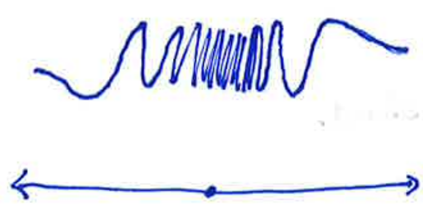
Are the following functions continuous or not continuous? Sketch an argument using the ϵ-δ definition of continuity and an argument using the open set definition of continuity.

(1) $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

Let $\varepsilon = 1/2$. Then if $x = 0$, for all $\delta > 0$, $y = \frac{\delta}{2}$ has $|x-y| = \frac{\delta}{2} < \delta$, but $|f(x) - f(y)| = 1 \geq \frac{1}{2} = \varepsilon$.

The preimage of $(\frac{1}{2}, \frac{3}{2})$ under f is $\{0\}$, which is not open

(2) $g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



Let $x = 0$

Let $\varepsilon = 1$. For each positive integer n, let $y = \frac{1}{2\pi n + \pi/2}$.

For all $\delta > 0$ there is an n s.t. $0 < y_n < \delta$. We have $|g(x) - g(y_n)| = 1 \geq 1 = \varepsilon$.

Let $U = (-1, 1)$. Then $0 \in g^{-1}(U)$, $y_n \notin g^{-1}(U)$ for all n.
Since $\forall \varepsilon > 0\ \exists n$ s.t. $y_n \in B(0, \varepsilon)$, so $B(0,\varepsilon) \not\subseteq U$,
Therefore U is not open.

(3) Prove or give a counterexample: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then whenever $U \subseteq \mathbb{R}$ is open, $f(U) \subseteq \mathbb{R}$ is open.

False: Let $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto 0$.

Then $f((0,1)) = \{0\}$ is not open.

(4) Prove or give a counterexample: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then whenever $Z \subseteq \mathbb{R}$ is closed, $f(Z) \subseteq \mathbb{R}$ is closed.

False. Let $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto e^x$.

Then $f(\mathbb{R}) = (0, \infty)$ is not closed.

We will show the following on homework:

Theorem 1. *If $f_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^{m_1}$ and $f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^{m_2}$ are continuous, then the function*

$$\begin{aligned} f_1 \times f_2 : \mathbb{R}^n &\longrightarrow \mathbb{R}^{m_1+m_2} \\ \mathbf{x} &\mapsto (f_1(\mathbf{x}), f_2(\mathbf{x})) \end{aligned}$$

is continuous.

(5) Let's show that subtraction is continuous without ϵ's and δ's.

(a) Show that the constant function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $x \mapsto -1$ is continuous using the open sets definition of continuity.

Let $U \subseteq \mathbb{R}$ be open.

$$f^{-1}(U) = \begin{cases} \emptyset & \text{if } -1 \notin U \\ \mathbb{R} & \text{if } -1 \in U. \end{cases}$$

In either case, $f^{-1}(U)$ is open, so f is continuous.

(b) Show that the identity function $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous using the open sets definition of continuity.

Let $U \subseteq \mathbb{R}$ be open.

$\mathrm{id}_{\mathbb{R}}^{-1}(U) = U$.

So $\mathrm{id}^{-1}(U)$ is open, so f is continuous.

(c) Prove that

$$-: \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x - y$$

is continuous by writing it as a composition of continuous functions.

m instead of · to make it clearer

We know $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$, $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto y$, $+ : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x+y$, $m : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto xy$

are continuous.

Then $+ \circ (\pi_1 \times (m \circ (f \times \pi_2))) : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} (x,y) \mapsto\ & x + (m \circ (f \times \pi_2))(x,y) \\ &= x + (m(-1, y)) \\ &= x + -1 \cdot y \\ &= x - y \end{aligned}$$

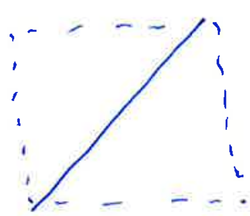
is continuous.

(d) Recall that the graph of a function $f : X \to Y$ is the subset

$$\text{graph}(f) = \{(x, y) \in X \times Y \mid f(x) = y\}.$$

Is the graph of $\text{id}_{\mathbb{R}}$ is a closed subset of $\mathbb{R}^2$? Is it an open subset of $\mathbb{R}^2$?

$\text{graph}(\text{id}_{\mathbb{R}}) = \{(x,x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$



it's closed, but not open.

(6) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then for each closed subset Z of $\mathbb{R}^m$, $f^{-1}(Z)$ is a closed subset of $\mathbb{R}^n$. (Hint: $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.)

Let Z be a closed subset of $\mathbb{R}^n$. Then $U = Z^c$ is open.

$$f^{-1}(Z) = f^{-1}((Z^c)^c) = f^{-1}(U^c) = \underbrace{(\underbrace{f^{-1}(U)}_{\text{open, since } f \text{ continuous}})^c}_{\text{closed, since complement of open.}}$$

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Prove that the function $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto y - f(x)$ is continuous.

$\pi_2 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto y$, $\quad f \circ \pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x)$ continuous

so $\quad - \circ (\pi_2 \times (f \circ \pi_1)) : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto (y - f(x))$ is continuous.

(b) Use part (a) to show that the graph of f is a closed subset of $\mathbb{R}^2$.

write $h : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto y - f(x)$.

By 6, $h^{-1}(\{0\}) = \{(x,y) \mid y - f(x) = 0\}$

$= \text{graph}(f)$, is closed.

(8) Consider the function

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

again. We will show that the graph of g is not closed. (And therefore g is not continuous!)

(a) For each $y \in [-1,1]$, show that $(0,y)$ is a limit point of graph(g). (Hint: what are the x's so that $\sin(1/x) = y$?)

Consider $x_n = \dfrac{1}{2\pi n + \arcsin(y)}$.

Then $\sin\left(\frac{1}{x_n}\right) = y$ for all n, so

$(x_n, y) \in \text{graph}(g)$ for all n.

For all $\varepsilon > 0$, there is n s.t. $0 < x_n < \varepsilon$, so

$\left(B((0,y), \varepsilon) \cap \text{graph}(g) \neq \emptyset \text{ for all } \varepsilon. \right.$
$\left. - \{0\} \right)$

$\therefore$ $(0,y)$ is a limit point of graph (g).