4CCS1ELA–Elementary Logic with Applications

3–Important Semantical Notions

3.1–Complete sets of connectives, truth-functions and substitutions

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Outline

1. Complete Sets of Connectives
2. Truth-functions
3. Substitution Instances

Complete Sets of Connectives

Complete Sets of Connectives

A set of connectives is called *complete* (or *adequate*) if every formula of propositional logic is equivalent to a formula using only connectives from this set.

Since every formula has a disjunctive normal form, the set $\{\neg, \wedge, \vee\}$ **is complete.**

From the De Morgan's laws we have

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$
$$P \wedge Q \equiv \neg(\neg P \vee \neg Q).$$

Therefore **both** sets of connectives $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are **complete**.

Complete Sets of Connectives (cont)

To show that a given set of connective is complete all we need to do is to **express it in terms of a known complete set of connectives**.

Example. The set $\{\neg, \rightarrow\}$ is complete because $P \rightarrow Q \equiv \neg P \vee Q$. Thus, the set $\{\neg, \rightarrow\}$ is expressed in terms of the complete set $\{\neg, \vee\}$.

No singleton set from the *standard set of connectives* $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is complete.

However, the (non-standard) sets containing the *Sheffer stroke* $\{\,|\,\}$ (Tutorial list 2) and Pierce's arrow $\{\downarrow\}$ (Tutorial list 3) are complete.

TRUTH-FUNCTIONS

Truth-functions

A **truth-function** is a function whose arguments can take only the values *true* (or 1) and *false* (or 0) and return either the value *true* (or 1) or the value *false* (or 0).

Formally, a **truth-function** f is a function $f : \{0,1\}^n \mapsto \{0,1\}$.

Any wff defines a truth-function, and vice-versa.

Example. Let f be the truth-function defined as follows:

$$f(P,Q,R) = 1 \text{ iff either } P = Q = 0 \text{ or } Q = R = 1$$

Then f is equal to 1 in exactly the following four cases:

$$f(0,0,0), f(0,0,1), f(0,1,1), f(1,1,1)$$

Truth-functions and Normal Forms

In terms of a truth-table, f can be computed as follows:

interpretation	P	Q	R	f
v_0	0	0	0	1
v_1	0	0	1	1
v_2	0	1	0	0
v_3	0	1	1	1
v_4	1	0	0	0
v_5	1	0	1	0
v_6	1	1	0	0
v_7	1	1	1	1

And from our conversion of a formula to DNF from its truth-table, f can be represented by the formula

$$(\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge Q \wedge R)$$

SUBSTITUTION INSTANCES

Substitution

Uniform substitution of formulae for propositional variables

Let $W, H_1, \ldots, H_n$ be formulae and $P_1, \ldots, P_n$ be propositional variables.

Then the expression $W(P_1/H_1, \ldots, P_n/H_n)$ denotes the formula obtained by replacing *simultaneously* all occurrences of P_1 in W by the formula H_1, all occurrences of P_2 in W by the formula H_2, …, and all occurrences of P_n by the formula H_n.

Example

Let W be $P \to (Q \to P)$, then $W(P/\neg P \vee R, Q/\neg P)$ is the formula $\neg P \vee R \to (\neg P \to \neg P \vee R)$.

We say that the formula $\neg P \vee R \to (\neg P \to \neg P \vee R)$ is a *substitution instance* of $P \to (Q \to P)$.

Questions.

What type of formula is $P \to (Q \to P)$?
What does that make $\neg P \vee R \to (\neg P \to \neg P \vee R)$?

Counter-example. . .

The formula $(\neg P \vee R \to \neg P) \to \neg P \vee R$ is **not** a substitution instance of $P \to (Q \to P)$!

- The two formulas have different structures!
- If we replace P by $(\neg P \vee R \to \neg P)$, then we would have to replace Q by $\neg P \vee R$, but we would still miss the consequent of $Q \to P$...
- Notice that $(\neg P \vee R \to \neg P) \to ((\neg P \vee R) \to (\neg P \vee R \to \neg P))$ **is** a substitution instance!

Substitution Properties

From $F \equiv G$ we can conclude

$$F(P_1/H_1, \dots, P_n/H_n) \equiv G(P_1/H_1, \dots, P_n/H_n).$$

Example. We can obtain a new equivalence as a substitution instance of the De Morgan law:

From the logical equivalence $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$, we have

$$\neg(\underline{(P \to R)} \vee \underline{(R \leftrightarrow Q)}) \equiv \neg\underline{(P \to R)} \wedge \neg\underline{(R \leftrightarrow Q)}.$$

Thus, a new tautology is generated:

$$\neg(\underline{(P \to R)} \vee \underline{(R \leftrightarrow Q)}) \leftrightarrow \neg\underline{(P \to R)} \wedge \neg\underline{(R \leftrightarrow Q)}.$$

To know more...

The material in this part of the lecture is discussed in Section 6.2 of Hein's “Discrete Structures, Logic, and Computability”, 4th edition.