MATH 65, FALL 2021, HOMEWORK QUESTIONS

Contents

1. First Homework, Sets, due September 16

Question 1.1. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{6, 7\}$. Please determine
(a) $A \cup B$
(b) $A \cap B$
(c) $A - B$
(d) $B - A$
(e) $A \times B$
(f) $\mathcal{P}(B)$

Question 1.2. Define two sets S, T by $S = \{x \in \mathbb{R} \mid x^2 < x\}$, $T = \{x \in \mathbb{R} \mid 0 < x < 1\}$. Show that $S = T$. Remember that in order to prove an equality of sets, you need to show that each set is contained in the other set.

Question 1.3. For any natural number $n \geq 1$, define the following intervals of the real line

$$A_n = (\frac{-1}{n}, \frac{1}{n}) = \{x \in \mathbb{R} | -\frac{1}{n} < x < \frac{1}{n}\}, \quad B_n = (n - \frac{1}{2}, n + \frac{1}{2}) = \{x \in \mathbb{R} | n - \frac{1}{2} < x < n + \frac{1}{2}\}$$

(a) Find $\bigcup_{n \in \mathbb{N}-\{0\}} A_n$ and $\bigcap_{n \in \mathbb{N}-\{0\}} A_n$.
(b) Find $\bigcup_{n \in \mathbb{N}-\{0\}} B_n$ and $\bigcap_{n \in \mathbb{N}-\{0\}} B_n$.

Question 1.4. Prove or disprove: $A \cup B = A \cap B$ if and only if $A = B$.

Question 1.5. Let

$$A = \{x \in \mathbb{R} \,|\, x^3 - 4x \geq 0\}, \;\; B = \{x \in \mathbb{R} \,|x^3 - x < 0\}.$$

(1) Describe A as a union of intervals on the real line.
(2) Describe B as a union of intervals on the real line.
(3) Determine $A \cap B$. Write it as a union of disjoint intervals in the real line.
(4) Determine $A \cup B$. Write it as a union of disjoint intervals in the real line.

Question 1.6. Let $\mathcal{P}(X)$ denote the power set of a set X, that is, the set of subsets of X

(a) Show that it is not in general true that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ (give an example of some A, B for which this is not true).
(b) If possible, give an example of A, B for which $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
(c) Determine the conditions on A, B under which $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ or show that this never happens.

2. Second homework, Proofs, due September 23

Question 2.1. (a) Prove that an integer is odd if and only if it is a sum of two consecutive integers.

(b) Is every even integer a sum of two consecutive even integers? Prove or disprove.

Question 2.2. Let a, b, c be integers. Show that if a does not divide bc, then a does not divide b. Hint: Use contrapositive.

Question 2.3. Prove that the two statements $(P \wedge Q) \vee (P \wedge \neg Q)$ and P are logically equivalent.

Question 2.4. Write the following sentence using the quantifier notation (that is the symbols $\exists$, $\forall$..) We do not claim these statements are true, so please do not try to prove them

(a) Every integer is prime
(b) There is an integer that is neither prime nor composite.
(c) For every integer x there exists an integer y such that $xy = 1$.
(d) There is an integer x such that for every integer y, $xy = 1$.
(e) For every integer x and every integer y, $x + y = y + x$.
(f) There is an integer x and an integer y such that $\frac{x}{y}$ is an integer.

Question 2.5. Write the negation of each of the sentences above, so that the negation appears as late in the sentence as possible. For example, in the negation of (e), the last piece of your statement should be $x + y \neq y + x$.

Question 2.6. The symbol $\exists!$ means " there exists a unique". So the statement $\exists! x\ P(x)$ (there exists a unique x satisfying $P(x)$) will fail if either there is no x satisfying $P(x)$ or if there is more than one x satisfying $P(x)$. Decide whether each of the following claims is true or false and prove your assertion. Here $\mathbb{N}$ denotes the set of natural numbers

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

(a) $\exists! x \in \mathbb{N}$ such that $x^2 = 4$.
(b) $\exists! x \in \mathbb{N}$ such that $x^2 = 3$.
(c) $\exists! x \in \mathbb{Z}$ such that $x^2 = 4$.
(d) $\exists! x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}\ xy = x$.
(e) $\exists! x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}\ xy = y$.

Question 2.7. Let a, b, c be odd integers. Prove that the equation

$$ax^2 + bx + c = 0$$

has no rational solution, i.e. there is no rational number (p/q) which can be plugged in for x to make the equation true. Hint: Follow the ideas in the proof of the fact that $\sqrt{2}$ is not rational.

3. Homework 3, Induction, due September 30

Question 3.1. If n is a positive integer, prove using induction that

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}$$

Question 3.2. Find and prove a formula for $\sum_{i=1}^{n} \frac{1}{i(i+1)}$.

Question 3.3. Determine the set of natural numbers n for which $3^{n+1} > n^4$. Then prove your assertion.

Question 3.4. (a) for $p \in \mathbb{N} - \{0, 1\}$, define what it means that p is prime.

(b) Show using strong induction that every natural number greater than 1 is a product of primes .

Question 3.5. In a hotdog line of $n \geq 2$ people at Fenway, everybody is wearing either Red Sox or NY Yankees apparel (and not both). If the first person in the line is a Red Sox fan and the last one a NY Yankees fan, show that somewhere in the line there is a Red Sox fan standing immediately before a NY Yankees fan.

Question 3.6. (a) Assume that b is a real number different from zero and that the sequence $a_n = b^n$ satisfies $a_n = 6a_{n-1} - 9a_{n-2}$. Find all possible values of b that work.

(b) Show that the sequence $b_n = n3^n$ satisfies the recurrence $b_n = 6b_{n-1} - 9b_{n-2}$.

(c) Find as many sequences as possible satisfying all of the conditions below

$$c_n = 6c_{n-1} - 9c_{n-2},\ c_0 = -2,\ c_1 = 6$$

Question 3.7. Recall that a basic L-shape (left image in picture below)is made of three identical squares shaped like an L. We say a board admits an L-tiling if it is possible to completely cover it with basic non-overlapping L-shapes.

(1) Prove that a $2^k \times 2^k$ chessboard with a single square in the lower left corner deleted admits an L-tiling, for any $k \in \mathbb{N}$.
(2) Prove that a $2^k \times 2^k$ chessboard with *any* single square deleted admits an L-tiling, for any $k \in \mathbb{N}$.

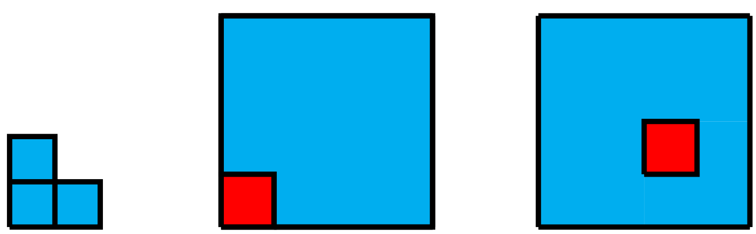

Figure 1. Basic L-shape, square with missing left corner and with missing arbitrary square respectively.

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4. Homework 4, Functions, due Thursday October 7

Question 4.1. Let $f : A \to B$ be a function. Show that f is onto if and only if there exists a function $g : B \to A$ such that $f \circ g = I_B$.

Question 4.2. Let

$$\begin{array}{rcccc} f & : & \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \times \mathbb{Z} \\ & & (a, b) & \to & (a + b, a - b) \end{array}$$

(a) Prove or disprove that f is one to one.
(b) Prove or disprove that f is onto

Question 4.3. Let $f : A \to B$ be a function. We are NOT assuming that f is a bijection, so f^{-1} is not defined as a function. If $X \subseteq A$, we define the subset of elements in B that are images of elements in X that is denoted with the notation $f(X)$ as follows

$$f(X) = \{y \in B \text{ such that } \exists x \in X,\ y = f(x)\}.$$

If $Y \subseteq B$, we define the subset of elements in A that map to Y that is denoted with the (somehow confusing) notation $f^{-1}(Y)$ as follows

$$f^{-1}(Y) = \{x \in A. \text{ such that } f(x) \in Y\}.$$

(a) If $Y \subseteq B$, show that $f((f^{-1}(Y)) \subseteq Y$.
(b) Show that if $\forall Y \subseteq B,\ f((f^{-1}(Y)) = Y$, then f is onto.
(c) Show that if f is onto, then $\forall Y \subseteq B,\ f((f^{-1}(Y)) = Y$.

Question 4.4. (a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 4x^3 + 13$. Is f a bijection? Carefully justify your answer.
(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 4x^3 + 13$. Is f a bijection? Carefully justify your answer.

Question 4.5. Let $f : S \to T$ be a function, and let A and B be subsets of S. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. Give an example to show that the reverse inclusion need not hold.

Question 4.6. Let A, B, C be three sets, f, g, h functions

$$f : A \to B,\ g : B \to C,\ h = g \circ f : A \to C$$

where h is the the composition of f, g given by $h(a) = g(f(a))$.

(1) Prove or disprove if f, g are one-to one, then h is one-to one.
(2) Prove or disprove if f, g are onto, then h is onto.
(3) Prove or disprove if f, g are bijections, then h is a bijection.
(4) Prove or disprove if h is one-to one then both f, g are one-to one.

5. Homework 5, Cardinality, due October 14

Question 5.1. Let A and B be finite sets

(a) Prove that if there is a one to one function $f : A \to B$ then B has at least as many elements as A ($|A| \leq |B|$).
(b) Prove that if there is an onto function $f : A \to B$ then B has at most as many elements as A ($|B| \leq |A|$).

Question 5.2. Let A and B be finite sets both with n elements.

(a) Prove that a function $f : A \to B$ is injective if and only if it is surjective.
(b) Prove that the equivalence is false if A is infinite. In particular, give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is injective, but not surjective, and a function $g : \mathbb{N} \to \mathbb{N}$ which is surjective, but not injective.

Question 5.3. In this question, for a, b real numbers, we use the notation you were accustomed to in Calculus for intervals on the real line and write

$$(a, b) = \{r \in \mathbb{R} \text{ such that } a < x < b\}$$

(a) If a, b, c, d are real numbers, $a < b, c < d$, construct a bijection between the intervals (a, b) and (c, d). Make sure that you prove that your function is a bijection.
(b) If a, b are real numbers, $a < b$, construct a bijection between the interval (a, b) and the whole real line $\mathbb{R}$. Make sure that you prove that your function is a bijection.

Question 5.4. (a) Let T be a set, and let S_1, S_2 be subsets of T. Suppose that $T = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Prove that if S_1 and S_2 are countable, then T is countable.
(b) Let $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. Prove that $\mathbb{I}$ is uncountable.

Question 5.5. Let S be a finite set of characters, and let T be the set of all finite (but arbitrarily long) sequences of characters in S. Thus, if $S = \{A, G, C, T\}$, then T is the set of all possible DNA sequences. If S is the set of keys on your keyboard, then T is the set of all possible sentences in the english language. Prove that T is countable.

Question 5.6. For a finite set A, the number of elements in the set is denoted by $|A|$.

(a) Assume that A, B are finite sets. Prove that

$$|A \cup B| = |A| + |B| \Leftrightarrow A \cap B = \emptyset$$

(b) Assume that A, B, C are finite sets. Denote by $|A|$ the number of elements in the set A. Prove or disprove

$$|A \cup B \cup C| = |A| + |B| + |C| \Leftrightarrow A \cap B = \emptyset,\ B \cap C = \emptyset, \text{ and } \quad C \cap A = \emptyset$$

6. Homework 6, Combinations, due October 28

Question 6.1. (a) How many sequences of length n can one make with the digits $0, 1$ containing exactly k 1's? (we assume $k \leq n$).

(b) How many sequences of length n can one make with the digits $0, 1, 2$ containing exactly k 1's? (we assume $k \leq n$).

Question 6.2. In how many ways can we partition a set with n elements into 2 part so that one part has 4 elements and the other part has all of the remaining elements (assume $n \geq 4$).

Question 6.3. (a) Prove algebraically that

$$\binom{n}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-2}{2} + \binom{n-1}{2}$$

Hint : you can use that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$.

(b) Use a combinatorial argument to show that

$$\binom{n}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-2}{2} + \binom{n-1}{2}$$

Hint: If your set is $\{1, 2, \ldots, n\}$ and you choose 3 elements out of it, think of the largest element in your subset.

Question 6.4. How many solutions are there to the equation

$$x + y + z = 85, \text{ such that } x, y, z \in \mathbb{N}, \text{ and } x \geq 5, \text{ and } z \leq 10$$

Question 6.5. Show that $\left(\binom{n}{k}\right) = \left(\binom{k+1}{n-1}\right)$ both algebraically and combinatorially.

Question 6.6. The bakery in your supermarket offers 4 different types of cookies: chocolate, peanut, sugar and oatmeal-raisin.

(a) In how many ways can you pick a dozen cookies so that at least three are chocolate?
(b) In how many ways can you pick a dozen cookies so that at most two are peanut?
(c) In how many ways can you pick a dozen cookies so that at least three are chocolate and at most two are peanut?
(d) In how many ways can you pick a dozen cookies so that at least three are chocolate or at most two are peanut?

Question 6.7. Show that for $n \geq 1$, the number of ways to group $2n$ people into n distinct pairs is $\frac{n!}{2^n}\binom{2n}{n} = \frac{(2n)!}{n!2^n}$.

Hint: Think instead about the problem of choosing n ordered pairs by choosing one element for each pair first.

7. Homework 7, Relations, due November 4

Question 7.1. We will say that an integer a is near an integer b if they are not more than two units apart. For example, 4 is near 6 and 4 is near 2. But 4 is not near 7. As the distance between two points can be computed with the absolute value, $R = \{(x, y) |\ |x - y| \leq 2\}$ or $x \sim y \Leftrightarrow\ |x - y| \leq 2$.
(a) Is $\sim$ reflexive?
(b) Is $\sim$ symmetric?
(c) Is $\sim$ antisymmetric?
(d) Is $\sim$ transitive?

Question 7.2. Prove or disprove: there are relations that are both symmetric and transitive but not reflexive.

Question 7.3. Determine if the following is an equivalence relation and if it is, describe the partition it induces

$$x, y \in \mathbb{Z}\quad x\mathcal{R}y \text{ if and only if x,y have the same number of digits}$$

Question 7.4. How many equivalence relations are there in the set $A = \{1, 2, 3\}$? Hint: How many partitions are there of the set $A = \{1, 2, 3\}$?

Question 7.5. (a) Consider the assignment $f_1 : \mathbb{Z} \to \mathbb{Z}_5$ given by $f_1(z) = [z]_5$. Is this a well defined function? Why or why not?
(b) Consider the assignment $f_2 : \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $f_2([z]_{10}) = [z]_5$. Is this a well defined function? Why or why not?
(c) Consider the assignment $f_3 : \mathbb{Z}_5 \to \mathbb{Z}_{10}$ given by $f_3([z]_5) = [z]_{10}$. Is this a well defined function? Why or why not?

Question 7.6. (a) Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y + [2]_{10} = [8]_{10}$ or show that there aren't any.
(b) Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y[2]_{10} = [8]_{10}$ or show that there aren't any.
(c) Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y[2]_{10} = [3]_{10}$ or show that there aren't any.

Question 7.7. Let $A = (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ be the set of pairs of strictly positive natural numbers. Define a relation in A by

$$(a, b), (c, d) \in A, \quad (a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

(a) Show that $\sim$ is an equivalence relation.
(b) Denote by B the set of equivalence classes of elements of A by the relation $\sim$ defined in part (a). Define an addition in B by

$$[(a, b)] + [(c, d)] = [a + c, b + d]$$

Show that this is a well defined addition among the cosets.
(c) Show that the function $f : A \to \mathbb{Z}$ given by $f((a, b)) = a - b$ gives rise to a well defined function on cosets $\bar{f} : B \to \mathbb{Z}$ given by $\bar{f}([(a, b)]) = a - b$ and that $\bar{f}$ is a bijection.
(d) Show that the addition of cosets that we defined in part (b) corresponds under the above function f to the addition of integers.(that is, show that $\bar{f}([(a, b)]) + \bar{f}([(c, d)]) = \bar{f}([a + c, b + d])$
(e) EXTRA CREDIT Could you define a product in B corresponding by $\bar{f}$ to the product in $\mathbb{Z}$?

8. Homework 8, Partial Orders and Probability, due November 11

Question 8.1. Let $A = \{1, 2, 3, 4, 5, 6, 9, 10, 12\}$ with the ordering by divisibility

(a) Draw the Hasse diagram for this poset.
(b) Find a minimal element or explain why none exists.
(c) Find a minimum element or explain why none exists.
(d) Find a maximal element or explain why none exists.
(e) Find a maximum element or explain why none exists.

Question 8.2. Let A be any set, $\mathcal{P}(A)$ the set of all subsets of A. Consider the inclusion relation

$$A_1, A_2 \in \mathcal{P}(A), \quad A_1 \preceq A_2 \;\Leftrightarrow A_1 \subseteq A_2$$

(a) Show that $(\mathcal{P}(A), \preceq)$ is a poset.
(b) Let $(X, \preceq)$ be a poset. A *chain* is a collection of distinct elements $x_i \in X$ such that every two elements are related. ($x_i \preceq x_j$ or $x_j \preceq x_i$). If A is a set with n elements and we order $\mathcal{P}(A)$ by inclusion, find a chain of maximal length.
(c) How many chains of maximal length are there in $\mathcal{P}(A)$?
(d) Let $(X, \preceq)$ be a poset. A *anti chain* is a collection of distinct elements $x_i \in X$ such that no two elements are related. For A a set and $\mathcal{P}(A)$ the set of all subsets of A ordered by inclusion, show that the set of subsets of A with fixed number of elements k form an antichain.
(e) If $A = \{1, 2, 3\}$, describe all possible antichains for $\mathcal{P}(A)$.

Question 8.3. In bridge, a player is dealt a 13-card hand from a 52-card deck. There are 13 kinds of cards: Ace, King, Queen, Jack, 10, 9, $\dots 2$, and 4 suits: hearts, clubs, spades, and diamonds. A pair consists of exactly 2 cards of the same kind, e.g., 2 Queens.

(a) What is the probability of being dealt a hand with no Jack, Queen, or King?
(b) What is the probability of being dealt a hand with all 4 Kings and exactly 3 Queens?
(c) What is the probability of being dealt a hand with exactly 5 pairs and 3 other cards, each of a different kind from the pairs and from each other?

Question 8.4. Let U be the sample space of randomly generated bit strings of length four. Let E be the event that the bit string has an even number of 1's. Let F be the event that the bit string starts with a 0.

(a) Compute $P(E), P(F), P(E \cap F)$.
(b) What is $P(E|F)$?
(c) What is $P(F|E)$?
(d) Are E and F independent

Question 8.5. Prove Bayes' Theorem: Suppose that S is a finite probability space, $A \subset S$ is an arbitrary event, and $B_1, \dots, B_r \subset S$ are mutually exclusive events which partition S, i.e. $S = B_1 \cup \ldots \cup B_r$. Suppose you are given that $P(B_i) = b_i$ and $P(A|B_i) = a_i$. Prove that

$$P(B_i|A) := \frac{a_i b_i}{\sum_{i=1}^{r} a_i b_i}$$

Question 8.6. You hold a bag of ten coins. Nine of them are fair, but one is loaded - it shows heads with probability 9/10. You draw out a coin at random and begin flipping it. The first five tosses are $HHHTH$. What is the probability that you are flipping one of the fair coins?

Question 8.7. Consider a dial having a pointer that is equally likely to point to each of n regions numbered $1, 2, \ldots, n$ in cyclic order. When the selection is k, the gambler receives 2^k dollars.

(1) What is the expected payoff per spin of the dial?
(2) Suppose that the gambler has the following option. After rate spin, the gambler can accept that payoff or flip a coin to change it. If the coin shows heads, the pointer movies one spot counterclockwise; if tails, it moves one spot clockwise. When should the gambler flip the coin? What is the expected payoff under the optimal strategy?

9. Homework 9, Graphs, due November 18.

Question 9.1. Show that in any graph, there is an even number of vertices of odd degree.

Question 9.2. (a) Prove that in any simple graph with two or more vertices, there are at least two vertices of the same degree.
(b) Is the result in (a) true for non-simple graphs?

Question 9.3. A graph is called r-regular if all vertices have degree r. If G is an r-regular graph, find a relationship between the number of vertices, the number of edges and r.

Question 9.4. If $G = (V, E, f)$ is a graph and $V_1 \subseteq V$ is a subset of the set of vertices of G, then the induced subgraph is the subgraph that has vertex set V_1 and edge set those edges in E such that both vertices in the edge are in V_1.

(a) If G is the complete graph K_n with n vertices, how many induced subgraphs does it have? How many non-isomorphic induced subgraphs does it have?
(b) If G is the complete graph K_n with n vertices, how many spanning subgraphs does it have? A subgraph is called a spanning subgraph if the set of vertices of the subgraph is the same as the set of vertices of the graph

Question 9.5. Recall that two graphs $(V_1, E_1, f_1), (V_2, E_2, f_2)$ are said to be isomorphic if there exists bijections $g_V : V_1 \to V_2,\ g_E : E_1 \to E_2$ compatible with the incidence morphisms f_1, f_2 meaning that if $g_E(e_1) = e_2$ and the vertices of e_1 are a, b, then the vertices of e_2 are $g_V(a), g_V(b)$.

(a) Show that" being isomorphic to" is an equivalence relation.
(b) Describe the isomorphism classes of simple graphs with 3 vertices. Note: a graph with vertex set $V = \{a, b, c\}$ and edge set $E = \{\{a, b\}\}$ is considered a different graph from the graph with vertex set $V = \{a, b, c\}$ and edge set $E' = \{\{a, c\}\}$

10. Homework 10, Subgraphs and Paths due December 2

Question 10.1. If G is a graph, a clique is a subgraph of G such that there is an edge between every two vertices. The clique number $\omega(G)$ is the number of vertices in the largest clique in G. An independent set is a subgraph of G such that there are no edges between any two vertices of the set. The independence number $\alpha(G)$ is the number of vertices in the largest independent set in G. Assume that G is a graph and H a subgraph.

(a) Prove or disprove $\alpha(G) \leq \alpha(H)$.
(b) Prove or disprove $\alpha(G) \geq \alpha(H)$.
(c) Prove or disprove $\omega(G) \leq \omega(H)$.
(d) Prove or disprove $\omega(G) \geq \omega(H)$.

Question 10.2. Find a graph G with $\alpha(G) = \omega(G) = 5$. What is the smallest number of vertices that G could have if it satisfies these conditions? Carefully justify your answer.

Question 10.3. Find a graph with 5 vertices that has clique number two AND independence number two or show that none exists.

Question 10.4. Given a simple graph G show that either G or its complement (or both) are connected.

Question 10.5. (a) Show that if a **simple** graph G with $n \geq 2$ vertices has at least $\binom{n-1}{2}+1$ edges, then G is connected.
(b) For every $n \geq 2$ give an example of a simple graph G with $\binom{n-1}{2}$ edges, such that G is not connected.

Question 10.6. Let $A_n = \{1, 2, \ldots, n\}$. Define a graph G_n whose vertex set V_n is the set of pairs of elements in A and whose edge set E_n is defined by "there is an edge between $\{a, b\}$ and $\{c, d\}$ if and only if $\{a, b\} \cap \{c, d\} = \emptyset$".

(a) Sketch G_2, G_3, G_4, G_5.
(b) How many vertices does G_n have? What is the degree of each vertex?
(c) How many edges does G_n have?
(d) Find the number of connected components of G_2, G_3, G_4, G_5, Find the number of connected components of G_n. Justify your answer.

Question 10.7. We will talk about Eulerian graphs on November 29.
Let G be a connected graph that is not eulerian (no Euler circuit).

(a) Show that we can add to G a single vertex and a few edges starting at this vertex so that the resulting graph is eulerian.
(b) Show that we can add to G a few edges (no new vertices) to make it eulerian, What is the smallest number of edges we need to add for this to happen?
(c) Let G be a simple connected graph that is not eulerian. Can we add to G a few edges (no new vertices) to make it eulerian, while keeping it simple?.

11. Homework 11, Trees and planar graphs, due December 9.

Question 11.1. Recall that a leaf in a tree is a vertex of the tree of degree one. We showed that if T is a tree and v a leaf, then $T - \{v\}$, the graph obtained by removing from T the vertex v and any edges incident with v is also a tree. Prove the converse: assume that T is a tree with at least 2 vertices, v a vertex in the tree such that $T - \{v\}$ is also a tree. Show that then v is a leaf.

Question 11.2. Recall that an n-cycle is a simple closed path of length n.

Let T be a tree, u, v vertices of T such that there is no edge between u, v. Consider the graph G obtained by adding to the edges of T a new edge e from u to v. Show that G has a unique cycle

Question 11.3. Let G be a connected graph in which every vertex has degree precisely 2. Is G then necessarily a cycle?

Question 11.4. Let G be a planar graph with v vertices, e edges, f faces and c connected components. Prove a relation between these numbers.

Question 11.5. Let G be a simple graph with 11 vertices. Show that at least one of G or its complement $\bar{G}$ is non-planar. Hint: If a graph is planar, the number of edges is bounded in terms of the number of vertices.

Question 11.6. Recall that the bipartite graph $K_{m,n}$ has a set of vertices V that can be written as the disjoint union of two subset V_1, V_2 with m and n vertices respectively and has one edge for every pair of vertices, the first from V_1 and the second from V_2.

For which values of m, n is the bipartite graph $K_{m,n}$ planar?

Question 11.7. (a) Show that there are exactly $2^{\binom{n}{2}}$ simple graphs with vertex set $\{v_1, \ldots, v_n\}$.

(b) Show that there are exactly $2^{\binom{n-1}{2}}$ simple graphs with vertex set $\{v_1, \ldots, v_n\}$ in which every vertex has even degree. (Hint: Show that there is a bijection between simple graphs with n vertices all of even degree and simple graphs with $n - 1$ vertices of arbitrary degree).

(c) Compute the probability that for a randomly chosen simple graph with vertex set $\{v_1, \ldots, v_n\}$ every connected component will have an Eulerian circuit .