Math 135 HW 1

1 a)

A	B	$A \Rightarrow B$	$\sim(A \to B)$	$\sim B$	$A \wedge (\sim B)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

b) For every $x \in \mathbb{R}$, $x^2 \le 5$ and $x \ge -3$

$\forall x \in \mathbb{R}, x^2 \le 5 \wedge x \ge -3$

2 a) If $\forall \varepsilon > 0$, $|x| \le \varepsilon$, then $x = 0$

b) If $\exists \varepsilon > 0$ such that $|x| \le \varepsilon$, then $x \neq 0$

c) To show first direction:

if $\forall \varepsilon > 0$, $|x| < \varepsilon$ then $x = 0$.

If $\varepsilon > 0$, then $|x| < \varepsilon \to -\varepsilon < x < \varepsilon$, since $\varepsilon \in \mathbb{R}^+$ the only x that satisfies the inequality is $x = 0$.

To show opposite direction:

If $x = 0$, then $\forall \varepsilon > 0$ $|x| < \varepsilon$

If $x = 0$, $|x| < \varepsilon \to 0 < \varepsilon$, since $\varepsilon > 0$, this is true.

Both implications are proven, showing $\forall \varepsilon > 0$, $|x| < \varepsilon$ iff $x = 0$.

3 By definition, for $\sup S$, where S is a set, $\forall s \in S$, $\sup S \ge s$.

By definition of $\inf S$, where S is a set, $\forall s \in S$ $\inf S \le s$.

Combining these gives us: $\inf S \le s \le \sup S$ $\forall s \in S$ so $\inf S \le \sup S$

4) If $\sqrt{10}$ is rational, then $\sqrt{10} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and have no common factors.

$$\left(\sqrt{10} = \frac{a}{b}\right)^2 \rightarrow 10 = \frac{a^2}{b^2} \rightarrow 10b^2 = a^2 \rightarrow 2(5b^2) = a^2$$

Therefore a^2 is even, and by definition, a is even

Let $a = 2K$ where $K \in \mathbb{N}$

$10b^2 = 4K^2 \rightarrow 5b^2 = 2K^2$. $2K^2$ is even so $5b^2$ is even. Since 5 is odd, b^2 must be even, therefore b is even.

If a and b are even, they share a common factor of 2, contradicting the initial assumption, so $\sqrt{10}$ is irrational.

5 Base case: $n = 0$

$$\sum_{k=0}^{0} r^k = 1 \quad \text{By formula: } \frac{1-r^{0+1}}{1-r} = 1$$

Now we assume true to $n = J$, to show for $n = J+1$:

$$\sum_{k=0}^{J+1} r^k = \sum_{k=0}^{J} r^k + r^{J+1} = \frac{1-r^{J+1}}{1-r} + r^{J+1}\left(\frac{1-r}{1-r}\right)$$

$$= \frac{1-\cancel{r^{J+1}} + \cancel{r^{J+1}} - r^{J+2}}{1-r}$$

$$\sum_{k=0}^{J+1} = \frac{1-r^{(J+1)+1}}{1-r}$$

Since the formula holds, $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

6a) By definition, $f^{-1}(B) = A$ by $f(a) = b$ for some $b \in C \cap D$, then $b \in C \cap b \in D$. Let $a \in f^{-1}(C \cap D)$

Since $b \in C \cap b \in D$, then $a \in f^{-1}(C) \cap f^{-1}(D)$

Therefore, $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$

b) $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$

Let $a \in f^{-1}(C) \cap f^{-1}(D)$. Therefore $a \in f^{-1}(C)$ and $a \in f^{-1}(D)$. By def, $f^{-1}(b) = a$.

So $f(b) \in C$ and $f(b) \in D$, so $f(b) \in C \cap D$

So $a \in f^{-1}(C \cap D)$.

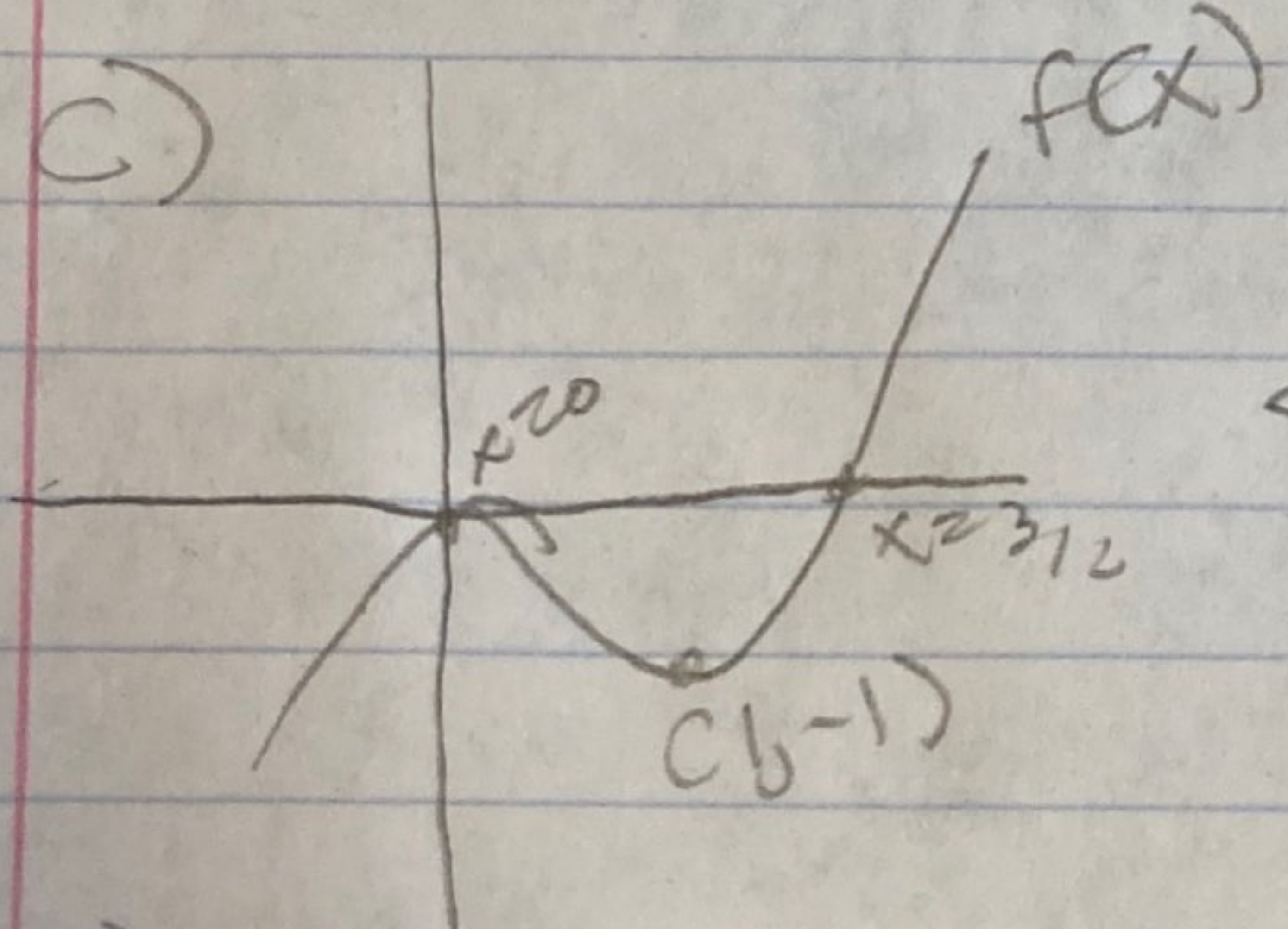
c) Yes, as $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$ and $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$ so they are equal.

7a) No, let $f(x) = 0$ $\quad 0 = 2x^3 - 3x^2$ $\quad 0 = x^2(2x-3)$

so $x = 0, 3/2$ meaning $f(0) = f(3/2)$ and f is not injective

b) Yes, $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$, so range of $f(x)$ is $(-\infty, \infty)$ and f is surjective

c)



min $f(x)$: $f'(x) = 6x^2 - 6x = 0$ $\quad x = 1$

min at $x = 1$ of $x = -1$

So $f([0, 3/2]) = [-1, 0]$

d) From the graph, $f^{-1}([0, \infty)) = [3/2, \infty)$