Chapter Seven

Directed Graphs

7.1 directed graphs :-

Def :- A directed graph D or digraph consists of two things :

(i) A set V whose elements are called vertices . points or nodes .
(ii) A set A of order pairs of vertices called arcs .we denote the digraph by D (V , A)

EX(1) :- D (V , A) , V = 4 , A = 7

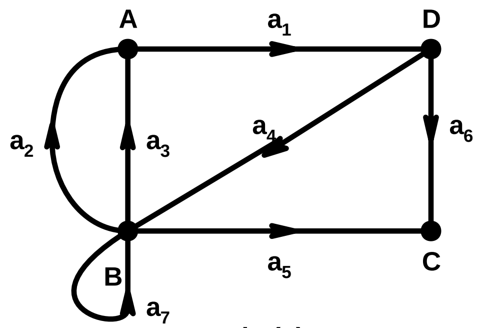


Fig (1)

a_1 = (A , D) , a_2 = (B , A)

a_3 = (B , A) , a_4 = (D , B)

a_5 = (B , C) , a_6 = (D , C)

a_7 = (B , B)

a_7 is a loop a_2 and a_3 are called parallel arcs :

7.2 Basic Definitions :-

* A closed walk has the same first and last vertices

* - The length of walk is n , its number of arcs

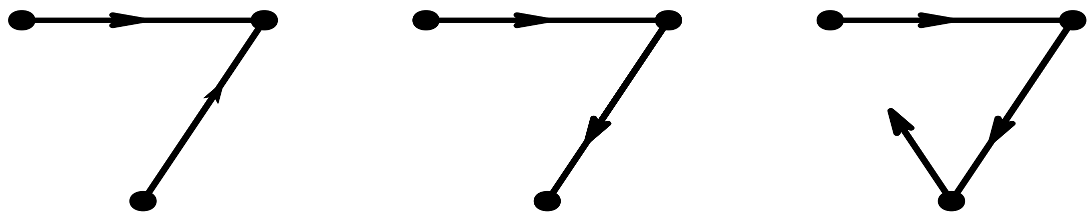
- A spanning walk contains all the vertices of the digraph

- A trail is a walk with distinct arcs

- A path is a walk with distinct vertices

- A cycle is a closed walk with distinct vertices (except the first and last)

EX (2)



strictly weak strictly unilateral strong

- The out degree and in degree of a vertex v are equal respectively to the number of arcs beginning and ending at v
- A vertex with zero in degree is called a source .
- A vertex with zero out degree is called a sink .

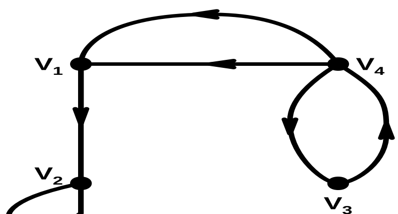
7.3 Digraphs Matrices :-

Let D be a directed graph with vertices $v_1 , v_2 , \ldots\ldots\ldots v_m$ The matrix of D is the m x m matrix

M_D = (mij) where

mij = the number of arcs beginning at V_i and ending at V_j

EX (3) :- Find M_D of digraph in Fig (2) .



D
Fig (2)

$$M = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & 0 & & 0 \end{pmatrix} \end{array}$$

EX (4) :- consider the digraph D pictured in Fig (3)

(a) Describe D formally .
(b) Find the number of paths from X to Z .
(c) Find the number of paths from Y to Z .
(d) Are there any sources or sinks ?
(e) Find the matrix M_D of the digraph D .
(f) Is D weakly connected ? unitaterraly connected ? strongly connected ?

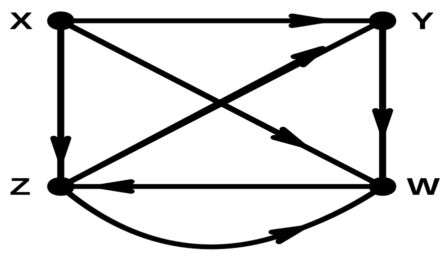


Fig (3)

Solution :-

(a) There are four vertices : x , y , z , and there are seven arcs : (x , y) , (x , w) , (x , z) , (y , w) , (z , y) , (z , w) , (w , z)
(b) There are three paths from X to Z : (x , z) , (x , w , z) and (x , y , w , z)
(c) There is only one path from y to z : (y , w , z)
(d) X is a source since it is not the terminal point of any arc , i. e . its in degree is zero . There are no sinks since every vertex has a non zero out degree , i.e . each vertex is the initial point of some arcs .
(e) The matrix M_D of D follows :-
(Here the rows and columns of M_D are labeled by x , y , z , w respectively)
The entry mij denotes the number of arcs from the ith vertex to the jth vertex

$$M_D = \begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(f) The digraph is not strongly connected since X is a source and hence there is no path from any other vertex , say y , to x .However , D is unilaterally connected since the path
(x , y , w , z) passes through all the vertices , and so there is a sub path connecting any pairs of vertices

7.4 Pruning Algorithm for minimal path :-

During the algorithm , each vertex v' is assigned a number ℓ (v') denoting the current minimal length from the initial point u to v' and a path p (v') from u to v' which has length ℓ (v') .

At each step of the algorithm , we examine an arcs (v' , v) from v' to v which say , has length k . If this is first time that we enter v then we assign :-

$$\ell (v) = \ell (v') + K \qquad \text{and} \qquad p (v) = p (v') v$$

If we have been at v before , then we check to see whether

$$\ell (v') + K < \ell (v)$$

If it is , then we have found a shorter path to v , and we change ℓ (v) and p (v) , otherwise we have ℓ (v) and p (v) alone , we begin by setting ℓ (4) = 0 and p (u) = u . where u is the initial point .

EX(5) :-

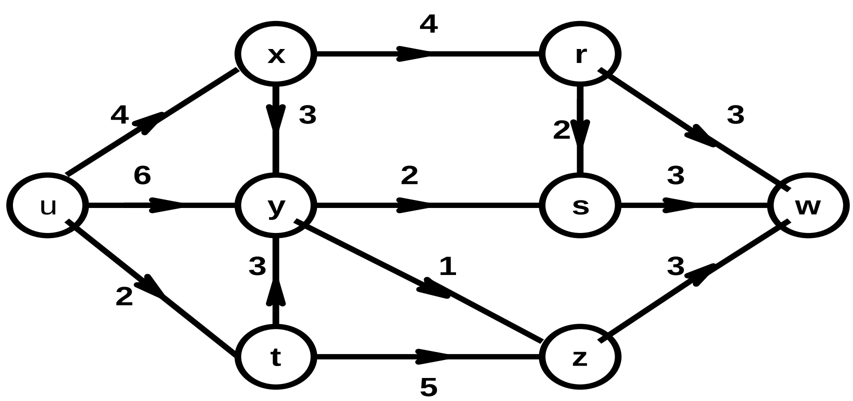


Fig (4)

Sol . :- we apply the above

Algorithm to the above grap

Find the shortest path between u and w .

(1) From u :- ℓ (x) = 4 , p (x) = u x , ℓ (y) = 6 , p (y) = u y , ℓ (z) = 2 , p (z) = u z

(2) From x :- ℓ (r) = 4 + 4 = 8 and p (r) = (x) r = u x r and ℓ(x) + k = 4 + 3 is not less than
ℓ(y) = 6 Thus we leave ℓ(y) and p(y) alone :-

(3) From z :- ℓ(t) 2 + 5 = 7 and p(t) = p(z) t = u z t and ℓ(z) + k = 2 + 3 = 5 is less than
ℓ(y) = 6 Thus we found a shorter path to y and we change ℓ(y) and p(y)
ℓ(y) = ℓ(z) + k = 2 + 3 = 5 and p(y) = p(z) y = u z y

(4) From y :- ℓ(s) = ℓ(y) + k = 5 + 2 = 7 and p(s) = p(y) s = u z y s we next check that .
ℓ(y) + k = 5 + 1 = 6 is less than ℓ(t) = 7 Hence we change
ℓ(t) and p(t) to read :- ℓ(t) = ℓ(y) + 1 =6 and p(t) = p(y) t = u z y t

(5) From r :- ℓ(w) = ℓ(r) + 3 = 11 and p(w) = u x r w
We next check that
ℓ(r) + k = 8 + 2 = 10 is not less than ℓ(5) = 7
Hence we change neither ℓ(s) nor p (s)

(6) From s :- ℓ(s) +k = 7 + 3 is less than ℓ(w) = 11
Hence we change ℓ(w) and p(w) to read :-
ℓ(w) = ℓ(s) +3 = 10 and p (w) = p(s)w = u z y s w

(7) From t :- ℓ(t) + k = 6 + 3 = 9 is less than ℓ(w) = 10
Hence we change ℓ(w) and p(w) to read :-
ℓ(w) = ℓ(t) + 3 = 9 and p(w) = p(t) w = u z y t w
Thus p(w) has been determined and is the required shortest path : it has length ℓ(w) = 9.