MATH 135 Exam 2 November 21, 2022

(100 points) 12 noon–1:20 p.m.

•**Carefully PRINT your full name:** Solutions

•**CIRCLE your section:** Section 1 (Tu) Section 2 (Tu) Section 3 (Hasselblatt)

Instructions: No books, notes, calculators, or external help from any person or device are allowed. Except in the true-false questions or when instructed otherwise, justify all of your steps.
Write only in the space provided and do not attach any extra page.

•**Please sign the following pledge:**
I pledge that in this exam I have neither given nor received assistance or cheated in any other way.

Signature: ____________________

1. (16 points) Circle either True or False. You do not need to justify your choice.

   (a) **True** / **False:** Every continuous function on a closed set has a maximum.
   ($f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$, has no maximum.)

   (b) **True** / **False:** Uniform convergence implies pointwise convergence, but the converse is not true.

   (c) **True** / **False:** Suppose $A \subset B \subset \mathbb{R}^n$. If A is closed and B is sequentially compact, then A is sequentially compact. (B bounded $\Rightarrow$ A bounded, A is also closed.)

   (d) **True** / **False:** If $A, B \subset \mathbb{R}^n$, then $\text{int}A \cup \text{int}B = \text{int}(A \cup B)$.
   0 1 2
   $A = [0,1]$
   $B = [1,2]$
   $\text{int } A \cup B = (0,2)$ $A \cup B = [0,2]$
   $\text{int} A \cup \text{int} B = (0,1) \cup (1,2)$

   (e) **True** / **False:** If $A, B \subset \mathbb{R}^n$, then $\text{bd}A \cup \text{bd}B = \text{bd}(A \cup B)$.
   $\text{bd } A \cup B = \{0, 2\}$, $\text{bd} A \cup \text{bd} B = \{0, 1, 2\}$.

   (f) **True** / **False:** If the sets A_i are open in $\mathbb{R}^n$ for all $i \in \mathbb{N}$, then the intersection $\bigcap_{i=1}^{\infty} A_i$ is open in $\mathbb{R}^n$.
   $\bigcap_{i=1}^{\infty} \left(-\frac{1}{i}, 1+\frac{1}{i}\right) = [0,1]$

   (g) **True** / **False:** Let A be a subset of $\mathbb{R}^n$ and let $F\colon A \to \mathbb{R}^m$ be continuous. If A is sequentially compact, then $F(A)$ is sequentially compact.

   (h) **True** / **False:** Let A be a subset of $\mathbb{R}^n$ and let $F\colon A \to \mathbb{R}^m$ be continuous. If A is closed in $\mathbb{R}^n$, then $F(A)$ is closed in $\mathbb{R}^m$.
   $F: \mathbb{R} \to \mathbb{R}$, $F(x) = e^x$, $F(\mathbb{R}) = (0, \infty)$.

2. (10 points) Fill in the blanks (no words). You do not need to justify your answer.
   Let $A = [0,1] \setminus \mathbb{Q}$, where $\mathbb{Q}$ denotes the rationals. Thus, A is the set of irrationals in $[0,1]$.

   (a) $\text{int}A =$ ϕ (Every open interval $(x-\varepsilon, x+\varepsilon)$ contains a rational point, so cannot be a subset of A.)

   (b) $\text{bd}A =$ $[0,1]$ (For $x \in [0,1]$, every open interval $(x-\varepsilon, x+\varepsilon)$ contains a rational point and an irrational point.)

   (c) $\text{ext}A =$ $(-\infty, 0) \cup (1, \infty)$ (the complement of (a) $\cup$ (b))

   (d) $\text{cl}A =$ $[0,1]$ ($\text{int} A \cup \text{bd} A$)

   (e) $\mathbb{R} \setminus A =$ $(-\infty, 0) \cup (1, \infty) \cup (\mathbb{Q} \cap [0,1])$

Carefully PRINT your full name:

3. (10 points) Let $A \subset \mathbb{R}^n$.

(a) State the definition (using sequences) of **continuity** of a mapping $F\colon A \to \mathbb{R}^m$ at a point $\mathbf{u} \in A$ and on the whole domain A.

$F\colon A \to \mathbb{R}^m$ is **continuous at** $u \in A$ if

$\forall\, u_k \in A$,

$u_k \to u \Rightarrow F(u_k) \longrightarrow F(u)$.

$F\colon A \to \mathbb{R}^m$ is **continuous on** A if

F is continuous at all $u \in A$.

(b) State the ϵ-δ criterion for **uniform continuity** of a mapping $F\colon A \to \mathbb{R}^m$. (Note that it is **continuity** in (a), but **uniform continuity** in (b).)

$\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $\forall\, u, v \in A$,

$\|u - v\| < \delta \Rightarrow \|F(u) - F(v)\| < \varepsilon$.

4. (14 points)

(a) State the definition of **uniform convergence** in $\mathbb{R}$: Let f_n and f be functions from $\mathbb{R}$ to $\mathbb{R}$. Then f_n converges uniformly to f on $\mathbb{R}$ if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in \mathbb{R}$,

$$|f_n(x) - f(x)| < \varepsilon.$$

(b) Let $\{f_n\}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$. Assume that $f_n \to f$ uniformly on $\mathbb{R}$. Prove using the definition of uniform convergence that $3f_n + 2 \to 3f + 2$ uniformly on $\mathbb{R}$.

Let $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in \mathbb{R}$,

$$|3f_n(x) + 2 - (3f(x) + 2)| < \varepsilon,$$

which is equivalent to $3|f_n(x) - f(x)| < \varepsilon$

or $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$.

Since $f_n \to f$ uniformly on $\mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in \mathbb{R}$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3},$$

or as shown above,

$$|3f_n(x) + 2 - (3f(x) + 2)| < \varepsilon.$$

This proves that $3f_n + 2 \longrightarrow 3f + 2$ uniformly on $\mathbb{R}$. $\square$

Carefully PRINT your full name:

5. (10 points)

(a) State the definition of a Cauchy sequence in $\mathbb{R}$: a sequence $\{x_n\}$ in $\mathbb{R}$ is *Cauchy* if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, $|x_m - x_n| < \varepsilon$.

(m could be replaced by $n+k$ for $k \in \mathbb{N}$.)

(b) Suppose $|x_{n+k} - x_n| < 1/n$ for all $n, k \in \mathbb{N}$. Prove that $\{x_n\}$ converges. (You may use whatever theorem about convergence you think is appropriate.)

Let $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ s.t. $\forall n \geq N$ and $k \in \mathbb{N}$,

$$|x_{n+k} - x_n| < \frac{1}{n} < \varepsilon.$$

Scratch work: $\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$.

Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then $\forall n \geq N$,

$n > \frac{1}{\varepsilon}$ or $\frac{1}{n} < \varepsilon$. Thus,

$$|x_{n+k} - x_n| < \frac{1}{n} < \varepsilon.$$

This proves that $\{x_n\}$ is Cauchy. Since every Cauchy sequence converges, $\{x_n\}$ converges. $\square$

6. (10 points) Determine the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{kx^k}{3^k}$.

Let $a_k = \frac{kx^k}{3^k}$. Consider the ratio

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)\,x^{\cancel{k}+1}}{3^{\cancel{k}+1}} \cdot \frac{\cancel{3^k}}{k\,\cancel{x^k}}\right|$$

$$= \frac{k+1}{k}\frac{|x|}{3} \longrightarrow \frac{|x|}{3}$$

By the ratio test the series converges for $\frac{|x|}{3} < 1$ or $|x| < 3$ and diverges for $\frac{|x|}{3} > 1$ or $|x| > 3$.

Therefore, the radius of convergence is 3. $\square$

7. (10 points) Prove that $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly on $\mathbb{R}$. (*Hint*. You may assume the p-test: $\sum_{n=1}^{\infty} 1/n^p$ converges for $p > 1$.)

For all $x \in \mathbb{R}$, since $x^2+n^2 \geq n^2$,

$$\left|\frac{1}{x^2+n^2}\right| \leq \frac{1}{n^2} = M_n.$$

By the p-test, $\sum \frac{1}{n^2}$ converges.

By the Weierstrass M-test, $\sum \frac{1}{x^2+n^2}$ converges uniformly on $\mathbb{R}$. $\square$

Carefully PRINT your full name:

8. (10 points) Let $A = \{(x,y) \in \mathbb{R}^2 \mid x^6 + y^6 = 1\}$. Prove that A is sequentially compact.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^6 + y^6 - 1$.

Then $A = f^{-1}(\{0\})$.

Since f is a polynomial, it is continuous.

Since $\{0\}$ is closed in $\mathbb{R}$ and under a continuous map on $\mathbb{R}^2$, the inverse image of a closed set is closed, $A = f^{-1}(\{0\})$ is closed in $\mathbb{R}^2$.

$x^6 \le x^6 + y^6 \le 1 \Rightarrow x^6 \le 1 \Rightarrow |x| \le 1$

Similarly, $|y| \le 1$.

So $x^2 + y^2 \le 1 + 1 = 2$.

This proves that $A \subset \overline{B}_{\sqrt{2}}(0)$ = closed ball of radius $\sqrt{2}$ centered at 0.

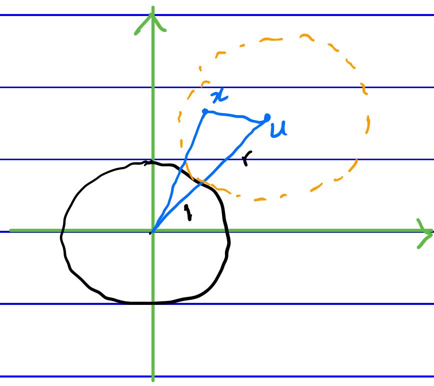
Hence, A is bounded.

Being closed and bounded, A is sequentially compact. $\square$

9. (10 points) Define

$$\mathcal{O} = \{u \in \mathbb{R}^n \mid \|u\| > 1\}.$$

Prove that $\mathcal{O}$ is open in $\mathbb{R}^n$ by showing that every point in $\mathcal{O}$ is an interior point of $\mathcal{O}$. (*Hint*. Pictures may help but do not suffice.)



Let $u \in \mathcal{O}$. Define

$r = \|u\| - 1$.

We will show that $B_r(u) \subset \mathcal{O}$. This will prove that u is an interior point of $\mathcal{O}$.

Proof that $B_r(u) \subset \mathcal{O}$.

Let $x \in B_r(u)$. We want $\|x\| > 1$. This means in the triangle inequality $\|x\|$ should be on the RHS. Try

$$\begin{aligned} \cancel{\|u\|} &\le \|u-x\| + \|x\| \\ &< r + \|x\| \\ &= \cancel{\|u\|} - 1 + \|x\|. \end{aligned}$$

Hence, $0 < -1 + \|x\|$ or $\|x\| > 1$.

This proves that $x \in \mathcal{O}$, so $B_r(u) \subset \mathcal{O}$. □

Since an arbitrary point $u \in \mathcal{O}$ is an interior point, $\mathcal{O}$ is open. □

Bonus question (3 points, of which 1 for correct spelling—so write clearly!):
Name a mathematician who made important contributions to real analysis.
[No credit for naming Fitzpatrick, Sandwich, or Tufts faculty.]