Tufts University
Department of Mathematics

Math 136 **Homework 3** [1] **Spring, 2023**

Due date: 11:59 pm, Sunday, February 12, 2023 on Gradescope.

You are encouraged to work on problems with other Math 136 students and to talk with your professor and TA but your answers should be in your own words.

Reading assignment: Read Sections 14.1, and 15.2 by Wednesday, February 8. Section 15.1 can be helpful to remind you of the background on linear algebra.

This homework covers Sections 14.2, 14.3, 15.2, and 15.3.

Problems:

1 (20 points) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=x^3+xy+1$. Explain why the graph of $z=f(x,y)$ has a tangent plane at $(x,y)=(1,1)$ and find an equation of this tangent plane.

2 (20 points) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x^2\sin(1/x) & x\neq 0\\ 0 & x=0\end{cases}$

(a) Is f differentiable at all points of $\mathbb{R}$. Why or why not?

(b) Is f continuously differentiable on $\mathbb{R}$, i.e., is $f\in C^1(\mathbb{R})$?

The next problems involves this definition that we gave in class.

Definition 1 *Let $\mathcal{O}\subset\mathbb{R}^n$ and let $f:\mathcal{O}\to\mathbb{R}$ Let $\mathbf{x}_0\in\mathcal{O}$. Then, f* is differentiable at $\mathbf{x}_0$ *if there is a vector $\mathbf{b}\in\mathbb{R}^n$ such that*

$$\lim_{\mathbf{h}\to\mathbf{0}}\frac{|f(\mathbf{x}_0+\mathbf{h})-[f(\mathbf{x}_0)+\langle\mathbf{b},\mathbf{h}\rangle]|}{\|\mathbf{h}\|}=0 \tag{1}$$

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3 (40 points) Assume $f:\mathcal{O}\to\mathbb{R}$ is differentiable at $\mathbf{x}_0\in\mathcal{O}$ and assume $\mathbf{b}\in\mathbb{R}^n$ satisfies (1) for f at $\mathbf{x}_0$.

(a) Explain why $g(\mathbf{x})=f(\mathbf{x}_0)+\langle\mathbf{b},\mathbf{x}-\mathbf{x}_0\rangle$ is an affine first-order approximation to f at $\mathbf{x}_0$. (Short answer.)

(b) Let $\mathbf{b}=(b_1,b_2,\dots,b_n)$. Show that f has all first partial derivatives at $\mathbf{x}_0$ and $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)=b_i$ for each $i=1,\dots,n$.
[HINT: Let $i\in\{1,2,\dots,n\}$. Consider a sequence in $\mathbb{R}$, $\{t_k\}$ such that

- $t_k\to 0$
- for all $k\in\mathbb{N}$, $t_k\neq 0$
- for all $k\in\mathbb{N}$, $\mathbf{x}_0+t_k\mathbf{e}_i\in\mathcal{O}$

and use the sequence $\{\mathbf{h}_k=t_k\mathbf{e}_i\}$ in the definition of the limit for (1).]

(c) Explain why $\nabla f(\mathbf{x}_0)=\mathbf{b}$. (Short answer.)

This problem continues on the next page

(d) Prove that if f and g are differentiable at $\mathbf{x}_0$ then $f + g$ is differentiable at $\mathbf{x}_0$ and $\nabla(f + g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$.

To summarize, if f is differentiable at $\mathbf{x}_0$, then the vector $\mathbf{b}$ in (1) is unique and equal to $\nabla f(\mathbf{x}_0)$. Furthermore, the First Order Approximation Theorem holds for differentiable functions: if f is differentiable at $\mathbf{x}_0$, then

$$g(\mathbf{x}) = f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle$$

is the unique affine first order approximation to f at $\mathbf{x}_0$. IN part (d), you showed that the gradient is additive, and in class we will show that differentiable functions are continuous.

4 (20 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = xy + 2x$.

(a) Use calculus rules to find $\nabla f(x, y)$.

(b) Use your calculation of $\mathbf{b} = \nabla f(x, y)$ from part (a) and the definition of differentiability, Definition 1, to show f is differentiable at (x, y).

HINT: You plug in $\mathbf{x}_0 = (x_0, y_0)$ and $\mathbf{b} = (h, k)$ into equation (1) along with your guess for $\mathbf{b} = \nabla f(x, y)$ and do algebra. You may use the following limits

$$\lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2 + k^2}} = 0$$

If your limit doesn't work out, check your calculation of $\nabla f(x, y)$ in (a).