Chapter Six

Planar Graphs

6.1 Planar Graphs :-

Def :- A graph or multi graph which can be drawn in the plane so that its edges do not cross is said to be planar .

EX (1) :-

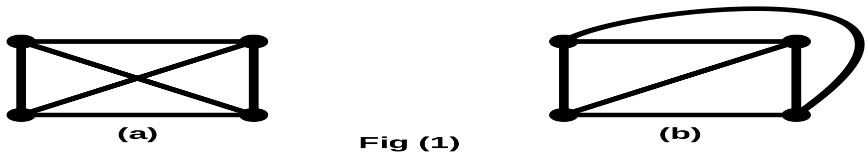


Fig (1)

*Maps :-

Def :- A particular planar representation of a finite planar multi graph is called a map . A given map divides the plane into various regions .

EX (2) :-

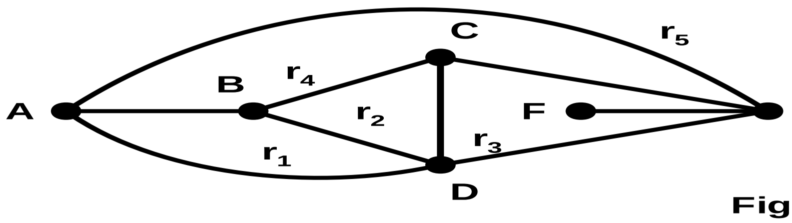


Fig (2)

The map with six vertices and nine edges divides the plane into five regions .

Theorem 6.1 :- The sum of the degrees of the regions of a map is equal to twice the number of edges .

The degree of the regions of fig (2) :-

deg (r_1) = 3 , deg (r_2) = 3 , deg (r_3) = 5 , deg (r_4) = 4 , deg (r_5) = 3

The sum of the degree is 18 , which , as expected , is twice the number of edges .

6.2 Euler's Formula :-

Theorem :- (Euler formula)

$$V - E + R = 2$$

Where V is the number of vertices , E is the number of edges , and R is the number of

regions For Fig (2) . $6 - 9 + 5 = 2$

Theorem 6.3 :- let G be a connected planar graph with p vertices and q edges , where

$$P \geq 3 \text{ . Then } q \leq 3p - 6$$

Proof :- let r be the number of regions in a planar representation of G by Euler formula .

$$P - q + r = 2$$

Now the sum of the degree of the regions equals 2q by .

Theorem (6.1) :- [The sum of the degree of the regions of a map is equal to twice the number of edges)

But each regions has degree 3 or more : hence .

$$2q \geq 3r$$

Thus $r \leq 2q/3$ **. substituting this in Euler s formula gives .**

$$2 = p - q + r \leq p - q = 2q / 3$$

Or $2 \leq p - q / 3$

Multiplying the inequality by 3 gives $6 \leq 3P - q$

Which gives us our result . $q \leq 3P - 6$

EX(3) :- For Fig (3) : k_s

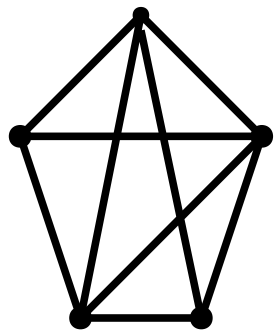


Fig (3)

P = 5 , q = 10

$q \leq 3p - 6 \rightarrow 10 \leq 3(5) - 6 = 9$

which is impossible . Thus is non planar .

Theorem : (kurato wsk I) :-

A graph is non planar if and only if it contains a sub graph homeomorphic to $k_{3,3}$ or k_5.

6.3 Colored Graphs :-

A vertex coloring , or simply coloring , of a graph G is an assignment of colors to the vertices of G such that adjacent vertices have different colors.

Def :- **we say that G is n – colorable if there exists a coloring of G which uses n colors (since the word " color " is used a noun).**

Theorem :- **A planar graph G is 5 – colorable .**

6.4 Four color theorem :-

Consider a map M (i.e a planar representation of a finite planar multi graph) . Two regions of M are said to be adjacent if they have an edge in common .

<u>EX(4):-</u> In the Fig (4)

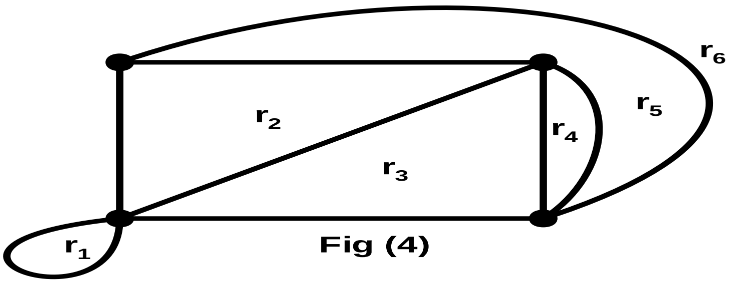


Fig (4)

The regions r_2 and r_3 are adjacent but the regions r_3 and r_5 are not . By a coloring of M we mean an assignment of a color to each region of M such that adjacent regions have different colors . A map M is n – colorable if there exists a coloring of M which uses n colors . The map in Fig (4) is 3 – colorable since the region could be painted as follows :-

r_1 red , r_2 white , r_3 red , r_4 white , r_5 red , r_6 blue .

<u>6.5 Trees :-</u>

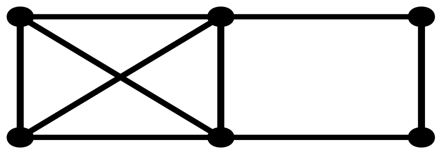
<u>Def :-</u> A tree is a connected graph with no cycles .

<u>Theorem :-</u> Trees are 2 – colorable

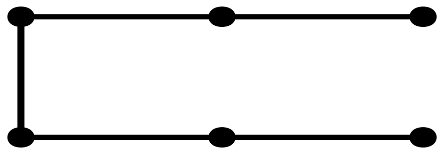
<u>Def :-</u> A sub graph T of a graph G is called a spanning tree of G if T is a tree and T include all the vertices of G

<u>Def :-</u> The minimal spanning tree of G is a spanning tree of G such that the sum of the length of the edges is minimal among all spanning trees of G .

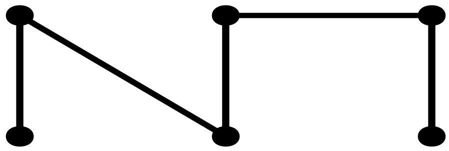
EX (5) :-



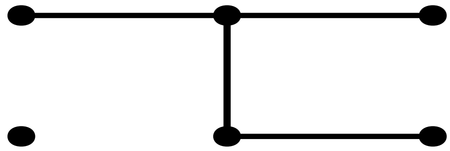
G



T_1



T_2

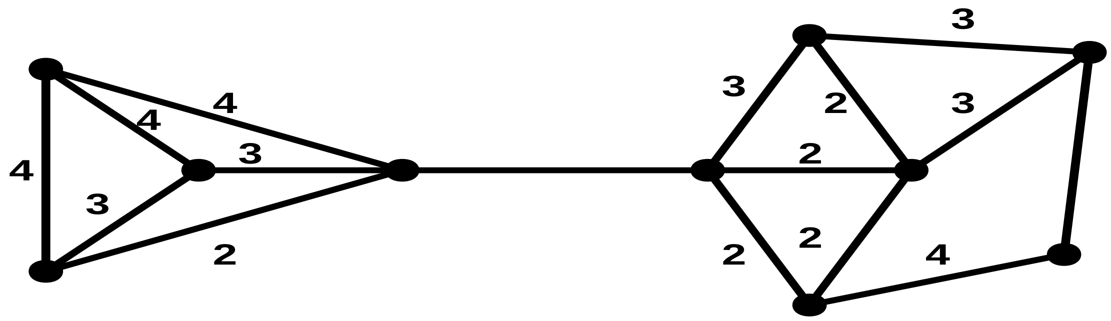


T_3

G : is a graph

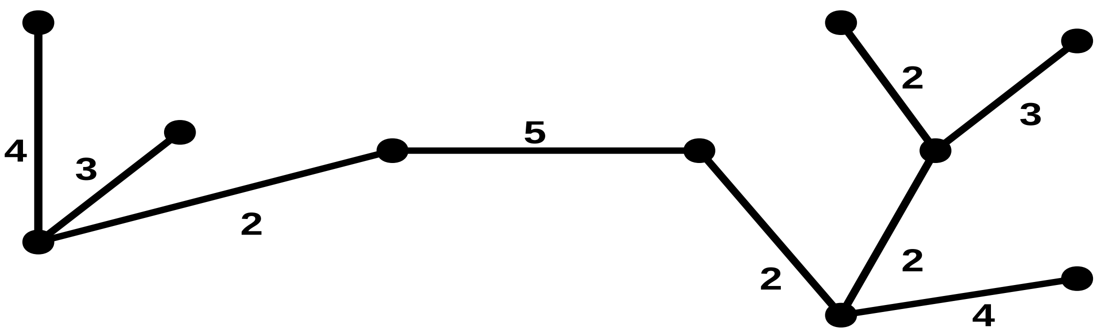
T_1 , T_2 , T_3 : are spanning tree of G

EX (6) :- **Find the minimal spanning tree of G**



G (graph)

Sol :-



T (minimal spanning tree)

6.6 Rooted Trees :-

Def :- A rooted tree R consists of a tree graph together with a designated vertex r called the root of the tree .

***Level :-** The length of the path from the root r to v is called the level or depth of v

***Leaf :-** Those vertices with degree one , other than r , are called the leaves of the rooted tree .

***Branch :-** A directed path from a vertex to a leaf is called a branch .

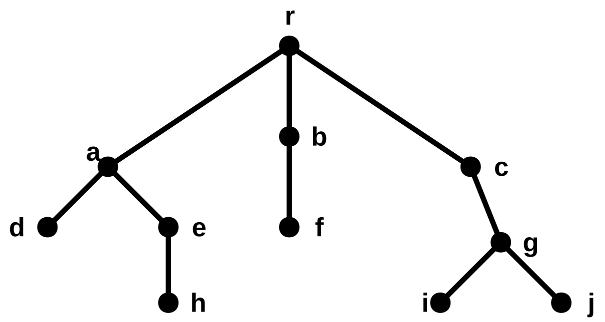


Fig (5)

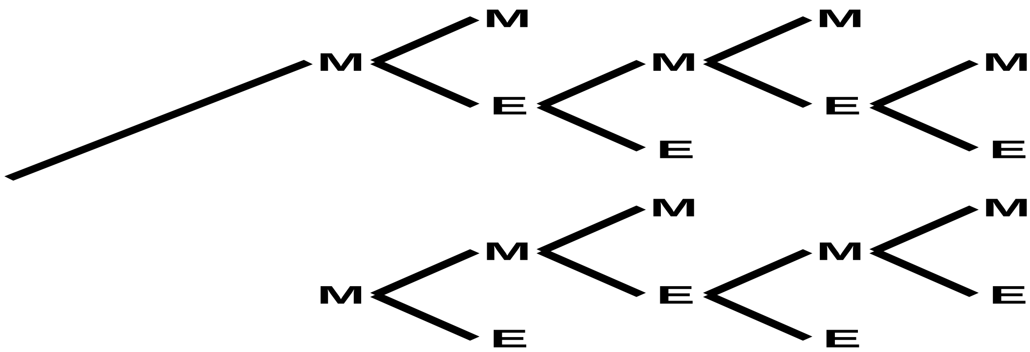
- Fig (5) shows a rooted tree :-

The root r is at the top of the tree .

The tree has five leaves , d , f , h , i , and j .

- **The level of a is 1 , the level of f is 2 and the level of j is 3 .**

<u>EX (7) :- </u>Fig (6) shows the various ways , the tournament can proceed find the leaves which correspond to the ways .



<u>Sol :-</u>

Observe that there ten leaves which correspond to the ten ways

MM , MEMM , MEMEM , MEME E , MEE , EMM , EMEMM , EMEME , EMEE , EE