Summary of natural deduction rules

Basic rules

(∧I)	$\dfrac{A, B}{A \wedge B}$	**Conjunction**	$\dfrac{A, B}{B \wedge A}$	(∧I)
(∧E)	$\dfrac{A \wedge B}{A}$		$\dfrac{A \wedge B}{B}$	(∧E)
(∨I)	$\dfrac{A}{A \vee B}$	**Disjunction**	$\dfrac{B}{A \vee B}$	(∨I)
(∨E)	$\dfrac{A \to C, B \to C, A \vee B}{C}$			
(→I)	If $\dfrac{\text{assumptions, } A}{B}$ then $\dfrac{\text{assumptions}}{A \to B}$ (antecedent of the "If . . . then" shown in a subcomputation box)	**Implication**	$\dfrac{A, A \to B}{B}$ (also known as modus ponens)	(→E)
(¬I)	$\dfrac{A \to B, A \to \neg B}{\neg A}$	**Negation**	$\dfrac{\neg A \to B, \neg A \to \neg B}{A}$	(¬E)
(↔I)	$\dfrac{(A \to B) \wedge (B \to A)}{A \leftrightarrow B}$	**Equivalence**	$\dfrac{A \leftrightarrow B}{(A \to B) \wedge (B \to A)}$	(↔E)

Variant rules

(∨E1)	$\dfrac{A \vee B, \neg A}{B}$ (disjunctive syllogism)	**Disjunction**	$\dfrac{A \vee B, \neg B}{A}$ (disjunctive syllogism)	(∨E2)
(CD)	$\dfrac{A \to C, B \to D, A \vee B}{C \vee D}$ (constructive dilemma)		$\dfrac{A \to C, B \to D, \neg C \vee \neg D}{\neg A \vee \neg B}$ (destructive dilemma)	(DD)
(→I1)	$\dfrac{\neg A}{A \to B}$	**Implication**	$\dfrac{B}{A \to B}$	(→I2)
(→E1)	$\dfrac{A \to B}{\neg A \vee B}$			
(¬E1)	$\dfrac{\neg\neg A}{A}$	**Negation**	$\dfrac{\neg A \to B, A \to B}{B}$	(¬E2)
(¬I1)	$\dfrac{A}{\neg\neg A}$		$\dfrac{A \to B, \neg A \to B}{B}$	(¬2)

Proofs of the Derived Rules from the Basic Ones

(∨E1) $\dfrac{\boldsymbol{A \vee B, \neg A}}{\boldsymbol{B}}$

1. $A \vee B$ data
2. $\neg A$ data
3. $A \to B$ (→I) (subcomputation box below)

		$\underline{B}$
3.1 A	assume	
3.2 $\neg B \to \neg A$	(→I) (subcomputation box below)	
		$\underline{\neg A}$
3.3.1 $\neg B$	assume	
3.3.2 $\neg A$	from 1.	
3.3 $\neg B \to A$	(→I) (subcomputation box below)	
		$\underline{A}$
3.3.1 $\neg B$	assume	
3.3.2 A	from 3.1	
3.4 B	3.2, 3.3 and (¬E)	

4. $B \to B$ (→I) (subcomputation box below)

		$\underline{B}$
4.1 B	assume	
4.2 B	from 4.1	

5. B from 1., 3., 4., and (∨E)

(CD) $\dfrac{\boldsymbol{A \to C, B \to D, A \vee B}}{\boldsymbol{C \vee D}}$

1. $A \to C$ data
2. $B \to D$ data
3. $A \vee B$ data
4. $A \to (C \vee D)$ (→I) (subcomputation box below)

		$\underline{C \vee D}$
4.1 A	assume	
4.2 C	1., 4.1, and (→E)	
4.3 $C \vee D$	4.2, and (∨I)	

5. $B \to (C \vee D)$ (→I) (subcomputation box below)

		$\underline{C \vee D}$
5.1 B	assume	
5.2 D	2., 5.1, and (→E)	
5.3 $C \vee D$	5.2, and (∨I)	

6. $C \vee D$ from 3., 4., 5., and (∨E)

(DD) $\dfrac{\boldsymbol{A\to C},\ \boldsymbol{B\to D},\ \boldsymbol{\neg C\vee\neg D}}{\boldsymbol{\neg A\vee\neg B}}$

1. $A\to C$ data
2. $B\to D$ data
3. $\neg C\vee\neg D$ data
4. $\neg C\to(\neg A\vee\neg B)$ ($\to$I) (subcomputation box below)

		$\underline{\neg A\vee\neg B}$
4.1 $\neg C$	assume	
4.2 $A\to\neg C$	($\to$I) (subcomputation box below)	
		$\underline{\neg C}$
	4.2.1 A	assume
	4.2.2 $\neg C$	from 4.1
4.3 $\neg A$	1, 4.2 and ($\neg$I)	
4.4 $\neg A\vee\neg B$	4.3 and ($\vee$I)	

5. $\neg D\to(\neg A\vee\neg B)$ ($\to$I) (subcomputation box below)

		$\underline{\neg A\vee\neg B}$
5.1 $\neg D$	assume	
5.2 $B\to\neg D$	($\to$I) (subcomputation box below)	
		$\underline{\neg D}$
	5.2.1 B	assume
	5.2.2 $\neg D$	from 5.1
5.3 $\neg B$	2, 5.2, and ($\neg$I)	
5.4 $\neg A\vee\neg B$	5.3 and ($\vee$I)	

6. $\neg A\vee\neg B$ from 3., 4., 5., and ($\vee$E)

($\to$I1) $\dfrac{\boldsymbol{\neg A}}{\boldsymbol{A\to B}}$

1. $\neg A$ data
2. $A\to B$ ($\to$I) (subcomputation box below)

		$\underline{B}$
2.1 A	assume	
2.2 $\neg B\to\neg A$	($\to$I) (subcomputation box below)	
		$\underline{\neg A}$
	2.2.1 $\neg B$	assume
	2.2.2 $\neg A$	from 1
2.3 $\neg B\to A$	($\to$I) (subcomputation box below)	
		$\underline{A}$
	2.2.1 $\neg B$	assume
	2.2.2 A	from 2.1
2.4 B	from 2.2, 2.3 and ($\neg$E)	

(→I2) $\dfrac{\boldsymbol{B}}{\boldsymbol{A\to B}}$

1. B data
2. $A\to B$ (→I) (subcomputation box below)

		$\underline{B}$
2.1	A	assume
2.2	B	from 1.

(→E1) $\dfrac{\boldsymbol{A\to B}}{\boldsymbol{\neg A\vee B}}$

1. $A\to B$ data
2. $\neg(\neg A\vee B)\to(\neg A\vee B)$ (→I) (subcomputation box below)

			$\underline{\neg A\vee B}$
2.1	$\neg(\neg A\vee B)$	assume	
2.2	$A\to(\neg A\vee B)$	(→I) (subcomputation box below)	
	2.2.1 A	assume	$\underline{\neg A\vee B}$
	2.2.2 B	from 2.2.1, 1., and (→E)	
	2.2.3 $\neg A\vee B$	from 2.2.2 and (∨I)	
2.3	$A\to\neg(\neg A\vee B)$	(→I) (subcomputation box below)	
	2.3.1 A	assume	$\underline{\neg(\neg A\vee B)}$
	2.3.2 $\neg(\neg A\vee B)$	from 2.1	
2.4	$\neg A$	2.2, 2.3, (¬I)	
2.5	$\neg A\vee B$	2.5 and (∨I)	

3. $\neg(\neg A\vee B)\to\neg(\neg A\vee B)$ (→I) (subcomputation box below)

		$\underline{\neg(\neg A\vee B)}$
3.1	$\neg(\neg A\vee B)$	assume
3.2	$\neg(\neg A\vee B)$	from 3.1

4. $\neg A\vee B$ from 2., 3., and (¬E)

(¬E1) $\dfrac{\neg\neg A}{A}$

1. $\neg\neg A$ data
2. $\neg A\to\neg A$ ($\to$I) (subcomputation box below)

		$\underline{\neg A}$
2.1 $\neg A$	assume	
2.2 $\neg A$	from 2.1	

3. $\neg A\to\neg\neg A$ ($\to$I) (subcomputation box below)

		$\underline{\neg\neg A}$
3.1 $\neg A$	assume	
3.2 $\neg\neg A$	from 1	

4. A from 2., 3., and ($\neg$E)

(¬E2) $\dfrac{\neg A\to B,\ A\to B}{B}$

1. $\neg A\to B$ data
2. $A\to B$ data
3. $\neg B\to\neg A$ ($\to$I) (subcomputation box below)

		$\underline{\neg A}$
3.1 $\neg B$	assume	
3.2 $A\to\neg B$	($\to$I) (subcomputation box below)	
		$\underline{\neg B}$
	3.2.1 A assume	
	3.2.2 $\neg B$ from 3.1	
3.3 $\neg A$	from 2., 3.2, and ($\neg$I)	

4. $\neg B\to A$ ($\to$I) (subcomputation box below)

		$\underline{\neg B}$
4.1 $\neg B$	assume	
4.2 $\neg A\to\neg B$	($\to$I) (subcomputation box below)	
		$\underline{\neg B}$
	4.2.1 $\neg A$ assume	
	4.2.2 $\neg B$ from 4.1	
4.3 A	from 1., 4.2 and ($\neg$E)	

5. B from 3., 4., and ($\neg$E)

(¬I1) $\dfrac{\boldsymbol{A}}{\boldsymbol{\neg\neg A}}$

1. A data
2. $\neg A \to \neg A$ ($\to$I) (subcomputation box below)

		$\underline{\neg A}$
2.1 $\neg A$	assume	
2.2 $\neg A$	from 2.1	

3. $\neg A \to A$ ($\to$I) (subcomputation box below)

		$\underline{A}$
3.1 $\neg A$	assume	
3.2 A	from 1	

4. $\neg\neg A$ from 2., 3., and (¬I)

(¬2) $\dfrac{\boldsymbol{A \to B, \neg A \to B}}{\boldsymbol{B}}$

1. $A \to B$ data
2. $\neg A \to B$ data
3. $\neg B \to B$ ($\to$I) (subcomputation box below)

		$\underline{B}$
3.1 $\neg B$	assume	
3.2 $A \to \neg B$	($\to$I) (subcomputation box below)	

		$\underline{\neg B}$
3.2.1 A	assume	
3.2.2 $\neg B$	from 3.1	

3.3 $\neg A$	from 1., 3.2, and (¬I)	
3.4 B	from 3.3, 2., and ($\to$E)	

4. $\neg B \to \neg B$ ($\to$I) (subcomputation box below)

		$\underline{\neg B}$
4.1 $\neg B$	assume	
4.2 $\neg B$	from 4.1	

5. B from 3., 4., and (¬E)