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Small-Sample Statistics

Gamma and Beta Functions and Distributions

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Outline

The gamma function

- Definition of gamma function:

$$\boxed{\Gamma(r) := \int_0^\infty du\ e^{-u} u^{r-1}}$$

- Special case

$$\boxed{\Gamma(1) = \int_0^\infty du\ e^{-u} = 1,}$$

and using substitution $u = w^2$,

$$\boxed{\Gamma\left(1/2\right) = \int_0^\infty du\ \frac{e^{-u}}{\sqrt{u}} = 2\int_0^\infty dw\ e^{-w^2} = \sqrt{\pi}}$$

Recurrence formula for gamma function

- Definition of gamma function:

$$\boxed{\Gamma(r) := \int_0^\infty du\ e^{-u} u^{r-1}}$$

- Recurrence formula

$$\Gamma(r+1) := \int_0^\infty du\ e^{-u} u^{r} = -e^{-u} u^{r}\Big|_0^\infty + r\int_0^\infty du\ e^{-u} u^{r-1}$$

$$\boxed{\Gamma(r+1) = r\Gamma(r)}$$

- Then $\Gamma(2) = 1 \cdot \Gamma(1) = 1!$, $\Gamma(3) = 2 \cdot \Gamma(2) = 2!$, etc., so

$$\boxed{\Gamma(r+1) = r! \quad \text{if } r \in \mathbb{Z}^+}$$

The beta function

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- Definition

$$\boxed{B(r,s) := \int_0^1 dt\ t^{r-1}(1-t)^{s-1}}$$

- Symmetry follows from the substitution $u = 1 - t$

$$\boxed{B(r,s) = B(s,r)}$$

- Relationship to gamma function

$$\boxed{B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}}$$

Proof of relationship between B and Γ functions

- First note

$$\begin{aligned}\Gamma(r)\Gamma(s) &= \int_0^\infty du\ e^{-u} u^{r-1} \int_0^\infty dv\ e^{-v} v^{s-1}\\ &= \int_0^\infty du \int_0^\infty dv\ e^{-u-v} u^{r-1} v^{s-1}\end{aligned}$$

- Change variables $u = zt$ and $v = z(1-t)$
- Hence $z = u + v$ and $t = u/(u+v)$
- Jacobian is $J = \begin{vmatrix} t & z \\ 1-t & -z \end{vmatrix} = -z$, so $|J| = z$ and

$$\begin{aligned}\Gamma(r)\Gamma(s) &= \int_0^\infty dz \int_0^1 dt\ z\ e^{-z} (zt)^{r-1} \left[z(1-t)\right]^{s-1}\\ &= \int_0^\infty dz\ e^{-z} z^{r+s-1} \int_0^1 dt\ t^{r-1}(1-t)^{s-1}\\ &= \Gamma(r+s)B(r,s),\end{aligned}$$

from which desired result immediately follows. $\square$

The gamma pdf I

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- Definition of the two-parameter gamma pdf for $y \geq 0$

$$\boxed{f_Y(y) = \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y}}$$

- Normalization follows immediately from definition of Γ.
- Expectation:

$$E(Y) = \int_0^\infty dy\ \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y} y = \frac{r}{\lambda} \int_0^\infty dy\ \frac{\lambda^{r+1}}{\Gamma(r+1)} y^{(r+1)-1} e^{-\lambda y}$$

$$\boxed{E(Y) = \frac{r}{\lambda}}$$

The gamma pdf II

- Mean square:

$$E(Y^2) = \int_0^\infty dy\ \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y} y^2 = \frac{r(r+1)}{\lambda^2} \int_0^\infty dy\ \frac{\lambda^{r+2}}{\Gamma(r+2)} y^{r+1} e^{-\lambda y}$$

$$\boxed{E(Y^2) = \frac{r(r+1)}{\lambda^2}}$$

- Variance:

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{r(r+1)}{\lambda^2} - \left(\frac{r}{\lambda}\right)^2$$

$$\boxed{\mathrm{Var}(Y) = \frac{r}{\lambda^2}}$$

The beta pdf I

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- Definition of the two-parameter beta pdf for $0 \leq \theta \leq 1$

$$\boxed{f_\Theta(\theta) = \frac{1}{B(r,s)} \theta^{r-1} (1-\theta)^{s-1}}$$

- Normalization follows immediately from definition of B.
- Expectation:

$$E(\Theta) = \frac{B(r+1,s)}{B(r,s)} = \frac{\Gamma(r+1)\Gamma(s)}{\Gamma(r+s+1)} \cdot \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{r\Gamma(r)\Gamma(r+s)}{(r+s)\Gamma(r+s)\Gamma(r)}$$

$$\boxed{E(\Theta) = \frac{r}{r+s}}$$

The beta pdf II

- Mean square:

$$E(\Theta^2) = \frac{B(r+2,s)}{B(r,s)} = \frac{\Gamma(r+2)\Gamma(s)}{\Gamma(r+s+2)} \cdot \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}$$
$$= \frac{r(r+1)\Gamma(r)\Gamma(r+s)}{(r+s)(r+s+1)\Gamma(r+s)\Gamma(r)}$$

$$\boxed{E(\Theta^2) = \frac{r(r+1)}{(r+s)(r+s+1)}}$$

- Variance:

$$\mathrm{Var}(\Theta) = E(\Theta^2) - [E(\Theta)]^2 = \frac{r(r+1)}{(r+s)(r+s+1)} - \left(\frac{r}{r+s}\right)^2$$

$$\boxed{\mathrm{Var}(\Theta) = \frac{rs}{(r+s)^2(r+s+1)}}$$

Sums of random variables

- $U, V \in \mathbb{R}$ are independent r.v.s with pdfs $f_U(u)$ & $f_V(v)$
- We wish to find pdf of sum $U + V$. Begin with cdf

$$\begin{aligned} F_{U+V}(t) =& P(U+V<t) \\ =& \int_{-\infty}^{+\infty} du \int_{-\infty}^{t-u} dv\; f_U(u) f_V(v). \end{aligned}$$

- Differentiating both sides with respect to t yields

$$\boxed{f_{U+V}(t) = \int_{-\infty}^{+\infty} du\; f_U(u) f_V(t-u).}$$

Sums of gamma-distributed r.v.s

- U gamma-distributed with parameters (r, λ)
- V gamma-distributed with parameters (s, λ)
- Then $f_{U+V}(t)$ is given by convolution

$$
\begin{aligned}
f_{U+V}(t) &= \int_{-\infty}^{+\infty} du\ f_U(u) f_V(t-u) \\
&= \int_0^t du\ \left[\frac{\lambda^r}{\Gamma(r)} u^{r-1} e^{-\lambda u}\right]\left[\frac{\lambda^s}{\Gamma(s)} (t-u)^{s-1} e^{-\lambda(t-u)}\right] \\
&= e^{-\lambda t} \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} \int_0^t du\ u^{r-1}(t-u)^{s-1} \\
&= e^{-\lambda t} \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)}\, t\ t^{r-1} t^{s-1} \int_0^1 dz\ z^{r-1}(1-z)^{s-1} \\
&= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} t^{r+s-1} B(r,s) e^{-\lambda t} \\
&= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} t^{r+s-1} \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} e^{-\lambda t} \\
&= \frac{\lambda^{r+s}}{\Gamma(r+s)} t^{r+s-1} e^{-\lambda t}
\end{aligned}
$$

Sums of gamma-distributed r.v.s

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- Hence if
  - U gamma-distributed with r df,
  - V gamma-distributed with s df,
- Then $U + V$ also gamma-distributed with $r + s$ df.

$$\boxed{f_{U+V}(t) = \frac{\lambda^{r+s}}{\Gamma(r+s)} t^{r+s-1} e^{-\lambda t}}$$

Summary

- We defined Γ and worked out key properties.
- We defined B and worked out key properties.
- We related B and Γ.
- We defined gamma pdf, and worked out key properties.
- We defined beta pdf, and worked out key properties.
- We showed sum of two gamma-distributed r.v.s is also gamma-distributed.
- Degrees of freedom are additive.