Bruce M. Boghosian

Uniform distribution
Definition
Likelihood and maximum likelihood
Estimators

The normal distribution
Definition
Likelihood and maximum likelihood
Estimators

Summary

Maximum Likelihood Estimation:

The Uniform Distribution and the Normal Distribution

Bruce M. Boghosian

Department of Mathematics

Tufts University

Outline

Bruce M. Boghosian

1 Uniform distribution
- Definition
- Likelihood and maximum likelihood
- Estimators

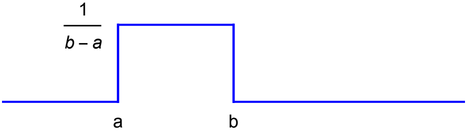
2 The normal distribution
- Definition
- Likelihood and maximum likelihood
- Estimators

3 Summary

The uniform distribution

- $X \in \mathbb{R}$ is a *continuous random variable*
- $X \in \mathbb{R}$ has the *uniform probability density function*,

$$\boxed{f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}}$$



- Given experimental results $\vec{x} = \{x_1, x_2, \ldots, x_n\}$, can we estimate a and b?

Moments of the uniform distribution

- Normalization:

$$\int_{\mathbb{R}} dx\ f_X(x) = \int_a^b dx\ \frac{1}{b-a} = \frac{b-a}{b-a} = 1$$

- Mean:

$$\int_{\mathbb{R}} dx\ f_X(x)x = \int_a^b dx\ \frac{x}{b-a} = \frac{b+a}{2}$$

- Variance:

$$\int_{\mathbb{R}} dx\ f_X(x)x^2 - \left(\frac{b+a}{2}\right)^2 = \int_a^b dx\ \frac{x^2}{b-a} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}$$

- Distribution has two parameters, a and b.

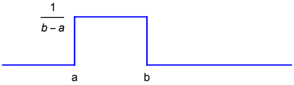
MLE with the uniform distribution

- We can be certain that $a \leq x_j \leq b$ for all $j = 1, \ldots, n$.
- We seek two estimators, $\hat{a}(\vec{x})$ and $\hat{b}(\vec{x})$.
- Neither a nor b is the mean in this case, so these estimators should be unrelated to the average of the data.

Defining the likelihood for n samples

- Given n *iid* samples of X from uniform distribution,

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$



- The *likelihood* is given by

$$L(a,b;\vec{x}) = \prod_{j=1}^{n} f_X(x_j) = \begin{cases} \left(\frac{1}{b-a}\right)^n & \text{if } x_j \in [a,b] \text{ for all } j = 1,\ldots,n \\ 0 & \text{if } x_j \notin [a,b] \text{ for any } j = 1,\ldots,n \end{cases}$$

- Choose $a \le \min_j x_j$ and $b \ge \max_j x_j$ so result is $\left(\frac{1}{b-a}\right)^n$.
- Maximize result by choosing *estimates*

$$a_e = \min_j x_j \quad \text{and} \quad b_e = \max_j x_j.$$

Maximum likelihood estimation of a and b

Bruce M. Boghosian

- Maximize result by choosing *estimates*

$$a_e = \min_j x_j \quad \text{and} \quad b_e = \max_j x_j.$$

- The maximum likelihood *estimators* for a and b are then

$$\boxed{\hat{a}(\vec{x}) = \min_j x_j}$$

$$\boxed{\hat{b}(\vec{x}) = \max_j x_j}$$

- Note that $\hat{a}(\vec{x})$ always overestimates the true a.
- Note that $\hat{b}(\vec{x})$ always underestimates the true b.

The standard normal distribution

- $X \in \mathbb{R}$ is a *continuous random variable*
- $X \in \mathbb{R}$ has the *normal probability density function*,

$$f_X(x) = \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(x-\mu)^2}{2v}\right],$$

which is normalized, with mean μ and variance $v = \sigma^2$.

- Distribution has two parameters μ and v.
- Given experimental results $\vec{x} = \{x_1, x_2, \ldots, x_n\}$, can we estimate μ and v?

Defining the likelihood for *n* samples

- The *likelihood* is given by

$$L(\mu, v; \vec{x}) = \prod_{j=1}^{n} f_X(x_j) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(x_j-\mu)^2}{2v}\right]$$

- The *log likelihood* is

$$\begin{aligned} \ln L(\mu, v; \vec{x}) &= \sum_{j=1}^{n}\left[-\frac{1}{2}\ln(2\pi v) - \frac{(x_j-\mu)^2}{2v}\right] \\ &= -\frac{n}{2}\ln(2\pi v) - \frac{1}{2v}\sum_{j=1}^{n}(x_j-\mu)^2 \end{aligned}$$

- We must find the maximum with respect to *both* μ and v.

Maximum likelihood estimation of μ and v

Bruce M. Boghosian

Uniform distribution
Definition
Likelihood and maximum likelihood
Estimators
The normal distribution
Definition
Likelihood and maximum likelihood
Estimators
Summary

- *Log likelihood* is

$$\ln L(\mu, v; \vec{x}) = -\frac{n}{2} \ln (2\pi v) - \frac{1}{2v} \sum_{j=1}^{n} (x_j - \mu)^2$$

- Set partial derivatives to zero

$$0 = \frac{\partial}{\partial \mu} \ln L(\mu, v; \vec{x}) = \frac{1}{v} \sum_{j=1}^{n} (x_j - \mu) = \frac{1}{v} \left(\sum_{j=1}^{n} x_j - n\mu \right)$$

$$0 = \frac{\partial}{\partial v} \ln L(\mu, v; \vec{x}) = -\frac{n}{2v} + \frac{1}{2v^2} \sum_{j=1}^{n} (x_j - \mu)^2$$

- Solving for location of maximum (μ_e, v_e) yields

$$\mu_e = \frac{1}{n} \sum_{j=1}^{n} x_j \quad \text{and} \quad v_e = \frac{1}{n} \sum_{j=1}^{n} \left[x_j - \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right) \right]^2$$

Maximum likelihood estimation of μ and v

- Maximum likelihood estimators for a and b are

$$\boxed{\hat{\mu}(\vec{x}) = \overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j}$$

$$\boxed{\hat{v}(\vec{x}) = \frac{1}{n}\sum_{j=1}^{n}\left[x_j - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]^2}$$

Summary

- We have extended our study of likelihood and maximum likelihood estimation to continuous random variables.
- We have applied the method to both the uniform distribution and the normal distribution.