Tufts University
Department of Mathematics

Math 135 | **Homework 1 v. 3** [1] | **Fall 2022**

(Corrections in red)

Readings for Problem Set 1

Preliminaries: ”Logic and Proof” (notes), ”Proofs, Countability” (notes)
§**1.1:** Mathematical induction and the completeness axiom
§**1.2:** Density of rational and irrational numbers, pp. 15–16 only, starting with definition of “dense”

Readings for Problem Set 2

Notes:: Countability and Cantor’s diagonal argument
§**1.2:** Completeness
§**1.3:** Inequalities and identities
§**2.1:** The convergence of sequences

Problem Set 1
(Due Wednesday, September 14, 2022, 11:59 p.m. on Gradescope)

You may hand in your problem set one day late, by Thursday, 11:59 p.m., but if you do so, you lose 10% of your grade on the assignment. You are encouraged to work together with other students in the class, but the work you hand in must be your own. You may not copy answers from the internet or from other books or students.

1. (10 points) (**Logic**)
   (a) Let A and B be logical statements. Use truth tables to show that the statement “$\sim (A \Rightarrow B)$” is equivalent to “$A \wedge (\sim B)$.”
   (b) Find the negation of

   $$\text{“For some } x \in \mathbb{R},\ x^2 > 5 \text{ and } x < -3,\text{”}$$

   equivalently,

   $$\text{“}\exists x \in \mathbb{R} \text{ such that } (x^2 > 5) \wedge (x < -3).\text{”}$$

2. (10 points) (**Contrapositive, converse, equivalence**) Let $x \in \mathbb{R}$. Consider the statement: *“If $x \neq 0$, then $\exists \varepsilon > 0$ such that $|x| \geq \varepsilon$.”*
   (a) Write the contrapositive of this statement without using the word “not.”
   (b) Write the converse of this statement.
   (c) Let $x \in \mathbb{R}$. Prove the if and only if statement “$\forall \varepsilon > 0,\ |x| < \varepsilon$ *iff* $x = 0$” by proving both implications.

3. (10 points) (**Inf and sup**) Fitzpatrick, §1.1, p. 11, #13. Suppose that S is a nonempty set of real numbers that is bounded. Prove that $\inf S \leq \sup S$.

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4. (10 points) (**Irrationality**) Prove that $\sqrt{10}$ is not a rational number. You may use the unique factorization property of integers.

5. (10 points) (**Mathematical induction**) Prove by mathematical induction the summation formula for geometric series: For $r \in \mathbb{R}$, $r \neq 1$: $\displaystyle\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$.

6. (15 points) (**Subsets, equality of sets**) Let $f : A \to B$ and let C and D be subsets of B.
   (a) (7 points) Prove or provide a counterexample that $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$. To prove this, you need to show every element of $f^{-1}(C \cap D)$ is in $f^{-1}(C) \cap f^{-1}(D)$. To provide a counterexample, you give a *specific* A, B, $f : A \to B$ and subsets C and D of B, such that the assertion is not true for this specific A, B, C, D and f.
   (b) (7 points) Prove or provide a counterexample to the assertion $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$ and prove your answer. To prove the assertion, you need to show every element of $f^{-1}(C) \cap f^{-1}(D)$ is in $f^{-1}(C \cap D)$. To provide a counterexample, you give a *specific* A, B, $f : A \to B$ and subsets C and D of B, such that the assertion is not true for this specific A, B, C, D and f.
   (c) (1 point) Does $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ in general?

7. (10 points) (**Injection, surjection, image, preimage**) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - 3x^2$. You may, for purposes of this problem, use what you know from Calc I-II to determine the location of maxima and minima and intervals of monotonicity. Use the definition of f and the shape of the graph of f to answer the questions.
   (a) Is f injective? Why or why not?
   (b) Is f surjective? Why or why not?
   (c) What is $f([0, 3/2])$?
   (d) what is $f^{-1}([0, \infty))$?

(End of Problem Set 1)