Tufts University
Department of Mathematics

Math 136 **Answers to Homework 3** [1] **Spring, 2023**

Reading assignment: Read Sections 14.1, and 15.2 by Wednesday, February 8. Section 15.1 can be helpful to remind you of the background on linear algebra.

This homework covers Sections 13.3, 14.1 and the definition of derivative for $f : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$.

Problems:

1 (20 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^3 + xy + 1$. Explain why the graph of $z = f(x,y)$ has a tangent plane at $(x,y) = (1,1)$ and find an equation of this tangent plane.

Solution: As f is a polynomial, f is in $C^1(\mathbb{R}^2, \mathbb{R})$. Therefore, f satisfies the First Order Approximation Theorem and so f has a tangent plane at any $(x,y) \in \mathbb{R}^2$.

First, we need to calculate the gradient of f at arbitrary points, so $\nabla f(x,y) = (3x^2 + y, x)$ and $\nabla f(1,1) = (4,1)$. So, the tangent plane is the plane with equation the Affine first order approximation of f at $(1,1)$:

$$z = f(1,1) + \langle \nabla f(1,1), (x-1, y-1) \rangle = 3 + 4(x-1) + (y-1) = 4x + y - 2.$$

2 (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) Is f differentiable at all points of $\mathbb{R}$. Why or why not?

Solution: For $x \neq 0$, f is the product of differentiable functions x^2 and $\sin(1/x)$, so f is differentiable for all $x \neq 0$. Now, we consider $x = 0$. We have to use the definition here since f is defined differently at $x = 0$ than elsewhere. If the limit exists:

$$f'(0) = \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin(1/h) = 0$$

because $\lim_{h \to 0} h = 0$ and $\sin(1/h)$ is bounded for $h \neq 0$ by Problem 7 on HW 1. Therefore, f is differentiable for all $x \in \mathbb{R}$.

(b) Is f continuously differentiable on $\mathbb{R}$, i.e., is $f \in C^1(\mathbb{R})$? **Solution:** For $x \neq 0$, f is the product of continuously differentiable functions x^2 and $\sin(1/x)$, so we can use the product rule to find $f'(x) = -\cos(1/x) + 2x \sin(1/x)$ for $x \neq 0$. Since f' is continuous for $x \neq 0$, $f \in C^1(\mathbb{R} \setminus \{\mathbf{0}\})$.

To see if f is continuously differentiable on $\mathbb{R}$ we need to check if f' is continuos at 0, that is: is $\lim_{x \to 0} f'(x) = 0 = f'(0)$. The answer is no because $x_k = \frac{1}{2\pi k}$ is a sequence that converges to zero and $x_k \neq 0$ for all k. However, $f'(x_k) = -1 + 0 = -1 \to -1 \neq 0$. Therefore f' is not continuous at 0 and $f \notin C^1(\mathbb{R})$.

This gives an example of a differentiable function that is not C^1.

The next problems involves this definition that we gave in class.

Definition 1 *Let $\mathcal{O} \subset \mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$ Let $\mathbf{x}_0 \in \mathcal{O}$. Then, f* is differentiable at $\mathbf{x}_0$ *if there is a vector $\mathbf{b} \in \mathbb{R}^n$ such that*

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{x}_0+\mathbf{h}) - [f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{h}\rangle]|}{\|\mathbf{h}\|} = 0 \tag{1}$$

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3 (40 points) Assume $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$ and assume $\mathbf{b} \in \mathbb{R}^n$ satisfies (1) for f at $\mathbf{x}_0$.

(a) Explain why $g(\mathbf{x}) = f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle$ is an affine first-order approximation to f at $\mathbf{x}_0$. (Short answer.)
Solution: Let $g(\mathbf{x}) = f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{x} - \mathbf{x}_0\rangle$. This is an affine function by definition. The limit (1) *is* the condition in the definition of the function g being a first order approximation to f.

(b) Let $\mathbf{b} = (b_1, b_2, \dots, b_n)$. Show that f has all first partial derivatives at $\mathbf{x}_0$ and $\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = b_i$ for each $i = 1, \dots, n$.
[HINT: Let $i \in \{1, 2, \dots, n\}$. Consider a sequence in $\mathbb{R}$, $\{t_k\}$ such that

- $t_k \to 0$
- for all $k \in \mathbb{N}$, $t_k \neq 0$
- for all $k \in \mathbb{N}$, $\mathbf{x}_0 + t_k\mathbf{e}_i \in \mathcal{O}$

and use the sequence $\{\mathbf{h}_k = t_k\mathbf{e}_i\}$ in the definition of the limit for (1).]
Solution: To solve this, it helps to first review (1) and the definitions of partial derivative and directional derivative: $\frac{\partial f}{\partial x_i}(\mathbf{x}) = \frac{df}{d\mathbf{e}_i}(\mathbf{x})$.
Also, recall that

$$\lim_{\mathbf{y}\to\mathbf{y}_0} f(\mathbf{y}) = 0 \text{ if and only if } \lim_{\mathbf{y}\to\mathbf{y}_0} |f(\mathbf{y})| = 0. \tag{2}$$

So, to check the directional derivative in direction $\mathbf{e}_i$, we take a real sequence $\{t_k\}$ such that $t_k \to 0$ and for all k, $t_k \neq 0$ and $\mathbf{x} + t_k\mathbf{e}_i \in \mathcal{O}$. Then, we plug the sequence $\{\mathbf{x} + t_k\mathbf{e}_i\}$ into the quotient in (1).

$$\lim_{k\to\infty} \frac{|f(\mathbf{x}_0 + t_k\mathbf{e}_i) - [f(\mathbf{x}_0) + \langle \mathbf{b}, t_k\mathbf{e}_i\rangle]|}{|t_k|} = 0$$

After a little simplification using the fact that $\langle \mathbf{b}, t_k\mathbf{e}_i\rangle = t_k b_i$ and (2), this becomes

$$\lim_{k\to\infty} \frac{f(\mathbf{x}_0 + t_k\mathbf{e}_i) - f(\mathbf{x}_0)}{t_k} - b_i = 0$$

This proves $\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = b_i$.

(c) Explain why $\nabla f(\mathbf{x}_0) = \mathbf{b}$. (Short answer.)
Solution: By definition $\nabla f(\mathbf{x}_0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}_0), \dots \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\right)$, and we showed $\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = b_i$ in the last part. Therefore, $\nabla f(\mathbf{x}_0) = \mathbf{b}$!

(d) Prove that if f and g are differentiable at $\mathbf{x}_0$ then $f + g$ is differentiable at $\mathbf{x}_0$ and $\nabla(f + g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$.
Solution: we guess that the vector $\mathbf{b}$ for $f + g$ is $\nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$. In the proof, we first use the triangle inequality on the numerator in the quotient in (1) for $f + g$ and break the

expression up into two expressions, one that is the quotient for f and the other that is the quotient for g. Here goes!

$$\begin{aligned}0 \leq & \frac{|(f+g)(\mathbf{x}_0+\mathbf{h}) - [(f+g)(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0), \mathbf{h}\rangle]|}{\|\mathbf{h}\|} \\ \leq & \frac{|f(\mathbf{x}_0+\mathbf{h}) - [f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{h}\rangle]|}{\|\mathbf{h}\|} + \frac{|g(\mathbf{x}_0+\mathbf{h}) - [g(\mathbf{x}_0) + \langle \nabla g(\mathbf{x}_0), \mathbf{h}\rangle]|}{\|\mathbf{h}\|}.\end{aligned}$$

Now, we use the fact that f and g are differentiable at $\mathbf{x}_0$ to see that the right-most two terms in the above inequalities go to zero as $\mathbf{h} \to 0$. Then, by the Sandwich Theorem, the middle term

$$\frac{|(f+g)(\mathbf{x}_0+\mathbf{h}) - [(f+g)(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0), \mathbf{h}\rangle]|}{\|\mathbf{h}\|} \to 0$$

so $f+g$ is differentiable!

To summarize, if f is differentiable at $\mathbf{x}_0$, then the vector $\mathbf{b}$ in (1) is unique and equal to $\nabla f(\mathbf{x}_0)$. Furthermore, the First Order Approximation Theorem holds for differentiable functions: if f is differentiable at $\mathbf{x}_0$, then

$$g(\mathbf{x}) = f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0\rangle$$

is the unique affine first order approximation to f at $\mathbf{x}_0$. In part (d), you showed that the gradient is additive, and in class we will show that differentiable functions are continuous.

4 (20 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = xy + 2x$.

(a) Use calculus rules to find $\nabla f(x, y)$.

Solution: $\nabla f(x, y) = (y + 2, x)$.

(b) Use your calculation of $\mathbf{b} = \nabla f(x, y)$ from part (a) and the definition of differentiability, Definition 1, to show f is differentiable at (x, y).

HINT: You plug in $\mathbf{x} = (x, y)$ and $\underline{\mathbf{h} = (h, k)}$ into equation (1) along with your guess for $\mathbf{b} = \nabla f(x, y)$ and do algebra. You may use the following limits

$$\lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2+k^2}} = 0$$

If your limit doesn't work out, check your calculation of $\nabla f(x, y)$ in (a).

Solution: Let's first calculate the numerator in (1):

$$\begin{aligned}f(x+h, y+k) - [f(x,y) + \langle (y+2, x), (h,k)\rangle \\ &= (x+h)(y+k) + 2(x+h) - xy - 2x - hy - 2h - xk \\ &= xy + hy + xk + hk + 2x + 2h - xy - 2x - hy - 2h - xk \\ &= hk.\end{aligned}$$

So,

$$\lim_{(h,k)\to(0,0)} \frac{f(x,y) + \langle (y+2, x), (h,k)\rangle}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = 0$$

by the hint. Therefore, f is differentiable at (x, y) with derivative $(y + 2, x)$!