Math 135 Solutions to Problem Set 11

1. (a) because the function $|f(x) - g(x)|$ doesn't necessarily have a max eg. $f(x) = x$, $g(x) = 0$
$\sup\{|f(x) - g(x)| \mid x \in (0,1)\} = 1$
but $\max\{|f(x) - g(x)| \mid x \in (0,1)\}$ dne
as $|f(x) - g(x)| = x < 1$ $\forall x \in (0,1)$

b. holds as $\forall x \in (a,b)$ $|f(x) - g(x)| = |g(x) - f(x)|$

c. $d(f,g) \geq 0$ as $\forall x \in (a,b)$ $|f(x) - g(x)| \geq 0$
if $f = g$ then $\forall x \in (a,b)$, $|f(x) - g(x)| = 0$
so $\sup\{|f(x) - g(x)| \mid x \in (a,b)\} = 0.$
if $f \neq g$ then for some $x_0 \in (a,b)$,
$f(x_0) \neq g(x_0)$ so
$0 < |f(x_0) - g(x_0)| \leq \sup\{|f(x) - g(x)| \mid x \in (a,b)\}$
$= d(f,g)$

(d) by the triangle ineq on $\mathbb{R}$
$\forall x_0 \in (a,b)$ $|f(x_0) - g(x_0)| \leq |f(x_0) - h(x_0)| + |h(x_0) - g(x_0)|$

(e) as $d(f,h) = \sup\{|f(x) - h(x)| \mid x \in (a,b)\}$
$d(f,h) \geq |f(x_0) - h(x_0)|$ $\forall x_0 \in (a,b)$
so $\forall x_0 \in (a,b)$, $|f(x_0) - g(x_0)| \leq d(f,h) + d(h,g)$

(f) So as the number $d(f,h) + d(h,g)$ is an upper bound for $\{|f(x) - g(x)| \mid x \in (a,b)\}$
the sup of this set, $d(f,g)$, is $\leq d(f,h) + d(h,g)$
This proves (f).

2. (a) let $x \in \text{int}(A)$ so $\exists \varepsilon > 0$

st $B_\varepsilon(X) \subset A$.

I claim $B_\varepsilon(X) \subset \text{int}(A)$

pf. let $y \in B_\varepsilon(X)$. As $B_\varepsilon(X)$ is open $\subset A$
$\exists\, \delta > 0$ st $B_\delta(y) \subset B_\varepsilon(X)$
So $B_\delta(y) \subset A$ and $y \in \text{int}(A)$

so $B_\varepsilon(X) \subset \text{int}(A)$ and $\text{int}(A)$ is open

b. $\text{ext}(A) = \text{int}(A^c)$ is open by (a)

c (this should have been put after d and e, sorry)

<u>Claim!</u> $\text{bd}(A) = X \setminus (\text{int}(A) \cup \text{ext}(A))$

Assuming the claim, as $\text{int}(A) \cup \text{ext}(A)$ is open, by the complementary characterization of closed sets, $\text{bd}(A)$ is closed.

this is self thy and is provided for completeness

proof of claim

let $x \in \text{bd}(A)$ then by (e) $x \notin \text{int}(A)$ and $x \notin \text{ext}(A)$

$\therefore x \in X \setminus (\text{int}(A) \cup \text{ext}(A))$

Now let $x \in X \setminus (\text{int}(A) \cup \text{ext}(A))$

As $x \in X$, $x \in \text{int}(A) \cup \text{bd}(A) \cup \text{ext}(A)$ by (d)

However as $x \notin \text{int}(A)$ and $x \notin \text{ext}(A)$
x must be in $\text{bd}(A)$ so $\text{bd}(A) = X \setminus (\text{int}(A) \cup \text{ext}(A))$

d. Recall

i $x \in \text{int}(A)$ iff $\exists\, \varepsilon > 0$ st $B_\varepsilon(x) \subset A$

ii $x \in \text{bd}(A)$ iff $\forall\, \varepsilon > 0$ $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$

iii $x \in \text{ext}(A)$ iff $\exists\, \varepsilon > 0$ st $B_\varepsilon(A) \subset A^c$.

let $x \in X$. if $x \notin \text{int}(A)$

then $\forall\, \varepsilon > 0$ $B_\varepsilon(X) \not\subset A$.

i.e. $\forall\, \varepsilon > 0$ $B_\varepsilon(X) \cap A^c \neq \emptyset$ *

if $x \notin \text{int}(A) \cup \text{bd}(A)$

if $x \notin int(A) \cup bd(A)$

Then as * holds, it must be true that
$\neg \; \forall \varepsilon > 0 \;\; B_\varepsilon(x) \cap A \neq \emptyset$

ie $\exists\, \varepsilon > 0$ st $B_\varepsilon(x) \cap A = \emptyset$
ie $\exists\, \varepsilon > 0$ st $B(x) \subset A^c$
ie if $x \notin (int(A) \cup bd(A))$ then $x \in ext(A)$

This proves $X = int(A) \cup bd(A) \cup ext(A)$

(e) The conditions i, ii, iii are mutually exclusive so $int(A)$, $bd(A)$, $ext(A)$ are mutually disjoint.

3. (a) as $X = int(A) \cup bd(A) \cup ext(A)$ and these 3 sets are disjoint.

$cl(A) = int(A) \cup bd(A)$ and $ext(A)$ are disjoint. So as $ext(A)$ is open $cl(A)$ is closed.

b. let B be a closed subset of X and assume $A \subset B$.

Then B^c is open (compl. char) and as $A \subset B$, $B^c \subset A^c$

ie B^c is an open subset of A^c

Useful Claim let $D \subset X$ if U is an open set and $U \subset D$ then $U \subset int(D)$

let $x \in U$ as U is open $\exists\, \varepsilon > 0$ st $B_\varepsilon(x) \subset U$. As $U \subset A$, $B_\varepsilon(x) \subset D$ and so $x \in int(D)$ So, $U \subset int(D)$.

Using this claim B^c is open and $B^c \subset A^c$ So $B^c \subset int(A^c) = ext(A)$.

$\therefore \; (ext(A))^c \subset (B^c)^c = B$

$Int(A) \cup bd(A)$ by 2 (d) and (e)
$\overset{\shortparallel}{cl(A)}$

4. — #3 on p 321.

Claim $f_n \to 0$ pointwise on $[0,1]$
Pf let $x \in [0,1)$ then $x^k \to 0$
so $f_k(x) = (1-x)x^k \to 0$

for $x = 1$ $f_n(x) = 0 \to 0$.

To see if $f_n \to 0$ unif, we show
max $f_n \to 0$

let $k \in \mathbb{N}$ then $\max(f_k)$ is at

the point on $(0,1)$ with

$$f_k'(x) = kx^{k-1} - (k+1)x^k$$
$$= x^{k-1}(k - (k+1)x) = 0$$

i.e at $x_k = \frac{k}{k+1}$ (Note: f_k is

non negative on $[0,1]$ and $f_n(0) = f_n(1) = 0$

So the Max of f_n, which exists
by the EVT is on $(0,1)$)

So $d(f_n, 0) = f_k\left(\frac{k}{k+1}\right) = \frac{1}{k+1}\left(\frac{k}{k+1}\right)^k \leq \frac{1}{k+1}$

and by the squeeze thm

$$\underset{\underset{0}{\downarrow}}{0} \leq d(f_n, 0) \leq \underset{\underset{0}{\downarrow}}{\frac{1}{k+1}}$$ so $d(f_n, 0) \to 0$

So, $f_n \to 0$ in $C([0,1], \mathbb{R})$

5. (15 points) (**Topology of a function space**) Define $A = \{f \in C([0,1],\mathbb{R}) \mid |f(x)| \le x^2+1, \forall x \in [0,1]\}$.

(a) (5 points) Prove A is closed in $C([0,1])$.

(b) Prove that $\text{int}(A) = \{f \in C([0,1],\mathbb{R}) \mid |f(x)| < x^2+1, \ \forall x \in [0,1]\}$.

(c) ~~Write A^c in set notation,~~ $A^c = \{f \in C([0,1],\mathbb{R}) \mid \text{????} \}$.

(d) Find $\text{bd}(A)$ and prove your result. ~~See Definition 1 for the definition of bd(A).~~

(e) Find $\text{ext}(A)$ and prove your result. See Definition 1 for the definition of $\text{ext}(A)$.

Solution. (a) Let $\{f_k\}$ be a sequence in A that converges to $f \in X := C([0,1],\mathbb{R})$. Then $\forall x \in [0,1]$, $|f_k(x)| \le x^2+1$ or

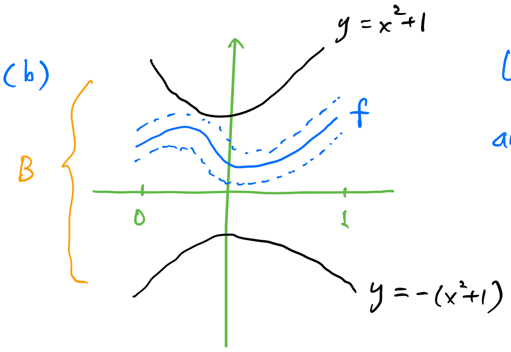
$$-(x^2+1) \le f_k(x) \le x^2+1.$$

Moreover, $f_k \to f$ uniformly and hence pointwise on $[0,1]$. Since the limit preserves $\le$, $\forall x \in [0,1]$,

$$-(x^2+1) \le f(x) \le x^2+1.$$

Therefore, $f \in A$. This proves that A is closed.

(b)



Let $B = \{f \in C([0,1],\mathbb{R}) \mid |f(x)| < x^2+1\}$ and let $f \in B$. Since $x^2+1-f(x)$ and $f(x)+(x^2+1)$ are continuous on $[0,1]$, by the extreme-value theorem, they both have a minimum on $[0,1]$. Let

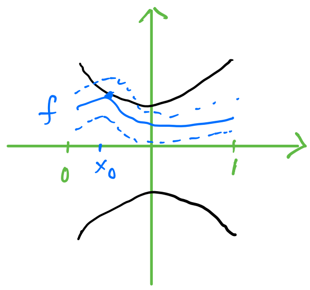
$$\varepsilon = \min\left\{ \min_{x\in[0,1]} (x^2+1)-f(x), \ \min_{x\in[0,1]} f(x)+(x^2+1) \right\}.$$

Then the ε-band $B_\varepsilon(f)$ lies between $y=-(x^2+1)$ and $y=x^2+1$ so that $B_\varepsilon(f) \subset A$. This proves that $f \in \text{int}\, A$. Hence, $B \subset \text{int}\, A$. (1)

To prove that $\text{int}\, A \subset B$, we will prove $B^c \subset (\text{int}\, A)^c$, where the complement is taken in $X = C([0,1],\mathbb{R})$. Let $f \in B^c$. Then $\exists x_0 \in [0,1]$ such that $|f(x_0)| \ge x_0^2+1$, i.e.,

$$f(x_0) \ge x_0^2+1 \quad \text{or} \quad f(x_0) \le -(x_0^2+1).$$

W.l.o.g, suppose $f(x_0) \ge x_0^2+1$. Then any ε-band $B_\varepsilon(f)$



will have a function g such that

$$g(x_0) > f(x_0) \geq x_0^2+1,$$

so $B_\varepsilon(f) \not\subset A$. This proves that $f \notin \operatorname{int} A$. Hence, $B^c \subset (\operatorname{int} A)^c$, so $\operatorname{int} A \subset B$. (2)

(1) and (2) together imply that $\operatorname{int} A = B$. □

(c) To get A^c, it suffices to negate the condition that defines A. Hence,

$$\boxed{A^c = \{ f \in C([0,1], \mathbb{R}) \mid \exists\, x_0 \in [0,1] \text{ such that } |f(x_0)| > x_0^2+1 \}}$$

(d) Since A is closed, $A = \operatorname{cl} A = \operatorname{int} A \cup \operatorname{bd} A$.

Therefore, $\operatorname{bd} A = A \setminus \operatorname{int} A = A \cap (\operatorname{int} A)^c$.

The condition that defines $(\operatorname{int} A)^c$ is

$$\exists\, x_0 \in [0,1] \text{ such that } |f(x_0)| \geq x_0^2+1.$$

Hence, the conditions that define $\operatorname{bd} A$ is

$$\boxed{\forall x \in [0,1],\ |f(x)| \leq x^2+1 \text{ and } \exists\, x_0 \in [0,1] \text{ s.t. } |f(x_0)| = x_0^2+1.}$$

(e) $\operatorname{ext} A = (\operatorname{int} A \cup \operatorname{bd} A)^c = (\operatorname{cl} A)^c$

$= A^c$ (because A is closed)

$= \boxed{\{ f \in C([0,1], \mathbb{R}) \mid \exists\, x_0 \in [0,1] \text{ s.t. } |f(x_0)| > x_0^2+1 \}}.$

6. p327 #12

Let $\varepsilon > 0$. As $c^k \to 0$

(here we ~~use~~ that $c \in (0,1)$),

$$\frac{c^k}{1-c} d(T(p_0), p_0) \to 0$$

So $\exists N \in \mathbb{N}$ st $\forall k \geq N$

$$0 \leq \frac{c^k}{1-c} d(T(p_0), p_0) < \varepsilon.$$

ie for $k \geq N$, $m \geq k$

$$d(p_m, p_k) \leq \frac{c^k}{1-c} d(T(p_0), p_0) < \varepsilon.$$

$\therefore \{p_k\}$ is Cauchy

7. p327 #1

(a) $f(x) = \frac{x}{2} = x$ iff $x = 0$.

but $0 \notin (0,1)$. The Contraction mapping principle is not contradicted

as $(0,1)$ is not complete
(The sequence $\{\frac{1}{2n}\}$ is Cauchy in $(0,1)$ but $\frac{1}{2n} \to 0 \notin (0,1)$)

b if $f(x)=x$ then $x+1=x$, $1=0$, which we know is false.
$\therefore$ $f(x)=x+1$ is not a contraction as $\forall x,y$ in $\mathbb{R}$ $|f(x)-f(y)| = |(x+1)-(y+1)|$
$= |x-y|$
and for no $c \in (0,1)$ is $|f(x)-f(y)| \le c|x-y|$ for $x \neq y$.

c. if $f(x,y) = (-y,x) = (x,y)$

$x=-y$, $y=x$ so $x=-x$, $x=0=y$
but $(0,0) \notin X$. So f does not have a fixed point in X

However $\|f(x_1,y_1) - f(x_2,y_2)\|$
$= \sqrt{(-y_1-(-y_2))^2 + (x_1-x_2)^2} = \|(x_1,y_1)-(x_2,y_2)\|$

and, if $(x_1,y_1) \neq (x_2,y_2)$ for no $c \in (0,1)$ is $\|f(x_1,y_1) - f(x_2,y_2)\| \le c\|(x_1,y_1)-(x_2,y_2)\|$

8. p327 #10

i) $f_n \to 1$ pointwise as $e^{x/n} \to e^0 = 1$.

ii) $f_n \to 1$ uniformly $\therefore$ in $C([0,1],\mathbb{R})$

pf let $x \in [0,1]$ then
$|f_n(x) - 1| = e^{x/n} - 1 \le e^{\frac{1}{n}} - 1$
as $e^{x/n} \ge 1$ as $\frac{x}{n} \le \frac{1}{n}$
and exp is increasing

As $e^{1/n} - 1 \to 0$, $f_n \to 1$ uniformly by the Theorem

Thm. $f_n \to f$ uniformly on X if $\exists$ seq $\{a_n\}$ of real numbers st $a_n \to 0$ and $\forall x \in X$ $d(f_n,f) \le a_n$.

So as $\{f_n\}$ converges uniformly $\{f_n\}$ converges in $C([0,1], \mathbb{R})$ so $\{f_n\}$ is Cauchy in $C([0,1], \mathbb{R})$.

9. P 327 #16

Let $X \subset \mathbb{R}^n$ $T: X \to \mathbb{R}^m$ Lipschitz.
Let $C > 0$ be s.t.
$\forall$ x, y in X $\|T(x) - T(y)\| \leq C \|x - y\|$.

By the defn of boundedness,
$\exists$ $x_0 \in X$ and $M > 0$ s.t.
$\forall$ $y \in X$ $d(x_0, y) = \|x_0 - y\| \leq M$.

$\therefore$ for any $y \in X$, $d(T(y), T(x_0))$

$$\leq C \|y - x_0\|$$
$$\leq CM$$

and so every point in the range $T(X)$ is at most CM units from $T(x_0)$ $\therefore$ $T(X)$ is bounded.

If the mapping T is only assumed to be continuous on X , the result is not true, because a continuous mapping can send a bounded set to an unbounded set. For example, the function T: (0,1) -> R, T(x) = 1/x, sends the bounded interval (0,1) to the unbounded set (1, \infty).