# Homework 3 Problem 2 Solution

**Problem 1.** Let $X_1, X_2$ be sets. Note that there are "inclusion" functions $j_1 : X_1 \cap X_2 \to X_1$, $j_2 : X_1 \cap X_2 \to X_2$, $i_1 : X_1 \to X_1 \cup X_2$ and $i_2 : X_2 \to X_1 \cup X_2$, all defined by the rule $x \mapsto x$, and the following diagram commutes:

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "i_2"] \\
X_1 \arrow[r, "i_1"] & X_1 \cup X_2.
\end{tikzcd}$$

Prove that $X_1 \cup X_2$ has the following universal property:

For all sets $B$ and functions $f_1 : X_1 \to B$ and $f_2 : X_2 \to B$ such that

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"] & B.
\end{tikzcd}$$

commutes, there exists a unique function $f : X_1 \cup X_2 \to B$ such that

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "i_2"] \arrow[ddr, bend left, "f_2"] & \\
X_1 \arrow[r, "i_1"] \arrow[drr, bend right, "f_1"] & X_1 \cup X_2 \arrow[dr, dashed, "f"] & \\
 & & B
\end{tikzcd}$$

commutes. (This is a special case of the "Universal property of the pushout.")

*Proof.* Suppose $B$ is a set and $f_1 : X_1 \to B$ and $f_2 : X_2 \to B$ are functions such that

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "f_2"] \\
X_1 \arrow[r, "f_1"] & B.
\end{tikzcd}$$

commutes. We want to show there exists a unique function $f : X_1 \cup X_2 \to B$ such that

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "i_2"] \arrow[ddr, bend left, "f_2"] & \\
X_1 \arrow[r, "i_1"] \arrow[drr, bend right, "f_1"] & X_1 \cup X_2 \arrow[dr, dashed, "f"] & \\
 & & B
\end{tikzcd}$$

commutes.

Let's begin by assuming that there is such a function $f$, and we'll see what has to be true about it.

Since

$$\begin{tikzcd} X_1 \arrow[r, "i_1"] \arrow[drr, bend right, "f_1"'] & X_1 \cup X_2 \arrow[dr, dashed, "f"] & \\ & & B \end{tikzcd}$$

commutes, $f \circ i_1 = f_1$. So for all $x \in X_1$, we have

$$f_1(x) = f(i_1(x)) = f(x).$$

This tells us what $f$ must be on elements of $X_1$. Symmetrically, since

$$\begin{tikzcd} X_2 \arrow[d, "i_2"] \arrow[dr, bend left, "f_2"] & \\ X_1 \cup X_2 \arrow[r, dashed, "f"] & B \end{tikzcd}$$

commutes, $f \circ i_2 = f_2$. So for all $x \in X_2$, we have

$$f_2(x) = f(i_1(x)) = f(x).$$

Altogether

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in X_1 \\ f_2(x) & \text{if } x \in X_2. \end{cases}$$

Our assumptions have determined the value of $f(x)$ for all $x \in X_1 \cup X_2$. This means that if there is a function $f$ that makes the diagram commute, it is uniquely determined by $f_1$ and $f_2$, as desired.

To complete the proof let's show that there exists such a function $f$. We'll have to write down a definition for $f$, check that it defines a function, and show that it makes all the necessary diagrams commute. From our proof of uniqueness, we have a good idea what $f$ should be. Let's try

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in X_1 \\ f_2(x) & \text{if } x \in X_2. \end{cases}$$

Now, this assigns a value to each element of $X_1 \cup X_2$, but for it to be well-defined we need exactly one value for each input. In other words, we need to check that for all $x \in X_1 \cap X_2$ (the $x$'s that land in both "if"s), we have $f_1(x) = f_2(x)$. Fortunately, we know

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"] & B. \end{tikzcd}$$

commutes, i.e. $f_1 \circ j_1 = f_2 \circ j_2$. Then for all $x \in X_1 \cap X_2$,
$$f_1(x) = f_1(j_1(x)) = f_2(j_2(x)) = f_2(x).$$
Therefore $f$ is well-defined.

To conclude the proof, let's check that the diagram

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"] & X_2 \arrow[d, "i_2"] \arrow[ddr, bend left, "f_2"] & \\
X_1 \arrow[r, "i_1"] \arrow[drr, bend right, "f_1"'] & X_1 \cup X_2 \arrow[dr, dashed, "f"] & \\
& & B
\end{tikzcd} \tag{1}$$

commutes. We know that the square commutes from the problem statement. To finish showing that the diagram commutes, we need to show

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"] \arrow[dr, bend right, "f_1"'] & X_1 \cup X_2 \arrow[d, dashed, "f"] \\
& B
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
X_2 \arrow[d, "i_2"] \arrow[dr, bend left, "f_2"] & \\
X_1 \cup X_2 \arrow[r, dashed, "f"] & B
\end{tikzcd}$$

commute. That is, we want to show
$$f \circ i_1 = f_1 \quad \text{and } f \circ i_2 = f_2.$$
(There are other paths in the big diagram (1) to worry about - for example, there are 4 paths from $X_1 \cap X_2 \to B$, and we technically need these to all agree for the big diagram to commute. However, they automatically agree once the square and two triangles commute. It may be worth convincing yourself that's true.)

By definition of $f$, for any $x \in X_1$, $f(i_1(x)) = f(x) = f_1(x)$. Therefore $f \circ i_1 = f_1$. Symmetrically, for any $x \in X_2$, $f(i_2(x)) = f(x) = f_1(x)$. Therefore, there exists an $f$ making the required diagram commute, and we are done. $\square$