Tuesday, November 2

1. Explore the theme of Picard iteration a bit more. Suppose you want to find a solution of
$$\frac{dx}{dt} = x, \quad x(0) = 1.$$
You know the solution, of course:
$$x(t) = e^t.$$
Suppose though that you wrote the initial value problem as a fixed point equation:
$$x(t) = \int_0^t x(s)ds + 1.$$
Suppose you started with the guess
$$x^{(0)}(t) = 0 \quad \text{for all } t.$$
What would be $x^{(1)}(t)$? $x^{(2)}(t)$? $x^{(3)}(t)$? $x^{(k)}(t)$ in general? Is it true that
$$\lim_{k\to\infty} x^{(k)}(t) = e^t \quad ?$$
(Hint: Look up the Taylor expansion of e^t around $t = 0$.)

2. In problem 1, what if we start with something crazy, like
$$x^{(0)}(t) = \cos t \quad \text{for all } t?$$
What would be $x^{(1)}(t)$? $x^{(2)}(t)$? $x^{(3)}(t)$? $x^{(k)}(t)$ in general? Is it true that
$$\lim_{k\to\infty} x^{(k)}(t) = e^t \quad ?$$
(Hint: Look up the Taylor expansions of $\sin t$ and $\cos t$ around $t = 0$.)

3. (for those with more mathematical training) Suppose that f is a continuous function from $\mathbb{R}$ into $\mathbb{R}$. Think about initial value problems of the form
$$\frac{dx}{dt} = f(x), \quad x(t) = x_0.$$
   (a) Explain why this is equivalent to
   $$x = F(x)$$
   where $F(x)$ is the function defined by
   $$F(x(t)) = \int_{t_0}^t f(x(s))ds + x_0.$$

(b) Introduce the following way of measuring the distance between two continuous functions, $x = x(t)$ and $y = y(t)$, defined for $t \in [a,b]$:

$$d_{[a,b]}(x,y) = \max_{a \le t \le b} |x(t) - y(t)|.$$

Assume that f is Lipschitz continuous, meaning: There is a constant $L \ge 0$ so that

$$|f(x) - f(y)| \le L|x - y| \quad \text{for all } x \text{ and } y.$$

Let $T > t_0$. (Think of T as the "target time" up to which we want to compute the solution $x(t)$.) Prove:

$$d_{[t_0,T]}(F(x), F(y)) \le (T - t_0) L d_{[t_0,T]}(x,y). \tag{1}$$

(c) Explain why (1) implies that there can be no more than one solution of $x = F(x)$ when $(T - t_0)L < 1$, so $T < t_0 + \frac{1}{L}$.

(d) In fact (1) implies that there is exactly one solution of $x = F(x)$ when $T < t_0 + \frac{1}{L}$. Ask me if you want to know why.

(e) Try to deduce from part (d) that

$$\frac{dx}{dt} = 1.2 + \cos x, \quad x(0) = 0,$$

has exactly one solution $x = x(t)$ defined for *all* $t \ge 0$. (You have to still do some work. Part (d) only tells you that there is exactly one solution on a small interval $[0, T]$.)