3. If $T_n \to T$ (see Exercise 2), show that $T_n' \to T'$. Discuss and compare with §5.3.

4. Find a sequence of continuous functions g_n such that $g_n \to \delta'$.

Theorem Proofs for Chapter 8

We will prove Darboux's theorem and Riemann's condition together.

8.1.2 Darboux's Theorem *Let $A \subset \mathbb{R}^n$ be bounded and lie in some rectangle S. Let $f : A \to \mathbb{R}$ be bounded and be extended to S by defining $f = 0$ outside A. Then f is integrable with integral I iff for any $\varepsilon > 0$ there is a $\delta > 0$ such that if P is any partition of S into rectangles $S_1, \ldots, S_N$ with sides of length $< \delta$ and if $x_1 \in S_1, \ldots, x_N \in S_N$, we have*

$$\left| \sum_{i=1}^{N} f(x_i)v(S_i) - I \right| < \varepsilon.$$

We call $\sum_{i=1}^{N} f(x_i)v(S_i)$ a ***Riemann sum.***

8.1.3 Riemann's Condition *f is integrable iff for any $\varepsilon > 0$ there is a partition P_ε of S such that $0 \le U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.*

Proof We will show that the conditions "f integrable," "f satisfies Riemann's condition," and "f satisfies Darboux's condition" are equivalent. This will be done in four steps.

Step 1 *If f is integrable, then f satisfies Riemann's condition.*

Proof Given $\varepsilon > 0$, there is a partition P_ε' such that

$$U(f, P_\varepsilon') < I + \frac{\varepsilon}{2},$$

where $I = \int_A f$. We can do this, since $I = \inf\{U(f,P) \mid P \text{ is a partition}\}$. If P is finer than P'_ε, then we know that

$$U(f,P) \leq U(f,P'_\varepsilon) < I + \frac{\varepsilon}{2}.$$

Similarly, choose P''_ε such that for P finer than P''_ε, we have $L(f,P) > I - \varepsilon/2$. Let $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$. If P is finer than P_ε, then

$$I - \frac{\varepsilon}{2} < L(f,P) \leq U(f,P) < I + \frac{\varepsilon}{2},$$

and so

$$0 \leq U(f,P) - L(f,P) < \varepsilon,$$

which is Riemann's condition.

Step 2 *If f satisfies Riemann's condition, then f is integrable.*

Proof For any $\varepsilon > 0$, there is a P_ε such that

$$0 \leq U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

This implies that $S = s$. Indeed, for each P, we have

$$L(f,P) \leq s \leq S \leq U(f,P),$$

and so if $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$, we also have $S - s < \varepsilon$ for every $\varepsilon > 0$ and hence $S = s$.

Step 3 *If f satisfies Darboux's condition, then f is integrable.*

Proof We will show that the I given in Darboux's condition will be the same as $S = \inf\{U(f,P) \mid P \text{ is a partition}\}$ and also the same as s. To accomplish this, given $\varepsilon > 0$, we produce a partition P such that

$$|U(f,P) - I| < \varepsilon,$$

which will show that $S \le I$. Similarly, we will have $I \le s$, and then $I \le s \le S \le I$ will imply $s = S = I$. To do this, choose $\delta > 0$ such that if P is a partition with sides $< \delta$, then

$$\left|\sum f(x_i)v(S_i) - I\right| < \frac{\varepsilon}{2},$$

where $S_1, \dots, S_N$ are the rectangles making up the partition P. Choose x_i such that

$$|f(x_i) - \sup_{S_i}(f)| < \frac{\varepsilon}{v(S_i)2N}.$$

Then

$$|U(f,P) - I| \le \left|U(f,P) - \sum f(x_i)v(S_i)\right| + \left|\sum f(x_i)v(S_i) - I\right|.$$

Now

$$\left|U(f,P) - \sum f(x_i)v(S_i)\right| < \sum \frac{\varepsilon v(S_i)}{v(S_i)2N} = \frac{\varepsilon}{2},$$

so that $|U(f,P) - I| < \varepsilon$, as required. The case for lower sums is similar.

Step 4 *If f is integrable, then f satisfies Darboux's condition.*

Proof Suppose f is integrable with integral I. We will show, in two steps, that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if P is any partition into rectangles $S_1, \dots, S_N$ with sides $< \delta$, and if $x_1 \in S_1, \dots, x_N \in S_N$, we have

$$\left|\sum_{i=1}^{N} f(x_i)v(S_i) - I\right| < \varepsilon.$$

Step 4A Let P be a partition of the rectangle $B \subset \mathbb{R}^n$. Given $\varepsilon > 0$, we shall show that *there exists a $\delta > 0$ such that for any partition P' into subrectangles with sides less than δ, the sum of the volumes of the subrectangles of P' that are not entirely contained in some rectangle of P is less than ε.*

To see this, we examine the cases $n = 1$ and $n > 1$ separately. First, suppose that we are working on the interval $[a, b]$; suppose that the partition P consists of N points. We assert that the δ that is needed is simply given by ε/N. Indeed, the length of the intervals in P' that are not contained in an interval of P is $N \times \delta$ = (maximum number of intervals not contained entirely in an interval of P) $\times$ (maximum length of each such interval of P') $= \varepsilon$. Turning to the general case, let the partition P consist of rectangles $V_1, \dots, V_M$. We denote the total "area" of the faces lying between any two rectangles by T. Let $\delta = \varepsilon/T$ and let P' be any partition of B into subrectangles of sides less than δ. For any rectangle

$S \in P'$ such that S is not contained in one of the V_i, S intersects two adjacent rectangles. One can see that $v(S) \le \delta A$, where A is the total area of faces between two subrectangles contained in S (see Figure 8.P-1). Thus $\sum_{S \in P'} v(S) < \delta T = \varepsilon$.

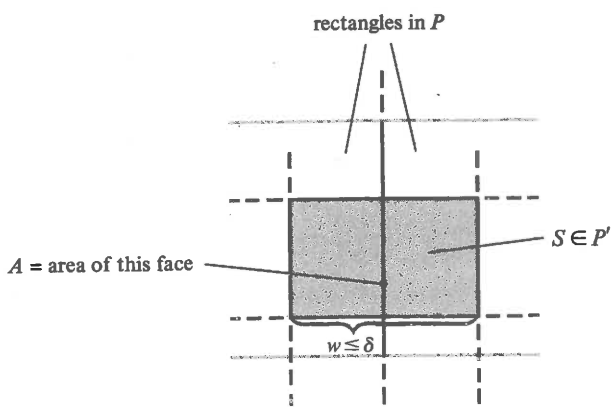

FIGURE 8.P-1 Showing that $v(S) = wA \le \delta A$

Step 4B Since f is bounded, there exists an $M > 0$ such that $|f(x)| < M$ for all $x \in S$. There are partitions P_1 and P_2 of S such that $I - L(f, P_1) < \varepsilon/2$ and $U(f, P_2) - I < \varepsilon/2$. Choose a partition P that refines both P_1 and P_2. Then $U(f, P) - I < \varepsilon/2$ and $I - L(f, P) < \varepsilon/2$. By step 4A, there exists a $\delta > 0$ such that for any partition of P into rectangles of sides $< \delta$, the sum of the volumes of the subrectangles not contained in some subrectangle of P is less than $\varepsilon/2M$. Let $S_1, \ldots, S_N$ be a partition into subrectangles of side less than δ, let $S_1, \ldots, S_K$ be the subrectangles contained in some subrectangle of P, and let $S_{K+1}, \ldots, S_N$ be the remaining subrectangles. If $x_1 \in S_1, \ldots, x_N \in S_N$, then

$$\begin{aligned}\sum_{i=1}^{N} f(x_i)v(S_i) &= \sum_{i=1}^{K} f(x_i)v(S_i) + \sum_{i=K+1}^{N} f(x_i)v(S_i) \\ &\le U(f, P) + M \cdot \frac{\varepsilon}{2M} \\ &= U(f, P) + \frac{\varepsilon}{2} < I + \varepsilon.\end{aligned}$$

Similarly,

$$\sum_{i=1}^{N} f(x_i)v(S_i) \ge L(f, P) - \frac{\varepsilon}{2} > I - \varepsilon.$$

Therefore,

$$\left|\sum_{i=1}^{N} f(x_i)v(S_i) - I\right| < \varepsilon. \quad \blacksquare$$

In some later proofs it will be convenient to have the following technical point at hand: *In the definition of measure zero, one can use either closed or open rectangles.*

Proof Let $A \subset \mathbb{R}^n$. First, suppose that, given $\varepsilon > 0$, there are open rectangles $V_1, V_2, \ldots$ covering A of total volume $< \varepsilon$. Let $B_i = \mathrm{cl}(V_i)$. Then $B_1, B_2, \ldots$ are closed rectangles covering A with the same total volume $< \varepsilon$.

Conversely, given $\varepsilon > 0$, suppose we have a covering by closed rectangles $B_1, B_2, \ldots$ with total volume $< \varepsilon/2^n$. Let V_i be the open rectangle containing B_i with twice the side. Then $v(V_i) = 2^n v(B_i)$, and so

$$\sum_{i=1}^{\infty} v(V_i) = 2^n \sum_{i=1}^{\infty} v(B_i) < \varepsilon.$$

This same argument also works for content zero. See Exercise 11 at chapter's end. $\blacksquare$

8.2.4 Theorem *Suppose that the sets $A_1, A_2, \ldots$ have measure zero in $\mathbb{R}^n$. Then $A_1 \cup A_2 \cup \cdots$ has measure zero in $\mathbb{R}^n$.*

Proof Since all of the A_i have measure zero, there is a covering of the A_i with rectangles $B_{i1}, B_{i2}, \ldots$ such that $\sum_{j=1}^{\infty} v(B_{ij}) < \varepsilon/2^i$. Since the collection $B_{i1}, B_{i2}, \ldots$ covers the A_i, the countable collection of all B_{ij} covers $A_1 \cup A_2 \cup \cdots$. But

$$\sum_{i,j=1}^{\infty} v(B_{ij}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v(B_{ij}) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

Since ε is arbitrary, $A_1 \cup A_2 \cup \cdots$ has measure zero. $\blacksquare$

Note. That we can sum up the $v(B_{ij})$ first by j, then by i, follows from the fact that the terms can be rearranged in an absolutely convergent double series. See Exercise 51, Chapter 5.

8.3.1 Lebesgue's Theorem *Let $A \subset \mathbb{R}^n$ be bounded and let $f : A \to \mathbb{R}$ be a bounded function. Extend f to all of $\mathbb{R}^n$ by letting it be zero at points* *not contained in A. Then f is (Riemann) integrable iff the points at which the extended f is discontinuous form a set of measure zero.*

Proof Consider some rectangle B that contains A. Then we must show that the function f is integrable on A iff the set of discontinuities of the function g, which equals f on A and zero elsewhere, has measure zero.

It is useful for the proof to have a measure of how "bad" a discontinuity is. To do this, we define the ***oscillation of a function h at*** x_0, written $O(h, x_0)$, to be

$$O(h, x_0) = \inf\{\sup\{|h(x_1) - h(x_2)| \mid x_1, x_2 \in U\} \mid U \text{ is a neighborhood of } x_0\}.$$

Note that the inf is taken over all neighborhoods U of x_0 and that $O(h, x_0) \geq 0$. We claim that $O(h, x_0) = 0$ iff h is continuous at x_0. To see this, note that h is continuous at x_0 iff for any $\varepsilon > 0$, there is a neighborhood U of x_0 such that $\sup\{|h(x_0) - h(x_1)| \mid x_1 \in U\} < \varepsilon$, and this is equivalent to $O(h, x_0) = 0$.

We are now ready to continue the proof—for convenience, it is broken into two steps. Let $g : B \to \mathbb{R}$ be defined by $g(x) = f(x)$ if $x \in A$, and $g(x) = 0$ if $x \notin A$.

Step 1 We assume that the set of discontinuities of g has measure zero. Thus, if we let $D_\varepsilon = \{x \mid O(g, x) \geq \varepsilon\}$ for $\varepsilon > 0$ and $D = \{\text{discontinuities of } g\}$, then $D_\varepsilon \subset D$. If y is an accumulation point of D_ε, every neighborhood of y contains a point of D_ε. Then every neighborhood U of y is a neighborhood of a point of D_ε, and by construction of D_ε, $\sup\{|g(x_1) - g(x_2)| \mid x_1, x_2 \in U\} \geq \varepsilon$. This implies $O(g, y) \geq \varepsilon$, and so $y \in D_\varepsilon$. This proves that D_ε is a closed set. Since $D_\varepsilon \subset B$, D_ε is bounded and therefore compact. Now D_ε has measure zero, since $D_\varepsilon \subset D$, and so by definition there is a collection $B_1, B_2, \ldots$ of (open) rectangles that cover D_ε such that $\sum_{i=1}^{\infty} v(B_i) < \varepsilon$. We know that a finite number of the B_i cover D_ε, since D_ε is compact. Suppose $B_1, \ldots, B_N$ cover D_ε. Certainly, $\sum_{i=1}^{N} v(B_i) < \varepsilon$.

Now pick a partition of B. By refining the partition, we may assume that each rectangle of it is either disjoint from D_ε or contained in one of the rectangles $B_1, B_2, \ldots, B_N$ that cover D_ε. Thus the rectangles of the partition fall into two (not necessarily disjoint) collections: $C_1 = \{\text{those rectangles that are contained in one of the } B_k\}$ and $C_2 = \{\text{those rectangles that do not intersect } D_\varepsilon\}$. We now use compactness to subdivide the rectangles in C_2 to obtain a further refinement of our partition. For each rectangle S that does not intersect D_ε, the oscillation of g at each point of the rectangle is less than ε. Hence we can find a neighborhood U_x of each point x of the rectangle such that $M_{U_x}(g) - m_{U_x}(g) < \varepsilon$, where $M_{U_x}(g) = \sup\{g(y) \mid y \in U_x\}$ and $m_{U_x}(g) = \inf\{g(y) \mid y \in U_x\}$. Since S is compact, a finite collection of the open sets U_x covers S. Pick a refined partition in S such that each rectangle of the partition is contained in U_{x_i} for some U_{x_i} in the finite collection that covers S. If we do this for each S in C_2, we get a partition P such that

$$\begin{aligned} U(g,P) - L(g,P) &\le \sum_{S\in C_1}(M_S(g) - m_S(g))v(S) + \sum_{S\in C_2}(M_S(g) - m_S(g))v(S) \\ &\le \varepsilon v(B) + \sum_{S\in C_1} 2Mv(S), \quad \text{where } |f(x)| < M \text{ on } A \\ &\le \varepsilon v(B) + 2M\varepsilon, \qquad \text{since } \sum_{S\in C_1} v(S) < \sum_{i=1}^{N} v(B_i) < \varepsilon. \end{aligned}$$

Since ε is arbitrary, Riemann's condition shows that g and hence f are integrable.

Step 2 Suppose g is integrable. The set of discontinuities of g is the set of points of oscillation greater than zero. Hence $\{\text{discontinuities of } g\} = D_1 \cup D_{1/2} \cup D_{1/3} \cup \cdots$, where $D_{1/n} = \{x \in B \mid O(g,x) \ge 1/n\}$. By Theorem **8.1.2,** there is a partition of B such that $U(g,P) - L(g,P) = \sum_{S\in P}(M_S(g) - m_S(g))v(S) < \varepsilon$. Now $D_{1/n} = \{x \in D_{1/n} \mid x \text{ lies on the boundary of some } S\} \cup \{x \in D_{1/n} \mid x \in \text{interior } (S) \text{ for some } S\} = S_1 \cup S_2$. The first of these sets, S_1, has measure zero, since we can cover the boundary of a rectangle with arbitrarily thin rectangles. Let C denote the collection of rectangles of the partition that have an element of $D_{1/n}$ in their interior. Then, if $S \in C$,

$$M_S(g) - m_S(g) \ge \frac{1}{n}$$

and

$$\frac{1}{n}\sum_{S\in C} v(S) \le \sum_{S\in C}(M_S(g) - m_S(g))v(S) \le \sum_{S\in P}(M_S(g) - m_S(g))v(S) < \varepsilon.$$

Hence C is a collection of rectangles that covers S_2 and $\sum_{S\in C} v(S) < n\varepsilon$. We can find a collection C' of rectangles that covers S_1 with $\sum_{S\in C'} v(S) < \varepsilon$. Then $C \cup C'$ covers $D_{1/n}$ and $\sum_{S\in C\cup C'} v(S) < (n+1)\varepsilon$. Since ε is arbitrary, $D_{1/n}$ has measure zero. Finally, $\{\text{discontinuities of } g\} = D_1 \cup D_{1/2} \cup D_{1/3} \cup \cdots$ has measure zero, by Theorem **8.2.4.** ■

8.3.2 Corollary *A bounded set $A \subset \mathbb{R}^n$ has volume iff the boundary of A has measure zero.*

Proof By Theorem **8.3.1,** it suffices to show that the set of discontinuities of 1_A, where

$$1_A(x) = \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A, \end{cases}$$

is the boundary of A. But if $x \in \operatorname{bd}(A)$, then any neighborhood of x intersects A and $\mathbb{R}^n \setminus A$. Hence there are points y in the neighborhood such that $|1_A(x) - 1_A(y)| = 1$. Thus 1_A is not continuous at x. If $x \notin \operatorname{bd}(A)$, then there is a neighborhood of x that lies entirely in A or $\mathbb{R}^n \setminus A$. In either case, 1_A is constant on this neighborhood, and so 1_A is continuous at x. ■

8.3.3 Corollary *Let $A \subset \mathbb{R}^n$ be bounded and have volume. A bounded function $f : A \to \mathbb{R}$ with a finite or countable number of points of discontinuity is integrable.*

Proof The discontinuities of the extended function g, which is equal to f on A and zero at points outside A, are simply the discontinuities of f possibly together with some discontinuities of g on the boundary of A, for the same reason as in the proof of **8.3.2.** But $\operatorname{bd}(A)$ has measure zero, by Corollary **8.3.2.** Hence it is sufficient to show that a countable set has measure zero. But this follows from Theorem **8.2.4** and from the fact that a point has measure zero. ■

8.3.4 Theorem

i. *Let $A \subset \mathbb{R}^n$ be bounded and have measure zero and let $f : A \to \mathbb{R}$ be any (bounded) integrable function. Then $\int_A f(x)\,dx = 0$.*

ii. *If $f : A \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all x in A and $\int_A f(x)\,dx = 0$, then the set $\{x \in A \mid f(x) \neq 0\}$ has measure zero.*

Proof

i. We claim that a set of measure zero cannot contain a nontrivial rectangle, that is, a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ such that $a_i < b_i$ for each i. The reason is that a subset of a set of measure zero must be of measure zero and a nontrivial rectangle does not have measure zero. Let S be a rectangle enclosing A, and extend f to S by setting it equal to 0 on $S \setminus A$; let P be any partition of S into subrectangles $S_1, \ldots, S_N$, and let M be such that $f(x) \leq M$ for all $x \in A$.

Then

$$L(f,P) = \sum_{i=1}^{N} m_{S_i}(f)v(S_i) \le M \sum_{i=1}^{N} m_{S_i}(1_A)v(S_i).$$

Suppose $m_{S_i}(1_A) \neq 0$ for some i, such that S_i is a (nontrivial) rectangle. This means that $S_i \subset A$, which contradicts the opening remarks of the proof. Thus, for any nontrivial S_i, $m_{S_i}(1_A) = 0$, while for any trivial S_i, $v(S_i) = 0$. Hence $\sum_{i=1}^{N} m_{S_i}(1_A)v(S_i) = 0$, or $L(f,P) \le 0$. Now $\sup_{x\in S_i} f(x) = -\inf_{x\in S_i}(-f(x))$, and so

$$U(f,P) = \sum_{S_i \in P} \sup_{x\in S_i} f(x)\, v(S_i) = -\sum_{S_i\in P} \inf_{x\in S_i}(-f(x))\, v(S_i) = -L(-f,P),$$

and by the same argument again, $L(-f,P) \le 0$. Hence $-L(-f,P) = U(f,P) \ge 0$. Since P was arbitrary, $U(f,Q) \ge 0 \ge L(f,Q)$ for any partition Q of S, and hence

$$\overline{\int_A} f \ge 0 \ge \underline{\int_A} f,$$

and so, since f is integrable,

$$\overline{\int_A} f = \underline{\int_A} f = \int_A f = 0.$$

i. Consider the set $A_m = \{x \in A \mid f(x) > 1/m\}$; we first show that A_m has content zero. Given $\varepsilon > 0$, let S be a rectangle enclosing A, extend f to S by setting it equal to 0 on $S\backslash A$, and let P be a partition of the rectangle S such that $U(f,P) < \varepsilon/m$. Such a partition exists by the fact that $\int_A f = 0$. If $S_1, \ldots, S_K$ are the subrectangles of the partition P that have nonempty intersection with A_m, then, if $M_{S_i}(f)$ is the sup of f on S_i,

$$\sum_{i=1}^{K} v(S_i) \le \sum_{i=1}^{K} mM_{S_i}(f)v(S_i) < \varepsilon,$$

since $mM_{S_i}(f) > 1$. Therefore $S_1, \ldots, S_K$ is a cover by closed rectangles of the set A_m such that $\sum_{i=1}^{K} v(S_i) < \varepsilon$. Hence A_m has content zero. Since A_m has content zero, it also has measure zero.

Finally, observe that

$$\{x \in A \mid f(x) \neq 0\} = \bigcup_{m=1}^{\infty} A_m.$$

Thus, by Theorem **8.2.4**, this set has measure zero. ■

8.4.1 Theorem *Let A, B be bounded subsets of $\mathbb{R}^n$, $c \in \mathbb{R}$, and let $f, g : A \to \mathbb{R}$ be integrable. Then*

i. *$f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.*

ii. *cf is integrable and $\int_A (cf) = c\int_A f$.*

iii. *$|f|$ is integrable and $|\int_A f| \leq \int_A |f|$.*

iv. *If $f \leq g$, then $\int_A f \leq \int_A g$.*

v. *If A has volume and $|f| \leq M$, then $|\int_A f| \leq Mv(A)$.*

vi. **Mean Value Theorem for Integrals** *If $f : A \to \mathbb{R}$ is continuous and A has volume and is compact and connected, then there is an $x_0 \in A$ such that $\int_A f(x)\,dx = f(x_0)v(A)$. The quantity*

$$\frac{1}{v(A)} \int_A f$$

*is called the **average** of f over A.*

vii. *Let $f : A \cup B \to \mathbb{R}$. If the sets A and B are such that $A \cap B$ has measure zero and $f|(A \cap B)$, $f|A$, and $f|B$ are all integrable, then f is integrable on $A \cup B$ and $\int_{A\cup B} f = \int_A f + \int_B g$.*

Note. If f is integrable on A and B, then it is integrable on $A \cap B$. Indeed, if A and B have volume, so does $A \cap B$, since $\mathrm{bd}(A \cap B) \subset \mathrm{bd}\, A \cup \mathrm{bd}\, B$.

Proof

i. Let S be a rectangle enclosing A and let f and g be extended to S by setting them equal to zero on $S \backslash A$. Suppose $\varepsilon > 0$ is given. By Theorem **8.1.2**, there is a $\delta_1 > 0$ such that if P_1 is any partition of S into subrectangles $S_1, \ldots, S_N$ with sides less than δ_1 and if $x_1 \in S_1, \ldots, x_N \in S_N$, then

$$\left| \sum_{i=1}^{N} f(x_i)v(S_i) - \int_A f \right| < \frac{\varepsilon}{2}.$$

Similarly, there is a $\delta_2 > 0$ such that if P_2 is any partition of S into subrectangles $R_1, \ldots, R_M$ with sides less than δ_2 and if $x_1 \in R_1, \ldots, x_M \in R_M$, then

$$\left| \sum_{i=1}^{M} g(x_i)v(R_i) - \int_A g \right| < \frac{\varepsilon}{2}.$$