M 125 HW 2

1 a) $K(x) = \frac{|xf'(x)|}{|f(x)|} = \frac{|x \cdot \frac{1}{x}|}{|\ln x|} = \boxed{\frac{1}{|\ln x|} \text{ so } f(x) \text{ is ill conditioned for } x \approx 1.}$

b) $K(x) = \frac{|xf'(x)|}{|f(x)|} = \frac{\left|\frac{-x}{(x-1)^2}\right|}{\left|\frac{x}{x-1}\right|} = \left|\frac{-1}{x-1}\right| = \frac{1}{|x-1|}$

$\boxed{K(x) = \frac{1}{|x-1|} \text{ so } f(x) \text{ is ill-conditioned for } x \text{ close to } 1.}$

c) $a_i \xrightarrow{g} x_J \qquad g(a_i) = x_J$

$$\frac{\left|\frac{g(a_i+\Delta)-g(a_i)}{a_i}\right|}{\left|\frac{\Delta}{a_i}\right|} = \left|\frac{g(a_i+\Delta)-g(a_i)}{\Delta} \cdot \frac{a_i}{g(a_i)}\right|$$

$$= \left|g'(a_i) \cdot \frac{a_i}{x_J}\right|$$

$$= \left|\frac{a_i g'(a_i)}{x_J}\right|$$

$\frac{da_i}{dx_J}$: $a_0 + a_1 x_J + a_2 x_J^2 + \dots + a_n x_J^n = 0$

$a_1 + 2a_2 x_J + \dots + i a_i x_J^{i-1} + \frac{\partial a_i}{\partial x_J} x_J^i + \dots + a_n x_J^n$

$$\frac{\partial a_i}{\partial x_J} = \frac{-a_1 - 2a_2 - \dots - n a_n x_J^{n-1}}{x_J^i} = \frac{-p'(x_J)}{x_J^i}$$

Want reciprocal so $\frac{x_J^i}{-p'(x_J)}$

$$K(a_i) = \left|\frac{x_J^i}{-p'(x_J)} \cdot \frac{a_i}{x_J}\right| = \left|\frac{x_J^{i-1} a_i}{p'(x_J)}\right|$$

So condition number is $\left|\frac{x_J^{i-1} a_i}{p'(x_J)}\right|$ and so $P(x)$ is ill conditioned for large a_i or large x_J^{i-1}, meaning as i increases $p(x)$ is more ill conditioned.

1d) Check back for write up of answers.

relative error is $\dfrac{\dfrac{|x-r|}{|r|}}{\left|\dfrac{2^{-23}}{210}\right|}$

2 $\tilde{f}(x) = \dfrac{|f(\tilde{x})-f(x)|}{|f(x)|} \cdot K(x)$

rel. error of $\tilde{x}$ is $\dfrac{|\tilde{x}-x|}{|x|} < \varepsilon_{machine}$

$$\frac{\dfrac{|f(\tilde{x})-f(x)|}{|f(x)|}}{\dfrac{|\tilde{x}-x|}{|x|}} \leftarrow K(x) \quad \cdot \frac{|\tilde{x}-x|}{|x|} < \varepsilon$$

$\dfrac{|\tilde{x}-x|}{|x|} < \varepsilon_m$ $\quad K(x)\cdot \varepsilon_m < \varepsilon$

So this is bounded above by $K(x)\varepsilon$

and $\left|\dfrac{\tilde{f}(x)-f(x)}{|f(x)|}\right| = O(K(x)\varepsilon)$

3 Since $f'(x) > 0\ \forall x$, and some r s.t. $f(r)=0$ exists, then by definition, r is unique as f is increasing.

For convergence $e_{i+1} = e_i^2 \dfrac{f''(c_i)}{2f'(x_i)}$ Since $f''(x) > 0\ \forall x$ and $f'(x) > 0\ \forall x$

$e_i^2 \dfrac{f''(c_i)}{2f'(x_i)} > 0$ so $e_{i+1} = x_{i+1} - r > 0$

$x_{i+1} > r$

So $\forall n > 0$, $x_n > r$ and $f(x_n) > f(r) = 0$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ $f(x_n) > 0\ \forall x \geq 1$, so $\dfrac{f(x_n)}{f'(x_n)} > 1$

So x_{n+1} is decreasing each iteration, but bounded below by $f(r)=0$

So by monotonic convergence theorem, $x_{n+1} \to r$.

4a) Let $g(a) = \frac{1}{a} - x$ $g'(a) = \frac{-1}{a^2}$

$$a_{n+1} = a_n - \frac{g(a)}{g'(a)}, \quad a_{n+1} = a_n - \left(\frac{\frac{1}{a_n} - x}{-\frac{1}{a_n^2}}\right)$$

$$a_{n+1} = a_n + a_n^2\left(\frac{1}{a_n} - x\right)$$

$$a_{n+1} = a_n + a_n - a_n^2 x$$

$$a_{n+1} = 2a_n - a_n^2 x = \boxed{a_n(2 - a_n x) = a_{n+1}}$$

b) $\varepsilon_{k+1} = -\varepsilon_k^2$ $\varepsilon_k = a x_k - 1$ where a is value converge to

$$(a x_{k+1} - 1) = -(a x_{k+1} - 1)^2$$

$$a x_{k+1} - 1 = -(a^2 x_{k+1}^2 - 2a x_k + 1)$$

$$\cancel{a} x_{k+1} \cancel{-1} = -\cancel{a}^2 x_{k+1}^2 + 2\cancel{a} x_k \cancel{-1}$$

$$x_{k+1} = -a x_{k+1}^2 + 2x_k$$

$x_{k+1} = x_k(2 - a x_k)$ which is equivalent to the Newton's method scheme derived above

c) $\varepsilon_1 = -\varepsilon_0^2$
$\varepsilon_2 = -\varepsilon_0^4$

$\varepsilon_k = -\varepsilon_0^{2k}$

Want $|\varepsilon_k| \leq 2^{-d}$

$$|-\varepsilon_0^{2k}| < 2^{-d}$$

$$\log_2 \varepsilon_0^{2k} < 2^{-d}$$

$$2k \log_2 \varepsilon_0 < -d$$

$$k < \frac{-d}{2k \log_2 \varepsilon_0}$$

It takes $\frac{-d}{2\log_2 \varepsilon_0}$ iterations to get within d binary decimals

Note this is positive as $|\varepsilon_0| < 1$ so $\log_2 \varepsilon_0 < 0$

4d) No, $f''(x) = -\frac{1}{x^2}$, $f''(x) < 0$ for $x < 0$, so as shown in q3, we cannot guarantee convergence.

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5) Check back for results and code for all parts.

6 a) $f'(x) = \frac{2}{3}(2x-1)^{-2/3}$, $f'(1) = \frac{2}{3} < 1$, so $f(x)$ is locally convergent at $r=1$

b) $f'(x) = \frac{3x^2}{2}$, $f'(1) = 3/2 > 1$, so $f(x)$ is not locally convergent at $x=1$.

c) $f'(x) = \cos x + 1$ $f'(0) = 2 > 1$, so $f(x)$ isn't locally convergent to the fixed point at $x=0$.

7 a) $x = \frac{4}{5}x + \frac{1}{x}$, $\rightarrow \frac{1}{5}x = \frac{1}{x}$, $\frac{x^2}{5} = 1$, $x = \sqrt{5}$ so this converges to $\sqrt{5}$

$g(x) = \frac{4}{5}x + \frac{1}{x}$, $g'(x) = \frac{4}{5} - \frac{1}{x^2}$, $g'(\sqrt{5}) = \frac{3}{5}$

$\boxed{g(x) \text{ converges to } \sqrt{5} \text{ and } |g'(\sqrt{5})| = \frac{3}{5} < 1}$

b) $x = \frac{x}{2} + \frac{5}{2x}$, $\frac{x}{2} = \frac{5}{2x}$, $x^2 = 5$, $x = \sqrt{5}$. So converges to $\sqrt{5}$.

$g(x) = \frac{x}{2} + \frac{5}{2x}$, $g'(x) = \frac{1}{2} - \frac{5}{2x^2}$, $g'(\sqrt{5}) = 0 < 1$

$\boxed{g(x) \text{ converges to } \sqrt{5} \text{ and } |g'(\sqrt{5})| = 0 < 1.}$

c) $x = \frac{x+5}{x+1}$, $x^2 + x = x + 5$, $x^2 = 5$, $x = \sqrt{5}$, so converges to $\sqrt{5}$

$g'(x) = \frac{x+1-(x+5)}{(x+1)^2} = \frac{-4}{(x+1)^2}$ $g'(\sqrt{5}) = \left|\frac{-4}{(\sqrt{5}+1)^2}\right| < 1$

So, $g(x)$ converges to $\sqrt{5}$ and $|g'(\sqrt{5})| = .38 < 1$.