Tufts University
Department of Mathematics

Math 135 | **Homework 3 v2**[1] | **Fall 2022**

(Changes in red)

Readings for the week of September 19, 2022

§**2.1:** Uniqueness of a limit, limit properties
§**2.2:** Boundedness, sequential density
§**2.3:** The monotone convergence theorem

Problem Set 3
(Due Wednesday, September 28, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

In # 4, you may assume the square root rule: If $a_n \to a$, then $\lim \sqrt{a_n} = \sqrt{a}$. You will be asked to prove this rule in Problem Set 4.

Definition. A sequence $\{a_n\}$ is said to ***diverge*** to ∞ if for each $M \in \mathbb{R}$, there exist an $N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n > M$.

1. (10 points) **Product rule**.
   Using the ε-N definition of convergence, prove that if $\lim\ a_n = 2$ and $\lim\ b_n = 5$, then $\lim\ a_n b_n = 10$.

2. (10 points) **The ε-N definition of convergence**. (*This is a restatement of p. 34, problem 11*)
   For each statement below, show that it is *not* equivalent to the ε-N definition of convergence ($a_n \to a$), either by finding an example of a sequence a_n which converges to a but does *not* satisfy the statement, or an example of a sequence a_n and a number a which *do* satisfy the statement but for which the sequence a_n does *not* converge to a.
   (a) For some ε, there is an index N such that $|a_n - a| < \varepsilon$ for all indices $n \geq N$
   (b) For each $\varepsilon > 0$ and each index N, $|a_n - a| < \varepsilon$ for all indices $n \geq N$.
   (c) There is an index N such that for every number $\varepsilon > 0$, $|a_n - a| < \varepsilon$ for all indices $n \geq N$.
   (*Hint*. In this problem, you can use the sequence $1/n \to 0$ to answer all three parts.)

3. (10 points) **Infinite series**. §2.1, p. 34, #14.
   Define the sequence $\{s_n\}$ by
   $$s_n = \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \cdots + \frac{1}{(n+1)n} \qquad \text{for all } n \in \mathbb{N}.$$
   Prove that $\lim s_n = 1$. (Hint: try to get a simpler formula for s_n, for example by writing
   $$\frac{1}{2\cdot 1} = \frac{1}{1} - \frac{1}{2}, \ \frac{1}{3\cdot 2} = \frac{1}{2} - \frac{1}{3},$$
   and so on. This is called a **telescoping sum**.)

4. (10 points) **Convergence of a sequence**.
Using any method, including limit rules, discuss the convergence of divergence of the following sequences. If the sequence converges, find the limit. If the sequence diverges, does it diverge to infinity, minus infinity, or oscillate?
   (a) $\sqrt{n+1}-\sqrt{n}$
   (b) $(\sqrt{n+1}-\sqrt{n})\sqrt{n}$
   (c) $(\sqrt{n+1}-\sqrt{n})n$
   (*Hint*: Rationalize $\sqrt{a}-\sqrt{b}$ by multiplying $(\sqrt{a}+\sqrt{b})/(\sqrt{a}+\sqrt{b})$.)

5. (10 points) **Closed sets**. §2.2, p. 37, #4.
Show that the set of irrational numbers fails to be closed.

6. (10 points) **Closed sets**. §2.2, p. 37, #2
Show that the set $(-\infty, 0]$ is closed.

7. (10 points) **Dense closed sets**.
Describe all the dense closed subsets of $\mathbb{R}$ and prove your result. (*Hint*. Write down the sequential denseness theorem (Prop. 2.19, p. 36) and the definition of a closed set (p. 37) and see whether you can put the two together. If a dense set S is missing $\sqrt{2}$, can it be closed?)

8. (10 points) **Monotone sequences and bounded sequences**. §2.3, p. 42, #1
For each of the following statements, determine whether it is true or false and justify your answer.
   (a) The sum of monotone sequences is monotone.
   (b) The product of monotone sequences is monotone.
   (c) Every bounded sequence converges.
   (d) Every monotone sequence converges.

9. (10 points) **ε-criteria for sup and inf**.
In this problem, you will prove some useful properties of suprema and infima:
   (a) Prove the ***ε-criterion for infima***:
   Suppose $A \subset \mathbb{R}$ is a nonempty set which is bounded below and $\ell \in \mathbb{R}$.
   Then $\ell = \inf A$ *if and only if*
      (i) ℓ is a lower bound for A, *and*
      (ii) for every $\varepsilon > 0$ there is a point $x \in A$ such that
   $$x < \ell + \varepsilon.$$
   (b) Let $A \subset \mathbb{R}$ be bounded below and let $\ell = \inf(A)$. Prove there is a sequence $\{a_n\}$ in A that converges to ℓ.
   (c) Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. State the ε-criterion for sup (i.e., l.u.b.). That is, state the version of (i) and (ii) for sup.