Tufts University
Department of Mathematics

Math 136 | **Solutions to Homework 2**[1] | **Spring, 2023**

You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

Reading assignment: Read Sections 14.1, 14.2, 14.3 by Wednesday, February 1. This homework covers Sections 13.2, 13.3, and 14.1.

Problems:

1 (15 points) Let $A \subset \mathbb{R}^n$ and let $\mathbf{x}_*$ be a limit point of A that is in A. Let $f : A \to \mathbb{R}^m$. Prove that f is continuous at $\mathbf{x}_*$ if and only if $\lim\limits_{\mathbf{x}\to\mathbf{x}_*} f(x) = f(\mathbf{x}_*)$. [HINT: one proof uses $\epsilon - \delta$ condition for limit in Theorem 13.7 and the $\epsilon - \delta$ condition for continuity in Theorem 11.11.]

Solution: First, assume f is continuous at $\mathbf{x}_*$. We check the definition of limit at $\mathbf{x}_*$, so let $\{\mathbf{x}_k\}$ be a sequence in $A \setminus \{\mathbf{x}_*\}$ that converges to $\mathbf{x}_*$. Since $\{\mathbf{x}_k\}$ is a sequence in A that converges to $\mathbf{x}_*$, $f(\mathbf{x}_k) \to f(\mathbf{x}_*)$ as f is continuous at $\mathbf{x}_*$. Therefore, $\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x}) = f(\mathbf{x}_*)$.

Now, assume $\lim\limits_{\mathbf{x}\to\mathbf{x}_*} f(x) = f(\mathbf{x}_*)$. Here, it is easier to use the $\epsilon - \delta$ characterization of limits (Theorem 13.7) and continuity (Theorem 11.11). By Theorem 13.7,

$$\forall \epsilon > 0,\ \exists \delta > 0, \text{ such that, if } 0 < \|\mathbf{x} - \mathbf{x}_*\| < \delta \text{ and } \mathbf{x} \in A, \text{ then } \|f(\mathbf{x}) - f(\mathbf{x}_*)\| < \epsilon. \tag{1}$$

From Theorem 11.11, f is continuous at $\mathbf{x}_*$ if and only if

$$\forall \epsilon > 0,\ \exists \delta > 0, \text{ such that, if } \|\mathbf{x} - \mathbf{x}_*\| < \delta \text{ and } \mathbf{x} \in A, \text{ then } \|f(\mathbf{x}) - f(\mathbf{x}_*)\| < \epsilon. \tag{2}$$

We can assume (1). Since the condition in (2) holds for any ϵ and δ if $\mathbf{x} = \mathbf{x}_*$, this shows that f is continuous at $\mathbf{x}_*$.

2 (20 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(a) Let $\mathbf{p} = (a,b)$ Calculate $D_{\mathbf{p}} f(0,0)$. This shows that f has all directional derivatives at 0.

Solution: NOTE: We showed this in class and you can just refer to that result. This is a repeat of the proof from class.

For part (a), we need to show $f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ has directional derivatives at $(0,0)$ in all directions. Let $\mathbf{p} = (a,b) \neq (0,0)$. First, assume $b \neq 0$. If the limit exists,

$$D_{\mathbf{p}} f(0,0) = \lim_{t\to 0} \frac{f(at,bt) - 0}{t} = \lim_{t\to 0} \frac{1}{t} \frac{a^2 b t^3}{a^4 t^4 + b^2 t^2} = \lim_{t\to 0} \frac{a^2 b}{a^4 t^2 + b^2} = \frac{a^2}{b}. \tag{3}$$

(*) By letting $\mathbf{p} = (0,1)$ we see $0 = D_{(0,1)} f(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

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Now assume $b = 0$. The proof for this case is different since the denominator in the last expression (3) is zero. In this case, $f(at, bt) = 0$ for all $t \neq 0$ so in this case,

$$D_{(a,0)}f(0,0) = \lim_{t \to 0} \frac{f(at,0) - 0}{t} = 0.$$

(*) From this we see $\frac{\partial f}{\partial x}(0,0) = D_{(1,0)}f(0,0) = 0$.

(b) Does f have all directional derivatives at all point $x \in \mathbb{R}^2$? Why or why not?

Solution: yes.

In part (a), we showed f has directional derivatives in all directions at $(0,0)$.

If $(x,y) \neq 0$ then $f(x,y)$ is a rational function with a non-zero denominator and so has continuous first partial derivatives at (x,y). One can show this directly by calculating $\partial f/\partial x$ and $\partial f/\partial y$ using the quotient rule. This calculation shows for $(x,y) \neq (0,0)$ that

$$\frac{\partial f}{\partial x}(x,y) = 2xy\frac{y^2 - x^4}{(x^4 + y^2)^2} \qquad \frac{\partial f}{\partial y}(x,y) = \frac{x^4(x^4 - y^2)}{(x^4 + y^2)^2}.$$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are rational functions with nonzero denominator for $(x,y) \neq (0,0)$ they are continuous so $f \in C^1(\mathbb{R}^2 \setminus \{\mathbf{0}\}$. Therefore, f has directional derivatives at all points $(x,y) \neq 0$ by the Directional Derivative theorem.

(c) Does f satisfy the conclusion of the Directional Derivative Theorem(Theorem 13.16 in Fitzpatrick) at $x_0 = 0$? Why or why not?

Solution: no.

We showed in part (a) that $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$ (see the (*)'ed statements above).

Therefore, *if the conclusion of the Directional Derivative Theorem held*, for all $\mathbf{p} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, $D_{\mathbf{p}}f(0,0)$ would equal $\langle \nabla f(0,0), \mathbf{p} \rangle = \langle (0,0), \mathbf{p} \rangle = 0$. However, this is not true as, for example, $D_{(1,1)}f(0,0) = 1$ by equation (3).

(d) Is f continuously differentiable on $\mathbb{R}^2$? Why or why not?

Solution: no.

In class we showed that f is not continuous at $(0,0)$ so f cannot be continuously differentiable because all continuously differentiable functions are continuous.

3 (15 points) Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$. Assume $f \in C^2(\mathcal{O})$; that is, f has all first and second order partial derivatives on $\mathcal{O}$ and the second order partial derivatives are continuous on $\mathcal{O}$.

(a) Prove that f is continuous on $\mathcal{O}$.

Solution: In class, we showed that C^1 functions are continuous. Let i and j be integers between 1 and n. If $f \in C^2(\mathcal{O})$ then $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial f}{\partial x_j}\right)$ is continuous.

Fix $j \in \{1, \ldots, n\}$. Since i is arbitrary, this says that all first derivatives of $\frac{\partial f}{\partial x_j}$ are continuous therefore $\frac{\partial f}{\partial x_j}$ is C^1 and so continuous.

Since j was arbitrary, this shows that all first partial derivatives of f are continuous. Therefore, $f \in C^1(\mathcal{O})$ so f is continuous.

(b) ~~Is f differentiable on $\mathcal{O}$? Why or why not~~? You didn't need to do this part.

(c) Is f continuously differentiable on $\mathcal{O}$? Why or why not? **Solution:** Yes.it is continuously differentiable. We showed f has continuous first partials in part (a) so $f \in C^1(\mathcal{O})$, as noted in (a).

4 (20 points) Let f and g be functions from $\mathcal{O}$ to $\mathbb{R}$ and assume g is a k^{th} order approximation to f at $\mathbf{x}_0 \in \mathcal{O}$. Prove the following

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that if } \mathbf{x} \in \mathcal{O} \setminus \{\mathbf{x}_0\} \text{ and } \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ then}$$
$$|f(\mathbf{x}) - g(\mathbf{x})| \le \epsilon \|\mathbf{x} - \mathbf{x}_0\|^k .$$

[HINT: Theorem 13.7 on p. 352 would be helpful.]
This shows that as $\mathbf{x} \to \mathbf{x}_0$, $f(\mathbf{x}) - g(\mathbf{x})$ goes to zero (eventually) faster than any multiple of $\|\mathbf{x} - \mathbf{x}_0\|^k$.

Solution: Let $\epsilon > 0$. Then, by Theorem 13.7, there is a $\delta > 0$ so that if $\mathbf{x} \in \mathcal{O} \setminus \{\mathbf{x}_0\}$ and $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, then

$$\frac{|f(\mathbf{x}) - g(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|^k} < \epsilon.$$

Multiplying by $\|\mathbf{x} - \mathbf{x}_0\|^k$, which is nonzero (since we are assuming $\mathbf{x} \neq \mathbf{x}_0$),

$$|f(\mathbf{x}) - g(\mathbf{x})| < \epsilon \|\mathbf{x} - \mathbf{x}_0\|^k$$

which implies the result.

(NOTE: We did not need $\le$ in the conclusion of the statement of the problem. If we had written $\mathbf{x} \in \mathcal{O}$ (as opposed to what we had: $\mathbf{x} \in \mathcal{O} \setminus \{\mathbf{x}_0\}$) in the hypothesis of the problem, we would have needed to allow equality since if $\mathbf{x} = \mathbf{x}_0$, we have $|f(\mathbf{x}) - g(\mathbf{x})| = \epsilon \|\mathbf{x} - \mathbf{x}_0\|^k = 0$.)