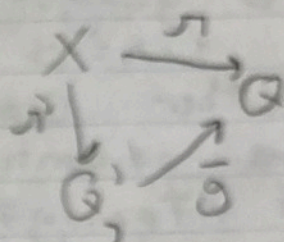
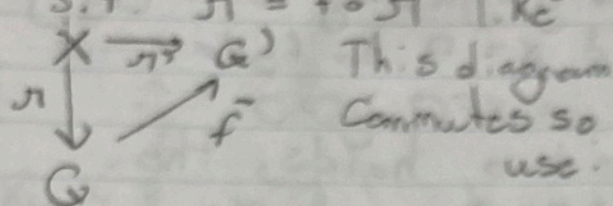


1 First, need to show that π respects the equivalence relation. To show, let $\bar{f}: Q \rightarrow Q$ or $\bar{f} = \text{id}_Q$. By the UPQ $\bar{f} \circ \pi = \text{id}_Q \circ \pi = \pi$ so π satisfies the equivalence relation. The same can be said for π' and Q' .

$\pi: X \rightarrow Q$ and $\pi': X \rightarrow Q'$ let $f: X \rightarrow Q'$ as in UPQ, B is arbitrary. So, by UPQ $\exists \bar{f}$ s.t. $\pi' = \bar{f} \circ \pi$ like



By UPQ, $\exists \bar{g}$ s.t. $\pi = \bar{g} \circ \pi'$

Since $\pi' = \bar{f} \circ \pi = \bar{f} \circ \bar{g} \circ \pi'$ so $\bar{f} \circ \bar{g} = \text{id}_{Q'}$ Can show the same way $\bar{g} \circ \bar{f} = \text{id}_Q$. So \bar{f} and \bar{g} are inverses and by UPQ is unique, so get \bar{f} is the unique bijection $Q \rightarrow Q'$ where $\bar{f} \circ \pi = \pi'$.

2 From the diagram $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$ Under the rule $x \mapsto x$, and as $X_1, AX_2 \in X_1$ and $X_1, AX_2 \in X_2$ then j_1 and j_2 have the same image and the same can be said for i_1, i_2 . So $i_1 \circ j_1 = i_2 \circ j_2$. To show uniqueness let $\bar{g}: X_1 \cup X_2 \rightarrow B$ then so

$$x \mapsto x$$

$f_2 = g \circ i_2$ but as $x \mapsto x$ and $f_2 = f \circ i_2$ $f \circ i_2 = g \circ i_2$, so $f = g$ and f is unique.

$$3a) \frac{x}{1} = x(\frac{1}{1}) = x \cdot 1 = x \quad \text{so } \frac{x}{1} = x$$

b) Contrapositive: If $xy = 0$ then $x = 0$ or $y = 0$

Case 1)

$$x \neq 0 \quad \frac{1}{x}(xy = 0) \Rightarrow \frac{1}{x}(xy) = \frac{1}{x} \cdot 0 \rightarrow (\frac{1}{x} \cdot x)y = 0$$

So as $\frac{1}{x} \cdot x = 1 \cdot y = 0$ and $y = 0$ so if $x \neq 0$ then $y = 0$

Case 2)

$$y \neq 0 \quad \frac{1}{y}(xy) = 0 \rightarrow \frac{1}{y}(yx) = \frac{1}{y} \cdot 0 \rightarrow (\frac{1}{y} \cdot y)x = 0$$

So $1 \cdot x = 0$ and $x = 0$ so if $y \neq 0$ then $x = 0$

The other case is $x = 0, y = 0$.

Therefore as contrapositive holds, the statement $x \neq 0$ and $y \neq 0 \rightarrow xy \neq 0$ holds

c) $1 = 1$ As $y, z \neq 0$ have inverses.

$$\text{So } yz \cdot \frac{1}{yz} = \frac{1}{y} \cdot y \cdot \frac{1}{z} \cdot z \quad \text{and } \frac{1}{y} \cdot y = 1 \text{ and } \frac{1}{z} \cdot z = 1$$

$$\rightarrow yz \cdot \frac{1}{yz} = y \cdot z \cdot \frac{1}{y} \cdot \frac{1}{z} \quad \text{by commutativity and as } yz \neq 0$$

$$= \frac{yz}{yz} = \frac{1}{y} \cdot \frac{1}{z} \quad \square \quad \square$$

$$d) w((\frac{1}{y})(\frac{1}{z})) = (\frac{1}{y})(\frac{1}{z}) \quad \text{Assume } y, z \neq 0$$

$$\rightarrow w(\frac{1}{y})w(\frac{1}{z}) = w(\frac{1}{y})(\frac{1}{z}) \quad \text{by def of division}$$

$$\rightarrow x[(\frac{1}{y})(\frac{w}{z})] = (\frac{w}{y})(\frac{1}{z})$$

$$\Rightarrow x(\frac{1}{y})(\frac{w}{z}) = x(\frac{w}{y})(\frac{1}{z})$$

$$\rightarrow x(\frac{1}{y})(\frac{w}{z}) = (\frac{w}{y})(x \cdot \frac{1}{z})$$

$$\text{So } (\frac{x}{y})(\frac{w}{z}) = (\frac{w}{y})(\frac{x}{z}) \quad \square$$

e) Since $x \neq 0$, it has an inverse $\frac{1}{x}$. If $\frac{1}{x} = 0$ then $x(\frac{1}{x}) = x \cdot 0, 1 = 0$ which isn't true so $\frac{1}{x} \neq 0$

3 f) $w, z \neq 0$ $\left(\frac{1}{\frac{w}{z}}\right)$ has an inverse $\frac{w}{z}$ so

$$\left(\frac{1}{\left(\frac{w}{z}\right)} \cdot \left(\frac{w}{z}\right) = 1\right) \frac{z}{w}$$

$$\frac{1}{\left(\frac{w}{z}\right)} \cdot \frac{w}{z} \cdot \frac{z}{w} = \frac{z}{w}$$

$$\frac{1}{\frac{w}{z}} \cdot \frac{w}{w} \cdot \frac{z}{z} = \frac{z}{w} \rightarrow \frac{1}{\frac{w}{z}} \cdot 1 = \frac{z}{w}$$

$$\text{So } \frac{1}{\frac{w}{z}} = \frac{z}{w}$$

$$g) \frac{xy}{z} = \left(\frac{1}{\frac{z}{xy}} = \frac{1}{\frac{z}{xy}}\right)$$

$$xy \left(\frac{1}{z}\right) = xy \left(\frac{1}{z}\right) \quad \text{by def. of division}$$

$$\frac{xy}{z} = xy \left(\frac{1}{z}\right) \quad \text{by associativity}$$

$$\frac{xy}{z} = x \left(y \cdot \frac{1}{z}\right) \rightarrow \frac{(xy)}{z} = x \left(\frac{y}{z}\right) \quad \square$$

4 a) $0 < 1$, so as shown in class $-1 < 0$
 so $-1 < 0 < 1 \quad \square$

b) \Rightarrow If $xy > 0$ then x and y are both positive/negative.

Suppose, $xy > 0$ but x and y have different signs. So, WLOG, $x > 0$ and $y < 0$.

Since $y < 0$ let $y = -z$ where $z > 0$, then

$x(-z) > 0$, but $\rightarrow -xz > 0$ and $x \cdot z > 0$ as $x, z > 0$

so contradiction as from class if $xz > 0$ then $-xz < 0$

So, x and y must both be positive/negative

\Leftarrow The positive case is shown from recitation.

If $xy < 0$ then $x = -w$ where $w > 0$ and $y = -z$ where $z > 0$

$$\text{So } xy = (-w)(-z) = (-1)w(-1)z = (-1)(-1)wz = (-1)^2 wz$$

$$\text{Therefore } xy = wz \text{ and as } wz > 0 \quad \square$$

c) $x > 0$ so as $x(\frac{1}{x}) = 1 > 0$ and as shown in part b, this means x and $\frac{1}{x}$ have the same sign. So, since $x > 0$, $\frac{1}{x} > 0$ \square

d) $x > y > 0$ as $x, y > 0$ they have multiplicative inverses

$$\text{So } \frac{1}{y} (0 < y < x) \rightarrow 0 < 1 < \frac{x}{y}$$

$$\frac{1}{x} (0 < 1 < x \cdot \frac{1}{y}) \rightarrow 0 < \frac{1}{x} < \frac{1}{x} \cdot x \cdot \frac{1}{y} \rightarrow 0 < \frac{1}{x} < \frac{1}{y} \quad \square$$

a) Induction on m so n is fixed

Base case: $m = 1$

$$a^n \cdot a^1 = a^{n+1} \text{ which is how we define exponents}$$

Assume true up to m , here so $a^n a^m = a^{n+m}$

Show for $m+1$

$$a^n a^{m+1} = a^{n+m+1}$$

$$\rightarrow a^n a^m a = a^{n+m+1}$$

$$a^{n+m} a = a^{n+m+1}$$

$$a^{n+m+1} = a^{n+m+1} \quad \square$$

by definition get

5 b) Induction on m so n is fixed

Base case $m=1$

$$(a^n)^1 = a^n \quad \text{as define } a^1 = a$$

$$a^n = a^n$$

Assume it holds up to m , so $(a^n)^m = a^{nm}$
Show for $m+1$ it holds.

$$(a^n)^{m+1} = a^{n(m+1)}$$

$$\rightarrow (a^n)^m a^n = a^{nm+n} \quad \text{by part a)}$$

$$(a^n)^m a^n = a^{nm} a^n \quad \text{as } a^n \neq 0.$$

$$\rightarrow a^{nm} a^n = a^{nm+n} \quad \text{as true to } m$$

$$(a^n)^m = a^{nm} \quad \text{So } a^{nm} a^n = a^{nm+n} \quad \text{and proves}$$

c) Base case for induction on m

$$m=1$$

$$a^1 b^1 = a \cdot b = (ab)^1 \quad \text{So base case holds}$$

Assume true up to m , show it holds for $m+1$

$$\text{So need to show } a^{m+1} b^{m+1} = (ab)^{m+1}$$

$$a^{m+1} b^{m+1} = a^m a b^m b = a^m b^m ab \quad \text{as true up to } m$$

$$a^m b^m = (ab)^m \quad \text{and } ab = (ab)^1 \text{ so}$$

$$(ab)^m (ab) = (ab)^{m+1} \quad \text{and by def}$$

$$(ab)^m (ab) = (ab)^m ab \quad \text{so this holds}$$

and is true

