assignment 4 - due Oct 2

Graded

Student

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Total Points

6 / 6 pts

Question 1

6.7

1 / 1 pt

- ✔ **– 0 pts** Correct

- **– 0.3 pts** Did not explain why proof generalizes to non-cycles
- **– 0.1 pts** Did not say what the final two permutations are in the non-cycles case, but did give a method to find them (I know this is nitpicky but it is important to write things out explicitly)
- **– 0.1 pts** Your permutations are slightly off.
- **– 0.5 pts** Your permutations are incorrect

Question 2

6.5

1 / 1 pt

- ✔ **– 0 pts** Correct

Question 3

6.6

1 / 1 pt

- ✔ **– 0 pts** Correct

- **– 0.2 pts** Trouble with counting
- **– 0.2 pts** Unclear presentation of logic

Question 4

6.12

1 / 1 pt

- ✔ **– 0 pts** Correct

Question 5

7.5

1 / 1 pt

- ✔ **– 0 pts** Correct

- **– 0.5 pts** To prove G is abelian you need to show $xy = yx$ for **all** $x, y \in G$

Question 6

7.7 **1** / 1 pt

✔ **− 0 pts** Correct

− 0.1 pts Did not prove that ϕ is a bijection/invertible

− 0.1 pts Did not do example in A_4

− 0.1 pts Incorrect computation in the A_4 example

6.7) First show for cyclic permutation of $n \geq 4$, $\alpha = (\alpha_1\ \alpha_2\ \alpha_3 \ldots \alpha_n) = \alpha$ can be written and $\alpha = \delta B$ where $|B| = |\delta| = 2$.

Note transpositions order 2 and product of disjoint transpositions is order 2.

Define $B = (\alpha_1\ \alpha_n)(\alpha_2\ \alpha_{n-1})(\alpha_3\ \alpha_{n-2}) \ldots\ldots$

If n = odd is left alone. So, this puts everything in reverse order. So set is $[a_n\ \alpha_{n-1} \ldots \alpha_2\ \alpha_1]$ after B. α_1 in right place, nothing else is.

Define $\delta = (\alpha_2\ \alpha_n)(\alpha_3\ \alpha_{n-1})(\alpha_4\ \alpha_{n-2}) \ldots$

If n is even, middle is left alone. So, applying this orders the list: $[a_2\ a_3\ a_4 \ldots a_n\ a_1]$ which is the cyclic permutation. So

$$\underset{\alpha\ =}{(a_1\ a_2 \ldots a_n)} = \underset{\delta}{(\alpha_2\ a_n)(a_3\ \alpha_{n-1}) \ldots} \circ \underset{B}{(\alpha_1\ \alpha_n)(\alpha_2\ \alpha_{n-1}) \ldots}$$

Now, to show can be done for any $\sigma \in S_n$

σ can be written as product of disjoint cycles. WLOG $\sigma = \alpha B \delta \ldots C$ disjoint.

By above can write $\alpha = \alpha_1 \alpha_2$ $B = B_1 B_2$ $\delta = \delta_1 \delta_2$ ... where $|a_1| = 2$ $|a_2| = 2$ and so on by above. However, as each cycle is disjoint, can say

$$\alpha = (\alpha_1\ \alpha_2)(B_1\ B_2)(\delta_1\ \delta_2) \ldots$$
$$= (\alpha_1 B_1 \delta_1)(\alpha_2 B_2 \delta_2)$$

and as each of these $\alpha_1 B_1 \delta_1$ are disjoint transpositions are order 2.

This is WLOG, and holds for any number of disjoint cycles making σ. But we've written σ as product of 2 permutations and as disjoint transpositions, are order 2, proving the statement.

Note if $n \leq 3$ can't write as disjoint transpositions so where it needs $n \geq 4$.

Questions assigned to the following page: 2 and 3

6.5) $P(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)$
$(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$

$\alpha = (143)$ so

$\alpha P = (x_4 - x_2)(x_4 - x_1)(x_4 - x_3)(x_2 - x_1)(x_2 - x_3)$
$\cdot (x_3 - x_1) = \cancel{P} P$ so $(143)P = P$

$\alpha = (23)(412) = (1324)$

$\alpha P = (x_3 - x_4)(x_3 - x_2)(x_3 - x_1)(x_4 - x_2)(x_4 - x_1)$
$\cdot (x_2 - x_1) = -P$

So $\boxed{\begin{array}{l}(143)P = P \\ (23)(142)P = -P\end{array}}$

6.6) $H \subset S_n$ and $H \not\subset A_n$ so $\exists \sigma \in H$ where σ is odd. Consider
$f: H \to H$, $x \to \sigma x$. f is bijective so

as $x, x' \in H$ $f(x) = \sigma x$ $f(x') = \sigma x'$ as $\sigma \in H$ and H is subgroup $\sigma^{-1} \in H$ so $\sigma^{-1}(\sigma x = \sigma x') \Rightarrow x = x'$ and f is injective

For surjective, let $x \in H$ show $\exists y \in H$ s.t. $f(y) = x$ well $\sigma^{-1}(\sigma y = x)$ $y = \sigma^{-1} x$ so f is onto. Therefore, f is bijective.

Multiplying odd permutation by odd makes even and even by odd makes odd permutation.

So f is a bijection that pairs up each odd permutation with even permutation and as $f: H \to H$ that means half of the permutations in H are odd, so the remaining half is even

6.12) So for a cycle $\alpha = (\alpha_1 \, \alpha_2 \dots \alpha_n)$ $|\alpha| = n$ as move elements n times until return to original position.
Suppose for $\sigma \in S_n$ $\sigma = \alpha B \delta \dots \Upsilon$ where α, B, δ etc are disjoint cycles. For simplicity, assume $\bar{\alpha} = \alpha B \delta Y$ but as n is finite, can induct this out to any finite product of cycles.
WLOG $|\alpha| = m$ $|B| = J$ $|\delta| = k$ $|Y| = l$

So want order of $(\alpha B \delta Y)$. Let $x \in \mathbb{Z}$, so want $(\alpha B \delta Y)^x = \alpha^x B^x \delta^x Y^x = e$.
So, we need an x s.t. $\exists \, z_1, z_2, z_3, z_4 \in \mathbb{Z}$ s.t. $mz_1 = x$ $Jz_2 = x$ $kz_3 = x$ $lz_4 = x$
as then have:

$$\alpha^x B^x \delta^x Y^x = (\alpha^m)^{z_1} (B^J)^{z_2} (\delta^k)^{z_3} (Y^l)^{z_4}$$
$$= e^{z_1} e^{z_2} e^{z_3} e^{z_4} = e$$

So, we want the smallest x where $\exists \, z_1, z_2, z_3, z_4$ s.t. $mz_1 = x$ $Jz_2 = x$ $kz_3 = x$ $lz_4 = x$.
This x is then by def, $LCM(m, J, k, l)$

So $|\alpha B \delta Y| = LCM(|\alpha|, |B|, |\delta|, |Y|)$ which shows the statement.

Again, this is easily generalizable to any product of disjoint cycles in S_n, as finite # of cycles would just have more variables to work with in argument. So kept at 4 for simplicity. No step is dependent on 4 disjoint cycles, just that there are finitely many which follows again as n is finite.

Question assigned to the following page: 5

7.5) $\Rightarrow$ $\phi: G \to G$ is isomorphism
$x \to x^{-1}$

show G is Abelian

$\phi(ab) = \phi(a)\phi(b) = a^{-1}b^{-1} = (ba)^{-1}$

$\phi(ab) = (ab)^{-1}$

So $(ab)^{-1} = (ba)^{-1}$ meaning $ab = ba$ and G is Abelian

$\Leftarrow$ G is Abelian so show ϕ is isomorphism

First bijective

Injective: let $x, y \in G$ s.t. $\phi(x) = \phi(y)$

then $x(x^{-1} = y^{-1})$ $e = y^{-1}x$ so $x = y$

Surjective: Let $y \in G$, show $\exists x \in G$ s.t. $\phi(x) = y$. As $y \in G$, then $y^{-1} \in G$ and $\phi(y^{-1}) = (y^{-1})^{-1} = y$ so ϕ is surjective and therefore bijective.

To check isomorphism show

$\phi(ab) = \phi(a)\phi(b)$

$\phi(ab) = (ab)^{-1} = b^{-1}a^{-1} = \phi(b)\phi(a)$

$\phi(ba) = (ba)^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b)$

As G is abelian $ab = ba$ so

$\phi(ab) = \phi(ba) = \phi(a)\phi(b) = \phi(b)\phi(a)$

So $\phi(ab) = \phi(a)\phi(b)$ and ϕ is an isomorphism.

Question assigned to the following page: 6

7.7)

Injective: $x, y \in G$ $\phi(x) = \phi(y)$

So $g^{-1}(gxg^{-1} = gyg^{-1}) \Rightarrow (xg^{-1} = yg^{-1})g \Rightarrow x = y$

Surjective: $y \in G$ show $\exists x \in G$ s.t. $\phi(x) = y$

$gxg^{-1} = y$ $g g^{-1}(gx = yg)$ $x = g^{-1}yg$

So $\phi(g^{-1}yg) = y$ and ϕ is surjective

So it is bijective

To show isomorphism. Let $x, y \in G$

$\phi(xy) = gxyg^{-1}$

$\phi(x) = gxg^{-1}$ $\phi(y) = gyg^{-1}$

$\phi(x)\phi(y) = gxg^{-1}gyg^{-1} = gxyg^{-1} = \phi(x)\phi(y)$

So $\phi(xy) = \phi(x)\phi(y)$

So $A_4 = \{e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$

and $g = (123)$ so $g^{-1} = (321)$

So $\phi(e) = e$

$\phi(123) = (123)$

$\phi(321) = (321)$

$\phi(124) = (342)$

$\phi(142) = (243)$

$\phi(134) = (142)$

$\phi(431) = (241)$

$\phi(234) = (143)$

$\phi(432) = (341)$

$\phi(12)(34) = (23)(14)$

$\phi(13)(24) = (12)(34)$

$\phi(14)(23) = (24)(13)$

Scrap work on back. Just solved all of them by plugging in. Also used fact that $\phi(x) = (\phi(x^{-1}))^{-1}$ which is shown on back as well.

Gets all 12 elements of A_4, no overlap and see clear bijection.

messy scrap wk for 7.7

$(123)(124)(321)$

$\phi(x)$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix} = (342) \quad (342)$$

$1\ 3\ 4\ 2 \qquad 1342$

So $\phi(x^{-1}) = g x^{-1} g^{-1}$

$= g(gx)^{-1}$

$\cancel{gxg^{-1}} \cdot g(gx)^{-1} \qquad gxg^{-1}$ so

$(gx)(gx)^{-1} = e$

$$\phi(134) \quad \begin{bmatrix} 3 & 1 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 4 & 1 & 3 & 2 \end{bmatrix} = (142)$$

$1\ 2\ 3\ 4$

$$\phi(234) = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 4 & 1 & 3 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix} = (143)$$

$1\ 2\ 3\ 4$

$$\phi(12)(34) = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix} = (23)(14)$$

$1\ 2\ 3\ 4$

$$\phi(13)(24) = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & 3 & 4 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix} = (12)(34)$$

$1\ 2\ 3\ 4$

$$\phi(14)(32) = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix} = (24)(13$$

Calculation scrapwork Just did them all out.

In general, just do out permutations, applying $(123)^{-1} = (321)$ first.

Also $\phi(x) = (\phi(x^{-1}))^{-1}$

as $\phi(x^{-1}) = gx^{-1}g^{-1} = g(gx)^{-1}$

$g(gx)^{-1} \cdot gxg^{-1} = gx^{-1}g^{-1}gxg^{-1} = e$

$gxg^{-1} \cdot g(gx)^{-1} = gxg^{-1}gx^{-1}g^{-1} = e$ so leverage that fact