1. (10 points) (**Continuity of the scalar product in the first variable**) §10.2, p. 281: # 1.
Let $\{\vec{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to the point $\vec{u}$. Prove that
$$\lim_{k\to\infty} \langle \vec{u}_k, \vec{v}\rangle = \langle \vec{u}, \vec{v}\rangle.$$

Solution. Since $\vec{u}_k \to \vec{u}$, by the componentwise convergence criterion, $u_k^i \to u^i$ for $i = 1, \dots, n$. Then

$$\begin{aligned} \lim \langle \vec{u}_k, \vec{v}\rangle &= \lim \sum u_k^i v^i \\ &= \sum u^i v^i \quad \text{(sum and product rules)} \\ &= \langle \vec{u}, \vec{v}\rangle. \end{aligned}$$
$\square$

Alternative Solution.

$$\begin{aligned} &|\langle \vec{u}_k, \vec{v}\rangle - \langle \vec{u}, v\rangle| \\ &= |\langle \vec{u}_k - \vec{u}, \vec{v}\rangle| \quad \text{(linearity in the first argument)} \\ &\le \|\vec{u}_k - \vec{u}\| \, \|\vec{v}\| \quad \text{(by Cauchy-Schwarz inequality)} \end{aligned}$$

As $k \to \infty$, $\|\vec{u}_k - \vec{u}\| \to 0$.

By the sandwich lemma, $\lim |\langle \vec{u}_k, \vec{v}\rangle - \langle \vec{u}, \vec{v}\rangle| = 0$.

By HW 5, #5, we may remove the absolute value in a limit converging to 0, so

$$\lim \langle \vec{u}_k, \vec{v}\rangle - \langle \vec{u}, \vec{v}\rangle = 0$$

or

$$\lim \langle \vec{u}_k, \vec{v}\rangle = \langle u, \vec{v}\rangle.$$
$\square$

2. (10 points) (**Cauchy sequences in** $\mathbb{R}^n$) §10.2, p. 282: # 8.
A sequence of points $\{\vec{u}_k\}$ in $\mathbb{R}^n$ is said to be *Cauchy* provided that $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $\forall k, \ell \geq K$,

$$\operatorname{dist}(\vec{u}_k, \vec{u}_\ell) < \varepsilon.$$

(a) Prove that $\{\vec{u}_k\}$ is Cauchy if and only if each component sequence is Cauchy.
(b) Prove that a sequence in $\mathbb{R}^n$ converges if and only if it is Cauchy. (*Hint*: For sequences of real numbers, this was proved in Section,9.1.)

a) For any $u = (u^1, \ldots, u^n)$ in $\mathbb{R}^n$,

$$|u^i| \leq \|u\| = \sqrt{(u^1)^2 + \cdots + (u^n)^2}.$$

Thus,

$$|u_k^i - u_\ell^i| \leq \|u_k - u_\ell\| = \sqrt{(u_k^1 - u_\ell^1)^2 + \cdots + (u_k^n - u_\ell^n)^2}$$

Suppose $\{u_k\}$ is Cauchy in $\mathbb{R}^n$. Let $\varepsilon > 0$. Then $\exists K$ such that $\forall k, \ell \geq K$ and $\forall i = 1, \ldots, n$,

$$|u_k^i - u_\ell^i| \leq \|u_k - u_\ell\| < \varepsilon$$

Hence, $\{u_k^i\}$ is Cauchy.

Conversely, suppose $\{u_k^i\}$ is Cauchy. Let $\varepsilon > 0$. $\exists K_i \in \mathbb{N}$ such that $\forall k, \ell \geq K_i$,

$$|u_k^i - u_\ell^i| < \frac{\varepsilon}{\sqrt{n}} \qquad (*)$$

Choose $K = \max(K_1, \ldots, K_n)$. Then for all $k, \ell \geq K$, $(*)$ is true for all $i = 1, \ldots, n$. Hence, $\forall k, \ell \geq K$,

$$\|u_k - u_\ell\| \leq \sqrt{\sum (u_k^i - u_\ell^i)^2} < \sqrt{\sum \frac{\varepsilon^2}{n}} = \varepsilon.$$

This implies that $\{u_k\}$ is Cauchy. $\square$

(b) $\{\vec{u}_k\}$ in $\mathbb{R}^n$ converges to $\vec{u}$
iff $\forall i = 1, \ldots, n$, the ith component u_k^i converges to u^i
iff $\forall i = 1, \ldots, n$, $\{u_k^i\}$ is Cauchy in $\mathbb{R}$
iff $\{\vec{u}_k\}$ is Cauchy (by part (a)). $\square$

3. (10 points) (**Open and closed sets**) §10.3, p. 288: # 2. For each of the following subsets A of $\mathbb{R}^2$, state whether it is open in $\mathbb{R}^2$, closed in $\mathbb{R}^2$, or neither open nor closed in $\mathbb{R}^2$. Justify your conclusions with the aid of pictures. (You will learn later a rigorous way to justify your conclusions using continuity.)

(a) $A = \{\vec{u} = (x,y) \mid x^2 > y\}$.

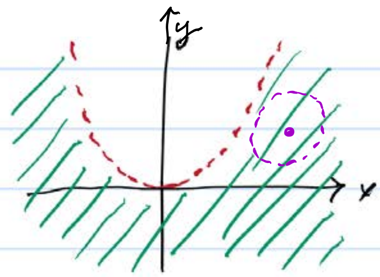
(b) $A = \{\vec{u} = (x,y) \mid x^2 + y^2 = 1\}$.

(c) $A = \{\vec{u} = (x,y) \mid x \text{ is rational}\}$.

(d) $A = \{\vec{u} = (x,y) \mid x \geq 0,\ y \geq 0\}$.

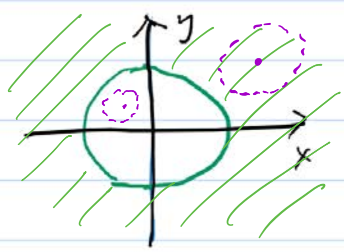
a. $A = \{u = (x,y) \mid x^2 > y\}$

Open, because every point in A is an interior point.



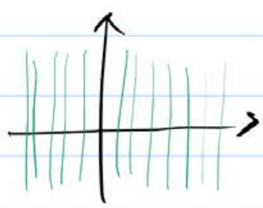
b. $A = \{u = (x,y) \mid x^2 + y^2 = 1\}$

Closed, because the complement is open



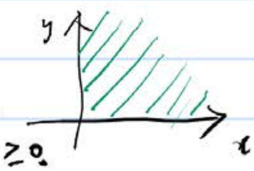
c. $A = \{u = (x,y) \mid x \text{ is rational}\}$

Neither open nor closed, because no open ball is contained in A or $\mathbb{R}^2 \setminus A$.



d. $A = \{u = (x,y) \mid x \geq 0,\ y \geq 0\}$

Closed: if $(x_k, y_k) \in A$ and $(x_k, y_k) \to (x, y)$, then $x_k \geq 0$, $y_k \geq 0$, so by Lemma 2.21, $x \geq 0$, $y \geq 0$.



4. (10 points) (**A closed ball is closed**)

(a) (9 pts)§10.3, p. 288: # 3.

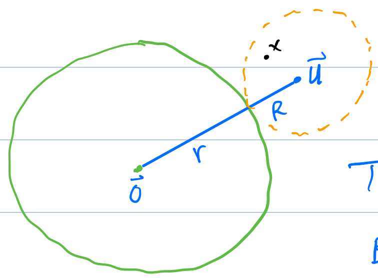
Let r be a positive number and define

$$\mathscr{O} = \{u \in \mathbb{R}^n \mid \|u\| > r\}.$$

Prove that $\mathscr{O}$ is open in $\mathbb{R}^n$ by showing that every point in $\mathscr{O}$ is an interior point of $\mathscr{O}$. (*Hint*: Draw a picture. For each $u \in \mathscr{O}$, let $R = \|u\| - r$ and prove that $B_R(u) \subset \mathscr{O}$.)

(b) (1 pt) Let r be a positive number and define $F = \{u \in \mathbb{R}^n \mid \|u\| \le r\}$. Prove that F is closed.

(a)

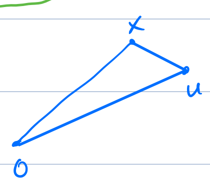


Let $u \in \mathscr{O}$ and $R = \|u\| - r$.

To prove $B_R(\vec{u}) \subset \mathscr{O}$, let $x \in B_R(u)$.

Then $\|u - x\| < R$.

By the triangle inequality,



$$\cancel{\|u\|} \le \|u - x\| + \|x\|$$

$$< R + \|x\| = \cancel{\|u\|} - r + \|x\|,$$

$$r < \|x\|.$$

Hence, $x \in \mathscr{O}$. This proves that $B_R(u) \subset \mathscr{O}$, so

u is an

(b) Since $F^c = \mathscr{O} = \{u \in \mathbb{R}^n \mid \|u\| > r\}$ is open,

F is closed. $\square$

5. (10 points) (**Closure**) §10.3, p. 289, # 12. For a subset A of $\mathbb{R}^n$, the *closure* of A, denoted by $\mathrm{cl}A$, is defined by

$$\mathrm{cl}A = \mathrm{int}A \cup \mathrm{bd}A.$$

Prove that

(a) $A \subset \mathrm{cl}A$;

(b) $A = \mathrm{cl}A$ if and only if A is closed in $\mathbb{R}^n$.

(a) $\boxed{A \subset \mathrm{cl}A}$

The set A can be decomposed as

$$A = \mathrm{int}A \cup (A \cap \mathrm{bd}A)$$

because a point of A is either an interior point or a boundary point. Hence,

$$A = \mathrm{int}A \cup (A \cap \mathrm{bd}A) \subset \mathrm{int}A \cup \mathrm{bd}A = \mathrm{cl}A.$$

(a) Alternative Proof. Since A is disjoint from $\mathrm{ext}A$, from $\mathbb{R}^n = \mathrm{int}A \cup \mathrm{bd}A \cup \mathrm{ext}A$, we can conclude that

$$A \subset \mathrm{int}A \cup \mathrm{bd}A = \mathrm{cl}A.$$

(b) $\boxed{A = \mathrm{cl}A \text{ if and only if } A \text{ is closed in } \mathbb{R}^n.}$

($\Rightarrow$) Suppose $A = \mathrm{cl}A = \mathrm{int}A \cup \mathrm{bd}A$. Since $A \supset \mathrm{bd}A$, by Proposition 10.19 (ii), A is closed.

($\Leftarrow$) Suppose A is closed. Then $A \supset \mathrm{bd}A$.

Then $A \supset \mathrm{int}A \cup \mathrm{bd}A = \mathrm{cl}A$.

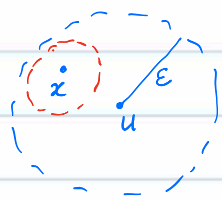
By (a), $A \subset \mathrm{cl}A$. This proves that $A = \mathrm{cl}A$. □

(b)($\Rightarrow$) Alternative proof. $A = \mathrm{cl}A$ iff $A^c = \mathrm{ext}A$, which is open by #6 (b). Hence, A is closed. □

6. (10 points) (**Interior, exterior, and boundary**) §10.3, p. 289, # 13.

(a) Prove that $\text{int}A$ is an open subset of $\mathbb{R}^n$.
(b) Use (a) to show that $\text{ext}A$ is also an open subset of $\mathbb{R}^n$.
(c) Use (a) and (b) together with the decomposition (10.11) on p. 287 to show that $\text{bd}A$ is a closed subset of $\mathbb{R}^n$.

(a) Let $u \in \text{int}A$. Then $\exists\, \varepsilon > 0$ such that $B(u,\varepsilon) \subseteq A$.
We need to show that $B(u,\varepsilon) \subseteq \text{int}A$.
Let $x \in B(u,\varepsilon)$. Since $B(u,\varepsilon)$ is open,
$\exists\, \delta > 0$ such that $B(x,\delta) \subseteq B(u,\varepsilon) \subseteq A$.
Therefore, x is also an interior point of A.
This proves that $B(u,\varepsilon) \subseteq \text{int}A$;
so $\text{int}A$ is open.



(b) By definition, $\text{ext}A = \text{int}(\mathbb{R}^n \setminus A)$.
By (a), $\text{ext}A$ is open in $\mathbb{R}^n$.

(c) By (10.11), $\mathbb{R}^n = \text{int}A \cup \text{ext}A \cup \text{bd}A$.
So $(\text{bd}A)^c = \text{int}A \cup \text{ext}A$.
By (a) and (b), $\text{int}A$ and $\text{ext}A$ are both open.
Since the union of open sets is open, $(\text{bd}A)^c$ is open.
Therefore, $\text{bd}A$ is closed. $\square$

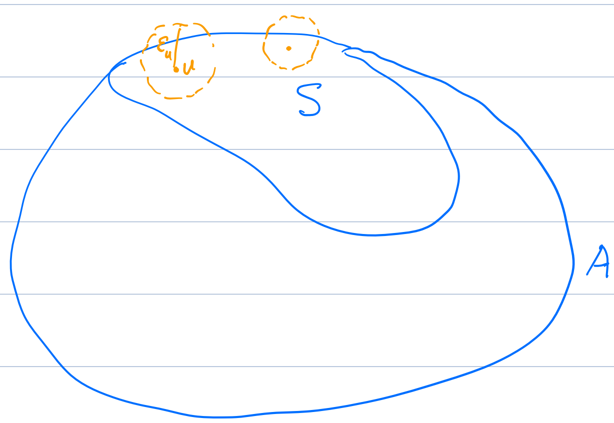
7. (10 points) (**Relatively open sets**) Prove that a set $S \subset A$ is relatively open in A if and only if for every u in S, there is an $\varepsilon > 0$ such that the relatively open ball $B_A(u, \varepsilon)$ is contained in S.

Proof. ($\Rightarrow$) Suppose S is relatively open.

Then $S = U \cap A$ for some open set U in $\mathbb{R}^n$.

Let $u \in S$. Then $u \in U$. Since U is open, $\exists\, \varepsilon > 0$ such that $B(u, \varepsilon) \subseteq U$. Then

$$B_A(u, \varepsilon) = B(u, \varepsilon) \cap A \subseteq U \cap A = S.$$



($\Leftarrow$) Suppose $\forall u \in S$, $\exists\, \varepsilon_u > 0$ such that $B_A(u, \varepsilon_u) \subseteq S$.

Then $B_A(u, \varepsilon_u) = B(u, \varepsilon_u) \cap A$.

Let $U = \bigcup_{u \in S} B(u, \varepsilon_u)$.

Then U is open because it is a union of open balls.

$S \subseteq U$ because every $u \in S$ is in $B(u, \varepsilon_u) \subseteq U$.

Since $S \subseteq A$, $S \subseteq U \cap A$.

On the other hand,
$$U \cap A = \Big(\bigcup_{u \in S} B(u, \varepsilon_u) \Big) \cap A = \bigcup \big(B(u, \varepsilon_u) \cap A \big) \subseteq S.$$

Therefore, $S = U \cap A$, which shows that S is relatively open in A. $\square$