2a) Let $A \subset \mathbb{R}$ be a subspace in the finite complement topology. Let $\{u_i\}_{i \in I}$ form an open cover of A so $\bigcup_{i \in I} u_i \supseteq A$

Let $u \in \{u_i\}$, then by def $\mathbb{R} - U = \{u_1, u_2 \dots u_n\}$
WLOG, suppose $u_1, u_2 \dots u_n \in A$. As $\bigcup u_i \supseteq A$, then at most, there exists $U_1 \dots U_n$ s.t. $u_1 \in U_1, u_2 \in U_2 \dots u_n \in U_n$
This is at most n U of these, and ~~can get~~
Therefore $\bigcup u\, U_1 \cup U_2 \dots \cup U_n \supseteq A$. So, this is a finite subcover since n is finite, so A is compact

$\square$

b) Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3 \dots\}$ be the ordered set of rationals on $[0,1]$.
Consider $U_i = (\mathbb{R} \setminus (\mathbb{Q} \cap [0,1]) \cup \{q_i\}$
U_i is open as $U_i^c = \mathbb{Q} \cap [0,1] - \{q_i\}$ which is countable.
Then, consider $\bigcup_{i \in I} U_i = \bigcup_{i \in I} (\mathbb{R} \setminus (\mathbb{Q} \cap [0,1]) \cup \{q_i\})$

$= \mathbb{R} \supseteq [0,1]$

So $\{U_i\}$ forms an open cover of $[0,1]$, but has no finite subcover.
If it did, $\exists n > 0$ s.t. $\bigcup_{i=1}^{n} (\mathbb{R} \setminus \mathbb{Q} \cap [0,1]) \cup \{q_i\} \supseteq [0,1]$
But for any n we pick, $\exists \{q_n\}$ that's not in the set since if we say stop at q_n, then $\exists q_{n+1} \in \mathbb{Q}$ between $(q_n, 1)$ but isn't included in the cover so the finite subcover isn't all of $[0,1]$

Therefore, $[0,1]$ is not compact. $\square$

3 Suppose τ and τ' are comparable and WLOG $\tau' \subseteq \tau$.

Consider: $Id_X\ (X,\tau') \longrightarrow (X,\tau)$

$x \longrightarrow x$

Since τ is finer, Id_X is continuous. Also, this is a homeomorphism as Id_X is bijective, (X,τ') is compact and (X,τ) is Hausdorff, so Id_X is a homeomorphism.

So, for $U \in \tau'$ then as $Id_X(U) = U$, U is open in τ since homeomorphism. We can use symmetric logic for $U \subseteq \tau$ and Id_X^{-1}.

Therefore, since $U \subseteq \tau' \Rightarrow U \subseteq \tau$ and $U \subseteq \tau \Rightarrow U \subseteq \tau'$, then τ and $\tau = \tau'$. So, if τ and τ' are comparable, then $\tau = \tau'$.

$\square$

4 Let $Z \subseteq X \times Y$ be closed. Consider $X - \pi_1(Z)$. Let $x_0 \in X - \pi_1(Z)$, Since Y is compact, we can use the Tube lemma on $x_0 \times Y$.

By the Tube Lemma, $\exists$ open nbd of x_0, W_{x_0} s.t. $W_{x_0} \times Y \subseteq N$ where N is open in $X \times Y$.

However, it then follows by definition that $(W_{x_0} \times Y) \cap Z = \emptyset$. So as W is an open nbd around x_0.

Note that $\bigcup_{x \in X - \pi_1(Z)} W_{x_0} = X - \pi_1(Z)$

W_{x_0} is open and union of open sets is open, so $X - \pi_1(Z)$ is open and $\pi_1(Z)$ is closed and therefore, is a closed map. $\square$