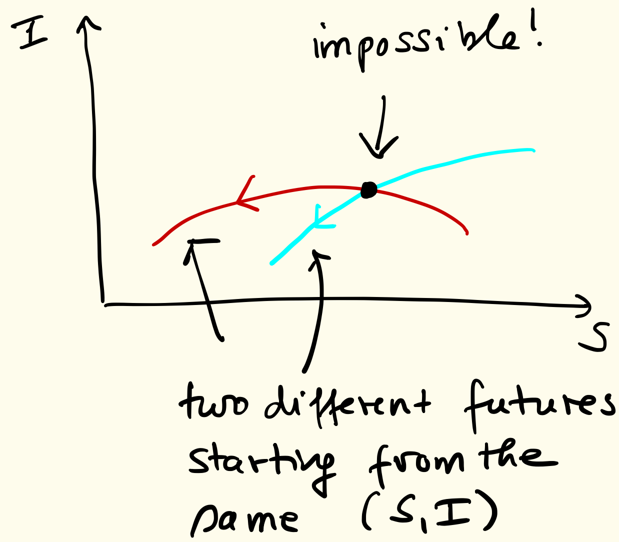
I am going to make some suggestions of things you could with your journal. You can't follow them all, and you need not follow any of them.

Tuesday, September 21

1. **Two solutions curves crossing in the phase plane.** Our differential equations model:

$$\frac{dS}{dt} = -\alpha IS$$
$$\frac{dI}{dt} = \alpha IS - \delta I$$

I explained that solutions can be visualized as curves in the phase plane, the (S, I) plane. Then I asked "Can two such curves cross?" Sophia said — perfectly correctly — "No, because if they crossed, then from that point on, two different futures would be possible, but the differential equations are supposed to predict one future only."



That was an excellent argument, but I am going to make an objection. One solution might have got to the crossing point at a different *time* than the other solution. Just because you see the trails of two jet planes intersect in the sky doesn't mean that there was a plane collision. One plane might have been in that location earlier, the other later.

Can you defend Sophia's reasoning against my objection? (If you aren't sure how, then first think about point 2 below.)

2. **Shifting time in the Kermack-McKendrick model.** Suppose that $(S(t), I(t))$ is a solution of

$$\begin{aligned} \frac{dS}{dt} &= -\alpha IS, \\ \frac{dI}{dt} &= \alpha IS - \delta I. \end{aligned}$$

Let $c \in \mathbb{R}$, and define

$$\tilde{S}(t) = S(t+c), \quad \tilde{I}(t) = I(t+c).$$

Prove that $(\tilde{S}(t), \tilde{I}(t))$ is also a solution.

Then think about what that means. Explain why you could summarize it as follows: "What happens in a Kermack-McKendrick epidemic doesn't change just because you change your calendar."

3. **Shifting time in general.** Suppose that $x = x(t)$ is a column vector with n entries, and $f(x)$ is also a column vector with n entries, and think about

$$\frac{dx}{dt} = f(x) \quad \text{for all } t.$$

Example: $n = 2$,

$$x = \begin{bmatrix} S \\ I \end{bmatrix}, \quad f(x) = \begin{bmatrix} -\alpha IS \\ \alpha IS - \delta I \end{bmatrix}.$$

Let $c \in \mathbb{R}$, and define

$$\tilde{x}(t) = x(t+c).$$

Prove:

$$\frac{d\tilde{x}}{dt} = f(\tilde{x}) \quad \text{for all } t.$$

Then think about what that means. Explain why you could summarize it as follows: "What happens in the world doesn't change just because you change your calendar."

4. **Autonomous vs. non-autonomous.** The equation

$$\frac{dx}{dt} = -2x \tag{1}$$

is an example of the form discussed in point 3 above. A solution is $x(t) = e^{-2t}$. Check that $\tilde{x}(t) = e^{-2(t+5)}$ is also a solution, in agreement with point 3. But now think about

$$\frac{dx}{dt} = -2tx. \tag{2}$$

Verify that $x(t) = e^{-t^2}$ is a solution, and verify that $\tilde{x}(t) = e^{-(t+5)^2}$ is *not* a solution. What's the difference? In (1), the right-hand side depends on x only. In (2) it depends on x and t. In general,

$$\frac{dx}{dt} = f(x)$$

is called an *autonomous* equation (or system of equations, of x and $f(x)$ are column vectors), and

$$\frac{dx}{dt} = f(x,t)$$

is called *non-autonomous*. Shifting time in a solution gives a new solution if the system is autonomous, not if it isn't autonomous.

Most systems that arise as models of scientific processes are autonomous. Can you see why that would be so, and when it might not be so?