Due date: 11:59 pm, Friday, April 14, 2023 on Gradescope.

You are encouraged to work on problems with other Math 136 students and to talk with Todd and our TA Wentao, but your answers should be in your own words.

A proper subset of the problems will be selected for grading.

Reading assignment for the week of April 3:

- Please read section 19.1 in Fitzpatrick (Fubini's Theorem in $\mathbb{R}^2$).
- Please read sections 8.2 and 10.1.and proofs in *Elementary Classical Analysis* Marsden-Hoffman. You won't be responsible for the proof of Lebesgue's Theorem.

  You will find Marsden-Hoffman, section 8.2 in Canvas in *Files/Marsden-Hoffman/Ch 8: Integration and Lebesgue's Theorem*. Chapter 10 is also in *Files/Marsden-Hoffman*.

Here are some useful theorems and definitions. Refer to them by number if you use them.

Theorem 1 (Lebesgue's Theorem) *Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let f be a bounded function from $\mathbb{I}$ to $\mathbb{R}$. Then, f is integrable if and only if the set of discontinuities of f, $D(f,\mathbb{I}) = \{\mathbf{x} \in \mathbb{I} \mid f \text{ is discontinuous at } \mathbf{x}\}$ has measure zero.*

Definition 2 *Let $A \subset \mathbb{R}^n$ and let f and g be functions from A to $\mathbb{R}$. Then, $f = g$* almost everywhere, *$f = g$ a.e., if $\{\mathbf{x} \in A \mid f(x) \neq g(x)\}$ has measure zero.*

Theorem 3 *Let A be a bounded subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded integrable function. If A has measure zero then $\int_A f = 0$.*

Theorem 4 *Let A be a bounded subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded integrable function.*

(a) If $f = 0$ a.e. on A, then $\int_A f = 0$.

(b) If $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in A$, then $\int_A f = 0$ if and only if $f = 0$ a.e. on A.

Problems:

1. (20 points) Let $\mathbb{I} = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$ and let $h : \mathbb{I} \to \mathbb{R}$ be a continuous function. For each $y \in [c,d]$ define
$$g(y) = \int_a^b h(x,y)\,dx.$$
Prove that the function $g : [c,d] \to \mathbb{R}$ is continuous.

   HINT: First, note that h is uniformly continuous. You may assume that this implies that for each $\epsilon > 0$ there is a $\delta > 0$ such that if y and y_0 are points in $[c,d]$ and $|y - y_0| < \delta$, then for all $x \in [a,b]$, $|h(x,y) - h(x,y_0)| < \epsilon$. Use this to bound $|g(y) - g(y_0)|$ above.

2. (15 points) Show that a.e. is an equivalence relation. That is, if $A \subset \mathbb{R}^n$ and f, g, and h are functions from A to $\mathbb{R}$ (or $\mathbb{C}$), then

   (a) $f = f$ a.e.; (b) if $f = g$ a.e. then $g = f$ a.e.; (c) if $f = g$ a.e. and $g = h$ a.e. then $f = h$ a.e.

HW continues on the next page.

The next problems are about the set $\mathcal{L}^2([a,b],\mathbb{R})$ which is the set of bounded integrable functions from $[a,b]$ to $\mathbb{R}$ with the inner product $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$.

This is a warm-up for $\mathcal{L}^2([a,b],\mathbb{C})$, which we will consider in class.

3. (20 points) Let f and g be functions in $\mathcal{L}^2([a,b],\mathbb{R})$. We will let fg be the product function $fg : [a,b] \to \mathbb{C}$ defined by $fg(x) = f(x)g(x)$.

   (a) Prove that $D(fg,[a,b]) \subset D(f,[a,b]) \cup D(g,[a,b])$.
   HINT: You know that if f and g are both continuous at a point $x \in [a,b]$ then fg continuous at x. What is the contrapositive of this statement?

   (b) Prove that the product $fg : [a,b] \to \mathbb{R}$ is integrable (thus, $fg \in \mathcal{L}^2([a,b],\mathbb{R})$).

   (c) Explain why the inner product $\langle f,g\rangle$ is defined on $\mathcal{L}^2([a,b],\mathbb{R})$ (i.e., why is the integral defined).

   (d) Prove that $\langle f,f\rangle = 0$ if and only if $f = 0$ a.e.

4. (25 points) Let f, g, and h be functions in $\mathcal{L}^2([a,b],\mathbb{R})$ and let $c \in \mathbb{R}$.

   (a) Prove that $f+g$ and cf are in $\mathcal{L}^2([a,b],\mathbb{R})$. Since the zero function is in $\mathcal{L}^2([a,b],\mathbb{R})$, this shows that $\mathcal{L}^2([a,b],\mathbb{R})$ is a vector space (as it is a subspace of the vector all functions from $[a,b]$ to $\mathbb{R}$).

   (b) Prove that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

   (c) Show that $\langle f, g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$, $\langle f,g\rangle = \langle g,f\rangle$, and $\langle cf,g\rangle = c\langle f,g\rangle$.

   (d) Is $\mathcal{L}^2([a,b],\mathbb{R})$ an inner product space? Why or why not?

5. (20 points) Let $L^2([a,b],\mathbb{R})$ be the set $\mathcal{L}^2([a,b],\mathbb{R})$ with equality defined as $f = g$ in $L^2([a,b],\mathbb{R})$ if $f = g$ a.e.

   (a) Show that under this definition of equality, if $f = g$ in $L^2([a,b],\mathbb{R})$ then $\int_a^b f = \int_a^b g$.

   (b) Let f_1, f_2, g_1, and g_2 be functions in $\mathcal{L}^2([a,b],\mathbb{R})$. Assume $f_1 = f_2$ a.e. and $g_1 = g_2$ a.e. Prove $\langle f_1,g_1\rangle = \langle f_2,g_2\rangle$. This shows that $\langle f,g\rangle$ is well-defined in $L^2([a,b],\mathbb{R})$.

   (c) Show that under this definition, the inner product $\langle f,g\rangle = \int_a^b fg$ is positive definite (that is, $\langle f,f\rangle \geq 0$ and $\langle f,f\rangle = 0$ if and only if $f = 0$ in $L^2([a,b],\mathbb{R})$.
   HINT: What does equality in $L^2([a,b],\mathbb{R})$ mean?

Here are optional extra-credit challenge problems. Todd will grade them.

1. (2 points) Use the fact that $f : \mathbb{I} \to \mathbb{R}$ is uniformly continuous to prove the hint to problem 1 of this homework.

2. (2 points–this should be worth much more) Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \cap [0,1] \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \cap [0,1] \text{ and } \frac{m}{n} \text{ is in lowest form} \end{cases}.$$

   So, for example, $f(0.75) = \frac{1}{4}$ and $f(\sqrt{2}/2) = 0$.

   (a) Show that $D(f,[0,1]) = \mathbb{Q} \cap [0,1]$

   (b) Show that f is integrable on $[0,1]$.