Homework 8

Early problem due on Gradescope at 11:59 pm on Wednesday, March 29th.
Due on Gradescope at 11:59 pm on Friday, March 31st.

(1) (Early problem, but consider doing (2) first) What's wrong with the following "proof" that $\overline{\bigcup_{i\in I} A_i} \subseteq \bigcup_{i\in I} \overline{A_i}$?

If $\{A_i\}_{i\in I}$ is a collection of subsets of X and if $x \in \overline{\bigcup_{i\in I} A_i}$, then every neighborhood U of x intersects $\bigcup_{i\in I} A_i$. Thus U must intersect some A_i, so that x must belong to the closure of some A_i. Therefore $x \in \bigcup_{i\in I} \overline{A_i}$.

(2) Let A and B be subsets of a topological space X. Let $\{A_i\}_{i\in I}$ be a collection of subsets of X. Prove the following:
  (a) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
  (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
  (c) $\overline{\bigcup_{i\in I} A_i} \supseteq \bigcup_{i\in I} \overline{A_i}$; give an example where equality fails.

(3) If $A \subseteq X$, we define the *boundary* of A by the equation

$$\operatorname{Bd} A = \overline{A} \cap \overline{(X - A)}.$$

  (a) Show that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint, and $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$.
  (b) Show that $\operatorname{Bd} A = \emptyset$ if and only if A is both open and closed.
  (c) Show that U is open if and only if $\operatorname{Bd} U = \overline{U} - U$.
  (d) If U is open, is it true that $U = \operatorname{Int}(\overline{U})$? Justify your answer.

(4) Let X, X', Y be topological spaces. Let $f : X \to Y$ be a function and let $g : X' \to X$ be a homeomorphism.
  (a) Prove that f is continuous if and only if $f \circ g$ is continuous.
  (b) Prove that f is a homeomorphism if and only if $f \circ g$ is a homeomorphism.

(5) X and Y be topological spaces. Let $\{U_i\}_{i\in I}$ be a collection of open subsets of X so that $X = \bigcup_{i\in I} U_i$.
  (a) Prove that a bijective function $f : X \to Y$ is a homeomorphism if and only if $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism for each i.
  (b) Find an example of spaces X, Y and a function $f : X \to Y$ so that $f|_{U_i} : U_i \to f(U_i)$ is a homeomorphism for each i, but f is not a homemorphism.

(6) (Optional, no credit) Consider the collection of all subsets A of the topological space X. The operations of closure $A \mapsto \overline{A}$ and complementation $A \mapsto X - A$ are functions from this collection to itself.
  (a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.
  (b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.