Tufts University
Department of Mathematics
Spring 2022

MA 166: Statistics

Solutions to Homework 3 (v1.0) [1]

Assigned Monday 7 February 2022
Due Monday 14 February 2022 at 11:59 pm EDT.

1. **Larsen & Marx, Problem 5.4.6, page 313: Let $Y_{\min}$ be the smallest order statistic in a random sample of size n drawn from the uniform pdf, $f_Y(y;\theta) = 1/\theta$, $0 \le y \le \theta$. Find an unbiased estimator for θ based on $Y_{\min}$.**

   First note that the cdf of Y is

$$F_Y(y;\theta) = \int_0^y dz\ f_Y(z;\theta) = \int_0^y dz\ \frac{1}{\theta} = \frac{y}{\theta}.$$

   Then, from page 193 of L&M, we know that for $0 \le y \le \theta$ we have

$$\begin{aligned} f_{Y_{\min}}(y) =& n\left[1 - F_Y(y)\right]^{n-1} f_Y(y) \\ =& n\left(1 - \frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} \\ =& \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1}. \end{aligned}$$

   From this we can calculate

$$\begin{aligned} E\left(Y_{\min}\right) =& \int_0^\theta dy\ f_{Y_{\min}}(y)y \\ =& \int_0^\theta dy\ \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1} y \\ =& n\theta\int_0^1 du\ \left(1-u\right)^{n-1} u \\ =& n\theta\int_0^1 dw\ w^{n-1}(1-w) \\ =& n\theta\int_0^1 dw\ \left(w^{n-1} - w^n\right) \\ =& n\theta\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ =& \frac{\theta}{n+1}, \end{aligned}$$

   and hence

$$\theta = (n+1)E\left(Y_{\min}\right).$$

Now suppose that we take m samples of $Y_{\min}$, and obtain the results $y_1, y_2, \ldots, y_m$. Note carefully that *each* of these measurements is the minimum of n measurements taken from the uniform distribution. The sample mean is then $\frac{1}{m}\sum_{j=1}^{m} y_j$. The *estimator* of θ based on $Y_{\min}$ is then

$$\boxed{\hat{\theta}(\vec{Y}_{\min}) = (n+1)\frac{1}{m}\sum_{j=1}^{m} Y_{\min j},}$$

and hence the *estimate* of θ is

$$\boxed{\theta_e(\vec{y}) = (n+1)\frac{1}{m}\sum_{j=1}^{m} y_j.}$$

Finally, note that

$$E\left[\hat{\theta}(\vec{Y}_{\min})\right] = (n+1)\frac{1}{m}\sum_{j=1}^{m} E\left[Y_{\min j}\right] = (n+1)\frac{1}{m}\sum_{j=1}^{m}\frac{\theta}{n+1} = (n+1)\frac{1}{m}\,m\frac{\theta}{n+1} = \theta,$$

demonstrating that the estimator $\hat{\theta}(\vec{Y}_{\min})$ is unbiased.

2. **Larsen & Marx, Problem 5.4.10, page 313: A sample of size 1 is drawn from the uniform pdf defined over the interval $[0, \theta]$. Find an unbiased estimator for θ^2. (Hint: Is $\hat{\theta^2} = Y^2$ unbiased?)**

   Following the hint, we examine

   $$E\left(Y^2\right) = \int_0^\theta dy\,\frac{1}{\theta}y^2 = \frac{1}{\theta}\,\frac{\theta^3}{3} = \frac{\theta^2}{3}.$$

   It follows that if we take $\hat{\theta^2} = Y^2$, as we are told to do, we will find

   $$E\left(\hat{\theta^2}\right) = E\left(Y^2\right) = \frac{\theta^2}{3},$$

   so, as an estimator for θ^2, this is clearly not unbiased.

   While the above observation demonstrates that θ^2 is not unbiased, it also suggests the solution. If we define the new estimator

   $$\boxed{\hat{\theta_1^2} := 3\hat{\theta^2} = 3Y^2,}$$

   we find that

   $$E\left(\hat{\theta_1^2}\right) = E\left(3Y^2\right) = 3\frac{\theta^2}{3} = \theta^2,$$

   so that $\hat{\theta_1^2}$ is an unbiased estimator for θ^2.

3. **Larsen & Marx, Problem 5.4.18, page 316: Suppose that $n = 5$ observations are taken from the uniform pdf, $f_Y(y;\theta) = 1/\theta$, $0 \le y \le \theta$, where θ is unknown. Two unbiased estimators for θ are**

$$\hat{\theta}_1 = \frac{6}{5} Y_{\max} \quad \text{and} \quad \hat{\theta}_2 = 6Y_{\min}.$$

Which estimator would be better to use? (Hint: What must be true of $\operatorname{Var}(Y_{\max})$ and $\operatorname{Var}(Y_{\min})$ given that $f_Y(y;\theta)$ is symmetric?) Does your answer as to which estimator is better make sense on intuitive grounds?

First, the restriction to $n = 5$ is unnecessary and annoying, so let's generalize the problem. For any n, let's define

$$\hat{\theta}_1 = \frac{n+1}{n} Y_{\max} \quad \text{and} \quad \hat{\theta}_2 = (n+1)Y_{\min}.$$

When we are finished, we can obtain the answer to the problem at hand by simply setting $n = 5$. There are times in life when working with variables is preferable to working with numbers, and this is one of them.

We begin by noting that the cdf of Y is

$$F_Y(y;\theta) = \int_0^y dz\; f_Y(z;\theta) = \int_0^y dz\; \frac{1}{\theta} = \frac{y}{\theta}$$

for $0 \le y \le \theta$. Then, from page 193 of L&M, we have

$$f_{Y_{\max}}(y) = n\,[F_Y(y)]^{n-1}\, f_Y(y) = \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1}.$$

and

$$f_{Y_{\min}}(y) = n\,[1 - F_Y(y)]^{n-1}\, f_Y(y) = \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1}.$$

for $0 \le y \le \theta$.

At this point, there is an easy way to solve this problem. We may notice that

$$f_{Y_{\min}}(y) = f_{Y_{\max}}(\theta - y),$$

so one pdf is just a reflection of the other about the point $\theta/2$. It follows that

$$E(Y_{\min}) = \int_0^\theta dy\; f_{Y_{\min}}(y)y = \int_0^\theta dy\; f_{Y_{\max}}(\theta - y)y = \int_0^\theta du\; f_{Y_{\max}}(u)(\theta - u)$$

$$= \theta - E(Y_{\max}),$$

and that

$$E\left(Y_{\min}^2\right) = \int_0^\theta dy\; f_{Y_{\min}}(y)y^2 = \int_0^\theta dy\; f_{Y_{\max}}(\theta - y)y^2 = \int_0^\theta du\; f_{Y_{\max}}(u)(\theta - u)^2$$

$$= \theta^2 - 2\theta E(Y_{\max}) + E\left(Y_{\max}^2\right).$$

From the above it follows that

$$\begin{aligned}
\operatorname{Var}\left(Y_{\min}\right) =& E\left(Y_{\min}^2\right) - \left[E\left(Y_{\min}\right)\right]^2 \\
=& \theta^2 - 2\theta E\left(Y_{\max}\right) + E\left(Y_{\max}^2\right) - \left[\theta - E\left(Y_{\max}\right)\right]^2 \\
=& E\left(Y_{\max}^2\right) - \left[E\left(Y_{\max}\right)\right]^2 \\
=& \operatorname{Var}\left(Y_{\max}\right),
\end{aligned}$$

so the variances of $Y_{\max}$ and $Y_{\min}$ are equal. Given that the pdfs are reflections of one another, this result is perhaps not surprising, but it's nice to see it proven. Now, defining

$$V := \operatorname{Var}\left(Y_{\min}\right) = \operatorname{Var}\left(Y_{\max}\right),$$

we may notice that

$$\operatorname{Var}\left(\hat{\theta}_1\right) = \operatorname{Var}\left(\frac{n+1}{n} Y_{\max}\right) = \left(\frac{n+1}{n}\right)^2 V,$$

and

$$\operatorname{Var}\left(\hat{\theta}_2\right) = \operatorname{Var}\left((n+1) Y_{\max}\right) = (n+1)^2 V.$$

Hence, the relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is

$$\boxed{\frac{\operatorname{Var}\left(\hat{\theta}_2\right)}{\operatorname{Var}\left(\hat{\theta}_1\right)} = \frac{(n+1)^2 V}{\left(\frac{n+1}{n}\right)^2 V} = n^2 \geq 1.}$$

so if $n > 1$, the first estimator is more efficient than the second. This is actually intuitively plausible, since $Y_{\max}$ is much closer to θ than $Y_{\min}$. The fact that you have to multiply it by a smaller factor to obtain the estimate for θ makes all the difference.

There is a more tedious but more complete way to solve this problem, and that is to actually compute the mean and variance of $Y_{\max}$ and $Y_{\min}$. I include this solution here for completeness (and because, inevitably, some of you will do the problem in this way). There are advantages to it. Along the way, we will be able to prove that both estimators are unbiased, which is something that was not required for this problem, but still interesting and reassuring.

First, let's consider $Y_{\max}$. We can calculate the mean as follows

$$\begin{aligned}
E\left(Y_{\max}\right) =& \int_0^\theta dy\ f_{Y_{\max}}(y) y \\
=& \int_0^\theta dy\ \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1} y \\
=& n\theta \int_0^1 du\ u^n \\
=& \frac{n}{n+1}\theta.
\end{aligned}$$

From this we see that

$$E\left(\hat{\theta}_1\right) = E\left(\frac{n+1}{n} Y_{\text{max}}\right) = \frac{n+1}{n} E\left(Y_{\text{max}}\right) = \frac{n+1}{n}\frac{n}{n+1}\theta = \theta,$$

thus verifying that $\hat{\theta}_1$ is unbiased. In similar fashion, we can calculate the second moment of the pdf

$$\begin{aligned} E\left(Y_{\text{max}}^2\right) =& \int_0^\theta dy\ f_{Y_{\text{max}}}(y) y^2 \\ =& \int_0^\theta dy\ \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1} y^2 \\ =& n\theta^2 \int_0^1 du\ u^{n+1} \\ =& \frac{n}{n+2}\theta^2. \end{aligned}$$

From this we see that the variance of Y_{max} is

$$\text{Var}\left(Y_{\text{max}}\right) = E\left(Y_{\text{max}}^2\right) - \left[E\left(Y_{\text{max}}\right)\right]^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+1)^2(n+2)}\theta^2.$$

Next, let's turn our attention to Y_{min}. We can calculate the mean as follows

$$\begin{aligned} E\left(Y_{\text{min}}\right) =& \int_0^\theta dy\ f_{Y_{\text{min}}}(y) y \\ =& \int_0^\theta dy\ \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1} y \\ =& n\theta \int_0^1 du\ \ (1-u)^{n-1} u \\ =& n\theta \int_0^1 dw\ w^{n-1}(1-w) \\ =& n\theta \int_0^1 dw\ \ \left(w^{n-1} - w^n\right) \\ =& n\theta \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ =& \frac{1}{n+1}\theta. \end{aligned}$$

From this we see that

$$E\left(\hat{\theta}_2\right) = E\left((n+1)Y_{\text{max}}\right) = (n+1)E\left(Y_{\text{max}}\right) = (n+1)\frac{1}{n+1}\theta = \theta,$$

thus verifying that $\hat{\theta}_2$ is unbiased. In similar fashion, we can calculate the second moment of the pdf

$$\begin{aligned}
E\left(Y_{\min}^2\right) &= \int_0^\theta dy\ f_{Y_{\min}}(y) y^2 \\
&= \int_0^\theta dy\ \frac{n}{\theta}\left(1-\frac{y}{\theta}\right)^{n-1} y^2 \\
&= n\theta^2 \int_0^1 du\ \left(1-u\right)^{n-1} u^2 \\
&= n\theta^2 \int_0^1 dw\ w^{n-1}(1-w)^2 \\
&= n\theta^2 \int_0^1 dw\ \left(w^{n-1}-2w^n+w^{n+1}\right) \\
&= n\theta^2 \left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right) \\
&= \frac{2}{(n+1)(n+2)}\theta^2.
\end{aligned}$$

The variance of $Y_{\min}$ is then

$$\mathrm{Var}\left(Y_{\min}\right) = E\left(Y_{\min}^2\right) - \left[E\left(Y_{\min}\right)\right]^2 = \frac{2}{(n+1)(n+2)}\theta^2 - \left(\frac{1}{n+1}\theta\right)^2 = \frac{n}{(n+1)^2(n+2)}\theta^2,$$

which is precisely equal to $\mathrm{Var}\left(Y_{\max}\right)$, as expected. The $n = 5$ case yields

$$\mathrm{Var}\left(Y_{\min}\right) = \mathrm{Var}\left(Y_{\min}\right) = \frac{5}{252}\theta^2.$$

The rest of the calculation proceeds as described in the first derivation above, and the conclusion is the same: For $n > 1$, the first estimator is more efficient than the second.

4. **Larsen & Marx, Problem 5.4.20, page 316: Given a random sample of size $\boldsymbol{n}$ from a Poisson distribution, $\hat{\boldsymbol{\lambda}}_1 = \boldsymbol{X}_1$ and $\hat{\boldsymbol{\lambda}}_2 = \overline{\boldsymbol{X}}$ are two unbiased estimators for $\boldsymbol{\lambda}$. Calculate the relative efficiency of $\hat{\boldsymbol{\lambda}}_1$ to $\hat{\boldsymbol{\lambda}}_2$.**

   The Poisson distribution is

$$p_X(k) = P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!},$$

   where k is a nonnegative integer. As a warmup exercise, let's review some properties of this distribution. It is normalized, since

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{+\lambda} = 1,$$

   where we have used the Maclaurin series for the exponential in the last step.

To find higher moments of the distribution, first note that

$$\left(\lambda\frac{d}{d\lambda}\right)\lambda^k = \lambda k \lambda^{k-1} = k\lambda^k$$

and by repeating this and making a very easy proof by mathematical induction,

$$\left(\lambda\frac{d}{d\lambda}\right)^n \lambda^k = k^n\lambda^k.$$

It follows that if we now rewrite the result in the last paragraph as

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{+\lambda},$$

and apply the operator $\left(\lambda\frac{d}{d\lambda}\right)^n$ to both sides, we obtain

$$\sum_{k=0}^{\infty} k^n\frac{\lambda^k}{k!} = \left(\lambda\frac{d}{d\lambda}\right)^n e^{+\lambda}.$$

Premultiplying both sides by $e^{-\lambda}$, this becomes the remarkably powerful result

$$E\left(X^n\right) = \sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}k^n = e^{-\lambda}\left(\lambda\frac{d}{d\lambda}\right)^n e^{+\lambda}.$$

For $n = 1$ the result of the previous paragraph is

$$E(X) = e^{-\lambda}\left(\lambda\frac{d}{d\lambda}\right)e^{+\lambda} = e^{-\lambda}\lambda e^{+\lambda} = \lambda,$$

and for $n = 2$ it is

$$E\left(X^2\right) = e^{-\lambda}\left(\lambda\frac{d}{d\lambda}\right)\left(\lambda\frac{d}{d\lambda}\right)e^{+\lambda} = e^{-\lambda}\left(\lambda\frac{d}{d\lambda}\right)\left(\lambda e^{+\lambda}\right) = e^{-\lambda}\lambda\left(\lambda e^{+\lambda} + e^{+\lambda}\right) = \lambda^2+\lambda.$$

From the above, it follows that the variance is

$$\mathrm{Var}\left(X^2\right) = E\left(X^2\right) - \left[E\left(X\right)\right]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda,$$

and the standard deviation is

$$\sigma = \sqrt{\mathrm{Var}\left(X^2\right)} = \sqrt{\lambda}.$$

With the above established, we turn our attention to the problem at hand, and consider the estimators $\hat{\lambda}_1 = X_1$ and $\hat{\lambda}_2 = \overline{X}$. The problem states that they are unbiased, so it is not necessary to prove this, but just for the sake of completeness let's do so. For the first estimator, we have

$$E\left(\hat{\lambda}_1\right) = E\left(X_1\right) = \lambda,$$

and for the second we have

$$E\left(\hat{\lambda}_2\right) = E\left(\overline{X}\right) = E\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n}\sum_{j=1}^{n} E\left(X_j\right) = \frac{1}{n}\sum_{j=1}^{n}\lambda = \frac{1}{n}n\lambda = \lambda,$$

and so both estimators are indeed unbiased.

We now compute the variances of our two estimators. For the first estimator, we have

$$\mathrm{Var}\left(\hat{\lambda}_1\right) = \mathrm{Var}\left(X_1\right) = \lambda,$$

and for the second we have

$$\mathrm{Var}\left(\hat{\lambda}_2\right) = \mathrm{Var}\left(\overline{X}\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n^2}\sum_{j=1}^{n} \mathrm{Var}\left(X_j\right) = \frac{1}{n^2}\sum_{j=1}^{n}\lambda = \frac{1}{n^2}n\lambda = \frac{\lambda}{n},$$

so the relative efficiency of $\hat{\lambda}_1$ with respect to $\hat{\lambda}_2$ is

$$\boxed{\frac{\mathrm{Var}\left(\hat{\lambda}_2\right)}{\mathrm{Var}\left(\hat{\lambda}_1\right)} = \frac{\left(\frac{\lambda}{n}\right)}{n} = \frac{1}{n}.}$$

Since this is less than one, we see that, unsurprisingly, $\hat{\lambda}_1$ is substantially less efficient than $\hat{\lambda}_2$.

5. **Larsen & Marx, Problem 5.5.4, page 319: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $f_Y(y;\theta) = \frac{1}{\theta}e^{-y/\theta}$, $y > 0$. Compare the Cramér-Rao lower bound for $f_Y(y;\theta)$ to the variance of the maximum likelihood estimator for θ, $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Is $\overline{Y}$ a best estimator for θ?**

   First, let's compute all of the moments that we will need from the exponential distribution. We could obtain them using integration by parts, but let me show you an easier way. First, we can verify that the pdf is normalized

$$\int_0^\infty dy\; f_Y(y;\theta) = \int_0^\infty dy\; \frac{1}{\theta}e^{-y/\theta} = \int_0^\infty du\; e^{-u} = 1.$$

   Rewrite this as

$$\int_0^\infty dy\; e^{-y/\theta} = \theta,$$

   and define the variable $\beta = \frac{1}{\theta}$ so that this becomes

$$\int_0^\infty dy\; e^{-\beta y} = \frac{1}{\beta}.$$

   Now take the nth derivative of both sides with respect to β to obtain

$$\int_0^\infty dy\; e^{-\beta y}(-y)^n = \frac{d^n}{d\beta^n}\left(\frac{1}{\beta}\right).$$

Multiply both sides by $(-1)^n\beta$ to obtain

$$\int_0^\infty dy\ \beta e^{-\beta y}y^n = (-1)^n\beta\frac{d^n}{d\beta^n}\left(\frac{1}{\beta}\right),$$

and replace β by $1/\theta$ to conclude

$$E\left(Y^n\right) = \int_0^\infty dy\ \frac{1}{\theta}e^{-y/\theta}y^n = (-1)^n\beta\frac{d^n}{d\beta^n}\left(\frac{1}{\beta}\right)\bigg|_{\beta=1/\theta}.$$

We can simplify the right-hand side by noting that

$$\frac{d}{d\beta} = \frac{d\theta}{d\beta}\frac{d}{d\theta} = -\frac{1}{\beta^2}\frac{d}{d\theta} = -\theta^2\frac{d}{d\theta}.$$

The very elegant result is then

$$E\left(Y^n\right) = \frac{1}{\theta}\left(\theta^2\frac{d}{d\theta}\right)^n\theta.$$

For $n=1$ the above result is

$$E\left(Y\right) = \frac{1}{\theta}\left(\theta^2\frac{d}{d\theta}\right)\theta = \theta,$$

and for $n=2$ it is

$$E\left(Y^2\right) = \frac{1}{\theta}\left(\theta^2\frac{d}{d\theta}\right)\left(\theta^2\frac{d}{d\theta}\right)\theta = \frac{1}{\theta}\left(\theta^2\frac{d}{d\theta}\right)\theta^2 = \frac{1}{\theta}\theta^2\left(2\theta\right) = 2\theta^2.$$

The more general result for the exponential distribution, $E\left(Y^n\right) = n!\ \theta^2$, can then be proven by induction. Finally, the variance is

$$\text{Var}(Y) = E\left(Y^2\right) - \left[E\left(Y\right)\right]^2 = 2\theta^2 - (\theta)^2 = \theta^2.$$

Of course, you can quote all of the above results from the book, so the above derivation is not required for the solution to this problem, but the results will be necessary.

With the above established, we can attack the problem at hand. First note that the maximum likelihood estimator is unbiased since

$$E\left(\hat{\theta}\right) = E\left(\frac{1}{n}\sum_{j=1}^n Y_j\right) = \frac{1}{n}\sum_{j=1}^n E\left(Y_j\right) = \frac{1}{n}\sum_{j=1}^n\theta = \frac{1}{n}\left(n\theta\right) = \theta.$$

The variance of the maximum likelihood estimator is then

$$\text{Var}\left(\hat{\theta}\right) = \text{Var}\left(\frac{1}{n}\sum_{j=1}^n Y_j\right) = \frac{1}{n^2}\sum_{j=1}^n\text{Var}\left(Y_j\right) = \frac{1}{n^2}\sum_{j=1}^n\theta^2 = \frac{1}{n^2}\left(n\theta^2\right),$$

or

$$\boxed{\operatorname{Var}\left(\hat{\theta}\right) = \frac{\theta^2}{n}.}$$

Now let's check the Cramér-Rao bound. We have

$$\ln f_Y(y;\theta) = -\ln\theta - \frac{y}{\theta},$$

whence

$$\frac{\partial}{\partial\theta}\ln f_Y(y;\theta) = -\frac{1}{\theta} + \frac{y}{\theta^2},$$

and

$$\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta) = \frac{1}{\theta^2} - \frac{2y}{\theta^3},$$

and

$$-n\frac{\partial^2}{\partial\theta^2}\ln f_Y(Y;\theta) = -\frac{n}{\theta^2} + \frac{2nY}{\theta^3}.$$

Taking the expectation value of this yields

$$E\left[-n\frac{\partial^2}{\partial\theta^2}\ln f_Y(Y;\theta)\right] = -\frac{n}{\theta^2} + \frac{2nE(Y)}{\theta^3} = -\frac{n}{\theta^2} + \frac{2n\theta}{\theta^3} = \frac{n}{\theta^2},$$

and inverting the above yields the Cramér-Rao bound,

$$\boxed{\left\{E\left[-n\frac{\partial^2}{\partial\theta^2}\ln f_Y(Y;\theta)\right]\right\}^{-1} = \frac{\theta^2}{n}.}$$

It is seen that the given estimator achieves the Cramér-Rao bound, so it is a minimum-variance estimator.

6. **Larsen & Marx, Problem 5.5.6, page 319: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the pdf**

$$f_Y(y;\theta) = \frac{1}{(r-1)!\theta^r}\, y^{r-1}e^{-y/\theta} \qquad \text{for } y > 0$$

   (a) **Show that $\hat{\theta} = \frac{1}{r}\overline{Y}$ is an unbiased estimator for θ.**
   The pdf is called the *gamma distribution*, and the first step is to calculate its moments. This is covered in Section 4.6 of our book, and I am going to assume that you are at least conversant with this material from your probability course. The nth moment is given by

$$E\left(Y^n\right) = \int_0^\infty dy\ f_Y(y;\theta)y^n = \int_0^\infty dy\ \frac{1}{(r-1)!\theta^r}\, y^{r-1}e^{-y/\theta}y^n = \frac{\Gamma(r+n)}{\Gamma r}\theta^n,$$

   where

$$\Gamma(z) := \int_0^\infty dt\ e^{-t}t^{n-1}$$

is the *gamma function*. This has a recurrence relation and its value for integer arguments are related to the factorial function as follows

$$\Gamma(r+1) = r\Gamma(r) = r!.$$

Applying the above for $n = 1$ yields

$$E(Y) = \frac{\Gamma(r+1)}{\Gamma r}\theta = r\theta,$$

and for $n = 2$ we find

$$E\left(Y^2\right) = \frac{\Gamma(r+2)}{\Gamma r}\theta = r(r+1)\theta^2.$$

The variance is then

$$\mathrm{Var}(Y) = E\left(Y^2\right) - [E(Y)]^2 = r(r+1)\theta^2 - (r\theta)^2 = r\theta^2.$$

We are now ready to solve the problem at hand. We have the estimator $\hat{\theta}(Y) = \frac{1}{r}\overline{Y}$, so

$$\boxed{E\left(\frac{1}{r}\overline{Y}\right) = E\left(\frac{1}{r}\,\frac{1}{n}\sum_{j=1}^{n} Y_j\right) = \frac{1}{nr}\sum_{j=1}^{n} E\left(Y_j\right) = \frac{1}{nr}\sum_{j=1}^{n} r\theta = \frac{1}{nr}\left(nr\theta\right) = \theta,}$$

demonstrating that the estimator is unbiased.

(b) **Show that $\boldsymbol{\hat{\theta} = \frac{1}{r}\overline{Y}}$ is a minimum-variance estimator for $\boldsymbol{\theta}$.**
The variance of our estimator is

$$\mathrm{Var}\left(\hat{\theta}\right) = \mathrm{Var}\left(\frac{1}{r}\,\frac{1}{n}\sum_{j=1}^{n} Y_j\right) = \frac{1}{r^2n^2}\sum_{j=1}^{n} \mathrm{Var}\left(Y_j\right) = \frac{1}{r^2n^2}\sum_{j=1}^{n} r\theta^2 = \frac{1}{r^2n^2}\left(nr\theta^2\right),$$

or

$$\boxed{\mathrm{Var}\left(\hat{\theta}\right) = \frac{\theta^2}{rn}.}$$

Finally, we need to compute the Cramér-Rao bound. We find

$$\ln f_Y(Y;\theta) = -\ln\left[(r-1)!\right] - r\ln\theta + (r-1)\ln Y - \frac{Y}{\theta},$$

whence

$$\frac{\partial}{\partial\theta}\ln f_Y(Y;\theta) = -\frac{r}{\theta} + \frac{Y}{\theta^2},$$

and

$$\frac{\partial^2}{\partial\theta^2}\ln f_Y(Y;\theta) = \frac{r}{\theta^2} - \frac{2Y}{\theta^3}.$$

Multiplying by $-n$ and taking the expectation value yields

$$-nE\left[\frac{\partial^2}{\partial\theta^2}\ln f_Y(Y;\theta)\right] = -n\left(\frac{r}{\theta^2} - \frac{2E(Y)}{\theta^3}\right) = -n\left(\frac{r}{\theta^2} - \frac{2r\theta}{\theta^3}\right) = \frac{rn}{\theta^2}.$$

Inverting this yields the Cramér-Rao bound,

$$\boxed{\left\{-nE\left[\frac{\partial^2}{\partial\theta^2}\ln f_Y(Y;\theta)\right]\right\}^{-1} = \frac{\theta^2}{rn}.}$$

It is seen that the given estimator achieves the Cramér-Rao bound, so it is a minimum-variance estimator.