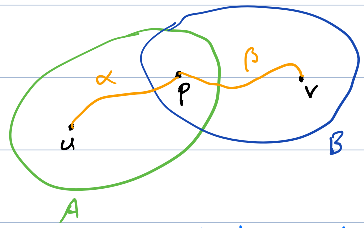
Math 135   **Solutions to Problem Set 10**   Fall 2022

1. (10 points) (**Path-connectedness of a union**) §11.3, p. 309, #2.
Let A and B be path-connected subsets of $\mathbb{R}^n$ whose intersection is nonempty. Prove that the union $A \cup B$ is also path-connected.

Solution.

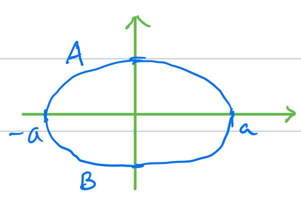


Let p be a point in the nonempty intersection $A \cap B$. If $u \in A$ and $v \in B$, then since A is path-connected, there is a path $\alpha : [a, b] \to A$ joining u and p in A. Similarly, since B is path-connected, there is a path $\beta : [b, c] \to B$ joining p and v in B. The concatenation of α followed by β is a path $: [a, c] \to A \cup B$ joining u and v in $A \cup B$. $\square$

2. (10 points) (**Path-connectedness of an ellipse**) §11.3, p. 309, #3.
Let a and b be positive real numbers. Use the path-connectedness of a graph on a path-connected domain and the previous problem to prove that the ellipse

$$\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$$

is path-connected. (*Hint*: Write the ellipse as the union of its closed upper half and its closed lower half. Then use #1.)



Let $A = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ y \geq 0\}$
$B = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ y \leq 0\}$.

Then $A \cap B = \{(\pm a, 0) \in \mathbb{R}^2\} \neq \varnothing$ and ellipse $= A \cup B$.

Since A is the graph of $z = b\sqrt{1 - \frac{x^2}{a^2}}$ and B is the graph of $z = -b\sqrt{1 - \frac{x^2}{a^2}}$ over the path-connected domain $[a, b]$, both A and B are path-connected. By #1, the ellipse, which is $A \cup B$, is path-connected. $\square$

3. (10 points) (**Path-connected subsets of $\mathbb{Q}$**)
   (a) Describe all path-connected subsets of $\mathbb{Q}$.
   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and let $A \subset \mathbb{R}^n$. Assume A is path-connected *and* assume $f(A) \subset \mathbb{Q}$. What can you say about the image $f(A)$?

(a) Pathwise connected subsets of $\mathbb{Q}$ are single points, since between any two rational numbers there is an irrational numbers so that it is not possible to connected two rational numbers with a path in $\mathbb{Q}$.

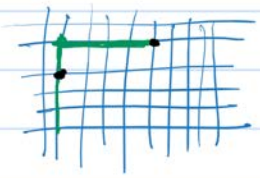
(b) Since $f(A)$ is a path-connected subset of $\mathbb{Q}$, it is a single point.

4. (10 points) (**Connectedness**) §11.4, p. 313, #3.
   Let A be a connected subset of $\mathbb{R}^3$. Suppose that the points $(0,0,1)$ and $(4,3,0)$ are in A.
   (a) Prove that there is a point in A whose second component is 2.
   (b) Prove that there is a point in A whose norm is 4.
   (*Hint*: Use the intermediate-value property of connectedness. In (a), which function is 2 an intermediate value of?)

a) Let $f: A \to \mathbb{R}$ be $f(x,y,z) = y$. Since A is connected and f is continuous, the intermediate value theorem holds for f. Note that $f(0,0,1) = 0$ and $f(4,3,0) = 3$. Since 2 is an intermediate value, there is a point p in A such that $f(p) = 2$.

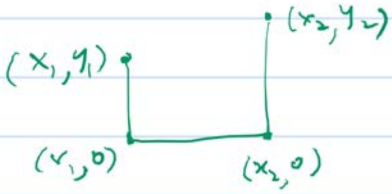
b) The norm $\| \ \| : A \to \mathbb{R}$, given by $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$, is a continuous function. Note that $\|(0,0,1)\| = 1$ and $\|(4,3,0)\| = 5$. Since 4 is an intermediate value, there is a point $p \in A$ such that $\|p\| = 4$. $\square$

5. (10 points) (**Rational and irrational numbers**) §11.3, p. 309, #6.
Show that the set $S = \{(x, y) \in \mathbb{R}^2 \mid x \text{ or } y \text{ is rational}\}$ is path-connected. (*Hint*: You may draw a picture and use it to describe paths between any two points in S.)



Given two points (x_1, y_1) and (x_2, y_2) in S, if $x_1 \in \mathbb{Q}$ and $y_2 \in \mathbb{Q}$, then both points connect to (x_1, y_2) by a path in S.

If $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}$, then the piecewise-linear path from (x_1, y_1) to $(x_1, 0)$ to $(x_2, 0)$ to (x_2, y_2) connects (x_1, y_1) and (x_2, y_2).



The other two cases are similar. $\square$

6. (20 points) (**Connectedness and the intermediate-value property**)
In this problem, you will show some useful facts about continuous functions. Let A be a subset of $\mathbb{R}^n$ and let $F : A \to \mathbb{R}^m$ be continuous.

(a) Let B be a nonempty subset of A. Prove using the definition that the function F on the smaller domain B, $F : B \to \mathbb{R}^m$ is continuous.

Solution.

Let $\{u_k\}$ be a sequence in B that converges to $b \in B$. Since $B \subseteq A$, $\{u_k\}$ is also a sequence in A that converges to $b \in A$. Since $F : A \to \mathbb{R}^n$ is continuous at a, $F(u_k) \to F(b)$. Therefore, $F : B \to \mathbb{R}^n$ is continuous.

(b) Let U and V be *disjoint open* subsets of $\mathbb{R}^n$. Prove that the function $f : U \cup V \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$$ is continuous.

In a neighborhood of $u \in U$, f is a constant, so it is continuous. Similarly, f is also a constant in a neighborhood of $v \in V$. Thus, f is continuous on $U \cup V$.

(c) Now let A be a set in $\mathbb{R}^n$ that is not connected. Find a function $f : A \to \mathbb{R}$ that is continuous and such that $f(A)$ is not an interval.

If A is not connected, then A has a separation U, V in $\mathbb{R}^n$.
Define $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for } x \in U \cap A \\ 1 & \text{for } x \in V \cap A. \end{cases}$$

Then f is continuous at $x \in U \cap A$ and at $x \in V \cap A$, so f is continuous on A, but $f(A) = \{0, 1\}$ is not an interval.

(d) (2 points) Does a disconnected set A have the intermediate-value property? Why or why not?

A does not have the IVP, because on it there is a continuous function as in (c), with $f(A) = \{0, 1\}$, so $1/2$ is an intermediate value that is not in $f(A)$. $\square$

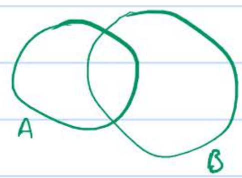
7. (10 points) (**Convex sets**)

(a) Let A and B be convex sets in $\mathbb{R}^n$. Is the intersection, $A \cap B$ convex? Either prove this or draw a counterexample and explain why it is a counterexample.

(b) Let A and B be pathwise connected sets in $\mathbb{R}^n$. Is the intersection $A \cap B$ pathwise connected? Either prove this or draw a counterexample and explain why it is a counterexample.

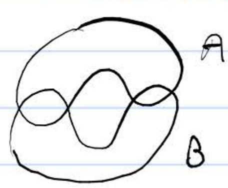
(c) (3 points) Why can't the argument from part (a) be used to prove $A \cap B$ is pathwise connected in part (b)?

(a) Is the intersection of convex sets convex? Prove.



Suppose A, B convex. Let $x, y \in A \cap B$.
Since A is convex and $x, y \in A$, the line segment $\overline{xy}$ joining x and y lies in A.
For the same reason, the line segment $\overline{xy}$ lies in B. Therefore, $\overline{xy}$ lies in $A \cap B$.
So $A \cap B$ is convex.

(b) Is the intersection of path-connected sets A, B path-connected?

No, see the example 

. The intersection is  .

(c) Why can't the argument from (a) be used to prove $A \cap B$ path-connected in (b)?

The line segment joining two points x, y is unique.
The parametrized path joining two points x and y is not unique.

8. (10 points) (**Topologist's sine curve**) §11.4, p 313, #7
Let K be the closed interval $\{0\} \times [-1,1]$ and $G = \{(x, \sin 1/x) \mid x \in (0,1]\}$. The topologist's sine curve is the union $A = K \cup G$ (Example 11.38, p. 312). Show that the topologist's sine curve is not path-connected. (*Hint*: Suppose that there is a parametrized path $\gamma\colon [0,1] \to \mathbb{R}^2$ in A joining $(0,1)$ to $(1, \sin 1)$. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Define t_* to be the supremum of the points t in $[0,1]$ such that γ maps the interval $[0,t]$ into K. This means $\gamma_1(t) \in K$ for all $t < t_*$, so $\gamma_1(t) = 0$ for all $t < t_*$. By continuity, $\gamma_1(t_*) = 0$. Express $\gamma_2(t)$ for $t > t_*$ in terms of $\gamma_1(t)$ and then show that $\gamma_2(t)$ is not continuous at t_*.)

Proof.

Let $u = (0,1)$ and $v = (1, \sin 1)$. Suppose there is a parametrized path $\gamma : [0,1] \to A$ joining u to v.
Let
$$t_* = \sup\{ t \in [0,1] \mid \gamma([0,t]) \subset K \}.$$
This means $\gamma(t) \in K$ for all $t < t_*$, so if $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, then $\gamma_1(t) = 0$ for all $t < t_*$. By continuity, $\gamma_1(t_*) = 0$.
For each $n \in \mathbb{N}$, $\gamma([0, t_* + \frac{1}{n}]) \not\subset K$, so $\exists\ t_n \in [t_*, t_* + \frac{1}{n}]$ such that $\gamma(t_n) \notin K$. Since $\gamma(t_n) \in A$, we must $\gamma(t_n) \in G = \{(x, \sin \frac{1}{x}) \mid x \in (0,1]\}$.
By the sandwich theorem, $t_n \to t_*$.
By continuity, $\gamma(t_n) \to \gamma(t_*)$.
This is impossible, because $\gamma(t_n) = (\gamma_1(t_n), \gamma_2(t_n))$, where $\gamma_2(t_n) = \sin \frac{1}{\gamma_1(t_n)}$, but since $\gamma_1(t_n) \to \gamma_1(t_*) = 0$, $\gamma_2(t_n) = \sin \frac{1}{\gamma_1(t_n)}$ does not have a limit as $n \to \infty$.

This contradiction proves that there is no path in A joining u and v. $\square$