

Instruction: In addition to the solution of the problems, include with your submission a brief report that summarizes in non-technical terms the problem, the method, merits of the method e.g. simplicity, scalability, convergence, stability, limitations of the method and limitations of the model if one was to use the proposed numerical methods in a practical setting. For both problems, include a printout of your code with your project submission. You should submit the project on Gradescope.

Orbit of Pallas

For it is the duty of an astronomer to record celestial motions through careful observation. Then, turning to the causes of these motions, he must conceive and devise hypotheses about them, since he cannot in any way attain the true cause. But, from whatever assumptions he adopts, the motions can be correctly calculated from the principles of geometry for the past as well as the future.

- Andreas Osiander, author of the introduction to the 1543 book *De Revolutionibus orbium coelestium* by Copernicus. Source: [Cunningham, 2016].

In March 28, 1802, the astronomer Heinrich Wilhelm Matthäus Olbers discovered the asteroid Pallas. It was initially conjectured that Pallas was a small planet between Mars and Jupiter. Pallas is the second asteroid to be discovered after the asteroid Ceres. Before discussing further the observational data of Pallas, we will note the definition of the word “celestial sphere”. In astronomy, it is usually the case that we have very little information on exact distances to celestial objects. In this setting, one defines a reference coordinate system and specifies information about celestial objects using direction. The celestial sphere is a sphere of arbitrarily large radius which is concentric with Earth. This sphere is fictitious and its role is that it allows us to project celestial bodies. Using the Earth’s coordinate system of longitude and latitude and projecting it onto the celestial sphere, we get the celestial coordinates right ascension and declination.

The declination of Pallas as a function of the right ascension is noted below.

Ascension (θ)	0	30	60	90	120	150	180	210	240	270	300	330
Declination	408	89	-66	10	338	807	1238	1511	1583	1462	1183	804

Table 1: Ascension and declination of Pallas. Ascension is in units of degrees and declination is in minutes.

Given this data, consider using the following 360° -trigonometric polynomial to approximate the observations.

$$f(\theta) = a_0 + \sum_{i=1}^k \left[a_i \cos\left(\frac{2i\pi\theta}{360}\right) + b_i \sin\left(\frac{2i\pi\theta}{360}\right) \right].$$

- Propose a least squares approach to fit the data to $f(\theta)$ for $k = 1$. Plot your fit against the data.
- Use your approach in (a) and apply it to $k = 2$. Plot your fit against the data.
- Extra credit:** Define $g(\theta)$ as follows:

$$g(\theta) = a_0 + \sum_{i=1}^5 \left[a_i \cos\left(\frac{2i\pi\theta}{360}\right) + b_i \sin\left(\frac{2i\pi\theta}{360}\right) \right] + a_6 \cos\left(\frac{12\pi\theta}{360}\right)$$

Fit the observational data exactly to $g(\theta)$.

Implied volatility

Volatility is a measure of the rate at which the price of a stock or index of stocks changes over a fixed period of time. Volatility is one metric that is considered to assess the risk in an investment. Generally, low volatility is associated with positive returns while high volatility is associated with negative returns. One way to measure volatility is by considering the past prices of a stock or index of stock. This is called *historical volatility*. In contrast, implied volatility is based on the dynamics of the market itself and is a measure of what the markets think volatility will be over a given period of time. In short, historical volatility is based on recent and past data while implied volatility is forward looking. Before defining implied volatility more precisely, we review some basic terminologies in finance.

Options: There are two common types of option contracts, *calls* and *puts*. A call option gives you the option to buy while a put option gives you an option to sell. In a call option, the holder/buyer has the right but not the obligation to buy a stock at a specified price. In contrast, in a put option, the holder/seller has the right but not the obligation to sell a stock at a specified price. In both cases, the right lasts on a particular date which is known as the *maturity* or *expiration* date. The specified or agreed price for the call or put is known as the *exercise* or *strike price*. There are two main types of options depending when the rights can be exercised. In *American options*, the options can be exercised any time before or on the expiration date. For *European options*, the options can only be exercised at the expiration date. Few notations are in order:

- $S(t)$: the stock price at time t
- K : the strike price
- T : the expiration time
- r : interest rate
- σ : volatility

One of the most popular approach to price an option is based on the Black-Scholes formula. For an European call option, the Black-Scholes formula for the price of the option is

$$C = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where d_1 and d_2 are given by

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = d_1 - \sigma\sqrt{T - t},$$

and Φ is the cumulative distribution function for the standard normal distribution. Note that, with the exception of the volatility parameter, all the variables in the Black-Scholes formula can be inferred directly. One approach to estimate the volatility is to use market data. Assume that we have a quote for a call option denote by C^* . Then, we can apply the Black-Scholes formula and find a σ that leads to C^* . This estimate of volatility which is implied by the market price of an option is known as *implied volatility*. Formally, the goal is to find σ such that

$$S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = C^*.$$

Here on, consider C to be a function of σ where we assume that all other parameters are known.

- (a) What value does $C(\sigma)$ approach as $\sigma \rightarrow \infty$?
- (b) Assume $S - Ke^{-r(T-t)} > 0$. What value does $C(\sigma)$ approach as $\sigma \rightarrow 0^+$?
- (c) Assume $S - Ke^{-r(T-t)} < 0$. What value does $C(\sigma)$ approach as $\sigma \rightarrow 0^+$?
- (d) Assume $S - Ke^{-r(T-t)} = 0$. What value does $C(\sigma)$ approach as $\sigma \rightarrow 0^+$?
- (e) Using (b), (c) and (d), evaluate $\lim_{\sigma \rightarrow 0^+} C(\sigma)$.
- (f) The derivative of C with respect to σ is known as *vega* and is given by

$$C'(\sigma) = S\sqrt{T-t}N'(d_1),$$

where $N'(d_1)$ is the standard normal probability density function. Prove that

$$\max(S - Ke^{-r(T-t)}, 0) < C^* < S.$$

- (g) **Extra credit:** Prove that $C'(\sigma)$ has a unique maximum value at σ^* where

$$\sigma^* = \sqrt{2 \left| \frac{\ln(\frac{S}{K}) + r(T-t)}{T-t} \right|}.$$

Hint: You can use the fact that $C''(\sigma) = \frac{d_1 d_2}{\sigma} C'(\sigma)$.

- (h) Argue that estimating the implied volatility can be formulated as a root finding problem. What methods can be employed?
- (i) **Extra credit:** Prove that using the initial guess $\sigma^0 = \sigma^*$, Newton's method converges quadratically to the implied volatility.
- (j) We now compute the implied volatility for call options using data taken from [Higham, 2004]. The call options were traded on the London International Financial Futures and Options Exchange (LIFFE) and reported in the Financial Times on August 22 2001. The data is for the FTSE 100 index which is an average of 100 equity shares quoted on the London Stock Exchange. The expiry date for these options was December 2001. The initial asset price was 5420.3. Let the interest rate be $r = 0.05$ and the duration of the option be $4/12$. Using your proposed method in (h), compute implied volatility for the following eight different exercise prices and option prices.

Exercise price	Option price
5125	475
5225	405
5325	340
5425	280.5
5525	226
5625	179.5
5725	139
5825	105

- (k) Briefly comment on your result in (j).

References

- [Cunningham, 2016] Cunningham, C. (2016). *Studies of Pallas in the Early Nineteenth Century: Historical Studies in Asteroid Research*. Springer International Publishing.
- [Higham, 2004] Higham, D. J. (2004). *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation*. Cambridge University Press.