Small-Sample Statistics

The F distribution

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Outline

The pdf of a quotient I

- **Lem.:** Let X and Y have pdfs $f_X(x)$ and $f_Y(y)$, respectively. Assume that $X = 0$ for at most a set of isolated points. Let $W = Y/X$. Then

$$f_W(w) = \int_{-\infty}^{+\infty} dx\ |x|\, f_X(x) f_Y(wx).$$

- **Pf.:** First compute the cdf:

$$\begin{aligned}
F_W(w) &= P(Y/X \le w) \\
&= P(Y/X \le w,\, X \ge 0) + P(Y/X \le w,\, X < 0) \\
&= P(Y \le wX,\, X \ge 0) + P(Y \ge wX,\, X < 0) \\
&= P(Y \le wX,\, X \ge 0) + 1 - P(Y \le wX,\, X < 0) \\
&= \int_0^{\infty} dx \int_{-\infty}^{wx} dy\ f_X(x) f_Y(y) + 1 - \int_{-\infty}^{0} dx \int_{-\infty}^{wx} dy\ f_X(x) f_Y(y)
\end{aligned}$$

The pdf of a quotient II

- **Pf. (continued):** We have

$$F_W(w) = \int_0^\infty dx \int_{-\infty}^{wx} dy\ f_X(x) f_Y(y) + 1 - \int_{-\infty}^0 dx \int_{-\infty}^{wx} dy\ f_X(x) f_Y(y)$$

- Differentiate with respect to w

$$f_W(w) = \int_0^\infty dx\ x\ f_X(x) f_Y(wx) + \int_{-\infty}^0 dx\ (-x)\ f_X(x) f_Y(wx)$$

- So we finally have that which was to be shown

$$\boxed{f_W(w) = \int_{-\infty}^{+\infty} dx\ |x|\ f_X(x) f_Y(wx)} \qquad \square$$

Definition of the F distribution

- **Def.:** Suppose that U and V are independent chi squared r.v.s with n and m degrees of freedom, respectively. A random variable of the form $\frac{V/m}{U/n}$ is said to have an *F distribution with m and n degrees of freedom*.
- **Thm.:** Suppose $F_{m,n} = \frac{V/m}{U/n}$ denotes an F r.v. with m and n degrees of freedom. The pdf of $F_{m,n}$ has the form

$$f_{F_{m,n}}(w) = \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2} w^{(m/2)-1}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)(n+mw)^{(m+n)/2}} = \frac{m^{m/2} n^{n/2} w^{(m/2)-1}}{B\left(\frac{m}{2},\frac{n}{2}\right)(n+mw)^{(m+n)/2}}$$

for $w \geq 0$.

Proof of the form of the F distribution I

- **Thm.:** Pdf of r.v. $F_{m,n} = \frac{V/m}{U/n}$ is $f_{F_{m,n}}(w)$.
- **Pf.:** U and V are χ^2 distributed with n and m df, resp.

$$f_V(v) = \frac{1}{2^{m/2}\Gamma(m/2)} v^{(m/2)-1} e^{-v/2} \text{ and } f_U(u) = \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2}$$

- Find pdf of $\frac{V}{U}$, using theorem for pdf of quotient

$$\begin{aligned} f_{V/U}(w) &= \int_0^\infty du\, |u|\, f_U(u) f_V(wu) \\ &= \int_0^\infty du\, u\, \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2} \frac{1}{2^{m/2}\Gamma(m/2)} (wu)^{(m/2)-1} e^{-wu/2} \end{aligned}$$

Proof of the form of the F distribution II

- Continuing, we have

$$f_{V/U}(w) = \frac{w^{(m/2)-1}}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} \int_0^\infty du\, u^{\frac{n+m}{2}-1} e^{-[(w+1)/2]u}$$

$$= \frac{w^{(m/2)-1}\left(\frac{w+1}{2}\right)^{-(n+m)/2}}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} \int_0^\infty dz\, z^{\frac{n+m}{2}-1} e^{-z}$$

$$= \frac{w^{(m/2)-1}(w+1)^{-(n+m)/2}}{\Gamma(n/2)\Gamma(m/2)} \int_0^\infty dz\, z^{\frac{n+m}{2}-1} e^{-z}$$

$$= \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma(n/2)\Gamma(m/2)} \frac{w^{(m/2)-1}}{(w+1)^{(n+m)/2}}$$

Proof of the form of the F distribution II—

- We have

$$f_{V/U}(w) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma(n/2)\Gamma(m/2)} \frac{w^{(m/2)-1}}{(w+1)^{(n+m)/2}}$$

- Finally, include scaling to obtain

$$f_{\frac{V/m}{U/n}}(w) = f_{\frac{n}{m}\frac{V}{U}}(w) = \frac{1}{n/m} f_{V/U}\left(\frac{w}{n/m}\right) = \frac{m}{n} f_{V/U}\left(\frac{m}{n}w\right)$$

$$= \frac{m}{n} \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma(n/2)\Gamma(m/2)} \frac{\left(\frac{mw}{n}\right)^{(m/2)-1}}{\left(\frac{mw}{n}+1\right)^{(n+m)/2}}$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2} w^{(m/2)-1}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)(n+mw)^{(m+n)/2}} \quad \square$$

Plotting the F distribution

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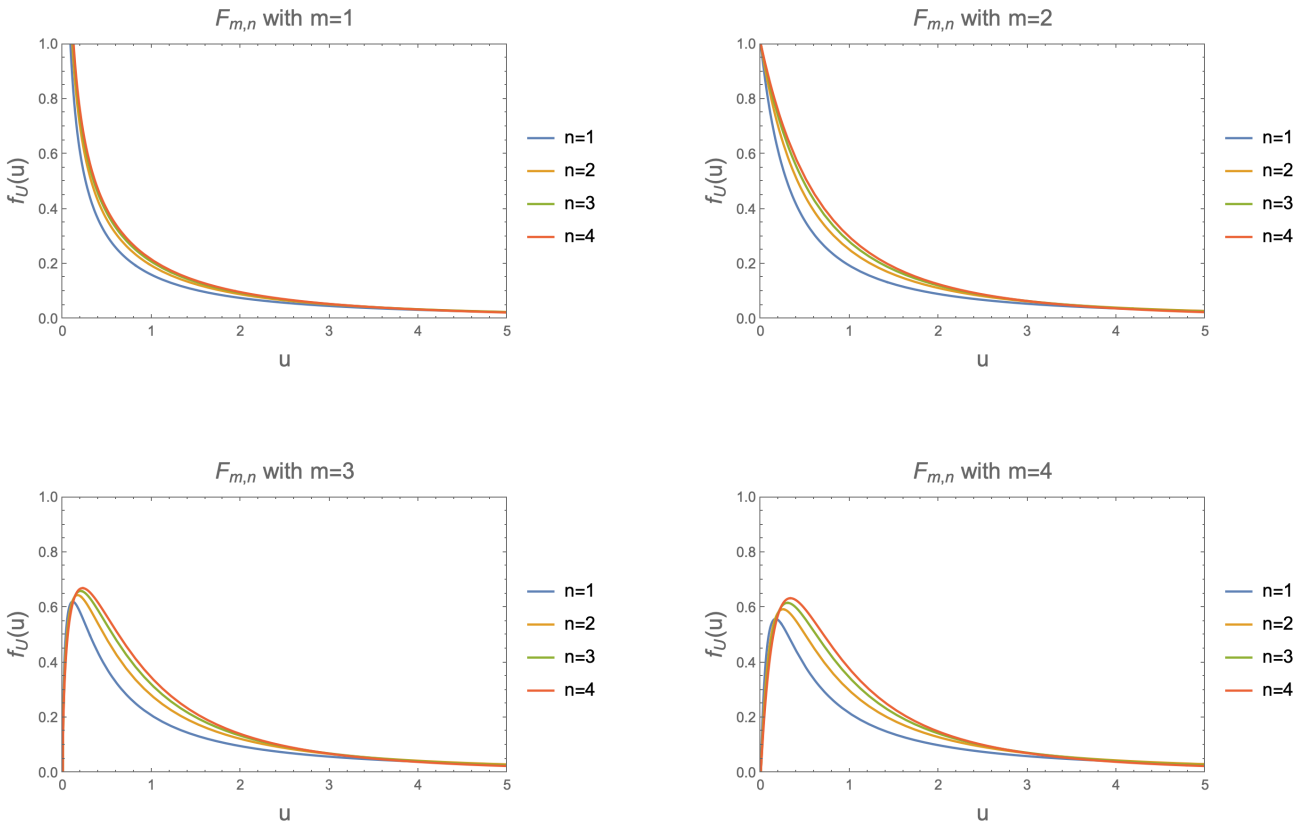

Tables in the appendices

- You are now in a position to understand yet another table in the back of the book.
- Table A.1 tabulates Z distributions for various α.
- Table A.3 tabulates χ^2 distributions for various α and n df.
- Table A.4 tabulates F distributions for various α and m and n df.

Summary

- We have derived Fisher's F distribution.
- It will be important in the derivation of the T distribution for small-sample statistics.
- It will also be important in its own right for other kinds of testing.