Chapter five

Graph theory

5.1 Graphs and Multi graphs :-

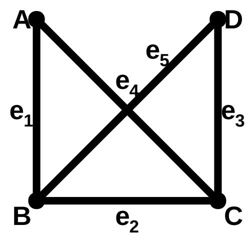
A graph G consists of two things :-

(i) A set V whose elements are called vertices , points or nodes .
(ii) A set E of unordered pairs of distinct vertices called edges .

We denote such a graph by G (V , E) .

EX (1) :- G (V , E) , See Fig (1).

(i) V consists of four vertices A , B , C , D
(ii) E consists of five edges



$e_1 = (A, B)$, $e_2 = (B, C)$, $e_3 = (C, D)$

$e_4 = (A, C)$, $e_5 = (B, D)$

Fig (1)

graph

***Multi graph :- is multiple edges or loops , See Fig (2) .**

e_4 & e_5 are multiple edges .

i.e . edges connecting the same end point . and e_6 is a loop

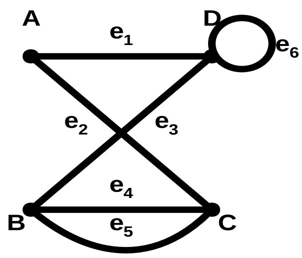


Fig (2) multi graph

<u>*Sub graph :-</u> let G (V , E) be a graph let V´ be a subset of V and let E´ be a subset of E , then G (V´ , E´) is a graph and is called a sub graph of G (V , E) .

<u>*A multi graph :-</u> is said to be finite if it has a finite number of vertices and finite number of edges .

<u>5.2 Degree :-</u> The degree of a vertex v , written deg (v) . equal to the number of edges which are incident on v

<u>Th</u> . The sum of the degree of the vertices of a graph is equal to twice the number of edges .

<u>EX (2) :-</u> for example 1 we have

Deg (A) = 2 , deg (B) = 3 , deg (C0 = 3 , deg (D) = 2

The sum of the degree = 10 = twice the number of deges .

<u>*Def :-</u> A vertex is said to be even if its degree is an even and odd if its degree is an odd

<u>5.3 Connectivity :-</u>

<u>* Def :-</u> A walk in a multi graph consists of all alternating sequence of vertices and edges of the form . v_0 , e_1 , v_1 , e_2 , v_2 ………. e_{n-1} , v_{n-1} , e_n , v_n .

<u>* Def :-</u> (1) The walk is said to be closed if $v_0 = v_n$

(2) The walk is said to be trail if all edges are distinct

(3) The walk is said to be path if all vertices are distinct

(4) The walk is said to be cycle if it is closed such that all vertices are distinct except

$$v_0 = v_n$$

(5) The number n of edges is called the length of the walk

<u>EX(3) :-</u> Consider the graph in Fig (3)

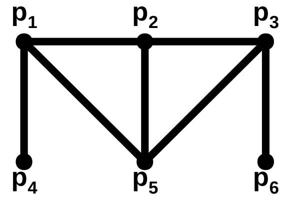


Fig (3)

Then . $\mathbf{p_1}$

(1) $(p_4, p_1, p_2, p_5, p_1, p_2, p_3, p_6)$ **is a walk from** p_4 **to** p_6 **it is not a trial since the edges**
S $\{p_1, p_2\}$ **is used twice .**

(2) $(p_4, p_1, p_5, p_2, p_6)$ **is not a walk since there is no edge** $\{p_2, p_6\}$

(3) $(p_4, p_1, p_5, p_2, p_3, p_5, p_6)$ **is a trial since no edge is used twice but not path since the vertex** p_5 **is used twice .**

(4) $(p_4, p_1, p_5, p_3, p_6)$ **is a path from** p_4 **to** p_6

(5) (p_4, p_5, p_6) **the shortest path has length 2 .**

Def :- **A graph is said to be connected if there is a path between any two of its vertices**

Th :- **There is a walk from a vertex u to a vertex** v **if and only if there is a path from u to** v

***The distance between vertices u and** v **of a connected graph G written d (u ,** v **) is the length of the shortest path between u and** v

***The diameter of a connected graph G is the maximum distance between any two of its vertices .**

***Def :-** **A vertex** v **in a connected graph G is called a cut point . if G –** v **is disconnected . the vertex in Fig (4) is a cut point .**

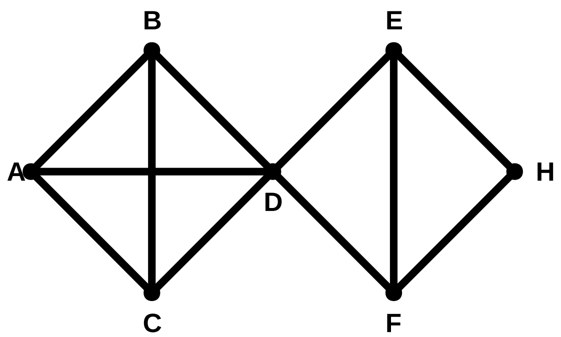


Fig (4)

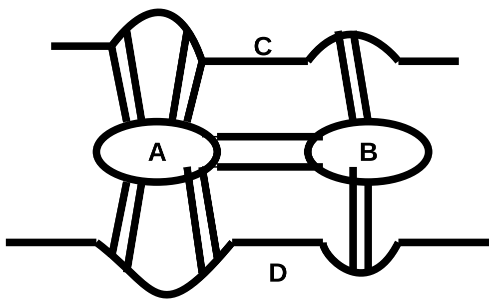
5.4 The Traversable Multi graphs :-

Def :- **A multi graph is said to be traversable if it can be drawn without any breaks in the curve and without repeating any edge .**

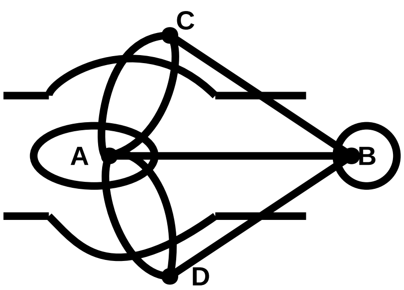
Theorem . (Euler) :- **A finite connected graph is eulerian if and only if each vertex has even degree .**

Corollary :- **Any finite connected graph with two odd vertices is traversable . A traversable trail may begin at either odd vertex and will end at the other odd vertex .**

***Hamiltonian graph :-** **it's a graph has a closed walk which includes each vertex exactly once .**



(a) konigsberg in 1736



(b) Eulers graphical representation

Fig (5)

EX (4) :- **Fig (6) shows a tranversable trail of the multi graph .**

(a) (b)

Fig (6)

EX (5) :-

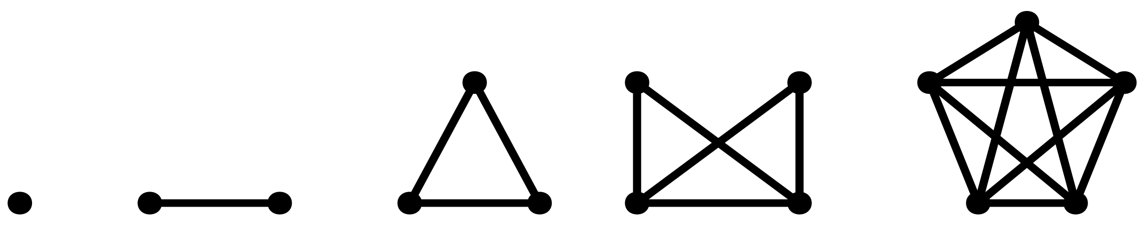
(a) hamiltonian & non Eulerian (b) Eulerian & non hamiltonian

Fig (7)

5.5 Special graphs :-

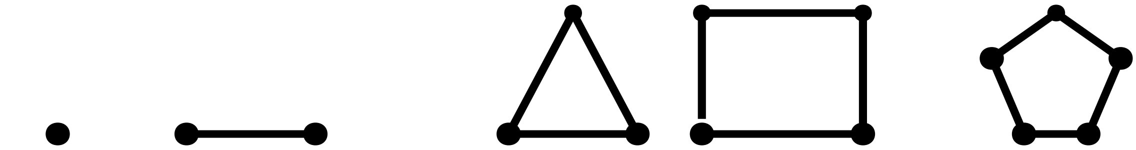
1. **Complete graph.**
2. **Regular graph .**
3. **Bipartite graph.**
4. **Tree graph .**

1 – Complete :- A graph is complete if each vertex is connected to every other vertex . The complete graph with n vertices is denote by K_n :-



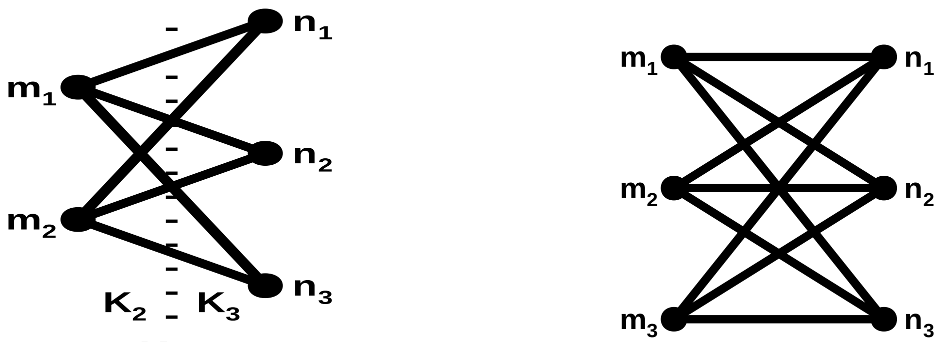
K_1 K_2 K_3 K_4 K_5

2- Regular :- A graph G is regular of degree K . K – regular if every has degree K . In other words a graph is regular if every vertex has the same degree .



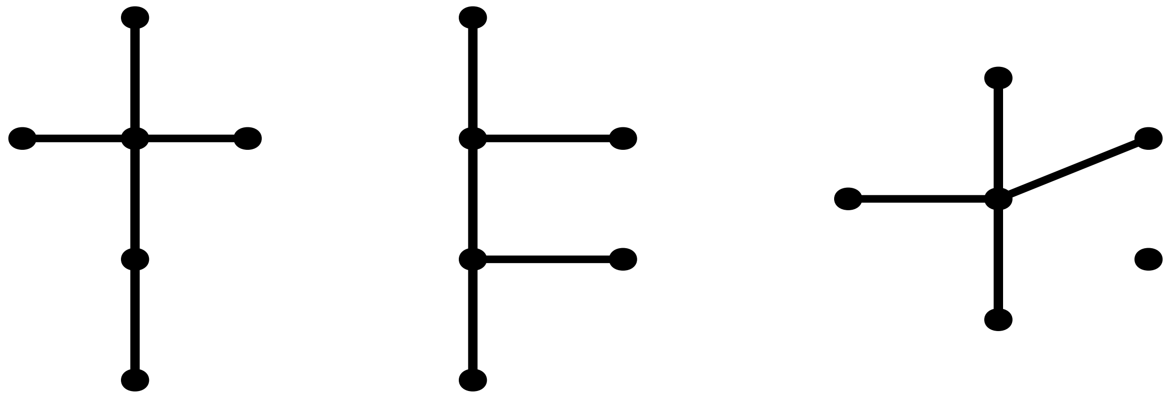
(i) 0-regular (ii) 1-regular (iii) 2-regular

<u>3- Bipartite :-</u> A graph G is said to be bipartite if its vertices V can be partitioned into two subset M and N such that each edge of G connects a vertex of M to a vertex of N , and denoted by $K_{m,n}$.



$K_{2,3}$ $K_{3,3}$

<u>4- Tree :-</u> A connected graph with no cycles is called a tree



<u>5.6 Matrices and Graphs :-</u>

<u>1- Adjacency matrix :-</u> let A = (aij) be the m x m matrix defined by :-

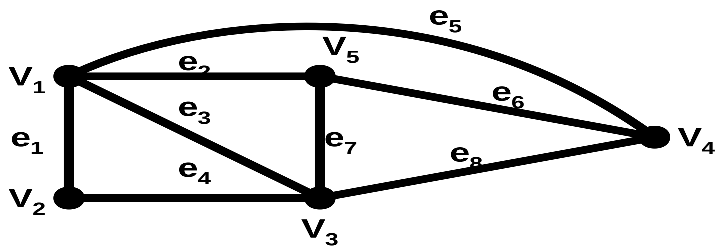
$$\text{aij} = \begin{cases} 1 \;\; if\ (V_i\ ,\ V_j)\ is\ an\ edge\ , i.e\ if\ V_i\ is\ adjacent\ to\ V_j \\ 0\ other\ wise\ . \end{cases}$$

The A is called the adjacency matrix of G .

<u>2- Incidence matrix :-</u> let M = (mij) be the m x m matrix defined by

mij = $\begin{cases} \mathbf{1} \;\; \boldsymbol{if\ the\ vertex\ \ V_i\ is\ incident\ on\ the\ edge\ ej} \\ \mathbf{0} \;\; \boldsymbol{other\ wice}\ . \end{cases}$

<u>Ex (6) :-</u>



$$A = \begin{array}{c|ccccc|} & v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & 0 & 1 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 0 & 1 & 1 & 0 \end{array}$$

$$M = \begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ v_1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ v_4 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ v_5 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

<u>5.7 Labeled graphs :-</u>

A graph G is called a labeled graph if its edges and / or vertices are assigned date of one kind or another .

A minimum path between p and Q in the following graph is ($P, A_1, A_2, A_5, A_3, A_6, Q$ **) , which has length 14 .**

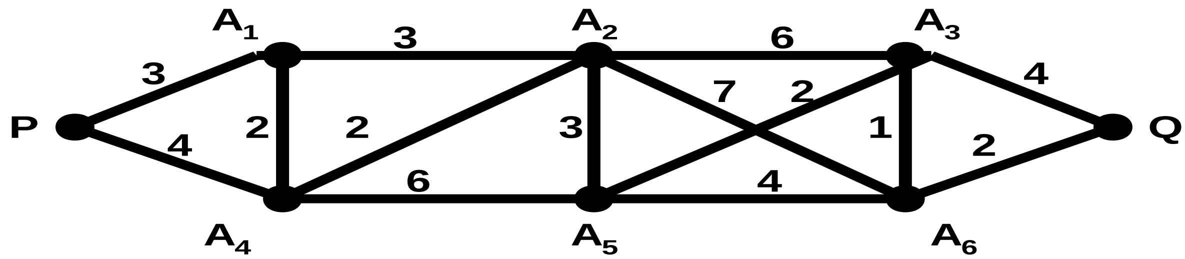
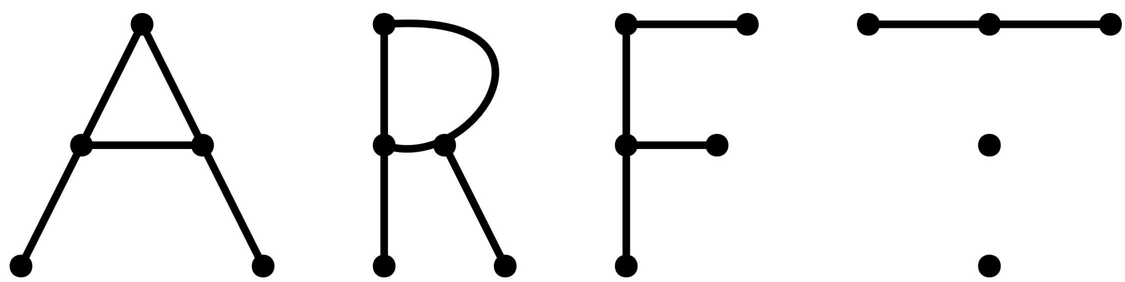


Fig (8)

5.8 Isomorphic Graphs :-

Suppose G (V , E) and $G^*(V^* , E^*)$ are graphs and f: V → V^* is a one – one correspondence between the sets of vertices such that { u , v } is an edge of G^* , then f is called an isomorphism between G and G^* .

EX (7) :- **A and R , F and T are isomorphic**

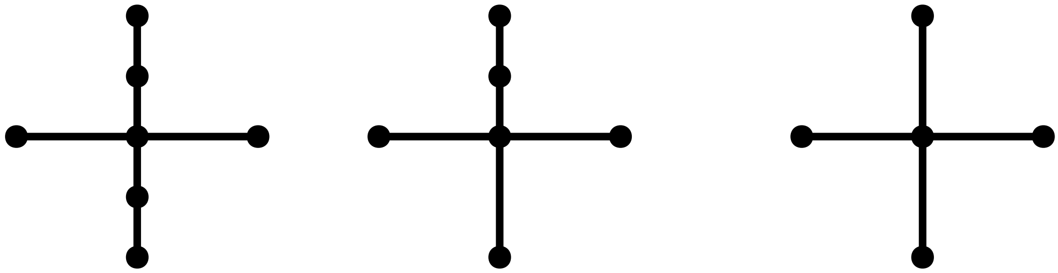


If G and G^* are isomorphic graphs , then corresponding vertices must have the same graphical properties , such as degree , being a cut point and so on .

*Homeomorphic :-

Given any graph G , we can obtain a new graph by dividing an edge of G with additional vertices . two graphs G and G^* are said to be homeomorphic if they can be obtained from isomorphic graphs by this method .

The graphs (a) and (b) in Fig (9) are not isomorphic ; but they are homeomorphic since each can be obtained from (c) by adding appropriate vertices .



(a) (b) (c)

<u>EX (8) :-</u> Draw the multi graph G whose adjacency matrix A = (aij)

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{vmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{vmatrix} \end{array}$$

<u>Sol.</u>