, we know that p^* is the new population in a year, so we just do $A * p$ and get that $p^* = \begin{bmatrix} p_n \\ 0.8p_n + 0.5p_0 \end{bmatrix}$
   This means that 80% of the new subscribers and 50% of the old subscribers will remain at the beginning of next year.

2. Make the (unrealistic) assumption that every new user the following year will sign up for the service after it is recommended to them by a current user. On average, how many new users next year will be recommended the service by a current user?

   For every successive year, the total number of subscribers are $(0.8p_n + 0.5p_0) + (p_n) = (1.8p_n + 0.5p_0)$, there are p_n new subscribers, but of those $0.8p_n + 0.5p_0$ were recommended, which means on average, $\frac{(0.8p_n+0.5p_0)}{p_n}$ came from being recommended by prior users.

3. It turns out that A has eigenvalues λ = 1,0.5. Find an eigenvector for the eigenvalue $\lambda = 1$ and use it to describe the long term behavior of the user population for this streaming service. In the long run, what will the ratio of new to old users be?

   Using Python, we can solve the equation and get the eigenvector $\begin{bmatrix} \frac{5}{8} \\ 1 \end{bmatrix}$, meaning in the long run, the ratio of new to old users will be $\frac{5}{8}$ to 1

4. Suppose that a different streaming service has the rate matrix, B = $\begin{bmatrix} 0.5 & 0 \\ 0.8 & 0.4 \end{bmatrix}$ which has λ = 0.5,0.4 as its eigenvalues. Using the eigenvector for the larger of the two eigenvalues, what is the long term behavior of this streaming service?

   For $\lambda = 0.5$, the eigenvector is $\begin{bmatrix} \frac{1}{8} \\ 1 \end{bmatrix}$ In the long term, the $\frac{8}{9}$ of the users will be prior members will $\frac{1}{9}$ will consist of new members.

5. Suppose that both streaming services (which we will refer to by their rate matrices A and B, start with the same population vector, p, this year, and that both charge $n for a yearly subscription. Write an expression (in terms of n,p_a,p_b) for the total income obtained by A and B after 4 years.

   If price is held constant, then the income after any year is equal to $n * (p_n + p_0)$. For streaming service A, revenue in each year:
   year 1: $(n * (p_n + p_0))$
   year 2: $(n * (p_n + p_0) + n * (1.8p_n + 0.5p_0))$
   year 3: $(n * (p_n + p_0) + n * (1.8p_n + 0.5p_0) + n * (2.2p_n + 0.25p_0))$

year 4: $(n*(p_n+p_0)+n*(1.8p_n+0.5p_0)+n*(2.7p_n+0.25p_0)+n*(3.15p_n+0.125p_0))$

For streaming service B, revenue is: year 1: $(n*(p_n+p_0)$
year 2: $(n*(p_n+p_0)+n*(1.8p_n+0.4p_0)$
year 3: $(n*(p_n+p_0)+n*(1.8p_n+0.4p_0)+n*(2.2p_n+0.20p_0)$
year 4: $(n*(p_n+p_0)+n*(1.8p_n+0.4p_0)+n*(2.7p_n+0.20p_0)+n*(3.15p_n+0.10p_0)$
year 4 revenue for streaming service B is $n(8.65p_n+1.870p_0)$