Math 135 **Real Analysis I** **Fall 2022**

Solutions to Final Exam Review[1]

In the following problems, unless otherwise specified, X and Y will denote metric spaces and A, B, etc. will denote subsets of X.

1. (a) Define what it means for a point $p \in X$ to be a *boundary point* of A (i.e., $a \in \text{bd}(A)$).

   Answer: *$p \in X$ is a boundary point of A if for all $\epsilon > 0$, the ball $B_\epsilon(p)$ contains a point x in A and a point y in the complement A^c.*

   (b) Define the *closure* $\text{cl}(A)$ of the set A.

   Answer: *The closure of A is the set $cl(A) = A \cup bd(A)$.*

   (c) Prove that a point $p \in X$ belongs to $\text{cl}(A)$ if and only if every ball $B_r(p)$ contains a point of A.

   Answer: *If $p \in cl(A)$, then $p \in A$ or $p \in bd(A)$. In either case, every ball $B_r(p)$ will contain a point of A.*

   Conversely, suppose that every ball $B_r(p)$ contains a point of A. We prove that either $p \in A$ or $p \in bd(A)$. (This will of course prove that $p \in cl(A)$. If $p \in A$, then we're done. If $p \notin A$, then every open ball $B_r(p)$ contains a point of A (by hypothesis) and a point not in A, namely p. Thus $p \in bd(A)$.

2. (a) Define what it means for $A \subset X$ to be *sequentially compact*.

   Answer: *A is* sequentially compact *provided that every sequence $\{p_k\}$ in A has a subsequence $\{p_{k_j}\}$ that converges to a point $p \in A$.*

   (b) Define what it means for $A \subset X$ to be *complete*.

   Answer: *A is* complete *provided that any Cauchy sequence $\{p_k\}$ in A converges to a point $p \in A$.*

   (c) Show that every sequentially compact set is complete.

   Answer: *Let A be a sequentially compact subset of X. Let us prove that A is complete. To this end, suppose that $\{p_k\}$ is a Cauchy sequence in A. Since A is sequentially compact, there is a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ that converges to a point $p \in A$.*

   We claim that the whole sequence $\{p_k\}$ converges to p. For this, let $\epsilon > 0$; since $\{p_k\}$ is Cauchy, there exists an index N_1 such that $d(p_k, p_l) < \epsilon/2$ for all $k, l \geq N_1$. Since $p_{k_j} \to p$ as $k_j \to \infty$, there is an index N_2 such that $d(p_{k_j}, p) < \epsilon/2$ for all $k_j \geq N_2$.

   Let $N = \max\{N_1, N_2\}$. Now fix some $k_j \geq N$. Then whenever $k \geq N$, we have

$$d(p_k, p) \leq d(p_k, p_{k_j}) + d(p_{k_j}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

3. (a) Define what it means for a subset B of A to be *relatively open* in A. Then define what it means for a subset C of A to be *relatively closed* in A.

   Answer: *B is* relatively open *in A provided that $B = A \cap U$ for some open set U in X. C is* relatively closed *in A provided that $C = A \cap D$ for some closed set D in X.*

   (b) Prove that a subset B of A is relatively closed in A if and only if $A \setminus B$ is relatively open in A.

   Answer: *By definition, B is relatively closed in A provided that $B = A \cap C$, for some closed set C in X.*

   Now suppose that B is relatively closed in A. Let C be a closed set with $B = A \cap C$, and put $U = C^c = X \setminus C$. Then U is open, and $A \setminus B = A \cap C^c = A \cap U$, is relatively open in A.

   Conversely, suppose that $A \setminus B$ is relatively open in A. Then $A \setminus B = A \cap U$, for some open set U in X. Then the set $C = U^c$ is closed, and since

$$B = A \setminus (A \setminus B) = A \setminus U = A \cap U^c = A \cap C,$$

   B is relatively closed in A.

   (c) Prove that A is disconnected if and only if A has a proper, nonempty subset B that is both relatively open and relatively closed in A.

   Answer: *Suppose that A is disconnected. Then there exists open sets U and V which separate A. Let $B = A \cap U$, and let $C = A \cap V$. B and C are clearly relatively open, and A is the disjoint union $A = B \sqcup C$. Since $B = A \setminus C$, Part (b) above shows that B and C are also relatively closed in A. Finally, since both B and C are nonempty, B is a nonempty proper subset of A.*

4. (a) Define what it means for a mapping $f : A \to Y$ to be *continuous* at a point $p_0 \in A$.

   Answer: *f is* continuous *at p_0 provided that whenever $\{p_k\}$ is a sequence in A that converges to p, the image sequence $\{f(p_k)\}$ converges in Y to $f(p_0)$.*

   (b) Define what it means for a mapping $f : A \to Y$ to be *uniformly continuous.*

   Answer: *f is* uniformly continuous *on A provided that whenever $\{p_k\}$ and $\{q_k\}$ are any two sequences in A for which $d(p_k, q_k) \to 0$ as $k \to \infty$, we have $d(f(p_k), f(q_k)) \to 0$ as $k \to \infty$.*

   (c) Prove that the function $f(x) = 1/(x^2 + 1)$ is continuous on $\mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

   Answer: *First we prove that f is continuous on $\mathbb{R}$ using the ϵ-δ criterion.*

Fix $x_0 \in \mathbb{R}$. For any pair of points $x \in \mathbb{R}$, we have

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \frac{1}{x^2+1} - \frac{1}{x_0^2+1} \right| \\ &= \left| \frac{x_0^2 - x^2}{(x^2+1)(x_0^2+1)} \right| \\ &= \left| \frac{x + x_0}{(x^2+1)(x_0^2+1)} \right| \cdot |x - x_0| \end{aligned}$$

Since $|x|/(x^2+1) \leq 1/2$ as $2|x| \leq x^2+1$ since $0 \leq (|x|^2-1)^2$. So, the first absolute value in (1) is equal to $1/2 + 1/2 = 1$ and so

$$|f(x) - f(x_0)| \leq 1|x - x_0| \tag{1}$$

for all $x \in \mathbb{R}$.

For continuity: Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$ and let $\delta = \epsilon$; $\delta > 0$ as $\epsilon > 0$. If $|x - x_0| < \epsilon$, then by (1), $|f(x) - f(x_0)| < \epsilon$.

For uniform continuity: You could use (1) to show f is Lipschitz, but here is a proof using the ϵ-δ condition for uniform continuity. Let $\epsilon > 0$ and let $\delta = \epsilon$. Let u and v be in $\mathbb{R}$ and $|u - v| < \delta$. Then, by (1), $|f(u) - f(v)| < \epsilon$, f is uniformly continuous.

5. Let X and Y be metric spaces and $A \subset X$. Let $f : A \to Y$ be continuous. Let $B \subset A$ be connected and open, and let $K \subset A$ be sequentially compact. Let $\mathcal{B} \subset Y$ be connected and open, and let $\mathcal{K} \subset Y$ be sequentially compact. What can you say about the following sets?

   (a) $f(A)$ (b) $f(B)$ (c) $f(K)$
   (d) $f^{-1}(\mathcal{K})$ (e) $f^{-1}(\mathcal{B})$ (f) $f^{-1}(f(B))$

   (When a statement doesn't follow directly from a theorem, try examples with specific functions.)

   Answers:

   (a) $f(A)$: nothing!

   (b) $f(B)$: connected.

   (c) $f(K)$: sequentially compact.

   (d) $f^{-1}(\mathcal{K})$: closed (since any sequentially compact subset of Y is closed in Y).

   (e) $f^{-1}(\mathcal{B})$: open.

   (f) $f^{-1}(f(B))$: contains B, but that's about it!

6. For each $k \in \mathbb{N}$, let $f_k(x) = \dfrac{x}{x+k}$, $f_k : [1, \infty) \to \mathbb{R}$.

(a) Prove $\{f_k\}$ converges pointwise to $f(x) = 0$ on $[1, \infty)$ using the definition of pointwise convergence and the definition of limits of sequences in $\mathbb{R}$.

Answer: *Fix any $x \in [1, \infty)$. Then*

$$\lim_{k\to\infty} \frac{x}{x+k} = \lim_{k\to\infty} \frac{\frac{x}{k}}{\frac{x}{k}+1} = \frac{0}{0+1} = 0.$$

Thus the sequence $\{f_k\}$ converges pointwise to 0 on $[1, \infty)$.

(b) Prove $\{f_k\}$ converges uniformly to $f(x) = 0$ on $[1, 2]$ using the definition of uniform convergence.

Answer: *Note that $f_k'(x) = \frac{k}{(x+k)^2} > 0$ for $x > 0$, so the function f_k is increasing on $[1, 2]$. In particular, we see that on $C([1, 2], \mathbb{R})$,*

$$d(f_k, 0) = \max_{x\in[1,2]} |f_k(x) - 0| = f_k(2) = \frac{2}{2+k} \to 0$$

as $k \to \infty$. Thus $\{f_k\}$ converges to 0 in $C([1, 2], \mathbb{R})$, and in particular $\{f_k\}$ converges uniformly to 0 on $[1, 2]$.

(c) Prove $\{f_k\}$ does not converge uniformly to $f(x) = 0$ on $[1, \infty)$.

Answer: *Suppose, to the contrary, that $\{f_k\}$ converges uniformly to 0 on $[1, \infty)$. Then for $\epsilon = \frac{1}{2}$, there exists an index N such that $|f_k(x) - 0| < \frac{1}{2}$ whenever $k \geq N$, for all $x \in [1, \infty)$. In particular, we would have $|f_N(x)| < \frac{1}{2}$ for all $x \in [1, \infty)$. This means that*

$$\frac{x}{x+N} < \frac{1}{2}$$

for all $x \geq 1$, which by a bit of algebra implies that $x < N$ for all $x \geq 1$, a contradiction.

7. Let $f_k(x) = 1/(kx + 1)$ and $g_k(x) = x/(kx + 1)$ on $[0, 1]$.

(a) Find the pointwise limit of the sequence $\{f_k\}$ on $[0, 1]$.

Answer: *Note first that $f_k(0) = 1$ for all k. If $x \in (0, 1]$, we have*

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \frac{1}{kx+1} = 0.$$

Thus, on $[0, 1]$, the sequence $\{f_k\}$ converges pointwise to the discontinuous function $g(x)$ given by

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

(b) Find the pointwise limit of the sequence $\{g_k\}$ on $[0, 1]$.

Answer: *Note that $g_k(0) = 0$ for all k. If $x \in (0, 1]$, we have*

$$\lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} \frac{x}{kx+1} = 0.$$

Thus the sequence $\{g_k\}$ converges pointwise on $[0, 1]$ to the constant function 0.

(c) Is $\{f_k\}$ Cauchy in $C([0,1],\mathbb{R})$?

Answer: *No, because $\{f_k\}$ were Cauchy, then $\{f_k\}$ would converge in $C([0,1],\mathbb{R})$, since $C([0,1],\mathbb{R})$ is complete. This would imply that $\{f_k\}$ converges uniformly to its pointwise limit $g(x)$, which as we have seen, is not continuous.*

(d) Is $\{g_k\}$ Cauchy in $C([0,1],\mathbb{R})$?

Answer: *Yes, because in fact $\{g_k\}$ converges to 0 in $C([0,1],\mathbb{R})$. To see this, note that for each k, $g_k'(x) = \frac{1}{(kx+1)^2} > 0$, so $g_k(x)$ is increasing (and nonnegative) on $[0,1]$. Hence*

$$d(g_k, 0) = \max_{x\in[0,1]} |g_k(x) - 0| = g_k(1) = \frac{1}{k+1} \to 0$$

as $k \to \infty$.

8. Give an example of a sequence of continuous functions $\{f_k\}$ that converges pointwise to a function that is not continuous. Why doesn't this contradict Theorem 9.31?

Answer: *Examples abound. For instance, see the solution to Problem 7a above. For another example, consider the sequence of functions $\{h_k\}$ on $[0,2]$, given by*

$$h_k(x) = \frac{x^k}{x^k+1}.$$

If $x \in [0,1)$, then

$$\lim_{k\to\infty} h_k(x) = \lim_{k\to\infty} \frac{x^k}{x^k+1} = \frac{0}{0+1} = 0.$$

If $x \in (1,2]$, we have

$$\lim_{k\to\infty} h_k(x) = \lim_{k\to\infty} \frac{x^k}{x^k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{x^k}} = 1.$$

Finally $h_k(1) = \frac{1}{2}$ for all k. Thus the sequence $\{h_k\}$ converges pointwise to the discontinuous function $h(x)$ given by

$$h(x) = \begin{cases} 0 & \textit{if } 0 \le x < 1 \\ \frac{1}{2} & \textit{if } x = 1 \\ 1 & \textit{if } 1 < x \le 2. \end{cases}$$

Our examples don't contradict 9.31 because none of the sequences above converges uniformly. (If the convergence were uniform, the limit functions would be continuous.)

9. $A \subset \mathbb{R}^n$ be connected and nonempty and let $f : A \to \mathbb{R}$ be continuous. Assume $f(x) \neq 0$ for all $x \in A$. Prove either that $f(x) > 0\ \forall x \in A$ or that $f(x) < 0\ \forall x \in A$

Answer: *By the Intermediate Value Theorem, the image $f(A)$ is an interval in $\mathbb{R}$. By hypothesis, this interval does not contain 0. Since intervals in $\mathbb{R}$ are convex, the interval $f(A)$ must either lie completely inside the positive axis $(0,\infty)$ on completely inside the negative axis $(-\infty,0)$.*

10. Recall that $C([a,b],\mathbb{R})$ denotes the vector space of all continuous functions $f : [a,b] \to \mathbb{R}$.

(a) Prove that

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

is a metric on $C([a,b],\mathbb{R})$.

Answer: *If f and g are in $C([a,b],\mathbb{R}$, then the function $|f(x)-g(x)|$ is continuous on $[a,b]$, and so attains a maximum value on $[a,b]$. Thus*

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|.$$

It is clear that $d(f,g) = d(g,f)$ and that $d(f,g) \geq 0$ for all f, $g \in C([a,b],\mathbb{R})$. Also

$$\begin{aligned} d(f,g) = 0 &\iff \max_{x\in[a,b]} |f(x) - g(x)| = 0 \\ &\iff |f(x) - g(x)| = 0 \textit{ for all } x \in [a,b] \\ &\iff f(x) = g(x) \textit{ for all } x \in [a,b] \\ &\iff f = g. \end{aligned}$$

Now for the triangle inequality. Suppose that f, g, and h are in $C([a,b],\mathbb{R}$. Then for any fixed $x \in [a,b]$, we have

$$\begin{aligned} |f(x) - h(x)| &\leq |f(x) - g(x)| + |g(x) - h(x)| \\ &\leq d(f,g) + d(g,h) \end{aligned}$$

Taking the maximum of the left hand side above over all $x \in [a,b]$, we obtain $d(f,h) \leq d(f,g) + d(g,h)$, as desired.

(b) Let

$$U = \{f \in C([a,b],\mathbb{R}) \mid f(x) > 0 \text{ for all } x \in [a,b]\}$$

Is U open in $C([a,b],\mathbb{R})$? Why or why not?

Answer: *U is open in $C([a,b],\mathbb{R})$. To show this, we will prove that any $f \in U$ is interior to U. Now if $f \in U$, then $f(x) > 0$ for all $x \in [a,b]$, so the minimum value m of f (which is attained at some point $x_0 \in [a,b]$, by the Extreme Value Theorem) must be positive.*

We claim that the ball $B_m(f)$ in $C([a,b],\mathbb{R})$ lies inside U. (This will prove that f is interior to U.) For any $g \in B_m(f)$, we have $d(g,f) < m$, so $|g(x) - f(x)| < m$ for all $x \in [a,b]$. In particular, $g(x) - f(x) > -m$ for all $x \in [a,b]$, whence

$$g(x) > f(x) - m > 0$$

for all $x \in [a,b]$. This shows that $g \in U$.

(c) Let

$$T = \{f \in C([a,b],\mathbb{R}) \mid f(x) \geq 0 \text{ for all } x \in [a,b]\}$$

Is T closed in $C([a,b],\mathbb{R})$? Why or why not?

Answer: *T is a closed subset of $C([a,b],\mathbb{R})$. To see this, suppose that $\{f_k\}$ is a sequence of functions in T that converges to a function $f \in C([a,b],\mathbb{R})$. We'll show that $f \in T$.*

Now, using the result of (d), we see that $\{f_k\}$ converges uniformly, and hence pointwise, to the function f on $[a,b]$. Fix a point $x \in [a,b]$. Then $f_k(x) \geq 0$ for each k, so $f(x) = \lim_{k\to\infty} f_k(x) \geq 0$. Since x is arbitrary, it follows that $f \in T$.

(d) Prove that a sequence $\{f_k\}$ in $C([a,b],\mathbb{R})$ converges to a point $f \in C([a,b],\mathbb{R})$ if and only if the sequence of functions $\{f_k\}$ converges uniformly to f on $[a,b]$.

Answer:

$$\begin{aligned}
&\{f_k\} \textit{ converges to } f \textit{ in } C([a,b],\mathbb{R}) \\
&\iff \textit{for every } \epsilon > 0, \textit{ there exists an index } N \textit{ such that } d(f_k, f) \leq \epsilon \textit{ for all } k \geq N \\
&\iff \textit{for every } \epsilon > 0, \textit{ there exists an index } N \textit{ such that } \max_{x\in[a,b]} |f_k(x) - f(x)| \leq \epsilon \\
&\hspace{30em} \textit{for all } k \geq N \\
&\iff \textit{for every } \epsilon > 0, \textit{ there exists an index } N \textit{ such that } |f_k(x) - f(x)| \leq \epsilon \\
&\hspace{20em} \textit{for all } k \geq N \textit{ and for all } x \in [a,b] \\
&\iff \{f_k\} \textit{ converges uniformly to } f \textit{ in } [a,b].
\end{aligned}$$

11. Use the facts that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x$ and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) = y$ are both continuous to prove that the square

$$\{(x,y) \in \mathbb{R}^2 \,\big|\, -1 < x < 1,\ -1 < y < 1\} = (0,1)^2$$

is open in $\mathbb{R}^2$.

Answer: *Let (x_1,y_1) and (x_2,y_2) be points in $\mathbb{R}^2$. Then,*

$$|f(x_1,y_1) - f(x_2,y_2)| = |x_1 - x_2| \leq \|(x_1,y_1) - (x_2,y_2)\|,$$

so f is Lipschitz with Lipschitz constants one. Similarly, g is Lipschitz and so f and g are continuous (really uniformly continuous).

Now Let $U = f^{-1}(-1,1)$ and $V = g^{-1}(-1,1)$. Since $(-1,1)$ is an open subset of $\mathbb{R}$, the inverse images U and V must be open subsets of $\mathbb{R}^2$. Hence $U \cap V$ is an open subset of $\mathbb{R}^2$. But this is precisely the square above!

12. Prove that the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k$ converges uniformly on $[-r,r]$ for any $r > 0$. Then prove that the limit function $f(x)$ is continuous on $(-\infty,\infty)$.

Answer: *We use the Weierstrass M test. For this, we need to obtain a suitable upper estimate of the absolute value of each term in the series above. Let $r > 0$. On $[-r, r]$,*

$$\left|\frac{1}{(k!)^2}x^k\right| = \frac{1}{(k!)^2}|x|^k \leq \frac{1}{(k!)^2}r^k := M_k.$$

Since

$$\frac{M_{k+1}}{M_k} = \frac{(k!)^2}{\big((k+1)!\big)^2}\frac{r^{k+1}}{r^k} = \frac{r}{(k+1)^2} \to 0 < 1 \text{ as } k \to \infty,$$

by the ratio test, $\sum_{k=0}^{\infty} M_k$ converges. By the Weierstrass M-test, the series $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(k!)^2}x^k$ converges uniformly on $[-r, r]$. Since each of the summands $\dfrac{1}{(k!)^2}x^k$ is a continuous function on $[-r, r]$, the partial sums are continuous. By Theorem 9.31, the limit of continuous functions under uniform convergence is continuous. Therefore, $f(x)$ is continuous on $[-r, r]$. Given any real number $x_0 \in \mathbb{R}$, there is a real number $r > 0$ such that $|x_0| < r$. Then $x_0 \in (-r, r) \subset [-r, r]$, so f is continuous at x_0. Since x_0 is an arbitrary real number, f is continuous on $\mathbb{R} = (-\infty, \infty)$.

13. Let $K \subset \mathbb{R}^n$ be closed and bounded. Prove that K is sequentially compact.

    Answer: *Let K be a closed and bounded subset of $\mathbb{R}^n$ and let $\{x_k\}$ be a sequence in K. As K is bounded and $\{x_k\}$ is a sequence in K, $\{x_k\}$ is a bounded sequence. By Theorem 11.17, every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence (fun fact: you don't need to know the number of this theorem, just its statement). Therefore, there is a convergent subsequence $\{x_{k_\ell}\}$ that converges to some $x \in \mathbb{R}^n$.*

    Now, we show $x \in K$. As K is closed and $\{x_{k_\ell}\}$ is a sequence in K that converges to x, the limit, x must be in K. Thus, every sequence in K has a subsequence that converges to a point in K and $K$is sequentially compact.

14. (a) Let K be a sequentially compact subset of a metric space X. Prove that K is closed and bounded.

    Answer: *K is closed: Let $\{p_k\}$ be any sequence in K which converges to a point $p \in X$. Since K is sequentially compact, there exists a subsequence $\{p_{k_j}\}$ of $\{p_k\}$ which converges to a point $p' \in K$. But since $\{p_k\}$ converges to p, its subsequence $\{p_{k_j}\}$ must also converge to p. Thus $p' = p$, and so $p \in K$.*

    K is bounded: Suppose, to the contrary, that K is not bounded. Then for any point $p \in X$, K is not contained in any open ball $B_r(p)$ centered at p. Now fix one point $p \in X$. For any $k \in \mathbb{N}$, K contains a point p_k which does not lie in the ball $B_k(p)$. (Note that the sequence $\{p_k\}$ is unbounded, since $d(p_k, p) \geq k$ for all k.)

    Since K is sequentially compact, the sequence $\{p_k\}$ has a convergent subsequence $\{p_{k_j}\}$. Any convergent sequence is bounded, so in particular, the subsequence $\{p_{k_j}\}$ is bounded. But $d(p_{k_j}, p) \geq k_j$ for all j, which implies that $\{p_{k_j}\}$ is unbounded. This is a contradiction.

(b) Is the converse to Part (a) above true? That is, is any closed and bounded subset of any metric space X sequentially compact?

Answer: *No. Let X be an infinite set with the discrete metric. Then X is closed; X is bounded because for any point $p \in X$, the open ball $B_2(p)$ coincides with X.*

On the other hand, X is not sequentially compact. For this, we note that since X is infinite, there exists a sequence $\{p_k\}$ in X consisting of distinct points. (Since X is infinite, it must have at least a countable number of distinct points, and those points provide such a sequence.) If X is sequentially compact, the sequence $\{p_k\}$ would have a subsequence $\{p_{k_j}\}$ which converges to a point $p \in X$. In particular, there is an index N such that $d(p_{k_j}, p) < 1$ whenever $j \geq N$. Thus $p_{k_j} = p$ for all $j \geq N$, contradicting the fact that $\{p_k\}$ consists of distinct points.

15. Let X and Y be metric spaces and let $f : X \to Y$ be continuous. Let K be a sequentially compact subset of X. Prove that the image $f(K)$ is sequentially compact.

Answer: *Consider any sequence $\{y_k\}$ in $f(K)$. For each k, choose any point $p_k \in K$ such that $f(p_k) = y_k$. Since K is sequentially compact, the sequence $\{p_k\}$ (which is a sequence in K) has a subsequence $\{p_{k_j}\}$ which converges to a point $p \in K$. Let $y = f(p)$. Then $y \in f(K)$, and since f is continuous at p, we have $y_{k_j} = f(p_{k_j}) \to f(p) = y$.*

This proves that $f(K)$ is a sequentially compact subset of Y.

16. Let X be a metric space and let $K \subset X$ be sequentially compact. Assume $f : K \to \mathbb{R}$ is continuous. Prove the Extreme Value Theorem for f and K.

Answer: *We'll just repeat the proof of Theorem 11.20 and Lemma 11.21 in the text, which states the same result with $X = \mathbb{R}^n$.*

Since $f : K \to \mathbb{R}$ is continuous (if $f : X \to Y$ is continuous then f is continuous on any subset of X) and K is sequentially compact, $f(K)$ is sequentially compact by problem 15

Since $f(K)$ is sequentially compact, $f(K)$ is closed and bounded. Hence the numbers $M = \sup f(K)$ and $m = \inf f(K)$ (which exist because $f(K)$ is bounded) belong to $f(K)$. In particular, there exist points p and q in K such that $f(p) = M$ and $f(q) = m$.

Therefore, the function f thus attains a maximum and a minimum value on K.

17. Let X be a metric space and let $A \subset X$ be connected. Assume $f : A \to \mathbb{R}$ is continuous. Prove the Intermediate Value Theorem for f and A.

Answer: *We want to prove that the image $f(A)$ is a convex subset of $\mathbb{R}$; i.e., is an interval. Suppose, to the contrary, that $f(A)$ is not convex. Then there exists three points a, b, and c in $\mathbb{R}$ such that $a < c < b$, with $a, b \in f(A)$ and $c \notin f(A)$. Then $f(A) \subset (-\infty, c) \cup (c, \infty)$, and the sets $f(A) \cap (-\infty, c)$, $f(A) \cap (c, \infty)$ are both nonempty.*

Let $B = f^{-1}(-\infty, c)$. Since $c \notin f(A)$, we also have $B = f^{-1}(-\infty, c]$. Since B is the inverse image of an open set as well as a closed set, B is both relatively open and relatively closed in A. B is nonempty, since $a < c$ and $a \in f(A)$, and B is proper since $c < b$ and $b \in f(A)$.

Since B is a nonempty proper subset of A that is both relatively open and relatively closed in A, Problem 3c shows that A is not connected, a contradiction.

18. Let A and B be closed subsets of $\mathbb{R}$. Prove that $A \times B = \left\{(a, b) \in \mathbb{R}^2 \,\middle|\, a \in A,\ b \in B\right\}$ is closed using the definition of closed set.

    Answer: *Recall that $p_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the first component, $p_1(a, b) = a$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the second, $p_2(a, b) = b$.*

    Let $(\mathbf{x}_k)$ be a sequence in $A \times B$ that converges to $(x_0, y_0) \in \mathbb{R}^2$. By the Componentwise Convergence Theorem, $p_1(\mathbf{x}_k) \to x_0$, and $p_2(\mathbf{x}_k) \to y_0$.

    Furthermore, the sequence $(p_1(\mathbf{x}_k))$ is a sequence in A since $\mathbf{x}_k \in A \times B$. Since A is closed in $\mathbb{R}$, the limit of this sequence, x_0, must be in A. Similarly, since the sequence $(p_2(\mathbf{x}_k))$ is a sequence in B and B is closed in $\mathbb{R}$, the limit of this sequence, y_0, must be in B. This shows that the limit of $(\mathbf{x}_k)$, (x_0, y_0) is in $A \times B$. Therefore, $A \times B$ is closed.