MATH235 HOMEWORK 3 SOLUTION

- Proof. 1.1. Since the rational numbers have measure 0, by monotonicity we have
$$1 \leq 0 + |[0,1] \cap S|$$
Therefore S has positive Lebesgue measure. Consider any two $a, b \in S$, assume without loss of generality that $a < b$, by density of rational numbers there exists some q such that $a < q < b$. Therefore S contains no interval.

  1.2. Consider some $x \in \partial B_i$ for some B_i. By assumption there exists some B_j, $j \neq i$ such that $x \in B_j$. For some $\epsilon > 0$ we know $B_\epsilon(x) \in B_j$ and therefore $B_\epsilon(x) \cap B_i \neq \emptyset$.

  1.3. We can construct two finite collections of boxes such that
$$\cup_{n=1}^{N} f(\frac{n+1}{N}) \cdot [\frac{n-1}{N}, \frac{n}{N}] \subseteq A \subseteq \cup_{n=1}^{N} f(\frac{n+1}{N}) \cdot [\frac{n}{N}, \frac{n}{N}]$$
by taking limit of $N \to \infty$, we obtain $|A|_e = \frac{1}{3}$. Since A is measurable, $|A| = \frac{1}{3}$.

  1.4. We know that A and $[0,1] \setminus A$ are disjoint subsets that form a union of $[0,1]$. Therefore the claim holds true. □
- 3.1.16. Let E be a subset of $\mathbb{R}^d$. Prove that if $f : E \to [-\infty, \infty]$ is a measurable function and $\{f = -\infty\}$ is a measurable set, then E is measurable.

  Proof. Notice that $\forall n \in \mathbb{R}$, $[-\infty, \infty] = (\cup(-n, \infty]) \cup \{\infty\}$. Therefore,
$$\begin{aligned} E = f^{-1}([-\infty, \infty]) &= f^{-1}\left((\cup(-n, \infty]) \cup \{\infty\}\right) \\ &= \left(\cup f^{-1}(-n, \infty]\right) \cup f^{-1}(\{-\infty\}). \end{aligned}$$
  Since $\{f = -\infty\}$ is a measurable set and E is a countable union of measurable sets hence measurable. □

- 3.1.18. (a). Prove that if $f : \mathbb{R}^d \to \mathbb{R}$ is a measurable function if and only if $f^{-1}(U)$ is a measurable set for every open set $U \subseteq \mathbb{R}$. (b). Prove that if $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function if and only if $f^{-1}(U)$ is a measurable set for every open set $U \subseteq \mathbb{C}$.

  Proof. (a)
  " $\Leftarrow$ ":
  Suppose that $f^{-1}(U)$ is measurable for each open set $U \subseteq \mathbb{R}$. Then for each $a \in \mathbb{R}$ we have that
$$\{f > a\} = \left\{x \in \mathbb{R}^d : a < f(x)\right\} = f^{-1}(a, \infty)$$
  is measurable, so f is measurable.
  " $\Rightarrow$ ":

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Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is measurable, and let $U \subseteq \mathbb{R}$ be any open set. Then we can write U as a countable disjoint union of open intervals (possibly including infinite open intervals), say $U = \cup (a_j, b_j)$. Since

$$f^{-1}(a_j, b_j) = \{a_j < f < b_j\} = \{a_j < f\} \cap \{f < b_j\},$$

we conclude that $f^{-1}(a_j, b_j)$ is measurable for each j, and hence $f^{-1}(U) = \cup f^{-1}(a_j, b_j)$ is measurable.

(b)

" $\Rightarrow$" :

Suppose that $f : \mathbb{R}^d \to \mathbb{C}$ is measurable. Then its real part f_r and its imaginary part f_i are both measurable. For simplicity let us identify $\mathbb{C}$ with $\mathbb{R}^2$. In particular, with this identification we write $f(x) = (f_r(x), f_i(x))$.

Given an open strip $(a, b) \times \mathbb{R}$ in $\mathbb{C}$, we have

$$f^{-1}((a, b) \times \mathbb{R}) = f_r^{-1}(a, b),$$

which is measurable since f_r is measurable. Similarly,

$$f^{-1}(\mathbb{R} \times (c, d)) = f_i^{-1}(c, d)$$

is measurable. Consequently the inverse image of the open rectangle

$$(a, b) \times (c, d) = ((a, b) \times \mathbb{R}) \cap (\mathbb{R} \times (c, d))$$

is measurable. Every open subset of $\mathbb{C}$ can be written as a countable union of open rectangles, so it follows that $f^{-1}(U)$ is measurable for every open set $U \subseteq \mathbb{C}$.

" $\Leftarrow$":

Suppose that the inverse image of any open subset of $\mathbb{C}$ is measurable. Again identifying $\mathbb{C}$ with $\mathbb{R}^2$, if we fix $a \in \mathbb{R}$ then the set $(a, \infty) \times \mathbb{R}$ is open in $\mathbb{C}$. Hence

$$\{f_r > a\} = f_r^{-1}(a, \infty) = f^{-1}((a, \infty) \times \mathbb{R})$$

is measurable. Therefore f_r is a measurable function, and similarly f_i is measurable, so we conclude that f is measurable.

$\square$

- 3.1.19. Let $E \subseteq \mathbb{R}^d$ be a measurable set with $|E| > 0$, and assume that $f : E \to \bar{F}$ is measurable. (a). Show that if f is finite a.e., then there exists a measurable set $A \subseteq E$ such that $|A| > 0$ and f is bounded in A. (b). Suppose that it is not the case that $f = 0$ a.e. Prove that there exists a measurable set $A \subseteq E$ and a number $\delta > 0$ such that $|A| > 0$ and $|f| \geq \delta$ on A.

Proof. (a). Consider $E = \cup_{n=1}^{\infty}\{|f| < n\} = \cup_{n=1}^{\infty} E_n$. Then there exists some n_0 such that $|E_{n_0}| > 0$ otherwise $|E| = 0$. Take $A = E_{n_0}$ which satisfies the statement.

(b). Consider $E_0 = \cup_{n=1}^{\infty}\{|f| \geq \frac{1}{n}\}$. If $|E_0| = 0$ then $f = 0$ a.e., but it is not the case. Therefore $\exists n_0 \geq 1$ such that $|\{|f| \geq \frac{1}{n_0}\}| > 0$. Take $\delta = \frac{1}{n_0}$ and $A = \{|f| \geq \frac{1}{n_0}\}$ we have the desired result. $\square$