Math 65 HW 11

1 If T is a tree and v a vertex, for $T-\{v\}$ to be a tree, $T-\{v\}$ must be connected and has non-cycles. Since T has no n cycles, it's impossible for $T-\{v\}$ to have any n-cycles. For $T-\{v\}$ to be connected, $\{v\}$ can only be connected to one edge, as otherwise we would have 2 components. Therefore, v is connected to one edge and of degree 1, so v is a leaf.

2 T is connected, meaning path from $u \to v$ / $v \to u$ exists. If add edge e' from $u \to v$, then path is now from $u \to v$ and $v \to u$, which forms a unique cycle.

3. Yes. Start at V_1, go on one edge and go $V_1 \to V_2 \to V_3 \to \dots V_n$ ($\to$ n vertices). When we go from $V_{n-1} \to V_n$, use both edges of V_{n-1} as we go to and then leave V_{n-1}, which requires 2 edges. When reach V_n and leave, only edge left is other edge of V_1, forming a cycle.

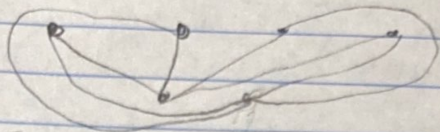
4 For each connected component, $v-e+f=2$ should hold true, as G is planar, so with c connected components $\sum_{i=1}^{c} v_i - e_i + f_i = 2c$, $\sum_{i=1}^{c} v_i = v$, $\sum_{i=1}^{c} e_i = e$, $\sum f_i = c + \sum (f_i - 1)$ for faces, as ~~account for exterior face which is double~~ counted, subtract one face and subtract 1 due to exterior, so $\sum f_i = c+f-1$

$v-e+f=2c$ becomes $v-e+c+f-1=2c$, $\boxed{v-e+f=1+c}$

5 Assume G is planar, let's show $\bar{G}$ is nonplanar.
For G $|E| \leq 3|V|-6$, $|V|=11$, so $|E_G| \leq 27$. There are $\binom{11}{2}=55$ possible edges among G and $\bar{G}$ and since $|E_G|+|E_{\bar{G}}|=55$ and at most $|E_G|=27$, then at minimum, $|E_{\bar{G}}| \geq 28$, meaning $\bar{G}$ is nonplanar.

6 For $K_{m,n}$, $K_{m,n}$ if ≤ 2 planar if $m, n \leq 2$
~~For any~~ $K_{1,n}$, b have 1 vertex connecting to all others w/o overlap.
For $K_{2,n}$ it's planar as can connect all n to 1 vertex using middle and connect to other vertex going around like this:

$K_{4,2}$



For any $m, n \geq 3$ K_{mn} not planar as proved in class K_{33} is nonplanar and every K_{mn} w/ $m, n \geq 3$ has K_{33} as a subgraph, so K_{mn} is never planar when $K_{m,n} \geq 3$.

7a) $|P(V)| = 2^n$ and for a simple graph there are $\binom{n}{2}$ possible edges so there are $2^{\binom{n}{2}}$ simple graphs

b) bijection between graph G of n vertices of even degree and n-1 of arbitrary vertices G'

$G \xrightarrow{\text{remove } v_n \text{ and it's edges}} G'$

$G \xleftarrow{\text{add } v_n \text{ and edge to vertices of odd degree}} G'$

Since bijection exists, $|G| = |G'|$ and $|G'| = 2^{\binom{n-1}{2}}$ so $|G| = 2^{\binom{n-1}{2}}$ and there are $2^{\binom{n-1}{2}}$ simple graphs w/ vertices of even degree

c) graph G has euler circuit when G has vertices of even degree, so $P(\text{Euler circuit}) = \dfrac{2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = \dfrac{1}{2^{\binom{n}{2} - \binom{n-1}{2}}}$

$\binom{n}{2} - \binom{n-1}{2} = n-1$ as all that's left is n^{th} object and paired once with each object before it, which is $n-1$

$\dfrac{1}{2^{\binom{n}{2} - \binom{n-1}{2}}} = \boxed{\dfrac{1}{2^{n-1}}} = \boxed{2^{1-n}}$