1. Math 65, Review Sheet for Exam 2 Fall 2021.

This review sheet is provided as a help tool when reviewing. It is a complement to and not a substitute for reviewing notes, homework problems and quizzes. Try to solve the problems without looking at the answers and ask for help only after you have devoted some time to think about it yourself. Please keep in mind that in the test, your explanations will be given at least as much credit as a numerical answer.

Question 1. The bookstore has 3 copies of the Algebra textbook, 10 copies of the Calculus book and 5 copies of the Discrete Math textbook.

(a) In how many ways can these books be placed in a shelf if they are all used copies, very different from each other?
(b) In how many ways can these books be placed in a shelf if they are all used copies, very different from each other and you want to keep the books for the same course together?
(c) In how many ways can these books be placed in a shelf if they are all new copies and so the books for each given course are indistinguishable ?
(d) In how many ways can these books be placed in a shelf if they are all new copies and so the books for each given course are indistinguishable and you want to keep the books for the same course together?

Question 2. How many (meaningless) words can you form with the letters of "committee on committees" (form just a single word, no spaces).

Question 3. Write the coefficient of x^9 in $(2x-3)^{20}$.

Question 4. Give a combinatorial proof of the following summation identities

(a) $\sum_{i=1}^{n}(i-1) = \binom{n}{2}$. Hint: when choosing two elements in $\{1, 2.3, ..., n\}$, distinguish according to the value of the largest of the two.
(b) $\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$. Hint: when choosing three elements in $\{1, 2.3, ..., n\}$, distinguish according to the value of the middle one..

Question 5. A store carries 5 different types of cookie mix (sugar, chocolate, chocolate chunk, peanut, and cheese swirl). A couple sends their children to the store. What is the number of packages the parents should ask their kids to bring back if they want to make sure that at least 3 packages are of the same kind (and therefore can be mixed together)?

Question 6. Let n be a positive integer and $x_0, \dots, x_n$ be integers. Show that there exists at least one pair x_i, x_j such that $x_i - x_j$ is divisible by n. Hint: Think about the possible remainders of x_i for the division by n.

Question 7. At 5 p.m., there are only 4 kinds of donuts left in the shop: jelly, coconut, plain and chocolate.

(a) In how many ways can you choose a dozen donuts so that at least 4 are chocolate AND at most 3 are coconut?
(b) In how many ways can you choose 12 donuts so that at least 4 are chocolate, at most 3 are coconut AND at most one is jelly?

Question 8. In how many ways can you place 50 books on 3 shelves if the first shelf can hold at most 10

(a) if all 50 books are identical.

(b) If all 50 books are different and you care about the order of the books on the shelves.

Question 9. Consider the following relation R on the set $A = \{a, b, c, d\}$.

$$R = \{(a, a), (a, b), (a, c), (c, a), (c, b), (c, c), (b, c), (d, d)\}.$$

Determine whether the relation is reflexive, symmetric, antisymmetric, and/or transitive. Give short explanations with your answers.

Question 10. Let $A = \{1, 2, 3\}$. Give the partition $A = \{1\} \cup \{2, 3\}$, determine the equivalence relation associated to this partition .

Question 11. Prove or disprove each of the following statements

(a) Consider the assignment $f : \mathbb{Q} \to \mathbb{Z}$ given by $f(\frac{a}{b}) = a+b$. Is this a well defined function?
(b) Consider the assignment $g : \mathbb{Q} \to \mathbb{Z}$ given by $g(\frac{a}{b}) = \frac{3a}{5b}$. Is this a well defined function?
(c) Consider the assignment $g : \mathbb{Q} \to \mathbb{Q}$ given by $g(\frac{a}{b}) = \frac{3a}{5b}$. Is this a well defined function?

Question 12. Prove or disprove each of the following statements

(a) If a poset has a maximal element, then it has to be unique.
(b) If a poset has only one maximal element, then it is a maximum.
(c) It is possible for a poset to have an element that is both maximal and minimal but it is not maximum or minimum.
(d) If x and y are incomparable then neither is a maximum.
(e) If x and y are both maximal and different, then they are incomparable.

Question 13. A bit string is a row of digits each being 0 or 1.

(a) When choosing a bit string of length 6 at random, what is the probability that it contains exactly three zeroes?
(b) When choosing a bit string of length 6 at random, what is the probability that it contains exactly three zeroes and either contain the string 000 or 111(or both)?

Question 14. Three dice are thrown simultaneously.

(a) Compute the probability that at least one of the dice shows a 3.
(b) Compute the probability that the sum of the numbers that show on the three dice is 5.
(c) Are the events in (a) and (b) independent?

Question 15. Three identical coins are thrown simultaneously. Consider the random variable that assigns to each possible output the number of heads H.

(a) Write the outputs of the random variable and the probability of each of them.
(b) Compute the expected value.
(c) Compute the variance.

2. Answers

Question 16. The bookstore has 3 copies of the Algebra textbook, 10 copies of the Calculus book and 5 copies of the Discrete Math textbook.

(a) In how many ways can these books be placed in a shelf if they are all used copies, very different from each other?
(b) In how many ways can these books be placed in a shelf if they are all used copies, very different from each other and you want to keep the books for the same course together?
(c) In how many ways can these books be placed in a shelf if they are all new copies and so the books for each given course are indistinguishable ?
(d) In how many ways can these books be placed in a shelf if they are all new copies and so the books for each given course are indistinguishable and you want to keep the books for the same course together?

Answer 1. (a) There are 18 books in all, hence 18! ways of placing them on the shelf.
(b) You need to order separately the books on each subject and then decide on the order of the three subjects among themselves. Hence, 3!10!5!3! ways of ordering.
(c) We can identify any of the orderings in a) that just switches books of a given subject. Hence $\frac{18!}{3!10!5!}$ choices.
(d) The only thing that matters is how to order the three subjects. There are 3! such choices.

Question 17. How many (meaningless) words can you form with the letters of "committee on committees" (form just a single word, no spaces).

Answer 2. How many (meaningless) words can you form with the letters of "committee on committees".

There is a total of 21 letters, n and s appear once, c and i appear twice, o appears three times, m, t and e appear four times. There are 21! ways of ordering all the letters but reordering identical letters does not change the result. Hence, the total number of options is $\frac{21!}{2\times2\times3!\times4!\times4!\times4!}$

Question 18. Write the coefficient of x^9 in $(2x-3)^{20}$.

Answer 3. By the binomial Theorem,

$$(2x-3)^{20} = \binom{20}{0}2^{20}x^{20} + \binom{20}{1}2^{19}x^{19}(-3) + \cdots + \binom{20}{11}2^9x^9(-3)^{11} + \cdots + \binom{20}{20}(-3)^{20}$$

The coefficient of x^9 in $(2x-3)^{20}$ is then

$$\binom{20}{9}2^9(-3)^{11}$$

Question 19. Give a combinatorial proof of the following summation identities

(a) $\sum_{i=1}^{n}(i-1) = \binom{n}{2}$. Hint: when choosing two elements in $\{1, 2.3, ..., n\}$, distinguish according to the value of the largest of the two.
(b) $\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$. Hint: when choosing three elements in $\{1, 2.3, ..., n\}$, distinguish according to the value of the middle one..

Answer 4. (a) The right hand side side tells us the number of ways to pick two elements from a set of size n. Suppose that set is $1, 2.3, ..., n$. We argue that the left hand side counts the same thing. We count each pair now by starting with what the largest element of the pair is. If the largest element is 1, there are zero possibilities for the second element, and so this case can't happen. If the largest element of a pair is 2, then there is one possibility for the second element of the pair, namely 1. If the largest element of the pair is 3, then there are two possibilities, 2 and 1. Continue in this fashion until we get to the final case, where the larger element if n, and there are $n-1$ possibilities for the second element. Summing up over all these cases we see there are

$$0+1+2+...+(n-1)$$

possibilities, and this is the left hand side of the identity.

(b) The right hand side tells us the number of ways to pick 3 element subsets from a set of size n. Suppose that set is $1,2,3,...,n$. As above, let's argue the left hand side counts the same thing. We want to count triples (x,y,z) where x,y,z are distinct integers in $1,2,3,...,n$. Order x,y,z from least to greatest, so y is always the middle number. The smallest y may be is 2, and the largest possible $n-1$. If $y=2$ there is one possibility for x, namely 1, and $n-2$ possibilities for z. The next possibility for y is 3. Then there are $3-1=2$ possibilities for x and $n-3$ possibilities for z. We can continue in this way. For each $y=i$ where $1<i<n$, there are $i-1$ possibilities for x and $n-i$ possibilities for z. Summing over each case gives us

$$(2-1)(n-2)+(3-1)(n-3)+...+(n-1-1)(n-(n-1))$$

possibilities. But this is just the left hand side of the summation identity.

Question 20. A store carries 5 different types of cookie mix (sugar, chocolate, chocolate chunk, peanut, and cheese swirl). A couple sends their children to the store. What is the number of packages the parents should ask their kids to bring back if they want to make sure that at least 3 packages are of the same kind (and therefore can be mixed together)?

Answer 5. We use the pigeonhole principle. The worst case scenario is that the kids bring two packages of each kind. As $5 \cdot 2 + 1 = 11$, if they buy 11 packages (yum!) at least 3 will be of the same kind.

Question 21. Let n be a positive integer and $x_0, \ldots, x_n$ be integers. Show that there exists at least one pair x_i, x_j such that $x_i - x_j$ is divisible by n. Hint: Think about the possible remainders of x_i for the division by n.

Answer 6. The remainder for the division of an integer by n is an integer between 0 and $n-1$. If we have the $n+1$ integers $x_0, \ldots, x_n$, there are at least two with the same remainder. Then the difference has remainder 0 and is therefore divisible by n.

Question 22. At 5 p.m., there are only 4 kinds of donuts left in the shop: jelly, coconut, plain and chocolate.

(a) In how many ways can you choose a dozen donuts so that at least 4 are chocolate AND at most 3 are coconut?
(b) In how many ways can you choose 12 donuts so that at least 4 are chocolate, at most 3 are coconut AND at most one is jelly?

Answer 7. (1) We need to choose 12 donuts among 4 different flavors. It is like getting a box with room for 12 donuts and three separators to be placed between the flavors.

We need to choose the spots for the separators. Therefore, there are $\left(\binom{4}{12}\right) = \binom{12+3}{3}$ options. If at least 4 are chocolate, we are only choosing the remaining 8. The number of ways of choosing a dozen donuts so that at least 4 are chocolate is

$$\left(\binom{4}{8}\right) = \binom{8+3}{3}$$

If on top of the 4 chocolate, we want 4 at least coconut, we would be choosing only 4 of the donuts . There are $\left(\binom{4}{4}\right) = \binom{4+3}{3}$ ways of doing this. We want to exclude the last possibility. Hence there are $\binom{11}{3} - \binom{7}{3}$ ways of picking 12 donuts with at least 4 chocolate and at most 3 coconut.

(2) We first count in how many ways we can pick 12 donuts so that at least 4 are chocolate and at least 2 are jelly. This amounts to choosing 6 donuts, so the number of options is $\left(\binom{4}{6}\right) = \binom{6+3}{3}$.

We now count in how many ways we can pick 12 donuts so that at least 4 are chocolate, at least 4 are coconut and at least 2 are jelly. This amounts to choosing 2 donuts, so the number of options is $\left(\binom{4}{2}\right) = \binom{2+3}{3}$.

Let us call U the choices of 12 donuts with at least 4 chocolate. We computed $|U| = \binom{11}{3}$.

Let us call A the choices of 12 donuts with at least 4 chocolate and at least 4 coconut. We computed $|A| = \binom{7}{3}$.

Let us call B the choices of 12 donuts with at least 4 chocolate and at least 2 jelly. We computed $|B| = \binom{9}{3}$.

Then, $A \cap B$ are the choices of 12 donuts with at least 4 chocolate and both at least 4 coconut AND at least 2 jelly. We computed $|A \cap B| = \binom{5}{3}$.

Note now that $A \cup B$ are the choices of 12 donuts with at least 4 chocolate and either at least 4 coconut or at least 2 jelly. Then

$$|A \cup B| = |A| + |B| - |A \cap B = \binom{7}{3} + \binom{9}{3} - \binom{5}{3}$$

The choices we are after are the complement of $A \cup B$ in U. The cardinality of this set is

$$|U| - |A \cup B| = \binom{11}{3} - \binom{7}{3} - \binom{9}{3} + \binom{5}{3} = 56$$

Question 23. In how many ways can you place 50 books on 3 shelves if the first shelf can hold at most 10

(a) if all 50 books are identical.
(b) If all 50 books are different and you care about the order of the books on the shelves.

Answer 8. (a) If all books are different, the only thing that matters is how many go on each shelf. We need to find the number x_i of books on each shelf

$$50 = x_1 + x_2 + x_3, \quad x_1 \leq 10$$

We compute the number of ways in which we can write $50 = x_1 + x_2 + x_3$ with no restrictions on x_1 and we subtract the number of ways in which we can write $50 = x_1 + x_2 + x_3$ with $x_1 \geq 11$.

Deciding how many books go on each shelf is like labeling each book with shelf 1, 2 or 3. The number of ways to do this equals the number of ways of choosing 50 objects of 3 different kinds. This number is $\left(\binom{3}{50}\right) = \binom{52}{2}$. The number of selections that would place 11 or more books on the top shelf is the same as the number of ways of distributing $50 - 11 = 39$ books among three shelves. This number is $\left(\binom{3}{39}\right) = \binom{41}{2}$. The number we are looking for is

$$\binom{52}{2} - \binom{41}{2}$$

(b) If the books are all different, we can start by ordering them in all possible ways, choosing how many go on each shelf and then placing them in the chosen order. Therefore, the number of choices is

$$50![\binom{52}{2} - \binom{41}{2}]$$

Question 24. Consider the following relation R on the set $A = \{a, b, c, d\}$.

$$R = \{(a, a), (a, b), (a, c), (c, a), (c, b), (c, c), (b, c), (d, d)\}.$$

Determine whether the relation is reflexive, symmetric, antisymmetric, and/or transitive. Give short explanations with your answers.

Answer 9. . Not reflexive (not (b, b)). Not symmetric (a, b) but not (b, a). Not transitive (b, c) and (c, a) but not (b, a). Not antisymmetric $c \neq b$ but (c, b) and (b, c).

Question 25. Let $A = \{1, 2, 3\}$. Give the partition $A = \{1\} \cup \{2, 3\}$, determine the equivalence relation associated to this partition .

Answer 10. Given the partition $A = \{1\} \cup \{2, 3\}$, the corresponding equivalence relation relates the elements in the same set of the partition. This gives

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\} \subset A \times A$$

Question 26. Prove or disprove each of the following statements

(a) Consider the assignment $f : \mathbb{Q} \to \mathbb{Z}$ given by $f(\frac{a}{b}) = a+b$. Is this a well defined function?
(b) Consider the assignment $g : \mathbb{Q} \to \mathbb{Z}$ given by $g(\frac{a}{b}) = \frac{3a}{5b}$. Is this a well defined function?
(c) Consider the assignment $g : \mathbb{Q} \to \mathbb{Q}$ given by $g(\frac{a}{b}) = \frac{3a}{5b}$. Is this a well defined function?

Answer 11. (a) This is not a well defined function as $\frac{1}{2} = \frac{2}{4}$ while $f(\frac{1}{2}) = 3 \neq 6 = f(\frac{2}{4})$.
(b) This is not a well defined function as $\frac{3a}{5b}$ is not an integer in general.
(c) This is a well defined function: if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, then $a_1 b_2 = a_2 b_1$. Then multiplying this identity with 15 and regrouping, we have

$$(3a_1)(5b_2) = (3a_2)(5b_1)$$

This equation is equivalent to the equality of the rational numbers

$$\frac{3a_1}{5b_1} = \frac{3a_2}{5b_2}$$

Therefore, the expression for the image of a rational number is independent of the representation of the rational number and g is a well defined function.

Question 27. Prove or disprove each of the following statements

(a) If a poset has a maximal element, then it has to be unique.
(b) If a poset has only one maximal element, then it is a maximum.
(c) It is possible for a poset to have an element that is both maximal and minimal but it is not maximum or minimum.
(d) If x and y are incomparable then neither is a maximum.
(e) If x and y are both maximal and different, then they are incomparable.

Answer 12. (a) If a poset has a maximal element, then it has to be unique. This is false. Take for instance the poset with elements $\{2, 3\}$ ordered by divisibility. As both 2, 3 are only related to themselves, they are both maximal.
(b) If a poset has only one maximal element, then it is a maximum. This is false: take the set $A = \mathbb{R} \cup \alpha$ with the order being that α is only related to itself and the elements in $\mathbb{R}$ are ordered by the regular $\leq$ relation. This set has only one maximal element, namely α but α is not a maximum as it is not larger or equal than the elements in $\mathbb{R}$.
(c) It is possible for a poset to have an element that is both maximal and minimal but it is not maximum or minimum. This is true, the example above works.
(d) If x and y are incomparable then neither is a maximum. This is true. By definition, a maximum is comparable to every other element.
(e) If x and y are both maximal and different, then they are incomparable. This is true. If they were comparable, say $x \preceq y$, by the condition that x is maximal, this would give us $x = y$.

Question 28. A bit string is a row of digits each being 0 or 1.
(a) When choosing a bit string of length 6 at random, what is the probability that it contains exactly three zeroes?
(b) When choosing a bit string of length 6 at random, what is the probability that it contains exactly three zeroes and either contain the string 000 or 111(or both)?

Answer 13. (a) There are 2^6 bit-strings of length 6 as for each digit you have two choices of a 0 or 1 in the given spot. There are $\binom{6}{3}$ bit strings that conntain exactly three zeroes, you only need to choose the spot where these zeroes would go. Therefore

$$P(\text{ exactly three zeroes }) = \frac{\binom{6}{3}}{2^6} = \frac{5}{16}$$

(b) One can place the 000 starting at the beginning or at the second spot, at the third or at at the fourth spot. Hence there are four such choices. Similarly, there are four such strings that contain 111. Two of these contain both 000 and 111. Hence, in total there are $4 + 4 - 2 = 6$ choices. Then, the probability

$$P(\text{ exactly three zeroes , 3 identical consecutive digits}) = \frac{6}{2^6} = \frac{3}{32}$$

Question 29. Three dice are thrown simultaneously.
(a) Compute the probability that at least one of the dice shows a 3.
(b) Compute the probability that the sum of the numbers that show on the three dice is 5.
(c) Are the events in (a) and (b) independent?

Answer 14. (a) It is easier to compute the probability that no three appears. This means that we have 5 options for the number that appears in each die. Hence, there are 5^3

options altogether. Hence, the probability that no three appears is $\frac{5^3}{6^3}$. The probability that at least one three appears is $1 - \frac{5^3}{6^3} = \frac{6^3-5^3}{6^3}$

(b) The sum of the numbers that show on the three dies is 5 if we have 1,1,3 or 1,2,2 in some order. Hence, there are 6 options and the probability is $\frac{6}{6^3} = \frac{1}{6^2}$.

(c) If both (a) and (b) occur at the same time, the numbers on the dies are $1, 1, 3$. There are three options and hence, the probability of the intersection is $\frac{3}{6^3} = \frac{1}{2.6^2}$

The two events are independent if and only if the probability of the intersection is the product of the probabilities. This is false in our case as

$$\frac{6^3 - 5^3}{6^3}\frac{1}{6^2} = \frac{1}{2.6^2}$$

is equivalent to $2(6^3 - 5^3) = 6^3$ which is equivalent to $6^3 = 2 \times 5^3$ and is clearly false.

Question 30. Three identical coins are thrown simultaneously. Consider the random variable that assigns to each possible output the number of heads H.

(a) Write the outputs of the random variable and the probability of each of them.
(b) Compute the expected value.
(c) Compute the variance.

Answer 15. (a) As we are assuming that the three coins are identical, we will consider only the number of heads, rather than the order in which they appear. So, the possible events with their probability are

$$P(HHH) = \frac{1}{8}, P(HHT) = \frac{3}{8}, P(HTT) = \frac{3}{8}, P(TTT) = \frac{1}{8}$$

Also, from the definition of the random variable,

$$X(HHH) = 3, X(HHT) = 2, P(HTT) = 1, P(TTT) = 0$$

(b) From the data above ,we can compute the expected value as

$$E(X) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} + \frac{0}{8} = \frac{12}{8} = \frac{3}{2}$$

(c) We compute the variance as

$$V(X) = \frac{1}{8}(3 - \frac{3}{2})^2 + \frac{3}{8}(2 - \frac{3}{2})^2 + \frac{3}{8}(1 - \frac{3}{2})^2 + \frac{1}{8}(0 - \frac{3}{2})^2 = \frac{6}{8} = \frac{3}{4}$$