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The Method of Moments:

The uniform distribution and wealth distributions

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Outline

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General methodology

- Make n measurements of Y, $Y_j = y_j$ for $j = 1, \ldots, n$.
- Posited distribution has s parameters, $f_Y(y; \theta_1, \ldots, \theta_s)$
- Set s *moments*, equal to corresponding *sample moments*

$$E(Y) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y = \frac{1}{n} \sum_{j=1}^{n} y_j$$

$$E(Y^2) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y^2 = \frac{1}{n} \sum_{j=1}^{n} y_j^2$$

$$\ldots$$

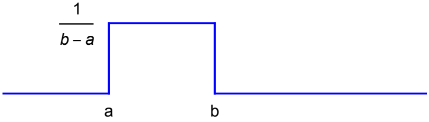
$$E(Y^s) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y^s = \frac{1}{n} \sum_{j=1}^{n} y_j^s$$

- Yields s simultaneous equations for the s parameters.

The uniform distribution and its moments

- Suppose $X \in \mathbb{R}$ has the *uniform pdf*,

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } X \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$



- *Moments* of the uniform pdf:

Moment	Expression	Result
Normalization	$E(1) = \int_{\mathbb{R}} dx\ f_X(x)$	$= 1$
Mean	$E(X) = \int_{\mathbb{R}} dx\ f_X(x)x$	$= \frac{b+a}{2}$
Mean square	$E(X^2) = \int_{\mathbb{R}} dx\ f_X(x)x^2$	$= \frac{a^2+ab+b^2}{3}$

Method of moments for uniform distribution

- There are two parameters, so set first two moments equal to sample moments,

$$E(X) = \frac{a+b}{2} = M_1 := \frac{1}{n}\sum_{j=1}^{n} x_j$$
$$E(X^2) = \frac{a^2+ab+b^2}{3} = M_2 := \frac{1}{n}\sum_{j=1}^{n} x_j^2$$

- Solve for a and b to obtain estimators

$$\hat{a}(\vec{x}) = M_1 - \sqrt{3}\sqrt{M_2 - M_1^2}$$
$$\hat{b}(\vec{x}) = M_1 - \sqrt{3}\sqrt{M_2 - M_1^2}$$

- Mean plus or minus $\sqrt{3}$ times standard deviation.
- This is very different from MLE estimates (min and max).

Comparison of both methods

- Sample n random points uniformly on interval $[3, 5]$

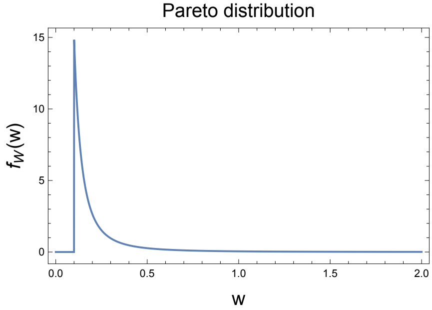
	MLE		MM	
n	a	b	a	b
10	3.07110	4.96605	3.44151	4.93097
50	3.03101	4.93344	3.11829	4.99247
100	3.00624	4.99622	2.91091	4.97592
500	3.01072	4.99533	2.94429	4.96859
1000	3.00406	4.99752	3.03049	5.03114
5000	3.00084	4.9993	3.01044	5.00693
10000	3.00013	4.99969	3.00128	5.00224

The Pareto distribution for wealth

- Vilfredo Pareto (1848-1923)
- Two-parameter *Pareto distribution* for wealth

$$f_W(w) = \begin{cases} \theta k^\theta w^{-\theta-1} & \text{for } w \geq k \\ 0 & \text{otherwise} \end{cases}$$

- *Pareto exponent* θ is exponent of power-law decay; cutoff at low wealth is k.



Moments of the Pareto distribution

- The mth moment exists only if $\theta > m$,

$$E(W^m) = \frac{k^m\theta}{\theta - m}$$

- Theoretical mean and mean square:

$$E(W) = \frac{k\theta}{\theta - 1}$$
$$E(W^2) = \frac{k^2\theta}{\theta - 2}.$$

- Problem: For most countries, $1 < \theta < 2$ so M_2 does not exist, due to extremely high-wealth individuals.

Estimation for the Pareto distribution

- Method of moments approach

$$E(W) = \frac{k\theta}{\theta - 1} = M_1 := \frac{1}{n}\sum_{j=1}^{n} w_j$$

$$E(W^2) = \frac{k^2\theta}{\theta - 2} = M_2 := \frac{1}{n}\sum_{j=1}^{n} w_j^2.$$

- Note M_1 and M_2 are known from data.
- Problem because $E(W^2)$ often does not exist.
- Hybrid (MLE / MM) approach may be best.

$$\hat{k} = \min_j w_j$$

$$\hat{\theta} = \frac{M_1}{M_1 - \min_j w_j}.$$

Summary

- We reviewed *method of moments* for finding estimators.
- We applied MM to the uniform distribution.
- We applied MM to the Pareto distribution for wealth.