MATH 65, FALL 2021, ANSWERS TO QUIZZES

Contents

1. First Quiz, Sets Sept 8

Question 1.1. (a) *Define* $A = \{n \in \mathbb{Z} |$ *exists* $k \in \mathbb{Z}, n = 2k+1\}$, $B = \{n \in \mathbb{Z} |$ *exists* $k \in \mathbb{Z}, n = 2k-1\}$. *Show that* $A = B$.

We need to show $A \subseteq B$ and $B \subseteq A$.

Take $n \in A$. We want to see $n \in B$. From the definition of A, there is $k \in \mathbb{Z}, n = 2k+1$. We can write $2k+1 = 2(k+1)-1$. As $k+1$ is also an integer, this shows that $n \in B$.

Take $n \in B$. We want to see $n \in A$. From the definition of B, there is $k \in \mathbb{Z}, n = 2k-1$. We can write $2k-1 = 2(k-1)+1$. As $k-1$ is also an integer, this shows that $n \in A$.

As every element in A is in B and every element in B is in A, the two sets are identical.

(b) *Define* $C = \{n \in \mathbb{Z} \mid$ *exists* $k \in \mathbb{N}, n = 2k+1\}$, $D = \{n \in \mathbb{Z} \mid$ *exists* $k \in \mathbb{N}, n = 2k-1\}$. *Is* $C = D$*?*

We could prove that $C \subseteq D$ in the same way that we proved before that $A \subseteq B$ in part (a). The converse inclusion though is not true. If we start the natural numbers at 0, then $k = 0 \in \mathbb{N}$ and $2 \times 0 - 1 = -1 \in D$. But $-1 \notin C$ as the only expression of 1 as $2k+1$ will be with $k = -1$ and -1 is not a natural number.

(c) *Consider a collection of identical coins distributed in piles. We take one coin from each pile and put them together to form a new pile. We can represent this situation by a set whose elements are the number of coins on each pile. For example, if we have two piles with one coin and one with 5 coins, the original set is* $\{1,1,5\}$*, the new set will be* $\{4,3\} = \{3,4\} =$ *(the order in which we list the elements of a set does not matter). Find the collections that are invariant under this operation (for instance* $\{1,1,5\}$ *is not but* $\{1,2\}$ *is. Once you come up with a good guess, make sure that you justify that these are the only choices that will work.*

Let us write our collection as $\{a_1, a_2 \dots, a_n\}$. The order does not matter, we can write $a_1 \le a_2 \le \dots \le a_n$. As we build a new pile with the coins we take away from each of the old piles, one and only one of the old piles needs to disappear so that the number of piles stays the same. So $a_1 = 1, a_2 > 1$. The new collection is then $\{a_2 - 1, \dots, a_n - 1, n\}$. Then, $a_2 - 1 \le a_3 - 1 \le \dots \le a_n - 1 < a_n$, that is all the terms of the new collection except possibly the new pile of stolen coins are smaller than a_n. If the collection is to be invariant, the height of the new pile of stolen coins must be a_n, that is $a_n = n$. then the new collection is also ordered as $a_2 - 1 \le a_3 - 1 \le \dots \le a_n - 1 < n$. By invariance,

$$a_2 - 1 = a_1 = 1, a_3 - 1 = a_2, a_4 - 1 = a_3, \dots a_n - 1 = a_{n-1}, a_n = n$$

This gives

$$a_1 = 1, a_2 = a_1 + 1 = 2, a_3 = a_2 + 1 = 3, \dots, a_n = n$$

Conversely, the set $\{1, 2, \dots, n\}$ is invariant under the operation.

2. Quiz 2, Properties of operations, due Sept. 13

Recall the following facts

Fact 2.1. In this course, we will freely use the basic properties of addition and product of numbers in the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ satisfy all the usual properties

(1) Associative, that is for all a, b, c in one of these sets $(a+b)+c = a+(b+c),\ a(bc) = (ab)c$.
(2) Commutative, that is for all a, b in one of these sets $a + b = b + a,\ ab = ba$.
(3) Existence of a 0 and a 1 with $a + 0 = a, a \times 1 = a$ for all a in these sets.
(4) Distributive property: for all a, b, c in any of these sets $a(b + c) = ab + ac$.
(5) Existence of inverses for addition in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, that is for all a on each of these sets, there is another element that we call $-a$ on the same set such that $a + (-a) = 0$.
(6) Existence of inverses for product in $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, that is for all $a \neq 0$ on each of these sets, there is another element that we call $\frac{1}{a}$ on the same set such that $a \times \frac{1}{a} = 1$.
(7) Order in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ but not in $\mathbb{C}$: for any two different a, b, either $a < b$ or $b < a$ (and conversely, if $a < b$, then $a \neq b$). Moreover, if $a < b$ and $b < c$, then $a < c$.
(8) If $a < b$ and c is in the same set, then $a + c < b + c$. If $a < b$ and $c > 0$, then $ac < bc$.

Using these facts and only these facts, see how many of the following you can prove:

Question. (a) For all $a \in \mathbb{R}$, if $a < 0$, then $-a > 0$. Hint: add $-a$ to inequality.
(b) For all $a \in \mathbb{R}$, $a \times 0 = 0$. Hint :start with $0 + 0 = 0$, multiply with a and use the distributive property.
(c) For all $a, b \in \mathbb{R}$, show that $-(a \times b) = a \times (-b) = (-a) \times b$..
(d) For all $a, b \in \mathbb{R}$, if $a \times b = 0$, then either $a = 0$ or $b = 0$.

Answer 1. (a) Assume $a < 0$. Using fact (7) from 2.1, we can add $-a$ to both sides and preserve the inequality: $a+(-a) < 0+(-a)$. Using the properties of 0 and inverse (facts (3) and (5)), we obtain

$$0 = a + (-a) < 0 + (-a) = -a$$

(b) From Fact (3) in 2.1, we know that for all real numbers a, $a + 0 = a$. In particular, taking $a = 0$, we have that $0 + 0 = 0$.. Multiplying this identity with a, we have that $a \times (0+0) = a \times 0$. From the distributive property, we have that $a \times (0+0) = a \times 0 + a \times 0$. But we also know that $a \times (0+0) = a \times 0$. So, we deduce that $a \times 0 = a \times 0 + a \times 0$. We know that every element has an additive inverse. This is true in particular for $a \times 0$. So, adding it to both sides of the equality, we have $a \times 0 + (-a \times 0) = [a \times 0 + a \times 0] + (-a \times 0)$. Now we will use the associative property, the property of 0 and the property of the additive inverse

$$\begin{aligned}
0 = && a \times 0 + (-a \times 0) && \text{by the property of the additive inverse}\\
= && [a \times 0 + a \times 0] + (-a) \times 0 && \text{as we saw before}\\
= && a \times 0 + [a \times 0 + (-a) \times 0)] && \text{by the associative property}\\
= && a \times 0 + 0 && \text{by the property of the additive inverse}\\
= && a \times 0 && \text{by the property of 0}
\end{aligned}$$

(c) From the existence of inverse for addition, $a + (-a) = 0$. Multiplying both sides with b and using the distributive property $ab + (-a)b = 0b = 0$ Adding $-(ab)$ to both sides and using the commutative property

$$(-a)b = 0 + (-a)b = ab + -(ab) + (-a)b = 0 + (-(ab)) = -(ab)$$

(d) To prove (d), it suffices to see that if $a \neq 0, b \neq 0$, then $ab \neq 0$. When two numbers are different, according to fact (g), one of them is smaller than the other. As $b \neq 0$ by assumption, either $0 < b$ or $b < 0$. Let us assume first, that $0 < b$. Then, if $0 < a$, multiplying both sides by b and using fact (h) and what we already proved that $b \times 0 = 0$, we obtain $0 = b \times 0 < b \times a$. If $a < 0$, proceeding similarly $b \times a < 0$, so in both cases $b \times a \neq 0$. If $a > 0$, we could reason similarly. If both $a < 0, b < 0$, we have shown in (a) that $-a > 0, -b > 0$. therefore, from what we have already proved, $(-a)(-b) > 0$. But then, using (c) twice, $(-a)(-b) = -a(-b) = -(-(ab))$. From the definition of additive inverse, the inverse of the inverse is the original number $-(-(ab)) = ab$. Therefore $ab > 0$. This proof works also in $\mathbb{Z}$

For a proof in $\mathbb{R}$ or $\mathbb{Q}$, we could reason as follows: Assume $ab = 0$ and $a \neq 0$, using (6), $\frac{1}{a}$ exists. Then

$$\begin{aligned}
b &= 1 \times b && \text{by the identity property of 1 for product} \\
&= (\frac{1}{a}a)b && \text{by the inverse property of product} \\
&= \frac{1}{a}(ab) && \text{by the associative property} \\
&= = \frac{1}{a}0 && \text{by the assumption } ab = 0 \\
&= \quad 0 && \text{as checked in (b)}
\end{aligned}$$

That is, $b = 0$ as claimed.

Question. (a) *Prove that $(p \wedge q) \vee r$ is logically equivalent to $(p \vee r) \wedge (q \vee r)$.*

It is enough to check that the truth values of the two propositions are the same for each possible choice of truth values of p, q, r.. We do this by writing the table

p	**q**	**r**	**p ∧ q**	**(p ∧ q)∨ r**	**p ∨ r**	**q ∨ r**	**(p ∨ r)∧ (q ∨ r)**
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	T	T	F	T	T
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

(b) *Prove that $\sqrt{3}$ is not rational.* Let p be the proposition $x = \sqrt{3}$. Let q be the proposition $x \notin \mathbb{Q}$. We want to prove the statement $p \Rightarrow q$. We will do a proof by contradiction. We know that $p \Rightarrow q$ is logically equivalent to $(p \wedge \neg q) \iff F$. The statement $p \wedge \neg q$ takes the form

$$x = \sqrt{3} \ \wedge \ x \in \mathbb{Q}$$

Equivalently,

$$\sqrt{3} \in \mathbb{Q}$$

Rational numbers are of the form $\frac{a}{b}$. So, with our assumptions, $\sqrt{3} = \frac{a}{b}$ and we can assume a, b do not have prime factors in common. Then squaring

$$3 = \frac{a^2}{b^2}$$

So, $3b^2 = a^2$. Note that 3 is a prime number and 3 divides a^2. The prime factors of a^2 are the same as the prime factors of a, just with twice the exponents. It follows that 3 is a prime factor of a. Then , $\exists c \in \mathbb{N},\ a = 3c$, so $3b^2 = a^2 = 9c^2$. Dividing both sides by 3, $b^2 = 3c^2$. Then 3 divides b which is not compatible with 3 divides a and a, b do not have prime factors in common. We got a false statement as we were aiming for.

3. Quiz 3, Induction, Sept. 20.

Question 3.1. (a) *Let $a \in \mathbb{R} - \{1\}$. Show that $S_n = 1 + a + a^2 + \cdots + a^{n-1} + a^n = \frac{a^{(n+1)}-1}{a-1}$.*

We will use induction

- Step 1: For $n = 0$ the sum only has the term : $S_0 = a^0 = 1$, so the sum is 1. On the other hand, the conjectured equation takes the form (plugging in $n = 0$), $\frac{a^1-1}{a-1} = 1$. The number computed directly from the sum and the number from the equation agree, so the conjecture works for $n = 0$.
- Assume that $S_{n-1} = \frac{a^{(n-1)+1}-1}{a-1} = \frac{a^n-1}{a-1}$
  Using this expression, we compute the sum for n.

$$S_n = 1 + a + a^2 + \cdots + a^{n-1} + a^n = S_{n-1} + a^n = \frac{a^n - 1}{a-1} + a^n = \frac{a^n - 1 + a^n(a-1)}{a-1} =$$

$$= \frac{a^n - 1 + a^{n+1} - a^n)}{a-1} = \frac{a^{n+1} - 1}{a-1}$$

  checking the result for n

Then, by the principle of induction, the result is true for every n.

An alternative proof goes as follows: From $S_n = 1 + a + a^2 + \cdots + a^{n-1} + a^n$, $aS_n = a + a^2 + \cdots + a^{n-1} + a^n + a^{n+1}$. Therefore $aS_n - S_n = a^{n+1} - 1$. As $aS_n - S_n = S_n(a-1)$, we obtain, this time without using induction, $S_n = \frac{a^(n+1)-1}{a-1}$.

(b) *The towers of Hanoi is a game consisting of a board with three pivots and a collection of n disks all of different sizes with a hole that fits in the pivots. You start with all the disks stacked in one pivot, all in decreasing order and must move them to one of the other pivots by moving one piece at a time and never placing a disk above a smaller one. What is the minimum number of moves that you need to make to solve the towers of Hanoi puzzle? Prove your answer.*

Let us call the minimum number of moves to move a tower of Hanoi with n pieces to a different pivot H_n.

Note that the only way to move the bottom piece is having a tower with the remaining pieces on a different pivot, moving that last piece to the free pivot and rebuilding the tower with the $n-1$ pieces on top of the now moved bottom piece. It follows that

$$H_n = 2H_{n-1} + 1$$

There are two ways to proceed now. If we do not know what the final expression is, we can compute a few values

$$H_1 = 1,\ H_2 = 2 + 1,\ H_3 = 2^2 + 2 + 1, \ldots H_n = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1$$

To get a closed form for this expression, we can use part (a) with a=2. Then, (note that we are adding up to 2^{n-1}, not 2^n) $H_n = 2^n - 1$.

On the other hand, if somebody gave us the expression $H_n = 2^n - 1$ and we only need to check it, we can proceed with the usual steps of induction:

- Step 1: For $n = 1$, there is only one piece to move, so one move suffices, that is $H_1 = 1$. Using the conjectured expression, $H_1 = 2^1 - 1 = 2 - 1 = 1$ which checks with what we found.

- Assume that $H_{n-1} = 2^{n-1} - 1$. Using this expression, and what we reasoned should be the recursive expression for H_n, namely $H_n = 2H_{n-1}+1$, we can find an expression for n as

$$H_n = 2H_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 2 + 1 = 2^n - 1$$

  checking the result for n

Then, by the principle of induction, $H_n = 2^n - 1$ for all n.

(c) *If your pieces start on pivot 1 and you want your final tower to be in pivot 3, where should you place your top piece in the first move?*

If you want the final tower to be in pivot 3, you need to be able to move the bottom piece to pivot 3. So you should be building the tower with $n-1$ pieces to pivot 2. This tells us that the pivot you need to use in the first move will keep alternating with n. When $n = 1$, there is only one piece to move, so you would start moving it to pivot 3. Hence, if n is odd, you would move the first piece to pivot 3. If n is even, you would make the first move to pivot 2.

Question. (1) We will call a shape made of 3-identical unit squares forming the shape of an L a basic L shape. An n-sized L shape will consist of $3n^2$ unit squares arranged as an L. The figure below shows a basic L shape and a 3-sized L-shape. Show

(a) A $2 \times n$ rectangle can be completely covered by non-overlapping basic L shapes if and only if n is divisible by 3.

(b) Every n-sized L-shape can be completely covered by non-overlapping basic L shapes.

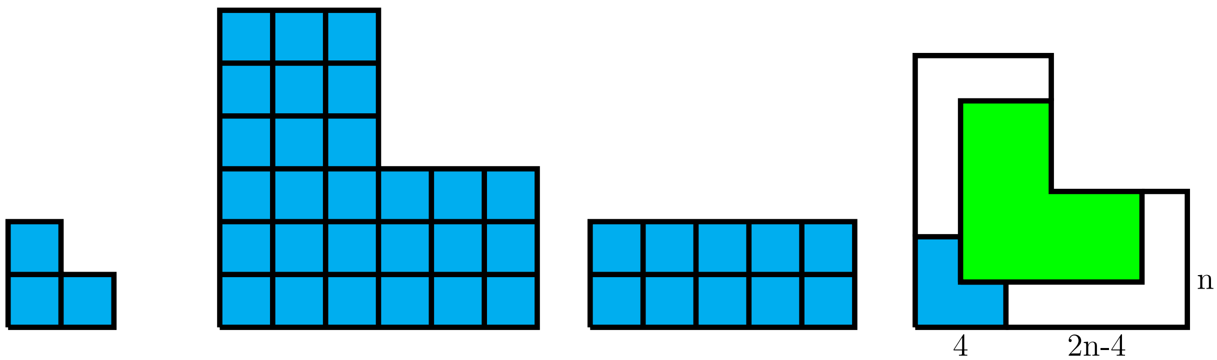


FIGURE 1. A basic L-shape and a 3-dimensional L-shape and a 2×5 rectangle.

Hint: look at the last picture. Remove a 2-sized and an n-2-sized L shapes at the angle

(a) A $2 \times n$ rectangle is made of $3n$ squares. A basic L-shape is made up of 3 squares. If the rectangle can be completely covered by non-overlapping basic L shapes, $2n$ needs to be divisible by 3. As $2, 3$ are relatively prime, this requires that n is divisible by 3.

To prove the converse, if $n = 3k$ is divisible by 3, a $2 \times n$ rectangle is made of k 2×3 rectangles. Each 2×3 rectangle can be covered by 2 basic Lshapes. Then the whole rectangle can also be covered with non-overlapping basic L shapes.

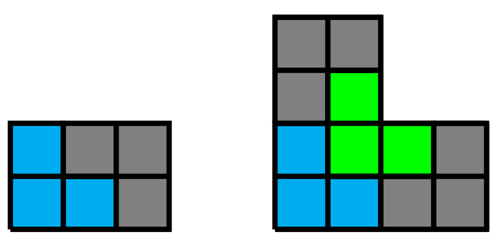


FIGURE 2. A 2×3 rectangle and a 2-sized L-shape covered by L-shapes

(b) In order to prove that every n-sized L-shape can be completely covered by non-overlapping basic L shapes, we use strong induction.

- Step 1 The case $n = 1$ is clear, as a 1-sized L-shape is a basic L shape.The case $n = 2$ is illustrated in the picture above.
- Step 2, We need to check that if the result is true for every $k < n$, then it is also true for n. We can assume $n > 2$, as the first two cases have already been

checked. Remove from the n-sized L-shape a 2-sized L-shape with corner on the lower left corner and an $n-2$-sized L-shape right above it (see Figure 3).

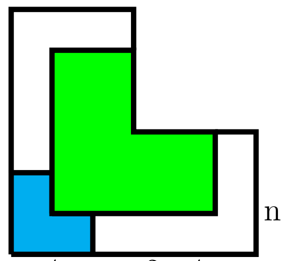


FIGURE 3. Remove from the L-shape a 2-L shape in the lower corner and an $n-2$ L-shape right above it.

The rest of the region consists of two pieces with similar shapes one at the top and the other at the right (white pieces in the picture). They can then be divided in $2 \times m$ rectangle with m divisible by 3 with suitable choices that depend on n. We describe the ones for the right white piece (see Figure 4):
If n is divisible by 3, so is $2n-6$. Form a vertical rectangle of height n and width 2 and a horizontal one of height 2 and width $2n-6$.
If $n-2$ is divisible by 3, so is $2n-4$. Form a vertical rectangle of height $n-2$ and width 2 and a horizontal one of height 2 and width $2n-4$.
If $n-1$ is divisible by 3, so are $n-4, 2n-8$. Form a vertical rectangle of height $n-4$ and width 2, a 2-dimensional L shape in the bottom left corner and a horizontal rectangle of height 2 and width $2n-8$.

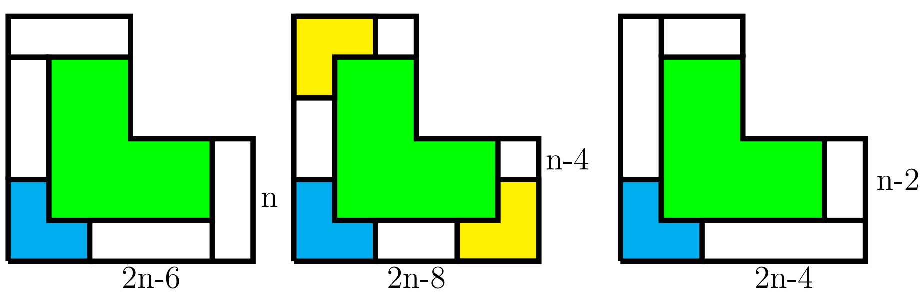


FIGURE 4. Subdivision of the left over rectangles corresponding to $n, n-1, n-2$ divisible by 3 respectively

(2) Consider the recurrence equation $a_n = a_{n-1} + 4a_{n-2} - 4a_{n-3}$ We would like to answer the following questions
   (a) Assume that b is a real number different from zero. Can the sequence $a_n = b^n$ be a solution to the recurrence? (and if so for which values of b).
   (b) Are there any solutions to the recurrence that satisfy $a_0 = 3, a_1 = 1, a_2 = 9$?
   (c) Are there any solutions to the recurrence that satisfy $a_0 = 3, a_1 = 1, a_2 = 9, a_3 = 2$?
   (a) We assume that the sequence $a_n = b^n$ satisfies $a_n = a_{n-1} + 4a_{n-2} - 4a_{n-3}$. Plugging in $a_n = b^n, a_{n-1} = b^{n-1}, a_{n-2} = b^{n-2}, a_{n-3} = b^{n-3}$, we obtain

$$b^n = b^{n-1} + 4b^{n-2} - 4b^{n-3} \iff b^n - b^{n-1} - 4b^{n-2} + 4b^{n-3} = 0 \iff b^{n-3}(b^3 - b^2 - 4b + 4) = 0$$

As we are assuming $b \neq 0$, this means $b^3 - b^2 - 4b + 4 = 0$. We can factor

$$b^3 - b^2 - 4b + 4 = (b-1)(b-2)(b+2).$$

Therefore, $b = 1, b = -2, b = 2$ are solutions of the stated form.

(b) We know that the sequences $a_n = 1, a_n = 2^n, a_n = (-2)^n$ are solutions of the recurrence. Then also any sequence of the form $a_n = x \times 1 + y \times 2^n + z \times (-2)^n$ for real numbers x, y, z is a solution. We can compute x, y, z so that $a_0 = 3, a_1 = 1, a_2 = 9$. We obtain $x = 1, y = 1, z = 1$.

(c) From the expression for a_n in terms of $a_{n-1}, a_{n-2}, a_{n-3}$ the first three terms of the sequence a_0, a_1, a_2 completely determine the sequence. So, as we saw before, if a solution were to exist with the given condition if would be given by $a_n = 1 + 2^n + (-2)^n$ which has the right values of a_0, a_1, a_2. But then $a_3 = 1 + 2^3 + (-2)^3 = 1 \neq 2$. Therefore, there is no sequence satisfying $a_n = a_{n-1} + 4a_{n-2} - 4a_{n-3}, a_0 = 3, a_1 = 1, a_2 = 9, a_3 = 2$.

4. Quiz 4, One to one and onto Functions, Sept. 28

Question 4.1. (a) *Define what it means for a function $f : A \to B$ to be one to one.*

A function $f : A \to B$ is one to one if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Equivalently $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

(b) *Define what it means for a function $f : A \to B$ to be onto.*

A function $f : A \to B$ is onto if every element in B is the image of at least one element in A. In symbols $\forall b \in B,\ \exists a \in A$ such that $b = f(a)$.

(c) *Consider the function* $\begin{array}{cccc} f & : & \mathbb{Z}\times\mathbb{Z} & \to & \mathbb{Z}\times\mathbb{Z} \\ & & (a,b) & \to & (3a+2b, a+b) \end{array}$ *Prove or disprove that f is one to one.*

We prove that f is one to one. Assume that $f(a_1, b_1) = f(a_2, b_2)$. From the definition of f,

$$(3a_1 + 2b_1, a_1 + b_1) = f(a_1, b_1) = f(a_2, b_2) = (3a_2 + 2b_2, a_2 + b_2)$$

Therefore, equating each component of the pair, we obtain,

$$3a_1 + 2b_1 = 3a_2 + 2b_2, \quad a_1 + b_1 = a_2 + b_2$$

Multiplying the second equation by 2 and subtracting from the first, we obtain $a_1 = a_2$. Multiplying the second equation by 3 and subtracting from the first, we obtain $-b_1 = -b_2$. Hence, $b_1 = b_2$. Therefore the pairs satisfy the equality $(a_1, b_1) = (a_2, b_2)$, proving that f is one to one.

(d) *Prove or disprove that the function f defined in (c) is onto.*

We prove that f is onto by showing that for all $(x, y) \in \mathbb{Z}\times\mathbb{Z}$, there exists $(a, b) \in \mathbb{Z}\times\mathbb{Z}$ such that $f(a, b) = (x, y)$. From the definition of f, one such (a, b) should satisfy

$$3a + 2b = x,\ a + b = y$$

Multiplying the second equation by 2 and subtracting from the first, we obtain $a = x-2y$. As, $x, y \in \mathbb{Z},\ x - 2y \in \mathbb{Z}$ Multiplying the second equation by 3 and subtracting from the first, we obtain $b = x - 3y$. Hence, $b = 3y - x \in \mathbb{Z}$. Then,

$$f(x - 2y, 3y - x) = (x, y)$$

and the map is onto

Question. *Let $f : A \to B$ be a function. We are NOT assuming that f is a bijection, so f^{-1} is not defined as a function.*

(a) *If $X \subseteq A$, define $f(X)$.*

$$f(X) = \{y \in B \text{ such that } \exists x \in X,\ y = f(x)\}.$$

(b) *If $Y \subseteq B$, define $f^{-1}(Y)$.*

$$f^{-1}(Y) = \{x \in A. \text{ such that } f(x) \in Y\}.$$

(c) *Show that for every subset $X \subseteq A$, $X \subseteq f^{-1}(f(X))$.*

Take $a \in X$. Then $f(a) \in B$ satisfies that there exists $x \in X$ (namely $x = a$ with $f(x) = f(a)$). Hence, from the definition of $f(X)$, $f(a) \in f(X)$. From the definition of $f^{-1}(Y)$ for $Y = f(X)$, $a \in f^{-1}(f(X))$. As this is true for every $a \in X$, we have the inclusion $X \subseteq f^{-1}(f(X))$.

(d) *Give an example of a function $f : A \to B$ and a subset $X \subseteq A$, such that $X \neq f^{-1}(f(X))$.*

Take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Take $X = \{2\}$. Then $f(X) = f(\{2\}) = \{4\}$. Hence $f^{-1}(f(X)) = f^{-1}(f(\{2\})) = f^{-1}(\{4\}) = \{2, -2\} \neq \{2\}$

(e) *Show that if f is one to one (injective), then for every subset X of A, $X = f^{-1}(f(X))$.*

We already know that $X \subseteq f^{-1}(f(X))$. We only need to show that $f^{-1}(f(X)) \subseteq X$.

Take $a \in f^{-1}(f(X))$. By definition of the inverse image of a set, this means that $f(a) \in f(X)$. By definition of image of a set, this implies that there exists $a' \in X$ such that $f(a') = f(a)$. Because f is one to one, $a = a'$.

(f) *Show that if for every subset X of A, $X = f^{-1}(f(X))$, then f is one to one (injective),*

Assume that $X = f^{-1}(f(X))$ for every subset X of A. Take $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$. Define $X = \{a_1\}$. Then $f(X) = f(\{a_1\}) = \{f(a_1)\}$ As $f(a_2) = f(a_1)$, $f(a_2) \in f(X)$. By definition of inverse image of a set, this implies $a_2 \in f^{-1}(f(X))$ and this set by assumption is $X = \{a_1\}$. Hence $a_2 \in \{a_1\}$ which implies $a_2 = a_1$.

We showed that if the images of two elements are the same, the elements are the same. This is the definition of the function being one to one.

5. Quiz 5, Countable sets, Oct. 5.

Question 5.1. (a) *In class, we showed that the set of pairs of positive integers is countable. Give a different way of counting the pairs of positive integers Describe your counting strategy both with a picture and algebraically.*

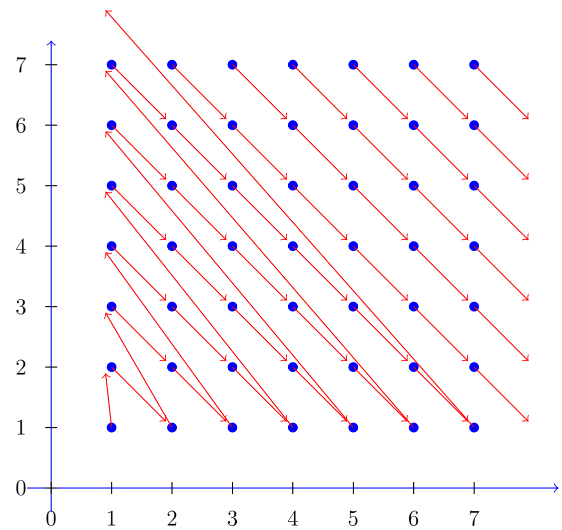


Figure 5. Ordering given in class.

We can order them in increasing order by their sum and within each sum in increasing order by the first term, so we would get a list such as

$$(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),(1,4)\dots$$

more specifically $(a,b)<(c,d)$ if and only if $a+b\leq c+d$ and if $a+b=c+d$ then $a<c$.

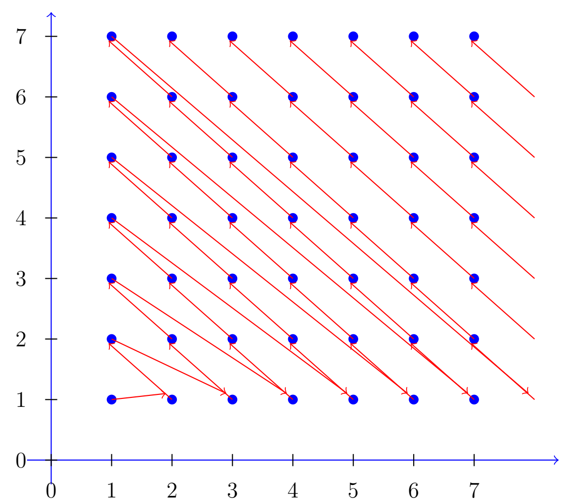


Figure 6. Another ordering.

We can order them in increasing order by their sum and within each sum in decreasing order by the first term, so we would get a list such as

$$(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),(1,4)\dots$$

more specifically $(a,b)<(c,d)$ if and only if $a+b\le c+d$ and if $a+b=c+d$ then $a>c$.

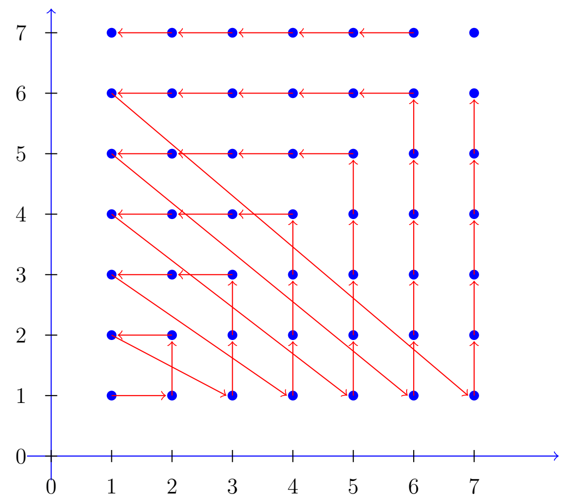


FIGURE 7. Another ordering.

In the ordering of figure 7,

$$\max\{a,b\}<\max\{c,d\} \text{ or } \max\{a,b\}=a=d=\max\{c,d\}$$

$$\text{or } \max\{a,b\}=a=c=\max\{c,d\} \text{ and } b<d \text{ or } \max\{a,b\}=b=d=\max\{c,d\} \text{ and } a>b$$

(b) *We know that the set of positive rational numbers is countable. Show that the set of all rational numbers is countable. Please be very precise and show how things follow from the definitions*

The set of negative rational numbers is in bijection with the positive rational numbers, so it is also countable. Because the set of even numbers is countable, there is a bijection from the positive rational with the even positive numbers and up to moving the even numbers one step up, with the strictly positive even numbers. The set of negative rational numbers is in bijection with the positive rational numbers, so it is also countable. Therefore, there is a bijection between the negative rationals and the natural odd numbers. As the set of rational numbers is the disjoint union of the positive, the negative and 0 and the natural numbers is a disjoint union of the positive even, the odd and 0, the three bijections together give a bijection between the rationals and the natural numbers.

Question. (a) *Let A_1, A_2, A_3 be finite sets. Find an equation for the cardinality of $A_1 \cup A_2 \cup A_3$ in terms of the cardinality of each of the sets and their intersections.*

We can write $A_1 \cup A_2 \cup A_3$ as the union of two sets, one being $A = A_1 \cup A_2$ and the other being $B = A_3$. Using the result that we know for two sets,

$$(*) \quad |A \cup B| = |A| + |B| - |A \cap B| = |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$$

We can apply again the rule for the union of two sets to $A_1 \cup A_2$ and get that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

We can rewrite $(A_1 \cup A_2) \cap A_3$ as a union of two sets as $(A_1 \cap A_3) \cup (A_2 \cap A_3)$. Then, apply again the rule for the union of two sets to $A = A_1 \cap A_3,\ B = A_2 \cap A_3$.

$$|(A_1 \cap A_3) \cup (A_2 \cap A_3)| = |A_1 \cap A_3| + |A_2 \cap A_3| - |(A_1 \cap A_3) \cap (A_2 \cap A_3)|$$

Now, $(A_1 \cap A_3) \cap (A_2 \cap A_3) = A_1 \cap A_2 \cap A_3$. Substitute now the values of the equations above in (*) to obtain

$$|A_1 \cup A_2 \cup A_3| = |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3| =$$
$$= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - [|A_1 \cap A_3| + |A_2 \cap A_3| - |(A_1 \cap A_2 \cap A_3|] =$$
$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

(b) *Assume that $A_1 \dots, A_n$ are finite sets. Find an equation for the cardinality of $A_1 \cup \dots \cup A_n$ in terms of the cardinality of each of the sets and their intersections.*

The equation below can be proved by induction as the case of $n = 3$.

$$|A_1 \cup \dots \cup A_n| = \Sigma_{i=1}^n |A|_i - \Sigma_{i \neq j} |A_i \cap A_j| + \Sigma |A_i \cap A_j \cap A_k| + \dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n|$$

(c) *In a survey of 500 household, all of them plan to buy something special for Halloween. A full 387 plan on buying candy, 153 will buy decorations and 147 will buy costumes. If only 64 will buy all three, how many will buy precisely two of the three things?*

Let us formulate the question as follows: Let A_1 be the set of households buying candy, A_2 the set of households buying decorations and A_3 be the set of households buying costumes. We have the following data

$$|A_1 \cup A_2 \cup A_3| = 500,\ |A_1| = 387,\ |A_2| = 153,\ |A_3| = 147, |A_1 \cap A_2 \cap A_3| = 64$$

We are being asked about the cardinality $|(A_1 \cap A_2) \cup (A_1 \cap A_3) \cup |(A_2 \cap A_3) - (A_1 \cap A_2 \cap A_3)|$. Using the equation for the number of elements of a union, we have

$$500 = 387 + 153 + 147 - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + 64$$

This does not allow us to compute the individual values of $|A_1 \cap A_2|, |A_1 \cap A_3|, |A_2 \cap A_3|$ but we can compute their sum $|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| = 251$. Note now that $A_1 \cap A_2 \cap A_3 \subseteq A_1 \cap A_2$ so we can write the sets as disjoint union

$$(A_1 \cap A_2) = [(A_1 \cap A_2) - (A_1 \cap A_2 \cap A_3)] \cup (A_1 \cap A_2 \cap A_3)$$

So, the number of elements of the set in the left is the sum of the number of elements of the two sets on the right and similarly with the other pairs. The set we are interested in as the union of three disjoint sets

$$(A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3) - (A_1 \cap A_2 \cap A_3) =$$

$$= [(A_1 \cap A_2) - (A_1 \cap A_2 \cap A_3)] \cup [(A_1 \cap A_3) - (A_1 \cap A_2 \cap A_3)] \cup [(A_2 \cap A_3) - (A_1 \cap A_2 \cap A_3)]$$

So, its number of elements is the sum of the number of elements of each of the three sets. Therefore,

$$|(A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3) - (A_1 \cap A_2 \cap A_3)| = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - 3|A_1 \cap A_2 \cap A_3|$$

Therefore, there are $251 - 3 \times 64 = 69$ household who are planning to buy precisely two of the types of Halloween items.

(d) *In a survey of 300 democratic leaning voters, 45 were young voters ages 18-25,103 identified as latinex including 27 young latinex, 162 identified as women including 59 latinas and everybody was either young, female or latinex. How many young latinas were in the group?*

Let us formulate the question as follows: Let A_1 be the set of latinex voters, A_2 the set of young voters and A_3 be the set of women. We have the following data

$$|A_1 \cup A_2 \cup A_3| = 300, \ |A_1| = 103, \ |A_2| = 45, \ |A_3| = 162, |A_1 \cap A_2| = 27, |A_2 \cap A_3| = 59$$

We are being asked about the cardinality $|A_1 \cap A_2 \cap A_3|$. We do not have enough data to determine this number as we do not know the number of young women $|A_2 \cap A_3|$. Knowing either of these numbers would allow us to determine the other using the equation for the number of elements of a union

6. Quiz 6, Combinations, Oct 18

Question. (a) *Give an algebraic proof of the expression*

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Hint: $2 = 1 + 1$.

Applying the binomial Theorem to the expression $(1+1)^n = 2^n$, we obtain

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

(b) *Give a combinatorial proof of the expression*

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Hint: 2^n *is the number of subsets of a set with* n *elements.*

We saw when discussing sets and subsets that the number of subsets of a set with n elements is 2^n. This gives us an interpretation for the left hand side. The terms in the right hand side $\binom{n}{k}$ is the number of ways of choosing k elements among n. So, it can be viewed as number of subsets with precisely k elements of a set with n elements. As a subsets of a set with n elements must have a number of elements k for some k with $0 \leq k \leq n$, this gives a combinatorial justification of why the two expressions should agree.

Question 6.1. (a) *You have 10 yellow beads and 10 purple beads all of slightly different shapes. In how many ways can you put them in a row? Give your answer in terms of factorials and explain how you get it.*

You have a total of 20 different beads. Hence, they can be arranged in 20! different way

(b) *You have 10 yellow beads and 10 purple beads all of slightly different shapes. In how many ways can you put them in a row if you do not want two beads of the same color to be adjacent?*

We can arrange the 10 yellow beads in 10! different ways and the purple beads in 10! different ways. Then we form the row by alternating colors without disturbing the order within a color. We can either start with yellow or with purple. This gives us two options. Hence, there are a total of $2(10!)^2$.

(c) *You have 10 yellow beads and 10 purple beads all of slightly different shapes. In how many ways can you make a necklace if you do not want two beads of the same color to be adjacent?*

Once your beads are in a row, you form the necklace by joining the two ends. We can identify each configuration with the ones we obtain by rotating. There are 20 different rotations before you get the identity. You can also flip the neck lace over (2 positions).

Hence the number of different necklaces is

$$\frac{2(10!)^2}{2 \times 20} = 5(9!)^2$$

(d) *In how many different ways can you load 25 identical parcels into 3 trucks if the first truck must carry at least 3 and the third truck can carry at most 6? Explain your reasoning.*

Three of the parcels go into the first truck. This leaves only 22 to distribute. We are looking for non-negative numbers a, b, c such that $a+b+c = 22$.There are $\left(\binom{3}{22}\right) = \binom{22+2}{2}$ choices. If the third truck were to carry 7 or more, only 15 parcels would need to be distributed. There are $\left(\binom{3}{15}\right) = \binom{15+2}{2}$ choices. The number we are looking for is the difference, namely

$$\left(\binom{3}{22}\right) - \left(\binom{3}{15}\right) = \binom{22+2}{2} - \binom{15+2}{2}$$

(e) *Give a combinatorial proof of the equation*

$$\left(\binom{n}{k}\right) = \left(\binom{n-1}{0}\right) + \left(\binom{n-1}{1}\right) + \left(\binom{n-1}{2}\right) + \cdots + \left(\binom{n-1}{k}\right)$$

Hint: think in how many ways you can choose k objects of n different .types if one of the types is special.

For a combinatorial proof, We want to choose k objects among n different types. Let us think of one type of object as different from the others. We can take all k objects of this special type, or take precisely $k-1$ objects of this special type, or take precisely $k-2$ objects of this special type, or ... or precisely one of the special type or none at all. This covers all possibilities and the various possibilities are disjoint.

If we take all k objects of this special type, we take 0 objects of the remaining $n-1$ types. This leads to $\left(\binom{n-1}{0}\right)$ choices.

If we take precisely $k-1$ objects of the special type, we take 1 object of the remaining $n-1$ types. This leads to $\left(\binom{n-1}{1}\right)$ choices.

If we take precisely $k-2$ objects of the special type, we take 2 objects of the remaining $n-1$ types. This leads to $\left(\binom{n-1}{2}\right)$ choices.

...

If we take precisely one object of the special type, we take $k-1$ objects of the remaining $n-1$ types. This leads to $\left(\binom{n-1}{k-1}\right)$ choices.

If we take no objects of the special type, we must take all k from the remaining $n-1$ types. This leads to $\left(\binom{n-1}{k}\right)$ choices.

Adding all these numbers, we get the total number of choices.

(f) *Show algebraically that*

$$\left(\binom{n}{k}\right) = \left(\binom{n-1}{0}\right) + \left(\binom{n-1}{1}\right) + \left(\binom{n-1}{2}\right) + \cdots + \left(\binom{n-1}{k}\right)$$

Hint: translate into regular choose numbers and use that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$.

We can translate this expression into regular choose numbers. We need to show that

$$\binom{n+k-1}{k} = \binom{n-2}{0} + \binom{n-1}{1} + \binom{n}{2} + \cdots + \binom{n+k-2}{k}$$

We will work with the right hand side. We know that $\binom{n-2}{0} = \binom{n-1}{0}$. Then, we use repeatedly that $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$ to obtain

$$\binom{n-2}{0}+\binom{n-1}{1}+\binom{n}{2}+\binom{n+1}{3}+\cdots+\binom{n+k-2}{k} = \binom{n-1}{0}+\binom{n-1}{1}+\binom{n}{2}+\cdots+\binom{n+k-2}{k} =$$

$$= \binom{n}{1}+\binom{n}{2}+\binom{n+1}{3}+\cdots+\binom{n+k-2}{k} = \binom{n+1}{2}+\binom{n+1}{3}+\cdots+\binom{n+k-2}{k} =$$

$$= \binom{n+2}{3}+\binom{n+2}{4}+\cdots+\binom{n+k-2}{k} = \binom{n+k-2}{k-1}+\binom{n+k-2}{k} = \binom{n+k-1}{k}$$

Alternatively, we could use induction on k:

- The smallest case to consider is $k=0$.the equation in this case is $\left(\binom{n}{0}\right) = \left(\binom{n-1}{0}\right)$ and this is correct as both sides are 0.
- Assume the result correct for k and prove it for $k+1$.

  $$\left(\binom{n}{k}\right) = \left(\binom{n-1}{0}\right) + \left(\binom{n-1}{1}\right) + \cdots + \left(\binom{n-1}{k}\right)$$

  Then, translating into regular choose numbers and using the identities we know $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$, $\binom{a}{b} = \binom{a}{a-b}$, we obtain

  $$\left(\binom{n}{k+1}\right) = \binom{n+k}{n-1} = \binom{n+k}{k+1} = \binom{n+k-1}{k} + \binom{n+k-1}{k+1} =$$

  $$= \left(\binom{n}{k}\right) + \left(\binom{n-1}{k+1}\right)$$

  Then, using the expression we are assuming for $\left(\binom{n}{k}\right)$, we obtain the expression we need for $\left(\binom{n-1}{k+1}\right)$.

7. Quiz 7, Equivalence relations, Oct 25.

Question 7.1. (a) *Define what it means for a relation $\sim$ to be an equivalence relation.*

- It is reflexive, that is, every element is related to itself $\forall a \in A,\ s \sim s$.
- It is symmetric $\forall a_1, a_2 \in A, a_1 \sim a_2 \Rightarrow a_2 \sim a_1$
- It is transitive $\forall a_1, a_2, a_3 \in A, a_1 \sim a_2$ and $a_2 \sim a_3 \Rightarrow a_1 \sim a_3$

(b) *Consider the following relation in the set of integers $a \sim b$ if and only if, there exists an integer k such that $a - b = 4k$. Show that $\sim$ is an equivalence relation.*

In order to show that $\sim$ is an equivalence relation, we need to show that it satisfies the three properties reflexive, symmetric and transitive.

- Given $a \in \mathbb{Z}$, $a - a = 0 = 4 \times 0$. Therefore, $\forall a,\ a \sim a$ and therefore, $\sim$ is reflexive.
- Assume aRb. By definition of R, there exists an integer x such that $a - b = 4x$. Then $b - a = -(a - b) = -4x = 4(-x)$. As $x \in \mathbb{Z}$, also $-x \in \mathbb{Z}$. Therefore $\forall a, b \in \mathbb{Z},\ a \sim b \Rightarrow b \sim a$ and $\sim$ is symmetric.
- Assume $a \sim b,\ b \sim c$. By definition of $\sim$, there exist integers x, y such that $a - b = 4x,\ b - c = 4y$. Then $a - c = (a - b) + (b - c) = 4x + 4y = 4(x + y)$. As $x \in \mathbb{Z}, y \in \mathbb{Z}$, also $x + y \in \mathbb{Z}$. Therefore $\forall a, b, c \in \mathbb{Z},\ a \sim b,\ b \sim c \Rightarrow a \sim c$ and $\sim$ is transitive.

(c) *Describe the equivalence classes or cosets of $\mathbb{Z}$ by this equivalence.*

By definition $a \sim b$ means that there exists an integer x such that $a - b = 4x$. Therefore, we can write $a = b + 4x$ for some integer x. Conversely, if a and b satisfy an equation of this form, then they are related. There are as many cosets as posssible remainders of long division by 4. So, there are four cosets. For example, [0] contains all integers divisible by 4, [1] contains al integers of the form $4x + 1, x \in \mathbb{Z}$ and so on.

8. Quiz 8, Equivalence classes, Oct. 27

Question 8.1. (a) *Consider the assignment $f_1 : \mathbb{Z} \to \mathbb{Z}_4$ given by $f_1(z) = [z]_4$. Is this a well defined function? Why or why not?*

The assignment $f_1 : \mathbb{Z} \to \mathbb{Z}_4$ given by $f_1(z) = [z]_4$. This a well defined function as to any integer z we assign a uniquely determined coset $[z]_4 \in \mathbb{Z}_4$

(b) *Consider the assignment $f_2 : \mathbb{Z}_4 \to \mathbb{Z}$ given by $f_2([z]_4) = z$. Is this a well defined function? Why or why not?*

This is not a well defined function as it assigns infinitely many integers to every equivalence class. For instance $[0]_4 = [4]_4 \in \mathbb{Z}_4$ but $0 \neq 4 \in \mathbb{Z}$.

(c) *Consider the assignment $f_3 : \mathbb{Z}_8 \to \mathbb{Z}_4$ given by $f_3([z]_8) = [z]_4$. Is this a well defined function? Why or why not?*

In order to determine if this is a well defined function, we need to see whether changing z without changing $[z]_8$ changes or preserves $[z]_4$.

Assume $[z_1]_8 = [z_2]_8$. By definition of the equivalence relation corresponding to cosets modulo 8, there exists $k \in \mathbb{Z}$ such that $z_1 - z_2 = 8k$. Then, $z_1 - z_2 = 4(2k)$. As $2k$ is an integer, $[z_1]_4 = [z_2]_4$, so f_3 is a well defined function.

(d) *Consider the assignment $f_4 : \mathbb{Z}_4 \to \mathbb{Z}_8$ given by $f_4([z]_4) = [z]_8$. Is this a well defined function? Why or why not?*

Consider the assignment $f_4 : \mathbb{Z}_4 \to \mathbb{Z}_8$ given by $f_3([z]_4) = [z]_8$. This is not well defined function as for instance $[1]_4 = [5]_4$ while $[1]_8 \neq [5]_8$

9. Quiz 9, Partial Orders, Nov.1.

Question 9.1. Let $(X, \preceq)$ be a poset

- *Define what it means for an element $x \in X$ to be minimal.*
  An element $x \in X$ is minimal if there is nothing strictly below it, that is if $y \in X$ satisfies $y \preceq x$, then $x = y$.
- *Define what it means for an element $x \in X$ to be minimum.*
  An $x \in X$ is a minimum if it is below everything else, that is $\forall y \in X,\ x \preceq y$.
- *Define what it means for an element $x \in X$ to be maximal.*
  An $x \in X$ is maximal if there is nothing strictly above it, that is if $y \in X$ satisfies $x \preceq y$, then $x = y$.

Prove or disprove each of the following statements

(a) *If a poset has a minimum element, then it has to be unique.*

Assume a, b are both minimums. By definition of minimum, a being a minimum means that , $\forall y \in X,\ a \preceq y$. As $b \in X$ this gives rise to $a \preceq b$

Similarly, b being a minimum means that , $\forall y \in X,\ b \preceq y$. As $a \in X$ this gives rise to $b \preceq a$.

Then, by the antisymmetric property, $a = b$. Hence, any two potential minimums are identical and hence the minimum is unique.

(b) *If a poset has a minimal element, then it has to be unique.*

If a poset has a minimal element, then it may not be unique.. For example, take the set $\{2, 3, 4, 6\}$ ordered by divisibility. Both $2, 3$ are minimal and they are certainly different.

(c) *If a finite poset has only one minimal element, then it is a minimum.*

Assume that $a \in X$ is the only minimal element and a is not a minimum. As a is not a minimum. there exists some x_0 such that $a \not\preceq x_0$ As a is the unique minimal element, x_0 is not minimal. Hence, there exists , some element x_1 that is strictly smaller than x_0, that is $x_1 \preceq x_0, x_1 \neq x_0$, also $x_1 \neq a$ from our assumptions.. As a is the unique minimal element, x_1 is not minimal. Hence, there exists , some element x_2 that is strictly smaller than x_1, that is $x_2 \preceq x_1, x_2 \neq x_1$. The process can be repeated and in this way we form a strictly decreasing chain. There are no repeats because of the antisymmetric property, therefore the chain is infinite contradicting the assumption that the poset is finite

Note though that an infinite poset may have only one minimal element, and not be a minimum. Take for instance the set of integers ordered by the usual $\leq$ and add to it i which is only related to itself. Then conditions for poset are satisfied, mostly because the only thing to check with i is the reflexive property. Then i is minimal, as there is nothing below it. There are no other minimal elements, as for every $x \in \mathbb{Z}, x - 1 < x$, so x is not minimal. On the other hand, i is not a minimum as it is not even related to the other elements in the set.

10. Quiz 10, Probability, Nov. 3

Question. *There are 4 kinds of donuts available in a store Boston cream, pumpkin spice, glazed and chocolate. Half a dozen donuts are picked at random.*

Important Remark We are assuming that the employees working the morning shift are Math 65 students. They figured out all possible ways of packing 6 donuts and created one stack for each assortment. Customers then pick one such pack at random or sometimes one pack from the section labeled as " at least two chocolate". If customers were to pick individual donuts at random to fill the box, the numerical answers would be different.

(a) *Do you think that the the events "at least 2 are chocolate" and " at most 1 is pumpkin spice," are independent? Do not make any computations yet, just discuss it with your peers.*

If at least two of the donuts are chocolate, this leaves less choice for the other 4, so it seems less likely that you are going to end up with two or more being pumpkin spice, Equivalently, it should be more likely that at most one is pumpkin spice,

(b) *What is the probability that at least 2 are chocolate?*

There are $\left(\binom{4}{6}\right) = \binom{6+3}{3} = \binom{9}{3}$ ways of choosing six donuts.

The way to choose six so that at least 2 are chocolate is the way to choose 4 donuts of 4 kinds, namely $\left(\binom{4}{4}\right) = \binom{4+3}{3} = \binom{7}{3}$.

The probability of choosing at least 2 chocolate in a random choice is the quotient to these two numbers

$$\frac{\binom{7}{3}}{\binom{9}{3}} = \frac{7 \times 6 \times 5}{9 \times 8 \times 7} = \frac{5}{12}$$

(c) *What is the probability when choosing 6 donuts that at most 1 is pumpkin spice?*

The complementary event is that at least 2 are pumpkin spice. The probability of choosing at least 2 pumpkin spice, is the same as the probability of choosing at least 2 chocolate namely $\frac{7\times6\times5}{9\times8\times7} = \frac{5}{12}$. The probability of choosing at most 1 pumpkin spice is then

$$1 - \frac{5}{12} = \frac{7}{12}$$

(d) *What is the probability that at least 2 are chocolate and most 1 is pumpkin spice?*

The probability of choosing at least 2 chocolate is $\frac{5}{12}$

The ways of choosing at least 2 chocolate and at least 2 pumpkin spice is $\left(\binom{4}{2}\right) = \binom{2+3}{3} = \binom{5}{3}$ The probability of choosing at least 2 chocolate and at least 2 pumpkin spice can be computed as the quotient of the ways of choosing at least 2 chocolate and at least 2 pumpkin spice divided by the total number of choices

$$\frac{\binom{5}{3}}{\binom{9}{3}} = \frac{5 \times 4 \times 3}{9 \times 8 \times 7} = \frac{5}{42}$$

The probability that at least 2 are chocolate and most 1 is pumpkin spice is then the difference between the probability of at least 2 chocolate and the probability of at least 2 chocolate and at least 2 chocolate and at least 2 pumpkin spice,

$$\frac{5}{12} - \frac{5}{42} = \frac{35 - 10}{84} = \frac{25}{84}$$

(e) *If at least 2 are chocolate, what is the probability that at most 1 is pumpkin spice?*

This is the conditional probability that at most one is chocolate given that at least two are chocolate We can compute it as the quotient of the probability of the intersection (at least two chocolate AND at most one pumpkin spice) divided by the probability of at least 2 chocolate

$$\frac{\frac{25}{84}}{\frac{5}{12}} = \frac{5}{7}$$

(f) *Are the events "at least 2 are chocolate" and " at most 1 is pumpkin spice" independent?*

They are not. The probability of choosing at most 1 pumpkin spice, is then $\frac{7}{12}$. The probability of choosing at most 1 is pumpkin spice, when two are chocolate is $\frac{5}{7} > \frac{7}{12}$.

Question. *A bag contains 5 coins. Four of them are fair coins that come up heads as often as tails when tossed. The fifth, while it looks and feels like the others comes up heads twice as often as tails.*

(a) *You pick a coin at random. You toss it and it comes up heads. How likely is it that it is the loaded coin.*

As there are five coins, one of which is loaded, the probability that a randomly picked coin is loaded is $P(L) = \frac{1}{5}$ while the probability of not being loaded, that is picking a fair coin is $P(F) = \frac{4}{5}$

For a fair coin, the probability of heads is $P(H|F) = \frac{1}{2}$. For the loaded coin, the probability of heads is $P(H|L) = \frac{2}{3}$.

As a coin is either loaded or fair, $H = (H \cap L) \cup (H \cap F)$ and this is a disjoint union so

$$P(H) = P(H \cap L) + P(H \cap F) = P(H|L)P(L) + P(H|F)P(F) = \frac{2}{3} \times \frac{1}{5} + \frac{1}{2} \times \frac{4}{5} = \frac{16}{30}$$

We are trying to compute $P(L|H)$

$$P(H|L) = \frac{P(H \cap L)}{P(L)},\ P(L|H) = \frac{P(H \cap L)}{P(H)}$$

we obtain

$$P(L|H) = \frac{P(H|L)P(L)}{P(H)} = \frac{\frac{2}{3} \times \frac{1}{5}}{\frac{16}{30}} = \frac{2 \times 30}{3 \times 5 \times 16} = \frac{1}{4}$$

(b) *You pick a coin at random. You toss it and it comes up heads. How likely is it that it is a fair coin.*

This is the complement of the previous option

$$P(F|H) = 1 - P(L|H) = \frac{3}{4}$$

(c) *You pick a coin at random. You toss it and it comes up tails. How likely is it that it its the loaded coin.*

We are now trying to compute $P(L|T)$

As Tails is the complement of heads, $P(T) = 1 - P(H) = \frac{14}{30}$. From

$$P(T|L) = \frac{P(T \cap L)}{P(L)},\ P(L|T) = \frac{P(T \cap L)}{P(T)}$$

we obtain

$$P(L|T) = \frac{P(T|L)P(L)}{P(T)} = \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{14}{30}} = \frac{30}{3 \times 5 \times 14} = \frac{1}{7}$$

(d) *You pick a coin at random from the given 5. You toss it 10 times. What is the expected value of the number of heads?*

If we were to toss one coin once, the expected value of the number of heads for one toss would be

$$E(X_H) = 1P(H) + 0P(T) = \frac{16}{30}$$

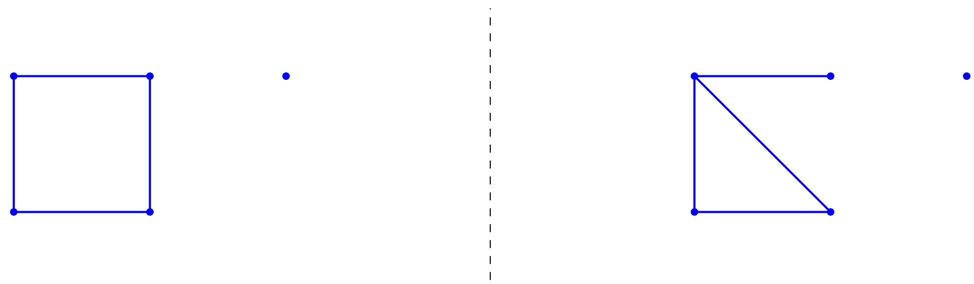
As the expected value is a linear function, the expected number of heads when tossing a coin 10 times is

$$E(10X_H) = 10E(X_H) = 10\frac{16}{30} = \frac{16}{3} = 5.333$$

11. Quiz 11, Graphs, Nov.15

Question. *For the following pairs of graphs below, determine if they are isomorphic. If they are isomorphic, give as many isomorphisms as possible between the two. If they are not isomorphic, give a reason why this is not possible.*

(a) The two graphs below, both have 4 vertices and 4 edges but they are not isomorphic as one has a vertex of degree 3 and the other doesn't.



(b) In the first graph, vertices b, e have degree 3, vertices a, f have degree 2 and vertices c, d have degree 1.

In the second graph, vertices B, F have degree 3, vertices A, E have degree 2 and vertices C, D have degree 1.

An isomorphism between the two graphs should send the pair of vertices $\{c, d\}$ to the pair of vertices $\{C, D\}$,as these are the only. ones with the same degree. But the vertices C, D are adjacent. The edge that connects them should be the image of an edge conectinng c, d. But no such edge exists. So no isomorphism is possible.

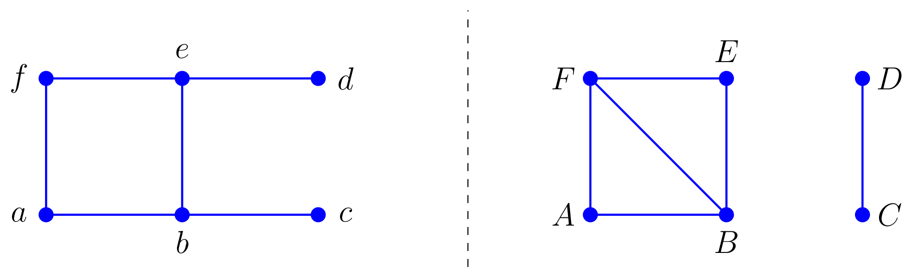


Figure 8. Two examples of graphs with with the same number of vertices of the same degree but not isomorphic.

(c) The two graphs below are isomorphic: In the first graph, vertices b, d have degree 4, vertices a, e have degree 3 and vertex c has degree 2.

In the second graph, vertices A, C have degree 4, vertices B, E have degree 3 and vertex C has degree 1.

An isomorphism between the two graphs should send the pair of vertices $\{b, d\}$ to the pair of vertices $\{A, C\}$, as these are the only vertices of degree 4. There are two ways in which this can be done. It should send the pair of vertices $\{a, e\}$ to the pair of vertices $\{B, E\}$, as these are the only vertices of degree 3.There are two ways in which this can be done. Moreover, it should send c to D, as these are the only vertices of degree 2..

As the graph is simple, the images of the vertices determine the images of the edges . It follows that there are at most 4 options for isomorphisms between the two graphs. We can check that all 4 options actually work, as they preserve the incidence of edges

$$\begin{array}{lll}
b & \rightarrow & A \\
d & \rightarrow & C \\
e & \rightarrow & E \\
a & \rightarrow & B \\
c & \rightarrow & D
\end{array} \qquad
\begin{array}{lll}
b & \rightarrow & C \\
d & \rightarrow & A \\
e & \rightarrow & E \\
a & \rightarrow & B \\
c & \rightarrow & D
\end{array} \qquad
\begin{array}{lll}
b & \rightarrow & A \\
d & \rightarrow & C \\
e & \rightarrow & B \\
a & \rightarrow & E \\
c & \rightarrow & D
\end{array} \qquad
\begin{array}{lll}
b & \rightarrow & C \\
d & \rightarrow & A \\
e & \rightarrow & B \\
a & \rightarrow & E \\
c & \rightarrow & D
\end{array}$$

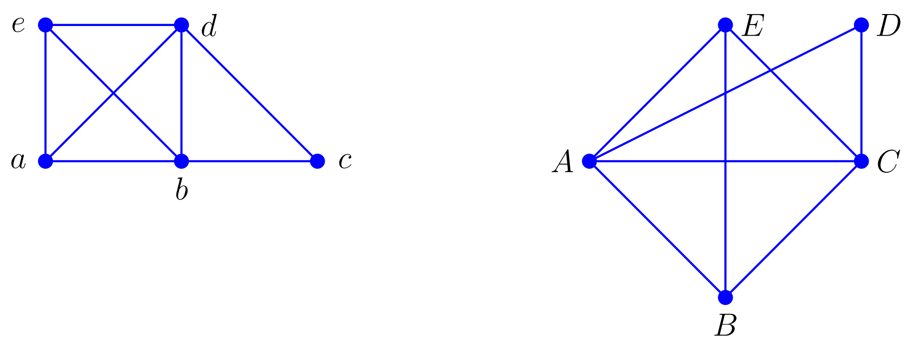


FIGURE 9. Two isomorphic graphs.

12. Quiz 12, Self Complementary graphs, Nov. 17.

Question 12.1. *The complement of a simple graph (V, E, f) is a graph with the same set of vertices V and with set of edges al the edges that do not appear in E. A graph is self complementary if it is isomorphic to its complement. Denote by n the number of vertices of the graph.*

(a) *Show that if a graph is self complementary, either n or $n-1$ are divisible by 4.* Hint: What is the number of edges of K_n.

The graph K_n has $\binom{n}{2}$ edges. Then, for any simple graph G

$$|E(G)| + |E(\bar{G})| = |E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$$

If a graph is self complementary, it is isomorphic to its complement. Hence, the number of edges of the graph and the complement is the same$|E(G)| = |E(\bar{G})|$. It follows that

$$|E(G)| = |E(\bar{G})| = \frac{1}{2}|E(K_n)| = \frac{n(n-1)}{4}$$

If n is even, $n-1$ is odd and vice versa. So, 4 must divide one of the two

(b) *Assume that a graph, is self complementary and write $d_1 \le d_2 \le \cdots \le d_{n-1} \le d_n$ for the degrees of its vertices. Show that $d_i + d_{n+1-i} = n-1$*

For any simple graph G, if a vertex v has degree d in G, then it has degree $n-1-d$ in $\bar{G}$. Therefore, if the degrees of the vertices of G are $d_1 \le d_2 \le \cdots \le d_{n-1} \le d_n$, the degrees of the vertices of $\bar{G}$ are

$$n-1-d_n \le n-1-d_{n-1} \le \cdots \le n-1-d_2 \le n-1-d_1.$$

If the graph is self complementary, it is isomorphic to its complement. In particular, the degrees of the vertices of G and $\bar{G}$ are the same. It follows that

$$d_1 = n-1-d_n,\ d_2 = n-1-d_{n-1} \dots d_{n-1} = n-1-d_2,\ d_n = n-1-d_1.$$

That is $d_i + d_{n+1-i} = n-1$

(c) *Assume that a graph, is self complementary and write $d_1 \le d_2 \le \cdots \le d_{n-1} \le d_n$ for the degrees of its vertices. Show that $d_n = d_{n-1}$.*

Write $v_1, v_2 \dots, v_{n-1}, v_n$ for the vertices of respective degree $d_1 \le d_2 \le \cdots \le d_{n-1} \le d_n$. Assume that $d_{n-1} < d_n$. So, there is only one vertex v_n in G of degree d_n and only one vertex v_1 in $\bar{G}$ of degree d_n.

Moreover, as $\bar{d}_i = n-1-d_{n+1-i}$, then also $d_1 = n-1-d_n < d_2 = n-1-d_n-1$. So, there is only one vertex v_1 in G of degree d_1 and only one vertex v_n in $\bar{G}$ of degree d_1.

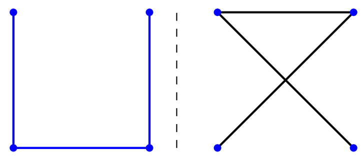
This implies that in the isomorphism between G and $\bar{G}$, the image of v_1 is v_n and the image of v_n is v_1. If there is one edge between v_1 and v_n in G, there is no edge between v_1 and v_n in $\bar{G}$. Therefore, no such isomorphism can exist as the edge would have nowhere to go if it is in G or nothing to come from if it is in $\bar{G}$

(d) *Find (up to isomorphism) all self complementary graphs with 4 vertices.*

We know that G must have $\frac{1}{2}\binom{4}{2} = 3$ edges. The degrees of its vertices must be

$$0 \le d_1 = d_2 \le d_3 = d_4 \le 3,\ d_1 + d_4 = 3, d_2 + d_3 = 3$$

There are in principle two options for degrees, with these conditions $0, 0, 3, 3$ and $1, 1, 2, 2$. If a simple graph with 4 vertices has one vertex of degree 0, then there are no edges coming out of this vertex. Therefore, the maximum degree of any other vertex is 2. This eliminates the option $0, 0, 3, 3$ and leaves only $1, 1, 2, 2$. As the vertices of degree 2need to be linked to two other vertices while the vertices of degree one can only be linked to one, the two vertices of degree two must b linked and also linked to one vertex of degree one. The only option up to isomorphism corresponds to the graph



(e) *Find (up to isomorphism) all self complementary graphs with 5 vertices.*

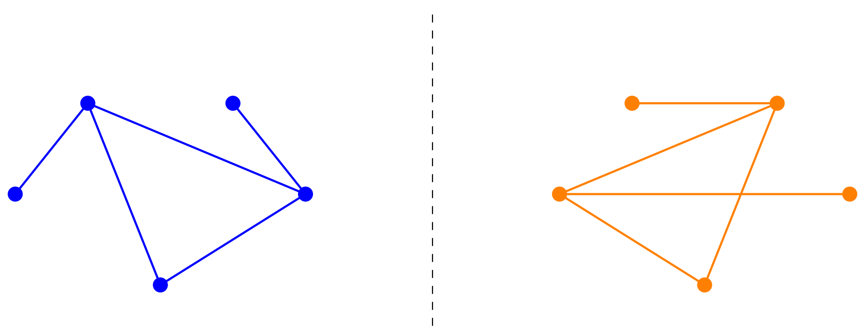
We know that G must have $\frac{1}{2}\binom{5}{2} = 5$ edges. The degrees of its vertices must be

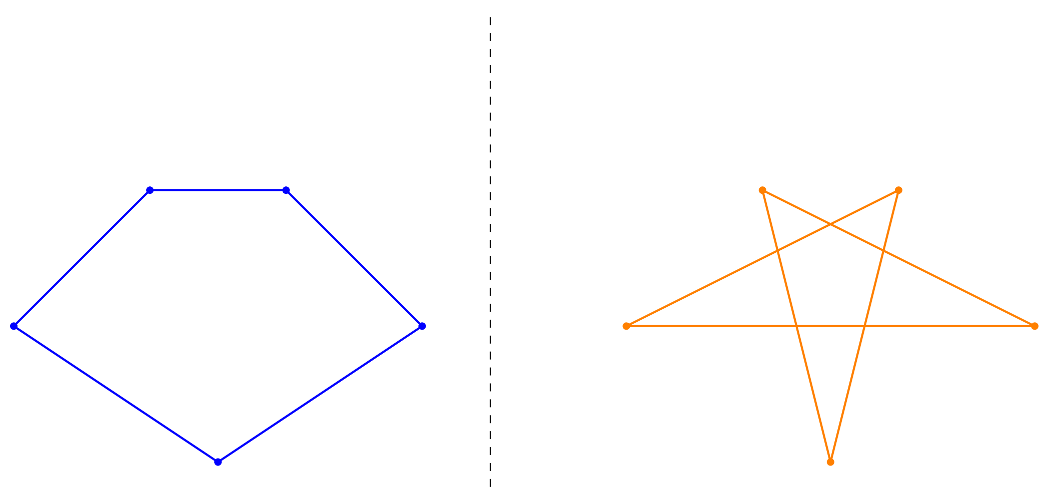
$$0 \leq d_1 = d_2 \leq d_3 \leq d_4 = d_5 \leq 4,\ d_1 + d_5 = 4, d_2 + d_4 = 4, 2d_3 = 4$$

Therefore, the only option for degrees are

$$0, 0, 2, 4, 4\ ;\quad 1, 1, 2, 3, 3;\quad 2, 2, 2, 2, 2$$

There is no simple with five vertices of respective degrees $0, 0, 2, 4, 4$: if there is one vertex of degree 4, there must be an edge from that vertex to every other vertex. Hence, there are no vertices of degree zero. The options for degrees are $1, 1, 2, 3, 3$ and $2, 2, 2, 2, 2$ pictured below



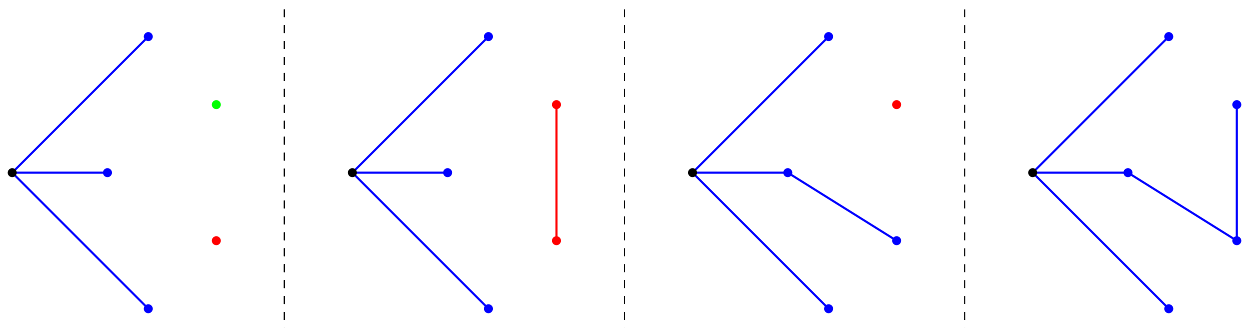


Question. *Let G be a simple graph with n vertices and at least one vertex of degree $n-3$.*

(1) *What is the largest number of components the graph can have, what is the smallest*

I will assume that $n-3 \geq 1$, that is $n \geq 4$. If one vertex has degree $n-3$, then it is connected to $n-3$ other vertices. Therefore these $n-2$ vertices are in a single component. We have then the following possibilities

- The remaining two vertices may have degree 0 and not be connected to anything else. In this case, there would be 3 components.
- The remaining two vertices are connected to each other but not to the rest. In this case, there would be 2 components.
- One of the remaining two vertices is connected to the large piece, the other has degree 0. In this case, there would be 2 components.
- Both remaining vertices are connected to the large piece. In this case, there would be 1 component.



(2) *What is the smallest number of edges you need for any of the possible number of components*

We answer the question in each of the above situations. Note first that to connect the first $n-2$ vertices we need only $n-3$ edges joining the vertex of degree $n-3$ to the remaining ones.

- The remaining two vertices may have degree 0. We only need $n-3$ edges
- The remaining two vertices are connected to each other but not to the rest. We need $n-2$ edges.
- One of the remaining two vertices is connected to the large piece, the other has degree 0.Again, we need $n-2$ edges.
- Both remaining vertices are connected to the large piece. In this case, we need at least n-1 edges.

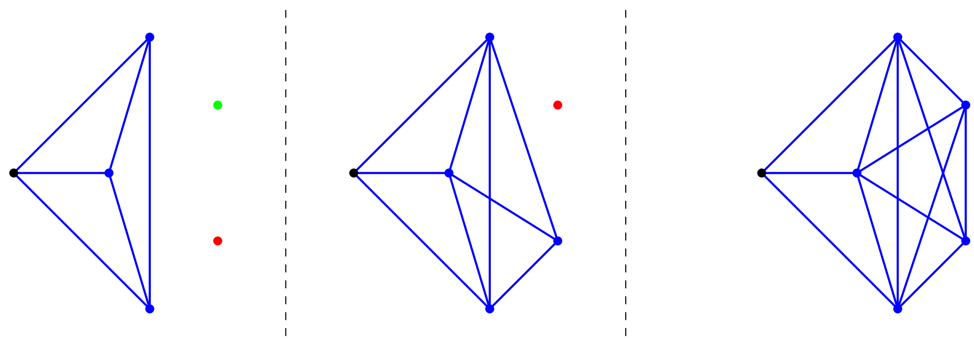
(3) *What is the largest number of edges you may have for any of the possible number of components (remember that one of the vertices has degree $n-3$)*

We answer the question for 3, 2 and 1 component.

- For 3 components, we can have two vertices of degree 0, a K_{n-3} and the vertex of degree $n-3$ linked to each of the vertices of the K_{n-3}. This gives $\binom{n-3}{2}+n-3$ edges.
- For two components, we can get the largest number of edges if we have one vertex on its own and as many vertices as possible in the one component We take a K_{n-2}, the vertex of degree $n-3$ linked to $n-3$ of the vertices of the K_{n-3} and one

vertex on its own, not linked to anything.. This gives $\binom{n-2}{2} + n - 3 = \binom{n-1}{2} - 1$ edges.

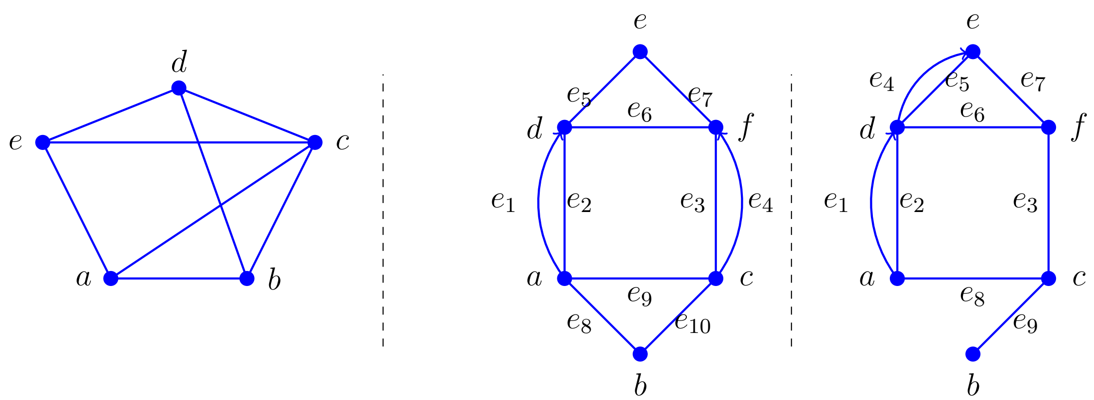
- For a single component, we can maximize the edges by taking a K_{n-1}, the vertex of degree $n-3$ linked to $n-3$ of the vertices of the K_{n-1}. This gives $\binom{n-1}{2} + n - 3 = \binom{n}{2} - 2$ edges.



Question. (a) *Show that every graph has an even number of vertices of odd degree.*

The sum of the degrees of all the vertices of a graph is twice the number of edges and therefore an even number. This requires that the sum of the degrees of the vertices of odd degree to be even. This can only happen if there is an even number of them.

(b) *Find the minimum number of paths that together contain all edges of the given graph, each edge appearing exactly once. Describe the paths by listing vertices and edges.*



The first graph has 4 vertices of odd degree. We then know that it has no Euler path or circuit. This means that we will need at least two paths to cover all edges. As the graph is simple, in order to give a path, it suffices to give the vertices. We have the following two paths containing between them all edges

$$P_1 = (a, e_{a,b}, b, e_{bc}, c, e_{a,c}, a, e_{a,e}, e, e_{ec}, c, e_{c,d}, d, e_{d,e}, e), P_2 = (b, e_{b,d}, d)$$

The second graph is eulerian. An Euler circuit is given as

$$P = (a, e_1, d, e_5, e, e_7, f, e_4, c, e_{10}, b, e_8, a, e_2, d, e_6, f, e_3, c, e_9, a)$$

The third graph has all six vertices of odd degree. We know that it has no Euler path or circuit. if it could be covered by two of them, it would have at most 4 vertices of odd degree. So, we will need at least 3 paths

$$P_1 = (a, e_1, d, e_4, e, e_7, f, e_3, c, e_8, a, e_2, d, e_5, e),), \ P_2 = (d, e_6, f), \ P_3 = (c, e_9, b)$$

(c) *Let G be a connected, non eulerian graph. Show that the smallest number of trails that together traverse each edge of G exactly once is half the number of vertices of odd degree*

Assume that we can cover all edges of G exactly once with the paths

$$P_1 = (v_0^1, e_1^1, \dots, e_{n_1}^1, v_{n_1}^1), \dots, P_k = (v_0^k, e_1^k, \dots, e_{n_1}^k, v_{n_k}^k)$$

and that no smaller number of paths covers all edges of G. We claim then that $v_0^1, v_{n_1}^1, \dots, v_0^k, v_{n_k}^k$ are all different. We will prove this by contradiction. Assume this is not the case. Then one of the following happens:

- $v_0^i = v_{n_i}^i$,that is the i^{th}path is closed. As the graph is connected, one of the vertices in the path P_i must appear in another of the paths. Then, as in the proof of Euler's Theorem, we can insert the circuit P_i in the middle of another circuit to form a shorter list of paths that still covers all edges of G.

- $v_0^j = v_{n_i}^i$,then the path P_i followed by the path P_j form a single path. Hence, there is a shorter list of paths that still covers all edges of G.
- $v_0^j = v_0^i$, or $v_{nn_j}^j = v_{n_i}^i$. As the reversal of a path is still a path, this is reduced to the previous case.

Therefore, for any minimal list of paths covering all edges, the endpoints of the paths are all different. The edges of a path contribute two to the degree of any intermediate vertex. but only one to the degree of the end points. Hence, if the list above gives a minimal set of paths covering the graph, $v_0^1, v_{n_1}^1, \ldots, v_0^k, v_{n_k}^k$ are the vertices of odd degree and there are twice as many as many as paths.

Alternative proof: Given a graph with odd degree vertices v_1, v_1', v_k, v_k' paired arbitrarily, add edges $e_1, \ldots, e_k$ each joining v_i, v_i'. The new graph has all vertices of even degree. Hence, it is Eulerian. Write an Euler circuit for the new graph. This circuit contains the edges $e_1, \ldots, e_k$ exactly once. Removing the edges $e_1, \ldots, e_k$, we break up the path in $k+1$ simple paths with the first vertex of the first path equal to the last vertex of the last path. Then, the first and last path can be joined together, and we obtain k paths beginning and ending at vertices of odd degrees.

Conversely, given paths with no repeated edges, if we assume that this is the minimum number, the endpoints are distinct, otherwise we could glue them and get fewer of them, We can now add edges from the end of one path to the beginning of the next to form an Eulerian circuit. In the new graph, every vertex has even degree, otherwise the Eulerian circuit would not exist. The new edges only add one degree to what used to be the endpoints of the paths. Hence, the end points of the paths were the vertices of old degree.

13. Quiz 13, Trees, Dec. 1.

Question 13.1. (a) *If T is a tree with $n \geq 2$ vertices and $d_1 \leq d_2 \leq \cdots \leq d_n$ are the degrees of the vertices, show that $d_1 = d_2 = 1$ and $\sum_{i=1}^n d_i = 2n - 2$.*

We know that a tree with n vertices has $n-1$ edges and the sum of the degrees is twice the number of edges. Therefore, $\sum_{i=1}^n d_i = 2n - 2$. A tree is connected by definition, so $d_i > 0$. We also showed that a tree has at least two leaves, hence $d_1 = d_2 = 1$. This also follows from the equation $\sum_{i=1}^n d_i = 2n - 2$: if $d_2 \geq 2$, then

$$\sum_{i=1}^{n} d_i \geq 2(n-1) + d_1 \geq 2(n-1) + 1 = 2n - 1 > 2n - 2$$

contradicting the equation we proved.

(b) *Find all possible trees (up to isomorphism) with 2, 3, 4 and 5 vertices. Explain why there aren't any more.*

For two vertices, the only options for degrees are $1, 1$. The graph has two vertices and a single edge between the two.

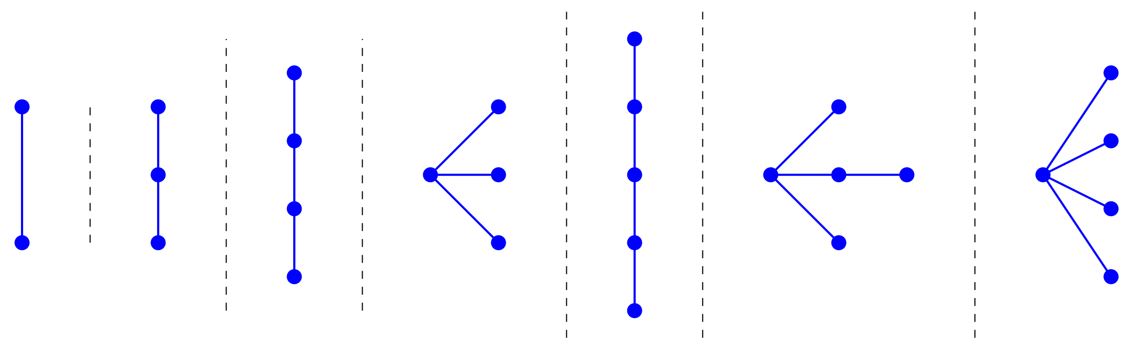


Figure 10. Trees with two, three, four and five vertices

Note that on a connected graph with more than two vertices, vertices of degree one cannot be attached to each other, otherwise, the graph would not be connected.

For three vertices, $d_1 = d_2 = 1$, hence $d_3 = 2 \times 3 - 2 - d_1 - d_2 = 2$. The only option for degrees is $1, 1, 2$. Using that the vertices of degree one cannot be attached to each other, the vertex of degree 2 must be attached to the two vertices of degree one with an edge each.

For four vertices, $d_1 = d_2 = 1$, hence $d_3 + d_4 = 2 \times 4 - 2 - d_1 - d_2 = 4$. There are two options for degrees $1, 1, 1, 3$ and $1, 1, 2, 2$. In the first case, the vertex of degree 3 is attached to each of the vertices of degree one. In the second case, using that the vertices of degree one cannot be attached to each other, the vertices of degree 2 are attached to each other and each is attached to one of the vertices of degree one.

For five vertices, $d_1 = d_2 = 1$, hence $d_3 + d_4 + d_5 = 2 \times 5 - 2 - d_1 - d_2 = 6$. There are three options for degrees

$$1, 1, 1, 1, 4; \quad 1, 1, 1, 2, 3; \quad 1, 1, 2, 2, 2.$$

In the first case, the vertex of degree 4 is attached to each of the vertices of degree one. In the second case, using that the vertices of degree one cannot be attached to each other, the vertices of degrees 2 and 3 are attached to each other and each is attached to one and two of the vertices of degree one respectively. In the third case, the vertices of degrees one are attached to two vertices of degree two and each of these is attached to the third vertex of degree two.

An alternative proof of why these are all is using that any tree with $n+1$ vertices can be obtained from a tree with n vertices by adding a leaf at some vertex.

In a tree with 2 vertices, there is an isomorphism that exchanges the vertices. Therefore, to get to a tree with 3 vertices there is only one way of attaching a leaf.

In a tree with 3 vertices, there is an isomorphism that exchanges the two vertices. of degree one. Therefore, to get to a tree with 4 vertices there are two ways of attaching a leaf, either to one vertex of degree one or to the vertex of degree two.

In a tree with 4 vertices that is a chain, there is an isomorphism that exchanges the two vertices. of degree one and the two vertices of degree two. Therefore, to get to a tree with 5 vertices from it there are two ways of attaching a leaf, either to one vertex of degree one or to one vertex of degree two. In a tree with 4 vertices that is not a chain, there are isomorphism that exchanges any two vertices. of degree one. Therefore, to get to a tree with 5 vertices from it there are two ways of attaching a leaf, either to one vertex of degree one or to the vertex of degree three. Attaching a leaf to one of the vertices of degree one gives the same tree that attaching a leaf to one of the middle vertices in the chain. So, in total, there are three non-isomorphic graphs with 5 vertices.

(c) *Assume that $d_1, d_2, \dots, d_n$. are integers, $d_i \geq 1$ and $\sum_{i=1}^n d_i = 2n-2$. Show that there exists a tree whose vertices have these degrees.*

We prove the result by induction on n.

If $n=1$, then $d_1 = 2\times 1 - 2 = 0 < 1$. So, there is no choice of integers satisfying the conditions and therefore nothing to prove. For $n=2$, $d_1 + d_2 = 2\times 2 - 2 = 2, d_1 \geq 1, d_2 \geq 1$ implies $d_1 = d_2 = 1$. Two vertices and an edge between the to satisfies the condition.

One single vertex with no edges is a tree and satisfies the condition. Assume that for every collection of integers $d_1, d_2, \dots, d_n, d_i \geq 1, n \geq 1$ satisfying $\sum_{i=1}^n d_i = 2n-2$ there exists a tree whose vertices have these degrees. Let now $d'_1, d'_2, \dots, d'_{n+1}$ be integers with $d'_i \geq 1$ satisfying $\sum_{i=1}^{n+1} d'_i = 2(n+1)-2$. Write these numbers in increasing order $1 \leq d'_1 \leq d'_2 \leq \cdots \leq d'_{n+1}$. We claim that $d'_1 = 1$, otherwise , the sum would be at least $2n$. Also, $d'_{n+1} > 1$ because if all $d'_i = 1$, , the sum would be n. Then $d'_2, d'_3, \dots, d'_n, d'_{n+1} - 1$ are integers greater than or equal to 1 with

$$d'_2, +d'_3 + \cdots + d'_n + d'_{n+1} - 1 = d'_1 + d'_2 + \cdots + d'_{n+1} - 2 = 2(n+1) - 2 - 2 = 2n$$

By the induction hypothesis, there exists a tree T' whose with n vertices whose degrees are $d'_2, d'_3, \dots, d'_n, d'_{n+1} - 1$. Now let T be the tree obtained by joining a single leaf to the last vertex. This adds one to the vertex degree of the last vertex (upgrading it from $d'_{n+1} - 1$ to d'_{n+1}), and creates a new vertex of degree $1 = d'_1$. Thus, the vertex degrees of T are exactly $d'_1, d'_2, \dots, d'_n, d'_{n+1}$. This completes the induction step.

(d) *Give an example of a collection of integers $d_1, d_2, \ldots, d_n, d_i \geq 1$ and $\sum_{i=1}^{n} d_i = 2n - 2$ such that there are two non-isomorphic trees whose vertices have these degrees.*

Take $n = 6$ and degrees $1, 1, 1, 2, 2, 3$ and the two graphs shown below.. The two graphs are non-isomorphic as for instance in one of the graphs the vertices of degree two are non-adjacent while in the other they are

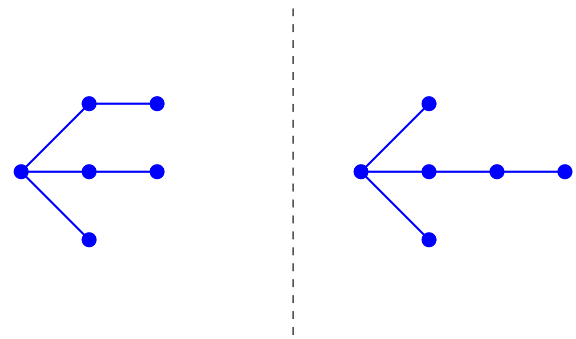


FIGURE 11. Two non-isomorphic trees with 6 vertices of degrees 1,1,1,2,2,3

Question. (a) *Sketch a simple planar graph with more than 3 vertices and such that* $|E| = 3|V| - 6$. *Sketch a simple planar graph with no triangles more than 4 vertices and such that* $|E| = 2|V| - 4$.

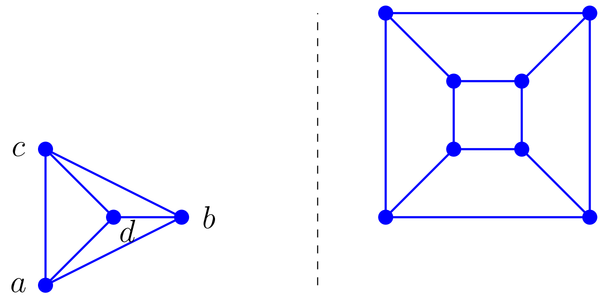


FIGURE 12. The left graph satisfies $|E| = 3|V| - 6$. The right graph has no triangles and satisfies $|E| = 2|V| - 4$.

(b) *Show that the graph* $K_{3,3}$ *can be immersed in the surface of a donut.*

The surface of a donut can be obtained by taking a rectangular flat piece of paper, or better a slightly stretchy material and gluing parallel edges. We show below how to sketch $K_{3,3}$ on the rectangle when we identify the edges to form a torus:

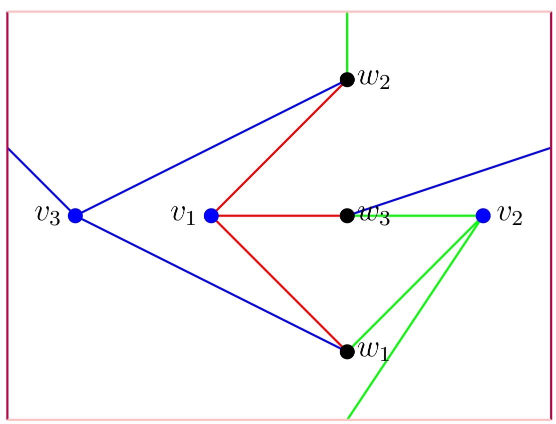


FIGURE 13. The graph $K_{3,3}$ in the torus.

14. Quiz 14, Rational and Real numbers, Dec 8

Question 14.1. (a) *Show that every rational number has a decimal expression that either terminates or repeats.* Hint: A rational number is the quotient of two integers, use long division. What are the possible remainders?

A rational number is of the form $\frac{a}{b}$ where a, b are integers. We can obtain the decimal expression by long division. The successive remainders should be less than b, so there is a finite number of potential remainders. This means that at some point, they should start repeating (possibly with zeroes). Therefore, the decimal expression of a rational number either terminates or repeats.

(b) *Show that $x = 3.746746746\ldots$ is a rational number, that is the quotient of two integers.* Hint Take look first at $y = x - 3$. Compute $1000y - y$

With $y = x - 3 = 0.746746746\ldots$, we get $999y = 1000y - y = 746$. Therefore $y = \frac{746}{999}$ which is rational. Then,

$$x = 3 + y = 3 + \frac{746}{999} = \frac{3 \times 999 + 746}{999}$$

(c) *Show that a decimal expression that either terminates or repeats is a rational number.* $x = a_n a_{n-1} \ldots a_1 a_0 \cdot b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m \ldots$

Assume given a decimal expression that either terminates or repeats. Let us see it comes from the quotient of two integers, and therefore corresponds to a rational number. If the expression terminates, we write a_i for the n^{th} digit of the integral part and b_i for the n^{th} digit of the decimal expression. Then, the number is of the form

$$a_n a_{n-1} \ldots a_1 a_0 \cdot b_1 b_2 \ldots b_m.$$

We can write this number as the quotient of two integers in the form

$$a_n a_{n-1} \ldots a_1 a_0 \cdot b_1 b_2 \ldots b_m = \frac{a_n a_{n-1} \ldots a_1 a_0 b_1 b_2 \ldots b_m}{10^m}$$

If the expression repeats, with similar notations as in the previous case, the number is of the form $x = a_n a_{n-1} \ldots a_1 a_0 \cdot b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m \ldots$. Then,

$$y = x - a_n a_{n-1} \ldots a_1 a_0 = \cdot b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m \ldots.$$

Notice now that

$$(10^m - 1)y = 10^m y - y = b_1 b_2 \ldots b_m$$

Therefore y is the quotient of two integers as we can write it in the form

$$y = \frac{b_1 b_2 \ldots b_m}{10^m - 1}$$

Then $x = a_n a_{n-1} \ldots a_1 a_0 + y$ is the sum of an integer and a rational number and therefore also rational

$$x = \frac{b_1 b_2 \ldots b_m + (10^m - 1) a_n a_{n-1} \ldots a_1 a_0}{10^m - 1}$$

(d) *Show that every rational number is the limit of a sequence of decimal numbers that terminate*

If the expression terminates, the rational number is the limit of the constant sequence where all terms are that rational number. Otherwise, $x = a_n a_{n-1} \ldots a_1 a_0 \cdot b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m \ldots$. Define the sequence (a_n) with a_n being the rational number that agrees with x on the first n digits after the decimal point and is 0 afterwards. Then, $x - a_n$ has zero integral part and the first n digits equal to zero. Therefore, $|x - a_n| < \frac{1}{10^n}$. Then $|a_n - x| < \epsilon$ if $\frac{1}{10^n} < \epsilon$ or equivalently $n \geq \frac{-\ln \epsilon}{\ln 10}$.

15. Quiz 15, Cauchy Sequences, Dec 13

Question 15.1. (a) *Write the definition of Cauchy sequence*

Choose $\epsilon \in \mathbb{Q}^+$

$$\exists m \in \mathbb{N}, \forall n_1, n_2 \geq m, |a_{n_1} - a_{n_2}| < \epsilon$$

(b) *Show that $a_n = \frac{(-1)^n}{n}$ is a Cauchy sequence.*

This sequence has limit 0 and every convergent sequence is a Cauchy sequence.

Alternatively, choose $\epsilon \in \mathbb{Q}^+$ and take $m \geq \frac{2}{\epsilon}$. Then, for $n_1, n_2 \geq m$

$$|a_{n_1} - a_{n_2}| \leq |\frac{n_2 + n_1}{n_1 n_2}| \leq \frac{2\max\{n_1, n_2\}}{n_1 n_2} \leq \frac{2}{\min\{n_1, n_2\}} \leq \epsilon$$

(c) *Show that $a_n = \frac{(-1)^n(n+3)}{n+2}$ is not a Cauchy sequence.*

The negation of the condition of Cauchy sequence is obtained by negating the statement (a). Recall that the negation of a quantifier such as "for all $x, P(x)$ is "there exists some x not $P(x)$". The negation of "exists $x, P(x)$ is "for all x not $P(x)$". Therefore, the negation of (a_n) being a Cauchy sequence is

$$\exists \epsilon_0 > 0,\ \forall m \in \mathbb{N}\ \exists n_1, n_2 \geq m,\ |a_{n_1} - a_{n_2}| > \epsilon$$

Take $\epsilon_0 = 2$. Choose any $m \in \mathbb{N}$ and choose $n_1 > m$ such that n_1. even, $n_2 = n_1 + 1$ For n_1 even, $a_{n_1} = \frac{(n_1+3)}{n_1+2} > \frac{(n_1+2)}{n_1+2} = 1$. Now $n_2 = n_1 + 1$ is odd and $a_{n_2} = -\frac{(n_2+3)}{n_2+2} < -1$. Hence, if n_1 is even , $n_2 = n_1 + 1$, then $|a_{n_1} - a_{n_1+1}| = a_{n_1} - a_{n_1+1} > 1 - (-1) = 2$ contradicting the condition for Cauchy sequence.

(d) *Write the condition* $\lim_{n\to\infty}(a_n) = 0$.

The condition $\lim_{n\to\infty}(a_n) = 0$ means that

$$\forall \epsilon > 0,\ \exists m \in \mathbb{N} \text{ such that } \forall n \geq m,\ |a_n - 0| = |a_n| < \epsilon$$

(e) *Let (a_n) be a Cauchy sequence. Write the condition that $([a_n)] \neq 0$. Your sentence should have a "greater than" sign (>) at the end.*

From the definition of real numbers $[(a_n)] \neq 0$ means that "It is not true that $\lim_{n\to\infty}(a_n) = 0$". So, we need to write the negation of the statement in (d). Recall that the negation of a quantifier such as "for all $x, P(x)$ is "there exists some x not $P(x)$". The negation of "exists $x, P(x)$ is "for all x not $P(x)$". Therefore, the negation of the limit of (a_n) being 0 becomes

$$\exists \epsilon_0 > 0,\ \forall m \in \mathbb{N}\ \exists n \geq m,\ |a_n| > \epsilon$$

(f) *Show that if $[(a_n)] \neq 0$, then there exists some m such that either for all $n \geq m$ $a_n > 0$ or for all $n \geq m$ $a_n < 0$.*

From (e), there exists an ϵ_0 such that for all m there exists $n_1 > m$ such that $|a_{n_1}| > \epsilon_0$.

Let us write the condition of Cauchy sequence for $\frac{\epsilon_0}{2}$.

$$\exists m_1 \in \mathbb{N}, \forall n_1, n_2 \geq m, |a_{n_1} - a_{n_2}| < \frac{\epsilon_0}{2}$$

This condition can be written as

$$\forall n_2 > m_1, -\frac{\epsilon_0}{2} < a_{n_1} - a_{n_2} < \frac{\epsilon_0}{2} \iff a_{n_1} - \frac{\epsilon_0}{2} < a_{n_2} < a_{n_1} + \frac{\epsilon_0}{2}$$

If $a_{n_1} > 0$, the condition $|a_{n_1}| > \epsilon_0$ translates to $a_{n_1} > \epsilon_0$. Then,

$$a_{n_2} > a_{n_1} - \frac{\epsilon_0}{2} > \epsilon_0 - \frac{\epsilon_0}{2} = \frac{\epsilon_0}{2} > 0.$$

If $a_{n_1} < 0$, then $a_{n_1} < -\epsilon_0$. Then,

$$a_{n_2} < a_{n_1} + \frac{\epsilon_0}{2} < -\epsilon_0 + \frac{\epsilon_0}{2} = -\frac{\epsilon_0}{2} < 0.$$

(g) *How do you reconcile your examples in (b) and (c) with your statement in (f)*

The example in (b) is a sequence with limit 0, so the corresponding real number is 0. This case is excluded in the statement of (f).

The example in (c) is not a Cauchy sequence, so there is nothing to reconcile it with .