Tufts University
Department of Mathematics

Math 135 | **Homework 8**[1] | **Fall 2022**

Readings for Problem Set 8

§10.2: Convergence of sequences in $\mathbb{R}^n$
§10.3: Open and closed sets, interior, boundary, exterior, and closure (defined in #12, p. 289)

Problem Set 8
(Due Wednesday, November 9, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

Let $A \subset \mathbb{R}^n$. A set $S \subset A$ is **relatively open** in A if it is the intersection of A with an open set U in $\mathbb{R}^n$: $S = A \cap U$. A **relatively open ball** in A is

$$B_A(u,\varepsilon) = \{a \in A \mid d(a,u) < \varepsilon\} = A \cap B(u,\varepsilon).$$

A relatively open ball in A is relatively open in A. A set $S \subset A$ is **relatively closed** in A if it is the intersection of A with a closed set F in $\mathbb{R}^n$: $S = A \cap F$. For example, $[0,1)$ is not open in $\mathbb{R}$, but it is relatively open in $[0,2]$, because $[0,1) = [0,2] \cap (-1,1)$.

1. (10 points) (**Continuity of the scalar product in the first variable**) §10.2, p. 281: # 1.
   Let $\{\vec{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to the point $\vec{u}$. Prove that
   $$\lim_{k\to\infty} \langle \vec{u}_k, \vec{v}\rangle = \langle \vec{u}, \vec{v}\rangle.$$

2. (10 points) (**Cauchy sequences in** $\mathbb{R}^n$) §10.2, p. 282: # 8.
   A sequence of points $\{\vec{u}_k\}$ in $\mathbb{R}^n$ is said to be *Cauchy* provided that $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $\forall k, \ell \geq K$,
   $$\operatorname{dist}(\vec{u}_k, \vec{u}_\ell) < \varepsilon.$$
   (a) Prove that $\{\vec{u}_k\}$ is Cauchy if and only if each component sequence is Cauchy.
   (b) Prove that a sequence in $\mathbb{R}^n$ converges if and only if it is Cauchy. (*Hint*: For sequences of real numbers, this was proved in Section 9.1.)

3. (10 points) (**Open and closed sets**) §10.3, p. 288: # 2. For each of the following subsets A of $\mathbb{R}^2$, state whether it is open in $\mathbb{R}^2$, closed in $\mathbb{R}^2$, or neither open nor closed in $\mathbb{R}^2$. Justify your conclusions with the aid of pictures. (You will learn later a rigorous way to justify your conclusions using continuity.)
   (a) $A = \{\vec{u} = (x,y) \mid x^2 > y\}$.
   (b) $A = \{\vec{u} = (x,y) \mid x^2 + y^2 = 1\}$.

(c) $A = \{\vec{u} = (x,y) \mid x \text{ is rational}\}$.
(d) $A = \{\vec{u} = (x,y) \mid x \geq 0,\ y \geq 0\}$.

4. (10 points) (**A closed ball is closed**)
   (a) (9 pts)§10.3, p. 288: # 3.
   Let r be a positive number and define
   $$\mathscr{O} = \{u \in \mathbb{R}^n \mid \|u\| > r\}.$$
   Prove that $\mathscr{O}$ is open in $\mathbb{R}^n$ by showing that every point in $\mathscr{O}$ is an interior point of $\mathscr{O}$. (*Hint*: Draw a picture. For each $u \in \mathscr{O}$, let $R = \|u\| - r$ and prove that $B_R(u) \subset \mathscr{O}$.)
   (b) (1 pt) Let r be a positive number and define $F = \{u \in \mathbb{R}^n \mid \|u\| \leq r\}$. Prove that F is closed.

5. (10 points) (**Closure**) §10.3, p. 289, # 12. For a subset A of $\mathbb{R}^n$, the *closure* of A, denoted by $\mathrm{cl}A$, is defined by
   $$\mathrm{cl}A = \mathrm{int}A \cup \mathrm{bd}A.$$
   Prove that
   (a) $A \subset \mathrm{cl}A$;
   (b) $A = \mathrm{cl}A$ if and only if A is closed in $\mathbb{R}^n$.

6. (10 points) (**Interior, exterior, and boundary**) §10.3, p. 289, # 13.
   (a) Prove that $\mathrm{int}A$ is an open subset of $\mathbb{R}^n$.
   (b) Use (a) to show that $\mathrm{ext}A$ is also an open subset of $\mathbb{R}^n$.
   (c) Use (a) and (b) together with the decomposition (10.11) on p. 287 to show that $\mathrm{bd}A$ is a closed subset of $\mathbb{R}^n$.

7. (10 points) (**Relatively open sets**) Prove that a set $S \subset A$ is relatively open in A if and only if for every u in S, there is an $\varepsilon > 0$ such that the relatively open ball $B_A(u,\varepsilon)$ is contained in S.

(End of Problem Set 8)