Tufts University
Department of Mathematics

Math 136 **Homework 4** [1] **Spring, 2023**

Due date: 11:59 pm, Sunday, February 26, 2023 on Gradescope.

You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

Reading assignment:

- *By Wednesday, February 15*, please read Sections 15.2, 15.3, 16.1, and the statement of the General Inverse Function Theorem in section 16.3. Also please read section 17.1 and Dini's theorem.

  The first part of this homework covers Sections 15.2, 15.3, the Inverse Function Theorem and Dini's Theorem.

- *By Thursday, February 23* We now start integration! Please read Sections 6.1-3.

  The second part of this homework covers section 6.1.

Facts from linear algebra:
Let A be an $m \times n$ matrix.

I. If $m < n$ then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

II. If $m > n$ then $A\mathbf{x} = \mathbf{y}$ does not have a solution for all $\mathbf{y} \in \mathbb{R}^m$.

III. If $m = n$ (so A is a square matrix), then the following are equivalent

- A is an invertible matrix
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- $A\mathbf{x} = \mathbf{y}$ has a solution for every $\mathbf{y} \in \mathbb{R}^n$.

Problems:

1. (10 points) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $F(x, y, z) = \begin{bmatrix} x^2y \\ e^{xy} \\ \sin(x^2 + yz) \end{bmatrix}$. Explain why $F \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ and find the derivative matrix $Df(x, y, z)$.

2. (15 points) Let f, g be twice continuously differentiable on $\mathbb{R}$ and define
$$F(x, t) = f(x + ct) + g(x - ct)$$
for $(x, t) \in \mathbb{R}^2$ and $c > 0$. Use the Chain Rule to show that F satisfies the *wave equation*:
$$c^2 \frac{\partial^2}{\partial x^2} F(x, y) = \frac{\partial^2}{\partial t^2} F(x, t).$$
If x is position and t is time, then the functions f and g are called travelling waves, g travelling to the right with speed c as t increases and f travelling to the left. Cool, eh!

3. (10 points) Problem # 5 on p. 428.

4. (10 points) Problem # 9 on p. 428. The function in this problem shows that a global version of the inverse function theorem is not true in general: This problem gives an example of a function $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ such that $JF(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$, but $F : \mathbb{R}^2 \to \mathbb{R}^2$ is not injective on $\mathbb{R}^2$. In contrast if I is an open interval in $\mathbb{R}$, $f \in C^1(I, \mathbb{R})$, and f' is never zero then f *is* globally injective as you proved on homework.

5. (20 points) Let B be an $m \times n$ matrix and let $\mathbf{a} \in \mathbb{R}^m$. Define $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(\mathbf{x}) = \mathbf{a} + B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. F is called an *affine function*

   (a) Find the derivative matrix DF and explain why F has continuous first derivatives (i.e., $F \in C^1(\mathbb{R}^n, \mathbb{R}^m)$).

   Now assume $m = n$. In this case, $F : \mathbb{R}^n \to \mathbb{R}^n$ and therefore $\mathbf{a} \in \mathbb{R}^n$ and $B \in M_{n \times n}$.

   (b) Find conditions on B for which F is bijective (one-to-one and onto).

   Now, let B be an $m \times n$ matrix, and let $\mathbf{a} \in \mathbb{R}^m$. Define $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(\mathbf{x}) = \mathbf{a} + B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (c) Assume $m > n$. Prove that F is not surjective.

   (d) Assume $m < n$. Prove that F is not injective.

   This explains why one can't expect an Inverse Function Theorem for $F \in C^1(\mathcal{O}, \mathbb{R}^m)$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and $m \neq n$.

6. (10 points) Problem #2 on p. 447. Note: The hypothesis of Dini's theorem fails at $(0,0)$ and there are two curves $g_1(x) = x$ and $g_2(x) = -x$ on the set $f(x,y) = 0$ that intersect at $(0,0)$. However, under the hypotheses of Dini's Theorem, there would be only one such curve (see class notes and the end of the statement of Dini's theorem in Fitzpatrick).

   The next problems are on integration for $f : [a,b] \to \mathbb{R}$.

7. (10 points) Problem # 3 on p. 141.

8. (15 points)

   (a) Do Problem # 5 on p. 141.

   (b) Use the result of part (a) to show that if f and g are integrable and $f(x) \leq g(x)$ $\forall x \in [a,b]$, then $\int_a^b g \leq \int_a^b f$.