Homework 4

Early problem due on Gradescope at 11:59 pm on Tuesday, February 14th.
Due on Gradescope at 11:59 pm on Friday, February 17th.

(1) (Early problem)

(a) Given a vector $\vec{x} = (x_1, \dots, x_{m_1+m_2}) \in \mathbb{R}^{m_1+m_2}$, show that

$$|(x_1, \dots, x_{m_1}, 0, \dots, 0)| \leq |\vec{x}|$$

and

$$|(0, \dots, 0, x_{m_1+1}, \dots, x_{m_1+m_2})| \leq |\vec{x}|$$

(b) Prove that if $f_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ are continuous, then the function

$$\begin{aligned} f_1 \times f_2 : \mathbb{R}^n &\to \mathbb{R}^{m_1+m_2} \\ \mathbf{x} &\mapsto (f_1(\mathbf{x}), f_2(\mathbf{x})) \end{aligned}$$

is continuous. (So for example, given that $x \mapsto (1, x)$ and $x \mapsto (x^2, x^3)$ are continuous functions, $x \mapsto (1, x, x^2, x^3)$ is also continuous.)

(2) Use the previous problem to prove that if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are continuous functions, then

$$\begin{aligned} f + g : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{x} &\mapsto f(\mathbf{x}) + g(\mathbf{x}) \end{aligned}$$

and

$$\begin{aligned} f \cdot g : \mathbb{R}^n &\to \mathbb{R} \\ \mathbf{x} &\mapsto f(\mathbf{x})g(\mathbf{x}) \end{aligned}$$

are continuous functions too.

(3) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Prove that f is continuous if and only if the following property holds:

($\star$) For each closed subset K of $\mathbb{R}^m$, $f^{-1}(K)$ is a closed subset of $\mathbb{R}^n$

(4) Let $\vec{x} \in \mathbb{R}^n$ and let $\epsilon \in \mathbb{R}_{>0}$. Prove that the open ball $B(\vec{x}, \epsilon)$ is an open subset of $\mathbb{R}^n$.

(5) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions Prove that

$$Z = \{(x, y) \in \mathbb{R}^2 \mid f(x) = g(y)\}$$

is a closed subset of $\mathbb{R}^2$ and

$$U = \{(x, y) \in \mathbb{R}^2 \mid f(x) \neq g(y)\}$$

is an open subset of $\mathbb{R}^2$.