Math 0235 **Fall 2020** **(Take Home) Midterm Exam**

1 Instructions

• Complete all the five problems in Section 2 and only two problems of your choice from the problems listed in Section 3.
• You may discuss the problems with peers. You must, however, write up your own solutions.
• Show work and be rigorous within reason.
• List all the references you might use.
• The exam is due by 7:00 p.m. on Friday December 11, 2020.
• If you need hints, clarifications, etc..., do not hesitate to come and talk to me.
• Good Luck!

2 Complete all the five problems in this section

Problem 1 (3 points) Let f and g be real-valued functions on $[0,1]$ with the property that for every $x \in [0,1]$, g is differentiable at x and $g'(x) = (f(x))^2$.

(1-1) Prove that $g \in BV([0,1])$ and is an increasing function. Conclude that $f \in L^1_m([0,1])$ where m is the Lebesgue measure restricted to $[0,1]$.

(1-2) Suppose, in addition, that f is bounded on $[0,1]$. Prove that

$$2\int_0^1 gf^2 dm = g^2(1) - g^2(0).$$

Problem 2 (3 points) Denote the Lebesgue measure restricted to the interval $[0,1]$ by m. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $\sum_{n\geq 1} \sqrt{n} a_n^2 < \infty$. Let $\{f_n\}_{n\geq 1}$ be an orthonormal basis for $L^2_m([0,1])$. Define $S_n = \sum_{k=1}^n a_k f_k$ and $S = \sum_{k=1}^\infty a_k f_k$.

Prove the following statements:

(2-1) $S \in L^2_m([0,1])$.

(2-2) $\sum_{k=1}^\infty a_k f_k(x)$ converges almost everywhere on $[0,1]$.

(2-3) $\sum_{k\geq 1} \|S - S_{k^2}\|_2^2 < \infty$.

Problem 3 (2 points) Assume that $f \in AC[0,1]$ and there is a function g continuous on $[0,1]$ such that $f' = g$ a.e. Show that f is differentiable everywhere on $[0,1]$, and that $f'(x) = g(x)$ for all $x \in [0,1]$. Show by an example that the hypothesis of absolute continuity is necessary.

Problem 4 (3 points) Let m denote the Lebesgue measure on $\mathbb{R}^d$ and let $f \in L^p_m(\mathbb{R}^d)$, where $1 \leq p < \infty$. For $\alpha > 0$, define

$$E_\alpha(f) = \{x \in \mathbb{R}^d : |f(x)| > \alpha\}.$$

(4-1) Show that E_α has finite Lebesgue measure.

(4-2) Use (a) to show that every $f \in L^p_m(\mathbb{R}^d)$, $1 \le p \le 2$, can be decomposed as $f_1 + f_2$ where $f_1 \in L^1_m(\mathbb{R}^d)$ and $f_2 \in L^2_m(\mathbb{R}^d)$.

Problem 5 (3 points)
Let m denote the Lebesgue measure restricted to the interval $[0,1]$, and let $f : [0,1] \to [0,\infty]$ be Lebesgue integrable. Assume that

$$\int_0^1 f^n dm = \int_0^1 f\, dm \quad \textit{for} \quad n = 1,2,3,\ldots.$$

Let $E = \{x \in [0,1] : f(x) > 1\}$ and $F = \{x \in [0,1] : 0 < f(x) < 1\}$.

Prove that $m(E \cup F) = 0$ and conclude that $f = 0$ or $f = 1$ a.e.

3 Complete two problems of your choice from this group

Problem 6 (3 points)
Let m denote the Lebesgue measure restricted to the interval $[0,1]$. In each of the following, either explain why the given conditions imply that

$$\lim_{n\to\infty} \int_0^1 f_n\, dm = \int_0^1 f\, dm$$

or provide a counterexample.

(6-1) f_n is continuous on $[0,1]$ and $f_n \to f$ a.e. on $[0,1]$.

(6-2) f_n is continuous on $[0,1]$ and $f_n \to f$ uniformly on $[0,1]$.

(6-3) f_n and f are continuous on $[0,1]$, $f_n(x) \ge f_{n+1}(x) \ge 0$ for all x, and $f_n \to f$ a.e. on $[0,1]$.

(6-4) f_n is continuous on $[0,1]$, $f_n(x) \le f_{n+1}(x)$ for all x, and $f_n \to f$ a.e. on $[0,1]$.

(6-5) f_n is continuous on $[0,1]$, and $f_n \to f$ in measure.

Problem 7 (3 points)
Suppose that $f \in L^1([0,1], dm)$ where m denotes the Lebesgue measure on $[0,1]$. Assume that $f \ge 0$ and that $\int_0^1 f dm = 1$. Prove that there exists a measurable set $A \subset [0,1]$, such that

$$m(A) = 1/2 \qquad \textit{and} \qquad \int_A f dm = \tfrac{1}{2}.$$

Problem 8 (3 points) Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with $m(A) < \infty$. Set $\varphi(x) = m(A \cap (x + A))$ for each $x \in \mathbb{R}$. Prove that $\lim_{|x|\to\infty} \varphi(x) = 0$.