Tufts University

Math 70 | **Department of Mathematics** | **March 5-7, 2021**

Linear Algebra | **Exam 1**

Instructions: This is an open-book exam: you may refer to our textbook, or any other linear algebra book or internet website. If you use another book, you must give the reference. If you use the internet, you must provide the URL of the website.

For theorems you use from the book, *you must give the theorem number or page number.* If you quote a theorem not in our book or class notes, you must prove it.

Your answer must be *in your own words*; a straight copy of some source or a close copy (some words changed) is not acceptable.

And of course, *during the exam, you may not consult anyone other than the instructors of the course, and you may not ask for help online (e.g., message boards, forums).*

If you have a question about something on the exam, email the four instructors. Emailing all of us will help insure a timely response. Here are the email addresses:

Instructors:

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The test will be available on Gradescope starting 12:01 a.m. Eastern Time on Friday March 5th. You will have 80 minutes **from the time you download the test** *to finish it. You will have an extra 10 minutes to upload it to Gradescope. Since it can take time to upload, do not wait until this 10 minutes is almost up.*

You must finish uploading your answers by 11:00p.m. on Sunday March 7th p.m. *When you upload your answers to Gradescope, please scan this signed signature page first and then scan all your answers into one PDF file. Please number the problems clearly and in order.*

Upload instructions are in the Gradescope module on your Math 70 Canvas site

Please sign the following pledge: I pledge that I have neither given nor received assistance on this exam and have used only the reference sources cited above.

Signature___________________

The test starts on the next page.

Problems start here.

1. (4 points) Indicate whether each statement is true or false. No justification is necessary. You should put your answers in a grid like the one given below.

Question:	1a	1b	1c	1d
True/False				

   (a) If $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_k \end{bmatrix}$ and if the matrix equation $A\mathbf{x} = \mathbf{v}$ is inconsistent, then $\mathbf{v}$ does not belong to the span of $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$. True

   (b) If a set of vectors in $\mathbb{R}^n$ is linearly dependent, then it must contain at least n vectors. False

   (c) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$ is one-to-one (injective). False

   (d) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one (injective) linear transformation and A is the standard matrix for T. Then the columns of A span $\mathbb{R}^n$? True

2. (8 points) Consider the following system

$$\begin{cases} x_1 + 2x_2 - x_3 = -3 \\ -x_1 - 4x_2 + x_3 = 5 \\ 2x_1 - x_2 - x_3 = 2 \end{cases}$$

   (a) Write this linear system of equation as a vector equation. Do not solve the equation.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

   (b) Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$. Do not solve.

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -4 & 1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

   (c) Find the vector $A \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, A is the matrix from part (b).

$$A \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -4 & 1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

(d) Solve the homogeneous equation $Ax = 0$. Are the columns of A linearly dependent or linearly independent?

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -4 & 1 \\ 2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The homogeneous equation $Ax = 0$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.It follows that the columns of A are linearly independent.

(e) Find the solution set of the original system of linear equations in parametric form. By part (c) $\mathbf{x_p} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a particular solution to the linear system and because the columns of A are linearly independent it is the ONLY solution.

3. (4 points) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, let $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and let $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Under what conditions is an arbitrary vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

We give two solutions. First, notice that $\mathbf{v_2} = \mathbf{v_1} + \mathbf{v_3}$, and that $\mathbf{v_1}$ and $\mathbf{v_3}$ are linearly independent. To show that you can see that there exists no scalar t such that $\mathbf{v_3} = \mathbf{tv_1}$. Thus the vectors $\mathbf{b}$ must be linear combination of $\mathbf{v_1}$ and $\mathbf{v_3}$, i.e., $\mathbf{b} = \mathbf{tv_1} + \mathbf{sv_3}$ where t, s are scalars.

$$\begin{bmatrix} 1 & 2 & 1 & b_1 \\ 1 & 0 & -1 & b_2 \\ -2 & 1 & 3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & -2 & -2 & -b_1 + b_2 \\ 0 & 5 & 5 & 2b_1 + b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & \frac{b_1 - b_2}{2} \\ 0 & 1 & 1 & \frac{2b_1 + b_3}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & b_2 \\ 0 & 1 & 1 & \frac{b_1 - b_2}{2} \\ 0 & 0 & 0 & \frac{-b_1 + 5b_2 + 2b_3}{10} \end{bmatrix}$$

We need to have $-b_1 + 5b_2 + 2b_3 = 0$.

4. (4 points) Suppose that $\{\mathbf{v}, \mathbf{u}\}$ are linearly independent. Show that $\{\mathbf{v}, \mathbf{u}, 2\mathbf{v} + 3\mathbf{u}\}$ are linearly dependent, but that any pair of vectors from $\{\mathbf{v}, \mathbf{u}, 2\mathbf{v} + 3\mathbf{u}\}$ are linearly independent.

$$-2\mathbf{v} - \mathbf{3u} + (\mathbf{2v} + \mathbf{3u}) = \mathbf{0}$$

so we have a nontrivial linear combination of the three vectors giving $\mathbf{0}$ so the vectors $\{\mathbf{v}, \mathbf{u}, 2\mathbf{v} + 3\mathbf{u}\}$ are linearly dependent. The pairs that we can form from this set are $\{\mathbf{v}, \mathbf{u}\}, \{\mathbf{v}, 2\mathbf{v} + 3\mathbf{u}\}, \{\mathbf{u}, 2\mathbf{v} + 3\mathbf{u}\}$.

$\{\mathbf{v}, \mathbf{u}\}$ are linearly independent, was given.

For the next set, suppose that $x_1\mathbf{v} + x_2(2\mathbf{v} + 3\mathbf{u}) = \mathbf{0}$ that is $(x_1 + 2x_2)\mathbf{v} + 3x_2\mathbf{v} = \mathbf{0}$. But because $\{\mathbf{v}, \mathbf{u}\}$ are linearly independent, we conclude that $x_1 + 2x_2 = 3x_2 = 0$. The only solution is $x_1 = x_2 = 0$. Thus, $\{\mathbf{v}, 2\mathbf{v} + 3\mathbf{u}\}$ are linearly independent.

The case of $\{\mathbf{u}, 2\mathbf{v} + 3\mathbf{u}\}$ is done similarly.

Note: If you don't use the fact that $\{\mathbf{v}, \mathbf{u}\}$ is a linearly independent set when proving that $\{\mathbf{v}, 2\mathbf{v} + 3\mathbf{u}\}$ and $\{\mathbf{v}, 2\mathbf{v} + 3\mathbf{u}\}$ are linearly independent then your explanation is incomplete.

More specifically if you claimed that $2\mathbf{v}+3\mathbf{u}$ is not a multiple of $\mathbf{v}$ (or $\mathbf{u}$) without saying $\{\mathbf{v},\mathbf{u}\}$ is a linearly independent set then you didn't say enough. This is because if $\mathbf{u}=k\mathbf{v}$ for some scalar k then $2\mathbf{v}+3\mathbf{u}$ is a multiple of $\mathbf{v}$ (or $\mathbf{u}$).

5. (8 points) Let $\mathbf{u}$ and $\mathbf{v}$ be fixed vectors in $\mathbb{R}^n$ and $T:\mathbb{R}^2\to\mathbb{R}^n$ by

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=x_1\mathbf{v}+x_2\mathbf{u}$$

(a) Show that $T(\mathbf{x})$ is a linear transformation.

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=x_1\mathbf{v}+x_2\mathbf{u}=\begin{bmatrix}\mathbf{v} & \mathbf{u}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

It follows that T is a matrix transformation thus it is linear.

Another solution:

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}\right)=(x_1+y_1)\mathbf{v}+(x_2+y_2)\mathbf{u}=(x_1\mathbf{v}+x_2\mathbf{u})+(y_1\mathbf{v}+y_2\mathbf{u})$$

Hence, $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)+T\left(\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)$

Similarly,

$$T\left(c\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=T\left(\begin{bmatrix}cx_1\\cx_2\end{bmatrix}\right)=cx_1\mathbf{v}+cx_2\mathbf{u}=c(x_1\mathbf{v}+x_2\mathbf{u})=cT\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$$

(b) Show that $\mathbf{v}$ and $\mathbf{u}$ are linearly independent if and only if $T(\mathbf{x})$ is one to one.

The standard matrix of T is the $n\times 2$ matrix $\begin{bmatrix}\mathbf{v} & \mathbf{u}\end{bmatrix}$. T is one-to-one if and only if each column of this matrix is a pivot column (in other words, there is no free variable). But this is equivalent to saying that the columns of the standard matrix are linearly independent. Hence $T(\mathbf{x})$ is one to one if and only if $\mathbf{v}$ and $\mathbf{u}$ are linearly independent.

6. (6 points) The following transformation is linear (you don't need to check this).

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_2\\x_1\\x_1+x_2\\x_3\end{bmatrix}$$

(a) Find the standard matrix for $T(\mathbf{x})$.

The standard matrix of this linear transformation is

$$\begin{bmatrix}0&1&0\\1&0&0\\1&1&0\\0&0&1\end{bmatrix}$$

(b) Is $T(x)$ one-to-one (injective)? Justify your answer.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column is a pivot column, so T is one-to-one.

(c) Is $T(x)$ onto (surjective)? Justify your answer.

Not every row contain a pivot, so T is not surjective.

7. (6 points) Suppose that a company has three plants, called Plants $1, 2,$ and 3 that produce milk y_1 and yogurt y_2. For every hour of operation,
- Plant 1 produces 20 units of milk and 15 units of yogurt.
- Plant 2 produces 30 units of milk and 5 units of yogurt.
- Plant 3 produces 0 units of milk and 40 units of yogurt.

(a) Suppose x_1, x_2, and x_3 record the amount of time that the three plants are operated. Find expressions for the number of units of milk y_1 and yogurt y_2 produced.

$y_1 = 20x_1 + 30x_2$, $y_2 = 15x_1 + 5x_2 + 40x_3$

(b) If we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, find the matrix A that defines the matrix transformation $T(\mathbf{x}) = \mathbf{y}$.

$$T(\mathbf{x}) = \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 20x_1 + 30x_2 \\ 15x_1 + 5x_2 + 40x_3 \end{bmatrix} = \begin{bmatrix} 20 & 30 & 0 \\ 15 & 5 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) If $\mathbf{x} = \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}$ describes the amount of time that the plants are operated, how much milk and yogurt is produced?

$$T\left(\begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}\right) = \begin{bmatrix} 20 & 30 & 0 \\ 15 & 5 & 40 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 1200 \\ 950 \end{bmatrix}$$

Thus, there will be 1200 unit of milk, and 950 units of yogurt produced.