MATH 125

Composite formula

$$\int_a^b f(x)\,dx$$

$h = x_{i+1} - x_i \quad ; \quad h = \frac{b-a}{m}$

Trapezoid

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right] - \frac{h^3}{12} f''(c_i)$$

$$\Longrightarrow \int_a^b f(x)\,dx = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$$

$$= \sum_{i=0}^{m-1} \left(\left[\frac{h}{2}\left(f(x_i) + f(x_{i+1})\right) \right] - \frac{h^3}{12} f''(c_i) \right)$$

$$= \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{m-1} f(x_i) \right] - \underbrace{\sum_{i=0}^{m-1} \frac{h^3}{12} f''(c_i)}_{*}$$

Let's simplify $*$

Theorem Let f be continuous on $[a, b]$. consider points $x_1, x_2, \ldots, x_n$ in $[a, b]$ and $a_1, \ldots, a_n > 0$. Then there exists a number c between a and b such that

$$(a_1 + a_2 + \ldots + a_n) f(c) = a_1 f(x_1) + a_2 f(x_2) + \ldots + a_n f(x_n)$$

Proof $f(x_i) \equiv$ Minimum value

$f(x_j) \equiv$ Maximum value

Note $a_1 f(x_1) + \ldots + a_n f(x_n) \geq a_1 f(x_i) + \ldots + a_n f(x_i)$

we also note $a_1 f(x_1) + \ldots + a_n f(x_n) \leq a_1 f(x_j) + \ldots + a_n f(x_j)$

$$f(x_i) \leq \frac{a_1 f(x_1) + \ldots + a_n f(x_n)}{a_1 + a_2 + \ldots + a_n} \leq f(x_j)$$

By the intermediate value theorem, there is a number c between x_i and x_j such that

$$f(c) = \frac{a_1 f(x_1) + \dots + a_n f(x_n)}{a_1 + \dots + a_n}$$

$$\sum_{i=0}^{m-1} \frac{h^3}{12} f''(c_i) = \frac{h^3}{12}\, m\, f''(c) \quad \text{for some } a < c < b$$

Trapezoid rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i \right) - \frac{h^3}{12}\, m\, f''(c)$$

$$\frac{h^2}{12}\, \underbrace{mh}_{(b-a)}\, f''(c)$$

$$= \frac{h}{2}\left(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i \right) - \frac{(b-a)h^2}{12} f''(c)$$

Composite Simpson's rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + y_{2m} + 4\sum_{i=1}^{m} y_{2i-1} + 2\sum_{i=1}^{m-1} y_{2i} \right] - \frac{(b-a)h^4 f''''(c)}{180}$$

We consider integrations $[x_{2i}, x_{2i+2}]$ for $i = 0, \dots, m-1$

Composite midpoint rule

$$\int_a^b f(x)\,dx = h\sum_{i=1}^{m} f(w_i) + \frac{(b-a)h^2}{24} f''(c) \qquad h = \frac{b-a}{m}$$

$w_i \equiv$ Midpoints of the subintervals of $[a, b]$

Gaussian quadrature

Degree of precision of quadrature: the degree for which all polynomial functions are integrated by the method with no error

Example Trapezoid $\to$ degree of precision $= 1$

Simpson $\to$ degree of precision $= 3$

Definition: The set of functions $\{P_0, P_1, \dots, P_n\}$ on the interval $[a, b]$ is orthogonal if

$$\int_a^b P_j(x)\, P_k(x)\,dx = \begin{cases} 0 & j \neq k \\ \neq 0 & j = k \end{cases}$$

Theorem: If $\{p_0, p_1, \dots, p_n\}$ is an orthogonal set of polynomials on $[a, b]$, where $\deg p_i = i$, then $\{p_0, p_1, \dots, p_n\}$ is a basis for the vector space of degree ~~n~~ at most n polynomials on $[a, b]$.

Proof PART I: use induction to prove span

PART II: Assume $\sum_{i=0}^{n} c_i p_i(x) = 0$

$$0 = \int_a^b p_k \left(\sum_{i=0}^{n} c_i \, p_i(x) \right) dx$$

$$0 = \sum_{i=0}^{n} c_i \int_a^b p_k \, p_i \, dx = c_k \int_a^b p_k^2 \, dx = 0 \Rightarrow c_k = 0$$

This holds for $0 \le k \le n$

Theorem: If $\{p_0, \dots, p_n\}$ is an orthogonal set of polynomials on $[a, b]$ and if $\deg p_i = i$, then p_i has i distinct roots in the interval (a, b).

Proof: Let $x_1, \dots, x_r$ be all distinct roots of $p_i(x)$ in (a, b) with odd multiplicity

Existence of root $\langle p_i, 1 \rangle = 0 \Rightarrow \int_a^b p_i(x) \, dx = 0$

$p_i(x)$ must change sign at least once in (a, b).
Therefore, $r \ge 1$.

Important remark The sign of $p_i(x)$ only changes at the roots

Let $q(x) = (x - x_1)(x - x_2) \cdots (x - x_r)$

Note that $p_i(x) \, q(x) \geq 0 \Rightarrow$ Exercise!

$$\int_a^b p_i(x) \, q(x) \neq 0 \quad *$$

If $r < i$, $q(x) = \sum_{j=1}^{k} c_j \, p_j$; $k < i$

However $\int_a^b p_i(x) \sum_{j=1}^{k} c_j \, p_j = 0$. This contradicts $*$

Therefore, $r = i$.
All roots of $P_i(x)$ lie in (a, b) and are distinct

Legendre polynomials

$$P_i(x) = \frac{1}{2^i i!} \frac{d^i}{dx^i} [(x^2-1)^i] \qquad 0 \le i \le n \text{ is orthogonal}$$

on $[-1, 1]$

* Note: Legendre polynomial of degree n has n roots $x_1, x_2, \dots x_n$ in $[-1, 1]$

Interpolation using Legendre roots:

$$Q(x) = \sum_{i=1}^{n} f(x_i)\, l_i(x)$$

Gaussian quadrature

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} \sum_{i=1}^{n} f(x_i)\, l_i(x)\,dx$$

$$= \sum_{i=1}^{n} c_i f(x_i) \qquad c_i = \int_{-1}^{1} l_i(x)\,dx$$

Theorem Gaussian quadrature using the degree n Legendre polynomials has degree of precision $2n-1$.

Proof $P(x) \equiv$ polynomial of degree atmost $2n-1$

$$P(x) = \underbrace{S(x) \quad P_n(x) + R(x)}_{\text{polynomials of degree less than } n}$$

Exercise $P(x_i) = ?$

$$\Rightarrow P(x_i) = R(x_i) \quad \text{since } P_n(x_i) = 0$$

$$\int_{-1}^{1} P(x)\,dx = \underbrace{\int_{-1}^{1} \cancel{S(x) P_n(x)}\,dx}_{=0} + \int_{-1}^{1} R(x)\,dx$$