M136 HW 4

1 $F \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ as each component function is in $C^2(\mathbb{R}^3, \mathbb{R}^3)$. We can see this as

$$Df(x,y,z) = \begin{bmatrix} \nabla F_1(x,y,z) \\ \nabla F_2(x,y,z) \\ \nabla F_3(x,y,z) \end{bmatrix}$$

$$Df(x,y,z) = \begin{bmatrix} 2xy & x^2 & 0 & 0 \\ ye^{xy} & xe^{xy} & 0 & 0 \\ 2x\cos(x^2+yz) & z\cos(x^2+yz & & y\cos(x^2+yz) \end{bmatrix}$$

Can see ∇F_i is continuous, and can see that second partials are continuous.

2 $$c^2 \frac{\partial^2}{\partial x^2} F(x,t) = c^2 \frac{\partial}{\partial x}(f'(x+ct)+g'(x-ct))$$
$$= c^2(f''(x+ct)+g''(x-ct))$$
$$\frac{\partial^2}{\partial t^2} F(x,t) = \frac{\partial}{\partial t}(cf'(x+ct)+cg'(x-ct))$$
$$\frac{\partial^2}{\partial t^2} F(x,t) = c^2 f''(x+ct)+c^2 g''(x-ct)$$

So $c^2 \frac{\partial^2}{\partial x^2} F(x,t) = \frac{\partial^2}{\partial t^2} F(x,t)$ $\square$

3 We showed if $f'(x) \neq 0$ $\forall x \in \mathbb{R}$ then f is one-to-one. Since $c > 0$ f is one-to-one. Need to show $f(x)$ is onto.
Since $f'(x) \geq c$, f is strictly increasing so $\lim_{x\to\infty} f(x) = \infty$. For $x < 0$, we can look at it essentially as going backwards. Starting at $x = 0$ going to $x_1 < 0$ as $f'(x) \geq c$ $\forall x \in \mathbb{R}$, so $f(x_1) < f(0)$. Can repeat this backward pattern and establish $\lim_{x\to-\infty} f(x) = -\infty$. This works as $f(x)$ is continuous.
Therefore, as $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, $\forall y \in \mathbb{R}$ $\exists x$ s.t. $f(x) = y$ so $f(x)$ is onto.
So, f is onto and one to one.

4 a) $DF(x_0,y_0) = \begin{bmatrix} e^{x_0}\cos y_0 & -e^{x_0}\sin y_0 \\ e^{x_0}\sin y_0 & e^{x_0}\cos y_0 \end{bmatrix}$

$\det DF(x_0,y_0) = (e^{x_0})^2 \cos^2 y_0 + (e^{x_0})^2 \sin^2 y_0$
$= (e^{x_0})^2(\cos^2 y_0 + \sin^2 y_0)$
$\det DF(x_0,y_0) = e^{2x_0} > 0 \quad \forall x_0 \in \mathbb{R}$ so IFT is applicable everywhere

b) If F is one-to-one $F((x_1,y_1)) = F((x_2,y_2))$ then $(x_1,y_1) = (x_2,y_2)$

Let $x_1 = 1$ $y_1 = 0$ $x_2 = 1$, $y_2 = 2\pi$

$F((1,0)) = (e,0)$

$F((1,2\pi)) = (e,0)$ but $0 \neq 2\pi$ so F isn't injective.

c) No, as the IFT states F is locally invertible around any (x_0,y_0). Since sine and cos, are periodic functions, we could still create a nbd where F is injective on that interval. However, we cannot create a global inverse.

5 a) $x \in \mathbb{R}^n$ so as $B = \begin{bmatrix} b_{11} & \cdots & \\ & \ddots & \\ & & b_{mn} \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

and $a = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ and $a + Bx = a + \begin{bmatrix} b_1 \cdot x \\ \vdots \\ b_m \cdot x \end{bmatrix}$ where b_i denotes the i^{th} row of B.

so $F = a + Bx = \begin{bmatrix} a_1 + b_1 \cdot x \\ \vdots \\ a_n + b_n \cdot x \end{bmatrix}$

$$DF = \begin{bmatrix} \nabla F_1 \\ \vdots \\ \nabla F_n \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = B$$

this follows as $\frac{\partial F_i}{\partial x_i}(b_i \cdot x) = b_1$ as $b_1 \cdot x = b_{11}x_1 + b_{12}x_1 + \ldots$

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5 b) For injectivity If $F(x_1)=F(x_2)$ then $x_1=x_2$

$$a+Bx_1 = a+Bx_2$$
$$Bx_1 = Bx_2$$
$$B(x_1-x_2)=0$$

If $x_1=x_2$ then only solution is $\vec{0}$ vector so only has trivial solution so B is invertible.

For onto, then $\forall y \in \mathbb{R}^m$ $\exists x \in \mathbb{R}^n$ where $y=a+Bx$ so $(y-a)=Bx$ and then for this to have a solution $\forall y$, B must be invertible.

Therefore, F is bijective if B is invertible.

c) $m>n$. If $F(x)$ is surjective, $\forall y \in \mathbb{R}^m$ $\exists x \in \mathbb{R}^n$ where $y=a+Bx$

So $y-a=Bx$, and by ii, there isn't necessarily a solution for all $(y-a)\in\mathbb{R}^m$ so, $\nexists x \in \mathbb{R}^n$ where a solution $(y-a)$ exists. So, $\forall y \in \mathbb{R}^m$ $\nexists x \in \mathbb{R}^n$ where $(y-a)=Bx$ so if $m \geq n$ F isn't surjective.

d) $m<n$. If F is injective then $F(x_1)=F(x_2) \Rightarrow x_1=x_2$

Let $x_1, x_2 \in \mathbb{R}^n$ $F(x_1)=F(x_2)$

$$a+Bx_1 = a+Bx_2$$

$$\Rightarrow Bx_1 = Bx_2 \Rightarrow Bx_1 - Bx_2 = 0 \Rightarrow B(x_1-x_2)=0$$

As $m<n$ $B(x_1-x_2)=0$ has nontrivial solutions. So, $x_1 - x_2 \neq \vec{0}$ and $x_1 \neq x_2$ and F isn't injective.

6 a) $$\frac{\partial f}{\partial x} = (2x)(x^2-y^2)+(x^2+y^2-2)(2x)$$

$$= 2x^3 - 2xy^2 + 2x^3 + 2xy^2 - 4x = 4x^3 - 4x$$

$$\frac{\partial f}{\partial y} = (2y)(x^2-y^2)+(x^2+y^2-2)(-2y)$$

$$= 2yx^2 - 2y^3 - 2yx^2 - 2y^3 + 4y = -4y^3 + 4y$$

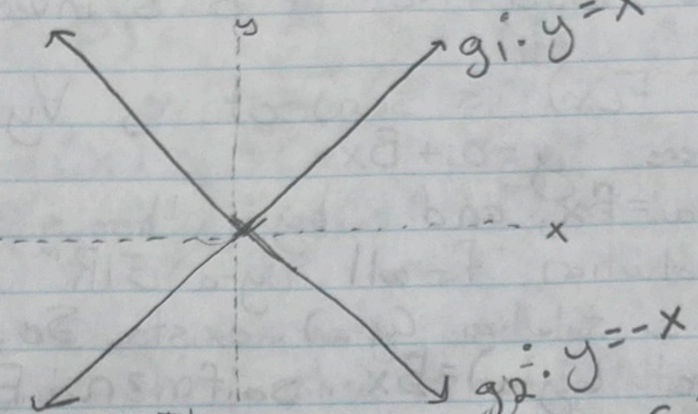
a) $\nabla f(x,y) = \begin{bmatrix} 4x^3-4x \\ -4y^3+4y \end{bmatrix}$

$\nabla f(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\nabla f(1,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\nabla f(1,-1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\nabla f(-1,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\nabla f(-1,-1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Since at each point $\nabla f = \vec{0}$, though it's $f(x,y)=0$ at each point, the assumption of Dini's Theorem doesn't hold.

b)



Under Dini's Theorem, such a $g(x)$ is continuously differentiable and these two curves crossing at $(0,0)$ which would counteract that.

7 $\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f$

To show $\underline{\int_a^b} f \leq 0$ as $\underline{\int_a^b} f = \sup\{L(f,p) \mid p$ is a partition$\}$, as $f(x)=0$ $\forall x \in \mathbb{Q}$, then for any partition, taking the lower integral means using the smallest value on that partition, which is ≤ 0

So $\underline{\int_a^b} f \leq 0$. To show $\overline{\int_a^b} f \geq 0$,

7 con. we know $\overline{\int}^b f = \inf\{U(f,P) \mid P \text{ is a partition}\}$
As $f(x) = 0$ $\forall x \in [a,b]$, taking $U(f,P)$ means on any P, the value used is ≥ 0
so $\inf\{U(f,P) \mid P \text{ is a partition}\} \geq 0$
Therefore $0 \leq \overline{\int_a^b} f$

Thus $\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f$ $\square$

8a) $L(g,P) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} g(x) (\Delta x_i)$

$L(f,P) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i$

As $g(x) \leq f(x)$ $\forall x \in [a,b]$, on any $[x_{i-1}, x_i]$ $g(x) \leq f(x)$ and $\inf g(x) \leq \inf f(x)$ on this interval. This holds at all intervals, and as using the same partition, then it follows

$\sum \inf_{x \in [x_{i-1}, x_i]} g(x) \Delta x_i \leq \sum \inf_{x \in [x_{i-1}, x_i]} f(x_i) \Delta x_i$

So $L(g,P) \leq L(f,P)$ $\square$

b) $L(g,P) \leq L(f,P)$ $\forall$ partition

$L(g,P) \leq L(f,P) \leq \sup\{L(f,P)\}$

So $L(g,P) \leq \underline{\int_a^b} f$ and this holds for all P

so $\sup\{L(g,P)\} \leq \underline{\int_a^b} f$

$\underline{\int_a^b} g \leq \underline{\int_a^b} f$

c/b) If f and g are integratable then $\underline{\int_a^b} f = \int_a^b f$ and $\underline{\int_a^b} g = \int_a^b g$ and by b)

$\underline{\int_a^b} g \leq \underline{\int_a^b} f$ so $\int_a^b g \leq \int_a^b f$ $\square$