Chapter four

Vectors & Matrices

4.1 Introduction :-

Date is frequently arranged in arrays formally a one-dimensional arrays is called a vector and a two – dimensional arrays is called a matrix

4.2 Vectors :-

- u = (u1 , u2 , ……….. un) , The numbers ui are called the components of u . If all the ui = 0 , then u is called the zero Vector .

- The sum of two vectors u & V

u + v = (u1 , u2 , ……….. un) + (v1 , v2 , ……… vn)

= (u1 + v1 , u2 + v2 , ………… un + vn)

-The product of a scalar K and a vector u

K.u = k (u1 , u2 , ……….. un) = (ku1 , ku2 , ……….. kun)

- 1.u = -u

-The dot product or inner product of vectors

u.v = u1 v1 + u2 v2 + ………… un vn $u1^2 + u2^2 + \dots\dots + un^2$

-The norm (or length) of u is denoted

II u II = $\sqrt{u.u} = \sqrt{\mathbf{u1}^2 + \mathbf{u2}^2 + \dots\dots\dots + \mathbf{un}^2}$

EX (4) :- **let u = (2 , 3 , -4) and V = (1 , -5 , 8) . Then**

u + v = (2 + 1 , 3 – 5 , -4 + 8) = (3 , -2 , 4)

5u = (5.2 , 5.3 , 5.(-4)) = (10 , 15 , -20)

-V = - 1.(1 , -5 , 8) = (-1 , 5 , -8)

2u – 3u = (4 , 6 , -8) + (-3 , 15 , -24) = (1 , 21 , -32)

u . v = 2.1 + 3.(-5) + (-4) .8 = 2 – 15 – 32 = -45

$\| u \| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$

<u>4.3 Matrices :-</u>

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}$$

n = vertical (column) , m = horizontal (row)

aij , i th row and the j th column .

The pair of numbers m and n is called the size of the matrix

<u>EX (2) :-</u> (a) the rectangular array $\begin{pmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{pmatrix}$ **is a 2X3 matrix . It s rows are (1 ,-3 , 4)**

and (0 , 5 , -2) and its columns are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ **and** $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$

(b) The 2 X 4 zero matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) The statement is equivalent to the system of equations . $\begin{pmatrix} x+y & 2Z+w \\ x-y & Z-w \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$

X + y = 3 , X – y = 1 , 2Z + w = 5 , Z – w = 4

The solution of the system of equation is X = 2 , y = 1 , Z = 3 , w = 1

<u>4.4 Matrix addition and scalar multiplication :-</u>

Let A and B be two matrices with the same size , i.e same number of rows and of columns

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots\cdots & b_{1n} \\ b_{21} & b_{22} & \cdots\cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots\cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots\cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots\cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots\cdots & a_{mn}+b_{mn} \end{pmatrix}$$

The product of a scalar k and a matrix A ,

$$K \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots\ldots\ldots & a_{mn} \end{pmatrix} = \begin{pmatrix} Ka_{11} & Ka_{12} & \ldots\ldots\ldots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \ldots\ldots\ldots & Ka_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ Ka_{m1} & Ka_{m2} & \ldots\ldots\ldots & Ka_{mn} \end{pmatrix}$$

-A = (-1)A & A – B = A + (-B)

The matrix -A is called the negative of the matrix A .

<u>EX (3) :-</u>

1. $$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1+3 & -2+0 & 3+(-6) \\ 0+2 & 4+(-3) & 5+1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & -3 \\ 2 & 1 & 6 \end{pmatrix}$$

2. $$3\begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & -5 \end{pmatrix} = \begin{pmatrix} 3.1 & 3.(-2) & 3.0 \\ 3.4 & 3.3 & 3.(-5) \end{pmatrix} = \begin{pmatrix} 3 & -6 & 0 \\ 12 & 9 & -15 \end{pmatrix}$$

3. $$2\begin{pmatrix} 3 & -1 \\ 4 & 6 \end{pmatrix} -5\begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 8 & 12 \end{pmatrix} + \begin{pmatrix} 0 & -10 \\ -5 & 15 \end{pmatrix} = \begin{pmatrix} 6 & -12 \\ 3 & 27 \end{pmatrix}$$

<u>4.5 Matrix multiplication :-</u>

If A and B two matrices such that the number of columns of A is equal to the number of rows of B .

A (m x p) x B (P x n) = AB (m x n) whose Ij entry is obtained by multiplying the elements of the ith rows of A by the correspond elements of the j th column of B and then adding

$$\begin{pmatrix} a_{11} \ldots \ldots . a_{1p} \\ a_{i1} \ldots \ldots a_{ip} \\ \ldots \ldots \ldots \ldots \ldots . . \\ a_{m1} \ldots \ldots . a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} \ldots b_{1j} \ldots . b_{1n} \\ \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots . \\ b_{pi} \ldots b_{pj} \ldots b_{pn} \end{pmatrix} = \begin{pmatrix} C_{11} \ldots \ldots \ldots C_{1n} \\ \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots . C_{1j} \ldots . . \\ C_{m1} \ldots \ldots C_{mn} \end{pmatrix}$$

(a) $$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} ra_1 + sb_1 & ra_2 + sb_2 & ra_3 + sb_3 \\ ta_1 + ub_1 & ta_2 + ub_2 & ta_3 + ub_3 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1.1 + 2.0 & 1.1 + 2.2 \\ 3.1 + 4.0 & 3.1 + 4.2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 3 & 11 \end{pmatrix}$$

(c) A system of linear equation such as : $\begin{Bmatrix} x+2y-3z=4 \\ 5x-6y+8z=8 \end{Bmatrix}$

Is equivalent to the matrix equation $\begin{pmatrix} 1 & 2 & -3 \\ 5 & -6 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

That is , any solution to the system of equations is also a solution to the matrix equation , and vice versa .

4.6 Transpose :-

The transpose of a matrix A , written A^T ,is the matrix obtained by writing the rows of A , in order , as columns .

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \\ C_1 & C_2 & \dots & C_n \end{pmatrix}^T = \begin{pmatrix} a_1 & b_1 & \dots & C_1 \\ a_2 & b_2 & \dots & C_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & \dots & C_n \end{pmatrix}$$

EX:- $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix}$

4.7 Square matrices :-

A matrix with the same number of rows as columns is called Square matrix

EX: let A = $\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ then A^2= $\begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$ If f(x) = $2x^2$- 3x + 5 then

F(A) = 2 $\begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$ - 3 $\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ + 5 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = $\begin{pmatrix} 16 & -18 \\ -27 & 61 \end{pmatrix}$

In the order hand , if g(x)= x^2+3x – 10 then

g (A) = $\begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$ + 3 $\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ - 10 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

4.8 Invertible matrices :-

A square matrix A is said to be invertible if there exists a matrix B with the property that

A B = B A = I , the identity matrix .

Such a matrix B is unique : It is called the inverse A and is denoted by A^{-1} . observe that B is the inverse A if and only if A is the inverse of B . For example , suppose .

A = $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ **and B =** $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ **then AB =** $\begin{pmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{pmatrix}$ **=** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

,and BA = $\begin{pmatrix} 6-5 & 15-5 \\ -2+2 & -5+6 \end{pmatrix}$ **=** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Thus A & B are inverses .

<u>EX:-</u>

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}\begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix} = \begin{pmatrix} -11+0+12 & 2+0-2 & 2+0-2 \\ -22+4+18 & 4+0-3 & 4-1-3 \\ -44-4+48 & 8+0-8 & 8+1-8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the two matrices are invertible and are inverses of each other .

<u>4.9 Determinants :-</u>

To each n-square matrix A = (aij) we assign a specific number called the determinant of A ,denoted by det (A) or I A I or

$$\begin{vmatrix} a_{11}\ a_{12} \dots\dots a_{1n} \\ a_{21}\ a_{22} \dots\dots a_{2n} \\ \dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\ a_{n2} \dots\dots\ a_{nn} \end{vmatrix}$$

The determinants of order one , two and three are defined as follows :

$$|a_{11}| = a_{11}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{12}\, a_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{22}\, a_{31} -$$
$$a_{12}\, a_{21} a_{33} \;-\; a_{11} a_{23}\, a_{32}$$

<u>EX:-</u>

(1) - $\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix}$ **= 5.3 – 4.2 = 15 – 8 = 7**

(2) – $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & -1 \\ 5 & 1 & 0 \end{bmatrix}$ **= 2 (6 x 0 – (-1) x 1) -1 (4 x 0 – (-1) x 5) + 3 (4 x 1 – 5 x 6)**

= 2 – 5 – 3 x 26 = -3 – 78 = -81

<u>4.10 Invertible matrices and determinants :-</u>

We can obtain the inverse of 2 X 2 matrix , with determinant non zero , by (i) inter changing the elements on the main diagonal (ii) taking the negative of the other elements , and

(iii) dividing the elements by the determinant of the original matrix .

$$A^{-1} = \frac{adj}{|\,A\,|} = \frac{|cof\ \ A\,|^{T}}{|\,A\,|}$$

<u>EX (1) :-</u>

A = $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ **then IAI = -2 and S o** $A^{-1} = \frac{1}{-2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix}$

<u>EX (2) :-</u>

A = $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ -3 & 0 & -2 \end{pmatrix}$

IAI = 1 (-6 -0) + 1 (-4 + 12)

= -6 + 8 = 2

IAI = 2

cof [aij] =$(-1)^{i+j}$

$$\text{c o f A} = \begin{pmatrix} \begin{bmatrix} 3 & 4 \\ 0 & -2 \end{bmatrix} & -\begin{bmatrix} 2 & 4 \\ -3 & -2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ -3 & 0 \end{bmatrix} \\ -\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -3 & -2 \end{bmatrix} & -\begin{bmatrix} 1 & -1 \\ -3 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 3 & 4 \end{bmatrix} & -\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \end{pmatrix}$$

$$\text{C o f A} = \begin{pmatrix} -6 & 8 & 9 \\ -2 & -2 & 3 \\ -4 & -4 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -6 & -2 & -4 \\ -8 & -2 & -4 \\ 9 & 3 & 5 \end{bmatrix}$$

<u>Theorem of matrices :-</u>

(1) (A + B) + C = A + (B + C) i.e . addition is associative
(2) A + B = B + A , i.e addition is commutative .
(3) A + 0 = 0 + A = A
(4) A + (-A) = (-A) + A = 0
(5) K (A + B) = K A + K B
(6) (K + K´) A = K A + K´ A
(7) (K K´) A = K (K´ A)
(8) 1A = A
(9) (A B) C = A (B C)

(10) A (B + C) = A B + A C

(11) (B + C) A = B A + C A

(12) K (A B) = (K A) B = A (K B) were K is a scalar

(13) $(\boldsymbol{A} + \boldsymbol{B})^{\boldsymbol{T}} = \boldsymbol{A}^{\boldsymbol{T}} + \boldsymbol{B}^{\boldsymbol{T}}$

(14) $(\boldsymbol{KA})^{\boldsymbol{T}} = \boldsymbol{KA}^{\boldsymbol{T}}$**, For K a scalar**

(15) $(\boldsymbol{AB})^{\boldsymbol{T}} = \boldsymbol{B}^{\boldsymbol{T}} \boldsymbol{A}^{\boldsymbol{T}}$

(16) $(\boldsymbol{A}^{\boldsymbol{T}})^{\boldsymbol{T}}$**= A**

(17) det (AB) = det (A) . det (B)

(18) A matrix is invertible if and only it has a non zero determinants