Chapter Eight

Combinatorial Analysis

8.1 Factorial notation :-

The product of the positive integers from 1 to n inclusive is denoted by n1 (read " n factorial "

$$n! = 1 . 2 . 3 \ldots\ldots (n-2)(n-1)\, n$$

In other words , n! is defined by

$$1! = 1 \qquad \text{and} \qquad n! = n . (n-1)!$$

It is also convenient to define 0! = 1

EX (1) :-

(a) $2! = 1 . 2 = 2$ $\qquad\qquad$ $4! = 1 . 2 . 3 . 4 = 24$

(b) $\frac{8!}{6!} = \frac{8 .7 .6!}{6!} = 8 . 7 = 56$ $\qquad\qquad$ $12 . 11 . 10 = \frac{12 .11 .10 .9!}{9!} = \frac{12!}{9!}$

$$\frac{12 . 11 . 10}{1 . 2 . 3} = 12 . 11 . 10 . \frac{1}{3!} = \frac{12!}{3!\ 9!}$$

(c) $\mathbf{n}(\mathbf{n}-1) \ldots\ldots\ldots (\mathbf{n}-\mathbf{r}+1) = \frac{\mathbf{n}(\mathbf{n}-1)\ldots(\mathbf{n}-\mathbf{r}+1)(\mathbf{n}-\mathbf{r})(\mathbf{n}-\mathbf{r}-1)\ldots\ldots 3.2.1}{(\mathbf{n}-\mathbf{r})(\mathbf{n}-\mathbf{r}-1)\ldots\ldots\ldots 3.2.1}$

$$= \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{r})!}$$

$$\frac{\mathbf{n}(\mathbf{n}-1)\ldots(\mathbf{n}-\mathbf{r}+1)}{1 . 2. 3 \ldots\ldots (r-1)r} = \mathbf{n}(\mathbf{n}-1)\ldots(\mathbf{n}-\mathbf{r}+1) . \frac{1}{\mathbf{r}!} = \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{r})!} . \frac{1}{\mathbf{r}!}$$

$$= \frac{\mathbf{n}!}{r!(n-r)!}$$

8.2 Binomial Coefficients :-

The symbol $\binom{\mathbf{n}}{\mathbf{r}}$ (read " n C r ") . where r and n are positive integers with r $\leq$ n , is defined as follows :-

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots.(n-r+1)}{1.2.3\ldots\ldots\ldots(r-1)r}$$

By example (1) – (C) . we see that :-

$$\binom{n}{r} = \frac{n(n-1)\ldots.(n-r+1)}{1.2.3\ldots\ldots\ldots(r-1)r} = \frac{n!}{r!\,(n-r)!}$$

But n – (n – r) = r ; hence we have the following important relation .

$\binom{n}{n-r} = \binom{n}{r}$ **or , in other words if a + b = n**

Then $\binom{n}{a} = \binom{n}{b}$

EX(2) :-

(a) $\binom{8}{2} = \frac{8.7}{1.2} = 28$

$$\binom{12}{5} = \frac{12.11.10.9.8}{1.2.3.4.5} = 792$$

Note that $\binom{n}{r}$ **has exactly r factors in both the numerator and the denominator .**

(b) Compute $\binom{10}{7}$ **by definition .**

$$\binom{10}{7} = \frac{10.9.8.7.6.5.4}{1.2.3.4.5.6.7} = 120$$

On the other hand , 10 – 7 = 3 and so we can also compute $\binom{10}{r7}$ **as follows :**

$$\binom{10}{7} = \binom{10}{3} = \frac{10.9.8}{1.2.3} = 120$$

8.3 Permutations :-

Any arrangement of a set of n objects is a given order is called a permutation of the objects (taken all at a time).

Any arrangement of any r ≤ n of these objects is a given order is called an r- permutation of the n object taken r at a time . for example :

(i) bdca , dcba and acdb are permutation of the four letters (taken all at a time):

(ii) bad , adb , cbd , and bca are permutation of the four letters taken three at a time .

(iii) ad , cb , da , and bd are permutation of the four letters taken two at a time .

we shall use p (n , r)

EX(3) :- How many permutation are there of six objects , say a , b , c , d , e and f , taken three at a time ? In other words . we want to find the number of "three – letter words" using the above six letters without repetitions .

Now the first letter can be chosen in six different ways ; following this , the second letter can be chosen in five different ways ; and following this , the last letter can be chosen in four different ways , write each number as follows :

Thus by the fundamental principle of counting there are 6.5.4 = 120 possible three – letter words without repetitions from the six letters . or there are 120 permutations of six objects taken three at a time :

$$P(6,3) = 120$$

$$P(n,r) = n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)$$

$$n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

Theorem : (8.2) $p(n,r) = \frac{n!}{(n-r)!}$

In the special case in which r = n . we have $p(n,n) = n(n-1)(n-2)\ldots.3.2.1 = n!$

(because $o! = 1$)

Corollary 8.3 :-

There are $n!$ permutations of n objects (taken all at a time)

For example , there are 3 ! = 1 . 2 . 3 = 6 permutations of the three letters a , b and c .

These are a b c , a c b , b a c , c a b , c b a , b c a

8.4 permutations and Repetitions :-

Theorem 8.6 :- let A contain n elements and let n1 , n2 , ……. nr be positive integers with n1 + n2 + ……… + nr = n then there exist !

$$\frac{n!}{n1!\ n2!\ n3!\ldots\ldots nr!}$$

different ordered partitions of A of the form (A_1 , A_2 … … A_r) where A ! contain n1 elements , A_2 contains n_2 elements , ……….. and A_r contain n_r elements .

we apply this theorem in the next example .

how many ways can nine toys be divided between four children if the youngest child is to received three toy and each of the others two toys ?

we wish to find the number of ordered partitions of the nine toys into four cells containing 3 , 2 , 2 and 2 toys respectively

By theorem (8.6) there are : such ordered partitions .

$$\frac{9!}{3!\ 2!\ 2!\ 2!} = 7560$$

EX(5) :-

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7!}{2!\ \ 5!} \cdot \frac{5!}{3!\ \ 2!} \cdot \frac{2!}{2!\ \ 0!} = \frac{7!}{2!\ \ 3!\ \ 2!}$$

Since each numerator after the first is canceled by the second term in the denominator of the previous factor .