Math 135 **Real Analysis I** **Fall 2022**
Professors Hasselblatt and Tu **Final Exam Review**[1]

The final examination will be held in **Braker 001** on **Thursday, December 15** from **8:30 to 10:30 a.m.** There will be a review session in JCC 265 on Tuesday afternoon, December 13, from 1:30 p.m. to 2:45 p.m.. The solutions to the final review problems will be posted Saturday evening, December 10.

The final exam will cover all topics taught in the course, but about 70% of the exam will pertain to the material covered after Exam 2—path-connectedness, connectedness, and metric spaces. The 70% figure may be deceptive, since metric spaces generalize earlier material about $\mathbb{R}^n$ so that the earlier material is also heavily represented.

The exam will include True/False questions, definitions, statements of theorems, and proofs and examples in which you justify your work. The proofs and examples will be from those discussed in class, or from the homework or these review problems, or they will be similar to one of these.

Justify every statement you make by either referring to theorems or definitions. On the test, you will get some credit in problems that you cannot finish if you quote relevant definitions and theorems.

What You Need to Know for the Final

I. Review Problem Sets 10 and 11.

II. Topics covered after Exam 2: Fitzpatrick, 11.3, 11.4, 12.1, 12.2, 12.4, 12.5.

(a) *Definitions*:

- the line segment $\{t\mathbf{u}+(1-t)\mathbf{v} \,|\, 0 \leq t \leq 1\}$ joining points $\mathbf{u}$ and $\mathbf{v}$, convex set (p. 305), parametrized path $\gamma\colon [a,b] \to \mathbb{R}^n$ and path joining points $\mathbf{u}$ and $\mathbf{v}$ (p. 305), pathwise connected set (p. 305), the intermediate-value property (p. 309), separation of a set A by open sets $\mathscr{U}$ and $\mathscr{V}$ and disconnected sets (p. 310), connected sets (p. 310).
- Metric space, $C([a,b],\mathbb{R})$, metric subspace, discrete metric, open balls, open and closed sets in a metric space, interior, boundary, and closure, convergence of a sequence $\{p_k\}$ in a metric space.
- Cauchy sequence in a metric space, complete metric space, Lipschitz and contraction mappings, fixed point, contraction mapping principle.
- Continuity of a mapping $f\colon A \to Y$ (where A is a subset of the metric space X): at a point $p \in A$ via the definition (which uses convergent sequences) , and on the set A.
- Sequentially compact set.
- Separation by open sets, connected set, bounded set, relatively open and relatively closed sets.

(b) *Theorems:* (know the proofs of theorems with a *)

- The image of a pathwise connected set under a continuous map is pathwise connected (Theorems 11.34 and 11.35)*, a set $A \subset \mathbb{R}^n$ is connected iff A satisfies the intermediate-value property (Theorem 11.36), every pathwise connected set A is connected (Corollary 11.37), a subset I of $\mathbb{R}$ is connected *iff* it is an interval (Theorem 11.31 and the intermediate-value property, the intermediate-value theorem for path-connected sets (Theorem 11.35) and for connected sets (follows from Theorem 11.36).

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a set A is disconnected if and only if it has a nonempty, proper subset that is both relatively open and relatively closed.

- The set $C([a,b],\mathbb{R})$ is a metric space*.
- Open balls in a metric space are open (12.8)*.
- Complementing characterization of open an closed sets (12.12)*.
- Finite intersections and arbitrary unions of open sets are open (12.13)*, finite unions and arbitrary intersections of closed sets are closed (12.14)*.
- Convergent sequences in a metric space are Cauchy (12.15)*.
- Convergence of a sequence $\{f_k\}$ in the metric space $C([a,b],\mathbb{R})$ is equivalent to uniform convergence of $\{f_k\}$ on $[a,b]$*.
- Cauchy in $C([a,b],\mathbb{R})$ is equivalent to uniformly Cauchy on $[a,b]$.
- $C([a,b],\mathbb{R})$ is complete (12.18)
- Subsets of complete metric spaces are closed iff they are complete (12.19)*.
- The contraction mapping principle (12.23) (parts of the proof including how to show the successive approximation sequence is Cauchy and why the limit is a fixed point of the contraction mapping)
- A map $f\colon X \to Y$ is continuous at p *iff* f satisfies the ε-δ criterion at p (12.34)*.
- A map $f\colon X \to Y$ is continuous on X *iff* $f^{-1}(V)$ is *open* in X for every *open* set V in Y (this is Theorem 12.35).
- A map $f\colon X \to Y$ is continuous on X *iff* $f^{-1}(C)$ is *closed* in X for every *closed* set C in Y (this follows from Theorem 12.35).
- If A is sequentially compact and $f\colon A \to Y$ is continuous on A then f is uniformly continuous on A.
- The equivalence of relatively open in A and open in the metric subspace A and equivalence of relatively closed in A and closed in the metric subspace A.
- The image of a sequentially compact set under a continuous map is sequentially compact (12.36)*.
- The extreme value theorem (12.37)*.
- The intermediate value theorem (12.39)*. Theorem 12.39 states that if (X,d) is a metric space and $A \subset X$, then A is connected if and only if it has the intermediate value property (if $f : A \to \mathbb{R}$ is continuous then the image $f(A)$ is an interval). This allows one to prove that the image of any connected set under a continuous function is connected.

(c) *Examples and Additional Concepts:*

- A sequentially compact set A in a metric space X is closed and bounded, but a closed and bounded set is not in general sequentially compact. (*Example:* if X is an infinite set with the discrete metric, then $A = X$ is closed and bounded but not sequentially compact.)
- If A is a subset of a metric space X, then $\mathrm{cl}(A)$ is the smallest closed set containing A; a point x belongs to $\mathrm{cl}(A)$ *iff* every ball $B_r(x)$ contains a point of A.
- A subset A of a metric space X is disconnected *iff* it has a proper, nonempty subset that is both relatively open and relatively closed in A.

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In the following problems, unless otherwise specified, (X,d) and (Y,ρ) will denote metric spaces and A, B, etc. will denote subsets of X.

1. (a) Define what it means for a point $p \in X$ to be a *boundary point* of A (i.e., $p \in \mathrm{bd}(A)$).
   (b) Define the *closure* $\mathrm{cl}(A)$ of the set A.
   (c) Prove that a point $p \in X$ belongs to $\mathrm{cl}(A)$ if and only if every ball $B_r(p)$ contains a point of A.

2. (a) Define what it means for $A \subset X$ to be *sequentially compact*.
   (b) Define what it means for $A \subset X$ to be *complete*.
   (c) Show that every sequentially compact set is complete.

3. (a) Define what it means for a subset B of A to be *relatively open* in A. Then define what it means for a subset C of A to be *relatively closed* in A.
   (b) Prove that a subset C of A is relatively closed in A if and only if $A \setminus C$ is relatively open in A.
   (c) Prove that A is disconnected if and only if A has a proper, nonempty subset B that is both relatively open and relatively closed in A.
   ◦ Recall that a subset B of A is relatively open in A if and only if B is open in the metric subspace A. Also, recall that a subset C of A is relatively closed in A *iff* C is a closed set in the metric subspace A.

4. (a) Define what it means for a mapping $f\colon A \to Y$ to be *continuous* at a point $p_0 \in A$.
   (b) Define what it means for a mapping $f\colon A \to Y$ to be *uniformly continuous*.
   NOTE: The book does not define uniform continuity in metric spaces, but guess the definition using the definition in Euclidean space. It will help to recall that X has metric d and Y has metric ρ.
   (c) Prove that the function $f(x) = 1/(x^2+1)$ is continuous on $\mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

5. Let X and Y be metric spaces and $A \subset X$. Let $f\colon A \to Y$ be continuous. Let $B \subset A$ be connected and open, and let $K \subset A$ be sequentially compact. Let $\mathscr{B} \subset Y$ be connected and open, and let $\mathscr{K} \subset Y$ be sequentially compact. What can you say about the following sets?
   (a) $f(A)$ (b) $f(B)$ (c) $f(K)$
   (d) $f^{-1}(\mathscr{K})$ (e) $f^{-1}(\mathscr{B})$ (f) $f^{-1}(f(B))$
   (When a statement doesn't follow directly from a theorem, try examples with specific functions.)

6. For each $k \in \mathbb{N}$, let $f_k(x) = \dfrac{x}{x+k}$, $f_k\colon [1,\infty) \to \mathbb{R}$.
   (a) Prove $\{f_k\}$ converges pointwise to $f(x) = 0$ on $[1,\infty)$ using the definition of pointwise convergence and the definition of limits of sequences in $\mathbb{R}$.
   (b) Prove $\{f_k\}$ converges uniformly to $f(x) = 0$ on $[1,2]$ using the definition of uniform convergence.
   (c) Prove $\{f_k\}$ does not converge uniformly to $f(x) = 0$ on $[1,\infty)$.

7. Let $f_k(x) = 1/(kx+1)$ and $g_k(x) = x/(kx+1)$ on $[0,1]$.
   (a) Find the pointwise limit of the sequence $\{f_k\}$ on $[0,1]$.
   (b) Find the pointwise limit of the sequence $\{g_k\}$ on $[0,1]$.

(c) Is $\{f_k\}$ Cauchy in $C([0,1],\mathbb{R})$?
(d) Is $\{g_k\}$ Cauchy in $C([0,1],\mathbb{R})$?

8. Give an example of a sequence of continuous functions $\{f_k\}$ that converges pointwise to a function that is not continuous. Why doesn't this contradict Theorem 9.31?

9. Let $A \subset \mathbb{R}^n$ be connected and nonempty and let $f\colon A \to \mathbb{R}$ be continuous. Assume $f(x) \neq 0$ for all $x \in A$. Prove either that $f(x) > 0$ for all $x \in A$ or that $f(x) < 0$ for all $x \in A$

10. Recall that $C([a,b],\mathbb{R})$ denotes the vector space of all continuous functions $f\colon [a,b] \to \mathbb{R}$.
   (a) Prove that the formula
   $$d(f,g) = \sup\left\{|f(x)-g(x)| \,\middle|\, x \in [a,b]\right\}$$
   defines a metric on $C([a,b],\mathbb{R})$.
   (b) Let
   $$U = \{f \in C([a,b],\mathbb{R}) \mid f(x) > 0 \text{ for all } x \in [a,b]\}$$
   Is U open in $C([a,b],\mathbb{R})$? Why or why not?
   (c) Describe the compliment of U in (b) in $C([a,b],\mathbb{R})$.
   (d) Let
   $$T = \{f \in C([a,b],\mathbb{R}) \mid f(x) \geq 0 \text{ for all } x \in [a,b]\}$$
   Is T closed in $C([a,b],\mathbb{R})$? Why or why not?
   (e) Prove that a sequence $\{f_k\}$ in $C([a,b],\mathbb{R})$ converges to a point $f \in C([a,b],\mathbb{R})$ if and only if the sequence of functions $\{f_k\}$ converges uniformly to f on $[a,b]$.

11. Use the facts that $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) = y$ are both continuous to prove that the square $\{(x,y) \in \mathbb{R}^2 \,|\, -1 < x < 1,\ -1 < y < 1\} = (-1,1)^2$ is open in $\mathbb{R}^2$.

12. Prove that the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k$ converges uniformly on $[-r,r]$ for any $r > 0$. Then prove that the limit function $f(x)$ is continuous on $(-\infty,\infty)$.

13. Let $K \subset \mathbb{R}^n$ be closed and bounded. Prove that K is sequentially compact.

14. (a) Let K be a sequentially compact subset of a metric space X. Prove that K is closed and bounded.
   (b) Is the converse to Part (a) above true? That is, is any closed and bounded subset of any metric space X sequentially compact? Justify your answer by giving either a proof or a counterexample.

15. Let X and Y be metric spaces and let $f : X \to Y$ be continuous. Let K be a sequentially compact subset of X. Prove that the image $f(K)$ is sequentially compact.

16. Let X be a metric space and let $K \subset X$ be sequentially compact. Assume $f\colon K \to \mathbb{R}$ is continuous. State and prove the extreme value theorem for f and K.

17. Let X be a metric space and let $A \subset X$ be connected. Assume $f\colon A \to \mathbb{R}$ is continuous. State and prove the intermediate value theorem for f and A.

18. Let A and B be closed subsets of $\mathbb{R}$. Prove that $A \times B = \left\{(a,b) \in \mathbb{R}^2 \,\middle|\, a \in A,\ b \in B\right\}$ is closed using the definition of closed set.

(End of Final Review Problems)