MATH 125 Lecture 19 Special systems

Recall important facts from last lecture:

LU decomposition theorem

Let $A \in \mathbb{R}^{m \times m}$ be a non-singular matrix. Carry out Gaussian elimination with no pivoting (and assume no division by zero occurs. The algorithm results L and U. The decomposition $A = LU$ is unique

Remark The constraint $l_{ii} = 1$ is important otherwise $A = LD^{-1}DU$ where D = invertible diagonal matrix

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PLU decomposition

ith canonical basis vector $e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ i^{th} position

Definition An $n \times n$ matrix P is called a permutation matrix if it is of the form

$$P = [e_{i_1} \ e_{i_2} \ e_{i_3} \ \dots \ e_{i_n}]$$

for some permutation $i_1, i_2, \dots, i_n$ of $1, 2, \dots, n$

Example $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $n = 4$, $(i_1, i_2, i_3, i_4) = (3, 1, 2, 4)$

$A \in \mathbb{R}^{n \times n}$

$$\begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} | & | & & | \\ e_{i_1} & e_{i_2} & \cdots & e_{i_n} \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_n} \end{pmatrix}$$

Theorem $P^T = P^{-1}$

proof $(PP^T)_{ij} = \sum_{k=1}^{n} P_{ik} P^T_{kj} = \sum_{k=1}^{n} P_{ik} P_{jk} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

$\therefore PP^T = I \Rightarrow P^T = P^{-1}$

What is the effect of PA?
(permutes rows)

first row $\longrightarrow$ i_1-st row

second row $\longrightarrow$ i_2-st row

Example

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

partial pivoting

Step 1 Interchange first and third row

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{P_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

Eliminate x_1

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/4 & 0 & 1 & 0 \\ -3/4 & 0 & 0 & 1 \end{pmatrix}}_{L_1} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{pmatrix} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & -1/2 & -3/2 & -3/2 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & 7/4 & 9/4 & 17/4 \end{pmatrix}$$

Step 2 Interchange second and fourth row

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{P_2} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & -1/2 & -3/2 & -3/2 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & 7/4 & 9/4 & 17/4 \end{pmatrix} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & -1/2 & -3/2 & -3/2 \end{pmatrix}$$

Eliminate x_2

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3/7 & 1 & 0 \\ 0 & 2/7 & 0 & 1 \end{pmatrix}}_{L_2} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & -1/2 & -3/2 & -3/2 \end{pmatrix} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -2/7 & 4/7 \\ 0 & 0 & -6/7 & -2/7 \end{pmatrix}$$

Step 3 Interchange third and fourth rows

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{P_3} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -2/7 & 4/7 \\ 0 & 0 & -6/7 & -2/7 \end{pmatrix} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -6/7 & -2/7 \\ 0 & 0 & -2/7 & 4/7 \end{pmatrix}$$

Eliminate x_3

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/3 & 1 \end{pmatrix}}_{L_3} \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -6/7 & -2/7 \\ 0 & 0 & -2/7 & 4/7 \end{pmatrix} = \begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -6/7 & -2/7 \\ 0 & 0 & 0 & 2/3 \end{pmatrix}$$

claim

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & -2/7 & 1 & 0 \\ 1/4 & -3/7 & 1/3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -6/7 & -2/7 \\ 0 & 0 & 0 & 2/3 \end{pmatrix}}_{U}$$

How ?

Recall our elimination process

$$L_3 P_3 L_2 P_2 L_1 P_1 A = U$$

Another claim $\quad L_3 P_3 L_2 P_2 L_1 P_1 = L_3' L_2' L_1' P_3 P_2 P_1$

$$L_3 P_3 L_2 P_2 L_1 P_1$$
$$(L_3)(P_3 L_2 P_3^{-1}) P_3 P_2 L_1 P_1$$
$$(L_3)\underbrace{(P_3 L_2 P_3^{-1})}_{\hat{L}_2}\underbrace{(P_3 P_2 L_1 P_2^{-1} P_3^{-1})}_{\hat{L}_1} P_3 P_2 P_1$$

$\hat{L}_k$ is equal to L_k with subdiagonal entries permuted.

Using claim, $\quad U = L_3' L_2' L_1' \underbrace{P_3 P_2 P_1}_{P} A$

$$(L_3' L_2' L_1')^{-1} U = PA$$

$\Downarrow$ PLU decomposition

<u>Theorem</u> Let $A \in \mathbb{R}^{m \times m}$ be an invertible matrix. Then there exist a permutation matrix P, a lower triangular matrix L with 1's on the main diagonal and a right upper triangular matrix such that

$$PA = LU$$

<u>How to solve $Ax = b$</u>

Multiply both sides by PA

$$PAx = Pb$$
$$LUx = Pb$$

Let $y = Ux$ $\quad Ly = Pb \Rightarrow$ Find y

Then solve for x from $Ux = y$

(Quite the same computational complexity as Gaussian elimination)