4CCS1RW1

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THEOREM 3–The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\frac{df^{-1}}{dx}\Big|_{x=b} = \frac{1}{\frac{df}{dx}\big|_{x=f^{-1}(b)}}$$

Theorem 3 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$\begin{aligned} f(f^{-1}(x)) &= x && \text{Inverse function relationship} \\ \frac{d}{dx} f(f^{-1}(x)) &= 1 && \text{Differentiation both sides} \\ f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) &= 1 && \text{Chain Rule} \\ \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))}. && \text{Solving for the derivative} \end{aligned}$$

EXAMPLE 1 The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f'1)(x) = 1/(2\sqrt{x})$.

Let's verify that Theorem 3 gives the same formula for the derivatives of $f^{-1}(x)$:

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} && f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x) \\ &= \frac{1}{2(\sqrt{x})}. \end{aligned}$$

Theorem 3 gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 3 at a specific point. We pick $x = 2$ (the number *a*) and $f(2) = 4$ (the value *b*). Theorem 3 says that the derivative of f at 2,which is $f'(2) = 4$, and the derivative of f^{-1} at$f(2)$, which is$(f^{-1})'(4)$,are reciprocals.It states that

$$f^{-1}(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}.$$

See Figure 3.36.

We will use the procedure illustrated in Example 1 to calculate formulas for the derivatives of many inverse functions throughout this chapter.Equation(1)sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

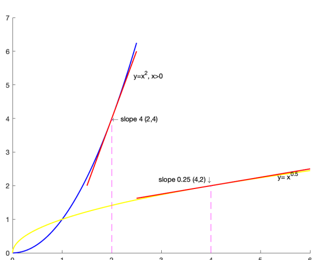

Figure 3.36

The derivative of $f^{-1} = \sqrt{x}$ at the point $(4, 2)$ is the reciprocal of the derivative of $f(x) = x^2$ at $(2, 4)$)(Example 1).

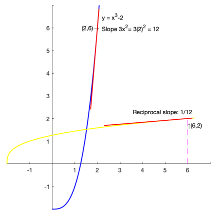

Figure 3.37

The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the the derivative of f^{-1} at $x = 6$)(Example 2).

EXAMPLE 2 Let $f(x) = x^3 - 2, x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution We apply Theorem 3 to obtain the value of the derrivative of f^-1 at $x = 6$:

$$\frac{df}{dx}\Big|_{x=2} = 3x^2\Big|_{x=2} = 12$$

$$\frac{df^{-1}}{dx}\Big|_{x=f(2)} = \frac{1}{\frac{df}{dx}\Big|_{x=2}} = \frac{1}{12}. \qquad \text{Eq (1)}$$

See Figure 3.37.

Derivative of the Natural Logarithm Function

Since we know the exponential function *f(x)* = e^x is differentiable everywhere, we can apply Theorem 3 to find the derivative of its inverse f^-1*(x)* = *ln x:*

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'((f^{-1})(x))} && \text{Theorem 3} \\
&= \frac{1}{e^{f^{-1}(x)}} && f'(u) = e^u \\
&= \frac{1}{e^{\ln x}} && x > 0 \\
&= \frac{1}{x}. && \text{Inverse function relationship}
\end{aligned}$$

Alternate Derivation Instead of applying Theorem 3 directly, we can find the derivative of $y = \ln x$ using implicit differentiation, as follows:

$$\begin{aligned}
y &= \ln x && x > 0 \\
e^y &= x && \text{Inverse function relationship} \\
\frac{d}{dx}(e^y) &= \frac{d}{dx}(x) && \text{Differentiate implicitly.} \\
e^y\frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. && e^y = x
\end{aligned}$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to x is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0.$$

The Chain Rule extends this formula to positive functions $u(x)$:

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, u > 0. \qquad (1)}$$