MATH 42 HOMEWORK 5

This homework is due at 11:59 p.m. (Eastern Time) on Wednesday, October 21. Scan the completed homework and upload it **as one pdf file** to Gradescope. The Canvas module "Written Assignments" has instructions for how to upload the assignment to Gradescope. This assignment covers §15.7 – 8, 16.1.

(1) Find the points (x, y) in the specified domains R where the function $f(x, y)$ is maximum and where $f(x, y)$ is minimum.
   (a) $f(x, y) = x^2 - y^2 + 1$; $R = \{(x, y) : x^2 + y^2 \leq 1\}$
   (b) $f(x, y) = x^2 + 2y^2 - 3xy$; $R = \{(x, y) : 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$
   (c) $f(x, y) = x^2 + y^2 - 2y + 1$; $R = \{(x, y) : x^2 + y^2 \leq 16\}$

(2) On the exam you were asked to find the dimensions x, y, z of a rectangular box with open top such that the volume of the box was 500in^3 and such the sum of the area of the 5 sides was minimized. Solve this problem with Lagrange multipliers. Do not reproduce the exam solution here, but compare this method to that one.

(3) Find the maximum and minimum values of the function $f(x, y, z) = x^2 + y^2 + z^2$ on the surface $x^2 + y^2 + z^2 - 4xy = 1$.

(4) This problem will explore the meaning of the Lagrange multiplier. Consider the ellipse $g(x, y) = (x^2/a^2) + (y^2/b^2) - k = 0$ where $k > 0$. When $k = 1$, we have the standard equation of the ellipse. For arbitrary, positive k, this term acts as a re-scaling of the semi-minor and semi-major axes by $\sqrt{k}$.
   (a) Find the maximum area of a rectangle that can be inscribed in the ellipse such that the edges of the rectangle are parallel to the coordinate axes.
      (i) Make a drawing of the problem. Indicate the values where the ellipse intersects the coordinate axes.
      (ii) Consider any inscribed rectangle (not necessarily of maximum area) such that its corners touch the ellipse. Denote the point in the positive quadrant where the corner of the inscribed rectangle and ellipse meet as (x, y), use the problem's symmetry to write an equation for the area of the rectangle, $f(x, y)$.
      (iii) Solve $\nabla F = \vec{0}$ where $F(x, y, \lambda) := f(x, y) - \lambda g(x, y)$ and the third component of ∇F is the partial derivative with respect to λ. Justify any algebra that involves dividing by yet-undetermined quantities by arguing that it would be nonsensical if such quantities were zero.
      (iv) Calculate the maximum area by using the points found when solving $\nabla F = \vec{0}$.
   (b) Let $H(k)$ denote your answer to the problem of finding the maximum area of a rectangle inscribed inside an ellipse of the equation

$$\frac{x^2}{(\sqrt{k}a)^2} + \frac{y^2}{(\sqrt{k}b)^2} = 1.$$

   Show that $\frac{\mathrm{d}H(k)}{\mathrm{d}k} = \lambda$.
   (c) Describe what the general relation $\frac{\mathrm{d}H(k)}{\mathrm{d}k} = \lambda$ demonstrates about the Lagrange multiplier, λ. Hint: The value k changes our constraint surface.

The Lagrange multiplier found during intermediate calculations is not *simply* a helpful tool; it has rich, problem-specific meanings in physics, engineering, and economics.

(5) Compute the following double integrals:

(a) $\iint_R x^3 + y^2 + 2xy\,dydx$ where R is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.

(b) $\iint_R e^{x+2y}\,dA$, where $R = \{(x,y) : 0 \le x \le \ln 2, 1 \le y \le \ln 3\}$

(c) $\iint_R (x^5 - y^5)^2 dA$, where R is the rectangle with vertices $(0,-1)$, $(0,1)$, $(1,1)$, and $(1,-1)$.

(d) $\int_0^1 \int_0^{\pi/2} x^5 \cos(x^3 y)\,dydx$