Monday, April 24

Saturday, April 22, 2023 14:50

The final is posted, and you need to hand it in by Thursday, May 4, 11:59 p.m. on Gradescope.

I realized it was fairer to give everyone the full time (not 5 or 6 days) to do the final.

<u>Student Hours for final:</u>
Wednesday, April 26, 11:00-12:00
Thursday, April 27, 1:30-3:00+
Thursday, April 27, 4;30-5:45, Review Session, Nelson Auditorium, Anderson Hall SEC
Friday, April 28, 1:30-2:30+ in the math lounge
Last class Wednesday April 26!

Please email Todd (todd.quinto@tufts.edu) or our TA, Wentao, (wentao.fan@tufts.edu) anytime, and we will answer your question as soon as possible. You can also ask on Piazza; students will not be able to see your question, but instructors will.

Please note that I will have intermittent internet access starting on Saturday, April 29. I will try my best to answer every question within 36 hours, and you can also email Wentao and ask us both using Piazza.

Rules:

- If you use information on the internet you must provide the complete URL, and if you use another book, you must provide a reference including page number or theorem number/name.
- You may not *directly copy* information from any source and you must make sure all answers are in your own words. If you happen to find one of the test problems in a book or on the internet, you may not just refer to the result but prove the result in your own words.
- If you discuss any problem on the test with *anyone* (including Todd and Wentao), you need to give their name and which problem(s) you discussed with them.

Misprint in problem 3: The function g should be:

$$g(x) = \begin{cases} 0 & 0 \le x \le \pi/3 \\ 1 & \pi/3 < x < 5\pi/3 \\ 0 & 5\pi/3 \le x \le 2\pi \end{cases}.$$

Math Department picnic, Tuesday, May 2, 12:00-4:00 on the Bromfield Pearson lawn!!!!

The standard Fourier sine and cosine series of a function $f \in L^2([0,2\pi],\mathbb{R})$

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad where$$
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx,$$

This follows (as we showed in class last time) from the fact that the orthonormal sine and cosine system is complete.

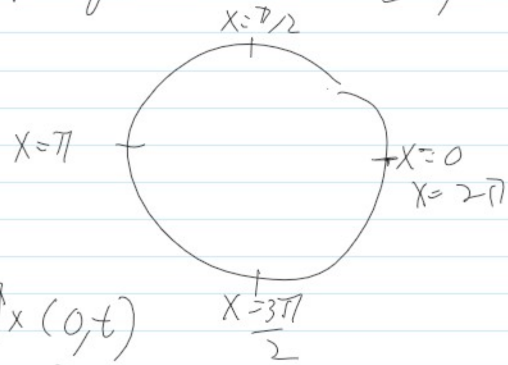
heat

trig-plot

Heat equation in insulated ring.

$T(x,t)$ = temp at posn $x \in [0, 2\pi]$
at time t

$k > 0$



PDE $k \dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial T}{\partial t}$

$(x,t) \in [0, 2\pi] \times (0, t)$

Boundary conds BC $T(0,t) = T(2\pi, t)$

$\dfrac{\partial T}{\partial x}(0,t) = \dfrac{\partial T}{\partial x}(2\pi, t)$ $\quad t \in (0, \infty)$

we can add an initial condition IC

$T(x,0) = f(x) \quad x \in [0, 2\pi]$

gives initial temp configuration

goal know how temp changes over tm

ie solve for $T(x,t)$ for $(x,t) \in [0,2\pi] \times (0,\infty)$

This is solved using Separation of variables
(Math 155 PDE)

for any constants $a_0, a_1, \ldots a_N$ $\quad N \in \mathbb{N}$

$b_1, b_2, \ldots b_N$

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) e^{-n^2 kt}$$

solves $k \dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}$

and bc

we want to use this to solve IC

$T(x,0) = f(x) \stackrel{?}{=} \dfrac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$

need an indefinite sum to do this $\forall f \in L^2$

So we need math to chan N to ∞!

Note if $f \in L^2$ all of its

Fourier coefficient are bounded.

Fourier coefficient are bounded.

pf for $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\langle f, \cos nx\rangle$

C S B

$\frac{1}{\pi}|\langle f,\cos nx\rangle| \le \frac{1}{\pi}\|f\|_2 \|\cos nx\|_2$ check this

$= \frac{1}{\pi}\|f\|_2 \sqrt{\pi}$

$\frac{1}{\pi}|\langle f, \cos nx\rangle| = |a_n| \le \frac{1}{\sqrt{\pi}}\|f\|_2$

Big Heat eqn Thm.

Let $\{a_n\}_{n=0}^{\infty}$ $\{b_n\}_{n=1}^{\infty}$ be bndd

sequences of real numbers

let $k>0$

Then $T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)e^{-n^2kt}$

solves Heat eqn BVP

$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $(x,t)\in[0,2\pi]\times(0,\infty)$

$T(0,t) = T(2\pi,t)$

$\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(2\pi,t)$ $\forall t\in(0,\infty)$

if $f\in L^2([0,2\pi],\mathbb{R})$ has

std Fourier series $f = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

Then $T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)e^{-n^2kt}$

where a_n & b_n are the Fourier coeffs of f

satisfies Heat eqn, BC and

IC $T(x,0) = f(x)$

$\left[\; a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx\right.$

Furthermore the soln $T(x,t)$ is a C^∞ has all partial derivs & they are all cont.

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fn for $(x,t) \in [0, 2\pi] \times (0, \infty)$

as soon as $t > 0$

and $\forall x \in [0, 2\pi]$ $\lim_{t \to \infty} T(x,t) = \frac{a_0}{2} = \frac{1}{2}\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

so $\lim_{t\to\infty} T(x,t)$ = ave value of f $\forall x$ = average value of f

eg solve Heat eqn

IBVP $\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $(x,t) \in [0, 2\pi] \times (0,\infty)$

BC $T(0,t) = T(2\pi, t)$

$\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(2\pi, t)$ $t \in (0,\infty)$

IC $T(x,0) = f(x) = \begin{cases} -1 & x \in (0, \pi] \\ 1 & x \in (\pi, 2\pi] \cup \{0\} \end{cases}$

Soln we know for any bdd (a_n) & (b_n)

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) e^{-n^2 t}$$

solves heat eqn BVP

now solve IC

$$f(x) = T(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx =$

$\frac{1}{\pi}\left(\int_0^{\pi} (-1)\sin nx\,dx + \int_{\pi}^{2\pi} (+1)\sin nx\,dx\right)$

$= \frac{2}{\pi}\int_0^{\pi} (-1)\sin nx\,dx = \frac{2}{\pi} \frac{\cos nx}{n}\Big|^{\pi}$

$$= \frac{2}{\pi} \int_0^{\pi} (-1) \sin nx \, dx = \frac{2}{\pi} \left. \frac{\cos nx}{n} \right|_{x=0}^{\pi}$$

$$= \frac{2}{n\pi} \left(\cos n\pi - \cos 0 \right) = \frac{2}{n\pi} \left((-1)^n - 1 \right)$$

So IBVP has soln

$$T(x,t) = 0 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \left((-1)^n - 1 \right) \sin nx \, e^{-n^2 t}$$

I will show partial sum of this

for all $x \in [0, 2\pi)$ specific T

$$S(N, x, t) = \sum_{n=1}^{N} \frac{2}{n\pi} \left((-1)^n - 1 \right) \sin nx \, e^{-n^2 t}$$