Wednesday, February 8

Sunday, February 5, 2023 16:43

TA Help session 10:30 Fridays, Math library, JCC 574
Student hours with Todd 1:30-3:00 my office JCC 575 (end of hall)

I added more get-to-know-you meeting slots. Please sign up!
https://docs.google.com/spreadsheets/d/1T8o6af3Oe3uA3aswPvv1pm0FdnmQ6oaiF5Le623wdLY/edit?usp=sharing

Definition 1 (Differentiability in $\mathbb{R}^n$) *Let $\mathcal{O} \subset \mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$ Let $\mathbf{x}_0 \in \mathcal{O}$. Then, f is* differentiable at $\mathbf{x}_0$ *if there is a vector $\mathbf{b} \in \mathbb{R}^n$ such that*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|f(\mathbf{x}) - [f(\mathbf{x}_0) + \langle \mathbf{b}, (\mathbf{x}-\mathbf{x}_0)\rangle]|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0 \quad \textit{or, equivalently:}$$
$$\lim_{\mathbf{h}\to\vec{0}} \frac{|f(\mathbf{x}_0+\mathbf{h}) - [f(\mathbf{x}_0) + \langle \mathbf{b}, \mathbf{h}\rangle]|}{\|\mathbf{h}\|} = 0. \tag{1}$$

Here are some properties of differentiable functions as well as connections between differentiability and other ideas we've discussed.

1. Let $f : \mathcal{O} \to \mathbb{R}$ and let $\mathbf{x}_0 \in \mathcal{O}$. Then, f is differentiable at $\mathbf{x}_0$ *if and only if* f has an affine first order approximation at $\mathbf{x}_0$ of the form $g(\mathbf{x}) = f(\mathbf{x}_0) + \langle \mathbf{b}, (\mathbf{x}-\mathbf{x}_0)\rangle$ for some $\mathbf{b} \in \mathbb{R}^n$.

   (This follows directly from the definitions of differentiability, of affine function, and of first order approximation. Try it!)

2. If $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$, then the affine function $g(\mathbf{x}) = f(\mathbf{x}_0) + \langle \mathbf{b}, (\mathbf{x}-\mathbf{x}_0)\rangle$ in (1) is unique (as affine first order approximations are unique). Therefore the vector $\mathbf{b}$ that makes (1) valid is unique.

   Definition 2 *If $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$, then we call the unique vector $\mathbf{b}$ in (1)* the derivative vector of f at $\mathbf{x}_0$ *and denote it by $Df(\mathbf{x}_0) = \mathbf{b}$.*

3. $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ then f has all first partials at $\mathbf{x}_0$ and

$$\mathbf{b} = Df(\mathbf{x}_0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}_0), \frac{\partial f}{\partial x_2}(\mathbf{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\right) = \nabla f(\mathbf{x}_0)$$

   You will prove this on HW 3.

4. If $f : \mathcal{O} \to \mathbb{R}$ is differentiable at $\mathbf{x}_0 \in \mathcal{O}$, then f is continuous at $\mathbf{x}_0$.

5. If $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable on $\mathcal{O}$ (i.e., $f \in C^1(\mathcal{O})$), then f is differentiable at all points in $\mathcal{O}$.

   (Try to prove this! The short proof uses the First Order Approximation theorem for C^1 functions.)

ex let $f(x,y) = x^2 + y$
let $(x_0, y_0) \in \mathbb{R}^2$ Show f is diff at (x_0, y_0) using defn.

Soln $\bar{x}_0 = (x_0, y_0)$

candidate for $\bar{b} = \nabla f(x_0, y_0) = (2x_0, 1)$ $\qquad f(x,y) = x^2 + y$

plug into limit ($\bar{h}$ version easier)

$$\lim_{(h,k)\to 0} \frac{|f(x_0+h, y_0+k) - (f(x_0,y_0) + \langle (2x_0,1), (h,k)\rangle)|}{\|(h,k)\|}$$

$\bar{h} = (h,k)$

$\cancel{x_0^2} + \cancel{2x_0h} + h^2 + \cancel{y_0} + \cancel{k} - \cancel{x_0^2} - \cancel{y_0} - \cancel{2x_0h} - \cancel{k}$

$$= \lim_{(h,k)\to\bar{0}} \frac{|(x_0+h)^2 + (y_0+k) - (x_0^2 + y_0 + 2x_0h + k)|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to\bar{0}} \frac{|h^2|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to \vec{0}} \frac{|h^2|}{\sqrt{h^2+k^2}}$$

$$0 \le \frac{|h^2|}{\sqrt{h^2+k^2}} \le |h| \frac{|h|}{\sqrt{h^2+k^2}} \le |h|$$

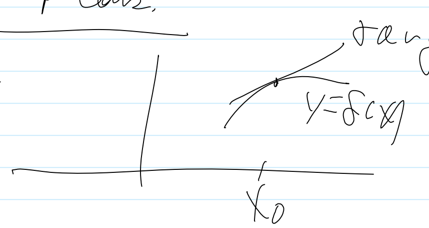
as $|h| \le \sqrt{h^2+k^2} = \|(h,k)\|$

$\to 0$ $\qquad$ $\to 0$ as $(h,k) \to 0$

So our orig limit $\to 0$

Tangent planes.

recall



apps f to 1^{st} order at x_0 and is affine.

Def $O \subset \mathbb{R}^2$ open $f : O \to \mathbb{R}$

the graph of f has a tangent plane at $(x_0,y_0) \in O$ if f has an affine 1^{st} order approx at (x_0,y_0) $\quad g(x,y) = f(x_0,y_0) + \langle \vec{b}, (x-x_0, y-y_0)\rangle$ for some $\vec{b} \in \mathbb{R}^2$

then the plane $z = g(x,y)$ is the tangent plane to graph of $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$

Notes ① graph of $z = f(x,y)$ has tangent plane at $(x_0, y_0, f(x_0,y_0))$

iff f is diff at (x_0,y_0)

So $f \in C^1(O) \to f$ has tangent plane at every $(x_0,y_0) \in O$ as f has affine 1^{st} order approx at every (x_0,y_0) in O

genl formula f diff at (x_0,y_0)

at every (x_0, y_0) in $\mathcal{O}$

genl formula f diff at (x_0, y_0)

tangent plane is $z = g(x,y) = f(x_0,y_0) + \langle \nabla f(x_0,y_0), (x-x_0, y-y_0) \rangle$

ex let $f(x,y) = x^2 + 2xy + 1$

Explain why f has tangent plane at $(1,1)$ and find an eqn for that plane

Soln as $f \in C^1$ f is diff $\therefore$ f has tangent plane at all points in $\mathbb{R}^2$

$$z = g(x,y) = f(x_0,y_0) + \langle \nabla f(x_0,y_0), (x-x_0, y-y_0) \rangle$$

$$g(x,y) = f(1,1) + \left\langle \left(\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial y}(1,1)\right), (x-1, y-1) \right\rangle$$

$f(x,y) = x^2 + 2xy + 1$ $\quad = 4 + \langle (4,2), (x-1, y-1) \rangle$

$\nabla f(x,y) = (2x+2y, 2x)$

$g(x,y) = 4 + 4x - 4 + 2y - 2 = -2 + 4x + 2y$

Derivatives for $F : \mathcal{O} \to \mathbb{R}^m$ where $\mathcal{O}$ is open in $\mathbb{R}^n$

Notation $F : \mathcal{O} \to \mathbb{R}^m$

$$F(\bar{x}) = \begin{bmatrix} F_1(\bar{x}) \\ F_2(\bar{x}) \\ \vdots \\ F_m(\bar{x}) \end{bmatrix}$$

call $F_i : \mathcal{O} \to \mathbb{R}$ $i = 1 \cdots m$ the i^{th} component fn. of F

We say F has all 1^{st} (2^{nd}) partial derivs on $\mathcal{O}$ if each F_i has all 1^{st} (2^{nd}) partial derivs on $\mathcal{O}$

$F \in C^1(\mathcal{O})$ (cont. diff) if each $F_i : \mathcal{O} \to \mathbb{R}$ is in C^1 (cont. diff)

$F \in C^2(\mathcal{O})$ if each component fn F_i is in $C^2(\mathcal{O})$

eg is $F(x,y) = \begin{bmatrix} |xy| \\ x^2+y^2 \end{bmatrix}$ cont. diff on $\mathbb{R}^2$?

nope as $F_1(x,y) = |xy|$ is not C^1.

— " $\lfloor x^2+y^2 \rfloor$ diff on $\mathbb{R}^2$?

nope as $F_1(x,y) = |xy|$ is not C^1.

Prop If $F \in C^1(\mathcal{O})$ then $F: \mathcal{O} \to \mathbb{R}^n$ is cont

~~pf~~ as component are cont diff / Thm.
∴ cont ∴ F is cont by
(cont iff component fns are cont)

Defn let $F: \mathcal{O} \to \mathbb{R}^n$ have 1^{st} order partials at all points $\bar{x} \in \mathcal{O}$.

Then we define the derivative fn $DF(\bar{x}_0)$ to be the $\bar{x}_0 \in \mathcal{O}$ $m \times n$ matrix with entry in i^{th} row j^{th} col

$$\frac{\partial}{\partial x_j} F_i(\bar{x}_0)$$

$$DF(\bar{x}_0) \quad \begin{array}{c|c} & \begin{array}{cccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{array} \\ \hline \begin{array}{c} F_1 \\ F_2 \\ \vdots \\ F_m \end{array} & \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\bar{x}_0) & \frac{\partial F_1}{\partial x_2}(\bar{x}_0) & \cdots & \frac{\partial F_1}{\partial x_n}(\bar{x}_0) \\ \frac{\partial F_2}{\partial x_1}(\bar{x}_0) & \cdots & & \frac{\partial F_2}{\partial x_n}(\bar{x}_0) \\ \vdots & & & \\ \frac{\partial F_m}{\partial x_1}(\bar{x}_0) & \cdots & & \frac{\partial F_m}{\partial x_n}(\bar{x}_0) \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \nabla F_1(\bar{x}_0) \\ \nabla F_2(\bar{x}_0) \\ \vdots \\ \nabla F_m(\bar{x}_0) \end{bmatrix}$$

ex let $F(x,y) = \begin{bmatrix} x^2 y \\ 2x+y \\ e^x \end{bmatrix}$ $F: \mathbb{R}^2 \to \mathbb{R}^3$

Find $DF(x,y)$

$$= \begin{array}{c|c} & \begin{array}{cc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array} \\ \hline \begin{array}{c} x^2y \\ 2x+y \\ e^x \end{array} & \begin{bmatrix} 2xy & x^2 \\ 2 & 1 \\ e^x & 0 \end{bmatrix} \end{array}$$

Defn If $F: \mathcal{O} \to \mathbb{R}^n$ has 1^{st} partials on $\mathcal{O}$

1^{st} partials on $\mathcal{O}$

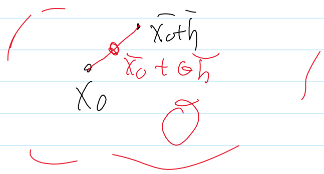
then we define the differential $dF(\bar{x}_0)$ for $\bar{x}_0 \in \mathcal{O}$ to be the Linear transformation with standard matrix $DF(\bar{x}_0)$

ie $dF(\bar{x}_0)(\bar{v}) = DF(\bar{x}_0)\,\bar{v}$

Lin transf $dF(\bar{x}_0)$ evaluated at $\bar{v} \in \mathbb{R}^n$ — matrix $DF(\bar{x}_0)$ times $\bar{v}$

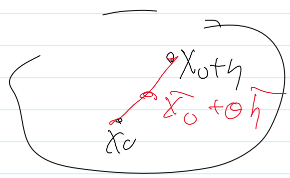
Mean Value Thm

$F : \mathcal{O} \to \mathbb{R}$ is C^1



$f(\bar{x}_0+\bar{h}) - f(x_0) = \langle \nabla f(\bar{x}_0+\theta\bar{h}), \bar{h} \rangle$ for some $\theta \in (0,1)$

Conjecture MVT in $\mathbb{R}^2$



$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

$$F(\bar{x}+\bar{h}) - F(\bar{x}) = \begin{bmatrix} F_1(x+\bar{h}) - F_1(x) \\ F_2(\bar{x}+\bar{h}) - F_2(x) \end{bmatrix} = \begin{bmatrix} \langle \nabla F_1(\bar{x}_0+\theta\bar{h}), \bar{h} \rangle \\ \langle \nabla F_2(\bar{x}_0+\theta\bar{h}), \bar{h} \rangle \end{bmatrix}$$

need diff θ's

need diff θ_j in general

MVT $F \in C^1(\mathcal{O}, \mathbb{R}^m)$ $F : \mathcal{O} \to \mathbb{R}^n$ (component fns are C^1)

Then $\exists$ $\theta_1, \theta_2, \ldots \theta_m$ in $(0,1)$

st $F_j(\bar{x}+\bar{h}) - F_j(x) = \langle \nabla F_j(\bar{x}+\theta_j\bar{h}), \bar{h} \rangle$

whenever $\bar{x} \in \mathcal{O}$ and seg ment between $\bar{x}$ and $\bar{x}+\bar{h}$ is in $\mathcal{O}$

Def $G : \mathbb{R}^n \to \mathbb{R}^m$ is affine $\bar{x}_0 \in \mathbb{R}^n$

if $G(\bar{x}) = \bar{a} + B(\bar{x} - \bar{x}_0)$

some $\bar{a}$ in $\mathbb{R}^m$ for some $m \times n$ matrix B