MATH 42 HOMEWORK 10 SOLUTIONS

Here are some general guidelines that may be helpful heading into the final – these are in some ways based on things noticed on homeworks and previous midterms during the course. On homework or an exam, **your goal should be to convince the grader that you fully understand the material**. Note that this burden increases when access to outside materials is granted. To this end:

- On an open-book, open-internet exam with access to Wolfram-Alpha and other powerful tools, **a correct numerical answer is not enough to convince the grader of complete command of the material**. Every single algebraic step need not be shown, but if u- or $trig$ substitution is used to evaluate an integral, give the substitutions. In short, **show your work!**

- **Avoid stream-of-consciousness math.** We tend to work in this way on scratch work, but it often leads to poor notation if not wholly incorrect statements, like integrals without dx's and equal symbols between scratch work. In time pressure, things can get messy, but it is generally good practice to do computations on scratch work and take the extra few minutes to write it out a bit cleaner for the final submission.

- **Use your words.** A couple of added words here and there between steps goes a long way in clarifying your process. In particular, **explicitly name a theorem when it is used**, and for complete rigor, be sure to check that the conditions under which the theorem may be applied are being met.

- **When plotting or graphing, label everything.** This one doesn't need any explaining.

SOLUTIONS

(1) **Let S be a surface parameterized by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ for some domain D in the two variables u, v.**

(a) **If $f(x,y,z)$ is a (scalar) function, write $\int\int_S f(x,y,z)\,dS$ as an integral in u, v. What does the dS mean here? How do you compute it?**

Let's start by thinking about what this integral is actually doing. We can think of f as a (surface) density, and the integral over the surface will tell us some amount(mass, temperature, etc.). For example, let's say we're constructing a building made entirely of windows, and each window is a little square tile. So, the building will be constructed with window tiles. If a storm is coming, we'll need to know the total force that the building can withstand without falling down. If f represents the amount of force that each pane can withstand (so it's a force/area unit), summing over all the tiles, $f\times$ area of the tile will give me the total force that the building can withstand. Now, suppose instead of window tiles, we want the building to be constructed so each side is just a wall of glass and we want to know the total force the building can withstand. This is equivalent to computing the surface integral of f (this all assumes that the joints between window panes are just as strong as the material of the windows themselves).

Using this analogy, we want f to represent some physical quantity (like the force/area), and dS to represent the area of each tile. More generally, we want dS to represent some small change in my surface, so a small change in area. To compute this is a bit more complicated than computing a small change of a curve, like the previous questions with line integrals. As we just saw with line integrals, given a curve $\vec{r}(t)$, we know $\vec{r}\,'(t)$ is tangent to the curve. Now, since we have a surface parameterized by u and v, it makes sense that $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are both tangent to the surface. Thinking way back to the beginning of the semester, given two vectors $\vec{a}$ and $\vec{b}$, we know that $|\vec{a} \times \vec{b}|$ gives the area of the parallelogram made by the two vectors. This means that $\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|$ will give us the area of a parallelogram on the surface. So,

$$dS = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dA$$

where $dA = dudv$ or $dA = dvdu$. The integral can be re-written

$$\int\int_S f(x, y, z)\, dS = \int\int_S f(u, v) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dA$$

(b) **If $\vec{F}$, is a vector field, write $\int\int_S \vec{F} \cdot \vec{n}\, dS$ as an integral in u, v. What is $\vec{n}$? Is the dS the same as before? How do we compute it in terms of u, v?**

If $\vec{F}$ is a vector field, the surface integral is computing the *flux* through the surface. In other words, $\vec{F} \cdot \vec{n}$ is the projection of $\vec{F}$ onto the normal vector to the surface. The surface integral gives us the amount of the vector field passing through the surface. We know $\vec{n}$ should be the unit normal vector, and we now need to figure out how to compute it. We know that $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are tangent to the surface, so if we take the cross product, it will give us a vector perpendicular to both of those. This mean the cross product will be perpendicular to the surface, or normal to the surface. To make it a unit vector, we simply divide by the magnitude.

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}$$

We are still integrating over the surface, so we still want dS to be a small change in surface area, $dS = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dA$. Putting these together, we re-write the surface integral as

$$\iint_S \vec{F} \cdot \vec{n} dS = \int\int_S \vec{F}(u, v) \cdot \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|} \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dA = \int\int_S \vec{F}(u, v) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) dA$$

The vector quantity $d\vec{S}$ is just $d\vec{S} = \vec{n} dS$, which we can think of as the unit normal vector scaled by the small change in area on the surface.

(2) **Can you parametrize the following surfaces? Can you give more than one answer to each?**

(a) **The surface $z = x + y$ for $1 \leq x \leq 2, 1 \leq y \leq 3$**

Because this surface is defined explicitly, we can parametrize with $x = u$, $y = v$, and $z = u + v$; that is,

$$\text{Explicit:} \quad \vec{r}(u, v) = \langle u, v, u + v\rangle \text{ for } \begin{array}{l} 1 \le u \le 2, \\ 1 \le v \le 3. \end{array}$$

There are many different ways to parametrize this plane, by let's see if we can parametrize in such a way that $0 \le u \le 1$ and $0 \le v \le 1$. For u, we map the original interval $[1, 2]$ to $[0, 1]$ by $u \to u - 1$. For v, we map the original interval $[1, 3]$ to $[0, 1]$ by $v \to \frac{v-1}{2}$. We can parametrize as follows:

$$\text{Explicit, version 2:} \quad \vec{r}(u, v) = \left\langle u - 1, \frac{v-1}{2}, u + \frac{v}{2} - \frac{3}{2} \right\rangle \text{ for } \begin{array}{l} 0 \le u \le 1, \\ 0 \le v \le 1. \end{array}$$

(b) **The portion of the paraboloid $z = x^2 + y^2$ that lies over the unit disk.**

Because the paraboloid is defined explicitly, we can parametrize with $x = u$, $y = v$, and $z = u^2 + v^2$. Because the paraboloid lies over the unit disk, we restrict the parameters such that $u^2 + v^2 \le 1$; that is,

$$\vec{r}(u, v) = \langle u, v, u^2 + v^2\rangle \text{ for } u^2 + v^2 \le 1.$$

Alternatively, because the paraboloid lies over the unit disk, we can parameterize using polar coordinates. This is actually the more natural choice. Let $x = v\cos(u)$ and $y = v\sin(u)$. Then, $z = x^2 + y^2 = v^2$. We get the following parametrization:

$$\text{Polar:} \quad \vec{r}(u, v) = \langle v\cos(u), v\sin(u), v^2\rangle \text{ for } \begin{array}{l} 0 \le u \le 2\pi, \\ 0 \le v \le 1. \end{array}$$

(c) **The portion of the paraboloid $z = x^2 + y^2$ that lies over the annulus centered at the origin with radius ranging from 1 to 2.**

Because the paraboloid is defined explicitly, we can parametrize with $x = u$, $y = v$, and $z = u^2 + v^2$. Because the paraboloid lies over the annulus, we restrict the parameters such that $1 \le u^2 + v^2 \le 2$; that is,

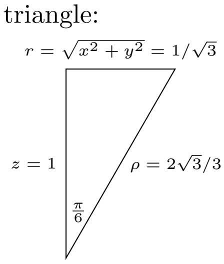
$$\text{Explicit:} \quad \vec{r}(u, v) = \langle u, v, u^2 + v^2\rangle \text{ for } 1 \le u^2 + v^2 \le 2.$$

Alternatively, because the paraboloid lies over the annulus, we can parameterize using polar coordinates. This is actually the more natural choice. Let $x = v\cos(u)$ and $y = v\sin(u)$. Then, $z = x^2 + y^2 = v^2$. We get the following parametrization:

$$\text{Polar:} \quad \vec{r}(u, v) = \langle v\cos(u), v\sin(u), v^2\rangle \text{ for } \begin{array}{l} 0 \le u \le 2\pi, \\ 1 \le v \le 2. \end{array}$$

(d) **A cone with vertex at the origin, opening up around the z-axis up to $z = 4$, with angle between the z-axis and the cone $\pi/6$.**

Let's first define this cone explicitly. The simplest elliptic cone is defined as $z^2 = x^2 + y^2$. We just have to make sure we take the angle $\pi/6$ into consideration. Consider the case when $z = 1$, we get the following triangle:



Therefore, the cone defined explicitly is $z^2 = 3(x^2+y^2)$. We can parametrize an explicit function with $x = u$, $y = v$, and $z = \sqrt{3(u^2+v^2)}$. To restrict our parameters, we look at the case when $z = 4$. This would be when $16 = 3(x^2+y^2)$. Therefore, our parameters live in a disk $u^2 + v^2 \leq 16/3$. We get the following parametrization:

$$\text{Explicit:} \quad \vec{r}(u,v) = \langle u, v, \sqrt{3(u^2+v^2)}\rangle \text{ for } u^2+v^2 \leq 16/3.$$

Clearly, this was not the most natural choice of parameterizing this cone. Instead, let's use cylindrical coordinates as our motivation. The simplest cone in cylindrical coordinates is $z = r$ (i.e., the radius expands as the height increases). Because $r = \sqrt{x^2+y^2}$, the cone with the angle $\pi/6$ is $z = \sqrt{3}r$. Thus, we can parametrize $x = v\cos(u)$, $y = v\sin(u)$, and $z = \sqrt{3}v$ where v ranges from 0 to the $\sqrt{16/3}$ and get the following parametrization:

$$\text{Cylindrical:} \quad \vec{r}(u,v) = \langle v\cos(u), v\sin(u), \sqrt{3}v\rangle \text{ for } \begin{array}{l} 0 \leq u \leq 2\pi, \\ 0 \leq v \leq 4\sqrt{3}/3. \end{array}$$

One other way to parametrize this cone is to use spherical coordinates. The cone can be parametrized in spherical coordinates as $\varphi = \pi/6$ and we only vary the parameters ρ and θ. Typically, the parametrize in spherical coordinates, $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$. Because $\varphi = \pi/6$, we get $x = 1/2\rho\cos\theta$, $y = 1/2\rho\sin\theta$, and $z = \sqrt{3}/2\rho$. We restrict ρ to range from 0 to $8\sqrt{3}/3$ (i.e., the hypotenuse of the triangle when $z = 4$):

$$\text{Spherical:} \quad \vec{r}(\rho,\theta) = \langle 1/2\rho\cos\theta, 1/2\rho\sin\theta, \sqrt{3}/2\rho\rangle \text{ for } \begin{array}{l} 0 \leq \rho \leq 8\sqrt{3}/3, \\ 0 \leq \theta \leq 2\pi. \end{array}$$

(3) **Evaluate the surface integral $\iint_S \vec{F}\cdot\vec{n}\,dS$ where $\vec{F} = 3x^2\vec{i} - 2xy\vec{j} + 8\vec{k}$ and S is the graph of the function $f(x,y) = 2x - y$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$, oriented so that the normal vectors point upward (i.e., in the positive z direction).**

When we have a surface where one variable may be explicitly defined in terms of the others, i.e. $z = f(x,y)$, we may find the normal vector simply by computing $\nabla g(x,y,z)$ where $g = z - f$. In other words, we have that $\nabla g = \langle -f_x, -f_y, 1\rangle = \langle -2, 1, 1\rangle$. Then we have that

$$\begin{aligned}
\int\int_S \vec{F}\cdot\vec{n}\,dS &= \int\int_S \vec{F}\cdot\frac{\nabla g}{\|\nabla g\|}dS \\
&= \int\int_S \vec{F}\cdot\frac{\nabla g}{\|\nabla g\|}\|\nabla g\|dxdy \\
&= \int_0^2\int_0^2 \langle 3x^2, -2xy, 8\rangle\cdot\langle -2,1,1\rangle\,dxdy \\
&= \int_0^2\int_0^2 -6x^2 - 2xy + 8\,dxdy \\
&= \int_0^2 -2(2)^3 - (2)^2y + 16\,dy \\
&= -8.
\end{aligned}$$

(4) **Compute $\iint_S \vec{F}\cdot\vec{n}\,dS$ where $\vec{F} = \langle y, -x, z\rangle$ and the surface S is given by the parametrization $\vec{r}(u,v) = \langle 2u, 2v, 5 - u^2 - v^2\rangle$, with $u^2 + v^2 \leq 1$. (Note that the parametrization allows us to orient the surface S, since the unit normals to S are proportional to the cross products $\vec{r}_u \times \vec{r}_v$.) Describe the direction of the**

net flux of $\vec{F}$ across the surface.

First, we need to compute the normal vector using the two tangent vectors $\vec{r}_u$ and $\vec{r}_v$. We have

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -2u \\ 0 & 2 & -2v \end{vmatrix} = \langle 4u, 4v, 4 \rangle$$

from which we may compute

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n}\, dS &= \iint_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv \\
&= \iint_S \langle 2v, -2u, 5 - u^2 - v^2 \rangle \cdot \langle 4u, 4v, 4 \rangle\, dudv \\
&= \iint_S 8uv - 8uv + 20 - 4u^2 - 4v^2\, dudv \\
&= \int_0^{2\pi} \int_0^1 20r - 4r^3 r\, drd\theta \\
&= 2\pi(10r^2 - r^4)\Big|_{r=0}^{r=1} \\
&= 18\pi
\end{aligned}$$

(5) **Find the average value of the temperature function $T(x,y,z) = 80 - 24z^2$ on the surface S defined by the cone: $z^2 = x^2 + y^2$ for $0 \le z \le 3$. The average temperature is found by adding up (integrating) all of the "temperature" and dividing by the surface area.**

Using $z = f(x,y) = \sqrt{x^2 + y^2}$, we may compute the average temperature as follows:

$$\frac{\iint_S T(x,y,z)\, dS}{\iint_S 1\, dS} = \frac{\iint_S T(x,y,f(x,z))\sqrt{f_x^2 + f_y^2 + 1} dA}{\iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA}$$

Let's do these separately. Adding up all of the"temperature" (internal energy) gives

$$\begin{aligned}
\iint_S T(x,y,f(x,z))\sqrt{f_x^2 + f_y^2 + 1}\, dA &= \iint_S (80 - 24(x^2 + y^2))\sqrt{\left(\tfrac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\tfrac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\, dA \\
&= \sqrt{2} \int_0^{2\pi} \int_0^3 80r - 24r^3\, drd\theta \\
&= 2\sqrt{2}\pi \left(40r^2 - 6r^4\right)\Big|_{r=0}^{r=3} \\
&= -256\sqrt{2}\pi
\end{aligned}$$

Now to compute surface area:

$$\begin{aligned}
\iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA &= \sqrt{2} \int_0^{2\pi} \int_0^3 r\, drd\theta \\
&= 9\sqrt{2}\pi
\end{aligned}$$

and therefore the average temperature is $-256/9 = -28$.

(6) **Compute the flux $\iint_S \vec{F}\cdot\vec{n}\,dS$ over the hemisphere S given by $x^2+y^2+z^2=9$, $x^2+z^2\leq 9$, and $y\geq 0$, where $\vec{F}=x\vec{i}+y^4\vec{j}+z\vec{k}$. Here S is oriented so that the normal vectors point rightward (i.e., in the positive y direction). Before evaluation, make an educated guess as to whether or not this integral will be positive, zero, or negative, based on the vector field and the geometry of the region.**

We have multiple ways of solving this problem. One is to cap the region and use the divergence theorem, while another is to evaluate the flux integral directly (one could also try to identify $\vec{F}$ as the flux of some vector field G and use Stokes' theorem). We will show two approaches.

Let's start by integrating directly. To do this, we need to find our normal vector. Rather than parametrizing the surface, we take notice that y can be written as a function of x and z on the entire surface, namely $y=f(x,y)=\sqrt{9-x^2-z^2}$. Then, the gradient of the function $g(x,y,z)=y-f(x,y)$ should give us a normal vector:

$$\nabla g=\langle -f_x,1,-f_z\rangle=\left\langle \frac{x}{\sqrt{9-x^2-z^2}},1,\frac{z}{\sqrt{9-x^2-z^2}}\right\rangle$$

(Hopefully this looks familiar from way back when we were finding tangent planes to surfaces, and characterizing the tangent planes by their normal vectors). Now, we have a double integral since we are integrating over a surface, and since y is everywhere defined by x and z, we will be integrating over a region in the xz plane. Therefore we have that

$$\begin{aligned}
\int\int_S \vec{F}\cdot\vec{n}dS &= \int\int_S \vec{F}\cdot\frac{\nabla g}{\|\nabla g\|}dS\\
&= \int\int_S \vec{F}\cdot\frac{\nabla g}{\|\nabla g\|}\|\nabla g\|\,dxdz\\
&= \int\int_S \vec{F}\cdot\nabla g\,dxdz\\
&= \int\int_S \langle x,(9-x^2-z^2)^2,z\rangle\cdot\left\langle \frac{x}{\sqrt{9-x^2-z^2}},1,\frac{z}{\sqrt{9-x^2-z^2}}\right\rangle dxdz\\
&= \int\int_S \frac{x^2+z^2}{\sqrt{9-x^2-z^2}}+(9-x^2-z^2)^2\,dxdz\\
&= \int_{-\pi}^{\pi}\int_0^3 \frac{r^3}{\sqrt{9-r^2}}+r(9-r^2)^2\,drd\theta
\end{aligned}$$

Ok, let's split this up and consider each term separately. The first component requires either some algebraic cleverness, or u-substitution with $u=9-r^2$ (and remembering to change the bounds for u!). We have

$$\begin{aligned}
\int_{-\pi}^{\pi}\int_0^3 \frac{r^3}{\sqrt{9-r^2}}\,drd\theta &= \int_{-\pi}^{\pi}\int_0^9 \frac{1}{2}\frac{9-u}{\sqrt{u}}\,dud\theta \\
&= \int_{-\pi}^{\pi}\left(9\sqrt{u}-\frac{1}{3}u^{3/2}\Big|_0^9\right)\,d\theta \\
&= \int_{-\pi}^{\pi} 18\pi\,d\theta \\
&= 36\pi
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\pi}^{\pi}\int_0^3 r(9-r^2)^2\,drd\theta &= \int_{-\pi}^{\pi}\left(-\frac{1}{6}(9-r^2)^3\Big|_0^3\right)\,d\theta \\
&= \int_{-\pi}^{\pi}\frac{243}{2}\,d\theta \\
&= 243\pi
\end{aligned}$$

and therefore we have that

$$\int\int_S \vec{F}\cdot\vec{n}\,dS \;=\; 243\pi+36\pi \;=\; 279\pi.$$

Now let's try it with the divergence theorem. The first thing to do is to cap our region with the disk $x^2+z^2\le 9$ in the xz plane. Let's call this surface S_2 and the part of the sphere we'll call S_1. Then by the divergence theorem

$$\int\int_{S_1}\vec{F}\cdot\vec{n}\,dS = \int\int\int_R \nabla\cdot\vec{F}\,dV - \int\int_{S_2}\vec{F}\cdot\vec{n}\,dS$$

We want to then compute both integrals on the right-hand side. Let's start with the surface integral over S_2. In the xz plane, $y=0$, and therefore $\vec{F}$ when restricted to the xz plane is $\langle x,0,z\rangle$. Now, the outward unit normal vector on this surface will be in the negative y direction, so $\vec{n}=\langle 0,-1,0\rangle$. This is orthogonal to $\vec{F}$ on S_2! In other words, the vector field $\vec{F}$ on the disk has only arrows along the disk, not pointing off of the disk. Thus, it must be true that

$$\int\int_{S_1}\vec{F}\cdot\vec{n}\,dS = \int\int\int_R \nabla\cdot\vec{F}\,dV$$

So now we want to compute the volume integral. The set-up is easy, but the execution takes some work.

$$\begin{aligned}
\int\int\int_R \nabla\cdot\vec{F}\,dV &= \int\int\int_R 1+4y^3+1\,dV \\
&= \int_0^{\pi}\int_0^{\pi}\int_0^3 (2\rho^2\sin\phi+4\rho^5\sin^4\phi\sin^3\theta)\,d\rho d\phi d\theta \\
&= \int_0^{\pi}\int_0^{\pi}\left(\frac{2}{3}\rho^3\sin\phi+\frac{4}{6}\rho^6\sin^4\phi\sin^3\theta\Big|_0^3\right)\,d\phi d\theta \\
&= \int_0^{\pi}\int_0^{\pi} 18\sin\phi+486\sin^4\phi\sin^3\theta\,d\phi d\theta
\end{aligned}$$

The first term here is a piece of cake, the second not so much, so let's tackle that one. This requires remembering how to integrate even and odd powers of sin, so let's look at these separately. For higher even powers, we have to make use of the trig identities involving squares of sin and cos:

$$\begin{aligned}
\int_0^{\pi} \sin^4 \phi \, d\phi &= \int_0^{\pi} (\sin^2 \phi)^2 \, d\phi \\
&= \int_0^{\pi} \left(\frac{1}{2} - \frac{\cos(2\phi)}{2}\right)^2 \, d\phi \\
&= \int_0^{\pi} \left(\frac{1}{4} - \frac{\cos(2\phi)}{2} + \frac{\cos^2(2\phi)}{4}\right) \, d\phi \\
&= \int_0^{\pi} \left(\frac{1}{4} - \frac{\cos(2\phi)}{2} + \frac{1}{4}\left(\frac{1}{2} + \frac{\cos(4\pi)}{2}\right)\right) \, d\phi \\
&= \frac{3}{8} - \frac{\sin(2\phi)}{4} + \frac{\sin(4\phi)}{32}\bigg|_0^{\pi} \\
&= \frac{3}{8}\pi
\end{aligned}$$

Phew. Now let's go back to the actual computation, and we'll work through the odd power of sin using the easier-to-remember identity $\cos^2 x + \sin^2 x = 1$ and some u substitution ($u = \cos\theta$).

$$\begin{aligned}
\int_0^{\pi} \int_0^{\pi} 18 \sin\phi + 486 \sin^4 \phi \sin^3 \theta \, d\phi d\theta &= \int_0^{\pi} 36 + \frac{729\pi}{4} \sin^3 \theta \, d\theta \\
&= \int_0^{\pi} 36 + \frac{729\pi}{4} (1 - \cos^2 \theta) \sin\theta \, d\theta \\
&= \int_0^{\pi} 36 \, d\theta + \int_{-1}^{1} \frac{729\pi}{4} (1 - u^2) \, du \\
&= 36\pi + \frac{729\pi}{4} \left(u - \frac{u^3}{3}\bigg|_{-1}^{1} \right) \\
&= 36\pi + \frac{729\pi}{4}\frac{4}{3} \\
&= 279\pi.
\end{aligned}$$