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The sign test and the Wilcoxon sign test

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Motivation

- Virtually all of the tests that we have used to date require some a priori knowledge of the form of the probability distribution of the data.
- Most also assume that the data is independent and identically distributed (iid).
  - Example: iid Bernoulli random numbers
  - Example: iid Normally distributed random numbers
- If W is the test statistic with actual pdf $f_W(w \mid H_0)$ when H_0 is true, then

$$\alpha' = P(W \in C) = \int_C dw\ f_W(w \mid H_0)$$

will not be equal to the desired α, because a different pdf was used to make the estimate of region C.

Robustness

- One solution we have found for this problem is to find *robust* methods of statistical analysis.
- For large n, we have invoked the CLT to find and test Z statistics, assuming only finiteness of the variance of the underlying distribution.
- Even for smaller n, it has been noted that T tests are also reasonably robust.

Nonparametric statistics

- The above raises the question of whether or not it is possible to find statistical tests that are truly independent of the form of the underlying distribution.
- Such *nonparametric statistics* are an active area of statistical research.
- In this module, we begin examination of some of the more general methods of nonparametric statistical research.

The sign test: Estimating the median

- Most of our tests to date have centered on estimating the mean μ, or testing the hypothesis $\mu = \mu_0$.
- Suppose instead we wish to devise a test about the median $\tilde{\mu}$, defined so

$$P(W < \tilde{\mu}) = \int_{-\infty}^{\tilde{\mu}} dw\ f_W(w) = \int_{\tilde{\mu}}^{+\infty} dw\ f_W(w) = P(W > \tilde{\mu})$$

- If the null hypothesis $H_0 :\ \tilde{\mu} = \tilde{\mu}_0$ is true, then no matter what the underlying distribution, we have

$$P\left(Y \le \tilde{\mu}_0\right) = P\left(Y \le \tilde{\mu}_0\right) = {}^1\!/_2.$$

The sign test: Estimating the median

- So, if H_0 is true and we take n samples, then the number of observations X exceeding μ_0 should be binomially distributed with $p = {}^1\!/_2$, and hence

$$E(X) = \frac{n}{2}$$
$$\mathrm{Var}(x) = \frac{1}{2}\left(1 - \frac{1}{2}\right) n = \frac{n}{4}$$

- Hence we would expect the statistic $\frac{X - n/2}{\sqrt{n/4}}$ to have approximately a standard normal distribution by the CLT (Laplace-DeMoivre Theorem), if n sufficiently large.
- Values of X much smaller or larger than $n/2$ would be evidence that $\tilde{\mu} \neq \tilde{\mu}_0$.

Hypothesis testing for the median

- Let $y_1, \ldots, y_n$ be a random sample of size n from any continuous distribution having median $\tilde{\mu}$, where $n \geq 10$. Let k denote the number of y_i's greater than $\tilde{\mu}_0$, and let $z = \frac{k - n/2}{\sqrt{n/4}}$. Then
  - To test $H_0: \ \tilde{\mu} = \tilde{\mu}_0$ versus $H_1: \ \tilde{\mu} > \tilde{\mu}_0$, at the αth level of significance, reject H_0 if $z \geq +z_\alpha$.
  - To test $H_0: \ \tilde{\mu} = \tilde{\mu}_0$ versus $H_1: \ \tilde{\mu} < \tilde{\mu}_0$, at the αth level of significance, reject H_0 if $z \leq -z_\alpha$.
  - To test $H_0: \ \tilde{\mu} = \tilde{\mu}_0$ versus $H_1: \ \tilde{\mu} \neq \tilde{\mu}_0$, at the αth level of significance, reject H_0 if either $z \leq -z_{\alpha/2}$ or $z \geq +z_{\alpha/2}$.

Example of hypothesis testing for the median

- Alice says the median number of birds that visit her feeder per day is 30.
- Bob says it is greater than 30.
- Bob collected data over the course of 10 days

Day	1	2	3	4	5	6	7	8	9	10
Birds	27	31	33	37	32	34	21	36	37	35

- Null hypothesis is to believe Alice $H_0: \ \tilde{\mu} = \tilde{\mu}_0 = 30$, so

Day	1	2	3	4	5	6	7	8	9	10
Sign	-	+	+	+	+	+	-	+	+	+

- There are 8 $+$ values and 2 $-$ values

$$z = \frac{8 - 10/2}{\sqrt{10/4}} = 1.897$$

- If $\alpha = 0.05$, then $z > z_\alpha = 1.645$, so we reject H_0.

Small-sample version of the example

- Tabulate the binomial distribution for $n = 10$, $p = 1/2$

j	$\binom{10}{j}\left(\frac{1}{2}\right)^{10}$
0	0.000976563
1	0.00976563
2	0.0439453
3	0.117188
4	0.205078
5	0.246094
6	0.205078
7	0.117188
8	0.0439453
9	0.00976563
10	0.000976563

- Note that $P(X \geq 8) = \sum_{j=8}^{10} \binom{10}{j}\left(\frac{1}{2}\right)^{10} = 0.5469$
- This would indicate that you have failed to reject H_0 at confidence level $\alpha = 0.05$.

Sign test for paired data

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Motivation

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Summary

- Suppose you have two streams of data, x_j and y_j, and you wish to examine pairs (x_j, y_j) for $j = 1, \ldots, n$.
- Let $p = P(X_j > Y_j)$ for $j = 1, \ldots, n$.
- Null hypothesis is that the two streams represent distributions with the same median, i.e., $H_0 : \ p = 1/2$.
- Let $W_j = \begin{cases} 0 & \text{if } X_j \leq Y_j \\ 1 & \text{if } X_j > Y_j \end{cases}$
- Measure $u = \sum j = 1^n w_j$ and compute statistic $\frac{u - n/2}{\sqrt{n/4}}$.
- If n sufficiently large, test this against z_α.

More on sign test for paired data

- We invoked the CLT above, so it may seem we are exploiting robustness, as in earlier Z tests, but note:
  - The distributions of X_i and Y_i do not need to be the same.
  - In fact, the distributions of X_i and X_j for $i \neq j$ do not need to be the same (and likewise for Y_i and Y_j).
  - None of these distributions needs to be symmetric.
  - All of these distributions could have different variances.
- The only essential requirements are that
  - X and Y have continuous pdfs (for continuous r.v.s).
  - The null hypothesis adds the requirement that $\tilde{\mu}_{X_i} = \tilde{\mu}_{Y_i}$ within each pair – i.e., for $i = 1, \ldots, n$.

From the median to the mean

- For a symmetric distribution, median and mean are equal.
- For a skewed distribution, they tend to be different.
- Example: Household wealth in the United States has...
  - a median of just under \$100,000.
  - a mean of about \$750,000.
- Fact that the mean is so much greater than the median is indicative of very wealthy households at the very top.
- In fact, the 400 wealthiest households in the US have as much wealth as the bottom 60% of the population – about 300,000,000 people – namely around \$3.5 trillion.

From the median to the mean

- If Elon Musk walked into our classroom,
  - the mean wealth would increase enormously.
  - the median wealth would hardly change.
- The median "cares" only about the number of households above and below its value:

$$\sum_{j=1}^{n} \mathrm{sgn}\left(X_i - \tilde{\mu}_X\right) = 0$$

- The mean "cares" about *by how much* those households are above or below its value:

$$\sum_{j=1}^{n} \left(X_i - \mu_X\right) = 0$$

Wilcoxon tests

- The only information retained by the sign test was whether data was greater than or less than the median.
- The amount by which the data was greater than or less than the median did not matter.
- The above approach worked for the median, but it will not work for the mean.
- If the statistic depended on the magnitude of the deviation from the median, on the other hand, it would require knowledge of the underlying distribution of the data.
- We can, however, allow the statistic to depend on the *rank* of the magnitude of the deviation from the median, in order to give more weight to higher deviations.

Example of Wilcoxon testing approach

- Suppose $n = 3$ and data is $y_1 = 6.0$, $y_2 = 4.9$, $y_3 = 11.2$.
- Objective is to test $H_0 : \mu = 10.0$ against $H_1 : \mu \neq 10.0$.
- Ranks are:

$$\begin{aligned} |y_1 - \mu_0| &= 4.0 & r_1 &= 2 \\ |y_2 - \mu_0| &= 5.1 & r_2 &= 3 \\ |y_3 - \mu_0| &= 1.2 & r_3 &= 1. \end{aligned}$$

- Signs of deviations from the mean are captured by

$$\begin{aligned} z_1 &= \frac{1}{2}\left(1 + \mathrm{sgn}(y_1 - \mu_0)\right) = 0 \\ z_2 &= \frac{1}{2}\left(1 + \mathrm{sgn}(y_2 - \mu_0)\right) = 0 \\ z_3 &= \frac{1}{2}\left(1 + \mathrm{sgn}(y_3 - \mu_0)\right) = +1 \end{aligned}$$

- *Wilcoxon rank statistic*

$$w = \sum_{j=1}^{n} r_j z_j = (2)(0) + (3)(0) = (1)(+1) = 1.$$

How does one turn this into a hypothesis test?

- If the null hypothesis $H_0: \ \mu = \mu_0$ is true, there should be as many zeros as ones among the z_js.
- **Thm.:** Let $y_1, \ldots, y_n$ be independent observations drawn from the continuous and symmetric (though not necessarily identical) pdfs, $f_{Y_i}(y)$ for $i = 1, \ldots, n$. Suppose that each of the $f_{Y_i}(y)$s has the same mean μ. If $H_0: \ \mu = \mu_0$ is true, the pdf of the data's signed rank statistic, $p_W(w)$ is given by

$$p_W(w) = P(W = w) = \frac{c(w)}{2^n},$$

where $c(w)$ is the coefficient of e^{wt} in the expansion of

$$\prod_{j=1}^{n} \left(1 + e^{jt}\right).$$

How does one turn this into a hypothesis test?

- **Pf.:** If H_0 is true, the pdf of the Wilcoxon rank statistic is equivalent to that of $U = \sum_j^n U_j$, where

$$U_j = \begin{cases} 0 & \text{with probability } 1/2 \\ j & \text{with probability } 1/2 \end{cases}$$

- Hence W and U have the same moment-generating function. Since the U_js are independent,

$$\begin{aligned} M_W(t) = M_U(t) &= \prod_{j=1}^n M_{U_j}(t) = \prod_{j=1}^n E\left(e^{U_j t}\right) \\ &= \prod_{j=1}^n \left(\frac{1}{2}e^{0t} + \frac{1}{2}e^{jt}\right) = \frac{1}{2^n}\prod_{j=1}^n \left(1 + e^{jt}\right) \end{aligned}$$

How does one turn this into a hypothesis test?

- At this point, we have

$$M_W(t) = \frac{1}{2^n}\prod_{j=1}^{n}\left(1+e^{jt}\right) = \frac{1}{2^n}\sum_{w=0}^{n(n+1)/2} c(w)e^{wt}.$$

since $1+2+\cdots+n = n(n+1)/2$.

- Next, by definition of generating functions,

$$M_W(t) = E\left(e^{Wt}\right) = \sum_{w=0}^{n(n+1)/2} p_W(w)e^{wt}$$

- Comparing the above, we see the desired result

$$p_W(w) = \frac{c(w)}{2^n}. \quad \square$$

Example of Wilcoxon test

- Suppose that $n = 4$ so $2^n = 16$ and $n(n+1)/2 = 10$, so

$$M_W(t) = \left(\frac{1+e^t}{2}\right)\left(\frac{1+e^{2t}}{2}\right)\left(\frac{1+e^{3t}}{2}\right)\left(\frac{1+e^{4t}}{2}\right)$$
$$= \frac{1}{16}\left(1 + e^t + e^{2t} + 2e^{3t} + 2e^{4t} + 2e^{5t} + 2e^{6t} + 2e^{7t} + e^{8t} + e^{9t} + e^{10t}\right)$$

- Hence, e.g., $P(W = 2) = {}^1\!/16$ and $P(W = 7) = {}^2\!/16$, etc.
- Note that $c(w)$ is the number of ways of adding subsets of the numbers $\{1, 2, \ldots, n\}$ to obtain w.

Example of application of Wilcoxon test

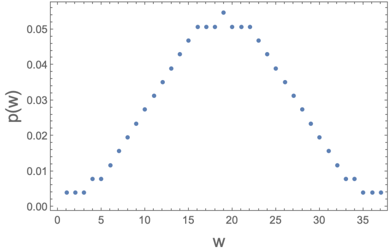
- Energy expenditures for women from heart rate (in kcal)
- During summer months and winter months
- Test difference D with $H_0: \mu = 0$ and $H_1: \mu \neq 0$

Subject	Summer, x_j	Winter, y_j	$d_j = y_j - x_j$	r_j	z_j
1	1458	1424	-34	1	0
2	1353	1501	148	5	1
3	2209	1495	-714	8	0
4	1804	1739	-65	2	0
5	1912	2031	119	4	1
6	1366	934	-432	7	0
7	1598	1401	-197	6	0
8	1406	1339	-67	3	0

- Wilcoxon rank statistic: $w = 5(1) + 4(1) = 9$

Example of application of Wilcoxon test

- Wilcoxon rank statistic: $w = 5(1) + 4(1) = 9$
- $\sum_{w=0}^{n(n+1)/2} p(w) e^{wt} = \frac{1}{2^n} \prod_{j=1}^{n} \left(1 + e^{jt}\right)$



- $\sum_{w=0}^{7} p(w) = \sum_{w=29}^{36} p(w) = \frac{19}{256} \approx 0.0742$
- Test is two-sided, so $2 \times \frac{19}{256} = \frac{19}{128} \approx 0.148$
- So for $\alpha = 0.15$, since $7 < w < 29$, we fail to reject H_0.

Example of application of Wilcoxon test

- $\sum_{w=0}^{n(n+1)/2} p(w) e^{wt} = \frac{1}{2^n} \prod_{j=1}^{n} \left(1 + e^{jt}\right)$
- Tables of cutoffs for $n = 4, \ldots, 12$ are in Larsen & Marx Appendix A, Table A.6.
- For $n = 8$ one can look up that

$$P(W \leq w_1^*) = P(W \geq w_2^*) = 0.074$$

corresponds to $w_1^* = 7$ and $w_2^* = 29$, as we calculated on the previous slide.

Summary

- We have motivated and discussed nonparametric statistics.
- We have learned about the sign test for estimations of medians $\tilde{\mu}$, and worked an example.
- We have learned about the Wilcoxon text for estimation of means μ, and worked an example.