Math 135 HW 8

1. $\lim_{k\to\infty} \langle \vec{u}_k, \vec{v} \rangle = \lim_{k\to\infty} u_k^{(1)} v^{(1)} + u_k^{(2)} v^{(2)} + \dots + u_k^{(n)} v^{(n)}$

By Componentwise criterion for convergence, as $\{\vec{u}_k\} \to u$, then $u_k^{(1)} \to u^{(1)}$ and so on. So our limit $= u^{(1)} v^{(1)} + u^{(2)} v^{(2)} + \dots + u^{(n)} v^{(n)} = \langle \vec{u}, \vec{v} \rangle$ $\square$

2a) $\Rightarrow$ $\{\vec{u}_k\}$ is Cauchy then each component sequence is Cauchy

Since $\{\vec{u}_k\}$ is Cauchy, $\forall \epsilon > 0, \exists K \in \mathbb{N}$ s.t. $\forall k, l \geq K$ that $\text{dist}(\vec{u}_k, \vec{u}_l) < \epsilon$

$\|\vec{u}_k - \vec{u}_l\| < \epsilon$

WLOG

$\|\vec{u}_k - \vec{u}_l\| = \sqrt{(u_k^{(1)} - u_l^{(1)})^2 + \dots + (u_k^{(n)} - u_l^{(n)})^2} > \sqrt{(u_k^{(i)} - u_l^{(i)})^2}$

So $|u_k^{(i)} - u_l^{(i)}| < \|\vec{u}_k - \vec{u}_l\|$ for where $1 \leq i \leq n$

As $|u_k^{(i)} - u_l^{(i)}| < \|\vec{u}_k - \vec{u}_l\| < \epsilon$ so each component sequence $< \epsilon$ and is Cauchy

$\Leftarrow$ If each component sequence = Cauchy then $\{\vec{u}_k\}$ is Cauchy

For each component sequence $\{u_k^{(i)}\}$ where $1 \leq i \leq n$,

$\exists K_i$ s.t. $\forall m, k \geq K$

$|u_k^{(i)} - u_m^{(i)}| < \frac{\epsilon}{\sqrt{n}}$ as they are each Cauchy

Take $K = \max\{K_1, K_2, \dots, K_n\}$

So $\forall k, m \geq K$, $\|\vec{u}_k - \vec{u}_m\| = \sqrt{\sum_{i=1}^{n} (u_k^{(i)} - u_m^{(i)})^2} < \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} = \epsilon$

So $\{\vec{u}_k\}$ is Cauchy.

This proves bothways and the iff statement.

2b) $\Rightarrow$ If $\{\vec{u}_k\} \in \mathbb{R}^n$ converges, it is Cauchy

Suppose $\{\vec{u}_k\} \to u$. Let $\epsilon > 0$, $\exists K_1 \in \mathbb{N}$ s.t. $\forall k \geq K$ $\|\vec{u}_k - \vec{u}\| < \frac{\epsilon}{2}$. $\exists K_2 \in \mathbb{N}$ s.t. $\forall m \geq K$ $\|\vec{u}_m - \vec{u}\| < \frac{\epsilon}{2}$

Note $\vec{u}_m - \vec{u}_n = (\vec{u}_m - \vec{u}) + (\vec{u} - \vec{u}_n)$ and let $K = \max\{K_1, K_2\}$

So $\forall m, k \geq K$: $\|(u_m - w) + (w - u_k)\| \leq \|u_m - w\| + \|u_k - w\|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$A_s \|u_m - u_k\| < \varepsilon$ $\forall m, k \geq K$ it is Cauchy

$\Leftarrow$ If it is Cauchy then it converges

Suppose $\{\vec{u}_k\}$ is Cauchy, $\exists K$, s.t. $\forall k, l \geq K$,
$\|\vec{u}_k - \vec{w}_l\| < \frac{\varepsilon}{2}$

Let $\vec{u}_{k_m}$ be a subsequence of $\vec{w}$. As every Cauchy sequence is bounded, $\vec{u}_{k_m} \to u$.

Since $\vec{w}_{k_m} \to \vec{w}$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\|\vec{u}_{k_m} - \vec{u}\| < \frac{\varepsilon}{2}$ where $\varepsilon > 0$

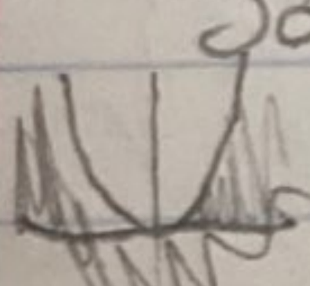
Choose $K = \max\{k, N\}$

$\|u_k - w\| = \|(u_k - u_{k_m}) + (u_{k_m} - w)\| \leq \|(u_k - u_{k_m})\| + \|(u_{k_m} - w)\|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

This proves both ways.

3a) $A = \{\vec{w} = (x, y) \mid x^2 > y\}$ Let $F(x, y) = y - x^2 > 0$

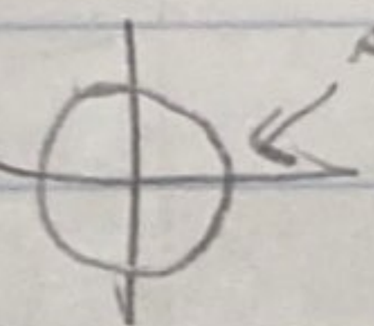
So $A = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) > 0\}$ or $F(x, y) \in (0, \infty)$



$S = f^{-1}((0, \infty))$ which is open so A is open

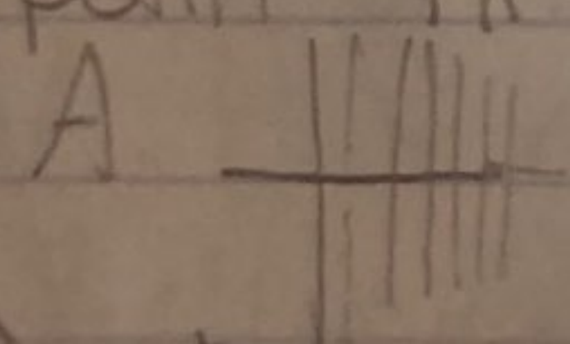
b) $A = \{\vec{w} = (x, y) \mid x^2 + y^2 = 1\}$ $F(x, y) = x^2 + y^2 - 1 = 0$

$A = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$

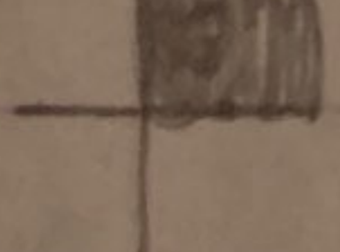


$f^{-1}([0])$ is closed so A is closed

c) A is neither open nor closed, as due to denseness of irrationals in $\mathbb{R}$ you cannot construct an open ball for $x \in A$, as $x \in \mathbb{Q}$. The opposite holds for any point in A^c so A is neither open nor closed.



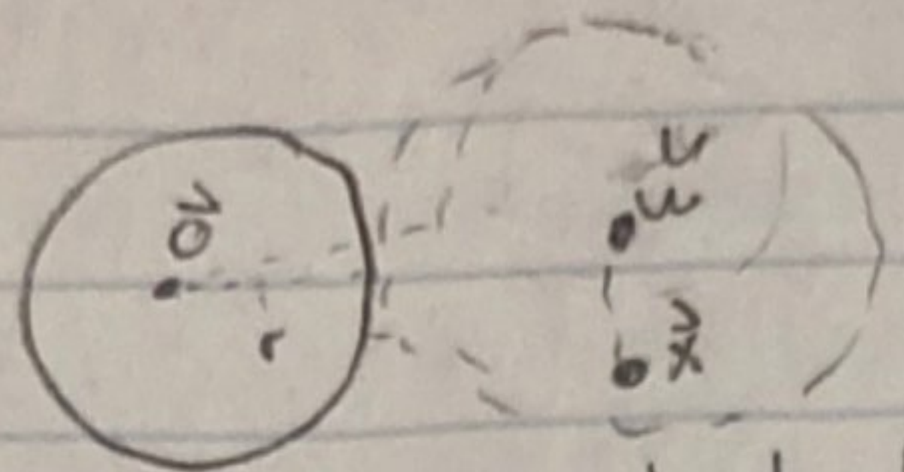
d)



$A = \{u = (x, y) \mid x \geq 0, y \geq 0\}$ is closed.

$\mathrm{Bd}A$ is $x = 0, y = 0$ and as $\mathrm{bd}A \subset A$ A is closed.

Scanned with CamScanner

4a) 

$dist(\vec{O}, w) = \|w\|$

Let $R = \|w\| - r > 0$ and suppose $x \in B_r(w)$

We want to show $x \in \mathcal{O}$ or that $r < dist(\vec{O}, x)$

By triangle inequality.

$dist(\vec{O}, w) \leq dist(\vec{O}, \vec{x}) + dist(x, w)$

$dist(\vec{O}, w) - dist(x, w) \leq dist(\vec{O}, x)$

so $\|w\| - dist(x, w) < \|w\| - R = r < dist(\vec{O}, x)$

so as $r < dist(\vec{O}, x)$, $x \in \mathcal{O}$ and $\mathcal{O}$ is open.

b) $F = \mathcal{O}^c$, and as $\mathcal{O}$ is open, $\mathcal{O}^c$ is closed so F is closed.

5a) $A \subset \text{Cl}A$ $\quad \text{Cl}A = \text{int}A \cup \text{bd}A$

$A = \text{int}A \cup (A \cap \text{bd}A) \subset \text{int}A \cup \text{bd}A = \text{Cl}A$ $\square$

b) $\Rightarrow$ $A = \text{Cl}A$ then A is closed in $\mathbb{R}^n$

If $A = \text{Cl}A$, then $A \supset \text{Cl}A \rightarrow A \supset \text{int}A \cup \text{bd}A$ and $A \supset \text{bd}A$ so A is closed.

$\Leftarrow$ A is closed in $\mathbb{R}^n$ then $A = \text{Cl}A$

Part a shows $A \subset \text{Cl}A$ so we need to show $A \supset \text{Cl}A$

$A \supset \text{int}A \cup \text{bd}A$. By definition of closed set, $A \supset \text{bd}A$ and $A \supset \text{int}A$, so $A \supset (\text{int}A \cup \text{bd}A)$ and $A \supset \text{Cl}A$.

Therefore $A = \text{Cl}A$.

6a) Let $x \in \text{int}A$. $\exists r > 0$ s.t. $B_r(x) \subset A$.

Let $y \in B_r(x)$. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(y) \subset B_r(x) \subset A$

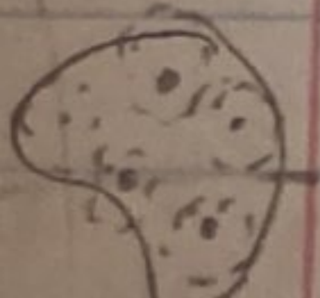
As $B_\varepsilon(y)$ is open, and by definition, $y \in \text{int}A$, so $B_r(x) \subset \text{int}A$ and $\text{int}A$ is open.

6b) Let $x \in \text{ext}A$. $\exists r>0$ s.t. $B_r(x) \subset \text{ext}A$
Let $y \in B_r(x)$, $\exists \epsilon>0$ and $<r$ s.t. $B_\epsilon(y) \subset B_r(x)$
So, $y \in \text{ext}A$. So if $y \in B_r(x)$ then $y \in \text{ext}A$
and $B_r(x) \subset \text{ext}A$ so $\text{ext}A$ is open.

c) $\mathbb{R}^n = \text{int}A \cup \text{ext}A \cup \text{bd}A$
From this, $\text{bd}A = (\text{int}A \cup \text{ext}A)^c$ as the
which is disjoint, so they share no elements.

$\text{int}A$ & $\text{ext}A$ is open so therefore $(\text{int}A \cup \text{ext}A)^c$
is closed, meaning $\text{bd}A$ is closed

7 ⇒ If $S \subset A$ is relatively open in A then $B_A(w, \epsilon) \subset S$
Let $S = U \cap A$ for $B_r(w)$
Take $w \in S$, then $w \in U$. As w is open, $\exists \epsilon > 0$
s.t. $B(w, \epsilon) \subset U$.
$B_A(w, \epsilon) = B(w, \epsilon) \cap A$
$B_A(w, \epsilon) = B(w, \epsilon) \cap A \subset U \cap A = S$

⇐ If $B_A(w, \epsilon) \subset S$ then $S \subset A$ is relatively open in A.

Suppose $\forall w \in S$, $\exists \epsilon_w > 0$ s.t. $B_A(w, \epsilon_w) \subset S$
By assumption, $B_A(w, \epsilon_w) \subset B(w, \epsilon_w) \cap A$
We want too look at the union of all relatively open balls so
we get $U = \bigcup_{w \in S} B(w, \epsilon_w)$

- Want look at $U \cap A = \left(\bigcup_{w \in S} B(w, \epsilon_w)\right) \cap A$

$= \bigcup_{w \in S} (B(w, \epsilon_w) \cap A) = \bigcup_{w \in S} B_A(w, \epsilon)$

$U \cap A = \bigcup_{w \in S} B_A(w, \epsilon) \subset S$ which means
$(U \cap A) \subset S$ so this proves both ways. □