The Method of Moments:

Motivation and a First Example

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Outline

Motivation

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- Mean and variance are often distribution parameters, e.g.,

$$f_X(x) = \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(x-\mu)^2}{2v}\right]$$

- Using MLE, we noticed
  - Estimate of mean often equal to mean of data.
  - Estimate of variance often equal to variance of data.
- Might it be possible to determine estimators in this way?
- Demand moments of posited dist. equal those of data.
- If there are s parameters, match s moments.
- Result is s simultaneous equations for parameters.
- We know that this will give MLE results some of the time.

General methodology

- Make n measurements of Y, $Y_j = y_j$ for $j = 1, \ldots, n$.
- Posited distribution has s parameters, $f_Y(y; \theta_1, \ldots, \theta_s)$
- Set s *moments*, equal to corresponding *sample moments*

$$E(Y) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y = \frac{1}{n} \sum_{j=1}^{n} y_j$$

$$E(Y^2) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y^2 = \frac{1}{n} \sum_{j=1}^{n} y_j^2$$

$$\cdots$$

$$E(Y^s) = \int dy\ f_Y(y; \theta_1, \ldots, \theta_s) y^s = \frac{1}{n} \sum_{j=1}^{n} y_j^s$$

- Yields s simultaneous equations for the s parameters.

Example: Bernoulli trial

- Define random variable for each coin toss,

$$X := \begin{cases} 1 & \text{if toss results in heads (with probability } p\text{)} \\ 0 & \text{if toss results in tails (with probability } 1-p\text{).} \end{cases}$$

- Discrete probability function for one toss, for $k \in \{0, 1\}$,

$$\boxed{p_X(k) = \text{Prob}(X = k) = p^k (1-p)^{1-k}}$$

Moments for a Bernoulli trial

- Discrete probability function for one toss, for $k \in \{0, 1\}$,

$$\boxed{p_X(k) = \text{Prob}(X = k) = p^k (1-p)^{1-k}}$$

- Normalization: $\sum_{k=0}^{1} p_X(k) = (1-p) + p = 1$
- Mean: $\sum_{k=0}^{1} p_X(k) k = (1-p)0 + p1 = p$

Method of moments for n Bernoulli trials

- Posited distribution has one parameter p.
- That parameter is the theoretical mean, $p = \sum_k p_X(k)k$
- Set theoretical mean to mean of data

$$p = \frac{1}{n}\sum_{j=1}^{n} k_j.$$

- Solve for p (trivially in this case) to obtain MM *estimate*

$$p_e = \frac{1}{n}\sum_{j=1}^{n} k_j.$$

- This is identical to our result from MLE.

Method of moments estimator for n Bernoulli trials

- The MM *estimate* for n Bernoulli trials is

$$p_e = \frac{1}{n}\sum_{j=1}^{n} k_j.$$

- The *estimator* is then

$$\hat{p}(\vec{k}) = \frac{1}{n}\sum_{j=1}^{n} k_j.$$

- This is identical to our result from MLE.

Summary

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- We have...
  - Learned the *method of moments* for finding estimators.
  - Compared it to *maximum likelihood estimation*.
  - Applied the method to *Bernoulli trials*.
  - Found MLE and MM give the same estimator in this case.