1. Math 65, Review Sheet for Exam I, Fall 2021.

Question 1.1. *Denote by $\mathbb{R}$ the set of real numbers. Define the function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ by $F(a, b) = (a + b, a - b)$.*

(a) Define what it means for a function $f : A \to B$ to be one to one.
(b) Define what it means for a function $f : A \to B$ to be onto.
(c) Define what it means for a function $f : A \to B$ to be a bijection.
(d) For the function F defined above, prove or disprove that F is one to one.
(e) For the function F defined above, prove or disprove that F is onto.
(f) For the function F defined above, prove or disprove that F is a bijection.

Question 1.2. *(a) Find a recurrence relation a_n for the number of strings of length n using the letters from the set $\{a, b, c, d, e\}$ that contain 2 consecutive e's.*
(b) What are the initial conditions, that is, the values of the first terms that together with the recurrence would allow you to find every other term
(c) What is a_4?

Question 1.3. *(1) Assume that $(a_n), (b_n)$ are sequences satisfying that*

$$a_n = 6a_{n-1} - 8a_{n-2} \quad b_n = 6b_{n-1} - 8b_{n-2}$$

Give conditions on a_0, a_1, b_0, b_1 that guarantee that for each sequence c_n satisfying $c_n = 6c_{n-1} - 8c_{n-2}$, one can find constants A, B such that $c_n = Aa_n + Bb_n$.
(2) Find two sequences of the form $c_n = x^n$ satisfying the recurrence.
(3) Find a non recursive expression for the sequence or sequences satisfying

$$c_n = 6c_{n-1} - 8c_{n-2} \qquad c_0 = 4, c_1 = 10$$

or show that no such sequence exists.

Question 1.4. *Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.*

Question 1.5. *Let $p(x)$ be the proposition: "I will go to the concert on day x."*
Let $q(x)$ be the proposition: "I have an exam on day x." Using p and q and logical connectives, write the propositions that follow. Write the negation of each of these statements both in mathematical symbols and in English.

(a) "I will not go to the concert today if I have an exam tomorrow."
(b) "If I do not have an exam tomorrow, I will go to the concert today."
(c) "If I do not go to the concert today, I will not have an exam tomorrow."
(d) "I will have the exam some day"
(e) "I will never go to the concert"

Question 1.6. *Determine the truth value of each of these statements where $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of natural numbers. If the statement is true, give a proof, if false give a counterexample. Write the negation of the statements and again give a proof or counterexample.*

(a) $\exists x \in \mathbb{N}\ (-x < -3)$
(b) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}\ \exists z \in \mathbb{N}\ (x = 2y + 4z)$
(c) $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ (y \geq x)$
(d) $\forall y \in \mathbb{N}\ \exists x \in \mathbb{N}\ (x < y)$
(e) $\forall x \in \mathbb{N}\ (-2x < -x)$

Question 1.7. *Assume that all the sets we are working with are contained in a set U. Then we will indicate with $\overline{A}$ the complement of A, that is the elements in U that are not in A, $\overline{A} = U - A$. Prove that $\overline{(A \cap B \cap C \cap D)} = \bar{A} \cup \bar{B} \cup \bar{C} \cup \bar{D}$ by showing each side is a subset of the other side.*

Question 1.8. *Let $A = \{x \in \mathbb{R} | x^2 \leq 8x\}, B = \{x \in \mathbb{R} | x^2 \leq 1\}$*
(a) Find $A \cap B$.
(b) Find $A \cup B$.

Question 1.9. *Assume that x is an integer.*
(a) Prove that if $x + 5$ is odd, then $3x + 2$ is even.
(b) Prove that if $x + 5$ is even, then $3x + 2$ is odd.
(c) Prove that (b) is equivalent to proving that if $3x + 2$ is even, then $x + 5$ is odd.

Question 1.10. *Use mathematical induction to prove that*

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

for all $n \geq 1$.

Question 1.11. *Show that any amount of postage of 12 cents or more can be made with 4 cents and 5 cent stamps.*

Question 1.12. *Assume that A, B are finite sets of cardinality m and n respectively. Show that $A \times B$ is finite of cardinality mn.*

Question 1.13. *Assume that $A \neq \emptyset$ is a set $f : A \to B$ a function. Show that f is one to one if and only if there exists a function $g : B \to A$ such that $g \circ f = I_A$.*

Question 1.14. *Let $f : A \to B$ be a function. We are NOT assuming that f is a bijection, so f^{-1} is not defined as a function.*
(a) If $X \subseteq A$, write the definition of $f(X)$.
(b) If $Y \subseteq B$, write the definition of $f^{-1}(Y)$.
(c) If $X \subseteq A$, show that $X \subseteq f^{-1}(f(X))$.
(d) Give an example of an $f : A \to B$ and an $X \subseteq A$ such that $X \neq f^{-1}(f(X))$.

Question 1.15. *(a) Show that the set of natural numbers divisible by 5 is countable.*
(b) Show that the real plane $\mathbb{R} \times \mathbb{R}$ is not countable.

Question 1.16. *How many four digit numbers contain the string* 123 *or the string* 234 *(or both)?*

2. Answers

Question 2.1. *Denote by $\mathbb{R}$ the set of real numbers. Define the function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ by $F(a, b) = (a + b, a - b)$.*
(a) Define what it means for a function $f : A \to B$ to be one to one.
(b) Define what it means for a function $f : A \to B$ to be onto.
(c) Define what it means for a function $f : A \to B$ to be a bijection.
(d) For the function F defined above, prove or disprove that F is one to one.
(e) For the function F defined above, prove or disprove that F is onto.
(f) For the function F defined above, prove or disprove that F is a bijection.

Answer 1. (a) A function $f : A \to B$ is one to one if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Equivalently $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(b) A function $f : A \to B$ is onto if every element in B is the image of at least one element in A. In symbols $\forall b \in B,\ \exists a \in A$ such that $b = f(a)$.

(c) A function $f : A \to B$ is a bijection if it is both one-to-one and onto. We saw that this is equivalent to the existence of an inverse function g: For any set X we use the symbol I_X to denote the identity function $I_X : X \to X$ given by $\forall x \in X,\ I_X(x) = x$. Then the inverse function g (if it exists) of a function f satisfies $f \circ g = I_B,\ g \circ f = I_A$. As we mentioned, the existence of the inverse of f is equivalent to f being a bijection.

(d) We prove that F is one to one. Assume that $F(a_1, b_1) = .F(a_2, b_2)$. From the definition of F,

$$(a_1 + b_1, a_1 - b_1) = F(a_1, b_1) = F(a_2, b_2) = (a_2 + b_2, a_2 - b_2)$$

Therefore, equating each component of the pair, we obtain,

$$a_1 + b_1 = a_2 + b_2, \quad a_1 - b_1 = a_2 - b_2$$

Adding the two equations, we obtain $2a_1 = 2a_2$ and subtracting them we obtain $2b_1 = 2b_2$. Dividing these equations by 2, we have $a_1 = b_1, a_2 = b_2$. Therefore the pairs satisfy the equality $(a_1, a_2) = (b_1, b_2)$, proving that F is one to one.

(e) We prove that F is onto by showing that for all $(x, y) \in \mathbb{R}\times\mathbb{R}$, there exists $(a, b) \in \mathbb{R}\times\mathbb{R}$ such that $F(a, b) = (x, y)$. From the definition of F, one such (a, b) should satisfy

$$a + b = x,\ a - b = y$$

Adding and subtracting the two equations, we obtain

$$2a = x + y, \quad 2b = x - y, \quad \text{or equivalently} \quad a = \frac{x + y}{2}, \quad b = \frac{x - y}{2}$$

Therefore, for any choice of $(x, y) \in \mathbb{R} \times \mathbb{R}$, $F(\frac{x+y}{2}, \frac{x-y}{2}) = (x, y)$ and F is onto.

(f) As the function F is both one to one and onto, it is a bijection. We essentially found the inverse of F in the previous point. The inverse of F is the function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $G(x, y) = (\frac{x+y}{2}, \frac{x-y}{2})$. We checked in (e) that

$$(F \circ G)(x, y) = F(G((x, y))) = F((\frac{x + y}{2}, \frac{x - y}{2})) = (x, y) = I_{\mathbb{R}\times\mathbb{R}}(x, y)$$

One can also check that

$$(G \circ F)((a, b)) = G(F((a, b)) = G(a + b, a - b) = (\frac{(a + b) + (a - b)}{2}, \frac{(a + b) - (a - b)}{2}) =$$
$$= (a, b) = I_{\mathbb{R}\times\mathbb{R}}(a, b).$$

Hence, $F \circ G = I_{\mathbb{R}\times\mathbb{R}}$, $G \circ F = I_{\mathbb{R}\times\mathbb{R}}$ and F, G are inverse of each other.

Question 2.2. *(a) Find a recurrence relation a_n for the number of strings of length n using the letters from the set $\{a, b, c, d, e\}$ that contain 2 consecutive e's.*

(b) What are the initial conditions, that is, the values of the first terms that together with the recurrence would allow you to find every other term

(c) What is a_4?

Answer 2. (a) We need to find a recurrence relation a_n for the number of strings of length n that contain 2 consecutive e's using the letters from the set $\{a, b, c, d, e\}$. If the last letter is a, b, c, d, the $(n-1)^{-th}$ string must have two e's. If the last 2 letters are ae, be, ce, de, the $n - 1^{-th}$ string must have two e's. If the last 2 letters are ee, then anything will do. So

$$a_n = 4a_{n-1} + 4a_{n-2} + 5^{n-2}$$

(b) The initial conditions are $a_1 = 0$, $a_2 = 1$. Because the recurrence allows you to compute a_n using the two previous terms, a_0, a_1 determine all of the terms of the sequence.

(c) From the recurrence and initial conditions, $a_3 = 9$ and $a_4 = 4 + 4 \cdot 9 + 5^2 = 65$.

Question 2.3. *(1) Assume that $(a_n), (b_n)$ are sequences satisfying that*

$$a_n = 6a_{n-1} - 8a_{n-2} \quad b_n = 6b_{n-1} - 8b_{n-2}$$

Give conditions on a_0, a_1, b_0, b_1 that guarantee that for each sequence c_n satisfying $c_n = 6c_{n-1} - 8c_{n-2}$, one can find constants A, B such that $c_n = Aa_n + Bb_n$.

(2) Find two sequences of the form $c_n = x^n$ satisfying the recurrence.

(3) Find a non recursive expression for the sequence or sequences satisfying

$$c_n = 6c_{n-1} - 8c_{n-2} \qquad c_0 = 4, c_1 = 10$$

or show that no such sequence exists.

Answer 3. (1) Assume that $(a_n), (b_n)$ are sequences satisfying that

$$a_n = 6a_{n-1} - 8a_{n-2} \quad b_n = 6b_{n-1} - 8b_{n-2}$$

If $c_n = Aa_n + Bb_n$, then $c_n = 6c_{n-1} - 8c_{n-2}$ as we can check by plugging in. A sequence satisfying an equation as this one, is determined by the value of its first two terms. The question then is whether given c_0, c_1, we can find A, B such that

$$c_0 = Aa_0 + Bb_0, \;\; c_1 = Aa_1 + Bb_1$$

Because a_0, a_1, b_0, b_1 have specific values and we are trying to figure out A, B no matter what the c_0, c_1, we need $\frac{a_0}{a_1} \neq \frac{b_0}{b_1}$ for the equation to always have a solution.

(2) We look for solutions of the form $c_n = x^n$. From $c_n = 6c_{n-1} - 8c_{n-2}$, we obtain the equation $x^n = 6x^{n-1} - 8x^{n-2}$. Dividing the equation by x^{n-2}, this produces the equation $x^2 - 6x + 8 = 0$. We factor it as as $(x-2)(x-4) = 0$. Hence, $c_n = 2^n, c_n = 4^n$, are solutions of the recurrence.

(3) From the previous two parts, the general solution is $c_n = e2^n + f4^n$. We now determine the values of e, f using the initial conditions

$$c_0 = 4 = e + f, \;\; c_1 = 10 = 2e + 4f$$

Subtracting from the first equation twice the second, we obtain $2f = 2$ and therefore $f = 1$. Substituting then in the first equation, we obtain $e = 3$. Hence $c_n = 3 \cdot 2^n + 4^n$.

Question 2.4. *Show that* $\neg(p \leftrightarrow q)$ *and* $\neg p \leftrightarrow q$ *are logically equivalent.*

Answer 4. $\neg(p \leftrightarrow q)$ is equivalent to $\neg p \leftrightarrow q$

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg p$	$\neg p \leftrightarrow q$
T	T	T	F	F	F
T	F	F	T	F	T
F	T	F	T	T	T
F	F	T	F	T	F

Same true values, hence equivalent.

Question 2.5. *Let* $p(x)$ *be the proposition: "I will go to the concert on day* x*."*

Let $q(x)$ *be the proposition: "I have an exam on day* x*." Using* p *and* q *and logical connectives, write the propositions that follow. Write the negation of each of these statements both in mathematical symbols and in English.*

(a) "I will not go to the concert today if I have an exam tomorrow."
(b) "If I do not have an exam tomorrow, I will go to the concert today."
(c) "If I do not go to the concert today, I will not have an exam tomorrow."
(d) "I will have the exam some day"
(e) "I will never go to the concert"

Answer 5. $p(x)$ "I will go to the concert on day x. $q(x)$ "I have an exam on day x." " Note that $r \to s$ is equivalent to $(\neg r) \vee s$ Therefore, the negation of $r \to s$ is $r \wedge (\neg s)$

(a) "I will not go to the concert today if I have an exam tomorrow." $q(\text{tomorrow}) \to (\neg p(\text{today}))$.

The negation of this statement is "I have an exam tomorrow and I will go to the concert today" that is $q(\text{tomorrow}) \wedge (p(\text{today}))$.

(b) "If I do not have an exam tomorrow, I will go to the concert today." $\neg q(\text{tomorrow}) \to p(\text{today})$

The negation is the statement "I do not have an exam tomorrow and I will not go to the concert today" $\neg q(\text{tomorrow}) \wedge (\neg p(\text{today}))$

(c) "If I do not go to the concert today, I will not have an exam tomorrow." $(\neg p(\text{today})) \to (\neg q(\text{tomorrow}))$

The negation is "I do not go to the concert today and I have an exam tomorrow" $(\neg p(\text{today})) \wedge q(\text{tomorrow})$

(d) . "I will have the exam some day" $\exists x,\ q(x)$

The negation is "I never have an exam" $\forall x,\ (\neg q(x))$

(e) . "I will never go to the concert" $\forall x,\ \neg p(x)$

The negation is "I will go to the concert some day" $\exists x, P(x)$.

Question 2.6. *Determine the truth value of each of these statements where* $\mathbb{N} = \{0, 1, 2, \dots\}$ *denotes the set of natural numbers. If the statement is true, give a proof, if false give a counterexample. Write the negation of the statements and again give a proof or counterexample.*

(a) $\exists x \in \mathbb{N}\ (-x < -3)$
(b) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}\ \exists z \in \mathbb{N}\ (x = 2y + 4z)$
(c) $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ (y \geq x)$
(d) $\forall y \in \mathbb{N}\ \exists x \in \mathbb{N}\ (x < y)$
(e) $\forall x \in \mathbb{N}\ (-2x < -x)$

Answer 6. (a) $\exists x \in \mathbb{N},\ -x < -3$ is true as $x = 4$ satisfies $-x < -3$. The negation is $\forall x \in \mathbb{N},\ -x \geq -3$

(b) $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ \exists z \in \mathbb{N},\ x = 2y + 4z$ F If x is odd, then for all y, z natural numbers $x \neq 2y + 4z = 2(y + 2z)$. The negation of the above is $\exists x \in \mathbf{N},\ \forall y \in \mathbb{N},\ \forall z \in \mathbb{N},\ x \neq 2y + 4z$ which is true with for example $x = 1$

(c) $(\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ y \geq x)$ is true where $x = 0$ works $0 = x \leq y$ for all y positive integers. The negation is $(\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ y < x)$. This is of course false as for $x = 0$ no such y exists.

(d) $\forall y \in \mathbf{N},\ \exists x \in \mathbf{N},\ x < y$ is false because if $y = 0$ no such x exists. Its negation can be written as $\exists y \in \mathbb{N},\ \forall x \in \mathbb{N},\ y \leq x$. This is true as $y = 0\ \forall x\ y = 0 \leq x$ (this is in fact part (c) with the variables interchanged).

(e) $\forall x \in \mathbb{N},\ -2x < -x$ is false as it is equivalent to $\forall x \in \mathbb{N},\ x < 2x$. This is in turn equivalent to $\forall x,\ 0 < x$ which is false for $x = 0$. Therefore, the statement is false

Question 2.7. *Assume that all the sets we are working with are contained in a set U. Then we will indicate with $\overline{A}$ the complement of A, that is the elements in U that are not in A, $\overline{A} = U - A$. Prove that $\overline{(A \cap B \cap C \cap D)} = \bar{A} \cup \bar{B} \cup \bar{C} \cup \bar{D}$ by showing each side is a subset of the other side.*

Answer 7. . We want to show that

$$\overline{A \cap B \cap C \cap D} = \bar{A} \cup \bar{B} \cup \bar{C} \cup \bar{D}.$$

We will prove that an element is in the left hand side if and only if it is in the right hand side

$$x \in \overline{A \cap B \cap C \cap D} \ \leftrightarrow \ x \notin \ A \cap B \cap C \cap D \ \leftrightarrow \neg(x \in A \wedge x \in B \wedge x \in C \wedge x \in D)$$
$$\leftrightarrow \ \neg(x \in A) \ \vee \neg(x \in B) \vee \neg(x \in C) \vee \neg(x \in D) \ \leftrightarrow x \in \bar{A} \ \vee x \in \bar{B} \ \vee x \in \bar{C} \ \vee x \in \bar{D}$$
$$\leftrightarrow x \in (\bar{A} \cup \bar{B} \cup \bar{C} \cup \bar{D})$$

Question 2.8. *Let $A = \{x \in \mathbb{R} | x^2 \leq 8x\}, B = \{x \in \mathbb{R} | x^2 \leq 1\}$*

(a) Find $A \cap B$.

(b) Find $A \cup B$.

Answer 8. By definition,

$$A = \{x \in \mathbb{R} | x^2 \leq 8x\} = \{x \in \mathbb{R} | x(x - 8) \leq 0\}.$$

A product is negative when exactly one of the factors is negative. This would require either $x \leq 0, x - 8 \geq 0$ or $x \geq 0, x - 8 \leq 0$. The first case cannot happen, hence

$$A = \{x \in \mathbb{R} | x(x - 8) \leq 0\} = \{x \in \mathbb{R} | 0 \leq x \leq 8\}.$$

Similarly,

$$B = \{x \in \mathbb{R} | x^2 \leq 1\} = \{x \in \mathbb{R} | x^2 - 1 \leq 0\} = \{x \in \mathbb{R} | -1 \leq x \leq 1\}$$

(a) Therefore, $A \cap B = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$.

(b) Find $A \cup B = \{x \in \mathbb{R} | -1 \leq x \leq 8\}$.

Question 2.9. *Assume that x is an integer.*

(a) Prove that if $x + 5$ is odd, then $3x + 2$ is even.

(b) Prove that if $x + 5$ is even, then $3x + 2$ is odd.

(c) Prove that (b) is equivalent to proving that if $3x + 2$ is even, then $x + 5$ is odd.

Answer 9. (a) $x + 5$ odd $\Rightarrow 3x + 2$ even.

By definition, $x+5$ odd means that $\exists k$ integer such that $x+5 = 2k+1$. Then we find $x = 2k+1-5 = 2k-4$. Using this expression for x, we obtain $3x+2 = 3(2k-4)+2 = 6k-12+2 = 2(3k-5)$. As k is an integer, $3k-5$ is also an integer, hence $3x+2$ is even.

(b) Assume $x + 5$ is even, we want to show $3x + 2$ is odd.

By definition, $x + 5$ even means $\exists k$ integer $x + 5 = 2k$. Then $x = 2k - 5$. Using this expression for x, we obtain $3x+2 = 3(2k-5)+2 = 6k-13 = 2(3k-7)+1$. As $3k-7$ is an integer, $3x + 2$ is odd.

(c) We know $p \to q$ is equivalent to $\neg q \to \neg p$. We also showed that the condition that a is related to b if and only if the difference is divisible by two is an equivalence relation that divides the integers into two equivalence classes, namely even and odd numbers. So a number is even if and only if it is not odd. Hence, (b) is equivalent to "If $3x+2$ is even, then $x + 5$ is odd.

Question 2.10. *Use mathematical induction to prove that*

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

for all $n \geq 1$.

Answer 10. We want to show that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

for all integer by induction

When $n = 1$,

$$\frac{1}{2} = \frac{2^1 - 1}{2^1} = \frac{2-1}{2} = \frac{1}{2}$$

which is true.

Assume now the result true for n, namely

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

We need to check that the corresponding result when replacing n by $n+1$ holds too. Adding $\frac{1}{2^{n+1}}$ to both sides of the above identity we obtain,

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{2^n - 1}{2^n} + \frac{1}{2^{n+1}} = \frac{2(2^n - 1)}{2^{n+1}} + \frac{1}{2^{n+1}} =$$

$$= \frac{2^{n+1} - 2 + 1}{2^{n+1}} = \frac{2^{n+1} - 1}{2^{n+1}}$$

which is the corresponding result for $n + 1$.

Question 2.11. *Show that any amount of postage of 12 cents or more can be made with 4 cents and 5 cent stamps.*

Answer 11. First

$$12 = 3 \times 4, 13 = 5 + 2 \times 4, 14 = 2 \times 5 + 4, 15 = 3 \times 5$$

Let now n be at least 16. Then $n - 4 \geq 12$. Hence, from our induction assumption, there exist integers $a, b \geq 0$ such that $n - 4 = 4a + 5b$. Then, $n = 4(a + 1) + 5b$. Hence, a postage of n can be made with 4 and 5 cents stamps. Then, by the induction principle, any amount of postage of 12 cents or more can be made with these stamps.

Question 2.12. *Assume that A, B are finite sets of cardinality m and n respectively. Show that $A \times B$ is finite of cardinality mn.*

Answer 12. From our assumptions, there exist bijections

$$f : A \to \{1, \ldots, m\}, \ g : B \to \{1, \ldots, n\}$$

We construct a bijection $h : A \times B \to \{1, \ldots, mn\}$ as follows

$$h(a, b) = m(g(b) - 1) + f(a).$$

This h is well defined: as $1 \leq f(a) \leq m, 1 \leq g(b) \leq n$

$$1 = m(1 - 1) + 1 \leq m(g(b) - 1) + f(a) \leq m(n - 1) + m = mn$$

and therefore the image of h is in the given codomain.

We show that h is one to one: assume that

$$h(a_1, b_1) = h(a_2, b_2).$$

Equivalently,

$$h(a_1, b_1) = m(g(b_1) - 1) + f(a_1) = m(g(b_2) - 1) + f(a_2) = h(a_2, b_2).$$

As $1 \leq f(a_1) \leq m$, the remainder of dividing $h(a_1, b_1)$ by m is $f(a_1)$ if $f(a_1) < m$ and 0 if $f(a_1) = m$. For the same reason, using that also $1 \leq f(a_2) \leq m$, the remainder of dividing $h(a_1, b_1) = h(a_2, b_2)$ by m is $f(a_2)$ if $f(a_2) < m$ and 0 if $f(a_2) = m$. As this remainder is unique, $f(a_1) = f(a_2)$. As f is one to one, this implies $a_1 = a_2$. From $f(a_1) = f(a_2)$ and $h(a_1, b_1) = m(g(b_1) - 1) + f(a_1) = m(g(b_2) - 1) + f(a_2) = h(a_2, b_2)$, subtracting $f(a_1) = f(a_2)$ from both sides, $m(g(b_1) - 1) = m(g(b_2 - 1))$. Dividing both sides by m and adding 1, $g(b_1) = g(b_2)$. Then, as g is one to one, $b_1 = b_2$. Therefore, h is one to one.

Let us now check that h is onto. Given $y, 1 \leq y \leq mn$, we can divide y by m and get a unique quotient and unique remainder $y = qm + r, \ 0 \leq r < m$. If $r = 0$ as $y > 0$, this would imply that $q > 0$. Then, we can rewrite $y = (q - 1)m + 0$. So, in any case, we can write $y = q'm + r', 1 \leq r' \leq m$. Moreover, as $1 \leq y \leq mnn$, we need to have $0 \leq q' \leq n - 1$ As f is onto and $1 \leq r' \leq m$, there exists some $a \in A$ with $f(a) = r'$. As $0 \leq q' \leq n - 1$, then $1 \leq q' + 1 \leq n$. As g is onto, there exists some $b \in B$ such that $q' + 1 = g(b)$. Therefore,

$$y = q'm + r' = (g(b) - 1)m + f(a) = h(a, b)$$

Therefore, h is onto. Then h is a bjjection and $|A \times B| = mn$

Question 2.13. *Assume that $A \neq \emptyset$ is a set $f : A \to B$ a function. Show that f is one to one if and only if there exists a function $g : B \to A$ such that $g \circ f = I_A$.*

Answer 13. Assume f is one to one. Define $g : B \to A$ as follows: if $b \in f(A)$, we can find a unique (as f is one to one) $a \in A$ such that $f(a) = b$. Define $g(b) = a$. If $b \in B - f(A)$, pick an arbitrary $a_0 \in A \neq \emptyset$ and define $g(b) = a_0$. Then, for $a \in A$, $(g \circ f)(a) = g(f(a)) = a$ by definition of g. Hence, $g \circ f = I_A$.

Conversely, assume there exists $g : B \to A$ such that $g \circ f = I_A$. Take $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Applying g to both sides of this equality and using that $g \circ f = I_A$, we obtain

$$a_1 = I_A(a_1) = g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2) = I_A(a_2) = a_2$$

Therefore. f is one to one.

Question 2.14. *Let $f : A \to B$ be a function. We are NOT assuming that f is a bijection, so f^{-1} is not defined as a function.*
(a) If $X \subseteq A$, write the definition of $f(X)$.
(b) If $Y \subseteq B$, write the definition of $f^{-1}(Y)$.
(c) If $X \subseteq A$, show that $X \subseteq f^{-1}(f(X))$.
(d) Give an example of an $f : A \to B$ and an $X \subseteq A$ such that $X \neq f^{-1}(f(X))$.

Answer 14. (a) If $X \subseteq A$, we define the subset of elements in B that are images of elements in X that is denoted as $f(X)$ as follows

$$f(X) = \{y \in B \text{ such that } \exists x \in X,\ y = f(x)\}.$$

(b) If $Y \subseteq B$, we define the subset of elements in A that map to Y that is denoted as $f^{-1}(Y)$ as follows

$$f^{-1}(Y) = \{x \in A. \text{ such that } f(x) \in Y\}.$$

(c) If $x \in X$, by definition of $f(X)$, $f(x) \in f(X)$. Then, with $Y = f(X)$, $x \in f^{-1}(Y)$ because $f(x) \in Y$. Therefore, substituting $Y = f(X)$, $x \in f^{-1}(f(X))$.

(d) We are assuming that $f(f^{-1}(Y)) = Y$ for all Y. In particular, we can take $Y = B$. This then says that $f(f^{-1}(B)) = B$. Hence, each element in B is in the image of an element in $f^{-1}(B)$. By definition of onto, the map is onto

(e) Take

$$A = \{a_1, a_2\},\ B = \{b\},\ f : A \to B \text{ given as } f(a_1) = b, f(a_2) = b, X = \{a_1\}$$

Then, $f(X) = \{f(a_1)\} = \{b\}$. Therefore,

$$f^{-1}(f(X)) = f^{-1}(\{b\}) = \{a \in A \text{ such that } f(a) \in \{b\}\} = \{a_1, a_2\}$$

This set is larger than $X = \{a_1\}$.

Question 2.15. *(a) Show that the set of natural numbers divisible by 5 is countable.*
(b) Show that the real plane $\mathbb{R} \times \mathbb{R}$ is not countable.

Answer 15. (a) We can prove this in at least two ways

- We know that any subset of a countable set is either finite or countable. The set of natural numbers divisible by 5 is a subset of $\mathbb{N}$ which by its very definition is countable. There are infinitely many natural numbers divisible by 5, so the set is not finite. It then follows that it is countable.
- We can imitate the proof that the set of even numbers is countable by giving an explicit bijection between $\mathbb{N}$ and $5\mathbb{N}$.

(b) As a subset of a countable set is finite or countable, if $\mathbb{R} \times \mathbb{R}$ were countable, then every infinite subset of it would be countable. The real numbers can be taken as a subset of the real plane (for example as the x axis). As $\mathbb{R}$ is not countable, $\mathbb{R} \times \mathbb{R}$ is not countable.

Question 2.16. *How many four digit numbers contain the string* 123 *or the string* 234 *(or both)?*

Answer 16. (a) There is exactly one number 1234 which contains both. If one of the two strings occupies the 3 first spaces, there are 10 choices for the units digit, if it occupies the last 3 spaces there are only 9 choices for the first digit (cannot be zero).

(b) Hence $10 + 10 + 9 + 9 - 1 = 37$ such numbers.