Tufts University
Department of Mathematics

Math 135 **Homework 4**[1] **Fall 2022**

Readings for the week of September 26, 2022

§**2.3:** p. 41, the nested interval theorem
§**2.4:** Sequential compactness in $\mathbb{R}$
§**3.1:** Continuous functions
§**3.5:** ε-δ implies continuity

Problem Set 4
(Due Wednesday, October 5, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (10 points) **Sequentially compact $\Rightarrow$ bounded**.
   Prove that if a set S in $\mathbb{R}$ is sequentially compact, then it is bounded. (*Hint*. Think of the examples $S = [0,\infty)$ or $S = \mathbb{N}$.)

2. (10 points) **Sequentially compact $\Rightarrow$ closed**.
   Prove that if a set S in $\mathbb{R}$ is sequentially compact, then it is closed. (*Hint*. Think of the example $S = (0,1]$.)

3. (15 points) **Square root rule**. Using the ε-N definition of the limit, prove that If $a_n \to a$, then $\lim \sqrt{a_n} = \sqrt{a}$.

4. (15 points) **Continuity**. §3.1, p. 57, # 6.
   Suppose $g\colon \mathbb{R} \to \mathbb{R}$ is defined by
   $$g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ -x^2 & \text{otherwise.} \end{cases}$$
   At what points is the function continuous? Justify your answer.

5. (25 points) **Sequential compactness**.
   For each of the following sets, say if it is sequentially compact and give a brief justification.
   (a) $\mathbb{Q} \cap [0,1)$
   (b) $\mathbb{Q} \cap [0,1]$
   (c) $[0,1] \cup [2,3]$
   (d) $\mathbb{Z} \cap [1,100]$
   (e) $[0,\infty)$

6. (10 points) **Continuity**. §3.1, p. 58, # 11.
   Suppose that the function $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and that $g(x) = 0$ if x is rational. Prove that $g(x) = 0$ for all x in $\mathbb{R}$.

7. (15 points) **Peak indices**. §2.4, p. 47, # 4 (a), (c), (d).
   For each of the following sequences, find the peak indices. Justify your conclusions.

   (a) $\left\{ \dfrac{1}{n} \right\}$ (c) $\{(-1)^n n\}$ (d) $\left\{ \dfrac{(-1)^n}{n} \right\}$.

(End of Problem Set 4)