Part II assignment 6: due Nov 13

Graded

Student

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Total Points

30 / 30 pts

Question 1

15.2

5 / 5 pts

✔ **– 0 pts** Correct

– 0 pts D_n has extra normal subgroups for each divisor of n, not just 2, but that is difficult to see from only D_4 and D_5

– 0.5 pts Did not find the subgroups of general D_n

Question 2

15.3

5 / 5 pts

✔ **– 0 pts** Correct

– 1 pt Did not show the subgroups are normal

Question 3

15.14

5 / 5 pts

✔ **– 0 pts** Correct

Question 4

15.15

5 / 5 pts

✔ **– 0 pts** Correct

– 0.5 pts It is unclear which groups your isomorphism is between, partly because you claim $G \cong \mathbb{Z}$ but also $H \cong \mathbb{Z}_2$ is a subgroup of G

– 0 pts If $\langle gH \rangle = G/H$ then $\langle g \rangle H = \mathbb{Z} \times \mathbb{Z}_2$ because g commutes with every element of H. It covers all of G because $|G| = |G/H| \cdot |H|$

– 0.5 pts When checking that a function $f : A \times B \to C$ is a homomorphism you need to show that for all $(a_1, b_1), (a_2, b_2) \in A \times B$, $f(a_1 a_2, b_1 b_2) = f(a_1, b_1) f(a_2, b_2)$. Otherwise you are not actually checking the group operation.

– 0.5 pts It is important to also explain why the $\mathbb{Z}_2$ subgroup is central

– 0.5 pts Problems defining isomorphism

Question 5

Kernel of. homomorphism is normal

5 / 5 pts

✔ **– 0 pts** Correct

– 0.5 pts Showed normal-ness but not subgroup-ness

– 0.5 pts Homomorphisms are not, in general, injective or surjective

– 0.5 pts Issues with proof that kernel is a subgroup

Question 6

Quotients of the infinite cyclic group

5 / 5 pts

✔ **– 0 pts** Correct

Question assigned to the following page: 1

M145 HW

15.2)
Am just showing for D_n, did conjugacy classes last HW.
If n is odd.
Well we know r^k is conjugate to r^{-k}
There, the normal subgroup is generated by $\langle r^k \rangle$ where k divides n. and $k \neq 1$, as gets full group

Additionally for odd n, we have the conjugacy class $\{r^k s \mid 0 \leq k \leq n-1\}$. However $|\{e\} \cup \{r^k s\}| = n+1$ which doesn't divide $2n$, so can't be normal subgroup.

For even n, we see how as r^k conjugate to r^{-k} we also have $\langle r \rangle$ but as even we can look at $\langle r^2 \rangle$
Well we know for even n, conjugacy is $\{r^{2i} s\}$ and $\{r^{2i+1} s\}$ So we can do is take $\langle r^2 \rangle \cup \{r^{2i} s\}$ and $\langle r^2 \rangle \cup \{r^{2i+1} s\}$ and this is a proper subgroup.

So in general (Also includ.
$n = \text{odd} \Rightarrow$ normal subgroup is $\langle r^k \rangle$ where k divides n.
$\wedge$
$n = \text{even} \Rightarrow \langle r^k \rangle$ where k divides n, along w/ $\{r^{2i} s \mid 0 \leq i \leq n/2 - 1\} \cup \langle r^2 \rangle$ and $\{r^{2i+1} s \mid 0 \leq i \leq n/2 - 1\} \cup \langle r^2 \rangle$

So for $D_4 = \langle r \rangle, \langle r^2 \rangle, \langle r^2, s \rangle, \langle r^2, rs \rangle$
for $D_5 = \langle r \rangle$

Questions assigned to the following page: 2 and 3

15.8) So the possible subgroups of Q are: (Cauchy's thm and table)
$\langle -1 \rangle, \langle i \rangle, \langle j \rangle, \langle K \rangle$

Now, as $|\langle i \rangle| = |\langle j \rangle| = |\langle K \rangle| = 4$, and $|Q| = 8$, then $8/4 = 2$ so the index of 3 subgroups is 2, so they are normal.

For $\langle -1 \rangle$, $g\langle -1 \rangle g^{-1} = \langle -1 \rangle$ as multiplication is commutative w/ $\mathbb{R}$ so commutes w/ every element and
$g\langle -1 \rangle g^{-1} = \langle -1 \rangle g g^{-1} = \langle -1 \rangle$
So $\langle -1 \rangle$ is normal, and all subgroups of Q are normal

15.14)
So $\mathbb{Q}/\mathbb{Z} = \{a + \mathbb{Z} \mid a \in \mathbb{Q}\}$
Let $x \in \mathbb{Q}/\mathbb{Z}$ so $x = a + \mathbb{Z}$ then $a = m/n$ where, $m, n \in \mathbb{Z}$ and $n \neq 0$
So $n(x = \frac{m}{n} + \mathbb{Z})$
$nx = m + \mathbb{Z}$, but as $m \in \mathbb{Z}$, then $m + \mathbb{Z} = \mathbb{Z}$ which is the identity so $|x| \leq n$, Meaning every element has finite order.

Now, $\mathbb{R}/\mathbb{Q} = \{a + \mathbb{Q} \mid a \in \mathbb{R}\}$ or more strongly $a \in \mathbb{Q}^c$ as if $a \in \mathbb{Q}$ then have $\mathbb{Q}$

Suppose $x \in \mathbb{R}/\mathbb{Q}$ and $|x| = n$, $n \in \mathbb{N}$ and $x = a + \mathbb{Q}$
then $x^n = (a + \mathbb{Q})^n \Rightarrow na + \mathbb{Q} = \mathbb{Q}$ as order n
So na is rational, this is a contradiction as a is irrational, so every non trivial element of $\mathbb{R}/\mathbb{Q}$ will have infinite order

Questions assigned to the following page: 4 and 5

15.15) So H = normal subgroup as $G \setminus H$ is infinite, cyclic then $G \setminus H \cong \mathbb{Z}$ also $H \cong \mathbb{Z}_2$ so w.t.s $G \cong G \setminus \mathbb{Z}_2 \times \mathbb{Z}_2$

Let $g \in G$, be the generator of of $G/\mathbb{Z}_2$, and $G/\mathbb{Z}_2 = \langle g \rangle$ s.t. $\angle$.

Now, as $H \cap \langle g \rangle = \{e\}$, we w.t.s $H\langle g \rangle = G$ called $\langle g \rangle K$

Let $x \in G$, then $xH \in G \setminus H$ so $xH = g^k H$ and $x^{-1} g^k = h \in H$ so $x = g^k h^{-1} \in KH$

Furthermore, as H is normal, we know it follows that $HK = KH = G$ so every element of H commutes w/ every element of K meaning that $G \cong H \times K$ so $G \cong \mathbb{Z} \times \mathbb{Z}_2$ $\square$

Extra 1)

$\text{Ker}\,\phi = \{x \in G \mid \phi(x) = e\}$

First w.t.s $\text{Ker}\,\phi$ is a group.

Closed: $a, b \in \text{Ker}\,\phi$ $ab \in \text{Ker}\,\phi$

$$\phi(ab) = \phi(a)\phi(b) = e$$

So $ab \in \text{Ker}\,\phi$

Associative: Trivial

Inverse: Let $a \in \text{Ker}\,\phi$, show $a^{-1} \in \text{Ker}\,\phi$

$$\phi(a^{-1}) = \phi(a)^{-1} = e^{-1} = e$$

So $a^{-1} \in \text{Ker}\,\phi$

So Kernel is a subgroup. To show normal, w.t.s for $h \in \ker\phi$ and $g \in G$ $ghg^{-1} \in \ker\phi$

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1})$$
$$= \phi(g)\phi(h)\phi(g)^{-1}$$
$$= \phi(g)\, e\, \phi(g)^{-1} = e$$

So $ghg^{-1} \in \ker\phi$ therefore $\ker\phi$ is a normal subgroup.

Extra 2)

By ~~(thm)~~ (5.3b) every subgroup of a cyclic subgroup must be cyclic.

So, the subgroups of $\mathbb{Z}$ are $n\mathbb{Z}$. $n \in \mathbb{Z}$

We see clearly $n\mathbb{Z}$ is cyclic, generated by $\langle n \rangle$.

These are the only cyclic subgroups, as this is only way to guarantee using single element.

Now, as $\mathbb{Z}$ is abelian, then all subgroups are normal, so $n\mathbb{Z}$ normal $\forall n \in \mathbb{Z}$, meaning all quotient groups on $\mathbb{Z}$ are of the form $\mathbb{Z}/n\mathbb{Z}$ where $n \in \mathbb{Z}$