MATH 70 WORKSHEET 4

[1]

Please give complete reasoning for all worksheet answers.

1. (4 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1 + 3x_2 \end{bmatrix}$. Decide whether T is a linear transformation. If T is linear, then prove it is using the definition of linear transformation. If not, find a specific counterexample (e.g., using specific values for x_1, x_2, x_3 and any other constants you use) to one of the properties, and explain why it is a counterexample.

2. (6 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^n$. Assume the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)$ are independent in $\mathbb{R}^m$. Prove that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent in $\mathbb{R}^n$.

   Solution:

   Suppose that there are real numbers c_1, c_2, and c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \vec{0}$. Applying the transformation T on both sides gives $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \vec{0}$ by linearity of T, and this implies that $c_1 = c_2 = c_3 = 0$ because vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)$ are independent. Hencefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent in $\mathbb{R}^n$.

3. (4 points) Given an $n \times m$ matrix A, answer the following.

   (a) The columns of A belong to which space ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3, \ldots$).

   Solution:

   $\mathbb{R}^n$

   (b) For the columns of A to be linearly independent there must be a pivot in each column .

   (c) For the columns of A to span the space they belong to (your answer to 1) there must be a pivot in each row .

   (d) If the columns of A span the space they belong to and the columns are also be linearly independent, what can you conclude about the dimension of A?

Solution:

If the columns of A span the space they belong to then there must be a pivot in each of the rows of A. If the columns of A are linearly independent then there must be a pivot in each column of A. Since the number of pivots is fixed, then the number of rows of A must be equal to the number of columns of A.

4. (6 points) Let $C = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 5 \end{bmatrix}$

   (a) We say B is a left inverse of A if $BA = I$ where I is the identity matrix of the correct size. Show that C is a left inverse of A.

   Solution:

   This is shown by the matrix calculation.

   (b) Show that C is not a right inverse of A (show $AC \neq I$)?

   Solution:

   This is shown by the matrix calculation.

   (c) How is this reconciled with *The invertible Matrix Theorem*?

   Solution:

   The *The invertible Matrix Theorem* only holds for square matrices

   (d) Does A have a right inverse? Justify your answer.

   Solution:

   No. Proof by contradiction. Suppose A has a right inverse, D, then D would have to be 2×3 matrix (because A is a 3×2), which means that there is a non-pivot column in D. So there is a nontrivial solution to $D\mathbf{x} = \mathbf{0}$ say $\mathbf{u}$. This means that $\mathbf{u} = I\mathbf{u} = AD\mathbf{u} = \mathbf{0}$ a contradiction.