Math 145 Practicum Questions

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Question 1. Consider the two lines k and ℓ on the hexagon below. The symmetry group of the regular hexagon D_6 is generated by rotation of $60°$ and a reflection. Does it matter which of k or ℓ is used for that reflection?

Question 2. Let k, ℓ be two lines in the plane, r the reflection across k and s the reflection across ℓ. Find the order of rs in each of the following cases:

(a) k and ℓ intersect at angle $\frac{p}{q}\pi$ radians where p, q are positive integers

(b) k and ℓ intersect at any other angle

(c) k and ℓ are parallel

Question 3. (a) Fix a positive integer n and let $G_n = \{z \in \mathbb{C} \,|\, z^n = 1\}$. Show that G_n forms a group under multiplication of complex numbers.

(b) Now let n vary; show G is a group under multiplication complex numbers where:

$$G = \bigcup_{n=1}^{\infty} \{z \in \mathbb{C} \,|\, z^n = 1\}$$

Question 4. (a) Which of the following sets form a group under multiplication modulo 14:

- $\{1, 3, 5\}$
- $\{1, 3, 5, 7\}$
- $\{1, 7, 13\}$
- $\{1, 9, 11, 13\}$

(b) Show that if a subset of $\{1, 2, \ldots, 21\}$ contains an even number or the number 11 then it cannot form a group under multiplication modulo 22.

Question 5. Prove that if p is a prime number then the set $\{1, 2, \ldots, n-1\}$ with multiplication modulo p is a group.

Question 6. Let G be a group and $x, y, g \in G$

(a) Show that x and gxg^{-1} have the same order.

(b) Show that xy and yx have the same order.

Question 7. Let $G = \{x \in \mathbb{Q} \mid 0 \leqslant x < 1\}$ and define

$$x +_1 y = \begin{cases} x+y, & 0 \leqslant x+y < 1 \\ x+y-1, & x+y \geqslant 1 \end{cases}$$

Show G with $+_1$ is a infinite group and every element of G has finite order.

Question 8. (a) Check that rs and r^2s together generate D_n.

(b) Find all subgroups of D_4 and D_5.

Question 9. Draw a diagonal in a regular hexagon. List those plane symmetries of the hexagon which leave the diagonal fixed, and those which send the diagonal to itself. Show that both collections of symmetries are subgroups of the group of all plane symmetries of the hexagon.

Question 10. Let f be an isometry from the real line to itself which sends the integers to integers.

(a) Show that when f has no fixed points, f is a translation by an integer distance.

(b) Show that when f leaves exactly one point fixed, this point is either an integer or lies midway between two integers and f is a reflection across this fixed point.

(c) Show that when f fixes more than one point that it is the identity map.

Question 11. Express each of the following elements of S_6 as a product of disjoint cyclic permutations, and as a product of transpositions.

(a)
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 1 & 6 & 4 \end{bmatrix}$$

(b) $(1235)(2134)$

(c) $(135)(234)(246)(345)$

Question 12. Show that the elements of S_5 which send the numbers $1, 3, 5$ among themselves form a subgroup of S_5. What is the order of this subgroup? Then do the same for the elements of S_5 sending $2, 4$ among themselves.

Question 13 (Challenging)**.** Show that:

(a) When n is odd (123) and $(12\cdots n)$ generate A_n.

(b) When n is even (123) and $(23\cdots n)$ generate A_n.

Question 14. Label the vertices of the tetrahedron $\{1, 2, 3, 4\}$. Check that each of the following are subgroups of the rotational symmetry group of the tetrahedron, and write as

(a) All rotational symmetries fixing the vertex 1 and the midpoint of vertices $2, 3, 4$.

(b) All rotational symmetries fixing the midpoint of vertices $1, 2$ and the midpoint of vertices $3, 4$.

(c) All rotational symmetries fixing vertex 4 and the midpoint of vertices $2, 3$.

Show that each of these collections form a subgroup of the group of rotational symmetries of the tetrahedron.

Question 15 (6.7 Part 1)**.** For each of the following permutations $p \in S_n$ find two order-2 permutations $\alpha, \beta \in S_8$ such that $p = \alpha\beta$:

(a) $p = (1234)$

(b) $p = (12345)$

(c) $p = (123456)$

(d) $p = (1234567)$

(e) $p = (12345678)$

Is there a pattern or strategy that you can use for longer and longer cycles?

Question 16 (6.7 Part 2)**.** Show that if $p, q, r \in S_n$ are disjoint cycles and each is the product of two order-2 permutations then pqr is also the product of two order-2 permutations.

Together with the previous question this allows you to do Problem 6.7 from the book, that if $n \geqslant 4$ then every element of S_n is the product of two order-2 permutations.

Question 17. Let $P(x_1, \ldots, x_n) = (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_n)(x_2 - x_3) \cdots (x_{n-1} - x_n)$ and for $\alpha \in S_n$, $\alpha P(x_1, \ldots, x_n) = P(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$. Show that if $\alpha, \beta, \gamma \in S_n$ satisfy $\alpha\beta = \gamma$, then $\alpha\beta P = \gamma P$. This will show that using P is a well-defined way of defining whether a permutation is even or odd.

Question 18. Show that S_n is generated by $\{(12), (23), \ldots, (n-1, n)\}$. Then show that it is generated by $\{(12), (123 \cdots n)\}$.

In this question I accidentally assigned something already proven in the book. I recommend trying to prove it yourself even if you've already read about it in the book, as practice. As a challenge you can try instead:

Find a generating set for the symmetry group of the cube.

Hint: You will need at least one reflection, let's say through the plane P. If S is your proposed generating set, then every symmetry needs to be written as a word in S, in particular every reflection. Check that if Q is the plane of a reflection that there is a word w in S such that $w(P) = Q$. If not, then S does not generate the whole symmetry group.

Question 19. Let $M_2(\mathbb{Z}/2\mathbb{Z})$ be the group of 2×2 matrices with entries in $\mathbb{Z}/2\mathbb{Z}$ with addition.

(a) What are the possible sizes of subgroups of $M_2(\mathbb{Z}/2\mathbb{Z})$?

(b) Which of those sizes are realized by subgroups?

(c) Can this group be written as a non-trivial product of smaller subgroups?

Non-trivial means other than $G = G \times \{e\}$ where $\{e\}$ is the trivial subgroup.

Question 20. Let $M_2(\mathbb{Z}/3\mathbb{Z})$ be the group of 2×2 matrices with entries in $\mathbb{Z}/3\mathbb{Z}$ with addition.

(a) What are the possible sizes of subgroups of $M_2(\mathbb{Z}/3\mathbb{Z})$?

(b) Which of those sizes are realized by subgroups?

(c) Can this group be written as a non-trivial product of smaller subgroups?

Non-trivial means other than $G = G \times \{e\}$ where $\{e\}$ is the trivial subgroup.

Question 21. (a) What are the possible sizes of subgroups of D_6?

(b) Which of those sizes are realized by subgroups?

(c) Can this group be written as a non-trivial product of smaller subgroups? *Non-trivial* means other than $G = G \times \{e\}$ where $\{e\}$ is the trivial subgroup.

Question 22. Let G be the full symmetry group of a pentagonal prism.

(a) What are the possible sizes of subgroups of G?

(b) Which of those sizes are realized by subgroups?

(c) Can this group be written as a non-trivial product of smaller subgroups?

Non-trivial means other than $G = G \times \{e\}$ where $\{e\}$ is the trivial subgroup.

Question 23. Let G be the full symmetry group of a pentagonal anti-prism, which is two pentagons A and B with each edge of pentagon A connected to a vertex of pentagon B by a triangular face, and vice versa.

(a) What are the possible sizes of subgroups of G?

(b) Which of those sizes are realized by subgroups?

(c) Can this group be written as a non-trivial product of smaller subgroups?

Non-trivial means other than $G = G \times \{e\}$ where $\{e\}$ is the trivial subgroup.

Question 24. Recall that the quaternion group Q is the set

$$\{\pm 1, \pm i, \pm j, \pm k\}$$

with multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1$$

(a) Find all the subgroups of Q

(b) For each subgroup H of Q find all the left cosets of H

(c) Determine which subgroups of Q are conjugate to one another.

Question 25. Let H be a subgroup of G. Prove the following:

(a) If $h \in H$ then $hH = H$

(b) If $g \notin H$ then $gH \cap H = \varnothing$

(c) If $f \in gH$ then $fH = gH$

(d) If $f \notin gH$ then $fH \cap gH = \varnothing$

(e) That the union of all left cosets of H equals G

Question 26. Recall that the group SO_3 is the group of 3×3 orthogonal real matrices with determinant $+1$. Show that SO_3 is exactly the group of rotations of $\mathbb{R}^3$ around axes that pass through the origin.

Hint: you can break this into the following steps:

(a) Find a matrix $A \in SO_3$ that represents a rotation about the z-axis by angle θ.

(b) Find a matrix $B \in SO_3$ that sends the z-axis to the axis containing the vector (a, b, c).

(c) Find a way to combine A and B to get a rotation of angle θ about the axis containing the vector (a, b, c).

Question 27. Find some finite groups that are *not* cyclic but which only have cyclic proper subgroups.

The more groups you find the more special bonus points you get!

Special bonus points are worthless and are not exchangeable for anything.