Final exam, MATH 125, Fall 2022

Instructions: This exam has 5 problems and is out of a total of 100 points. Each question is worth 20 points. You need to show all work or justify your results referring to results/theorems to receive a full credit. If an answer is given with no work shown, no credit will be given. You have 4 hours minutes to do the exam and upload to Gradescope. You can use a calculator for the exam to do computations e.g. basic calculations, computing eigenvalues/eigenvectors. You can also refer to the textbook and lecture notes. However, you can not discuss the questions with any person or perhaps more precisely any carbon form. Please scan and upload your test to Gradescope. Late exams receive no credit, with no exceptions. Good luck!

Name: ______________________________

Academic Honesty Certification: I certify that I have taken this exam without the aid of unauthorized people. I have read the Exam instructions and agree to the rules.

Signature: ____________________ Date: ____________________

1A [10 pts]. Consider the initial value problem

$$\begin{aligned} y' &= f(t,y) \\ y(a) &= y_0 \\ t &\in [a,b], \end{aligned}$$

where $f(t,y)$ is Lipschitz continuous in y on $[a,b]\times[-\infty,\infty]$. Let $a < t_1 < t_2 < ...t_{n-1} < b$ a set of $n+1$ points on the grid $[a,b]$ with uniform spacing h. Consider the following numerical method to approximate the solution of the initial value problem

$$\begin{aligned} w_0 &= y_0 \\ w_{i+1} &= w_i + \frac{h}{2}\left[f(t_i,w_i) + f(t_i+h, w_i + hf(t_i,w_i))\right] \quad \text{for } i = 1,2,... \end{aligned}$$

Above, w_i denotes the approximate solution of $y(t_i)$. Prove that the local truncation error of this method is $O(h^3)$.
Hint: Consider the Taylor expansion of $y(t_{i+1})$ about t_i. Use the fact that $y'(t) = f(t,y)$ and $y''(t) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)y'(t)$. Expand the term $f(t_i+h, w_i+hf(t_i,w_i))$ about (t_i,w_i) using the first order multidimensional Taylor expansion. Given a function $g(x,z)$, the first order Taylor expansion of $g(x+p, z+q)$ about (x,z) is given below

$$g(x+p,z+q) = g(x,z) + \frac{\partial g}{\partial x}p + \frac{\partial g}{\partial z}q + O(h^2)$$

1B [10 pts]. Consider the initial value problem $y'(t) = ty$, $y(0) = 1$ on $[0,1]$.

(a) Prove that there is a unique solution to the problem on the interval $[0,1]$.

(b) Assume that we use forward Euler method to solve the initial value problem using $n+1$ points on the grid $[0,1]$ with uniform spacing h such that $t_i = ih$ for $0 \le i \le n$. Recall the global error of forward Euler method at t_i

$$g_i = |y_i - w_i| \le \frac{Mh}{2L}\left[e^{L(t_i-t_0)} - 1\right],$$

where $y_i = y(t_i)$, w_i is the numerical approximation of $y(t_i)$, M is the upper bound of the second derivative of $|y''(t)|$ on [0,1] i.e. $M = \max_{0\le t\le 1} |y''(t)|$ and L is the Lipschitz constant of $f(t, y) = ty$ with respect to y on the interval $[0, 1]$. Find value/values of h such that the global error at t_i is at most 10^{-3}. [Hint: You can use the fact that $y(t) = e^{t^2/2}$ is the exact solution].

2A [7 pts]. Consider the following numerical integration (quadrature) rule to integrate a continuous function $f(x)$ on the interval $[0, h]$.

$$\int_0^h f(x)\,dx \approx h\left[af(0) + bf\left(\frac{h}{3}\right) + cf(h)\right]$$

(a) Determine a, b and c such that the formula above is exact for polynomials of as high degree as possible.

(b) Determine the error in the quadrature rule. [**Hint**: Consider Taylor expansion of $f(x)$].

2B [7 pts]. Let $f(x)$ be a continuous function on the interval $[x_0, x_1]$.

(a) Find the interpolating polynomial that passes through $(x_0, f(x_0))$ and $(x_1, f(x_1)$. Let $P(x)$ denote your result.

(b) We approximate the integral of $f(x)$ on the interval $[x_0, x_1]$ as follows

$$\int_{x_0}^{x_1} f(x)\,dx \approx \int_{x_0}^{x_1} P(x)\,dx.$$

Find the exact integration of the right hand side of the above approximation to propose a numerical integration method. What is the method you obtain?

2C [6 pts]. Let $S_n(x)$ be a polynomial of degree less than n and let P_n denote the n-degree Legendre polynomial. Prove that $\int_{-1}^{1} S_n(x)P_n(x)\,dx = 0$.

3A [7 pts]. Let $\boldsymbol{A}$ and $\mathbf{R}$ be two invertible $n \times n$ matrices. Assume that we have the LU decomposition of $\mathbf{R}\boldsymbol{A}$ as $\mathbf{R}\boldsymbol{A} = \mathbf{LU}$ where $\mathbf{L}$ is lower triangular and $\mathbf{U}$ is upper triangular. Propose an algorithm with computational complexity $O(n^2)$ to solve $\mathbf{A}\boldsymbol{x} = \mathbf{b}$.

3B [8 pts]. A matrix $\boldsymbol{A}$ is strictly column diagonally dominant if $A_{ii} > \sum_{j\neq i} A_{ji}$ i.e in each column of $\boldsymbol{A}$, the diagonal element is strictly larger than the sum of the other entries in that column. For any column diagonally dominant matrix $\boldsymbol{A}$, prove that Gaussian elimination can be applied with no need for partial pivoting. [**Hint**: Consider the first step of Gaussian elimination with A_{11} as the pivot. Try to argue the steps that follow using induction].

3C [5 pts]. Based on the methods we discussed in class, what method will you propose to solve $\boldsymbol{A}\boldsymbol{x} = \mathbf{b}$ efficiently where $\boldsymbol{A} \in \mathcal{R}^{n\times n}$ is an invertible matrix with the following structure

(a) $\boldsymbol{A}$ is tri-diagonal. What is the computational complexity of the proposed method?

(b) $\boldsymbol{A}$ is sparse with at most k non-zero entries in each row but not necessarily structured i.e. the non-zero entries could be located anywhere. What is the computational complexity of the proposed method?

4A [8 pts]. Consider the following iterative method to solve the linear system $\boldsymbol{Ax} = \mathbf{b}$.

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha\left(\boldsymbol{Ax}^k - \mathbf{b}\right) \quad k = 0, 1, 2, ...$$

The method starts with an input of an initial vector $\boldsymbol{x}^{(0)}$. Above, α denotes a scalar parameter.

(a) Write the condition that ensures that the iterative method converges to the solution of $\boldsymbol{Ax} = \mathbf{b}$ starting from any initial vector $\boldsymbol{x}^{(0)}$.

(b) Consider the following specific linear system $\boldsymbol{Ax} = \mathbf{b}$ where

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the value/values of α such that the iterative method in (a) converges to the solution of $\boldsymbol{Ax} = \mathbf{b}$ starting from any initial vector $\boldsymbol{x}^{(0)}$.

4B [7 pts]. Let $\boldsymbol{A} \in \mathcal{R}^{n\times n}$ be a diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$. We assume that $\lambda_1 > \lambda_2 > ... > \lambda_n$ i.e. the eigenvalues are ordered in strictly decreasing order. Assume that we know λ_1 and $\boldsymbol{v}_1$. Describe how you will use power method to find $\boldsymbol{v}_2$. Given $\boldsymbol{v}_2$, what is the best estimate of λ_2?

4C [5 pts]. Given $\boldsymbol{A} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$, define a matrix $\boldsymbol{B}$ as follows:

$$\boldsymbol{B} = \mathbf{I} + \alpha\mathbf{A} + \alpha^2\boldsymbol{A}^2,$$

where $\mathbf{I} \in \mathcal{R}^{2\times 2}$ is the identity matrix and α is a scalar. Consider the iterative algorithm

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{Bx}^{(k)} \quad k = 0, 1, 2, ...$$

Above, the method starts with an input of an initial vector $\boldsymbol{x}^{(0)}$. For what values of α does the iterative algorithm converge to the zero vector?

5A [10 pts]. Let $W = \text{span}\{1, x\}$ be a subspace of functions on the interval $[-1, 1]$ i.e. any function $f \in W$ can be written as $f(x) = a + bx$ for some constants a and b.

(a) Prove that the set $\{1, x\}$ is orthogonal on $[-1, 1]$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(b) Find the least square approximation of $f(x) = \sin(x)$ on the interval $[-1, 1]$ using the basis $\{1, x\}$ i.e. what is the closest element to $f(x) = \sin(x)$ in W?

5B [10 pts]. Assume that we have n data points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ where the y values are positive i.e. $y_i > 0$ for $1 \le i \le n$. We want to fit the the data to the following model

$$y = ae^{bx},$$

where a and b are constants to be determined with $a > 0$. Show that the problem of estimating a and b can be formulated as a least squares problem.