MATH 65, FALL 2021, HOMEWORK ANSWERS

Contents

1. First Homework, Sets, due September 16

Question 1.1. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{6, 7\}$. Please determine

(a) $A \cup B$
(b) $A \cap B$
(c) $A - B$
(d) $B - A$
(e) $A \times B$
(f) $\mathcal{P}(B)$

Answer 1.1.

(1) From the definition of union, it consists of all elements that are in at least one of the two sets $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.
(2) From the definition of intersection, it consists of all elements that are in both sets $A \cap B = \{6\}$.
(3) We want all of the elements which are in A but not in B. So, $A - B = \{1, 2, 3, 4, 5\}$.
(4) Similar to (c), we just want to look at elements which are in B but not A. $B - A = \{7\}$.
(5) In this part we want to write all of the ordered pairs (x, y) where $x \in A$ and $y \in B$. Thus,

$$A \times B = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7)\}.$$

(6) Here we want to write down all of the subsets of B. Directly, we see

$$\mathcal{P}(B) = \{\emptyset, \{6\}, \{7\}, \{6, 7\}\}.$$

Question 1.2. Define two sets S, T by $S = \{x \in \mathbb{R} \mid x^2 < x\}$, $T = \{x \in \mathbb{R} \mid 0 < x < 1\}$. Show that $S = T$. Remember that in order to prove an equality of sets, you need to show that each set is contained in the other set.

Answer 1.2. We first show $T \subseteq S$. Let $x \in T$. Then $x > 0$ and therefore multiplying an inequality with x preserves the inequality. As by assumption $x < 1$, multiplying with x we get $x^2 < x$. Therefore, $x \in S$.

Assume now $x \in S$. Then, by assumption $x^2 < x$. As for every integer x, $x^2 \geq 0$, we have $0 \leq x$. On the other hand, $0^2 = 0$ and we are assuming $x^2 < x$. It follows that $x > 0$. Now the inequality $x^2 < x$. can be written as $x - x^2 > 0$. Factoring the left hand side, we obtain $x(x-1) > 0$. We already know that x is positive. When the product of a positive number with another number is positive, the other number is also positive. Hence $x - 1 > 0$ and therefore $x \in T$.

Question 1.3. For any natural number $n \geq 1$, define the following intervals of the real line

$$A_n = (\frac{-1}{n}, \frac{1}{n}) = \{x \in \mathbb{R} | -\frac{1}{n} < x < \frac{1}{n}\}, \quad B_n = (n - \frac{1}{2}, n + \frac{1}{2}) = \{x \in \mathbb{R} | n - \frac{1}{2} < x < n + \frac{1}{2}\}$$

(a) Find $\bigcup_{n \in \mathbb{N} - \{0\}} A_n$ and $\bigcap_{n \in \mathbb{N} - \{0\}} A_n$.
(b) Find $\bigcup_{n \in \mathbb{N} - \{0\}} B_n$ and $\bigcap_{n \in \mathbb{N} - \{0\}} B_n$.

Answer 1.3. (a) The set A_n is a nested collection, whose first set is $(-1, 1)$ and whose sets get smaller and smaller, squeezing down to zero. The union of the sets in A_n is the open interval $(-1, 1)$. The intersection of the sets in A_n is $\{0\}$, because that one value is in every set $(-1/n, 1/n)$, but no other positive number is less than $1/n$ for all n. no other negative number is greater than $-1/n$ for all n.

(b) The set B_n is a collection of disjoint intervals of length one starting and ending half way between two integers. The union of the sets in B_n is the portion of the real line of numbers greater than $\frac{1}{2}$ except for the points that lie half way between two integers $\bigcup_{n \in \mathbb{N}} B_n = (\frac{1}{2}, \infty) - \{\frac{2k+1}{2}, k \in \mathbb{Z}\}$. The intersection of the sets in B_n is empty. We could even say more, the intersection of any two B_i is empty.

Question 1.4. Prove or disprove: $A \cup B = A \cap B$ if and only if $A = B$.

Answer 1.4. We claim that it is true. We need to prove the two implications

(a) We first prove $A = B \;\Rightarrow\; A \cup B = A \cap B$.

We are assuming $A = B$. This implies $A \cup B = A = B$, $A \cap B = A = B$. Hence, $A \cup B = A = A \cap B$ as we were trying to prove.

(b) We now prove $A \cup B = A \cap B \Rightarrow A = B$. For any arbitrary sets X, Y, we always have $X \cap Y \subseteq X \subseteq X \cup Y$. Therefore $A \cap B \subset A, B \subset A \cup B$. Then, from the assumption $A \cup B = A \cap B$, we obtain

$$B \subseteq A \cup B = A \cap B \subseteq A$$

It follows that $B \subseteq A$. The same sort of argument shows that $A \subseteq B$ and the two inclusion together give the equality of the two sets.

Question 1.5. Let

$$A = \{x \in \mathbb{R} \,|\, x^3 - 4x \geq 0\}, \; B = \{x \in \mathbb{R} \,| x^3 - x < 0\}.$$

(1) Describe A as a union of intervals on the real line.

(2) Describe B as a union of intervals on the real line.
(3) Determine $A \cap B$. Write it as a union of disjoint intervals in the real line.
(4) Determine $A \cup B$. Write it as a union of disjoint intervals in the real line.

Answer 1.5. (1) We can factor
$$x^3 - 4x = x(x-2)(x+2).$$
A product of three numbers is positive if all of them are positive or two of them are negative and one positive. Now, $x - 2 \geq 0$ is equivalent to $x \geq 2$, $x + 2 \geq 0$ is equivalent to $x \geq -2$. So, all three factors are positive for $x \geq 2$ while two of the factors are negative and the third positive for $-2 \leq x \leq 0$. Using interval notation as in Calculus, $A = [-2, 0] \cup [2, \infty)$. These are closed intervals that include the (finite) end points.

(2) We can factor
$$x^3 - x = x(x-1)(x+1).$$
A product of three numbers is negative if all of them are negaitive or only one of them is negative. Now, $x - 1 < 0$ is equivalent to $x < 1$, $x + 1 < 0$ is equivalent to $x < -1$. So, all three factors are negative for $x < -1$ while only one of the factors is negative for $0 < x < 1$. Using interval notation $B = (-\infty, -1) \cup (0, 1)$. These are open intervals that do not include the (finite) end points.

(3) From the descriptions of A, B, $A \cap B = [-2, -1)$.
(4) From the descriptions of A, B, $A \cup B = (-\infty, 1) \cup [2, \infty)$.

Question 1.6. Let $\mathcal{P}(X)$ denote the power set of a set X, that is, the set of subsets of X

(a) Show that it is not in general true that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ (give an example of some A, B for which this is not true).
(b) If possible, give an example of A, B for which $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
(c) Determine the conditions on A, B under which $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ or show that this never happens.

Answer 1.6. (a) I'll give a counterexample. If $A = \{a\}, B = \{b\}$. Then $A \cup B = \{a, b\}$. Therefore, $\{a, b\} \in \mathcal{P}(A \cup B)$ but it is neither in $\mathcal{P}(A)$ nor in $\mathcal{P}(B)$.
(b) If $A = B$, then $A \cup B = A = B$ and therefore $\mathcal{P}(A \cup B) = \mathcal{P}(A) = \mathcal{P}(B) = \mathcal{P}(A) \cup \mathcal{P}(B)$
(c) If $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$, then every subset of $A \cup B$ is either a subset of A or a subset of B. This implies that either $A \subseteq B$ or $B \subseteq A$. Otherwise, we can find $a \in A - B, b \in B - A$ and then $\{a, b\} \in \mathcal{P}(A \cup B)$ but $\{a, b\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$

When $A \subseteq B$, so $A \cup B = B$. Then any subset of A is also a subset of B, so we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ and therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

If the inclusion of sets is the other way around, the argument is the same.

2. Second homework, Logic and quantifiers, Due September 23

Question 2.1. (a) Prove that an integer is odd if and only if it is a sum of two consecutive integers.

(b) Is every even integer a sum of two consecutive even integers? Prove or disprove.

Answer 2.1. (a) By definition of odd, an integer is odd if and only if it can be written as $a = 2q + 1$ where q is an integer. Then we can write

$$a = 2q + 1 = q + (q + 1)$$

which displays a as the sum of two consecutive integers. Conversely, the sum of two consecutive integers $q, q + 1$ is $q + (q + 1) = 2q + 1$ is odd. Hence, the statement is true.

(b) We will prove that the sum of two consecutive even number is even but not every even number is the sum of two consecutive even numbers.

Assume that a is the sum of two consecutive even numbers. Then we can find an integer x so that the consecutive integers are $2x, 2x + 2$. Therefore,

$$a = 2x + (2x + 2) = 4x + 2 = 2(2x + 1)$$

As x is an integer, $2x + 1$ is an integer. This shows that a is even. On the other hand, we obtained the expression for the sum of two consecutive integers as $a = 4x + 2$. This means that the remainder of the division of a by 4 is 2. Therefore, the multiples of 4 (which are all even) are not the sum of two consecutive integers. This disproves the converse implication.

Question 2.2. Let a, b, c be integers. Show that if a does not divide bc, then a does not divide b. Hint: Use contrapositive.

Answer 2.2. Instead of proving that if a does not divide bc, then a does not divide b, we can prove the equivalent statement that if a divides b, then a divides bc.

The condition that a divides b is equivalent to saying that there exists an integer x such that $b = ax$. Then, $bc = (ax)c = a(xc)$. As x, c are integers, xc is an integer. Hence, bc is the product of a with an integer. By definition of divisibility, this means that bc is divisible by a.

Question 2.3. Prove that the two statements $(P \wedge Q) \vee (P \wedge \neg Q)$ and P are logically equivalent.

Answer 2.3. It suffices to check that he truth table for both statements is the same. We need to show all possible values of P and Q, so our table will have $2 \times 2 = 4$ rows.

P	**Q**	$\neg Q$	$P \wedge Q$	$P \wedge \neg Q$	$(P \wedge Q) \vee (P \wedge \neg Q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	F	F
F	F	T	F	F	F

Question 2.4. Write the following sentence using the quantifier notation (that is the symbols $\exists$, $\forall$..) We do not claim these statements are true, so please do not try to prove them

(a) Every integer is prime

(b) There is an integer that is neither prime nor composite.
(c) For every integer x there exists an integer y such that $xy = 1$.
(d) There is an integer x such that for every integer y, $xy = 1$.
(e) For every integer x and every integer y, $x + y = y + x$.
(f) There is an integer x and an integer y such that $\frac{x}{y}$ is an integer.

Answer 2.4. (a) $\forall x \in \mathbb{Z},\ x$ is a prime .
(b) $\exists x \in \mathbb{Z}$ x is not a prime and x is not composite .
(c) $\forall x \in \mathbb{Z}$ $\exists y \in \mathbb{Z}$ such that $xy = 1$.
(d) $\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$ $xy = 1$
(e) $\forall x \in \mathbb{Z}$ $\forall y \in \mathbb{Z}$ $x + y = y + x$.
(f) $\exists x \in \mathbb{Z},\ \exists y \in \mathbb{Z}$ such that $\frac{x}{y}$ is an integer.

Question 2.5. Write the negation of each of the sentences above, so that the negation appears as late in the sentence as possible. For example, in the negation of (e), the last piece of your statement should be $x + y \neq y + x$.

Answer 2.5. Recall that the negation of an "for all x, $p(x)$" results in "exists an x such that not $p(x)$"'. Similarly, the negation of an "exists an x such that $p(x)$" results in "for all x, not $p(x)$"'
(a) $\exists x \in \mathbb{Z}$ x is not a prime .
(b) $\forall x \in \mathbb{Z}$ x is a prime or x is composite .
Note here that the negation of an "and" results in an "or" of the negations.
(c) $\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$ $xy \neq 1$.
(d) $\forall x \in \mathbb{Z}$ $\exists y \in \mathbb{Z}$ such that $xy \neq 1$
(e) $\exists x \in \mathbb{Z}$ $\exists y \in \mathbb{Z}$ $x + y \neq y + x$.
(f) $\forall x \in \mathbb{Z},\ \forall y \in \mathbb{Z}$ $\frac{x}{y}$ is not an integer.

Question 2.6. The symbol $\exists!$ means " there exists a unique". So the statement $\exists! x\ P(x)$ (there exists a unique x satisfying $P(x)$) will fail if either there is no x satisfying $P(x)$ or if there is more than one x satisfying $P(x)$. Decide whether each of the following claims is true or false and prove your assertion. Here $\mathbb{N}$ denotes the set of natural numbers

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

(a) $\exists! x \in \mathbb{N}$ such that $x^2 = 4$.
(b) $\exists! x \in \mathbb{N}$ such that $x^2 = 3$.
(c) $\exists! x \in \mathbb{Z}$ such that $x^2 = 4$.
(d) $\exists! x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$ $xy = x$.
(e) $\exists! x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$ $xy = y$.

Answer 2.6. (a) The statement $\exists! x \in \mathbb{N}$ such that $x^2 = 4$ is true: the equation $x^2 = 4$ is equivalent to $x^2 - 4 = 0$. Factoring, this can be written as $(x - 2)(x + 2) = 0$ which has solutions $x = 2, x = -2$. Among these, only one is a natural number. So, the equation has a unique solution on the set of natural numbers.
(b) The statement $\exists! x \in \mathbb{N}$ such that $x^2 = 3$ is false. There is no natural number whose square is 3 as $1^1 = 1 < 3, 2^2 = 4 > 3$ and taking the square of a positive number is an increasing function as multiplying an inequality with a positive number preserves the inequality:

$$0 < x < y \Rightarrow 0 < x^2 < xy < y^2$$

Therefore, the square of any natural number greater than 1 is at least four and in particular greater than three.

(c) The statement $\exists! x \in \mathbb{Z}$ such that $x^2 = 4$ is false as the equation has two solutions, namely $x = 2, x = -2$, both integers. Hence the solution is not unique.

(d) The statement $\exists! x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}\ xy = x$ is true. Note that the equation $xy = x$ is equivalent to $x(y-1) = 0$. This equation has solutions $x = 0$ or $y = 1$. The question asks whether there is an x that works for all possible y's. This solution exists, namely $x = 0$. Moreover, it is the only one that works no matter what the y , as for $y \neq 1$ there are no other solutions.

(e) The statement $\exists! x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}\ xy = y$ is true. If we take $x = 1$, it satisfies the equation for every y. Moreover, this is the only x that would work for every y: if we take $y = 2$, the equation becomes $2x = 2$ which has unique solution $x = 1$. So, the solution exists and is unique.

Question 2.7. Let a, b, c be odd integers. Prove that the equation

$$ax^2 + bx + c = 0$$

has no rational solution, i.e. there is no rational number (p/q) which can be plugged in for x to make the equation true. Hint: Follow the ideas in the proof of the fact that $\sqrt{2}$ is not rational.

Answer 2.7. We suppose that p/q is a rational solution to the equation $ax^2 + bx + c = 0$, and derive a contradiction. If p/q is a solution, we have

$$a(p/q)^2 + b(p/q) + c = 0.$$

Clearing denominators by multiplying through by q^2, we get

$$ap^2 + bpq + cq^2 = 0.$$

We may assume that the rational number p/q is expressed in reduced form, so that p and q share no common factors. In particular, they cannot both be even. Thus, we have three cases to consider:

- If p even, q odd, then ap^2 and bpq are both even and cq^2 is odd, so the sum of all three terms sum is odd, and so not 0.
- If p odd, q even, then ap^2 is odd, and bpq and cq^2 are both even. Their sum then cannot be 0.
- If p odd, q odd, then all three terms are odd, so the sum of all three terms is odd, and so not 0.

All three possibilities lead us to a contradiction, so there is no rational solution p/q.

3. Homework 3, Induction, due September 30

Question 3.1. If n is a positive integer, prove using induction that

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}$$

Answer 3.1. For $n = 1$, the two sides of the expression are

$$1^3 = 1 \text{ and } \frac{n^2(n+1)^2}{4} = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Assume now that the equation is correct for n, that is $1^3+2^3+3^3+\cdots+(n-1)^3+n^3 = \frac{n^2(n+1)^2}{4}$. We need to prove then that

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 + (n+1)^3 = \frac{(n+1)^2((n+1)+1)^2}{4}$$

Using the induction assumption, we can wrtie

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 =$$

$$= (n+1)^2\frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\frac{n^2+4(n+1)}{4} = (n+1)^2\frac{n^2+4n+4)}{4} = (n+1)^2\frac{(n+2)^2}{4}$$

proving the result.

Question 3.2. Find and prove a formula for

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

Answer 3.2. Let $S(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)}$. Let's compute $S(n)$ for a few $n \in \mathbb{N}$ and see if we can conjecture a formula: $S(1) = 1/2$, $S(2) = 2/3$, $S(3) = 3/4$. We might conjecture that $S(n) = \frac{n}{n+1}$. Let's now prove our conjecture by induction.

First, the base case, $S(1) = 1/2$ which is indeed $\frac{n}{n+1}$ when $n = 1$. Now suppose $S(k) = \frac{k}{k+1}$ (this is our induction hypothesis). We want to prove now that $S(k+1) = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}$. Consider $S(k+1)$,

$$S(k+1) = \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{1}{(k+1)(k+2)} = S(k) + \frac{1}{(k+1)(k+2)}.$$

By the induction hypothesis we can rewrite $S(k)$ above with our formula, and then simplify:

$$S(k+1) = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2},$$

as desired.

We should note that not every problem of this flavor needs to be proven with induction. Here is an alternate solution which does not require induction. Observe that each summand can be written as a difference of fractions:

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

When taking the sum of the series, the negative portion of one summand cancels with the positive portion of the subsequent summand, resulting in nearly complete cancellation:

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i(i+1)} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}.\end{aligned}$$

Note that this agrees with our inductive proof above!

Remark: a *telescoping series* is a series with a large amount of cancellation. In fact, the Wikipedia page on telescoping series has this question in its introduction!

Question 3.3. Determine the set of natural numbers n for which $3^{n+1} > n^4$. Then prove your assertion.

Answer 3.3. By substituting small numbers for n, it can be checked that $3^{n+1} > n^4$ is false for $n = 3, 4$, but true for $n = 1, 2, 5, 6, 7$. This suggests that we could prove $3^{n+1} > n^4$ when $n \geq 5$. We will do so by induction.

The base case is $n = 5$. Because $3^{5+1} = 729 > 625 = 5^4$, the statement holds. Assume the inequality for $n = k \geq 5$ is true, i.e. that $3^{k+1} > k^4$. We want to show the inequality is true for $n = k + 1$. Start by

$$3^{k+2} = 3 \times 3^{k+1} > 3 \times k^4 \qquad \text{(by the inductive hypothesis).}$$

On the other hand, we assumed that $k \geq 5$ (because we start the induction from $n = 5$). Therefore, we have the inequalities

$$\begin{aligned} k^4 &\geq 5 \cdot k^3 > 4k^3; \\ \tfrac{1}{3}k^4 &\geq 5^2 \cdot \tfrac{1}{3}k^2 = \tfrac{25}{3}k^2 > 6k^2; \\ \tfrac{1}{3}k^4 &\geq 5^3 \cdot \tfrac{1}{3}k = \tfrac{125}{3}k > 4k; \\ \tfrac{1}{3}k^4 &\geq 5^4 \cdot \tfrac{1}{3} = \tfrac{625}{3} > 1. \end{aligned}$$

Hence,

$$\begin{aligned} 3^{k+2} &> 3k^4 \\ &= k^4 + k^4 + \tfrac{1}{3}k^4 + \tfrac{1}{3}k^4 + \tfrac{1}{3}k^4 \\ &> k^4 + 4k^3 + 6k^2 + 4k + 1 \\ &= (k+1)^4. \end{aligned}$$

Via induction, we have proved the inequality $3^{n+1} > n^4$ holds for $n \geq 5$. Checking when $n < 5$ shows that the inequality holds for all natural numbers n other than 3 or 4.

Question 3.4. (a) for $p \in \mathbb{N} - \{0, 1\}$, define what it means that p is prime.

(b) Show using strong induction that every natural number greater than 1 is a product of primes .

Answer 3.4. (a) If $p \in \mathbb{N} - \{0, 1\}$, then p is prime if its only divisors are $1, p$..

(b) We prove the result using strong induction. Let x be a natural number greater than 1. If x is prime, then $x = x$ is the factorization of x as product of primes. If x is not prime, by definition of prime, there exist natural numbers y, z different from $1, x$ such that $x = yz$. As $x \neq 0, x = yz$, then $y \neq 0$. As by assumption $y \neq 1$ and we showed $y \neq 0$, then $y > 1$. The assumption $x = yz$ then implies $z < x$. The induction assumption applies to z which is therefore a product of primes. Similarly, the induction assumption applies to y which is a product of primes. Then x is the product of the primes in y and z and therefore is a product of primes.

Question 3.5. In a hotdog line of $n \geq 2$ people at Fenway, everybody is wearing either Red Sox or NY Yankees apparel (and not both). If the first person in the line is a Red Sox fan and the last one a NY Yankees fan, show that somewhere in the line there is a Red Sox fan standing immediately before a NY Yankees fan.

Answer 3.5. We prove the result by induction on n. If there are only two people on the line, as the first is RS and the last NY , there is a RS (the first person) standing immediately in front of a NY (the second and last person.).

Assume now that the result is true for a line with n people in which the first is RS and the last NY. We will prove it for a line with $n+1$ satisfying the conditions. Consider the line that you obtain when you ignore the last ($n+1^{th}$ person). If in this line, the last person is NY, then we can apply the induction assumption and we know that there is one RS right before a NY fan. As these two people were part of the line with $n+1$ people, then the result is true also for the line with $n+1$ people. If the n^{th} person in the original line is RS, then we do not have the assumptions that would allow us to apply induction. But in this situation, the pair consisting of the n^{th} and $(n+1)^{th}$ persons satisfy the conditions we need. Hence, the result is true for any line with $n+1$ people and therefore, it is true for any n.

Question 3.6. (a) Assume that b is a real number different from zero and that the sequence $a_n = b^n$ satisfies $a_n = 6a_{n-1} - 9a_{n-2}$. Find all possible values of b that work.

(b) Show that the sequence $b_n = n3^n$ satisfies the recurrence $b_n = 6b_{n-1} - 9b_{n-2}$.

(c) Find as many sequences as possible satisfying all of the conditions below

$$c_n = 6c_{n-1} - 9c_{n-2},\ c_0 = -2,\ c_1 = 6$$

Answer 3.6. (a) We assume that the sequence $a_n = b^n$ satisfies $a_n = 6a_{n-1} - 9a_{n-2}$. Plugging in $a_n = b^n, a_{n-1} = b^{n-1}, a_{n-2} = b^{n-2}$, we obtain

$$b^n = 6b^{n-1} - 9b^{n-2} \iff b^n - 6b^{n-1} + 9b^{n-2} = 0 \iff b^{n-2}(b^2 - 6b + 9) = 0$$

As we are assuming $b \neq 0$, this means $b^2 - 6b + 9 = 0$. Solving the quadratic equation, we obtain $b = 3$.

(b) We show that the sequence $b_n = n3^n$ satisfies the recurrence $b_n = 6b_{n-1} - 9b_{n-2}$ by plugging in

$$b_n = n3^n,\ 6b_{n-1} - 9b_{n-2} = 6(n-1)3^{n-1} - 9(n-2)3^{n-2} = 3^n[2n - 2 - n + 2] = n3^n$$

As both sides are identical, the condition is satisfied

(c) Defining $c_n = 6c_{n-1} - 9c_{n-2}$, each term of the sequence is given in terms of the two previous ones. Then, one can prove using induction as we did before that there is a unique sequence with given initial values c_0, c_1. We know that the two sequences $a_n = 3^n, b_n = n3^n$ satisfy the equations $a_n = 6a_{n-1} - 9a_{n-2}, b_n = 6b_{n-1} - 9b_{n-2},$. Then, any sequence of the form $c_n = Aa_n + Bb_n$ also satisfies the equation as

$$6c_{n-1} - 9c_{n-2} = 6(Aa_{n-1} + Bn_{n-1}) - 9(Aa_{n-2} + Bn_{n-2}) = A(6a_{n-1} - 9a_{n-2}) + B(6b_{n-1} - 9b_{n-2}) =$$

$$= Aa_n + Bb_n = c_n$$

We only need to find the constants A, B such that $c_0 = -2,\ c_1 = 6$. From the equation $c_n = A3^n + Bn3^n$, plugging in $n = 0, n = 1$, we have

$$c_0 = -2 = A,\ c_1 = 6 = 3A + 3B = -6 + 3B$$

So, $A = -2, B = 4,\ c_n = 4n3^n - 2 \times 3^n$

Question 3.7. Recall that a basic L-shape (left image in picture below)is made of three identical squares shaped like an L. We say a board admits an L-tiling if it is possible to completely cover it with basic non-overlapping L-shapes.

(1) Prove that a $2^k \times 2^k$ chessboard with a single square in the lower left corner deleted admits an L-tiling, for any $k \in \mathbb{N}$.
(2) Prove that a $2^k \times 2^k$ chessboard with *any* single square deleted admits an L-tiling, for any $k \in \mathbb{N}$.

Answer 3.7. Part (a) of this problem is a consequence of part (b), so we will prove part (b) here with an induction argument. The base case is $k = 1$. In this situation, removing any one square (red squares in second sketch of the picture below) from the 2×2 chessboard leaves a single L-tile shaped board, which can be trivially tiled by L-shapes.

Assume for our inductive hypothesis that the $2^{k-1} \times 2^{k-1}$ chessboard, with any one square removed, can be tiled by L-shapes. Consider any $2^k \times 2^k$ chessboard with any single square removed.

The missing square must appear in one of the four $2^{k-1} \times 2^{k-1}$ sub-chessboards. Without loss of generality assume the missing square is in the bottom-left $2^{k-1} \times 2^{k-1}$ sub-chessboard. This sub-chessboard is missing a square, so by the inductive hypothesis, can be tiled by L-tiles.

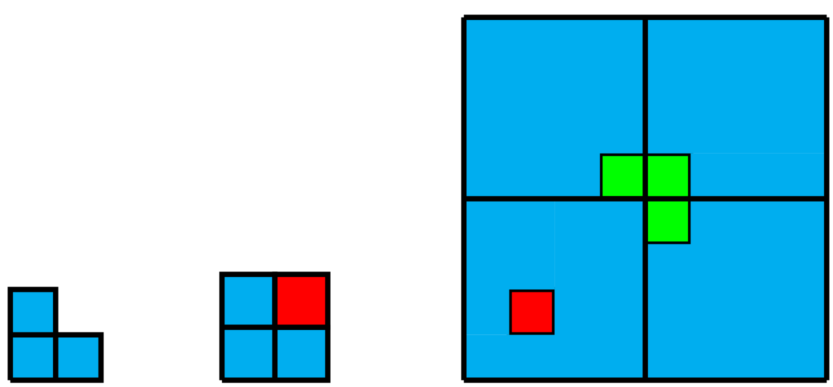

FIGURE 1. A basic L-shape, the case $n = 1$ and the inductive step

Afterwards, we can fit a single L-tile in the centre of the main chessboard which occupies a square in all three of the remaining untiled sub-chessboards. If we consider these three sub-chessboards as one corner piece missing, the inductive hypothesis can be used again to tile each sub-chessboard with L-tiles Hence every square of the original chessboard can be occupied with a square of a L-tiling. Thus the $2^k \times 2^k$ chessboard, with a missing square, can be L-tiled, completing the induction.

Note that this is a constructive method and could be used to tile any $2^k \times 2^k$ chessboard by successive subdivisions and tiling of an L at the center of the new subdivision.

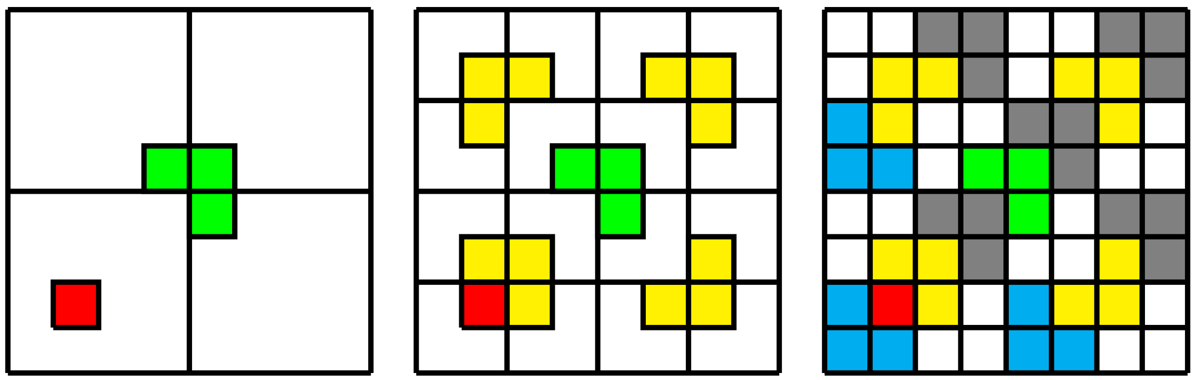

FIGURE 2. The inductive process of tiling an 8×8 square with the missing red tile

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4. Homework 4, Functions, due Thursday October 7

Question 4.1. Let $f : A \to B$ be a function. Show that f is onto if and only if there exists a function $g : B \to A$ such that $f \circ g = I_B$.

Answer 4.1. If f is onto, for every $b \in B$, we can find $a \in A$ such that $f(a) = b$. This a is not unique, but we just choose one of them for each b. We then define $g(b) = a$ for that chosen a depending on b. Then, for $b \in B$, $(f \circ g)(b) = f(g(b)) = f(a) = b$ where the a is the one we chose and the last equality comes from our choice of a.

Conversely, if there exists $g : B \to A$ such that $f \circ g = I_B$, then for each $b \in B, b = I_B(b) = (f \circ g)(b) = f(g(b))$. As $g(b) \in A$, we are displaying b as the image of an element in A, namely $g(b)$. Hence, f is onto.

Question 4.2. Let

$$\begin{array}{rcccc} f & : & \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \times \mathbb{Z} \\ & & (a, b) & \to & (a + b, a - b) \end{array}$$

(a) Prove or disprove that f is one to one.
(b) Prove or disprove that f is onto

Answer 4.2. (a) We prove that f is one to one. Assume that $f(a_1, b_1) = f(a_2, b_2)$. From the definition of f,

$$(a_1 + b_1, a_1 - b_1) = f(a_1, b_1) = f(a_2, b_2) = (a_2 + b_2, a_2 - b_2)$$

Therefore, equating each component of the pair, we obtain,

$$a_1 + b_1 = a_2 + b_2, \quad a_1 - b_1 = a_2 - b_2$$

Adding the two equations, we obtain $2a_1 = 2a_2$ and subtracting them we obtain $2b_1 = 2b_2$. Dividing these equations by 2, we have $a_1 = a_2, b_1 = b_2$. Therefore the pairs satisfy the equality $(a_1, a_2) = (b_1, b_2)$, proving that F is one to one.

(b) We prove that f is not onto by showing that there exists $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, such that for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $f(a, b) \neq (x, y)$. Choose $x = 1, y = 0$ and assume $f(a, b) = (1.0)$. From the definition of f, $f(a, b) = (a+b, a-b)$. Therefore, the assumption that $f(a, b) = (1.0)$ gives rise to the equations

$$a + b = 1, \ a - b = 0$$

Adding the two equations, we obtain $2a = 1$ which implies that 1 is even. This contradiction shows that f is not onto.

Question 4.3. Let $f : A \to B$ be a function. We are NOT assuming that f is a bijection, so f^{-1} is not defined as a function. If $X \subseteq A$, we define the subset of elements in B that are images of elements in X that is denoted with the notation $f(X)$ as follows

$$f(X) = \{y \in B \text{ such that } \exists x \in X, \ y = f(x)\}.$$

If $Y \subseteq B$, we define the subset of elements in A that map to Y that is denoted with the (somehow confusing) notation $f^{-1}(Y)$ as follows

$$f^{-1}(Y) = \{x \in A. \text{ such that } f(x) \in Y\}.$$

(a) If $Y \subseteq B$, show that $f((f^{-1}(Y)) \subseteq Y$.
(b) Show that if $\forall Y \subseteq B, \ f((f^{-1}(Y)) = Y$, then f is onto.
(c) Show that if f is onto, then $\forall Y \subseteq B, \ f((f^{-1}(Y)) = Y$.

Answer 4.3. (a) If $y \in f((f^{-1}Y))$, by definition of $f(X)$, there exists $x \in f^{-1}(Y)$ such that $f(x) = y$.) By definition of $f^{-1}(Y)$, the condition that $x \in f^{-1}(Y)$ means that $f(x) \in Y$. As $y = f(x), y \in Y$. This is true for arbitrary $y \in f(f^{-1}(Y))$, hence $f(f^{-1}(Y)) \subseteq Y$.

(b) We are assuming that $f(f^{-1}(Y)) = Y$ for all Y. In particular, we can take $Y = B$. This then says that $f(f^{-1}(B)) = B$. Hence, each element in B is in the image of an element in $f^{-1}(B)$. By definition of onto, the map is onto

(c) We now assume f is onto. Let $Y \subseteq B$. The condition $f((f^{-1}Y)) \subseteq Y$ is true for every function f as proved in (b). We need to prove the converse inclusion. Choose an arbitrary $y \in Y$. As f is onto, there exists $x \in X$ such that $f(x) = y \in Y$. From the definition of $f^{-1}(Y)$, $x \in f^{-1}(Y)$. Then, $f(x) \in f(f^{-1}(Y))$. As $f(x) = y$, $y \in f(f^{-1}(Y))$. This is true for every $y \in Y$, therefore $Y \subseteq f(f^{-1}(Y))$. As we already have the converse inclusion, we get equality.

Question 4.4. (a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 4x^3 + 13$. Is f a bijection? Carefully justify your answer.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 4x^3 + 13$. Is f a bijection? Carefully justify your answer.

Answer 4.4. (a) $f : \mathbb{Z} \to \mathbb{Z} \quad f(x) = 4x^3 + 13$ is not a bijection as the right hand side $4x^3 + 13 = 2(2x^3 + 6) + 1$ is odd, hence the map is not onto.

(b) $f : \mathbb{R} \to \mathbb{R} \quad f(x) = 4x^3 + 13$ is a bijection:

Using calculus we show that f is one to one:

$f'(x) = 12x^2 > 0$. Hence f is always increasing and therefore one to one.

Moreover $\lim_{x\to-\infty}, f(x) = -\infty \quad \lim_{x\to\infty} f(x) = +\infty$. Then by the intermediate value theorem, every real value is attained and f is onto.

We prove the result again using only the definition of one to one and onto:

One to one: assume $f(x_1) = f(x_2) \leftrightarrow 4{x_1}^3 + 13 = 4{x_2}^3 + 13$. Subtracting 13 from both sides, $4{x_1}^3 = 4{x_2}^3 \to {x_1}^3 = {x_2}^3$. Then, $x_1 = x_2$ as a real number has a unique real cubic root.

Onto: given y, need to find x. $f(x) = y \leftrightarrow 4{x_1}^3 + 13 = y$. Equivalently $4{x_1}^3 = y - 13$ ${x_1}^3 = \frac{y-13}{4}$ or $x_1 = \sqrt[3]{\frac{y-13}{4}}$ which is always defined. Hence, f is onto.

Question 4.5. Let $f : S \to T$ be a function, and let A and B be subsets of S. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. Give an example to show that the reverse inclusion need not hold.

Answer 4.5. To show that $f(A \cap B) \subseteq f(A) \cap f(B)$, we take an arbitrary element in $f(A \cap B)$, and show that it must be contained in $f(A) \cap f(B)$. To that end, let $y \in f(A \cap B)$. This means that $y = f(x)$, for some $x \in A \cap B$.

Now $x \in A \cap B$ means that $x \in A$ and $x \in B$. But $x \in A$ implies $f(x) \in f(A)$, and $x \in B$ implies $f(x) \in f(B)$. It follows that $f(x)$ is in both $f(A)$ and $f(B)$, that is, $f(x) \in f(A) \cap f(B)$. Since $y = f(x)$, we have $y \in f(A) \cap f(B)$ as desired.

An example which shows that the reverse inclusion doesn't hold: Consider the function

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2.$$

If we look at the subsets $A, B \subseteq S$ given by

$$A = \{-2\}$$
$$B = \{2\}$$

Then $f(A) = \{4\}$ and $f(B) = \{4\}$, so $f(A) \cap f(B) = \{4\}$. On the other side, we have $A \cap B = \emptyset$, so $f(A \cap B) = f(\emptyset) = \emptyset$. Thus, in this example, $f(A) \cap f(B) \not\subseteq f(A \cap B)$.

Question 4.6. Let A, B, C be three sets, f, g, h functions

$$f : A \to B, \; g : B \to C, \; h = g \circ f : A \to C$$

where h is the the composition of f, g given by $h(a) = g(f(a))$.

(1) Prove or disprove if f, g are one-to one, then h is one-to one.
(2) Prove or disprove if f, g are onto, then h is onto.
(3) Prove or disprove if f, g are bijections, then h is a bijection.
(4) Prove or disprove if h is one-to one then both f, g are one-to one.

Answer 4.6. (1) Assume that f, g are one-to one, we will show that h is one-to one. Assume that $a_1, a_2 \in A$ and that $h(a_1) = h(a_2)$. From the definition of h as the composition of g and f, we have

$$g(f(a_1)) = h(a_1) = h(a_2) = g(f(a_2))$$

As g is one to one, this implies that $f(a_1) = f(a_2)$. As f is one to one, this implies that $a_1 = a_2$. As this is true each time that $h(a_1) = h(a_2)$, this shows that h is one to one.

(2) Assume that f, g are onto. We will show that h is onto. Take an arbitrary element $c \in C$. We need to find $a \in A$ such that $h(a) = c$. As g is onto, there exist $b \in B$ such that $g(b) = c$. As f is onto, there exist $a \in A$ such that $f(a) = b$. From the definition of h as the composition of g and f,

$$h(a) = g(f(a)) = g(b) = c$$

We were able to find $a \in A$ mapping to c for each. $c \in C$. Hence, h is onto.

(3) If f, g are bijections, they are both one to one and onto. We already proved that then h is both one to one and onto. Hence, h is a bijection.

(4) We will show that if h is one-to one then f is one-to one but g may fail to be one to one.

Assume that h is one-to one, we will show that f is one-to one. Assume that $a_1, a_2 \in A$ and that $f(a_1) = f(a_2)$. Applying g to both sides of this equation, we have $g(f(a_1)) = g(f(a_2))$. From the definition of h as the composition of g and f, we have

$$h(a_1) = g(f(a_1)) = g(f(a_2) = h(a_2))$$

As h is one to one, this gives us $a_1 = a_2$. This is true for all $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Hence, f is one to one.

Take now $f : \mathbb{N} \to \mathbb{Z}$ to be the inclusion. Take $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = x^2$. The composition $h : \mathbb{N} \to \mathbb{Z}$ is given by $h(x) = g(f(x)) = g(x) = x^2$ which is one to one as $x \in \mathbb{N}$. But g is not one to one.

5. Homework 5, Cardinality, due October 14

Question 5.1. Let A and B be finite sets

(a) Prove that if there is a one to one function $f : A \to B$ then B has at least as many elements as A ($|A| \le |B|$).
(b) Prove that if there is an onto function $f : A \to B$ then B has at most as many elements as A ($|B| \le |A|$).

Answer 5.1. (a) Suppose $f : A \to B$ is injective. If A has n elements, there is a bijection h from the set $A_n = \{1, 2, \dots, n\}$ to the set A. Composing h with f, we obtain an injective map from A_n to B. The image of this composition is the same as the image of f and because the map is one to one, the image has n elements. We know from Proposition 8.4 in the Notes that any subset of a finite set has at most as many elements as the whole set. Therefore B has at least as many elements as the image of f, that is, as many elements as A.

(b) Assume there is an onto function $f : A \to B$. From question 4.1 in the homework, there is a function $g : B \to A$ such that $f \circ g = I_B$. Then g is one to one as $g(b_1) = g(b_2)$ implies $b_1 = f(g(b_1)) = f(g(b_2)) = b_2$. Then, by part (a), B has at most as many elements as A ($|B| \le |A|$).

Question 5.2. Let A and B be finite sets both with n elements.

(a) Prove that a function $f : A \to B$ is injective if and only if it is surjective.
(b) Prove that the equivalence is false if A is infinite. In particular, give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is injective, but not surjective, and a function $g : \mathbb{N} \to \mathbb{N}$ which is surjective, but not injective.

Answer 5.2. (a) ($\Rightarrow$) Suppose $f : A \to B$ is injective. Then the function $A \to Imf \subseteq B$ is injective. Therefore, from the previous question, the image of f has at least as many elements as A. On the other hand, the image of f is a subset of B, therefore it has at most as many elements as B with equality only if $Imf = B$. Therefore, $Imf = B$ which means that f is onto.

($\Leftarrow$) Suppose now that f is surjective. Note that this means the image of f has n elements. For the sake of contradiction, suppose f is not injective. Then there exist $a_1 \neq a_2$ in A such that $f(a_1) = f(a_2)$. Then the image of f coincides with the image of $A - \{a_2\}$ which has $n - 1$ elements. Therefore, the image of f contains at most $n - 1$ elements . But then f cannot be surjective, a contradiction. Hence f is injective.

(b) Let $f : \mathbb{N} \to \mathbb{N}$ be given by $n \mapsto 2n$. Suppose $f(a) = f(b)$, then $2a = 2b$, and so $a = b$. Thus f is injective. Now note that there are no odd numbers in the image of f, so f is not surjective.

Let $g : \mathbb{N} \to \mathbb{N}$ be given by $n \mapsto \lceil n/2 \rceil$. ($\lceil x \rceil$ is the ceiling of x, it is the integer m closest to $x/2$ such that $x \le m$. i.e. round x up to the next integer.) We have g is not injective since $g(3) = g(4) = 2$. But g is surjective: take any $y \in \mathbb{N}$ and set $x = 2y$, then $g(x) = g(2y) = \lceil 2y/2 \rceil = y$.

Question 5.3. In this question, for a, b real numbers, we use the notation you were accustomed to in Calculus for intervals on the real line and write

$$(a, b) = \{r \in \mathbb{R} \text{ such that } a < x < b\}$$

(a) If a, b, c, d are real numbers, $a < b, c < d$, construct a bijection between the intervals (a, b) and (c, d). Make sure that you prove that your function is a bijection.
(b) If a, b are real numbers, $a < b$, construct a bijection between the interval (a, b) and the whole real line $\mathbb{R}$. Make sure that you prove that your function is a bijection.

Answer 5.3. (a) Assume that a, b, c, d are real numbers, $a < b, c < d$. Define

$$f : (a, b) \to (c, d) \text{ as } f(t) = \frac{d-c}{b-a}t + \frac{bc-ad}{b-a}$$

As $a < b$, the denominator is not zero and this is a well defined expression. As $a < b, c < d$, the coefficient of t is positive. Hence, this is a strictly increasing function, in particular, it is one to one. By plugging in, we check that $f(a) = c, f(b) = d$. As the function is strictly increasing, the image of the interval (a, b) is the interval (c, d) and the function is onto by the intermediate value theorem. Hence, the function is a bijection

(b) As we already showed that there is a bijection between any two open intervals, it suffices to construct a bijection between the real line and one particular interval. You could choose the tangent function that transforms the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ to the whole real line. Alternatively, you can use a geometric construction to prove this. For example, if we consider the lower half of the circle $x^2 + (y-1)^2 = 1$, it is in bijective correspondence with the interval $(-1, 1)$ on the x-axis, just by projection. Project then this half circle from the center onto the x-axis, The expression for the resulting map is a s follows:

$$f : (-1, 1) \to \mathbb{R}, \quad t \to \frac{t}{\sqrt{1-t^2}}$$

As the absolute value of t is strictly less than one, $1 - t^2$ is strictly positive and we can compute the square root and divide by it to get a real number. The map that we obtain in that way is a bijection: its derivative is $\frac{1}{(\sqrt{1-t^2})^3}$ which is positive, hence the function is always increasing and therefore one to one. The limit of the function as t approaches 1 is $+\infty$, the limit of the function as t approaches -1 is $-\infty$. By the intermediate value Theorem, the map is onto.

Question 5.4. (a) Let T be a set, and let S_1, S_2 be subsets of T. Suppose that $T = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Prove that if S_1 and S_2 are countable, then T is countable.
(b) Let $\mathbb{I} = \mathbb{R} - \mathbb{Q}$ be the set of irrational numbers. Prove that $\mathbb{I}$ is uncountable.

Answer 5.4.

(1) If S_1, S_2 are countable, then there exist bijections $f_1 : \mathbb{N} \to S_1$ and $f_2 : \mathbb{N} \to S_2$. Using these given functions, we define a bijection $g : \mathbb{N} \to T$ as follows:

$$g(x) = \begin{cases} f_1(x/2) \text{ if } x \text{ is even,} \\ f_2((x-1)/2) \text{ if } x \text{ is odd.} \end{cases}$$

Thus $g(0) = f_1(0), g(1) = f_2(0)$, $g(2) = f_1(1)$, $g(3) = f_2(1)$, $g(4) = f_1(2)$, etc.

To show that g is surjective, let $t \in T$. Since $T = S_1 \cup S_2$, we have $t \in S_1$ or $t \in S_2$. Without loss of generality, let us suppose that $t \in S_1$. Then $t = f_1(k)$ for some integer $k \in \mathbb{N}$ since f_1 is surjective. Then $t = f_1(k) = g(2k)$, so t is in the image of g. This shows that g is surjective. To show that g is injective, suppose that $g(x) = g(y)$. We must show that this implies $x = y$. Note that $S_1 \cap S_2 = \emptyset$ implies that x and y must both be even or both be odd (so that $g(x)$, $g(y)$ are both in S_1 or both in S_2). Without loss of generality, suppose that x, y are both even, say $x = 2k$, $y = 2l$ for some $k, l \in \mathbb{N}$. Then $g(x) = f_1(k)$ and $g(y) = f_1(l)$. Since f_1 is injective $f_1(k) = f_1(l)$ implies $k = l$, which implies $x = y$. This shows g is injective as desired.

(2) Suppose that $\mathbb{I}$ were countable. We proved in class that $\mathbb{Q}$ is countable. Since $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, the result of (a) would then imply that $\mathbb{R}$ is countable. But this is a contradiction, since we showed that $\mathbb{R}$ is not countable. We conclude that $\mathbb{I}$ must not be countable.

Question 5.5. Let S be a finite set of characters, and let T be the set of all finite (but arbitrarily long) sequences of characters in S. Thus, if $S = \{A, G, C, T\}$, then T is the set of all possible DNA sequences. If S is the set of keys on your keyboard, then T is the set of all possible sentences in the english language. Prove that T is countable.

Answer 5.5. A set is countable if we can put it in a bijection with the natural numbers. This means we can write the elements of our set as a list, where we don't write any element more than once, and we don't miss out on any elements. Let's describe how to do this for the set T.

Take all the characters in S, and arrange them in an ordered (finite) list: $\{s_1, s_2, s_3, ..., s_n\}$. This list contains all the one-character sentences in the English language. We now add to the list all of the two-character sentences, ordered lexicographically as words in a dictionary : compared strings by looking at the first letter and order them based on which letter comes first. If the first letters are the same, look at the next letter, and so on. So $s_1s_1 < s_1s_2 < s_2s_1 < s_2s_2$, Our list becomes $\{s_1, s_2, s_3, ..., s_1s_1, s_1s_2, s_1s_3, ..., s_2s_1, ..., s_ns_n\}$. We now add all the three-character sentences, then four-character sentences, and k-character sentences, for all $k \in \mathbb{N}$. With this procedure, we obtain a list (that is, a bijection from $\mathbb{N}$ to T) of all the finite strings of characters in T.

This is onto because every sentence in T is somewhere on the list. If x is a string in T, then it is a finite sequence of letters, of length, say, k. By construction we have included every k-character sentence, *for all* $k \in \mathbb{N}$, so x must be in our list somewhere. It is also an one to one because we did not repeat any sentences.

Question 5.6. For a finite set A, the number of elements in the set is denoted by $|A|$.

(a) Assume that A, B are finite sets. Prove that

$$|A \cup B| = |A| + |B| \Leftrightarrow A \cap B = \emptyset$$

(b) Assume that A, B, C are finite sets. Denote by $|A|$ the number of elements in the set A. Prove or disprove

$$|A \cup B \cup C| = |A| + |B| + |C| \Leftrightarrow A \cap B = \emptyset,\ B \cap C = \emptyset, \text{ and } \quad C \cap A = \emptyset$$

Answer 5.6. (a) We should agree first that for any set X, the number of elements of X satisfies

$$|X| \geq 0 \text{ and } |X| = 0 \Leftrightarrow X = \emptyset$$

We know that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

As $|A \cap B| \geq 0$, this implies that $|A \cup B| \leq |A| + |B|$. Moreover,

$$|A \cup B| = |A| + |B| \Leftrightarrow |A \cap B| = 0 \Leftrightarrow A \cap B = \emptyset$$

(b) From $A \cup B \cup C = (A \cup B) \cup C$, using the case of two sets, it follows that

$$|A \cup B \cup C| \leq |A \cup B| + |C| \leq |A| + |B| + |C|$$

Equality implies that we have equality in each of the inequalities above. From the case of two sets, this means that $(A \cup B) \cap C = \emptyset$ and $A \cap B = \emptyset$ As $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, this union can only be empty if each of the sets of the union $A \cap C, B \cap C$ are empty. So, the pairwise intersections are empty..

Conversely, if the pairwise intersections are empty, as $A \cap B \cap C \subseteq A \cap B = \emptyset$ and any subset of the empty set is empty, we obtain that also $A \cap B \cap C = \emptyset$. Then, from the equation for the cardinality of the union of two sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| =$$

$$= |A| + |B| + |C| - |\emptyset| - |\emptyset| - |\emptyset| + |\emptyset| = |A| + |B| + |C| - 0 - 0 - 0 + 0 = |A| + |B| + |C|$$

Question 6.1. (a) How many sequences of length n can one make with the digits $0,1$ containing exactly k 1's? (we assume $k \leq n$).

(b) How many sequences of length n can one make with the digits $0,1,2$ containing exactly k 1's? (we assume $k \leq n$).

Answer 6.1. (a) If the sequences of length n has to contain precisely k 1's, it is completely determined once you pick the spot for the ones among the n possible spots. Hence, there are $\binom{n}{k}$ options.

(b) If the sequences of length n has to contain precisely k 1's, you need to pick the spot for the ones among the n possible spots and then choose the digits for the remaining spots among $0,2$. There are $\binom{n}{k}$ options for the spot for the 1's. There are two choices of filling for each remaining spot. Hence, there are $\binom{n}{k}2^{n-k}$ choices in all.

Question 6.2. In how many ways can we partition a set with n elements into 2 part so that one part has 4 elements and the other part has all of the remaining elements (assume $n \geq 4$).

Answer 6.2. Once we determine the set with 4 elements, we determine also the second set as the complement. Hence, there are as many partitions as there are subsets of 4 elements, namely $\binom{n}{4}$ unless $n = 8$.

If $n = 8$, we will have double counted as every partition will appear as having chosen the first 4 elements or having chosen the last 4. In this case, there are $\frac{\binom{8}{4}}{2}$ options.

Question 6.3. (a) Prove algebraically that

$$\binom{n}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-2}{2} + \binom{n-1}{2}$$

Hint : you can use that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$.

(b) Use a combinatorial argument to show that

$$\binom{n}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-2}{2} + \binom{n-1}{2}$$

Hint: If your set is $\{1, 2, \ldots, n\}$ and you choose 3 elements out of it, think of the largest element in your subset.

Answer 6.3. (a) We will use that use that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$. repeatedly. We have

$$\binom{n}{3} = \binom{n-1}{3} + \binom{n-1}{2} = \binom{n-2}{3} + \binom{n-2}{2} + \binom{n-1}{2} =$$

$$= \binom{n-3}{3} + \binom{n-3}{2} + \binom{n-2}{2} + \binom{n-1}{2} = \cdots = \binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-2}{2} + \binom{n-1}{2}$$

As $\binom{3}{3} = 1 = \binom{2}{2}$, we can replace the first term in the last summation with $\binom{2}{2}$ to get the desired result.

(b) We want to choose three elements out of a set of n. Any such set needs to contain 3 or a larger element.

If the highest value in the subset is 3, the remaining two need to be chosen among $\{1,2\}$. There are $\binom{2}{2}$ such choices.

If the highest value in the subset is 4, the remaining two need to be chosen among $\{1,2,3\}$. There are $\binom{3}{2}$ such choices. ...

If the highest value in the subset is n, the remaining two need to be chosen among $\{1,2,\ldots,n-1\}$. There are $\binom{n-1}{2}$ such choices.

Adding these numbers, we get the desired expression.

$$\binom{n}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-2}{2} + \binom{n-1}{2}$$

Question 6.4. How many solutions are there to the equation

$$x + y + z = 85, \text{ such that } x, y, z \in \mathbb{N}, \text{ and } x \geq 5, \text{ and } z \leq 10$$

Answer 6.4. Because of the condition $x \geq 5$, this is equivalent to finding the solutions of the equation (with $x' = x - 5$)

$$x' + y + z = 80, \text{ such that } x', y, z \in \mathbb{N}, \text{ and } z \leq 10$$

We will compute the order of the two sets

$$A = \{(x', y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ such that } x' + y + z = 80\}$$
$$B = \{(x', y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ such that } x' + y + z = 80, \quad z \geq 11\}$$

B is a subset of A and we are looking for the complement of B in A. The number of elements of A is the number of ways in which one can choose 80 objects of three different kinds (kinds x', y and z). There are $\left(\binom{3}{80}\right) = \binom{82}{2}$ choices. The set B is equivalent to (writing $z' = z - 11$) as

$$B = \{(x', y, z') \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ such that } x' + y + z' = 69\}$$

There are $\left(\binom{3}{69}\right) = \binom{71}{2}$ choices. Hence, the number of solutions of the given equation with stated conditions is

$$\binom{82}{2} - \binom{71}{2}.$$

Question 6.5. Show that $\left(\binom{n}{k}\right) = \left(\binom{k+1}{n-1}\right)$ both algebraically and combinatorially.

Answer 6.5. For the algebraic proof, we first translate this expression into regular choose numbers. We then use that $\binom{a}{b} = \binom{a}{a-b}$ reflecting the fact that choosing a set is equivalent to choosing its complement (the unchosen elements). We have

$$\left(\binom{n}{k}\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k} = \left(\binom{k+1}{n-1}\right).$$

For the combinatorial proof, recall the model of the big box which has space for $n+k-1$ objects. We choose the spot for the $n-1$ separators placed to indicate where each kind of objects will go in order (before the first separator, between second and third and so on). Instead of thinking of choosing the spot for the separators, we can think of choosing the spots for the objects.

Question 6.6. The bakery in your supermarket offers 4 different types of cookies: chocolate, peanut, sugar and oatmeal-raisin.

(a) In how many ways can you pick a dozen cookies so that at least three are chocolate?

(b) In how many ways can you pick a dozen cookies so that at most two are peanut?
(c) In how many ways can you pick a dozen cookies so that at least three are chocolate and at most two are peanut?
(d) In how many ways can you pick a dozen cookies so that at least three are chocolate or at most two are peanut?

Answer 6.6. Imagine choosing 12 cookies of four types and putting then in a long box that has room for the 12 cookies and equally shaped separators to put between the different types of cookies. We need to choose the spot for the separators. Therefore, one can choose the dozen cookies of the four varieties in $\binom{12+3}{3}$ ways.

(a) If three of the cookies must be chocolate, we are only choosing 9 cookies. There are $\binom{9+3}{3}$ ways of doing so.
(b) If at most two are peanut, we need to exclude the option of having three or more peanut. This leaves $\binom{12+3}{3} - \binom{9+3}{3}$ options.
(c) If three of the cookies must be chocolate, we are only choosing 9 cookies. If at most two are peanut, we need to exclude the option of having three or more peanut. This leaves $\binom{9+3}{3} - \binom{6+3}{3}$ options.
(d) If at least three are chocolate or at most two are peanut, we need to count separately how many options we have for each of these and subtract the number of options for the intersection, that is the ones in (c). This gives

$$\binom{9+3}{3} + \binom{12+3}{3} - \binom{9+3}{3} - (\binom{9+3}{3} - \binom{6+3}{3}) =$$
$$\binom{12+3}{3} - \binom{9+3}{3} + \binom{6+3}{3}$$

Question 6.7. Show that for $n \geq 1$, the number of ways to group $2n$ people into n distinct pairs is $\frac{n!}{2^n}\binom{2n}{n} = \frac{(2n)!}{n!2^n}$.

Hint: Think instead about the problem of choosing n ordered pairs by choosing one element for each pair first.

Answer 6.7. We will give a few different proofs

(a) We can order the $2n$ people in a row. There are $(2n)!$ ways to do this. We then pair first and second, third and four and so on. This provides us with n ordered pairs with every pair of elements ordered. We can reorder the 2 elements in each pair in 2^n different ways. We can also reorder the pairs among themselves in $n!$ ways. So, the total numbers of ways to form the pairs is $\frac{(2n)!}{n!2^n}$.
(b) Choose n people to be the firs of the n pairs. There are $\binom{2n}{n}$ ways to do so. Order them arbitrarily. Then order the remaining n people and assign them in order to the first n. There are $n!$ ways to do so. Reordering the two people in each pair would not change the answer. So, the total number of choices is $\frac{n!}{2^n}\binom{2n}{n}$.
(c) First, let's count the number of ways to divide the people into *ordered* pairs - that is, we're going to divide the group into n pairs and give one member of each pair a silly hat. We can do this in two different ways:
   - We first choose the n lucky hat recipients, which gives us $\binom{2n}{n}$ options. Then, we decide how to pair up the n hat recipients with the remaining n people, for which

there are $n!$ choices. Thus there are $\binom{2n}{n}n!$ possible divisions of the group into ordered pairs.

- We can first form the n pairs. We do not know in how many ways we can do this, this is precisely what we are trying to compute. So let us call this number x. Then, in each pair, we choose the hat recipient. There are 2 options in each pair, so 2^n options total. Therefore, there are $2^n x$ ways to choose the n ordered pairs.

As the two counts should be the same, $2^n x = \binom{2n}{n}n!$. Therefore,

$$x = \frac{n!}{2^n}\binom{2n}{n} = \frac{(2n)!}{n!2^n}.$$

7. Homework 7, Relations, due November 4

Question 7.1. We will say that an integer a is near an integer b if they are not more than two units apart. For example, 4 is near 6 and 4 is near 2. But 4 is not near 7. As the distance between two points can be computed with the absolute value, $R = \{(x, y) |\ |x - y| \leq 2\}$ or $x \sim y \Leftrightarrow\ |x - y| \leq 2$.

(a) Is $\sim$ reflexive?
(b) Is $\sim$ symmetric?
(c) Is $\sim$ antisymmetric?
(d) Is $\sim$ transitive?

Answer 7.1. (a) The relation $\sim$ reflexive because for all $x \in \mathbb{Z}$, $|x - x| = |0| = 0 \leq 2$.

(b) The relation $\sim$ is symmetric: assume $x \sim y$. By definition of the relation, this means that $|x - y| \leq 2$. As $|x - y| = |y - x|$, we also have $|y - x| \leq 2$ and therefore, $y \sim x$.
(c) The relation $\sim$ is not antisymmetric. To disprove a property, it suffices to find $x \neq y$ such that $x \sim y$ AND $y \sim x$ We have $1 \sim 2$ and $2 \sim 1$ as $|1 - 2| = 1 = |2 - 1|$. Nevertheless $1 \neq 2$.
(d) The relation $\sim$ is not transitive. To disprove a property, it suffices to provide a counterexample. We have $1 \sim 3$ and $3 \sim 5$ as $|1 - 3| = 2 \leq 2, |3 - 5| = 2 \leq 2$. Nevertheless $|1 - 5| = 4 > 2$ .Hence, 1 is not related to 5 and $\sim$ is not transitive.

Question 7.2. Prove or disprove: there are relations that are both symmetric and transitive but not reflexive.

Answer 7.2. One might be tempted to say that if a relation is symmetric and transitive, then it needs to be reflexive. An argument would go as follows: if xRy, by symmetry, then yRx. By transitivity, xRy and yRx implies xRx.

There is nothing wrong with this argument but notice that we have the conditional "IF xRy". If we try to find a counterexample, we should include an element in our set not related to any other element. Let us take our set to be $S = \{1, 2, 3\}$ and the relation to be

$$R = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$$

This relation is symmetric and transitive but not reflexive as 3 is not related to itself (or to anything else for that matter).

Question 7.3. Determine if the following is an equivalence relation and if it is, describe the partition it induces

$$x, y \in \mathbb{Z}\ \ x\mathcal{R}y \text{ if and only if x,y have the same number of digits}$$

Answer 7.3. This relation is reflexive as x has the same number of digits as itself, it is symmetric as x, y play the same role in the definition and it is transitive as if x, y have the same number of digits and y, z have the same number of digits, then all three of them have the same number of digits, in particular x, z do. As the relation is reflexive, symmetric and transitive, it is an equivalence relation.

There is an equivalence class for every strictly positive natural number n, with the elements in this class being the integers with n digits. For instance, for $n = 2$ the corresponding class contains the integers from 10 to 99.

Question 7.4. How many equivalence relations are there in the set $A = \{1, 2, 3\}$? Hint: How many partitions are there of the set $A = \{1, 2, 3\}$?

Answer 7.4. Every equivalence relation in a set gives rise to the partition determined by the equivalence classes. Conversely, every partition defines an equivalence relation by saying that two elements are related if and only if they belong to the same subset in the partition.

One has 5 different partitions of the set $A = \{1, 2, 3\}$ namely

- Three subsets with one element each , that is $A = \{1\} \cup \{2\} \cup \{3\}$
- One subset with 2 elements and one with one. There are three different ways to choose the one element, the other two are then completely determined:

$$A = \{1\} \cup \{2\,3\}, \quad A = \{1\,3\} \cup \{2\}, \quad A = \{1, 2\} \cup \{3\}$$

- A single subset with all the elements $A = \{1, 2, 3\}$.

As there are 5 different partitions of A, there are 5 different equivalence relations in A.

Question 7.5. (a) Consider the assignment $f_1 : \mathbb{Z} \to \mathbb{Z}_5$ given by $f_1(z) = [z]_5$. Is this a well defined function? Why or why not?

(b) Consider the assignment $f_2 : \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $f_2([z]_{10}) = [z]_5$. Is this a well defined function? Why or why not?

(c) Consider the assignment $f_3 : \mathbb{Z}_5 \to \mathbb{Z}_{10}$ given by $f_3([z]_5) = [z]_{10}$. Is this a well defined function? Why or why not?

Answer 7.5. (a) Consider the assignment $f_1 : \mathbb{Z} \to \mathbb{Z}_5$ given by $f_1(z) = [z]_5$. This a well defined function as to any integer z we assign a uniquely determined coset $[z]_5 \in \mathbb{Z}_5$

(b) Consider the assignment $f_2 : \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $f_2([z]_{10}) = [z]_5$. In order to determine if this is a well defined function, we need to see whether changing z without changing $[z]_{10}$ changes or preserves $[z]_5$.

Assume $[z_1]_{10} = [z_2]_{10}$. By definition of the equivalence relation corresponding to cosets modulo 10, there exists $k \in \mathbb{Z}$ such that $z_1 - z_2 = 10k$. Then, $z_1 - z_2 = 5(2k)$. As $2k$ is an integer, $[z_1]_5 = [z_2]_5$,so f_2 is a well defined function.

(c) Consider the assignment $f_3 : \mathbb{Z}_5 \to \mathbb{Z}_{10}$ given by $f_3([z]_5) = [z]_{10}$. This is not well defined function as for instance $[1]_5 = [6]_5$ while $[1]_{10} \neq [6]_{10}$

Question 7.6. (a) Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y + [2]_{10} = [8]_{10}$ or show that there aren't any.

(b) Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y[2]_{10} = [8]_{10}$ or show that there aren't any.

(c) Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y[2]_{10} = [3]_{10}$ or show that there aren't any.

Answer 7.6. (a) If $y \in \mathbb{Z}_{10}$, we can write $y = [x]_{10}$ for some integer x. Then, the equation $y + [2]_{10} = [8]_{10}$ becomes $[x]_{10} + [2]_{10} = [8]_{10}$ Using the definition of addition of cosets, $[x + 2]_{10} = [8]_{10}$. By definition of the equivalence relation modulo 10, this means that $x+2 = 8+10k$ for some integer k. So, $x = 6+10k$. Then, $y = [x]_{10} = [6]_{10}$ is the unique solution of the equation.

(b) If $y \in \mathbb{Z}_{10}$, we can write $y = [x]_{10}$ for some integer x. Then, the equation $y[2]_{10} = [8]_{10}$ becomes $[x]_{10}[2]_{10} = [8]_{10}$ Using the definition of product of cosets, $[2x]_{10} = [8]_{10}$. By definition of the equivalence relation modulo 10, this means that $2x = 8 + 10k$ for some integer k. So, $x = 4+5k$. Then, $y = [x]_{10} = [4]_{10}$ or $y = [x]_{10} = [9]_{10}$ are the two possible solutions of the equation.

An alternative way of finding the solution is to test all values in $\mathbb{Z}_{10}$

(c) If $y \in \mathbb{Z}_{10}$, we can write $y = [x]_{10}$ for some integer x. Then, the equation $y[2]_{10} = [3]_{10}$ becomes $[2x]_{10} = [3]_{10}$. By definition of the equivalence relation modulo 10, this means that $2x = 3 + 10k$ for some integer k or $2x - 10k = 3$. If k is an integer, the left hand side is even while the right hand side is odd. So, the equation does not have a solution.

Question 7.7. Let $A = (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ be the set of pairs of strictly positive natural numbers. Define a relation in A by

$$(a, b), (c, d) \in A, \quad (a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

(a) Show that $\sim$ is an equivalence relation.
(b) Denote by B the set of equivalence classes of elements of A by the relation $\sim$ defined in part (a). Define an addition in B by

$$[(a, b)] + [(c, d)] = [a + c, b + d]$$

Show that this is a well defined addition among the cosets.
(c) Show that the function $f : A \to \mathbb{Z}$ given by $f((a, b)) = a - b$ gives rise to a well defined function on cosets $\bar{f} : B \to \mathbb{Z}$ given by $\bar{f}([(a, b)]) = a - b$ and that $\bar{f}$ is a bijection.
(d) Show that the addition of cosets that we defined in part (b) corresponds under the above function f to the addition of integers.(that is, show that $\bar{f}([(a, b)]) + \bar{f}([(c, d)]) = \bar{f}([a + c, b + d])$
(e) EXTRA CREDIT Could you define a product in B corresponding by $\bar{f}$ to the product in $\mathbb{Z}$?

Answer 7.7. (a)
- The relation $\sim$ is reflexive that is $\forall (a, b) \in A,\ (a, b) \sim (a, b)$. This is equivalent to $a + b = b + a$ which is clearly true.
- The relation $\sim$ is symmetric that is $\forall (a, b),\ (c, d) \in A$, if $(a, b) \sim (c, d)$ then $(c, d) \sim (a, b)$. The condition $(a, b) \sim (c, d)$ is equivalent to $a + d = b + c$ which is equivalent to $c + b = a + d$ and this says $(c, d) \sim (a, b)$.
- The relation $\sim$ is transitive $\forall (a, b),\ (c, d),\ (e, f) \in A$, if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $(a, b) \sim (e, f)$. The conditions $(a, b) \sim (c, d),\ (c, d) \sim (e, f)$ are equivalent to $a + d = b + c, c + f = d + e$ adding these two equations, we obtain

$$a + d + c + f = b + c + d + e.$$

Simplifying, we obtain $a + f = b + e$ and this says $(a, b) \sim (e, f)$.

(b) Assume $[(a_1, b_1)] = [(a_2, b_2)], [(c_1, d_1)] = [(c_2, d_2)]$ We need to check that

$$[(a_1 + c_1, b_1 + d_1)] = [(a_2 + c_2, b_2 + d_2)].$$

From the definition of cosets, $[(a_1, b_1)] = [(a_2, b_2)]$ means that $[(a_1, b_1)] \sim [(a_2, b_2)]$ and from our definition of $\sim$ this can be translated to $a_1 + b_2 = a_2 + b_1$. Similarly $c_1 + d_2 = c_2 + d_1$. Adding these two equations, we obtain

$$(a_1 + c_1) + (b_2 + d_2) = a_1 + b_2 + c_1 + d_2 = a_2 + b_1 + c_2 + d_1 = (a_2 + c_2) + (b_1 + d_1).$$

This implies that $(a_1 + c_1, b_1 + d_1) \sim (a_2 + c_2, b_2 + d_2)$ as needed.

(c) Let us show that the function $f : A \to \mathbb{Z}$ given by $f((a, b)) = a - b$ depends only on the coset of the element (a, b) and not on the particular element in that coset. Assume $[(a_1, b_1)] = [(a_2, b_2)]$. This means that $a_1 + b_2 = a_2 + b_1$. Therefore, subtracting $b_1 + b_2$ from both sides of this inequality, we obtain

$$f(a_1, b_1) = a_1 - b_1 = a_2 - b_2 = f(a_2, b_2)$$

Therefore $\bar{f}$ is well defined. We now need to check that $\bar{f}$ is a bijection.

- It is one to one: If $a-b = \bar{f}([(a,b)]) = \bar{f}([(c,d)]) = c-d$, this means that $a-b = c-d$ or equivalently $a+c = b+d$. Then, by definition of $\sim$, $(a,b) \sim (c,d)$ or equivalently $[(a,b)]) = [(c,d)]$. So $\bar{f}$ is one to one.
- It is onto: given $a \geq 0, a \in \mathbb{Z}$, then $a \in \mathbb{N} - \{0\}$. Then, $\bar{f}([a+1,1)] = a+1-1 = a$. Given $a < 0, a \in \mathbb{Z}, a-1 < 0$, therefore $1-a > 0$ and then $1-a \in \mathbb{N} - \{0\}$, $\bar{f}([1, 1-a)] = 1-1+a = a$. So, every integer whether positive or negative is in the image and f is onto.

(d) From the definition of $\bar{f}$,

$$\bar{f}([(a,b)]) + \bar{f}([(c,d)]) = a-b+c-d = (a+c)-(b+d) = \bar{f}([a+c, b+d])$$

(e) We would like

$$\bar{f}([(a,b)])\bar{f}([(c,d)]) = (a-b)(c-d) = \bar{f}([(a,b)][(c,d])$$

From the distributive property of the product in $\mathbb{Z}$, $(a-b)(c-d) = ac+bd-ad-bc$. If we define the product in B as

$$[(a,b)][(c,d)] = [(ac+bd, ad+bc)]$$

it would be compatible with $\bar{f}$. We need to check that it is well defined Assume

$$[(a_1,b_1)] = [(a_2,b_2)], [(c_1,d_1)] = [(c_2,d_2)].$$

This can be translated to

$$a_1+b_2 = a_2+b_1, \ c_1+d_2 = c_2+d_1$$

Which we can rewrite as

$$a_1-b_1 = a_2-b_2, \ c_1-d_1 = c_2-d_2$$

Multiplying these two equalities, we obtain

$$a_1c_1+b_1d_1-(a_1d_1+b_1c_1) = (a_1-b_1)(c_1-d_1) = (a_2-b_2)(c_2-d_2) = a_2c_2+b_2d_2-(a_2d_2+b_2c_2)$$

or equivalently,

$$a_1c_1+b_1d_1+a_2d_2+b_2c_2 = a_2c_2+b_2d_2+a_1d_1+b_1c_1$$

and this means

$$[(a_1c_1+b_1d_1, a_1d_1+b_1c_1)] = [(a_2c_2+b_2d_2, a_2d_2+b_2c_2)]$$

So, the product of cosets is well defined.

8. Homework on Partial Orders and Probability, due November 11

Question 8.1. Let $A = \{1, 2, 3, 4, 5, 6, 9, 10, 12\}$ with the ordering by divisibility

(a) Draw the Hasse diagram for this poset.
(b) Find a minimal element or explain why none exists.
(c) Find a minimum element or explain why none exists.
(d) Find a maximal element or explain why none exists.
(e) Find a maximum element or explain why none exists.

Answer 8.1. (a) The Hasse diagram is pictured below. There should be vertical segments from 1 to 2, 3 and 5; from 2 to 4, 6 and 10; from 3 to 6 and 9; from 4 to 12; from 5 to 10; from 6 to 12.

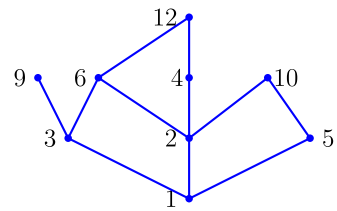


Figure 3. The Hasse diagram for "divides by" for the set $\{1, 2, 3, 4, 5, 6, 9, 10, 12\}$

(b) Only 1 is minimal as it divides every other number.
(c) Again 1 is minimum as it divides every other number.
(d) There are three maximal elements, namely 9, 10 and 12 as they do not divide anything else in the list.
(e) There is no maximum, as there is no number that is divisible by all others in the list.

Question 8.2. Let A be any set, $\mathcal{P}(A)$ the set of all subsets of A. Consider the inclusion relation

$$A_1, A_2 \in \mathcal{P}(A), \quad A_1 \preceq A_2 \ \Leftrightarrow A_1 \subseteq A_2$$

(a) Show that $(\mathcal{P}(A), \preceq)$ is a poset.
(b) Let $(X, \preceq)$ be a poset. A *chain* is a collection of distinct elements $x_i \in X$ such that every two elements are related. ($x_i \preceq x_j$ or $x_j \preceq x_i$). If A is a set with n elements and we order $\mathcal{P}(A)$ by inclusion, find a chain of maximal length.
(c) How many chains of maximal length are there in $\mathcal{P}(A)$?
(d) Let $(X, \preceq)$ be a poset. A *anti chain* is a collection of distinct elements $x_i \in X$ such that no two elements are related. For A a set and $\mathcal{P}(A)$ the set of all subsets of A ordered by inclusion, show that the set of subsets of A with fixed number of elements k form an antichain.
(e) If $A = \{1, 2, 3\}$, describe all possible antichains for $\mathcal{P}(A)$.

Answer 8.2. (a) As for every subset X, $X \subseteq X$, the relation is reflexive.

By definition of equality of sets , $X \subseteq Y$ and $Y \subseteq X$ implies $X = Y$. Hence the relation is antisymmetric.

If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$. So, inclusion is transitive. Therefore, $(\mathcal{P}(A), \preceq)$ is a poset.

(b) The relation is inclusion. Hence, a chain is a collection of subsets of A each strictly contained in the next. To go from one subset to the next, we need to add at least one element. The largest chains will be obtained when we add one element each time, we start with the empty set and end with the whole set A. Such a chain has $n+1$ steps.
(c) Giving a chain as described before is equivalent to giving an ordering of the elements in A that tells us which element we add at each step. Therefore, the number of elements is the number of orderings of A, namely $n!$.
(d) If two subsets of A have the same number of elements and are not equal, then neither of them can be contained in the other. Therefore, the set of subsets of a fixed order k form an antichain.
(e) Assume $A = \{1, 2, 3\}$ Any set of $\mathcal{P}(A)$ consisting of a single subset of A is an antichain. Any set of $\mathcal{P}(A)$ consisting of more than one subset of A that contain either the empty set, or A itself will not be an antichain. Therefore, an antichain with two terms will have either two subsets with one element (3 options), two subsets with two elements (3 options) or one subset with one element and the subset with the other two elements (three options).

An antichain with three terms will need to be either all three sets with one element or all. three sets with two elements.. Hence we compute the total number of antichains as

$$8 + 3 + 3 + 3 + 2 = 19$$

Question 8.3. In bridge, a player is dealt a 13-card hand from a 52-card deck. There are 13 kinds of cards: Ace, King, Queen, Jack, 10, 9, $\ldots 2$, and 4 suits: hearts, clubs, spades, and diamonds. A pair consists of exactly 2 cards of the same kind, e.g., 2 Queens.

(a) What is the probability of being dealt a hand with no Jack, Queen, or King?
(b) What is the probability of being dealt a hand with all 4 Kings and exactly 3 Queens?
(c) What is the probability of being dealt a hand with exactly 5 pairs and 3 other cards, each of a different kind from the pairs and from each other?

Answer 8.3. (1) All 13 cards should be chosen among the remaining 52-12=40. Hence, the probability is

$$\frac{\binom{52-12}{13}}{\binom{52}{13}} = \frac{\binom{40}{13}}{\binom{52}{13}}$$

(2) There is only one way of choosing all four kings while there are $\binom{4}{3}$ ways of choosing 3 queens and there are $\binom{52-8}{6}$ ways of choosing the remaining six cards among the cards of the remaining 11 suits. Hence the probability is

$$\frac{1\binom{4}{3}\binom{44}{6}}{\binom{52}{13}}$$

(3) We must choose which five kinds we want to use for each pair and then for each of them two (out of the possible four) cards of this kind. We must choose then 3 more kinds among the ones that have not been used and one card out of the possible four for each of these kinds. Hence, the probability is

$$\frac{\binom{13}{5}\binom{4}{2}^5\binom{13-5}{3}\cdot 4^3}{\binom{52}{13}}$$

Question 8.4. Let U be the sample space of randomly generated bit strings of length four. Let E be the event that the bit string has an even number of 1's. Let F be the event that the bit string starts with a 0.

(a) Compute $P(E), P(F), P(E \cap F)$.
(b) What is $P(E|F)$?
(c) What is $P(F|E)$?
(d) Are E and F independent

Answer 8.4. (a) If there is an even number of ones in four digits, there are either zero, two or four ones. If there are two ones, there are $\binom{4}{2}$ ways of choosing their spot. There is only one way of having none or four ones. Hence

$$P(E) = \frac{1 + \binom{4}{2} + 1}{2^4} = \frac{8}{16} = \frac{1}{2}.$$

If a bit string starts with a zero, the other three digits can be chosen freely. There are 2^3 choices of such bit strings. Then,

$$P(F) = \frac{2^3}{2^4} = \frac{1}{2}.$$

If both E, F hold, there is an even number of ones and the first bit is a zero. Hence there are no ones or two ones that are chosen among the last three bits. There are $1 + \binom{3}{2}$ ways of doing this. Hence,

$$P(E \cap F) = \frac{1 + \binom{3}{2}}{2^4} = \frac{4}{16} = \frac{1}{4}.$$

(b) From the equations for conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1}{2}$$

(c)

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{1}{2}.$$

(d) The two events are independent as $P(F|E) = P(E)$. Equivalently $P(E \cap F) = P(E)P(F)$.

Question 8.5. Prove Bayes' Theorem: Suppose that S is a finite probability space, $A \subset S$ is an arbitrary event, and $B_1, \ldots, B_r \subset S$ are mutually exclusive events which partition S, i.e. $S = B_1 \cup \ldots \cup B_r$. Suppose you are given that $P(B_i) = b_i$ and $P(A|B_i) = a_i$. Prove that

$$P(B_i|A) := \frac{a_i b_i}{\sum_{i=1}^{r} a_i b_i}$$

Answer 8.5. To start, recall that

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)}.$$

Rearranging yields

$$P(B_i \cap A) = P(B_i|A)P(A) = a_i b_i.$$

Notice that since the B_i partition S, they also partition A. That is $A = \bigcup_{i=1}^r A \cap B_i$. Thus,

$$P(A) = \sum_{i=1}^{r} P(A \cap B_i) = \sum_{i=1}^{r} a_i b_i.$$

Plugging the latter two expressions into the first formula give the answer we were looking for:

$$P(B_i|A) = \frac{a_i b_i}{\sum_{i=1}^r a_i b_i.}$$

Question 8.6. You hold a bag of ten coins. Nine of them are fair, but one is loaded - it shows heads with probability 9/10. You draw out a coin at random and begin flipping it. The first five tosses are $HHHTH$. What is the probability that you are flipping one of the fair coins?

Answer 8.6. Suppose A is the event "the coin is fair", and B is the event "the coin flips are $HHHTH$".

As there are 10 coins in the bag nine of which are fair, $P(A)$ is given as $\frac{9}{10}$ that is, before you consider the outcome of the tosses.

From the definition of conditional probability $P(A \mid B)$ of the coin being fair,

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B)}.$$

The probability $P(B \mid A)$ is the chance of obtaining $HHHTH$ from a fair coin, which is equal to $(\frac{1}{2})^5$. On the other hand,B can be written as a disjoint union

$$B = (B \cap A) \cup (B \cap \bar{A})$$

Therefore,

$$P(B) = P(B \cap A) + P(B \cap \bar{A}) = P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A}) = (\frac{1}{2})^5 \cdot \frac{9}{10} + (\frac{9}{10})^4 \frac{1}{10} \cdot \frac{1}{10}$$

Hence,

$$P(\text{coin is fair} \mid \text{flips are } HHHTH) = P(A \mid B) = \frac{(\frac{1}{2})^5 \cdot \frac{9}{10}}{(\frac{1}{2})^5 \cdot \frac{9}{10} + (\frac{9}{10})^4 \frac{1}{10} \cdot \frac{1}{10}} \approx 0.8108.$$

Question 8.7. Consider a dial having a pointer that is equally likely to point to each of n regions numbered $1, 2, \ldots, n$ in cyclic order. When the selection is k, the gambler receives 2^k dollars.

(1) What is the expected payoff per spin of the dial?
(2) Suppose that the gambler has the following option. After rate spin, the gambler can accept that payoff or flip a coin to change it. If the coin shows heads, the pointer movies one spot counterclockwise; if tails, it moves one spot clockwise. When should the gambler flip the coin? What is the expected payoff under the optimal strategy?

Answer 8.7.

(1) There are n disjoint outcomes which partition all possibilities, namely that one lands on $k \in [n]$ on the spinner. This happens with probability $1/n$. If one lands on k, the payoff is 2^k. Thus,

$$E[\text{one spin}] = \sum_{k \in \{1,\ldots,n\}} \frac{1}{n} 2^k = \frac{1}{n} \sum_{k \in \{1,\ldots,n\}} 2^k = \frac{2(2^n - 1)}{n}.$$

(2) Suppose the wheel has been spun and lands on $k \in \{1, \ldots, n\}]$. Let's compute the expected value of the coin flip given the payoff is already 2^k. With probability $1/2$ the payoff will become 2^{k-1} and with probability $1/2$ the payoff will become 2^{k+1} except in the case $k = 1$ when the payoffs become 2^n or 2^2 each with probability $1/2$ and the case $k = n$ when the payoffs becoem 2^{n-1} or 2^1 each with probability $1/2$.

Let us first suppose we are in the case $k \neq 1, n$. Then the expected value of flipping the coin is

$$\frac{1}{2} 2^{k-1} + \frac{1}{2} 2^{k+1} = 2^k \frac{1}{2} (\frac{1}{2} + 2) = \frac{5}{4} 2^k.$$

So in these cases where $k \neq 1, n$, the expected payoff is larger than the original payoff, so it is optimal to flip the coin.

Suppose now that $k = 1$, the expected payoff of flipping the coin is

$$\frac{1}{2} 2^n + \frac{1}{2} 2^2 = 2(2^{n-1} + 1).$$

Again the expected payoff is larger so long as $n \geq 2$.

Finally, suppose $k = n$, no matter the outcome of the coin flip, the payoff will be lower, so the coin should not be flipped in this case.

9. Homework on Graphs, due November 18.

Question 9.1. Show that in any graph, there is an even number of vertices of odd degree.

Answer 9.1. Every edge adds to the degree of two vertices. Hence, the sum of the degrees of all vertices is twice the number of edges, which is an even number. The sum of a bunch of even numbers and another bunch of odd numbers is even only when the number of odd numbers is even. Hence, there is an even number of vertices of odd degree.

Question 9.2. (a) Prove that in any simple graph with two or more vertices, there are at least two vertices of the same degree.
(b) Is the result in (a) true for non-simple graphs?

Answer 9.2. (a) As simple graphs does not have loops or multiple-edges. Therefore, the degree of any vertex is a number between 0 and $n-1$ where n is the number of vertices of the graph. There cannot be both vertices with degree 0 and vertices with degree $n-1$ because if there is a vertex with degree 0, then no edges go to this vertex and therefore there are at most $n-2$ edges coming out of any of the remaining vertices. We then have n vertices and at most $n-1$ possible degrees (either $\{0, 1, \dots n-2\}$ or $\{1, \dots, n-2, n-1\}$). By the pigeonhole principle, two of the vertices must have the same degree.
(b) The result is not necessarily true for non simple graphs as the example below shows: take a graph with vertices v_1, v_2, v_3, edges e_1, e_2, e_3 and incidence map

$$f(e_1) = \{v_1, v_2\} = f(e_2), f(e_3) = \{v_2, v_3\}.$$

Then $\deg v_1 = 2, \deg v_2 = 3, \deg v_3 = 1$ are all different.

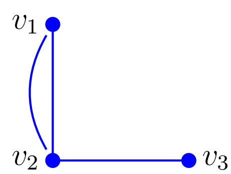


Figure 4. A non-simple graph with three vertices, all of different degree

Question 9.3. A graph is called r-regular if all vertices have degree r. If G is an r-regular graph, find a relationship between the number of vertices, the number of edges and r.

Answer 9.3. If a graph has n vertices, each of degree r, then the sum of the degrees is nk. This number is twice the number e of edges. Hence, $2e = nr$.

Question 9.4. If $G = (V, E, f)$ is a graph and $V_1 \subseteq V$ is a subset of the set of vertices of G, then the induced subgraph is the subgraph that has vertex set V_1 and edge set those edges in E such that both vertices in the edge are in V_1.
(a) If G is the complete graph K_n with n vertices, how many induced subgraphs does it have? How many non-isomorphic induced subgraphs does it have?
(b) If G is the complete graph K_n with n vertices, how many spanning subgraphs does it have? A subgraph is called a spanning subgraph if the set of vertices of the subgraph is the same as the set of vertices of the graph

Answer 9.4. (a) Given a graph G an induced subgraph is completely determined by the choice of a subset of its set of vertices. If the graph has n vertices, it has 2^n possible subset of the set of vertices (including the empty set and the whole set). Hence, the graph has 2^n subgraphs including the empty graph and the whole graph.

If we start with the complete graph on n. vertices, every induced subgraph will also be a complete graph. In particular, all induced subgraph with a fixed number k of vertices are isomorphic.

(b) A spanning subgraph is a subgraph whose set of vertices is the same as the set of vertices of the graph. Therefore, a spanning subgraph is obtained from a graph by choosing an arbitrary subset of the set of edges of G. If G is the complete graph with n vertices, it has $\binom{n}{2}$ edges. Therefore, there are $2^{\binom{n}{2}}$ subsets of the set of edges and correspondingly $2^{\binom{n}{2}}$ spanning subgraphs. This number includes again the original complete graph.

Question 9.5. Recall that two graphs $(V_1, E_1, f_1), (V_2, E_2, f_2)$ are said to be isomorphic if there exists bijections $g_V : V_1 \to V_2,\ g_E : E_1 \to E_2$ compatible with the incidence morphisms f_1, f_2 meaning that if $g_E(e_1) = e_2$ and the vertices of e_1 are a, b, then the vertices of e_2 are $g_V(a), g_V(b)$.

(a) Show that" being isomorphic to" is an equivalence relation.

(b) Describe the isomorphism classes of simple graphs with 3 vertices. Note: a graph with vertex set $V = \{a, b, c\}$ and edge set $E = \{\{a, b\}\}$ is considered a different graph from the graph with vertex set $V = \{a, b, c\}$ and edge set $E' = \{\{a, c\}\}$

Answer 9.5. (a) The condition' being isomorphic to" is an equivalence relation:

- It is reflexive as the identity map gives an isomorphism from a graph to itself: First, the identity map in both the set of vertices and the set of edges is a bijection. One way to show that a function is a bijection is by showing it has an inverse such that the composition both ways is the identity. The identity map is its own inverse, hence a bijection.
  Taking the identity map in both the set of vertices and the set of edges, we get trivial compatibility with incidence at the nodes: If an edge e has end points a, b, then $g_E(e) = Id(e) = e$ has end points $a = Id(a), b = Id(b)$.
- We show that the relation is symmetric. Assume that $(V_1, E_1, f_1) \sim (V_2, E_2, f_2)$ By definition of the relation, this means that there is an isomorphism from

  $$(V_1, E_1, f_1) \text{ to } (V_2, E_2, f_2) \text{ given by } g_V^1 : V_1 \to V_2, g_E^1 : E_1 \to E_2$$

  We then check that there is an isomorphism from (V_2, E_2, f_2) to (V_1, E_1, f_1) given by the inverse map $g_V^{-1} : V_2 \to V_1, g_E^{-1} : E_2 \to E_1$. These functions do exist because the inverse of a bijection is a well defined function and a bijection.
  We check compatibility with incidence at the nodes: If an edge e has end points a, b, then by assumption $g_E(e)$ has end points $g_V(a), g_V(b)$. As every edge in V_2 is of the form $e' = g_E(e)$, then we can check that $g_E^{-1}(e') = e$, has end points $a = g_V^{-1}(g_V(a)), b = g_V^{-1}(g_V(b))$.
  Therefore $g_V^{-1} : V_2 \to V_1, g_E^{-1} : E_2 \to E_1$ give an isomorphism of the second graph with the first showing that the relation is symmetric
- We prove that the relation is transitive. Assume that $(V_1, E_1, f_1) \sim (V_2, E_2, f_2)$ and $(V_2, E_2, f_2) \sim (V_3, E_3, f_3)$

By defnition of the relation, this means that there is an isomorphism from

$$(V_1, E_1, f_1) \text{ to } (V_2, E_2, f_2) \text{ given by } g_V^1 : V_1 \to V_2, g_E^1 : E_1 \to E_2$$

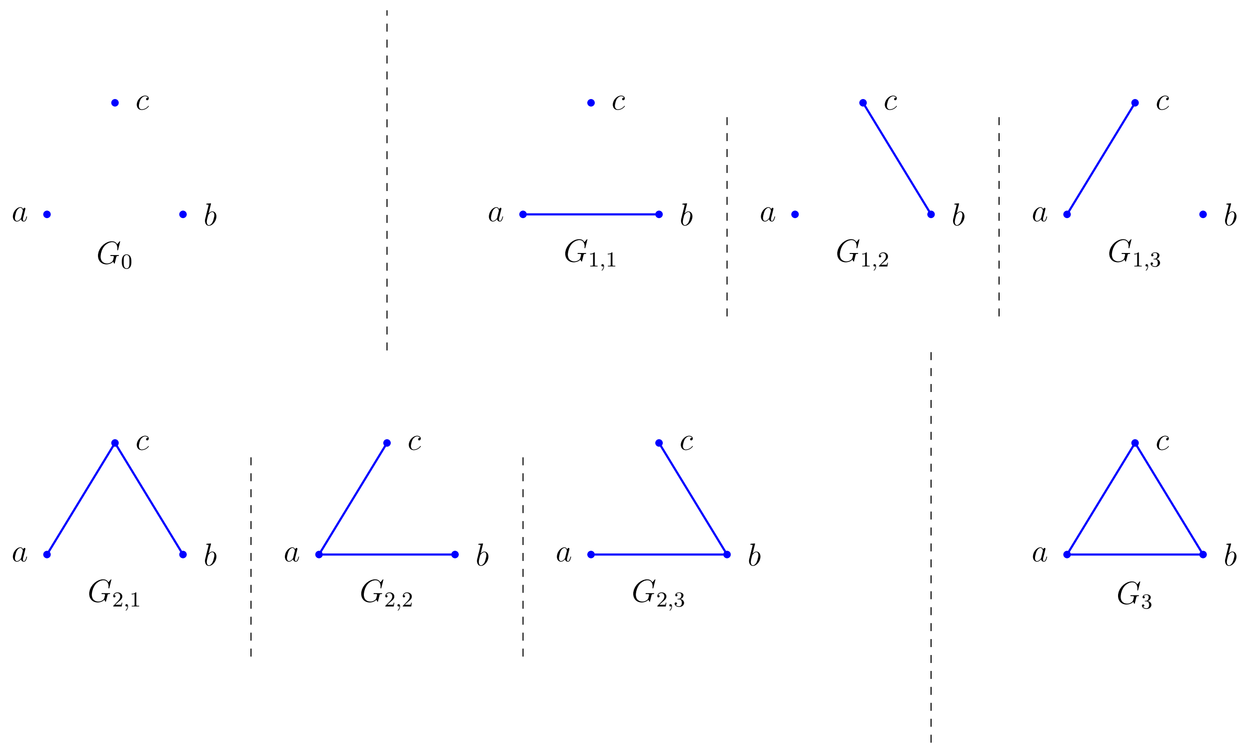
and an isomorphism

$$(V_2, E_2, f_2) \text{ to } (V_3, E_3, f_3) \text{ given by } g_V^2 : V_2 \to V_3, g_E^2 : E_2 \to E_3,$$

We show then that the composition

$$(V_1, E_1, f_1) \text{ to } (V_3, E_3, f_3) \text{ given by } g_V^2 \circ g_V^1 : V_1 \to V_3, g_E^2 \circ g_E^1 : E_1 \to E_3$$

is also an isomorphism of graphs. From the assumption that if a vertex $e \in E_1$ has endpoints a, b, then its image $g_E^1(e)$ has end points $g_V^1(a), g_V^1(b)$ and $g_E^2(g_E^1(e))$ has end points $g_V^2(g_V^1(a)), g_V^2(g_V^1(b))$,it follows that the composition preserves incidences and therefore, is an isomorphism of graphs.
By definition of the relation, this means that $(V_1, E_1, f_1) \sim (V_3, E_3, f_3)$



(b) There cannot be an isomorphism between two graphs with different number of edges. So,

$$[G_0] = \{G_0\}, \quad [G_{1,1}] = \{G_{1,1}, G_{1,2}, G_{1,3}\}, \quad [G_{2,1}] = \{G_{2,1}, G_{2,2}, G_{2,3}\}, \quad [G_3] = \{G_3\},$$

There are four isomorphism classes .

On a graph with no edges, every vertex has degree zero. As an isomorphism preserves the degree of corresponding vertices, this graph is only isomorphic to graphs in which all degrees of the vertices are zero degree zero. Hence, this graph G_0 is only isomorphic to itself.

The three possible graphs with one edge $G_{1,1}, G_{1,2}, G_{1,3}$ (see picture) are all isomorphic to each other. For instance, there are two isomorphisms f_1, f_2 from $G_{1,1}$ to $G_{1,2}$. For both of these morphisms $f_i(c) = a$ because the only vertex of degree 0 in $G_{1,1}$ must map to the only vertex of degree 0 in $G_{1,2}$. The other two vertices can be assigned arbitrarily by say $f_1(a) = b, f_1(b) = c, f_2(a) = c, f_1(b) = b$, With both assignments, the edge is preserved. Something similar can be said for morphisms from $G_{1,1}$ to $G_{1,3}$ or from $G_{1,2}$ to $G_{1,3}$.

Similarly, all graphs with two edges are isomorphic among themselves. In fact, $G_{2,1}$ is the complement of $G_{1,1}$, $G_{2,2}$ is the complement of $G_{1,2}$. The maps f_1, f_2 defined above among the vertices also define an isomorphism of graphs from $G_{2,1}$ to $G_{2,2}$..

Finally, the graph G_3 is only isomorphic to itself as there are no other graphs with thee edges and an isomorphism establishes a bijection between the set of edges.

Question 10.1. If G is a graph, a clique is a subgraph of G such that there is an edge between every two vertices. The clique number $\omega(G)$ is the number of vertices in the largest clique in G. An independent set is a subgraph of G such that there are no edges between any two vertices of the set. The independence number $\alpha(G)$ is the number of vertices in the largest independent set in G. Assume that G is a graph and H a subgraph.

(a) Prove or disprove $\alpha(G) \leq \alpha(H)$.
(b) Prove or disprove $\alpha(G) \geq \alpha(H)$.
(c) Prove or disprove $\omega(G) \leq \omega(H)$.
(d) Prove or disprove $\omega(G) \geq \omega(H)$.

Answer 10.1. (a) It is not always true that $\alpha(G) \leq \alpha(H)$. For the graph G_1 in Figure a, left $\alpha(G_1) = 2$ as the set $\{b, c\}$ is an independent set while the set $\{a, b, c\}$ is not. For its subgraph H_1 pictured center left, $\alpha(H_1) = 1$ as the set $\{b\}$ is an independent set while the set $\{a, b\}$ is not.

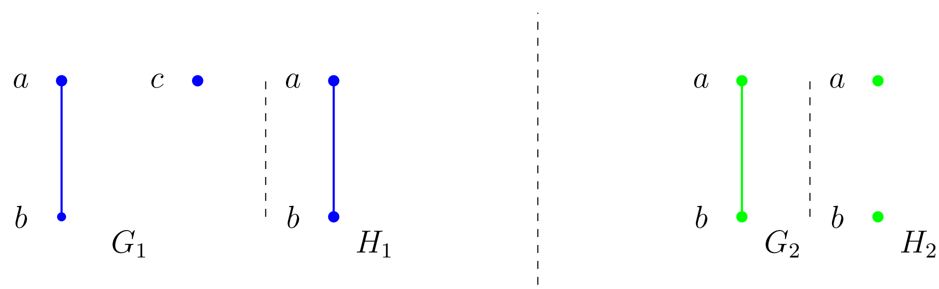


Figure 5. Left, the independence number is smaller in the subgraph than in the graph. Right the independence number is larger in the subgraph, the clique number is larger in the graph

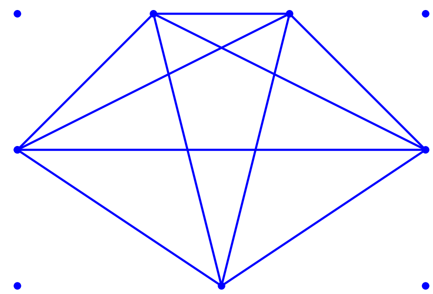
(b) It is not always true that $\alpha(G) \geq \alpha(H)$. For the graph G_2 pictured above, center right $\alpha(G_2) = 1$ as the set $\{a\}$ is an independent set while the set $\{a, b\}$ is not. For its subgraph H_2 pictured above right, $\alpha(H_2) = 2$ as the set $\{a, b\}$ is an independent set (there are no edges in the subgraph).

(c) It is not true that for every graph G and subgraph H $\omega(G) \leq \omega(H)$. For the graph G_2 pictured above, $\omega(G_2) = 2$ as the set $\{a, b\}$ is a clique, in this case containing all vertices of G_2. For its subgraph H_2 pictured above, $\omega(H_2) = 1$ as there are no edges in the subgraph.

(d) It is true that for every graph G and subgraph H, $\omega(G) \geq \omega(H)$: Assume that the largest clique in H has k vertices. There is then a set of k vertices in H such that between every two vertices there is an edge in H. Every edge in H is also an edge in G. Hence G has a clique with k vertices. From the definition of clique number, $\omega(G) \geq k = \omega(H)$.

Question 10.2. Find a graph G with $\alpha(G) = \omega(G) = 5$. What is the smallest number of vertices that G could have if it satisfies these conditions? Carefully justify your answer.

Answer 10.2. A graph containing a clique with 5 vertices in addition to four vertices of degree 0 satisfies the condition (see picture above). The four vertices of degree 0 together

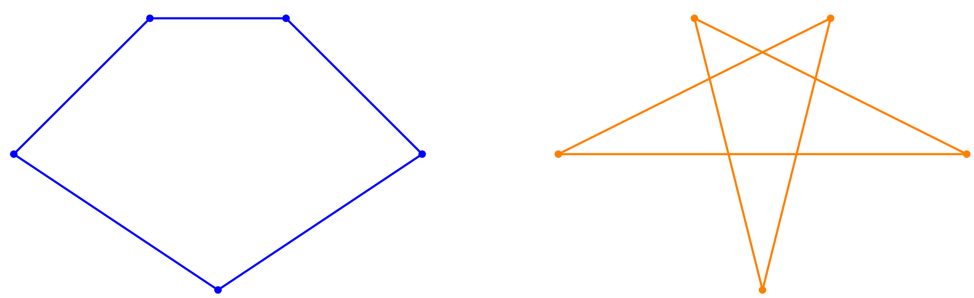


with one of he vertices in the clique form an independent set. Adding any other vertex (necessarily in the clique) the set is not independent, as there is an edge between any two vertices in the clique.

Nine is the smallest number of vertices in a graph satisfying the two conditions. We need five vertices for the clique, five vertices for the independent set and these two sets can have at most one element in common, as no more than one element in any clique may also be part of an independent set.

Question 10.3. Find a graph with 5 vertices that has clique number two AND independence number two or show that none exists.

Answer 10.3. Consider a pentagon, that is a graph with 5 vertices numbered $v_1, \ldots, v_5$ and edges from v_i to v_{i+1} $i = 1, \ldots, 4$ and from v_5 to v_1. This graph has clique number $\omega(G) = 2$, as there are edges but no triangles. This graph has independence number $\alpha(G) = 2$: starting at any vertex v_i, if you want a vertex not connected to this one, you need to skip the next. But then, skipping again you get back to the original. Alternatively, you can see that the dual graph (pictured below) has no triangles and therefore clique number 2



Question 10.4. Given a simple graph G show that either G or its complement (or both) are connected.

Answer 10.4. Let us assume that the graph G is not connected. We will then prove that its complement is connected. Write G as the union of its connected components

$$G = G_1 \cup G_2 \cup \cdots \cup G_k.$$

Choose one vertex $v_1 \in G_1, v_2 \in G_2$. Because there is no path between v_1 and v_2 in G, there is no edge $\{v_1, v_2\}$ in G. Hence, $\{v_1, v_2\}$ is an edge in $\bar{G}$.

Take now an arbitrary vertex v. If $v \notin G_1$, then there is no edge $\{v_1, v\}$ in G. Hence, $\{v_1, v\}$ is an edge in $\bar{G}$. If $v \in G_1$, then there is no edge $\{v_2, v\}$ in G. Hence, $\{v_2, v\}$ is an edge in $\bar{G}$ and $\{v_1, v_2\}$ followed by $\{v_2, v\}$ is a path in $\bar{G}$..

Therefore, every vertex can be joined to v_1 by a path in $\bar{G}$ proving that $\bar{G}$ is connected.

Question 10.5. (a) Show that if a **simple** graph G with $n \geq 2$ vertices has at least $\binom{n-1}{2}+1$ edges, then G is connected.

(b) For every $n \geq 2$ give an example of a simple graph G with $\binom{n-1}{2}$ edges, such that G is not connected.

Answer 10.5. (a) We will show that if a graph G has $n \geq 2$ vertices and is not connected, then it has fewer than $\binom{n-1}{2}+1$ edges. Assume then that G is not connected. It has then two disconnected pieces $G = G_1 \cup G_2$ with n_1, n_2 vertices, $n_1 \geq 1, n_2 \geq 1, n = n_1 + n_2$. Because there are no edges between vertices in G_1 and vertices in G_2, the largest possible number of edges is

$$\binom{n_1}{2} + \binom{n_2}{2} = \frac{(n_1-1)n_1 + (n_2-1)n_2}{2} = \frac{n_1^2 + n_2^2 - (n_1+n_2)}{2}$$

On the other hand, from $n = n_1 + n_2$, we get

$$\binom{n-1}{2}+1 = \binom{n_1+n_2-1}{2}+1 = \frac{(n_1+n_2-1)(n_1+n_2-2)+2}{2} = \frac{n_1^2+n_2^2+2n_1n_2-3(n_1+n_2)+4}{2}$$

We need to show that

$$n_1^2 + n_2^2 - (n_1+n_2) < n_1^2 + n_2^2 + 2n_1n_2 - 3(n_1+n_2) + 4$$

This is equivalent to

$$0 < 2(n_1n_2 - n_1 - n_2 + 2) = 2[(n_1-1)(n_2-1)+1]$$

which is true as $n_1 \geq 1, n_2 \geq 1$ imply $(n_1-1)(n_2-1) \geq 0$ and therefore

$$(n_1-1)(n_2-1)+1 \geq 1.$$

(b) For every $n \geq 2$, consider the complete graph with $n-1$ vertices. Add to it one vertex but no additional edges. Then G has $\binom{n-1}{2}$ edges but G is not connected as it has one vertex which does not belong to any edge.

Question 10.6. Let $A_n = \{1, 2, \ldots, n\}$. Define a graph G_n whose vertex set V_n is the set of pairs of elements in A and whose edge set E_n is defined by "there is an edge between $\{a, b\}$ and $\{c, d\}$ if and only if $\{a, b\} \cap \{c, d\} = \emptyset$".

(a) Sketch G_2, G_3, G_4, G_5.
(b) How many vertices does G_n have? What is the degree of each vertex?
(c) How many edges does G_n have?
(d) Find the number of connected components of G_2, G_3, G_4, G_5, Find the number of connected components of G_n. Justify your answer.

Answer 10.6. (a) In Figure 6, we sketched G_2, G_3, G_4, In Figure 7, we sketched G_5

(b) Because a vertex of G_n is a pair of numbers in the set $A_n = \{1, 2, \ldots, n\}$, there are $\binom{n}{2}$ vertices $v_{i,j}$ in G_n. A vertex $v_{i,j}$ is contiguos to $v_{k,l}$ if and only if $\{i,j\} \cap \{k,l\} = \emptyset$. Therefore, the vertices adjacent to $v_{i,j}$ are parameterized by pairs of indices of $\{1, 2, \ldots, n\} - \{i, j\}$. There are $\binom{n-2}{2}$ such vertices. Hence, the degree of each vertex is $\binom{n-2}{2}$

(c) Using that the sum of the degrees of the vertices is twice the number of edges, we obtain that the number of edges of G_n is

$$\frac{\binom{n}{2}\binom{n-2}{2}}{2} = \frac{n(n-1)(n-2)(n-3)}{8}$$

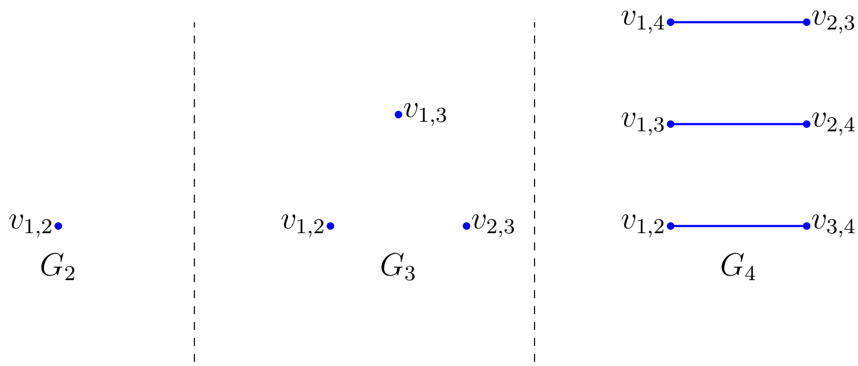


FIGURE 6. G_2, G_3, G_4

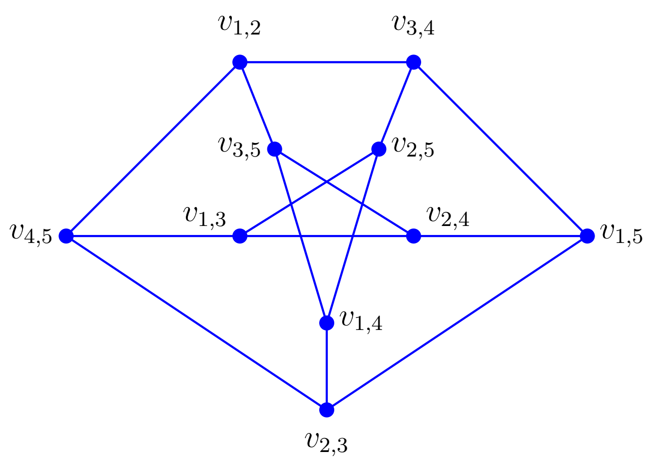


FIGURE 7. G_5

An alternative way of counting the edges is as follows: there is an edge between v_{i_1,i_2} and v_{i_3,i_4}, if and only if $\{i_1, i_2, i_3, i_4\}$ are all different. We can start by choosing 4 of the elements in $A_n = \{1, 2, \ldots, n\}$. Then we decide with which of the three remaining elements we pair the first of the elements we have. Therefore, we have

$$3\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{8}$$

(d) The graph G_2 has only one vertex and therefore one component. The graph G_3 has 3 vertices and no edges, therefore has 3 components. The graph G_4, has 6 vertices and three edges, each joining two different vertices. It has 3 components.

We claim that $G_n, n \geq 5$ is connected. Take two vertices $v_{i,j}, v_{k,l}$. If $\{i,j\} \cap \{k,l\} = \emptyset$, then there is an edge between $v_{i,j}$ and $v_{k,l}$. If $\{i,j\} \cap \{k,l\} \neq \emptyset$, as we are assuming the vertices to be different, we can assume $k = i, l \neq i, l \neq j$. As $n \geq 5$, there are two indices s, t different from i, j, l. There is then an edge $e_{i,j,s,t}$ from $v_{i,j}$ to $v_{s,t}$ and an edge $e_{s,t,i,l}$ from $v_{s,t}$ to $v_{i,l}$. Therefore, $v_{i,j}, e_{i,j,s,t}, v_{s,t}, e_{s,t,i,l}, v_{i,l}$ is a path connecting $v_{i,j}, v_{i,l}$. Hence, every two vertices are connected. So, the graph is connected.

Question 10.7. Let G be a connected graph that is not eulerian (no Euler circuit).

(a) Show that we can add to G a single vertex and a few edges starting at this vertex so that the resulting graph is eulerian.
(b) Show that we can add to G a few edges (no new vertices) to make it eulerian, What is the smallest number of edges we need to add for this to happen?
(c) Let G be a simple connected graph that is not eulerian. Can we add to G a few edges (no new vertices) to make it eulerian, while keeping it simple?.

Answer 10.7. (a) The sum of the degrees of the vertices of a graph is twice the number of edges and in particular it is even. Therefore, there needs to be an even number of vertices of odd degree. Add one new vertex to the graph . Add an edge from that vertex to every vertex of odd degree. The degree of the vertices of odd degree is increased by one and is therefore even now. The degree of the new vertex equals the number of vertices of odd degree in the old graph and is therefore even. As all the vertices in the new graph have even degree, the graph becomes eulerian.
(b) As we showed before, there is an even number of vertices of odd degree. If we pair them and add to the graph an edge between the two vertices in the pair, then the vertices will all have even degree.
(c) Take K_4 which is simple but not eulerian, as its vertices have degree 3. We cannot add any edges to K_3 and keep it simple ,as it already has all possible edges. So, this is not always possible.

Question 11.1. Recall that a leaf in a tree is a vertex of the tree of degree one. We showed that if T is a tree and v a leaf, then $T - \{v\}$, the graph obtained by removing from T the vertex v and any edges incident with v is also a tree. Prove the converse: assume that T is a tree with at least 2 vertices, v a vertex in the tree such that $T - \{v\}$ is also a tree. Show that then v is a leaf.

Answer 11.1. We will show that if v is not a leaf, then $T - \{v\}$ is not connected and therefore, not a tree.

The vertex v not being a leaf means that it has degree at least 2. Let v_1, v_2 be two vertices adjacent to v. Then, there is a path v_1, v, v_2 that joins v_1 to v_2. As T is a tree, this is the only path that joins v_1 to v_2. Therefore, in $T - \{v\}$ there is no path joining v_1 to v_2. Hence, $T - \{v\}$ is not connected.

Question 11.2. Recall that an n-cycle is a simple closed path of length n.

Let T be a tree, u, v vertices of T such that there is no edge between u, v. Consider the graph G obtained by adding to the edges of T a new edge e from u to v. Show that G has a unique cycle

Answer 11.2. As T is connected, there exist a simple path in T,

$$P = (u = v_0, e_{v_0,v_1}, v_1, \dots, v_{n-1}, e_{v_{n-1},v_n}, v_n = v).$$

Then, in the graph G obtained by adding the edge e to T, there is a closed simple path, which is therefore a cycle

$$P' = (u = v_0, e_{v_0,v_1}, v_1, \dots, v_{n-1}, e_{v_{n-1},v_n}, v_n = v, e, v_{n+1} = u)$$

We need to check that this is the only cycle. Assume that there is another simple closed path in G,

$$P'' = (w_0, e_{w_0,w_1}, w_1, \dots, w_{m-1}, e_{w_{m-1},w_m}, w_m = w_0).$$

As there are no circuits in T , this circuit necessarily includes the edge e, that is, there exist a j with $w_j = v$ and either $w_{j-1} = u$ or $w_{j+1} = u$. To fix ideas, let us say $w_{j+1} = u$. Then,

$$(u = w_{j+1}, e_{w_{j+1},w_{j+2}} w_{j+2} \dots w_m = w_0, e_{w_0,w_1}, w_1, \dots, w_j = v).$$

is a path in T from u to v. As a path in T is unique,

$$w_{j+1} = v_0, w_{j+2} = v_1 \dots w_m = v_{m-j-1}, w_1 = v_{m-j}, \dots, w_j = v_n$$

Therefore, the new given circuit $w_0, w_1, \dots w_{m-1}, w_m = w_1$ is actually the circuit $u = v_1, v_2, \dots, v_{n-1}, v_n = v, v_{n+1} = u$ we built before , up to starting somewhere in the middle.

Question 11.3. Let G be a connected graph in which every vertex has degree precisely 2. Is G then necessarily a cycle?

Answer 11.3. Choose an arbitrary vertex v_1 of G. As the degree of v_1 is 2, there is a vertex v_2 adjacent to v_1.

As the degree of v_2 is 2, in addition to v_1, there is a vertex v_3 adjacent to v_2.

As the degree of v_3 is 2, in addition to v_2, there is a vertex v_4 adjacent to v_3. We cannot have $v_4 = v_2$ because v_2 has degree 2 and we already know it is adjacent to v_1, v_3. If $v_4 = v_1$, we have a circuit.

If $v_4 \neq v_1$, we repeat the process for v_4, as the degree of v_4 is 2, in addition to v_3, there is a vertex v_5 adjacent to v_4. We cannot have $v_5 = v_2$ because v_2 has degree 2 and we already know it is adjacent to v_1, v_3. Similarly, we cannot have $v_5 = v_3$. If $v_5 = v_1$, we have a circuit.

As the number of vertices is finite, at some point, we will get $v_n = v_1$. Therefore, we have a circuit $v_1, v_2, v_3, \dots, v_{n-1}, v_n = v_1$.

Assume now that there is an additional vertex on G besides $v_1, v_2, v_3, \dots, v_{n-1}$. As G is connected, there is a path $v = w_1, w_2, \dots w_k = v_1$ from v to v_1. Let i be the smallest index such that $w_i \in \{v_1, v_2, v_3, \dots, v_{n-1}\}$, meaning

$$\exists j, \text{ such that } w_i = v_j \text{ while } w_l \neq v_t, \forall l < i, \forall t \in \{1, \dots, n-1\}.$$

From the fact that v is not in the set $\{v_1, v_2, v_3, \dots, v_{n-1}\}$ while $w_k = v_1$, these i, j exist. But then v_j is adjacent to $v_{j-1}, v_{j+1}, w_{i-1}$ which contradicts the assumption that v_j has degree 2. Hence, no additional vertices exist and G is a circuit.

Question 11.4. Let G be a planar graph with v vertices, e edges, f faces and c connected components. Prove a relation between these numbers.

Answer 11.4. Assume that G has connected components $G_1, \dots, G_c$. Denote by v_i, e_i, f_i the number of vertices, edges and faces of G_i. Then, $v_i - e_i + f_i = 2$. The number of vertices of the graphs is the sum of the vertices of each components and similarly for the edges $v = \sum v_i, e = \sum e_i$.

As the graph is planar, edges of the different components do not intersect. Hence, G_2 is on the unbounded face of G_1 and similarly for the other. Hence, G_2 subdivides one of the faces of G_1 into f_2 faces and the number of faces of the union is $f_1 + f_2 - 1$. Similarly, each new component adds $f_i - 1$ faces to the union. Hence, $f = \sum f_i - (c-1)$. Then,

$$v - e + f = \sum v_i - \sum e_i + \sum f_i - (c-1) = 2c - (c-1) = c + 1$$

In the case $c = 1$, we would obtain the usual equation $v - e + c = 2$.

Question 11.5. Let G be a simple graph with 11 vertices. Show that at least one of G or its complement $\bar{G}$ is non-planar. Hint: If a graph is planar, the number of edges is bounded in terms of the number of vertices.

Answer 11.5. If G has 11 vertices and e edges, then $\bar{G}$ has 11 vertices and $\bar{e} = \binom{11}{2} - e = 55 - e$ edges. For a planar graph, the number of vertices and edges satisfy $e \leq 3v - 6$. If both $G, \bar{G}$ were planar, we would have

$$e \leq 3v - 6 = 33 - 6 = 27, \quad 55 - e \leq 27$$

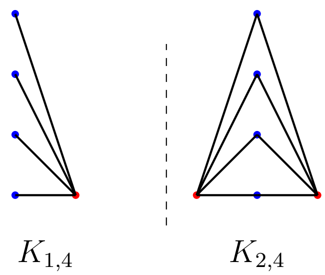
Adding these two equations, we obtain $55 \leq 54$ which is false. Hence, at least one of the two graphs is non-planar.

Question 11.6. Recall that the bipartite graph $K_{m,n}$ has a set of vertices V that can be written as the disjoint union of two subset V_1, V_2 with m and n vertices respectively and has one edge for every pair of vertices, the first from V_1 and the second from V_2.

For which values of m, n is the bipartite graph $K_{m,n}$ planar?

Answer 11.6. The graph $K_{1,n}$ is a tree with one vertex of degree n and n leaves, in particular it is planar.

The graph $K_{2,n}$ is planar. We can take the two vertices of the first set to be $(1, 0), (-1, 0)$ and the vertices of the other set to be $(0, i), i = 1, 2, \dots, n$. The segments that join $(1, 0)$ to



$(0, i), i = 1, 2, \dots, n$ are entirely contained on the half plane $x \geq 0$. They intersect each other only at $(1, 0)$. The segments that join $(-1, 0)$ to $(0, i), i = 1, 2, \dots, n$ are entirely contained on the half plane $x \leq 0$. They intersect each other only at $(-1, 0)$. The segments of the first set intersect the segments of the second set only at the vertices $(0, i)$. Hence, the graph is planar.

If both m, n are larger than 3, then the graph contains $K_{3,3}$ which is not planar. If a graph contains a non-planar subgraph, then the graph is not planar. One can easily prove the contrapositive: if a graph can be immersed in the plane without the edges intersecting then any subgraph obtained by removing some vertices and perhaps edges is already immersed in the plane without the edges intersecting.

If both $m, n \geq 3$, then $K_{m,n}$ contains $K_{3,3}$ and therefore it is non-planar.

Question 11.7. (a) Show that there are exactly $2^{\binom{n}{2}}$ simple graphs with vertex set $\{v_1, \dots, v_n\}$.

(b) Show that there are exactly $2^{\binom{n-1}{2}}$ simple graphs with vertex set $\{v_1, \dots, v_n\}$ in which every vertex has even degree. (Hint: Show that there is a bijection between simple graphs with n vertices all of even degree and simple graphs with $n - 1$ vertices of arbitrary degree).

(c) Compute the probability that for a randomly chosen simple graph with vertex set $\{v_1, \dots, v_n\}$ every connected component will have an Eulerian circuit .

Answer 11.7. (a) To make a graph on n labeled vertices, we only need to decide when to put an edge between any two vertices. There are 2 options (edge or no edge) and $\binom{n}{2}$ possible pairs of edges. So there are $2^{\binom{n}{2}}$ simple graphs on n labeled vertices.

(b) Define the two sets

$$A = \{ \text{ simple graphs with vertices } \{v_1, \dots, v_n\} \text{ all vertices even degree}\}.$$

$$B = \{ \text{ simple graphs with vertices } \{v_1, \dots, v_{n-1}\}$$

Define functions

$$f : A \to B, \quad h : B \to A$$

as follows: Let $G \in A$.Create a new graph G' as follows: remove the vertex v_n to G and all edges that connect to v_n. This new graph G' is a graph with vertices $\{v_1, ..., v_{n-1}\}$. Define $f(G) = G'$.

Let $G_1 \in B$. Create a new graph G'_1 as follows: Add a vertex v_n to G_1 and connect every odd degree vertex of G to v_n. This new graph G'_1 is a graph with vertices $\{v_1, ..., v_n\}$ in which every old vertex $\{v_1, ..., v_{n-1}\}$ has now even degree, as those who had odd degree before have now one more edge while those who had even degree stay with the same edges as in G_1. The new vertex also has even degree, as we know that the number of vertices of even degree in a graph is even. Define $h(G_1) = G_1'$

We claim that f, h are functions inverse of each other: For $G_1 \in B$, $f(h(G_1))$ is obtained by first adding one vertex and a few edges to G_1 and then removing that vertex and edges. So,$f(h(G_1)) = G_1$. Given $G \in A$, recall that by assumption, all vertices of G have even degree. Therefore, $f(G)$ is a graph with vertices $v_1, \ldots, v_{n-1}$ in which the vertices of odd degree are the ones that were joined to v_n in G. Then, by definition of h, $h(f(G)) = G$. When there is a biection between two sets, they both have the same nnumber of elements. By (a), here are $2^{\binom{n-1}{2}}$ elements inn B. Hence, there are $2^{\binom{n-1}{2}}$ graphs with vertex set $\{v_1, \ldots, v_n\}$ in which every vertex has even degree.

(c) A connected graph has a Eulerian circuit precisely when each of its vertices $\{v_1, ..., v_n\}$ has even degree. So the desired probability is

$$\frac{2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = 2^{\binom{n-1}{2} - \binom{n}{2}} = 2^{\frac{(n-1)(n-2)-(n-1)n}{2}} = 2^{\frac{(n-1)(n-2-n)}{2}} = \frac{1}{2^{n-1}}.$$