Math 125 Proj 2

1a) $L=\int_{-\pi}^{\pi}\sqrt{(-a\sin t)^2+(b\cos t)^2}\,dt=\int_{-\pi}^{\pi}a\sqrt{\cos^2 t+\sin^2 t}\,dt=2\pi a$

b) $L=\int_{-\pi}^{\pi}\sqrt{(-a\sin t)^2+(b\cos t)^2}\,dt=\int_{-\pi}^{\pi}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,dt$

$=\int_{-\pi}^{\pi}\sqrt{a^2(\sin^2 t+\frac{b^2}{a^2}\cos^2 t)}\,dt=a\int_{-\pi}^{\pi}\sqrt{\sin^2 t+(b^2/a^2)(1-\sin^2 t)}\,dt$

$=a\int_{-\pi}^{\pi}\sqrt{\frac{b^2}{a^2}+\sin^2 t-\frac{b^2}{a^2}\sin^2 t}\,dt=a\int_{-\pi}^{\pi}\sqrt{\frac{b^2}{a^2}+(1-\frac{b^2}{a^2})\sin^2 t}\,dt$

$1-\frac{b^2}{a^2}=k^2$

$=a\int_{-\pi}^{\pi}\sqrt{(1-k^2)+k^2\sin^2 t}\,dt=a\int_{-\pi}^{\pi}\sqrt{1+k^2(1-\sin^2 t)}\,dt$

$=a\int_{-\pi}^{\pi}\sqrt{1-k^2\cos^2 t}\,dt=a\int_{-\pi}^{\pi}\sqrt{1-k^2\sin^2 t}\,dt$ as both are positive only and $-\pi$ to π is one rotation so

$=2a\int_{0}^{\pi}\sqrt{1-k^2\sin^2 t}\,dt=4a\int_{0}^{\pi/2}\sqrt{1-k^2\sin^2 t}\,dt$ as splits cycle. we can do this again as positive

c) $12\int_{0}^{\pi/2}\sqrt{1-\frac{5}{9}\sin^2 t}\,dt$

Use trapezoid method, I use $n=10^6$ to get error $\le 10^{-6}$

So from class error is:

error analysis

$e_n \le \frac{\max_{x\in[a,b]}|f''(x)|\,(b-a)^3}{12n^2}$

$e_n \le \frac{|5/6|(\frac{\pi}{2})^3}{12(10^6)^2}=\frac{\frac{5\pi^3}{48}}{12\cdot 10^{12}}\le 10^{-6}$

$f(x)=\sqrt{1-\frac{5}{9}\sin^2 x}$

$f''(x)=\frac{-5(5\sin^4 x-9\sin^2 x+9\cos^2 x)}{3(9-5\sin^2(x))^{3/2}}$

max on $[0,\frac{\pi}{2}]=5/6$

1 c) con. Using my code I get
perimeter = 15.865439431894304. (code at end)

1 d) Alternative perimeter approximations

$2\pi\sqrt{\frac{a^2+b^2}{2}}$: 16.01904...

$\pi\left(3(a+b)-\sqrt{(3a+b)(a+3b)}\right)$: 15.86543

$\pi(a+b)\left(1+\frac{3h}{10+\sqrt{4-3h}}\right)$: 15.8646541

$h=\frac{(a-b)^2}{(a+b)^2}$

Ramanujan approximations, first one is from Matt Parker video, link is in code. Both Ramanujan approxs. are within .001 of the numerical integration.

1.2 a) You can use Gaussian quadrature to get weights and a polynomial that is orthogonal to do the integral.

b) For n=2 $\int_{-1}^{1} f(x)\,dx \approx \sum_i w_i f(x_i) = f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$

As $w_i = 1$

So $\int_{-1}^{1} f(x)\,dx \approx 1.693$

For n=3 $\int_{-1}^{1} f(x)\,dx \approx \sum_i w_i f(x_i) = \frac{5}{9} f\left(-\frac{\sqrt{3}}{5}\right)+\frac{8}{9}f(0)+\frac{5}{9}f\left(\frac{\sqrt{3}}{5}\right)$

$\int_{-1}^{1} f(x)\,dx \approx 1.712$

For n=4

Weights from wikipedia and roots from problem

$\int_{-1}^{1} f(x)\,dx \approx \sum_i w_i f(x_i) = 1.7112$ rounded since I'm writing this out.

1.2 c) Normal PDF is $\frac{1}{\sqrt{2\pi}} e^{-x^2}$ so we can take $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2} dx$

Using a numerical solver $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2} dx \approx 0.683$

In comparison to results from b, take answers and multiply by $1/\sqrt{2\pi}$.

$n=2$: $1.693/\sqrt{2\pi} \approx 0.675$, $e_2 = 0.0017$
$n=3$: $1.712/\sqrt{2\pi} \approx 0.792$ $e_3 = 0.00003$
$n=4$: $1.7111/\sqrt{2\pi} \approx 0.683$ $e_4 = 0.00001$

same rounding done

3 a) Note that A and A^T have the same eigenvalues. As $A^T v = \lambda v \rightarrow \det(A^T v - \lambda I v)$ $\rightarrow \det((Av - \lambda I v)^T) = \det(Av - \lambda I v)$

A^T is row stochastic and let $v = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$Av = \lambda v$ $A\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

The sum of each row is 1, and get

$\begin{bmatrix} \sum^{n} a_{1i} \\ \vdots \\ \sum^{n} a_{in} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ So $\lambda = 1$ is an eigenvalue of A.

b) Using the circle theorem on A^T, all eigenvalues fall within the circle centered at a_{ii} w/ radius $\sum_{i \neq n} a_{in}$ as $a_{ii} + \sum_{i \neq n} a_{in} = 1$, all eigenvalues will be ≤ 1, and as $\lambda = 1$ is an eigenvalue, it is the largest one

3c) $X^* = Ax^*$. As $\lambda = 1$, then we can say X^* is an eigenvector of A with eigenvalue of 1 as $Ax^* = 1 \cdot x^*$ when it has converged, and as it is the vector associated with the largest eigenvalue, it is the dominant eigenvector.

3d and e are on next page.