Tufts University
Department of Mathematics

Math 136 **Homework 1** [1] **Spring, 2023**

Due date: 11:59 pm, Sunday, January 29, 2023 on Gradescope.

Problems:

1 (15 points) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Answer the following without using the Fundamental Theorem of Calculus.

(a) Assume $f'(x) = 0$ for all $x \in (a,b)$. What can you conclude about f on $[a,b]$? Prove your result.

Solution: Let $x \in (a,b]$. Since f satisfies the hypothesis of the Mean Value Theorem, there is a $c \in (a,x)$ such that $\dfrac{f(x) - f(a)}{x-a} = f'(c)$. Since $f'(c) = 0$ for all $c \in (a,b)$, $f(x) = f(a)$.

(b) Now assume $f'(x) = 3$ for all $x \in (a,b)$. What can you conclude about f on $[a,b]$? Prove your result.

Solution: Let $x \in (a,b]$. Since f satisfies the hypothesis of the Mean Value Theorem, there is a $c \in (a,x)$ such that $\dfrac{f(x) - f(a)}{x-a} = f'(c)$. Since $f'(c) = 3$ for all $c \in (a,b)$, $f(x) - f(a) = 3(x-a)$ or $f(x) = f(a) + 3(x-a)$.

2 (20 points) Suppose that f is differentiable on (a,b), f' is continuous on (a,b), and $f'(x) \neq 0$ for all $x \in (a,b)$.

(a) Prove that f is one-to-one on (a,b).

Solution: Because $f' : (a,b) \to \mathbb{R}$ is continuous and (a,b) interval f' is either always positive or always negative (if $f'(x_1) < 0 < f'(x_2)$ for points $x_1$and x_2 in (a,b), then $f'(x) = 0$ for some x between x_1 and x_2 by the IVT).

Therefore, f is strictly monotonic on (a,b) and therefore f is one-to-one on (a,b).

(b) Prove that f maps (a,b) onto some open interval I NOTE: unbounded open intervals are allowed.

Solution: As (a,b) is connected and f is continuous, the image $I = f(a,b)$ is a connected subset of $\mathbb{R}$. The only connected subsets of $\mathbb{R}$ are intervals, so I is an interval.

We now need to show I is an open interval. To show this, we assume I is an interval that is not open. This means I contains at least one of its endpoints. Assume I contains its left endpoint, which we will label c. This means that, for some $x_0 \in (a,b)$, $f(x_0) = c$ *and* for all $x \in (a,b)$, $f(x) \geq c$. Since (a,b) is an open interval and f is differentiable on (a,b) and f has a minimum at x_0 (as $f(x_0) \leq f(x) \forall x \in (a,b)$, we see that $f'(x_0) = 0$. This contradicts the assumption that f' is never zero. A similar proof can be done if I contains its right endpoint. NOTE: This covers the cases of both bounded and unbounded intervals.

* The results of part (a) and (b) and the definition of inverse function imply that f^{-1} is defined from domain I onto (a,b). By Theorem 3.29, $f^{-1} : I \to (a,b)$ is continuous.

(c) Prove that the derivative $(f^{-1})'$ exists and is continuous on I. *Note:* in your proof, explain why you do not divide by zero.

Solution: First we prove ${f^{-1}}'$ exists. Let $x_0 \in (a,b)$ and let $y_0 = f(x_0)$. Therefore, $f^{-1}(y_0) = x_0$. Let $\{y_j\}$ be a sequence in I that converges to y_0 and is never equal to y_0. Let $x_j = f^{-1}(y_j)$, so $f(x_j) = y_j$. Then,

$$\frac{f^{-1}(y_j) - f^{-1}(y_0)}{y_j - y_0} = \frac{x_j - x_0}{f(x_j) - f(x_0)}. \tag{1}$$

Note that the expression in (1) is defined as f is injective and $x_j \neq x_0$, so $f(x_j) \neq f(x_0)$. As $y_j \to y_0$, $x_j \to x_0$ because f^{-1} is continuous by and $x_0 = f^{-1}(y_0)$ as noted in * above.

Therefore, the right-hand side of (1) converges to $1/f'(x_0)$. Therefore,

$$\lim_{j\to\infty} \frac{f^{-1}(y_j) - f^{-1}(y_0)}{y_j - y_0} = \frac{1}{f'(x_0)}$$

Since the sequence $\{y_j\}$ is an arbitrary sequence converging in $I \setminus \{y_0\}$ to y_0, ${f^{-1}}'(y_0) = 1/f'(x_0)$.

Now we use the assumption that f' is continuous; ${f^{-1}}'$ is continuous because ${f^{-1}}'(y) = 1/f'(f^{-1}(y))$ is the composition of continuous functions $y \mapsto f^{-1}(y) \mapsto f'(f^{-1}(y)) \mapsto 1/f'(f^{-1}(y))$ for $y \in I$.

3 (15 points) Let $A \subset \mathbb{R}^n$. Prove that A is closed if and only if A contains all its limit points.

NOTE: this is a standard theorem but please don't look the proof up. You'll learn more if you prove this using what you learned in 135 as well as discussions with other students in the class, Todd, and Wentao.

Solution: We use the definition of closed set in Fitzpatrick: A is closed if a sequence in A converges, it converges to a point in A.

First, assume A does not contain all of its limit points. Let $\mathbf{x_0} \notin A$ and assume $\mathbf{x_0}$ is a limit point of A. Let $\{\mathbf{x_k}\}$ be a sequence of points in A that converges to $\mathbf{x_0}$. This is possible because $\mathbf{x_0}$ is a limit point of A. Therefore A has a sequence, $\{\mathbf{x_k}\}$, that converges but the limit, $\mathbf{x_0}$ is not in A. Therefore A is not closed.

Now, assume A is not closed. This means there is a sequence $\{\mathbf{x_k}\}$ of points in A that converges to a point $\mathbf{x_0} \notin A$. Since $\mathbf{x_k} \neq \mathbf{x_0}$ for all k (as $\mathbf{x_0} \notin A$), $\mathbf{x_0}$ is a limit point of A that is not in A.

Therefore, A is closed if and only if A contains its limit points.

4 (20 points) Decide whether the following limits exist and prove your result. If the limit exists, also determine the value of the limit.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(x^3+y^6)}{3x^3+3y^6}$

Solution: FUN FACT: Since the domain was not specified we use the largest domain for this function possible. Since the denominator of this expression can be zero whenever $x = -y^2$, we need to assume the domain excludes those points.

So, to neaten the proof up, let's take a specific sequence in the domain for which the calculation will be simple.

We will show that the function $\frac{\cos(x^3+y^6)}{3x^3+3y^6}$ is unbounded Let $\mathbf{x_k} = (1/k^2, 1/k)$. Then the sequence of images of $\mathbf{x}_k$ we must consider to calculate the limit in this problem is $y_k = \dfrac{k^6\cos(2/k^6)}{6}$.

We will show that the sequence is unbounded, so let $M > 0$. As the cosine function is continuous and $\cos 0 = 1$, and $x_k^3 + y_k^6 \to 0$, there is an $N_0 \in \mathbb{N}$ such that $\cos(x_k^3 + y_k^6) \geq 1/2$ for all $k \geq N_0$. Therefore, for $K \geq N_0$

$$y_k = \frac{k^6\cos(2/k^6)}{6} \geq \frac{k^6}{12}$$

Now, use the Archimedean Property to choose $N_1 \geq \max\left\{N_0, \sqrt[6]{12M}\right\}$. Putting this all together, for $k \geq N_1$

$$y_k = \frac{k^6\cos(2/k^6)}{6} \geq \frac{k^6}{12} \geq M.$$

Since this sequence is unbounded and $\mathbf{x}_k \to \mathbf{0}$, $\mathbf{x}_k \neq \mathbf{0}$, the original limit doesn't exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+3y^2}$. **Solution:** This is similar to a problem from class. Let $f(x,y) = \frac{x^3y}{x^6+3y^2}$ We consider two sequences. For the first, let $\mathbf{x_k} = (0, 1/k)$, then $\mathbf{x_k} \to \mathbf{0}$ and $\mathbf{x_k} \neq \mathbf{0}$. By definition $f(\mathbf{x_k}) = 0 \to 0$.

Now, consider the sequence $\mathbf{y_k} = (1/k, 1/k^3)$. As we will see, the calculation allows us to simplify the denominator since, then both numerator and denominator have the same degree in k: $f(\mathbf{y_k}) = \frac{k^{-6}}{4k^{-6}} = 1/4$. Therefore $\lim_{k\to\infty} f(\mathbf{y_k}) = 1/4 \neq 0$ and our limit does not exist.

5 (10 points) Section 4.3, p. 110, Fitzpatrick: # 11.

Solution: Let $n \in \mathbb{N}$. We will show that if f has at least n roots, then f' has at least $n-1$ roots.

So, assume f has $n+1$ or more roots. Order the first $n+1$ roots $x_1 < x_2 < \cdots < x_n < x_{n+1}$ and let $i \in \{2, \ldots, n+1\}$. Then $f(x_{i-1}) = 0 = f(x_i)$ and f is differentiable on $\mathbb{R}$ so by Rolle's theorem there must be a root to f' in the interval (x_{i-1}, x_i). So, there is one root of f' in (x_1, x_2), one root of f' in (x_2, x_3) . . . one root of f' in (x_n, x_{n+1}). That is, f' has at least n roots.

Taking the contrapositive, if f' has at most $n-1$ roots, then f has at most n roots.

6 (10 points) Section 4.3, p. 110, Fitzpatrick: # 21.

Solution: Let $v \in \mathbb{R}$ and let $\{u_k\}$ be a sequence in $\mathbb{R} \smallsetminus \{v\}$ that converges to v. Using the assumption, we have

$$-|u_k - v| \leq \left| \frac{f(u_k) - f(v)}{u_k - v} - 0 \right| \leq |u_k - v|.$$

By the Sandwich Theorem, as the outside terms go to zero, the inside term must go to zero therefore $f'(v) = 0$.

7 (10 points) Section 13.1, p. 352, Fitzpatrick: # 7. Is the result of this problem true if g is unbounded. Why or why not?

Solution: First, assume $c > 0$ such that $|g(\mathbf{x})| \leq c$ for all $\mathbf{x} \in A$. For fun, let's use the $\epsilon - \delta$ condition for limits. Let $\epsilon > 0$ let $\delta > 0$ such that for all $\mathbf{x} \in A \smallsetminus \{\mathbf{x}_*\}$, if $\|\mathbf{x} - \mathbf{x}_*\| < \delta$ then $|f(\mathbf{x})| < \epsilon/c$. Here we use that $c > 0$.

Then, for $\mathbf{x} \in A \smallsetminus \{\mathbf{x}_*\}$, if $\|\mathbf{x} - \mathbf{x}_*\| < \delta$ then $|f(\mathbf{x})g(\mathbf{x})| \leq c|f(\mathbf{x})| < c\epsilon/c = \epsilon$. Therefore, $fg \to 0$ as $\mathbf{x} \to \mathbf{x}_*$.

This is not true if g is unbounded. The proof gives some clue to this because the bound $|g(x)| \leq c$ was crucial for the proof in the first part of this problem.

So, lets let $A = (0, \infty)$, let $f(x) = x$ and let $g(x) = 1/x$. Then, $f \to 0$ as $x \to 0$ but $f(x)g(x) = 1 \not\to 0$ as $x \to 0$.