# M 135 HW 5

1 Let $\varepsilon > 0$, want $\delta > 0$ s.t. $\forall x \in \mathbb{R}$,
$|x-2| < \delta \rightarrow |x^2-4| < \varepsilon$

Scratchwork:
$|x^2-4| = |(x-2)(x+2)| = |x-2||x+2| < \varepsilon$
Suppose $|x-2| < 1$, so $|x+2| = |x-2+4| \leq |x-2| + 4 < 5$
So $5|x-2| < \varepsilon$, $|x-2| < \frac{\varepsilon}{5}$
end scratch work

Choose $\delta = \min(1, \frac{\varepsilon}{5})$
$|x-2| < \delta$, and $|x-2| < 1$, $|x+2| = |x-2+4| \leq |x-2| + 4 < 5$.
$|x^2-4| \leq |x-2||x+2| < 5|x-2| < 5\delta \leq 5 \cdot \frac{\varepsilon}{5} < \varepsilon$

Thus $f(x) = x^2$ satisfies $\varepsilon$-$\delta$ criteria at $x=2$ $\square$

2 Let $\varepsilon > 0$, want $\delta > 0$ s.t. $\forall x \in D$,
$|x-y| < \delta \rightarrow |h(x) - h(y)| < \varepsilon$ : Verifies $\varepsilon$-$\delta$ at $y$

Scratch work:
$\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \left|\frac{x^2-y^2}{(1+x^2)(1+y^2)}\right| \leq \frac{|x-y||x+y|}{(1+x^2)(1+y^2)} \leq |x-y|\left(\frac{|x|}{(1+x^2)(1+y^2)} + \frac{|y|}{(1+x^2)(1+y^2)}\right)$
$\leq |x-y|\left(\frac{1}{1+x^2} + \frac{1}{1+y^2}\right)$ as $\frac{|x|}{(1+x^2)(1+y^2)} \leq \frac{1}{1+y^2}$ wlog
$< 2|x-y|$ as $\frac{1}{1+x^2} \leq 1$.
$2|x-y| < \varepsilon$, $|x-y| < \frac{\varepsilon}{2}$
end scratch work
Choose $\delta \leq \frac{\varepsilon}{2}$
$|x-y| < \frac{\varepsilon}{2}$, then $\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \left|\frac{x^2-y^2}{(1+x^2)(1+y^2)}\right| = \frac{|x-y||x+y|}{(1+x^2)(1+y^2)} \rightarrow$
$\rightarrow \leq \frac{|x-y|(|x|+|y|)}{(1+x^2)(1+y^2)} \leq |x-y|\left(\frac{|x|}{(1+x^2)(1+y^2)} + \frac{|y|}{(1+x^2)(1+y^2)}\right) < |x-y| \cdot \left(\frac{1}{1+y^2} + \frac{1}{1+x^2}\right)$

$|x-y|\left(\frac{1}{1+x^2}-\frac{1}{1+y^2}\right) < 2|x-y| = 2\cdot\frac{\varepsilon}{2} = \varepsilon$

So $h(x)$ satisfies $\varepsilon-\delta$ criteria on $\mathbb{R}$. $\square$

3 $f(x) = \frac{1}{\sqrt{x+x^2}} + x^2 - 2x$ is continuous on $x \geq 0$, $f$ is a combination of polynomials and rationals and $x > 0$.

$f(1) = \frac{1}{\sqrt{2}} - 1 < 0 \quad f(2) = \frac{1}{\sqrt{6}} + 2 > 0$

Since $f(1) < 0$ and $f(2) > 0$ and $f(x)$ is continuous, by the intermediate value theorem, $\exists\, c \in (1,2)$ s.t. $f(c) = 0$.

4 Let $g(x) = f(x) - x$

$g(a) = f(a) - a \geq 0$ as $\min f(x) = a$

$g(b) = f(b) - b \leq 0$ as $\max f(x) = b$

Since $g(a) \geq 0$ and $g(b) \leq 0$, by intermediate value theorem, $\exists\, c \in [a,b]$ s.t. $g(c) = 0$.
If $g(c) = 0$, then $f(c) - c = 0$, $f(c) = c$ and we have a fixed point.

5 $\Rightarrow$

If $\lim a_n = 0$ then $\lim |a_n| = 0$

Since $a_n \to 0$, Let $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n| < \varepsilon$

If $\lim |a_n| = 0$, then for $\varepsilon > 0$, $\exists\, N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$

$||a_n| - 0| < \varepsilon$

$||a_n|| = |a_n| < \varepsilon$ from above so $N_2 = N_1$ and $\lim |a_n| = 0$

$\Leftarrow$ If $\lim|a_n|=0$ show $\lim a_n = 0$.
Note $\forall \{x_n\} \in \mathbb{R}$, $-|x_n| \le x_n \le |x_n|$

$\lim -|a_n| = -\lim|a_n| = -1 \cdot 0 = 0$
Since $-|a_n| \le a_n \le |a_n|$ take the limit
$\lim -|a_n| \le \lim a_n \le \lim |a_n|$
$0 \le \lim a_n \le 0$ so $\lim a_n = 0$ $\square$
This proves both sides

6) If $f(x)$ is uniformly continuous, then $\forall u_n, v_n \in \mathbb{R}$, if $u_n - v_n \to 0$ then $f(u_n) - f(v_n) \to 0$
Let $u_n = n + 1/n$, $v_n = n$
$u_n - v_n = \frac{1}{n} \to 0$
$f(u_n) - f(v_n) = (n+\frac{1}{n})^3 - n^3$
$= (n+\frac{1}{n})(n^2+2+\frac{1}{n^2}) - n^3$
$= n^3 + 3n + \frac{3}{n} + \frac{1}{n^3} - n^3 = 3n + 3/n + 1/n^3$
$\lim(f(u_n) - f(v_n)) = \lim(3n + 3/n + 1/n^3) \not\to 0$ so $f(x)$ isn't uniformly continuous

7 $|f(u) - f(v)| \le C|u-v|$ Let $\{u_n\}$ and $\{v_n\} \in D$.
If $f: D \to \mathbb{R}$ is uniformly continuous, then $\forall u_n, v_n \in D$ if $u_n - v_n \to 0$ $f(u_n) - f(v_n) \to 0$.

Since $f$ is Lipschitz, for every $u \in u_n$ and $v \in v_n$
$|f(u) - f(v)| \le C|u-v|$
or $|f(u_1) - f(v_1)| \le C|u_1 - v_1|$, $|f(u_2) - f(v_2)| \le C|u_2 - v_2|$ and etc.
Since $u_n - v_n \to 0$, then we can say $C|u_n - v_n| \to 0$ for all $n$
$|f(u_n) - f(v_n)| \le C|u_n - v_n| \le 0$
$\lim(0 \le |f(u_n) - f(v_n)| \le C|u_n - v_n| \le 0)$
$\lim 0 \le \lim|f(u_n) - f(v_n)| \le 0$ (so) $\lim(|f(u_n) - f(v_n)|) = 0 = \lim f(u_n) - f(v_n)$
and $f$ is uniformly continuous.

8a) $f:[0,1]\to\mathbb{R}$ is continuous if it satisfies the $\epsilon$-$\delta$ criteria, so $\forall \epsilon>0, \exists \delta>0$ s.t. $\forall x,y\in D$ $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$

Scratch work)

$$|\sqrt{x}-\sqrt{y}|\cdot\frac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})}=\frac{|x-y|}{|\sqrt{x}+\sqrt{y}|}<\epsilon$$

Suppose $|x-y|<1$, so $y+1<x<1+y$, $\sqrt{x}<\sqrt{1+y}$

$\sqrt{x}+\sqrt{y}<\sqrt{y+1}+\sqrt{y}$, so $\frac{|x-y|}{\sqrt{x}+\sqrt{y}}<\frac{|x-y|}{\sqrt{y+1}+\sqrt{y}}<\frac{|x-y|}{\sqrt{y}}<\epsilon$

$|x-y|<\sqrt{y}\,\epsilon$

End Scratch work)

Choose $\delta=\min(1,\sqrt{y}\,\epsilon)$

$|x-y|<\delta$ and $|x-y|<1$, so $|\sqrt{x}-\sqrt{y}|=$
$|\sqrt{x}-\sqrt{y}|=\frac{|x-y|}{\sqrt{x}+\sqrt{y}}$ as $|x-y|<1$ $y+1<x<y+1$, $\sqrt{x}<\sqrt{y+1}$

So $\frac{|x-y|}{\sqrt{x}+\sqrt{y}}\leq\frac{|x-y|}{\sqrt{y}+\sqrt{y+1}}<\frac{|x-y|}{\sqrt{y}}=\frac{\delta}{\sqrt{y}}=\frac{\epsilon\sqrt{y}}{\sqrt{y}}=\epsilon$ $\square$

$f:[0,1]\to\mathbb{R}$ is continuous $\square$

b) If $f:D\to\mathbb{R}$ is continuous on a sequentially compact set, it is uniformly continuous. part a) shows $f$ is continuous, and $[0,1]$ is closed and bounded so it is sequentially compact. Therefore, $f: D\to\mathbb{R}$ is uniformly continuous.

c) If $f$ is Lipschitz, then $|\sqrt{u}-\sqrt{v}|\leq C|u-v|$ $\forall u,v\in D$. Let $v=0$, and $u\neq 0$, so $|\sqrt{u}|\leq C|u|$

$\sqrt{u}\leq Cu$, $\frac{1}{\sqrt{u}}\leq C$ as $u\to 0$ $C$ gets larger and is unbounded

So no $C$ exists, and $f(x)$ is not Lipschitz.

9 $f: D \to \mathbb{R}$ is uniformly continuous iff it satisfies the $\epsilon$-$\delta$ criteria on D, meaning $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall u, v \in D$ if $|u-v| < \delta$ then $|f(u) - f(v)| < \delta$

$\Rightarrow$ f is uniformly continuous so it satisfies the $\epsilon$-$\delta$ criteria.
Proof by contradiction:
Suppose f doesn't satisfy $\epsilon$-$\delta$ criteria.
Meaning $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, for $u, v \in D$ $|u-v| < \delta$ and $|f(u) - f(v)| \geq \epsilon$.
Since f is uniformly continuous if $u_n, v_n \in D$ and $u_n - v_n \to 0$, then $|f(u_n) - f(v_n) \to 0$ but if $u_n - v_n \to 0$, then $u_n \to v_n \to u$, $u \in \mathbb{R}$, $|f(u_n) - f(v_n)| \geq \epsilon$ is then not true, which is a contradiction, because $f(u_n) - f(u_n) = 0$
so f satisfies the $\epsilon$-$\delta$ criteria

$\Leftarrow$ f satisfies $\epsilon$-$\delta$ criteria over D so it is uniformly continuous. So $\forall \epsilon > 0$
Let $u_n, v_n \in D$ and $u_n - v_n \to 0$. We want to show $\lim f(u_n) - f(v_n) = 0$.
Let $\epsilon > 0$, then we want to see if $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|f(u_n) - f(v_n)| < \epsilon$.
By definition of $\epsilon$-$\delta$ criteria, $|u-v| < \delta \to |f(u) - f(v)| < \epsilon$.
Since $u_n - v_n \to 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|u_n - v_n| < \delta$
so by $\epsilon$-$\delta$ criteria, $|f(u_n) - f(v_n)| < \epsilon$.
Therefore $\lim f(u_n) - f(v_n) = 0$, so f is uniformly continuous.

Both ways are shown, thus the theorem is proven $\square$