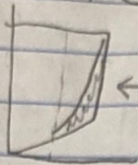
Math 165 HW 9

1 a) $P(X<1) = \frac{\text{area } x<1}{\text{area trapezoid}} = \frac{\frac{1}{2}(1)(3+4)}{\frac{1}{2}(2)(4+2)} = \boxed{\frac{7}{12}}$

b) $P(Y<X^2)$. $Y<X^2$ true on $(1,2)$



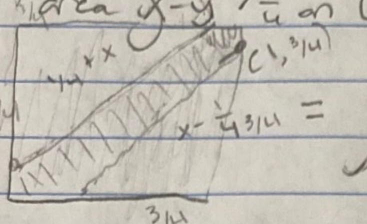
$\leftarrow$ this area $= \int_1^2 x^2 - x \, dx = \frac{x^3}{3} - \frac{x^2}{2} \Big|_1^2 = \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$

$\frac{2}{3} + \frac{1}{6} = \frac{5}{6}$

$P(Y<X^2) = \frac{\text{area}(Y<X^2)}{\text{area trapezoid}} = \frac{5/6}{6} = \boxed{\frac{15}{36}}$

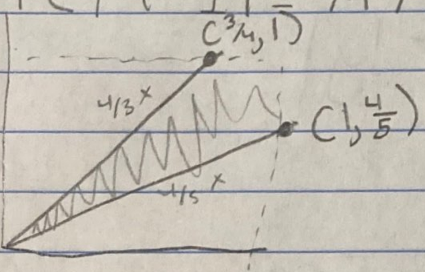
2 a) $P(|X-Y| \leq 0.25) = \frac{\text{area}(|X-Y| \leq 0.25)}{\text{area unit square}}$

area $|X-Y| \leq 0.25$ = area $y-x \leq 1/4$ on $[0,1]$ minus area $x-y > \frac{1}{4}$ on $[0,1]$



$= 1 - 2\left(\frac{1}{2}\left(\frac{3}{4}\right)\right)^2 = 1 - 2\left(\frac{9}{32}\right) = \frac{14}{32} = \boxed{\frac{7}{16}}$

b) $P\left(\left|\frac{X}{Y} - 1\right| \leq 1/4\right)$



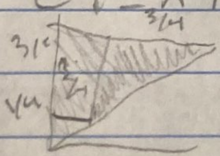
Want shaded region, can do 1 − nonshaded $= 1 - \frac{31}{40} = \frac{9}{40}$

nonshaded $= \frac{1}{2}(1)\left(\frac{4}{5}\right) + \frac{1}{2}(1)\left(\frac{3}{4}\right)$

$= \frac{4}{10} + \frac{3}{8} = \frac{16}{40} + \frac{15}{40} = \frac{31}{40}$

$\frac{\text{area shaded}}{\text{area whole}} = \frac{9/40}{1} = \boxed{\frac{9}{40}}$

c) $P(Y \geq X \mid Y \geq 1/4) = \frac{P(Y \geq 1/4 \cap Y \geq X)}{P(Y \geq \frac{1}{4})} = \frac{\frac{15}{32}}{\frac{3}{4}} = \boxed{\frac{5}{8}}$



$\frac{5}{4} \cdot \frac{3}{4} = \frac{15}{32}$

$\frac{15}{32} \cdot \frac{4}{3}$

Scanned with CamScanner

3a) For $(x,y): 0<|y|<x<1$, then
$\boxed{f_{X,Y}(x,y)=1}$, as triangle of area 1

b) $f_X(x)=\int_{-x}^{x} 1\,dy = y\Big|_{-x}^{x} = \boxed{2x=f_X(x)}$

c) $f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_{x=|y|}^{1} 1\,dx = x\Big|_{|y|}^{1} = \boxed{1-|y|=f_Y(y)}$

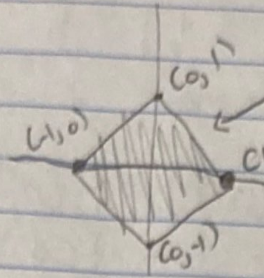
e) $1 \neq 2x(1-|y|)$ for $\forall\ x,y \in (0,1)$ so not independent

d) $E(X)=\int_0^1 x f_X(x)\,dx=\int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \boxed{\frac{2}{3}}$

$E(Y)=0$, due to symmetry present in the graph

Math 165 Hw 9

4 a)



$0<|x|+|y|<1$ area $=2$, so $\boxed{f(x,y)=\frac{1}{2}}$

b)

$$f_x(x)=\int_{-(1-|x|)}^{1-|x|}\frac{1}{2}\,dy=\frac{y}{2}\Big|_{-(1-|x|)}^{1-|x|}=1-|x|$$

$$f_y(y)=\int_{|y|-1}^{1-|y|}\frac{1}{2}\,dx=\frac{x}{2}\Big|_{|y|-1}^{1-|y|}=\boxed{\begin{array}{l}1-|y|=f_y(y)\\ 1-|x|=f_x(x)\end{array}}$$

c) No, as $(1-|y|)(1-|x|)\neq\frac{1}{2}$ for all x,y that fulfill $0<|x|+|y|<1$.

d) The distribution is symmetric over the x and y axis so $E(X)=0$, $E(Y)=0$

5 a)

$$P(X\le x, Y\le y)=\int_0^x\int_0^y 6e^{-2x-3y}\,dy\,dx$$

$$=\int_0^x 6e^{-2x}\cdot\left(-\frac{1}{3}e^{-3y}\right)\Big|_0^y\,dx$$

$$=\int_0^x 6e^{-2x}\left(-\frac{1}{3}e^{-3y}+\frac{1}{3}\right)dx$$

$$=(1-e^{-3y})\int_0^x 2e^{-2x}\,dx=\boxed{(1-e^{-3y})(1-e^{-2x})}$$

b) $f_x(x)=\int_0^\infty 6e^{-2x-3y}\,dy=-2e^{-2x-3y}\Big|_0^\infty=\boxed{2e^{-2x}=f_x(x)}$

c) $f_y(y)=\int_0^\infty 6e^{-2x-3y}\,dx=3e^{-2x-3y}\Big|_0^\infty=\boxed{3e^{-3y}=f_y(y)}$

d) Yes, as $f_x(x)f_y(y)=6e^{-2x-3y}=f(x,y)$ X and Y are independent

Scanned with CamScanner

6a) $E(X+Y) = E(X) + E(Y) = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$

X has mean $\frac{1-0}{2} = \frac{1}{2}$, Y has mean 1

b) Since X, Y independent, $E(XY) = E(X)E(Y)$

$= \frac{1}{2}(1)$

$= \boxed{\frac{1}{2}}$

c) $E[(X-Y)^2] = E(X^2) - 2E(X)E(Y) + E(Y^2)$

$E(X^2) = E(X)$ as is uniform, $E(Y^2) = E(Y)$ as $\mu = 1$

$= (\frac{1}{2})^2 - 2(\frac{1}{2})(1) + 1$

$= \frac{1}{4} - 1 + 1 = \boxed{\frac{1}{4}}$

d) $E(X^2 e^{2Y}) = E(X^2)E(e^{2Y}) = \frac{1}{4}E(e^{2y})$

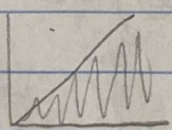
$E(e^{2y}) = \int_0^\infty e^{2y} \cdot e^{-y} dy = \infty$, so $\boxed{E(x^2 e^{2y}) = \infty}$

7a) X, Y are exponential distributions w/ parameters $\lambda_x = \frac{1}{20}$ and $\lambda_y = \frac{1}{25}$ respectively

$f_x(x) = \frac{1}{20} e^{-\frac{1}{20}x}$ $f_y(y) = \frac{1}{25} e^{-\frac{y}{25}}$ Since X, Y are independent, $f(x,y) = f_x \cdot f_y$, meaning

$f(x,y) = \frac{1}{20} e^{-\frac{1}{20}x} \cdot \frac{1}{25} e^{-\frac{1}{25}y} = \boxed{\frac{1}{500} e^{-\frac{1}{20}x - \frac{1}{25}y} = f(x,y)}$

b) $P(Y<X)$



← area we want

$$P(Y<X) = \int_0^\infty \int_0^x \frac{1}{500} e^{-\frac{1}{20}x - \frac{1}{25}y} dy\, dx$$

$$= \frac{1}{500} \int_0^\infty -25 e^{-\frac{x}{20} - \frac{y}{25}} \Big|_0^x dx$$

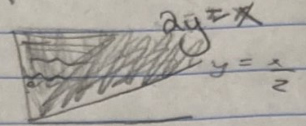
$$= \frac{1}{20} \int_0^\infty -e^{-\frac{9}{100}x} + e^{-\frac{x}{20}} dx = \frac{1}{20}\left(\frac{100}{9} e^{-\frac{100}{9}x} - 20e^{-\frac{x}{20}}\right)\Big|_0^\infty$$

$$= \frac{1}{20}\left(20 - \frac{100}{9}\right) = \boxed{\frac{4}{9}}$$

Scanned with CamScanner

Math 16S Hw 9

7c) $P(2Y \geq x)$



$y = \frac{1}{2}$

$$P(2Y \geq x) = \int_0^\infty \int_{\frac{1}{2}x}^\infty \frac{1}{500} e^{-\frac{x}{20} - \frac{y}{25}} \, dy \, dx$$

$$= \frac{1}{500} \int_0^\infty \left. -25 e^{-\frac{x}{20} - \frac{y}{25}} \right|_{\frac{x}{2}}^\infty dx$$

$$= \frac{1}{500} \int_0^\infty 25 e^{-\frac{x}{20} - \frac{x}{50}} \, dx$$

$$-\frac{5}{100} - \frac{2}{100} = -\frac{7}{100}$$

$$= \frac{1}{20} \int_0^\infty e^{-\frac{7}{100}x} \, dx = \frac{1}{20} \cdot \left. -\frac{100}{7} e^{-\frac{7}{100}x} \right|_0^\infty$$

$$= \frac{1}{20} \cdot \frac{100}{7} = \boxed{\frac{5}{7}}$$

Scanned with CamScanner