8.1.5 Example *Show that $\int_0^1 x\,dx = 1/2$, using the definition of the integral. Compare with a geometrical computation.*

Solution Break up $[0, 1]$ into n equal parts

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right].$$

Using this as a partition, the infimum of $f(x) = x$ on $[i/n, (i+1)/n]$ is i/n and the supremum is $(i+1)/n$. Thus, calling this partition P,

$$\begin{aligned} U(f, P) &= \sum_{i=0}^{n-1} \frac{i+1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{i=0}^{n-1} (i+1) \\ &= \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{1}{n^2} \cdot \frac{1}{2} n \cdot (n+1) \end{aligned}$$

and

$$L(f, P) = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2}(0 + 1 + \cdots + (n-1)) = \frac{1}{n^2}\left(\frac{1}{2}\right)(n-1)(n)$$

since $1 + 2 + \cdots + k = (1/2)k(k+1)$. Thus

$$U(f, P) = \frac{1}{2}\left(1 + \frac{1}{n}\right) \quad \text{and} \quad L(f, P) = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

These both converge to 1/2 as $n \to \infty$. Thus, from Riemann's condition (or Darboux's theorem), we see that f is integrable, with integral equal to 1/2. This is also geometrically obvious from Figure 8.1-3. ♦

Exercises for §8.1

1. Prove that if R is a Riemann sum for a function f and partition P, then $L(f, P) \le R \le U(f, P)$.
2. Let $f : [0, 2] \to \mathbb{R}$ be defined by $f(x) = 0$ for $0 \le x \le 1$, and by $f(x) = 1$ for $1 < x \le 2$. Compute, using the definition, $\int_0^2 f(x)\,dx$.
3. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ if $x \ne 1/2$ and $f(1/2) = 1$. Prove that f is integrable and $\int_0^1 f(x)\,dx = 0$.

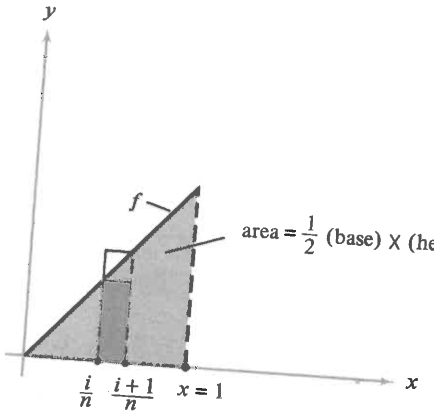

FIGURE 8.1-3 The integral of $f(x) = x$ is the area of a triangle

4. Let $A \subset \mathbb{R}^n$ and let $f(x) = 1$ for $x \in A$. What do you think $\int_A f$ should be?
5. Evaluate $\int_0^1 (3x + 4)\,dx$ using the definition and compare the answer with a geometrical computation of area.
6. Let $f : [a, b] \to \mathbb{R}$ be continuous. Use Riemann's condition and uniform continuity of f to prove that f is integrable.

§8.2 Volume and Sets of Measure Zero

On the real line, one usually integrates over intervals. However, in $\mathbb{R}^n$ we usually need to integrate over more complicated sets. We must be sure that the sets we are dealing with are restricted in such a way that the partitioning in the definition of integrability is reasonable. Here, "reasonable" means, roughly speaking, that the boundary of the set is not too complicated. Our immediate goal is to develop enough concepts to make these ideas precise. First, we define the volume of a set.

8.2.1 Definition *If $A \subset \mathbb{R}^n$ is a bounded set, let the **characteristic function** 1_A of A be the map $1_A : \mathbb{R}^n \to \mathbb{R}$ defined by $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. We say that A **has volume** if 1_A is integrable, and the **volume** of A is the number*

$$\int_A 1_A(x)\,dx = v(A).$$

This definition is natural because the region under the graph of 1_A is the "cylindrical" region with height 1 and base A (Figure 8.2-1). We shall also use the phrase "A has content" to mean the same as "A has volume." Sometimes a set that has volume is called ***Jordan measurable.***

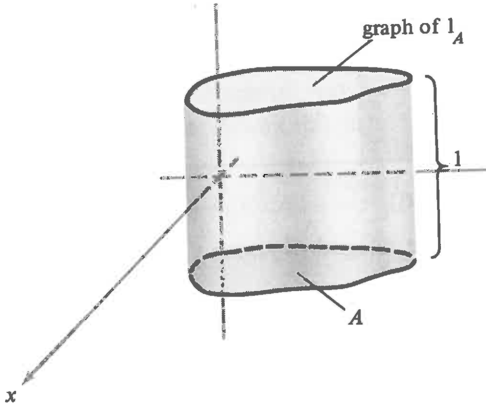

FIGURE 8.2-1 The characteristic function of a set

If $n = 1$, so that $A \subset \mathbb{R}$, we speak of $v(A)$ as the ***length*** of A, and when $A \subset \mathbb{R}^2$, we use the term ***area*** of A for $v(A)$.

We say that A has ***volume zero*** (or ***content zero***) if $v(A) = 0$. From the definition of the integral, this is equivalent to the statement that for every $\varepsilon > 0$ there is a finite covering of A by rectangles, say, $S_1, \ldots, S_m$, such that the total volume is less than ε; that is,

$$\sum_{i=1}^{m} v(S_i) < \varepsilon,$$

where $v(S_i)$ is computed for rectangles as before. (The details are worked out in Worked Example **8.1** at the end of the chapter.)

It is useful to allow *countable* coverings as well as finite ones. These ideas were systematically introduced by Henri Lebesgue around 1900.

8.2.2 Definition *A set $A \subset \mathbb{R}^n$ (not necessarily bounded) is said to have **measure zero** if for every $\varepsilon > 0$ there is a covering of A, say, $S_1, S_2, \ldots$, by a countable (or finite) number of rectangles such that the total volume $\sum_{i=1}^{\infty} v(S_i) < \varepsilon$. Recall that $S_1, S_2, \ldots$ are said to **cover** A when $\bigcup_{i=1}^{\infty} S_i \supset A$. Note that these rectangles may overlap.*

It is important to realize that these concepts depend on the space in which we are working. To illustrate the point, consider an example.

8.2.3 Example *Show that, regarded as a subset of $\mathbb{R}^2$, the real line has measure zero, but as a subset of $\mathbb{R}$ it does not.*

Solution To prove the first assertion, given $\varepsilon > 0$, we want to find rectangles $S_1, S_2, \ldots$ that enclose the x axis and have total area $< \varepsilon$. Considering Figure 8.2-2, let

$$S_i = [-i, i] \times \left[-\frac{\varepsilon}{2i \cdot 2^{i+1}}, \frac{\varepsilon}{2i \cdot 2^{i+1}}\right].$$

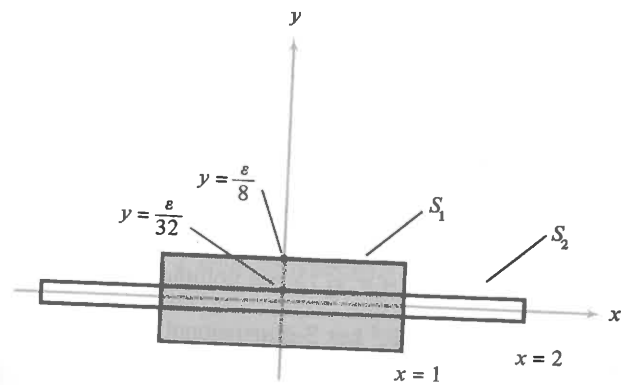

FIGURE 8.2-2 A family of rectangles that covers the horizontal axis

Since $v(S_i) = 2i[(2\varepsilon)/(2i \cdot 2^{i+1})] = \varepsilon/2^i$, we get $\sum_{i=1}^{\infty} v(S_i) < \sum_{i=1}^{\infty}(\varepsilon/2^i) = \varepsilon$, since $1/2 + 1/4 + 1/8 + \cdots = 1$. It is clear that the real line as a subset of itself cannot have measure zero, because in covering it by intervals, the total length of the intervals will be $+\infty$. ♦

This demonstration is typical of the way one proves that a set has measure zero. Another example of a set of measure zero is the sphere

$$S = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

From the definition of volume, it is clear that *if A has volume zero, then A has measure zero.* Indeed, if A has volume zero and $\varepsilon > 0$, we can even find a *finite* covering by rectangles for A with total volume $< \varepsilon$. Also, note that if A has measure zero and $B \subset A$, then B has measure zero as well.

The main advantage of measure zero over volume zero is indicated in the following theorem.

8.2.4 Theorem *Suppose that the sets $A_1, A_2, \ldots$ have measure zero in $\mathbb{R}^n$. Then $A_1 \cup A_2 \cup \cdots$ has measure zero in $\mathbb{R}^n$.*

From this we conclude, for example, that any set composed of a countable number of points has measure zero.

8.2.5 Example Consider the set A of rationals in $[0,1] \subset \mathbb{R}$. The set A does not have volume; that is, 1_A is not integrable. Indeed, the function that has the value 1 on rationals, 0 on the irrationals, is not integrable as we have seen in §4.8. Nevertheless, the set A does have measure zero, because a point has volume and measure zero and A consists of countably many points, and so Theorem **8.2.4** applies. ♦

Exercises for §8.2

1. Show that $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ has volume zero.
2. Show that the xy plane in $\mathbb{R}^3$ has 3-dimensional measure 0.
3. If $A \subset [a,b]$ has measure zero in $\mathbb{R}$, prove that $[a,b] \backslash A$ does *not* have measure zero in $\mathbb{R}$. (It follows from Exercise 10 at the end of the chapter that $[a,b]$ does not have measure zero.)
4. Use Exercise 3 to show that the irrationals in $[0,1]$ do not have measure zero.
5. Must the boundary of a set have measure zero?
6. Must the boundary of a set of measure zero have measure zero?

§8.3 Lebesgue's Theorem

We now consider one of the most important results in integration theory. We feel intuitively that most "decent" functions, such as continuous functions, ought to be integrable, since the area under their graphs should be definable. To settle the question of exactly how decent is "decent" we have the theorem of H. Lebesgue. With this theorem Lebesgue opened up new advances in integration theory by stressing the concept of measure zero.

8.3.1 Lebesgue's Theorem *Let $A \subset \mathbb{R}^n$ be bounded and let $f : A \to \mathbb{R}$ be a bounded function. Extend f to all of $\mathbb{R}^n$ by letting it be zero at points not contained in A. Then f is (Riemann) integrable iff the points at which the extended f is discontinuous form a set of measure zero.*

We can draw two important conclusions from this result as stated in the following two corollaries.

8.3.2 Corollary *A bounded set $A \subset \mathbb{R}^n$ has volume iff the boundary of A has measure zero.*

This is obtained by applying the theorem to the characteristic function 1_A. Since finite and countable sets have measure zero, the theorem also gives:

8.3.3 Corollary *Let $A \subset \mathbb{R}^n$ be bounded and have volume. A bounded function $f : A \to \mathbb{R}$ with a finite or countable number of points of discontinuity is integrable.*

This result shows that many of the common functions one meets in practice are integrable. For example, a continuous function on an interval $[a,b]$ is integrable because $[a,b]$ has volume (the boundary consists of two points). Functions with a finite number of points of discontinuity are similarly integrable.

Notice that integrability of f in Theorem **8.3.1** depends on the extension of f. For instance, if A is the set of all rationals in $[0,1]$ and f is identically 1, f restricted to A is continuous on A but the extended f is nowhere continuous and in fact is not integrable. In Corollary **8.3.3** it is *not* necessary to extend f. This is accounted for using the fact that A has volume and making use of Corollary **8.3.2**. Another useful result is as follows.

8.3.4 Theorem

i. *Let $A \subset \mathbb{R}^n$ be bounded and have measure zero and let $f : A \to \mathbb{R}$ be any (bounded) integrable function. Then $\int_A f(x)\,dx = 0$.*

ii. *If $f : A \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all x and $\int_A f(x)\,dx = 0$, then the set $\{x \in A \mid f(x) \neq 0\}$ has measure zero.*

This theorem is not unreasonable. Indeed, a set of measure zero is "small" with, essentially, zero volume, and so the integral of any function over it ought to be zero. The second part is likewise reasonable.

8.3.5 Example *Let*

$$f(x) = \begin{cases} x, & -1 \leq x \leq 0 \\ 3x+8, & 0 < x \leq 1. \end{cases}$$

Show that f is integrable on $[-1, 1]$.

Solution The set $[-1, 1]$ has volume, and f has only one discontinuity, at $x = 0$. Since f is bounded, **8.3.3** shows that f is integrable. ♦

8.3.6 Example *Let $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on $[-1, 1]$.*

Solution Here f has one point of discontinuity, at $x = 0$. Also, $|f(x)| \leq 1$, and so f is bounded. Thus, by Corollary **8.3.3**, f is integrable. ♦

8.3.7 Example Let $f(x, y) = x^2 + \sin(1/y)$ for $y \neq 0$ and $f(x, 0) = x^2$. Show that f is integrable on $A = \{(x, y) \mid x^2 + y^2 < 1\}$.

Solution *Here f is bounded on A = interior of unit disk in $\mathbb{R}^2$, and has discontinuities on the line $y = 0$, which is a set of zero measure in $\mathbb{R}^2$. Also, A has volume (its boundary has zero volume). Hence, by Theorem **8.3.1**, f is integrable.* ♦

Exercises for §8.3

1. Let $f(x) = x^3$ on $[-1, 1]$. Prove that f is integrable.
2. Let $f(x, y) = 1$ if $x \neq 0$ and $f(0, y) = 0$. Prove that f is integrable on $A = [0, 1] \times [0, 1] \subset \mathbb{R}^2$.

3. Compute $\int_A f$ where f and A are as in Exercise 2.
4. Let $A \subset \mathbb{R}^n$ be open and have volume, and let $f : A \to \mathbb{R}$ be continuous, $f(x) \geq 0$, and $f(x_0) > 0$ for some $x_0 \in A$. Show that $\int_A f > 0$.
5. Let $r_k = 1/k$, $k = 1, 2, \ldots$, and on $\mathbb{R}$ let U be the open set
$$U = \bigcup_{k=1}^{\infty} D\left(r_k, \frac{1}{2^k}\right).$$
Decide whether or not U has volume.
6. Let $f(x) = \cos(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on $[-1, 1]$.

§8.4 Properties of the Integral

We now present some of the elementary properties of the integral analogous to those for functions on an interval.

8.4.1 Theorem *Let A, B be bounded subsets of $\mathbb{R}^n$, $c \in \mathbb{R}$, and let $f, g : A \to \mathbb{R}$ be integrable. Then*

i. *$f + g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.*

ii. *cf is integrable and $\int_A (cf) = c\int_A f$.*

iii. *$|f|$ is integrable and $|\int_A f| \leq \int_A |f|$.*

iv. *If $f \leq g$, then $\int_A f \leq \int_A g$.*

v. *If A has volume and $|f| \leq M$, then $|\int_A f| \leq Mv(A)$.*

vi. **Mean Value Theorem for Integrals** *If $f : A \to \mathbb{R}$ is continuous and A has volume and is compact and connected, then there is an $x_0 \in A$ such that $\int_A f(x)\,dx = f(x_0)v(A)$. The quantity $(1/v(A))\int_A f$ is called the average of f over A.*

vii. *Let $f : A \cup B \to \mathbb{R}$. If the sets A and B are such that $A \cap B$ has measure zero and $f|(A \cap B)$, $f|A$, and $f|B$ are all integrable, then f is integrable on $A \cup B$ and $\int_{A \cup B} f = \int_A f + \int_B f$.*