MATH 70 WORKSHEET 2 SOLUTIONS

Instructions: This worksheet is due on Gradescope at 11:59 p.m. Eastern Time on Sunday, September 27. You are encouraged to work with others, but the final results must be your own.[1]

Please give complete reasoning for all worksheet answers.

1. (6 points) Let A be an 3×4 matrix, $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ where $\mathbf{a}_i \in \mathbb{R}^3$ is the i^{th} column of A for $i = 1, 2, 3, 4$. Use Theorem 3 of Chapter 1 of the book to prove that if $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is a linear combination of the columns of A.

   Solution:
   Let matrix $A \in \mathbf{R}^{3\times 4}$ have columns $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$. If $A\mathbf{x} = \mathbf{b}$ is consistent, it must have a solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

   By Theorem 3, the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as a linear combination of the columns of A. Thus, since $A\mathbf{x} = \mathbf{b}$ has a solution, $\mathbf{b}$ can be written as a linear combination of the columns of A, i.e. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b}$.

2. (4 points) Let A be an $m \times n$ matrix. Can the zero vector $\mathbf{x} = \mathbf{0}$ be in the solution set to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$? Explain your reasoning.

   Solution:

   No. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. If $\mathbf{x} = \mathbf{0}$, then

$$\begin{aligned} A\mathbf{x} &= x_1\mathbf{a}_1 + \cdots x_n\mathbf{a}_n \\ &= 0\mathbf{a}_1 + \cdots 0\mathbf{a}_n \\ &= \mathbf{0} \neq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \end{aligned}$$

3. (10 points) Let $\mathbf{v}$ and $\mathbf{w}$ be any two distinct nonzero vectors in $\mathbb{R}^n$. Is the set

$$W = \{\mathbf{v}, \mathbf{w}, 2\mathbf{v} - 3\mathbf{w}\}$$

linearly independent or linearly dependent? Prove your answer.

Solution:
W is linearly dependent. We can prove it with the following dependence relation:

$$2\mathbf{v} - 3\mathbf{w} - (2\mathbf{v} - 3\mathbf{w}) = \mathbf{0}.$$

4. (5 points) Let A be a 2×2 matrix, $\mathbf{u}, \mathbf{v}$ two linearly independent vectors in $\mathbb{R}^2$. Suppose that the equation $Ax = \mathbf{u}$ has a unique solution $\mathbf{u}'$. Prove that the equation $Ax = \mathbf{v}$ has also a unique solution $\mathbf{v}'$, and that the vectors $\mathbf{u}'$ and $\mathbf{v}'$ are linearly independent.

Solution:
Since $A\mathbf{x} = \mathbf{u}$ has a *unique* solution, every column of A must be a pivot column. (A non-pivot column would correspond to a free variable, so a consistent system would have infinitely many solutions.) Since A has two columns, this means A has two pivots, one for each row. Thus A has a pivot in every row, so Theorem 4 implies $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$. In particular, the equation $A\mathbf{x} = \mathbf{v}$ has a solution. This solution is unique since A has no non-pivot columns (and hence there are no free variables).

Finally, we must show that if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and $A\mathbf{u}' = \mathbf{u}$, $A\mathbf{v}' = \mathbf{v}$, then $\mathbf{u}'$ and $\mathbf{v}'$ are linearly independent. Suppose

$$c_1\mathbf{u}' + c_2\mathbf{v}' = \mathbf{0}.$$

By the definition of linear independence, it will suffice to show that $c_1 = c_2 = 0$. Multiplying both sides of the equation by A, we have

$$A(c_1\mathbf{u}' + c_2\mathbf{v}') = A\mathbf{0} = \mathbf{0}.$$

On the other hand, using Theorem 5 (properties of the matrix-vector product) we have

$$A(c_1\mathbf{u}' + c_2\mathbf{v}') = c_1 A\mathbf{u}' + c_2 A\mathbf{v}' = c_1\mathbf{u} + c_2\mathbf{v}.$$

Thus

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}$ are linearly independent, this is only possible if $c_1 = c_2 = 0$, which is what we wanted.