

Chapter 17

A reason to care about eigenvalues: stability of equilibria

In the upcoming chapters, we will think about computing eigenvalues and eigenvectors. To explain why eigenvalues and eigenvectors are a useful thing to think about, I'll devote one chapter to the analysis of the stability of dynamic equilibria. This is among the most important applications of eigenvalues.

17.1 Systems of ordinary differential equations

17.1.1 Notation

Suppose that

$$F : \mathbb{R}^m \rightarrow \mathbb{R}^m$$

is a given continuous function. So $F(x) \in \mathbb{R}^m$ for $x \in \mathbb{R}^m$. Think about a point moving in \mathbb{R}^m . Its location at time t is $x(t)$. Suppose that we know that it moves according to the equation

$$\frac{dx}{dt} = F(x). \quad (17.1)$$

This means, more explicitly, that at all times t ,

$$\frac{dx}{dt}(t) = F(x(t)). \quad (17.2)$$

We denote the components of x by x_1, x_2, \dots, x_m . They are functions of time. The components of F are denoted by F_1, \dots, F_m . So $F_i = F_i(x_1, x_2, \dots, x_m)$ for $1 \leq i \leq m$. Equation (17.1) is a compact representation of a system of m differential equations in m unknown functions. The first of these m differential equations, for instance, is

$$\frac{dx_1}{dt} = F_1(x_1, x_2, \dots, x_m),$$

the second is

$$\frac{dx_2}{dt} = F_2(x_1, x_2, \dots, x_m),$$

and so on.

One can hardly overstate the importance of (17.1) to the world. The components of x could be positions and velocities of planets and of a spacecraft, in which case (17.1) describes the motion of those objects. The components of x could also be population sizes, in which case (17.1) describes how populations evolve while interacting with each other. The components of x could describe the properties of a nerve membrane, in which case (17.1) describes the dynamics of a nerve cell. Almost always, when one studies quantities that change with time according to known laws, an equation of the form (17.1) is in the background, if not in the foreground. Equation (17.1) represents much of the natural sciences. The differential equations are called *ordinary* because the unknown functions x_i depend on a single variable only, namely time. (When the unknown functions depend on several variables, most typically time and space, one gets *partial* differential equations.)

Exercise 1. The simplest example is $m = 1$, in which case $x(t)$ is just a number for each t — maybe the size of a population at time t — and $F(x) = x$. So our equation is

$$\frac{dx}{dt} = x \quad (17.3)$$

or, using a prime to indicate the derivative,

$$x'(t) = x(t) \quad \text{for all } t.$$

As long as $x(t)$ is not zero, we can write this as

$$\frac{x'(t)}{x(t)} = 1 \quad \text{for all } t. \quad (17.4)$$

(a) Explain why (17.4) implies that

$$\ln |x(t)| = t + C \quad (17.5)$$

for a constant C .

(b) Explain why (17.5) implies

$$x(t) = ke^t$$

for a constant k .

(c) Explain why the constant k equals $x(0)$, so

$$x(t) = x(0)e^t.$$

So eq. (17.3) describes exponential growth of a population.

One of the first things that one learns in a course on differential equations is

that for “nice” functions F , and for any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^m$, there is exactly one solution $x = x(t)$ of

$$\frac{dx}{dt} = F(x), \quad x(t_0) = x_0. \quad (17.6)$$

Here “nice” might mean “continuously differentiable” for instance. One calls (17.6) an *initial value problem*. We prescribe where x is at time t_0 , which we think of as the “present” or “initial” time, and we prescribe the law by which x moves, the system $dx/dt = F(x)$. The theorem that I just cited (without stating it quite formally) says that this information is enough to predict the future, i.e., compute $x(t)$ for $t > t_0$, and also to reconstruct the past, i.e., compute $x(t)$ for $t < t_0$. Often we choose $t_0 = 0$. This is just a matter of how we define our calendar: Letting $t_0 = 0$ means that we call the present moment “time 0”.

I’ll note parenthetically that the solution $x(t)$ of (17.6) need not be defined for all t . For instance, suppose that $m = 1$, $F(x) = x^2$, $t = 0$, $x_0 = 1$:

$$\frac{dx}{dt} = x^2, \quad x(0) = 1. \quad (17.7)$$

The solution of this problem is

$$x(t) = \frac{1}{1-t}, \quad (17.8)$$

which becomes infinite at $t = 1$. This is called *blow-up in finite time*.

Exercise 2. Verify that (17.8) solves (17.7).

17.1.2 Example: competing species

Suppose that two species compete for the same food. Their population sizes (measured perhaps in thousands or millions) are called x_1 and x_2 , and change with time. The rate of growth of x_1 is proportional to x_1 , but with a constant of proportionality that decreases when x_2 increases. Similarly, the rate of growth of x_2 is proportional to x_2 , but with a constant of proportionality that decreases when x_1 increases. Let us assume, for instance, that

$$\frac{dx_1}{dt} = (1 - x_2)x_1, \quad (17.9)$$

$$\frac{dx_2}{dt} = (1 - x_1)x_2. \quad (17.10)$$

Here $m = 2$, and

$$F(x) = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} (1 - x_2)x_1 \\ (1 - x_1)x_2 \end{bmatrix}.$$

Exercise 3. Let $u = x_1 - x_2$. Explain why

$$u(t) = u(0)e^t.$$

Hint: Subtract eq. (17.10) from (17.9). Then compare Exercise 1.

What does this imply about the two competing species?

17.1.3 Example: predators and preys

Now assume that the first species is a predator, and the second a prey. The predators prosper when a lot of prey are around, but their number decreases in the absence of enough prey. The prey prosper when few predators (per prey) are around, unless their own numbers get too large. For instance, we might assume that

$$\frac{dx_1}{dt} = \left(-1 + \frac{2x_2}{1+x_2}\right)x_1, \quad (17.11)$$

$$\frac{dx_2}{dt} = \left(2 - \frac{x_1}{1+x_2} - \frac{x_2}{2}\right)x_2. \quad (17.12)$$

So here

$$F(x) = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} \left(-1 + \frac{2x_2}{1+x_2}\right)x_1 \\ \left(2 - \frac{x_1}{1+x_2} - \frac{x_2}{2}\right)x_2 \end{bmatrix}.$$

The constants in these equations are quite arbitrary. I am giving this example for illustration only, not to model two specific populations.

17.2 Equilibria

17.2.1 Definition of equilibrium

One of the simplest questions to ask is for which $x_* \in \mathbb{R}^m$ the constant $x(t) \equiv x_*$ is a solution of (17.1). In other words, we are asking where x can rest without change. The answer, quite obviously, is: $x(t) \equiv x_*$ is a solution of (17.1) if and only if

$$F(x_*) = 0.$$

One then calls x_* an *equilibrium* for (17.1), or a *steady state*, or a *fixed point*.

Exercise 4.

- (a) Show that for our competition model, eqs. (17.9) and (17.10), there are exactly two equilibria.
- (b) What does the model predict if the first species goes extinct, so $x_1 = 0$?

Exercise 5. Show that for our predator-prey model, eqs. (17.11) and (17.12), there are exactly three equilibria, corresponding to extinction of the predator (but not the prey), coexistence of predator and prey, or extinction of both predator and prey. Why is extinction of the prey impossible unless the predator goes extinct as well? Can you answer that question first using common sense, then using our equations?

17.2.2 Linearization near an equilibrium

We are especially interested in solutions near an equilibrium. Will they move away from the equilibrium, or towards it? If they always move towards the equilibrium, we call the equilibrium *stable*, or *attracting*. If some solutions starting out near the equilibrium move away from it, we call the equilibrium *unstable*. If *all* solutions starting out near the equilibrium move away from it, we call the equilibrium *repelling*.

Stability of equilibria is crucial in science. Is the earth's climate presently at an equilibrium that is robust to small perturbations? When the rabbits have a good year by chance, and multiply more than usual, will that cause the whole local ecosystem to go out of whack, or will things be back to normal next year? When the International Space Station is at a steady height above the earth, will slight perturbations of its height cause it to fly off into space? Will a brief rise in your heart rate bring your entire body out of equilibrium, or will the heart rate just return to normal?

Exercise 6. Think of other real-life examples where the stability of equilibria is important.

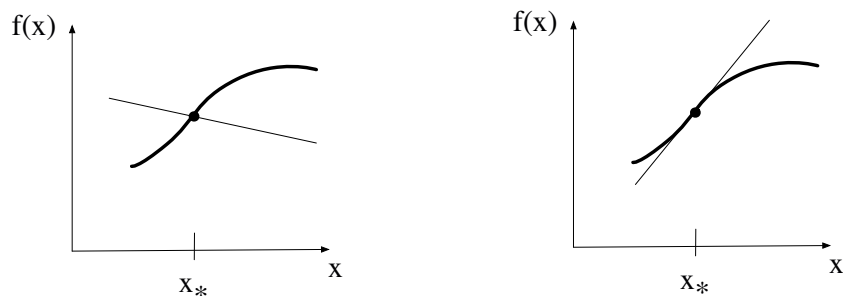
Exercise 7. Think about the example

$$\frac{dx}{dt} = e^{-x} - 1.$$

- (a) Plot the graph of the function $F(x) = e^{-x} - 1$.
- (a) Explain: $x_* = 0$ is an equilibrium.
- (b) Explain: solutions $x(t)$ with $x(t) < 0$ move right, towards $x_* = 0$.
- (c) Explain: solutions $x(t)$ with $x(t) > 0$ move left, towards $x_* = 0$.

So $x_* = 0$ is a stable equilibrium.

Exercise 8. Do all the steps of Exercise 7 for $F(x) = e^x - 1$. What changes?



getting the value right

getting the value and the derivative right

Figure 17.1: What's special about the tangent line. In this example, the tangent line *intersects* the graph. This only happens at inflection points.

Exercise 9. Think about the case $m = 1$. Assume that $F = F(x)$ is differentiable. Assume that $F(x_*) = 0$. Explain: If $F'(x_*) > 0$, then x_* is an unstable equilibrium. If $F'(x_*) < 0$, then x_* is a stable equilibrium.

We will use the *local linear approximation* of F near x_* :

$$F(x) \approx F(x_*) + F'(x_*)(x - x_*) \quad \text{for } x \approx x_*. \quad (17.13)$$

It is important to understand that in (17.13), we think of x as an independent variable, studying the dependence of $F = F(x)$ on x . By contrast, in (17.1), x is the *dependent* variable, and we are interested in understanding the dependence of $x = x(t)$ on t .

Approximation (17.13) is familiar from your first semester of calculus, but there you only studied functions of a single variable. So let us assume for a moment that $m = 1$. Then the right-hand side of the approximate equation in (17.13) is the familiar expression for the tangent to the graph of F at x_* . The equation

$$y = F(x_*) + F'(x_*)(x - x_*) \quad (17.14)$$

defines the tangent line. It is characterized by two properties: First, when $x = x_*$ then $y = F(x_*)$. Second, for all x , and therefore in particular for $x = x_*$, $dy/dx = F'(x_*)$. The tangent is a good approximation near $x = x_*$ because at x_* , it gets both the value and the derivative right; see Fig. 17.1.

Exercise 10. Find the local linear approximation to $F(x) = \ln x$ at $x_* = 1$.

Now back to the case when $m > 1$. In that case x , x_* , and $F(x)$ are all vectors. What is $F'(x_*)$? It is the $m \times m$ matrix of all first derivatives. The first

row contains the first partial derivatives of F_1 . The second row contains the first partial derivatives of F_2 . And so on. The matrix $F'(x_*)$ is also called the *Jacobi matrix* of F at x_* . For our model of two competing species, eqs. (17.9) and (17.10), we have at $x_* = (1, 1)$:

$$F'(x_*) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \quad (17.15)$$

Exercise 11. For our model of a predator and a prey, eqs. (17.11) and (17.12), when $x_* = (3, 1)$, verify that

$$F'(x_*) = \frac{1}{4} \begin{bmatrix} 0 & 6 \\ -2 & 1 \end{bmatrix}. \quad (17.16)$$

When $m > 1$, what is special about (17.13)? It is linear, and it gets the value and all m^2 first derivatives right. That is, when you insert x_* into the right-hand side of the approximate equation (17.13), you get $F(x_*)$, and when you take the partial derivative with respect to x_j of the i -th component of

$$F(x_*) + F'(x_*)(x - x_*),$$

you get

$$\frac{\partial F_i}{\partial x_j}(x_*).$$

The right-hand side of the approximate equation (17.13) is the only linear approximation of F that gets the value and all m^2 first-order derivatives at x_* right. It is in this sense the best linear approximation to F near x_* .

For a solution $x(t)$ with $x(t) \approx x_*$, we approximate

$$\frac{dx}{dt} = F(x)$$

by

$$\frac{dx}{dt} = F(x_*) + F'(x_*)(x - x_*). \quad (17.17)$$

Remembering that

$$F(x_*) = 0$$

and writing

$$J = F'(x_*),$$

we see that (17.17) becomes

$$\frac{dx}{dt} = J(x - x_*). \quad (17.18)$$

One more trick: We write

$$u = x - x_*.$$

So u is the difference between x and x_* . It is a function of t of course, since x is a function of t . Notice that

$$\frac{du}{dt} = \frac{dx}{dt}.$$

This is because x_* does not depend on t , so

$$\frac{d}{dt}x_* = 0.$$

So eq. (17.18) becomes

$$\frac{du}{dt} = Ju. \quad (17.19)$$

This is a system of m *linear* ordinary differential equations for m unknown functions u_1, u_2, \dots, u_m . Note that $u_* = 0$ is an equilibrium point. Remembering that u is the difference between x and x_* , our question is now whether $u(t) \rightarrow 0$ as $t \rightarrow \infty$ when $u(0) \approx 0$.

17.2.3 Solving the linearized system

Let's think about eq. (17.19) in the very simple case when $m = 1$ first.

Exercise 12.

(a) Explain why

$$\frac{du}{dt} = Ju \quad (17.20)$$

(where J is a number and $u = u(t)$ is a number) is equivalent to

$$u(t) = u(0)e^{Jt}. \quad (17.21)$$

(Compare Exercise 1.)

(b) Explain using (17.21): 0 is a stable fixed point if $J < 0$, and an unstable one if $J > 0$.

(c) Explain: You could have come to the same conclusion using Exercise 9.

Now let $m > 1$. Suppose that you knew m linearly independent eigenvectors u_1, u_2, \dots, u_m of J , and the associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. Then any solution $u(t)$ of $du/dt = Ju$ can be written in the form

$$u = c_1 u_1 + \dots + c_m u_m.$$

Let's be more explicit:

$$u(t) = c_1(t)u_1 + \dots + c_m(t)u_m. \quad (17.22)$$

The c_i are time-dependent. The u_i are constant vectors. We insert (17.22) into $du/dt = Ju$, and find:

$$c'_1(t)u_1 + \dots + c'_m(t)u_m = c_1(t)\lambda_1 u_1 + \dots + c_m(t)\lambda_m u_m.$$

Since u_1, u_2, \dots, u_m are linearly independent, this implies

$$c'_i(t) = \lambda_i c_i(t) \quad (17.23)$$

for all i with $1 \leq i \leq m$. The derivative of c_i is a constant multiple of c_i . That's the equation of Exercise 12, except that now we write " c_i " instead of " u " and " λ_i " instead of " J ". According to Exercise 12, eq. (17.23) is equivalent to

$$c_i(t) = c_i(0)e^{\lambda_i t}$$

for some constant k_i . So

$$u(t) = c_1(0)e^{\lambda_1 t}u_1 + \dots + c_m(0)e^{\lambda_m t}u_m.$$

We summarize what we just learned in a theorem.

Theorem 17.1. *Let $J \in \mathbb{R}^{m \times m}$, and let u_1, u_2, \dots, u_m be linearly independent eigenvectors of J with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. Then the solutions of $du/dt = Ju$ satisfy*

$$u(t) = c_1(0)e^{\lambda_1 t}u_1 + \dots + c_m(0)e^{\lambda_m t}u_m,$$

where k_1, k_2, \dots, k_m are constants.

We have deliberately not said that u_1, u_2, \dots, u_m are *real* vectors, and that $\lambda_1, \lambda_2, \dots, \lambda_m$ are *real* eigenvalues. They can be complex.

What happens if J does not have a set of linearly independent eigenvectors? A similar result still holds, but to state and prove it, we would need the *Jordan normal form theorem* and the notion of *generalized eigenvectors*. Since the reader probably doesn't know those concepts yet, and we don't have enough space to introduce them here, I will not discuss this case here.

17.2.4 Stability of equilibria

If all the functions $e^{\lambda_i t}$ in Theorem 17.1 converge to zero as $t \rightarrow \infty$, then $u(t)$ converges to zero. If any of the functions $e^{\lambda_i t}$ diverges to ∞ as $t \rightarrow \infty$, then there are solutions of $du/dt = Ju$ that start arbitrarily close to 0, but diverge to infinity.

When does $e^{\lambda t}$ converge to zero, and when does it converge to ∞ ? Write

$$\lambda = a + ib$$

with $a \in \mathbb{R}, b \in \mathbb{R}$. One calls a the *real part* of λ , and b the *imaginary part*. Then

$$e^{\lambda t} = e^{(a+ib)t} = e^{at}e^{ibt} = e^{at}(\cos bt + i \sin bt).$$

This implies that

$$|e^{\lambda t}| = e^{at} \rightarrow \begin{cases} 0 & \text{if } a = \operatorname{Re}(\lambda) < 0, \\ \infty & \text{if } a = \operatorname{Re}(\lambda) > 0. \end{cases}$$

Now recall that we really wanted to understand $dx/dt = F(x)$, and in particular the solutions near an equilibrium x_* . We had discovered that when x is a solution near x_* , then $u = x - x_*$ approximately satisfies $du/dt = Ju$. Our discussion suggests the following theorem.

Theorem 17.2. *Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a continuously differentiable function, $x_* \in \mathbb{R}^m$, $F(x_*) = 0$. Let J be the Jacobi matrix of F at x_* . If all eigenvalues of J have (strictly) negative real parts, then x_* is stable. If J has an eigenvalue with (strictly) positive real part, then x_* is unstable.*

We now apply this result to the two examples discussed earlier, the model of competition among species, and the model of predator-prey dynamics. The eigenvalues of (17.15) are $+1$ and -1 . This means that coexistence of the two competing species is unstable. In population biology, the “principle of competitive exclusion” is a widely discussed idea. According to this principle, two species competing for the same resources cannot co-exist in a stable way. Our example is a mathematical illustration of this principle.⁶

Figure 17.2 shows solutions of the system (17.9), (17.10). This sort of plot is called a *phase plane plot*. It does not show x_1 and x_2 as functions of time, but rather shows the paths that (x_1, x_2) can trace out while following the differential equations (17.9), (17.10). It’s like seeing the white streak that a jet plane leaves in the blue sky, rather than seeing the plane fly. The figure shows that indeed the equilibrium $(1, 1)$ is unstable.

The eigenvalues of (17.16) are approximately $0.125 \pm 0.875i$, so their real parts are positive. This implies that for our predator-prey model, the coexistence equilibrium $(3, 1)$ is not stable. Figure 17.3 shows solutions of the system (17.11), (17.12), demonstrating that indeed the equilibrium $(3, 1)$ is unstable, but also showing that there is a solution moving along a closed loop in the plane (x_1, x_2) . This corresponds to persistent oscillations in the predator and prey population sizes. Predator-prey oscillations are indeed observed in nature; see [2] and references given there.

In summary, to assess the stability of a dynamic equilibrium, one must find the eigenvalues of the Jacobi matrix, and see if there are any with positive real part. In our two examples, that was easy. It would be much less easy for an equilibrium involving 1000 interacting species, since then the eigenvalues of a 1000×1000 matrix must be examined.

⁶It is known that the principle is not universally correct. The most famous counter-example is the fact that different species of plankton survive in the ocean, co-existing even though they feed on the same resources.

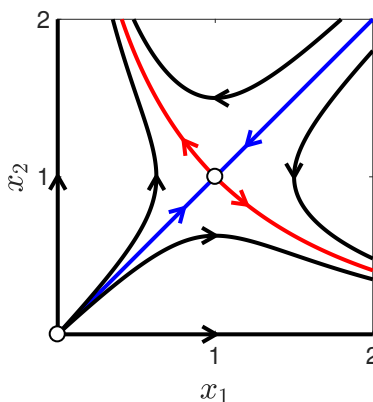


Figure 17.2: Solutions of eqs. (17.9), (17.10) near the coexistence fixed point $(1, 1)$.

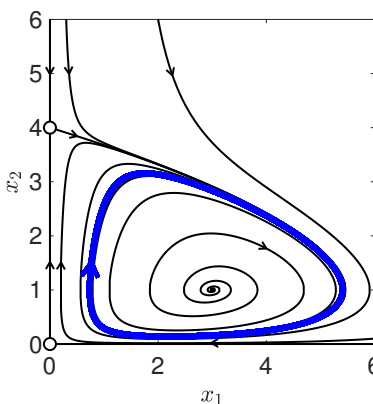


Figure 17.3: Solutions of (17.11), (17.12) near the coexistence fixed point $(3, 1)$.

Exercise 13. Suppose that two population sizes x_1 and x_2 are governed by

$$\begin{aligned}\frac{dx_1}{dt} &= (1 - x_2)x_1(1 - x_1), \\ \frac{dx_2}{dt} &= (2 - x_1)x_2(3 - x_2).\end{aligned}$$

- Does this model competition between species, or a predator and a prey? (Assume $0 < x_1 < 1$ and $0 < x_2 < 3$.)
- Find all equilibria. Pick one, and analyze its stability, if that's possible by studying the eigenvalues of the Jacobi matrix. (There are instances when studying the eigenvalues of the Jacobi matrix is inconclusive.)

Exercise 14. Give an example of a system $\frac{dx}{dt} = F(x)$ for which 0 is an equilibrium, the eigenvalues of $F'(0)$ have real parts ≤ 0 , one of the eigenvalues is equal to zero, and the equilibrium 0 is stable. Then give another example satisfying all of the above assumptions, except that the equilibrium is unstable. (Feel free to make life simple by choosing $m = 1$.)