Tuesday, September 28

1. Do problem 2.5.6 in Strogatz's book.

2. Think about an initial value problem

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0.$$

Euler's method works like this: Pick a small $\Delta t > 0$. Compute approximations

$$x_1, x_2, x_3, \ldots$$

for

$$x(t_0 + \Delta t), x(t_0 + 2\Delta t), x(t_0 + 3\Delta t), \ldots$$

from this equation:

$$\frac{x_{k+1} - x_k}{\Delta t} = f(x_k), \quad k = 0, 1, 2, \ldots.$$

(a) If your problem were

$$\frac{dx}{dt} = -x, \quad x(0) = 1,$$

what would be the exact solution? What would be the approximation x_1 for $x(\Delta t)$ computed by Euler's method? What would be x_2? Can you write a general formula for x_k?

(b) Don't assume any more that your problem is the one from part (a), but still assume that $n = 1$ for simplicity. Euler's method is based on the idea that

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \approx \frac{dx}{dt}(t)$$

when Δt is small. Approximately how large is the error in this approximation,

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} - \frac{dx}{dt}(t) \ ?$$

Your answer will have to do with the second derivative of x. (Use Taylor's theorem. If you don't remember Taylor's theorem, look it up, it's one of the most useful things you learn in Calculus II.)

(c) Explain (loosely) why part (b) makes it unsurprising that for a fixed $T > t_0$, with the property that $T - t_0 = N\Delta t$ for some positive integer N,

$$\max_{k=0,1,\ldots,N} |x_k - x(t_0 + k\Delta t)| \leq C\Delta t$$

for some constant C that depends on the solution $x(t)$ (so it depends on the function f and the number x_0), but not on Δt. In other words, the discrepancy between x_k and the true $x(t_0 + k\Delta t)$ converges to zero, as Δt tends to 0, proportionally to Δt.

3. Suppose that we apply Euler's method to the Kermack-McKendrick disease model:

$$\begin{aligned}\frac{dS}{dt} &= -\alpha IS \\ \frac{dI}{dt} &= \alpha IS - \beta I \\ \frac{dR}{dt} &= \beta I\end{aligned}$$

   How is what we get related to the "iterated map" model that the differential equations came from?