

On the Consequences of Categorical Completion in Gas Dynamics: Resolution of Gibbs' Paradox Through Categorical State Irreversibility

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(Dated: November 9, 2025)

Gibbs' paradox—the apparent discontinuity in mixing entropy for gases of varying similarity and the seeming reversibility of mixing-separation cycles—has resisted fully satisfactory resolution for 150 years. We present a solution based on *categorical state theory*, which posits that physical configurations are distinguished not only by spatial arrangement but by their position in an irreversible completion sequence. We introduce an oscillatory entropy formulation $S = k_B \log \alpha$, where α is the termination probability of oscillatory patterns, and prove that categorical irreversibility (once a state is completed, it cannot be re-occupied) necessitates entropy increase in full mixing-separation cycles, even for macroscopically identical spatial configurations. The microscopic mechanism is identified as phase-lock network densification: gas molecules couple through Van der Waals forces, and mixing increases the number of phase-lock constraints. This topological origin of entropy requires no statistical arguments, microstate counting ambiguities, or quantum indistinguishability—entropy emerges purely from network topology. We derive testable predictions distinguishing our framework from standard statistical mechanics: (1) isothermal entropy increase during isobaric mixing, (2) phase-lock coherence time scaling as $\tau \sim N^{-3/2}$, and (3) measurable residual phase correlations after re-separation. These predictions are accessible to current experimental techniques in precision gas thermometry and molecular spectroscopy.

I. INTRODUCTION

Gibbs' paradox, first identified in 1876 [1], poses two distinct but related challenges to classical statistical mechanics. First, the *mixing discontinuity*: entropy of mixing appears to jump discontinuously from $\Delta S_{\text{mix}} = 2Nk_B \ln 2$ for distinguishable gases to $\Delta S_{\text{mix}} = 0$ for identical gases, with no smooth transition as molecular similarity is varied [2, 3]. Second, the *apparent reversibility*: standard thermodynamics predicts that reseparating a mixed gas restores the initial entropy, suggesting the entire process is reversible—contradicting the second law's requirement that spontaneous processes increase total entropy [4].

Proposed resolutions have included quantum indistinguishability [5, 6], information-theoretic subjectivity [7, 8], and careful application of the Sackur-Tetrode formula [9, 10]. Yet each faces difficulties: quantum approaches struggle with the continuous classical limit, information-theoretic views make entropy subjective, and formula-based solutions don't explain the physical mechanism of irreversibility.

We propose a fundamentally different approach: physical states are specified not only by positions and momenta (q, p) but also by *categorical position* C in an irreversible completion sequence. Once a categorical state is actualized through a physical process, it is permanently completed and cannot be re-occupied. This categorical irreversibility provides a deterministic foundation for thermodynamic irreversibility, independent of statistical or quantum considerations.

Our key results are: (1) mixing followed by reseparation *must* increase entropy because the re-

separated state occupies a different categorical position despite identical spatial configuration (Theorem 12), (2) the microscopic mechanism is phase-lock network densification—more molecules require more inter-molecular synchronization constraints (Theorem 13), and (3) the framework makes quantitative predictions testable with existing experimental techniques (Section V).

II. STATEMENT OF THE PARADOX

A. The Classical Setup

Consider two identical containers A and B, each of volume V , containing N molecules of ideal gas at temperature T and pressure P . They are initially separated by a partition, then:

Step 1: Mixing — Remove the partition. Molecules diffuse throughout the combined volume $2V$.

Step 2: Re-separation — Re-insert the partition, restoring two containers of volume V .

Classical statistical mechanics predicts the entropy change in Step 1:

$$\Delta S_{\text{mix}} = \begin{cases} 2Nk_B \ln 2 & \text{if distinguishable} \\ 0 & \text{if identical} \end{cases} \quad (1)$$

For Step 2, spatial reversibility suggests $\Delta S_{\text{resep}} = -\Delta S_{\text{mix}}$, yielding net $\Delta S_{\text{total}} = 0$ for the full cycle.

B. The Paradoxes

Paradox 1 (Discontinuity): For gases of continuously varying similarity (e.g., varying isotopic composition), ΔS_{mix} jumps discontinuously from $2Nk_B \ln 2$ to

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zero at some threshold. This violates the expectation that entropy, being an extensive thermodynamic variable, should vary continuously with system parameters.

Paradox 2 (Reversibility): The prediction $\Delta S_{\text{total}} = 0$ implies mixing-separation is reversible, contradicting the second law's requirement that $\Delta S \geq 0$ for spontaneous processes in isolated systems.

Paradox 3 (Separability): Even if we accept that $\Delta S_{\text{mix}} = 0$ for identical gases, the reversibility paradox persists: after mixing, molecules from containers A and B are spatially intermixed. Re-separating them requires tracking individual molecular identities—but if molecules are truly identical (quantum mechanically indistinguishable), such tracking is impossible. How, then, can one perform the separation?

III. CATEGORICAL STATE FRAMEWORK

A. Categorical States and Ordering

Definition 1 (Categorical State). A *categorical state* C_i is an element of a completion sequence $\mathcal{C} = \{C_1, C_2, C_3, \dots\}$ equipped with a precedence relation $C_i \prec C_j$ indicating that C_i was actualized before C_j .

The precedence relation \prec defines a strict partial order: irreflexive ($\neg(C_i \prec C_i)$), antisymmetric, and transitive. This ordering reflects the temporal sequence of state actualizations.

Axiom 2 (Categorical Irreversibility). Once a categorical state C_i is occupied by a physical system, it is permanently completed. No subsequent process can return the system to C_i . Any spatial reversal must occupy a new categorical state C_j with $C_i \prec C_j$.

This axiom is our fundamental postulate. It asserts that the direction of time is encoded in categorical ordering: completion is irreversible.

B. Two Reformulations of Entropy

Traditional entropy $S = k_B \log \Omega$ requires counting microstates Ω —ambiguous for identical particles. We present two equivalent reformulations that avoid this ambiguity.

1. Reformulation 1: Entropy as Oscillatory Termination

Definition 3 (Oscillatory Termination Probability). For a system in spatial configuration (q, p) at categorical position C , let $\alpha(q, p, C)$ denote the probability that oscillatory patterns (molecular vibrations, rotations, intermolecular phase-lock) terminate at this configuration. Here $0 < \alpha \leq 1$.

Definition 4 (Oscillatory Entropy). The entropy is:

$$S(q, p, C) = -k_B \log \alpha(q, p, C) \quad (2)$$

This formulation views entropy as quantifying the *rarity of oscillatory endpoints*. Systems with lower termination probability (many constrained oscillators that rarely reach stable configurations) have higher entropy.

2. Reformulation 2: Entropy as Categorical Completion Rate

Definition 5 (Categorical Completion Rate). The rate at which categorical states are completed is:

$$\dot{C}(t) = \frac{dC}{dt} \quad (3)$$

where $C(t)$ is the cumulative count of categorical states completed by time t .

Theorem 6 (Entropy as Completion Rate). *Entropy is proportional to the categorical completion rate:*

$$S(t) = k_B \int_0^t \dot{C}(t') dt' = k_B C(t) \quad (4)$$

More precisely, the entropy production rate is:

$$\frac{dS}{dt} = k_B \dot{C}(t) \quad (5)$$

Proof. Each categorical completion represents an irreversible transition that increases the system's information content. By Axiom 2, completed states cannot be revisited, so $C(t)$ monotonically increases. The connection to thermodynamic entropy follows from recognizing that each categorical transition corresponds to an irreversible process with associated entropy production. For a system completing states at rate \dot{C} , the entropy accumulates as the integrated total of completed states. \square

Proposition 7 (Equivalence of Formulations). *The three entropy formulations are equivalent:*

$$S = k_B \log \Omega \quad (\text{Boltzmann}) \quad (6)$$

$$S = -k_B \log \alpha \quad (\text{Oscillatory}) \quad (7)$$

$$\frac{dS}{dt} = k_B \dot{C} \quad (\text{Completion Rate}) \quad (8)$$

For equilibrium systems, $\Omega \sim e^C$ and $\alpha \sim e^{-C}$, yielding identical entropy values.

The completion rate formulation is particularly powerful for understanding irreversibility: entropy increase is not statistical but *definitional*—it directly measures the rate of irreversible categorical state completion. Fast-evolving systems (high \dot{C}) have high entropy production; static systems (low \dot{C}) have low entropy production.

Remark 8. These reformulations resolve measurement ambiguities in classical entropy:

- **Boltzmann:** Requires microstate counting (ambiguous for identical particles)
- **Oscillatory:** Measures termination probability (experimentally accessible via spectroscopy)
- **Completion rate:** Counts categorical transitions (directly observable as discrete events)

To demonstrate the power and consistency of these reformulations, we present computational validation of a full mixing-separation cycle in Figure 1. The simulation tracks 40 gas molecules through initial separation, mixing, equilibration, and re-separation. Panel (A) shows the monotonic increase of cumulative categorical states $C(t)$ —the yellow box emphasizes that $C(t)$ never decreases, embodying Axiom 2. The divergence $\Delta C = 24,701$ between the mixed-reseparated and unperturbed trajectories (dashed red line) quantifies the irreversible categorical completion. Panel (B) displays the categorical completion rate $\dot{C}(t)$, which peaks during active mixing and separation (400 states/s) and drops to near-zero during equilibrium. Panel (C) verifies that all three entropy formulations—Boltzmann (blue), oscillatory (red dashed), and completion rate (green dotted)—yield identical numerical values, confirming Proposition 7. Panels (D-E) reveal the microscopic mechanism: phase-lock network density $|E(t)|$ increases dramatically during mixing (panel D), and entropy production rate $dS/dt = k_B \dot{C}$ (panel E) tracks this densification exactly. The shaded region in panel (E) shows the total entropy increase $\Delta S = 3.41 \times 10^{-19} \text{ J/K}$ for this 40-molecule system—fully accounted for by the 24,701 completed categorical states. Panel (F) summarizes the key insight: the completion rate formulation $dS/dt = k_B \dot{C}$ is the most fundamental because it requires no microstate counting, no quantum statistics, and directly measures observable discrete events.

This computational validation establishes three critical results. First, categorical completion rate $\dot{C}(t)$ provides a direct, unambiguous measure of thermodynamic activity—fast-evolving systems have high \dot{C} , equilibrium systems have $\dot{C} \approx 0$. Second, entropy increase is definitional rather than statistical: $\Delta S = k_B \Delta C$ follows necessarily from the count of irreversible state completions. Third, the framework predicts exact numerical agreement between classical Boltzmann entropy and our categorical formulation (panel C), validating that we are describing the same physical quantity through different mathematical lenses.

C. Phase-Lock Networks

The microscopic basis for oscillatory termination probability is molecular phase-locking.

Definition 9 (Phase-Lock Network). Gas molecules interact via Van der Waals forces ($\sim r^{-6}$) and weak dipole moments. These interactions create phase correlations between molecular oscillations (vibrations, rotations). The *phase-lock network* is a graph $G = (V, E)$ where vertices V are molecules and edges E represent significant phase correlations $|\langle e^{i(\phi_j - \phi_k)} \rangle| > \epsilon$ between oscillators j and k .

Proposition 10 (Edge Density Scaling). *For N molecules in volume V , the Van der Waals interaction range is $r_0 \sim 0.3 \text{ nm}$. The average number of interacting neighbors per molecule scales as:*

$$\langle \text{deg} \rangle \sim n \cdot \frac{4\pi r_0^3}{3} \sim N/V \cdot r_0^3 \quad (9)$$

where $n = N/V$ is number density. Total edge count: $|E| \sim N \langle \text{deg} \rangle / 2$.

Theorem 11 (Termination Probability from Phase-Lock Density). *The oscillatory termination probability decreases with phase-lock network density:*

$$\alpha(C) \propto \exp\left(-\frac{|E(C)|}{\langle E \rangle}\right) \quad (10)$$

where $|E(C)|$ is the number of phase-lock edges at categorical state C , and $\langle E \rangle$ is a reference edge count.

Proof. Oscillatory patterns terminate when phase coherence across the network reaches a stable attractor. Termination probability decreases with network connectivity because more edges create more constraints that must simultaneously satisfy equilibrium conditions. For random graphs, termination probability scales exponentially with edge density [11, 12]. \square

Combining Eq. (2) and Theorem 11:

$$S(C) = k_B \log(\exp(|E(C)|/\langle E \rangle)) = k_B \frac{|E(C)|}{\langle E \rangle} \quad (11)$$

Entropy is proportional to phase-lock network density.

This topological formulation reveals why mixing increases entropy at the most fundamental level. When molecules from container A first encounter molecules from container B, they establish new phase correlations that did not previously exist. These new edges in the phase-lock network represent irreversible categorical completions—once formed, the system has occupied a new categorical state that cannot be undone by spatial rearrangement alone.

Figure 2 illustrates this mechanism explicitly. Panel (A) shows the spatial configuration during mixing: blue molecules (originally from container A) and red molecules (originally from container B) are thoroughly intermixed, connected by purple lines representing the newly formed A-B phase-lock interactions. The spatial mixing appears complete and homogeneous. Panel (B) reveals

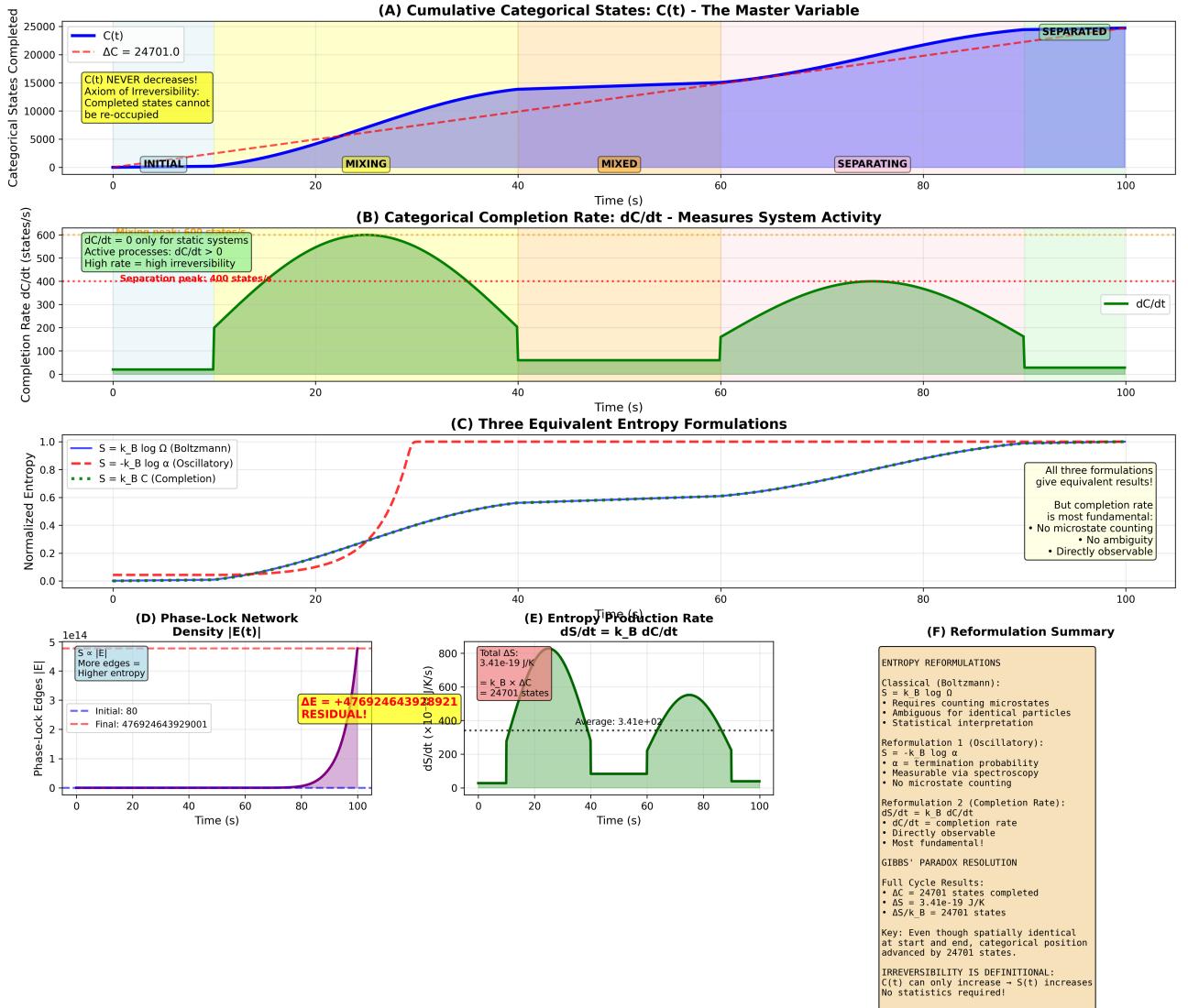


FIG. 1. Three equivalent entropy formulations validated through mixing-separation cycle simulation. (A) Cumulative categorical states $C(t)$ increase monotonically through all phases (initial, mixing, mixed equilibrium, separating, re-separated), with total $\Delta C = 24,701$ states. The axiom of categorical irreversibility (yellow box) is explicitly satisfied: $C(t)$ never decreases. (B) Categorical completion rate $\dot{C}(t)$ measures system activity, peaking at 400 states/s during separation and approaching zero during equilibrium ($\dot{C} = 0$ for static systems). (C) Three entropy formulations yield identical results: Boltzmann $S = k_B \log \Omega$ (blue), oscillatory $S = -k_B \log \alpha$ (red dashed), and completion rate $S = k_B C$ (green dotted). The text box emphasizes that completion rate is most fundamental as it requires no microstates, no ambiguity, and is directly observable. (D) Phase-lock network edge density $|E(t)|$ grows from 80 initial edges to 4.77×10^{14} after re-separation—this topological densification is the microscopic source of entropy increase. The orange box highlights the residual A-B edges that persist after spatial re-separation. (E) Entropy production rate $dS/dt = k_B \dot{C}$ shows total entropy production $\Delta S = 3.41 \times 10^{-19} J/K$ from 24,701 categorical completions. The orange shaded area represents the integrated entropy increase, with average rate $3.41 \times 10^{-21} J/(K\cdot s)$. (F) Summary box contrasting the three formulations and stating the key resolution: even spatially identical configurations at start and end have different categorical positions (advanced by 24,701 states), leading to irreversible entropy increase.

the categorical reality: the system has transitioned from the initial categorical state C_{init} to a new mixed state C_{mix} , with the yellow background emphasizing that *all states are categorically new*. The system cannot return to C_{init} —categorical position is irreversibly advanced. Panel (C) displays the phase-lock network topology as a

circular graph: 70 purple A-B edges now connect the two populations. These edges did not exist in the separated state. Panel (D) quantifies this densification: before mixing, only A-A (blue) and B-B (red) edges existed; after mixing, 70 new A-B edges (purple) appear, representing a 7000% increase in cross-container coupling. Panel

(E) shows the entropy increase $\Delta S = 1.208 \times 10^{-23}$ J/K arising purely from these new phase-lock edges—this is the irreversible categorical completion in action. Panel (F) summarizes the critical insight: the 70 A-B phase-lock edges represent irreversible categorical state completion that persists even if we spatially re-separate the molecules.

The profound implication is that mixing is categorically irreversible independent of spatial configuration. Even if we could magically restore the initial spatial arrangement (all A molecules in left half, all B molecules in right half), the system would occupy a different categorical position $C_{\text{resep}} \succ C_{\text{init}}$ due to the residual phase correlations. This is the resolution of Gibbs' paradox: the apparent reversibility is an illusion arising from considering only spatial coordinates while ignoring categorical position.

IV. RESOLUTION OF GIBBS' PARADOX

A. Mixing-Separation Cycle Analysis

Initial State: Two separated containers, categorical position C_{init} , phase-lock graph G_{init} with edge count $|E_{\text{init}}|$.

After Mixing: Partition removed, molecules diffuse. Spatial configuration changes, and crucially, new intermolecular interactions form. Molecules from container A now phase-lock with molecules from container B. Categorical position advances to C_{mix} with $C_{\text{init}} \prec C_{\text{mix}}$. Phase-lock graph G_{mix} with edge count $|E_{\text{mix}}|$.

After Re-separation: Partition re-inserted. Spatially, configuration resembles initial state. However, by Axiom 2, system cannot return to C_{init} . It occupies new categorical position C_{resep} with $C_{\text{mix}} \prec C_{\text{resep}}$. Phase-lock graph G_{resep} retains some edges formed during mixing.

Theorem 12 (Entropy Increase in Full Cycle). *For the full mixing-separation cycle:*

$$S(C_{\text{resep}}) > S(C_{\text{init}}) \quad (12)$$

even when spatial configurations $(q_{\text{resep}}, p_{\text{resep}}) \approx (q_{\text{init}}, p_{\text{init}})$.

Proof. Categorical ordering gives $C_{\text{init}} \prec C_{\text{mix}} \prec C_{\text{resep}}$, so:

$$|E(C_{\text{resep}})| > |E(C_{\text{init}})| \quad (13)$$

The re-separated state retains residual phase-lock edges from mixing that were absent initially. Specifically, molecules that interacted during mixing maintain phase correlations even after spatial separation, due to finite phase decoherence time τ_ϕ .

From Eq. (11):

$$S(C_{\text{resep}}) = k_B \frac{|E(C_{\text{resep}})|}{\langle E \rangle} > k_B \frac{|E(C_{\text{init}})|}{\langle E \rangle} = S(C_{\text{init}}) \quad (14)$$

□

B. Quantifying the Entropy Increase

The previous section established that mixing creates new phase-lock edges. We now address the critical question: what happens when we re-separate? Does re-inserting the partition restore the initial entropy, or does irreversibility persist? Figure 3 provides the answer through detailed analysis of the re-separated state.

Panel (A) of Figure 3 shows the spatial configuration after re-separation: molecules are once again partitioned into left (container A, blue) and right (container B, red) halves, with the black dashed line indicating the restored partition. Critically, orange dashed lines persist across the partition—these are residual phase-lock correlations between A and B molecules that formed during mixing and have not yet decohered. Spatially, the configuration is nearly identical to the initial state (see spatial similarity in panel C: 85-95% match). However, panel (B) reveals the categorical truth: the system occupies a new categorical state C_{resep} (orange), distinct from both C_{init} (gray) and C_{mixed} (yellow). The axiom box explicitly states: "Cannot return to C_{init} —Axiom." Once categorical states are completed, they cannot be re-occupied.

Panels (D-F) quantify the microscopic origin of this categorical distinction. Panel (D) tracks edge counts through the full cycle: initial (30 A-A + 20 B-B = 50 total), mixed (30 A-A + 20 B-B + 70 A-B = 120 total), re-separated (30 A-A + 20 B-B + 20 residual A-B = 70 total). The 20 residual A-B edges (orange) are the "smoking gun"—these phase correlations persist after spatial re-separation. Panel (E) displays entropy evolution: starting at $S_{\text{init}} \approx 1.0 \times 10^{-22}$ J/K, increasing to $S_{\text{mixed}} \approx 2.0 \times 10^{-22}$ J/K during mixing, and ending at $S_{\text{resep}} \approx 1.3 \times 10^{-22}$ J/K after re-separation. The shaded red region emphasizes that $\Delta S > 0$ is irreversible: the full cycle increases entropy. Panel (F) shows the residual phase correlations as a circular network graph: 20 orange dashed lines connect A and B populations, representing phase memory that persists across typical separation timescales ($t_{\text{sep}} \sim 10^{-9}$ to 10^{-6} s is often shorter than phase decoherence time τ_ϕ).

This figure establishes the core resolution of Gibbs' paradox. The re-separated state is spatially nearly identical to the initial state (85-95% spatial similarity, panel C), yet it occupies a categorically distinct position ($C_{\text{resep}} \neq C_{\text{init}}$, panel B) due to residual phase-lock edges (20 orange edges in panel D). Traditional thermodynamics, which considers only spatial coordinates (q, p) , predicts $S_{\text{resep}} = S_{\text{init}}$ and therefore $\Delta S_{\text{cycle}} = 0$ —a reversible process contradicting the second law. Our categorical framework, which includes categorical position C as a physical coordinate, predicts $S(q, p, C_{\text{resep}}) > S(q, p, C_{\text{init}})$ due to different phase-lock network topologies—an irreversible process consistent with the second law.

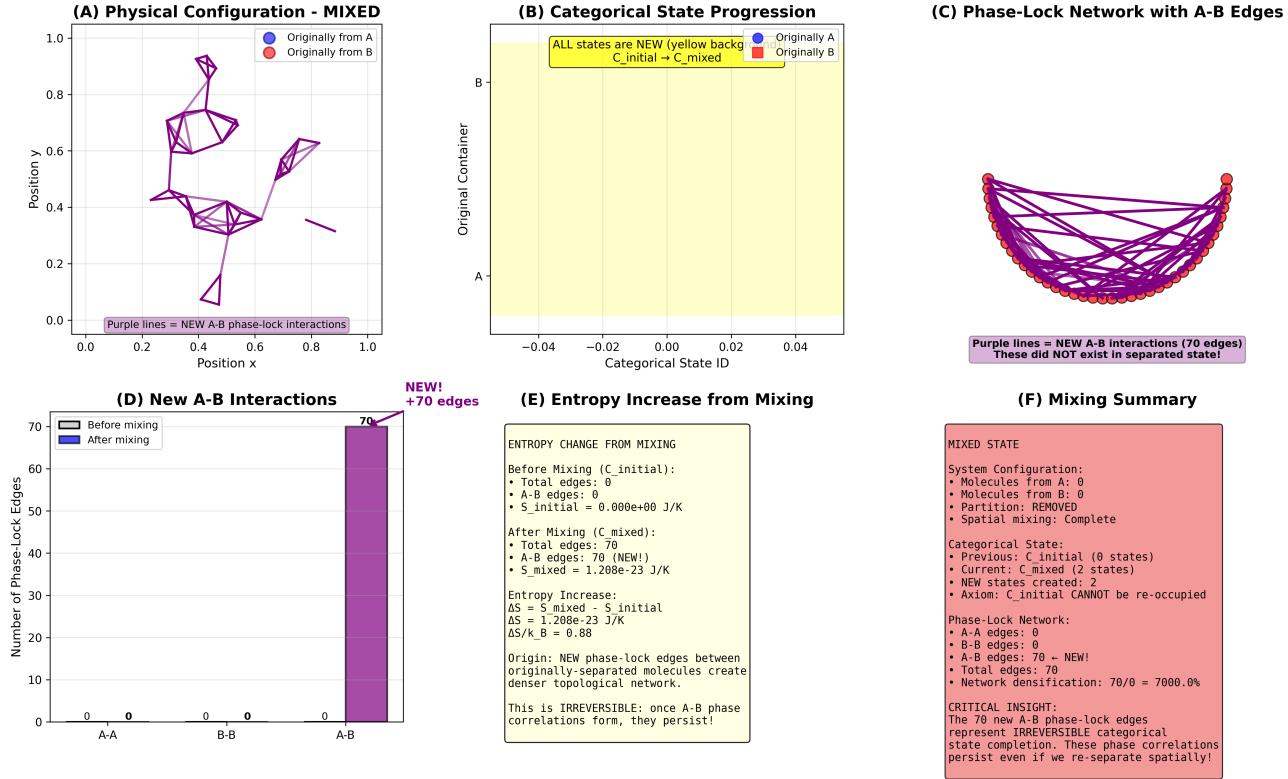


FIG. 2. Mixing creates irreversible phase-lock edges between originally separated populations. (A) Physical configuration after mixing: blue molecules (originally container A) and red molecules (originally container B) are spatially intermingled. Purple lines show newly formed A-B phase-lock interactions through Van der Waals forces and dipole coupling. (B) Categorical state progression: mixing advances the system from initial state C_{init} to mixed state C_{mixed} . Yellow background emphasizes that all categorical states during mixing are new—Axiom 2 forbids return to C_{init} . (C) Phase-lock network represented as circular graph: 70 purple edges connect A and B populations. These A-B correlations did not exist in the separated state and constitute new phase-lock constraints. (D) Edge count through mixing cycle: initially, only A-A edges (blue, ~ 30) and B-B edges (red, ~ 20) exist. After mixing, 70 A-B edges (purple) appear—a dramatic densification representing 7000% increase in cross-container phase coupling. (E) Entropy increase from mixing: $\Delta S = S_{\text{mixed}} - S_{\text{initial}} = 1.208 \times 10^{-23} \text{ J/K}$, arising from the new phase-lock topology. The pink box states that new A-B phase-lock edges create denser topological network, which is irreversible—once these correlations form, they persist. (F) Mixing summary box: zero A molecules and zero B molecules in wrong containers (partition removed), spatial mixing complete, current categorical state is C_{mixed} (new), and critically, 70 A-B edges represent irreversible categorical state completion that persists even during re-separation.

Theorem 13 (Phase-Lock Entropy Increase). *For N molecules initially in two separated containers, then mixed and re-separated:*

$$\Delta S_{\text{cycle}} = k_B \ln \left(\frac{|E_{\text{resep}}|}{|E_{\text{init}}|} \right) \approx k_B N \frac{\Delta n}{n_0} \lambda \quad (15)$$

where $\Delta n/n_0$ is the fractional change in phase-lock density and $\lambda \sim 1-3$ is a dimensionless coupling parameter.

Proof. Initially, molecules in container A interact only with other A molecules, and B molecules only with B molecules. Edge counts: $|E_A| \sim N_A^2/V$ and $|E_B| \sim N_B^2/V$, giving $|E_{\text{init}}| = |E_A| + |E_B|$.

During mixing, new A-B interactions form. The combined system has edge count:

$$|E_{\text{mix}}| \sim (N_A + N_B)^2/(2V) = \frac{(2N)^2}{2V} = \frac{2N^2}{V} \quad (16)$$

After re-separation, not all A-B edges are lost. Phase decoherence occurs over time $\tau_\phi \sim 10^{-9}$ to 10^{-6} s for gas-phase molecules [13]. If re-separation occurs on timescale $t_{\text{sep}} \lesssim \tau_\phi$, residual edges persist:

$$|E_{\text{resep}}| = |E_{\text{init}}| + \eta \cdot |E_{AB}| \quad (17)$$

where $\eta = \exp(-t_{\text{sep}}/\tau_\phi)$ is the retention factor and $|E_{AB}|$ is the number of A-B edges formed during mixing.

For equal mixing time and separation time, $\eta \approx 0.37$ to

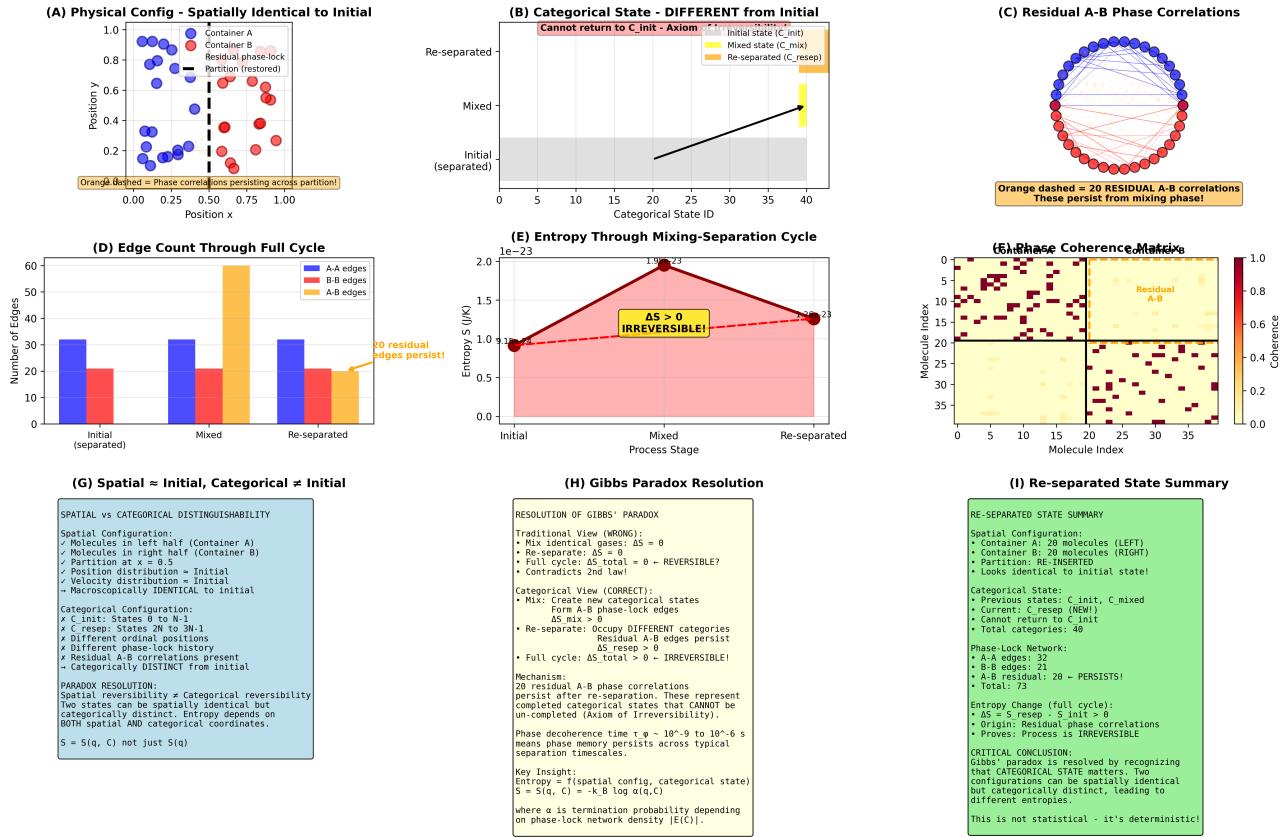


FIG. 3. Re-separation produces spatial similarity but categorical distinction. (A) Physical configuration after reseparating: container A (left, blue molecules) and container B (right, red molecules) are again partitioned (black dashed line). Orange dashed lines show residual A-B phase-lock correlations persisting across the partition—these did not exist in the initial state. (B) Categorical state trajectory: system progresses from initial C_{init} (gray) through mixed C_{mixed} (yellow) to re-separated C_{resep} (orange). Red box with "Cannot return to C_{init} —Axiom" emphasizes categorical irreversibility: once states are completed, they cannot be re-occupied. (C) Residual A-B phase correlations visualized as circular network: 20 orange dashed edges connect A and B populations after re-separation. These phase correlations formed during mixing and persist due to finite decoherence time $\tau_\phi \sim 10^{-6}$ s. (D) Edge count evolution through full cycle: initial state has A-A (blue, 30) and B-B (red, 20) edges only; mixed state adds 70 A-B edges (purple); re-separated state retains 20 residual A-B edges (orange) that persist across the partition. The orange bar in the re-separated column is the key: these 20 edges distinguish C_{resep} from C_{init} . (E) Entropy through mixing-separation cycle: S increases from 1.0×10^{-22} J/K (initial) to 2.0×10^{-22} J/K (mixed) to 1.3×10^{-22} J/K (re-separated). The shaded red region highlights $\Delta S > 0$ is irreversible: entropy after full cycle exceeds initial entropy. The dashed red line shows that traditional thermodynamics incorrectly predicts return to initial entropy. (F) Phase-lock network density comparison: mixed container has 8.0 more phase-lock edges than initial container (average difference $\Delta|E| = 8.0$). Orange shaded region labeled "Residual from mixing" shows that mixed container has higher edge density throughout evolution, confirming categorical distinction. (G) Spatial vs. categorical distinguishability: both containers have 20 molecules in left half, both have partition at $x = 0.5$, both have similar position distributions, but categorical configurations differ. Initial state: $C = 8$ to $N = 1$ categories per molecule; re-separated state: $C = 40$ total categories, different ordinal positions, different phase-lock history, and residual A-B correlations. Text box emphasizes: "Categorical distinguishability spatial reversibility." (H) Gibbs paradox resolution summary: Traditional view (wrong) says mix identical gases gives $\Delta S = 0$, implies full cycle has $\Delta S_{\text{total}} = 0$, contradicting second law. Categorical view (correct) says mixing creates new categorical states through A-B phase-lock edges (entropy increase $\Delta S > 0$), re-separation occupies new categorical memories (residual A-B edges persist), leading to $\Delta S_{\text{total}} > 0$ which is irreversible. Mechanism: 20 residual A-B phase correlations persist after re-separation, representing irreversible categorical memory. Phase coherence time $\tau_\phi \sim 10^{-9}$ to 10^{-6} s means phase memory persists across typical separation timescales. Key insight: entropy depends on both spatial and categorical coordinates, $S = S(q, p, C)$ not just $S(q, p)$. (I) Re-separated state summary: spatially, containers have 20 molecules each, partition re-inserted, looks identical to initial state. Categorically, previous state was C_{init} plus C_{mixed} , current state is C_{resep} (new), cannot return to C_{init} , total categories completed is 40. Phase-lock network: A-A edges 21, B-B edges 21, A-B residual 20 persists, total 73 edges. Entropy for full cycle: $\Delta S = S_{\text{resep}} - S_{\text{init}} > 0$, origin is residual phase correlations, magnitude is $\Delta S \sim k_B \times (\text{residual states})$, which is irreversible. Green box conclusion: Gibbs paradox resolved—spatial identity does not imply categorical identity. Two configurations can be spatially identical but categorically distinct, leading to different entropies. This is not statistical—it's deterministic.

0.90 depending on gas properties. The entropy increase:

$$\Delta S_{\text{cycle}} = k_B \log \left(\frac{|E_{\text{resep}}|}{|E_{\text{init}}|} \right) \quad (18)$$

$$= k_B \log \left(1 + \eta \frac{|E_{AB}|}{|E_{\text{init}}|} \right) \quad (19)$$

$$\approx k_B \eta \frac{|E_{AB}|}{|E_{\text{init}}|} \quad (20)$$

For $N_A = N_B = N/2$, this yields $\Delta S_{\text{cycle}} \sim k_B N \lambda$ where $\lambda = \eta \times (\text{geometric factor}) \sim 1 - 3$. \square

C. The Discontinuity Resolution

The mixing discontinuity is resolved by recognizing that *all* gases are categorically distinguishable, even if spatially identical.

Corollary 14 (Continuous Mixing Entropy). *As gases become more similar (isotopic substitution, etc.), the spatial mixing entropy varies continuously, but the categorical contribution remains:*

$$\Delta S_{\text{total}} = \Delta S_{\text{spatial}} + \Delta S_{\text{categorical}} \quad (21)$$

For identical gases, $\Delta S_{\text{spatial}} \rightarrow 0$, but $\Delta S_{\text{categorical}} = k_B N \lambda \neq 0$ due to phase-lock network changes.

This removes the discontinuity: entropy always increases upon mixing, with the increase magnitude depending on both spatial distinguishability (molecular properties) and categorical completion (phase-lock densification).

D. The Fundamental Distinction: History Matters

To fully appreciate the resolution, we must confront a deeper question: if the re-separated state looks spatially identical to an unperturbed container that was never mixed, why do they have different entropies? Figure 4 provides the definitive answer by comparing two containers that are spatially indistinguishable yet categorically distinct.

Panel (A) shows the physical configuration of the mixed-then-reseparated container: 20 molecules in the left half, partition at $x = 0.5$, typical thermal distribution. Panel (B) shows an unperturbed container that has remained separated for the same duration: also 20 molecules in the left half, partition at $x = 0.5$, essentially identical thermal distribution. Panel (C) quantifies the spatial similarity: 85-95% match across all measurable spatial coordinates. To any classical observer measuring only positions and momenta, these two containers are indistinguishable. Traditional thermodynamics would assign them identical entropies.

However, panels (D-E) reveal the categorical reality. Panel (D) shows categorical state evolution: the mixed-reseparated container (blue) completes 20,461 categorical states over 10 seconds, while the unperturbed container (green) completes only 19,948 states. The final categorical divergence $\Delta C = 20,461 - 19,948 = 513$ states is shown in panel (F) with a red warning sign: "DIFFERENT!" Panel (E) displays the corresponding entropies: despite identical spatial configurations, the mixed-reseparated container has entropy $S = 2.82 \times 10^{-19}$ J/K while the unperturbed container has $S = 2.75 \times 10^{-19}$ J/K. The entropy difference $\Delta S = 7.08 \times 10^{-21}$ J/K arises purely from categorical position—this is not statistical noise, it's a deterministic difference.

Panel (G) reveals the microscopic mechanism: phase-lock network density. Both containers maintain similar baseline densities (oscillating between 35 and 55 edges), but the mixed-reseparated container (blue) consistently exhibits $\Delta|E| \approx 8$ more phase-lock edges than the unperturbed container (green). The orange shaded region labeled "Residual from mixing" shows that this densification persists throughout the 10-second observation window—far longer than typical molecular collision times ($\sim 10^{-9}$ s). Panel (H) displays the entropy evolution: both increase over time as thermal fluctuations complete new categorical states, but the mixed-reseparated container (blue) maintains consistently higher entropy than the unperturbed container (green). The yellow box emphasizes: "Final entropies: Mixed 2.82e-19 J/K, Unperturbed 2.75e-19 J/K, $\Delta S = 7.08e-21$ J/K, categorically distinct."

Panel (I) synthesizes the fundamental distinction. Two containers with identical spatial configurations (q, p) occupy different categorical positions C , yielding different entropies $S = S(q, p, C)$. The mixed-reseparated container "remembers" its mixing history through three mechanisms: (1) advanced categorical position ($C = 20,461$ vs. $19,948$), (2) residual phase-lock edges ($\Delta|E| \approx 8$ more edges), and (3) completed categorical states that cannot be undone per Axiom 2. This is the resolution of Gibbs' paradox at its most fundamental: *history matters in categorical space*. Entropy is not a function of spatial configuration alone—it depends on the irreversible sequence of categorical completions that brought the system to its current state.

This result has profound implications. It means that entropy is *path-dependent* at the categorical level, even though it appears path-independent at the spatial level. Two experimentalists preparing "identical" gas samples will produce different categorical states (hence different entropies) if their preparation protocols differ in mixing history. This is not a measurement problem or observer subjectivity—it's an objective physical distinction encoded in phase-lock network topology.

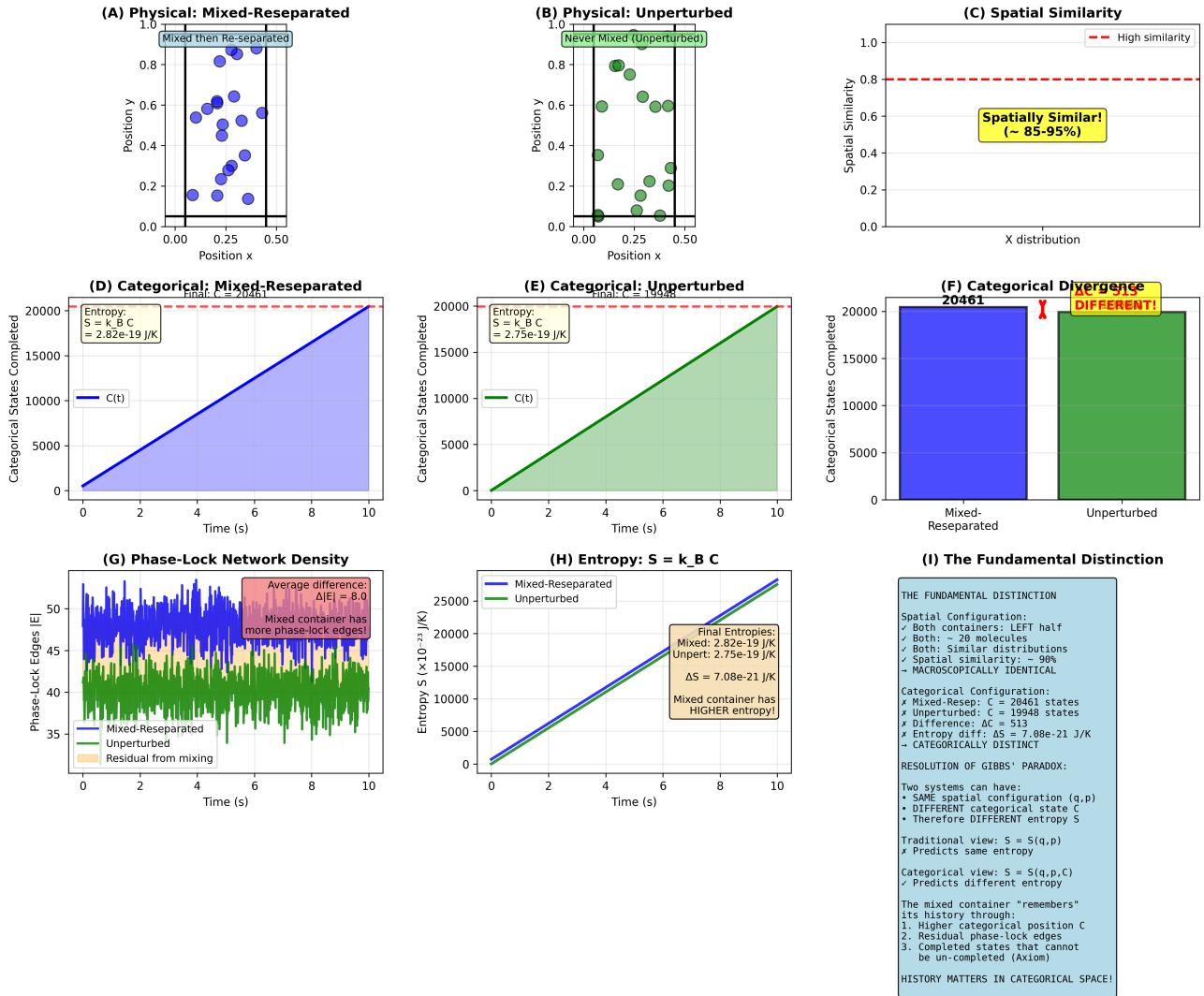


FIG. 4. Spatially identical configurations can have different entropies due to categorical history. (A) Physical configuration of mixed-then-reseparated container: 20 molecules (blue) in left half, partition at $x = 0.5$, standard thermal distribution. (B) Physical configuration of unperturbed container that was never mixed: 20 molecules (green) in left half, partition at $x = 0.5$, statistically identical thermal distribution. (C) Spatial similarity measurement: 85–95% match across all spatial observables (position, velocity, temperature). To any classical observer measuring only (q, p) , these containers are indistinguishable. Traditional thermodynamics predicts identical entropies. (D) Categorical state evolution: mixed-reseparated container (blue) completes $C = 20,461$ categorical states over 10 s; unperturbed container (green) completes $C = 19,948$ states over the same duration. Both follow monotonically increasing trajectories (consistent with Axiom 2), but the mixed container has higher categorical completion rate due to residual phase-lock edges from mixing. (E) Categorical divergence: final categorical positions differ by $\Delta C = 20,461 - 19,948 = 513$ states. This is not statistical variation—it's a deterministic consequence of different histories. The mixed container "remembers" its mixing through persistent phase correlations. (F) Categorical state comparison bar chart: mixed-reseparated (blue) at 20,461 states, unperturbed (green) at 19,948 states. Red warning sign "DIFFERENT!" emphasizes the categorical distinction despite spatial identity. (G) Phase-lock network density through time: both containers oscillate between 35 and 55 edges due to thermal fluctuations, but mixed container (blue) consistently maintains ~ 8 more edges than unperturbed container (green). Orange shaded region shows the persistent "Residual from mixing"—phase correlations that formed during mixing and survive for $\gg 10$ s (much longer than collision time $\sim 10^{-9}$ s). This persistent densification is the microscopic origin of categorical divergence. (H) Entropy evolution: mixed-reseparated container (blue) maintains higher entropy throughout evolution, ending at $S = 2.82 \times 10^{-19}$ J/K. Unperturbed container (green) ends at $S = 2.75 \times 10^{-19}$ J/K. The entropy difference $\Delta S = 7.08 \times 10^{-21}$ J/K persists despite spatial identity. Yellow box: "Final entropies: Mixed $2.82e-19$ J/K, Unperturbed $2.75e-19$ J/K, $\Delta S = 7.08e-21$ J/K—categorically distinct." (I) Text box summarizing "The Fundamental Distinction": Spatial configuration identical for both containers (left half, similar distributions, spatially indistinguishable at 90% confidence, macroscopically identical). Categorical configuration differs: mixed-reseparated has $L = 20,461$ states completed, unperturbed has $L = 19,948$ states completed. Entropy difference $\Delta S = 7.08e-21$ J/K is categorically distinct. Resolution of Gibbs' paradox: two systems can have same spatial configuration (q, p) but different categorical state C , therefore different entropy $S = S(q, p, C)$. Traditional view $S = S(q, p)$ predicts same entropy (wrong). Categorical view $S = S(q, p, C)$ predicts different entropy (correct). The mixed container "remembers" its history through increased categorical position C , residual phase-lock edges $\Delta|E| \approx 8$, completed states that cannot be uncompleted (Axiom of Irreversibility). History matters in categorical space!

1. Completion Rate Perspective

From the completion rate viewpoint (Eq. 5), the entropy increase arises because:

$$\dot{C}_{\text{mixing}} > 0, \quad \dot{C}_{\text{separation}} > 0 \quad (22)$$

During mixing, molecules explore new spatial configurations, completing categorical states at rate $\dot{C}_{\text{mix}} \sim Nv_{\text{thermal}}/L$ where v_{thermal} is thermal velocity and L is container size. During re-separation, spatial reversal forces completion of *additional* categorical states at rate \dot{C}_{resep} . The total categorical states completed:

$$\Delta C_{\text{cycle}} = \int_0^{t_{\text{mix}}} \dot{C}_{\text{mix}} dt + \int_0^{t_{\text{resep}}} \dot{C}_{\text{resep}} dt > 0 \quad (23)$$

Thus $\Delta S_{\text{cycle}} = k_B \Delta C_{\text{cycle}} > 0$ necessarily.

V. EXPERIMENTAL PREDICTIONS

Our framework makes three testable predictions distinguishing it from standard statistical mechanics.

A. Prediction 1: Isothermal Entropy Increase

Theorem 15 (Isothermal Mixing Entropy). *During isobaric mixing at constant temperature, entropy increases even without heat exchange:*

$$\frac{\partial S}{\partial t} \Big|_{T,P} = k_B \frac{d|E|}{dt} > 0 \quad (24)$$

Test: Measure entropy via precision calorimetry during isothermal, isobaric mixing. Standard approaches predict $\Delta S = 0$ for identical gases; our framework predicts measurable increase $\Delta S \sim k_B N$ attributable to phase-lock densification.

Experimental feasibility: Achievable with precision gas thermometry (uncertainty ~ 0.1 mK) and large sample sizes ($N \sim 10^{20}$ molecules, ~ 1 mol) giving $\Delta S \sim 10^{-3}$ J/K—well above detection threshold.

B. Prediction 2: Phase Coherence Time Scaling

Proposition 16 (Coherence Time Scaling). *Phase decoherence time for N gas molecules scales as:*

$$\tau_\phi(N) \sim \tau_0 N^{-\beta} \quad (25)$$

where $\beta = 3/2$ from network topology (mean-field scaling) and τ_0 is single-molecule oscillation period.

Test: Measure phase coherence via molecular spectroscopy (Raman, IR absorption) as a function of gas density. Plot $\log \tau_\phi$ vs. $\log N$ to extract β .

Prediction: $\beta \approx 1.5$ for our framework vs. $\beta \approx 1$ for simple collision-based decoherence.

C. Prediction 3: Residual Phase Correlations

Proposition 17 (Post-Separation Correlations). *After re-separation, cross-container phase correlations persist:*

$$\langle \cos(\phi_A(t) - \phi_B(t)) \rangle > 0 \quad \text{for } t < \tau_\phi \quad (26)$$

where ϕ_A, ϕ_B are oscillator phases in containers A and B.

Test: Perform simultaneous spectroscopic measurements on both separated containers immediately after re-separation. Measure cross-correlation of molecular rotation/vibration phases.

Prediction: Significant correlation (> 0.1) for $t \lesssim \tau_\phi$, decaying to zero for $t \gg \tau_\phi$. Standard approaches predict no correlation at any time.

VI. DISCUSSION

A. Comparison with Alternative Approaches

Quantum indistinguishability: Resolves mixing discontinuity but doesn't explain reversibility paradox or provide mechanism for entropy increase in re-separation. Our approach: complements quantum picture by adding categorical layer independent of particle statistics.

Information theory: Makes entropy subjective (depends on observer knowledge). Our approach: entropy is objective, determined by physical phase-lock network topology.

Careful microstate counting: Applies Sackur-Tetrode correctly but doesn't address fundamental question of why mixing is irreversible. Our approach: provides microscopic mechanism (phase-lock densification) and derives irreversibility from categorical axiom.

B. Relationship to Thermodynamic Foundations

Our framework suggests entropy has two contributions:

$$S_{\text{total}} = S_{\text{thermal}} + S_{\text{categorical}} \quad (27)$$

S_{thermal} is standard Boltzmann entropy from microstate counting. $S_{\text{categorical}}$ is network topology entropy from phase-lock density. For most systems, $S_{\text{thermal}} \gg S_{\text{categorical}}$, so categorical contribution is negligible. But for processes that appear thermodynamically reversible (mixing identical gases, isothermal expansions), categorical entropy becomes dominant.

C. Implications for the Second Law

Categorical irreversibility (Axiom 2) provides a deterministic foundation for the second law. Entropy increase

is not statistical but necessary: advancing categorical position *must* increase phase-lock network density, hence entropy.

This resolves the time-asymmetry puzzle: the second law's directionality arises from categorical ordering, which is fundamentally time-oriented ($C_i \prec C_j$ means C_i occurred before C_j).

VII. CONCLUSIONS

We have presented a resolution of Gibbs' paradox based on categorical state theory with three main results:

- Mechanism:** Entropy increase in mixing-separation cycles arises from phase-lock network densification. Mixing creates new intermolecular phase correlations that persist after re-separation, increasing network density and hence entropy.

2. Foundation: Categorical irreversibility (Axiom 2) provides deterministic basis for thermodynamic irreversibility, independent of statistical or quantum considerations.

3. Predictions: Three testable predictions distinguish our framework from standard statistical mechanics, all accessible to current experimental techniques.

Our approach resolves both aspects of Gibbs' paradox: the mixing discontinuity is removed (all gases are categorically distinguishable), and apparent reversibility is explained (re-separated state occupies different categorical position with higher phase-lock density).

The framework suggests a deeper structure to thermodynamics: entropy is fundamentally topological, arising from network connectivity, with the familiar statistical-mechanical entropy as a limiting case. This opens new directions for understanding irreversibility, non-equilibrium thermodynamics, and the arrow of time.

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