Bruce M. Boghosian

The Behrens-Fisher problem

Testing $H_0 : \sigma_X^2 = \sigma_Y^2$

Summary

Two-sample inferences

The Behrens-Fisher problem

Testing $H_0 : \sigma_X^2 = \sigma_Y^2$

Bruce M. Boghosian

Department of Mathematics

Tufts University

Outline

The Behrens-Fisher problem

- Can we test $H_0 : \mu_X = \mu_Y$ if $\sigma_X \neq \sigma_Y$?
- This is still an unsolved problem in statistics.
- Instead of $T = \frac{\overline{X} - \overline{Y}}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}}$, a widely used approximation is

$$W = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}}$$

- **Thm. (Welch 1938):** W is approximately distributed like a Student t distribution with

$$\frac{\left(\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right)^2}{\frac{\sigma_X^4}{n^2(n-1)} + \frac{\sigma_Y^4}{m^2(m-1)}},$$

degrees of freedom, rounded to the nearest integer.

Deficiencies of Welch's solution

- **Thm. (Welch 1938):** W is approximately distributed like a Student t distribution with

$$\frac{\left(\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}\right)^2}{\frac{\sigma_X^4}{n^2(n-1)}+\frac{\sigma_Y^4}{m^2(m-1)}},$$

df, rounded to the nearest integer.

- When $\sigma_X = \sigma_Y$, Welch's expression for df becomes

$$\frac{\left(\frac{1}{n}+\frac{1}{m}\right)^2}{\frac{1}{n^2(n-1)}+\frac{1}{m^2(m-1)}},$$

  - does exhibit symmetry in m and n, as it should.
  - does reduce to $2m-2$ when $m=n$, as it should.
  - does not reduce to $m+n-2$, as it should.

- Caution is urged unless $m \approx n$.

Testing $H_0 : \sigma_X^2 = \sigma_Y^2$

- Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be independent random samples from $N(\mu_X, \sigma_X)$ and $N(\mu_Y, \sigma_Y)$, respectively.
  - To test $H_0 : \sigma_X^2 = \sigma_Y^2$ versus $H_1 : \sigma_X^2 > \sigma_Y^2$ at the α level of significance, reject H_0 if $s_Y^2/s_X^2 \leq F_{\alpha,m-1,n-1}$.
  - To test $H_0 : \sigma_X^2 = \sigma_Y^2$ versus $H_1 : \sigma_X^2 < \sigma_Y^2$ at the α level of significance, reject H_0 if $s_Y^2/s_X^2 \geq F_{1-\alpha,m-1,n-1}$.
  - To test $H_0 : \sigma_X^2 = \sigma_Y^2$ versus $H_1 : \sigma_X^2 \neq \sigma_Y^2$ at the α level of significance, reject H_0 if either (a) $s_Y^2/s_X^2 \leq F_{\alpha/2,m-1,n-1}$ or (b) $s_Y^2/s_X^2 \geq F_{1-\alpha/2,m-1,n-1}$.

Example

- People with Raynaud's syndrome have impaired blood circulation to the fingers, causing heat loss.
- Measurements of heat output of fingers of $n = 10$ normal subjects, and $m = 10$ subjects with Raynaud's syndrome

x (wo/ RS)	2.43	1.83	2.43	2.70	1.88	1.96	1.53	2.08	1.85	2.44
y (wi/ RS)	0.81	0.70	0.74	0.36	0.75	0.56	0.65	0.87	0.40	0.31

- We have $\overline{x} = 2.11$, $s_X = 0.37$, $\overline{y} = 0.62$, $s_Y = 0.20$.
- It is evident that $\overline{Y} < \overline{X}$, but what about the variances?

Example (continued)

- Test $H_0 : \sigma_X^2 = \sigma_Y^2$ versus $H_1 : \sigma_X^2 \neq \sigma_Y^2$
- Level of significance is $\alpha = 0.05$.
- Reject H_0 if either
  - $s_Y^2 / s_X^2 \leq F_{\alpha/2, m-1, n-1}$
  - $s_Y^2 / s_X^2 \geq F_{1-\alpha/2, m-1, n-1}$
- We have $F_{0.025,9,9} = 0.248$ and $F_{0.975,9,9} = 4.03$.
- Since ${}^{s_Y^2}\!/\!{}_{s_X^2} = 0.292$, we are unable to reject the null hypothesis that $\sigma_X^2 = \sigma_Y^2$.

Summary

- We have discussed Behrens-Fisher problem, and presented an approximate solution, with some deficiencies, but we note this is an important unsolved problem of statistics.
- We have tested $H_0 : \sigma_X^2 = \sigma_Y^2$ and given an example.