4CCS1ELA–ELEMENTARY LOGIC WITH APPLICATIONS

5–PROVING WITH NATURAL DEDUCTION

5.2–EXAMPLES USING THE BASIC RULES

Dr. Odinaldo Rodrigues

odinaldo.rodrigues@kcl.ac.uk

Room BH(S) TBC, +44 (0)20 7848 2087

Department of Informatics

King's College London

Worked Examples

Sample proof 1

Show that $(A \vee B) \rightarrow C \vdash (A \rightarrow C) \wedge (B \rightarrow C)$

Sample proof 1

Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1.	$(A \vee B) \to C$	Data

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Show that $(A \vee B) \rightarrow C \vdash (A \rightarrow C) \wedge (B \rightarrow C)$

1.	$(A \vee B) \rightarrow C$	Data
2.	$A \rightarrow C$	From subcomputation box below

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Show that $(A \vee B) \rightarrow C \vdash (A \rightarrow C) \wedge (B \rightarrow C)$

1.	$(A \vee B) \rightarrow C$	Data
2.	$A \rightarrow C$	From subcomputation box below
3.	$B \rightarrow C$	From subcomputation box below

Sample proof 1

Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1. $(A \vee B) \to C$ — Data
2. $A \to C$ — From subcomputation box below
3. $B \to C$ — From subcomputation box below
4. $(A \to C) \wedge (B \to C)$ — From 2. and 3. and $(\wedge I)$

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Show that $(A \vee B) \rightarrow C \vdash (A \rightarrow C) \wedge (B \rightarrow C)$

1. $(A \vee B) \rightarrow C$ Data
2. $A \rightarrow C$ From subcomputation box below

$$\boxed{\quad \underline{C} \quad}$$

3. $B \rightarrow C$ From subcomputation box below
4. $(A \rightarrow C) \wedge (B \rightarrow C)$ From 2. and 3. and $(\wedge I)$

Sample proof 1

Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1. $(A \vee B) \to C$ Data
2. $A \to C$ From subcomputation box below

2.1	A	Assume	$\underline{C}$

3. $B \to C$ From subcomputation box below

4. $(A \to C) \wedge (B \to C)$ From 2. and 3. and $(\wedge I)$

Sample proof 1

Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1. $(A \vee B) \to C$ Data
2. $A \to C$ From subcomputation box below

			$\underline{C}$
2.1	A	Assume	
2.2	$A \vee B$	From 2.1 and $(\vee\mathrm{I})$	

3. $B \to C$ From subcomputation box below

4. $(A \to C) \wedge (B \to C)$ From 2. and 3. and $(\wedge\mathrm{I})$

Sample proof 1

Show that $(A \vee B) \rightarrow C \vdash (A \rightarrow C) \wedge (B \rightarrow C)$

1.	$(A \vee B) \rightarrow C$	Data
2.	$A \rightarrow C$	From subcomputation box below

$\underline{C}$

2.1	A	Assume
2.2	$A \vee B$	From 2.1 and ($\vee$I)
2.3	C	From 2.2, 1. and ($\rightarrow$E)

3.	$B \rightarrow C$	From subcomputation box below
4.	$(A \rightarrow C) \wedge (B \rightarrow C)$	From 2. and 3. and ($\wedge$I)

Sample proof 1

Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1.	$(A \vee B) \to C$	Data
2.	$A \to C$	From subcomputation box below

$\underline{C}$

2.1	A	Assume
2.2	$A \vee B$	From 2.1 and ($\vee$I)
2.3	C	From 2.2, 1. and ($\to$E)

3.	$B \to C$	From subcomputation box below

$\underline{C}$

4.	$(A \to C) \wedge (B \to C)$	From 2. and 3. and ($\wedge$I)

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Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1.	$(A \vee B) \to C$	Data
2.	$A \to C$	From subcomputation box below

		$\underline{C}$
2.1	A	Assume
2.2	$A \vee B$	From 2.1 and ($\vee$I)
2.3	C	From 2.2, 1. and ($\to$E)

3.	$B \to C$	From subcomputation box below

		$\underline{C}$
3.1	B	Assume

4.	$(A \to C) \wedge (B \to C)$	From 2. and 3. and ($\wedge$I)

Sample proof 1

Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1.	$(A \vee B) \to C$	Data
2.	$A \to C$	From subcomputation box below

			$\underline{C}$
2.1	A	Assume	
2.2	$A \vee B$	From 2.1 and $(\vee I)$	
2.3	C	From 2.2, 1. and $(\to E)$	

3.	$B \to C$	From subcomputation box below

			$\underline{C}$
3.1	B	Assume	
3.2	$A \vee B$	From 3.1 and $(\vee I)$	

4.	$(A \to C) \wedge (B \to C)$	From 2. and 3. and $(\wedge I)$

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1.	$(A \vee B) \to C$	Data
2.	$A \to C$	From subcomputation box below

		$\underline{C}$
2.1	A	Assume
2.2	$A \vee B$	From 2.1 and ($\vee$I)
2.3	C	From 2.2, 1. and ($\to$E)

3.	$B \to C$	From subcomputation box below

		$\underline{C}$
3.1	B	Assume
3.2	$A \vee B$	From 3.1 and ($\vee$I)
3.3	C	From 3.2, 1. and ($\to$E)

4.	$(A \to C) \wedge (B \to C)$	From 2. and 3. and ($\wedge$I)

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

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$$\frac{A \rightarrow C, B \rightarrow C, A \vee B}{C} \quad (\vee\mathrm{E})$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1. $A \vee B$ data
2. $\neg B$ data

$$\frac{A \to C, B \to C, A \vee B}{C} \quad (\vee E)$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1. $A \vee B$ data
2. $\neg B$ data
3. $A \to A$ From subcomputation box below
4. $B \to A$ From subcomputation box below
5. A From 1., 3., 4., and ($\vee$E)

$$\frac{A \to C,\ B \to C,\ A \vee B}{C}\ (\vee\mathrm{E})$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1. $A \vee B$ data
2. $\neg B$ data
3. $A \to A$ From subcomputation box below

			$\underline{A}$
3.1	A	Assume	
3.2	A	From 3.1	

4. $B \to A$ From subcomputation box below

5. A From 1., 3., 4., and ($\vee$E)

$$\frac{A \to C, B \to C, A \vee B}{C} \quad (\vee \mathrm{E})$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1. $A \vee B$ data
2. $\neg B$ data
3. $A \rightarrow A$ From subcomputation box below

		$\underline{A}$
3.1 A	Assume	
3.2 A	From 3.1	

4. $B \rightarrow A$ From subcomputation box below

		$\underline{A}$
4.1 B	Assume	

5. A From 1., 3., 4., and ($\vee$E)

$$\frac{A \rightarrow C, B \rightarrow C, A \vee B}{C} \quad (\vee \mathrm{E})$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1. $A \vee B$ data
2. $\neg B$ data
3. $A \rightarrow A$ From subcomputation box below

		$\underline{A}$
3.1 A	Assume	
3.2 A	From 3.1	

4. $B \rightarrow A$ From subcomputation box below

		$\underline{A}$
4.1 B	Assume	
4.2 $\neg A \rightarrow B$	From subcomputation below	
4.3 $\neg A \rightarrow \neg B$	From subcomputation below	

5. A From 1., 3., 4., and ($\vee$E)

$$\frac{A \rightarrow C, B \rightarrow C, A \vee B}{C} \quad (\vee E)$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1.	$A \vee B$	data	
2.	$\neg B$	data	
3.	$A \to A$	From subcomputation box below	
3.1	A	Assume	$\underline{A}$
3.2	A	From 3.1	
4.	$B \to A$	From subcomputation box below	
4.1	B	Assume	$\underline{A}$
4.2	$\neg A \to B$	From subcomputation below	
4.3	$\neg A \to \neg B$	From subcomputation below	
4.4	A	From 4.2, 4.3, and $(\neg E)$	
5.	A	From 1., 3., 4., and $(\vee E)$	

$$\frac{A \to C,\ B \to C,\ A \vee B}{C}\ (\vee E)$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1.	$A \vee B$	data	
2.	$\neg B$	data	
3.	$A \to A$	From subcomputation box below	
3.1	A	Assume	$\underline{A}$
3.2	A	From 3.1	
4.	$B \to A$	From subcomputation box below	
4.1	B	Assume	$\underline{A}$
4.2	$\neg A \to B$	From subcomputation below	
4.2.1	$\neg A$	Assume	$\underline{B}$
4.2.2	B	From 4.1	
4.3	$\neg A \to \neg B$	From subcomputation below	
4.4	A	From 4.2, 4.3, and $(\neg\text{E})$	
5.	A	From 1., 3., 4., and $(\vee\text{E})$	

$$\frac{A \to C,\ B \to C,\ A \vee B}{C}\ (\vee\text{E})$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1. $A \vee B$ — data
2. $\neg B$ — data
3. $A \to A$ — From subcomputation box below

		$\underline{A}$
3.1 A	Assume	
3.2 A	From 3.1	

4. $B \to A$ — From subcomputation box below

		$\underline{A}$
4.1 B	Assume	
4.2 $\neg A \to B$	From subcomputation below	
4.2.1 $\neg A$	Assume	$\underline{B}$
4.2.2 B	From 4.1	
4.3 $\neg A \to \neg B$	From subcomputation below	
4.3.1 $\neg A$	Assume	$\underline{\neg B}$
4.3.2 $\neg B$	From 2	
4.4 A	From 4.2, 4.3, and ($\neg$E)	

5. A — From 1., 3., 4., and ($\vee$E)

$$\frac{A \to C, B \to C, A \vee B}{C} \quad (\vee E)$$

Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1.	$A \vee B$	data
2.	$\neg B$	data
3.	$A \to A$	From subcomputation box below
	3.1 A	Assume
	3.2 A	From 3.1
		$\underline{A}$
4.	$B \to A$	From subcomputation box below
	4.1 B	Assume
	4.2 $\neg A \to B$	From subcomputation below
	4.2.1 $\neg A$	Assume
	4.2.2 B	From 4.1
		$\underline{B}$
	4.3 $\neg A \to \neg B$	From subcomputation below
	4.3.1 $\neg A$	Assume
	4.3.2 $\neg B$	From 2
		$\underline{\neg B}$
	4.4 A	From 4.2, 4.3, and ($\neg$E)
		$\underline{A}$
5.	A	From 1., 3., 4., and ($\vee$E)

$$\frac{A \to C,\ B \to C,\ A \vee B}{C}\ (\vee E)$$

$$\frac{A \vee B,\ \neg B}{A}\ (\vee E2)$$

will be introduced as a variant rule!

Sample proof 3

Show that $\vdash \neg(A \wedge \neg A)$ (This shows that $\neg(A \wedge \neg A)$ is a tautology)

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1. $A \wedge \neg A \rightarrow A$ From subcomputation box below

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Show that $\vdash \neg(A \wedge \neg A)$ (This shows that $\neg(A \wedge \neg A)$ is a tautology)

1. $A \wedge \neg A \rightarrow A$ From subcomputation box below

2. $A \wedge \neg A \rightarrow \neg A$ From subcomputation box below

Sample proof 3

Show that $\vdash \neg(A \wedge \neg A)$ (This shows that $\neg(A \wedge \neg A)$ is a tautology)

1. $A \wedge \neg A \rightarrow A$ From subcomputation box below

2. $A \wedge \neg A \rightarrow \neg A$ From subcomputation box below

3. $\neg(A \wedge \neg A)$ From 1. and 2. and $(\neg I)$

Sample proof 3

Show that $\vdash \neg(A \wedge \neg A)$ (This shows that $\neg(A \wedge \neg A)$ is a tautology)

1.	$A \wedge \neg A \rightarrow A$	From subcomputation box below
		$\underline{A}$
1.1	$A \wedge \neg A$	Assume
1.2	A	From 1.1 and ($\wedge$E)
2.	$A \wedge \neg A \rightarrow \neg A$	From subcomputation box below
3.	$\neg(A \wedge \neg A)$	From 1. and 2. and ($\neg$I)

Sample proof 3

Show that $\vdash \neg(A \wedge \neg A)$ (This shows that $\neg(A \wedge \neg A)$ is a tautology)

1. $A \wedge \neg A \rightarrow A$ — From subcomputation box below

		$\underline{A}$
1.1 $A \wedge \neg A$	Assume	
1.2 A	From 1.1 and ($\wedge$E)	

2. $A \wedge \neg A \rightarrow \neg A$ — From subcomputation box below

		$\underline{\neg A}$
2.1 $A \wedge \neg A$	Assume	
2.2 $\neg A$	From 2.1 and ($\wedge$E)	

3. $\neg(A \wedge \neg A)$ — From 1. and 2. and ($\neg$I)

Sample proof 4

Show that $A \wedge \neg A \vdash B$

(This shows you how to derive any conclusion from an inconsistent set)

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1. $A \wedge \neg A$ Data

Sample proof 4

Show that $A \wedge \neg A \vdash B$

(This shows you how to derive any conclusion from an inconsistent set)

1. $A \wedge \neg A$ Data
2. $\neg B \rightarrow A$ From subcomputation box below
3. $\neg B \rightarrow \neg A$ From subcomputation box below

Sample proof 4

Show that $A \wedge \neg A \vdash B$

(This shows you how to derive any conclusion from an inconsistent set)

1.	$A \wedge \neg A$	Data
2.	$\neg B \to A$	From subcomputation box below
3.	$\neg B \to \neg A$	From subcomputation box below
4.	B	From 2. and 3. and ($\neg$E)

Sample proof 4

Show that $A \wedge \neg A \vdash B$

(This shows you how to derive any conclusion from an inconsistent set)

1.	$A \wedge \neg A$	Data
2.	$\neg B \rightarrow A$	From subcomputation box below
2.1	$\neg B$	Assume
2.2	A	From 1. and ($\wedge$E)
		$\underline{A}$
3.	$\neg B \rightarrow \neg A$	From subcomputation box below
4.	B	From 2. and 3. and ($\neg$E)

Sample proof 4

Show that $A \wedge \neg A \vdash B$

(This shows you how to derive any conclusion from an inconsistent set)

1.	$A \wedge \neg A$	Data
2.	$\neg B \rightarrow A$	From subcomputation box below
	2.1 $\neg B$	Assume
	2.2 A	From 1. and ($\wedge$E)
		$\underline{A}$
3.	$\neg B \rightarrow \neg A$	From subcomputation box below
	3.1 $\neg B$	Assume
	3.2 $\neg A$	From 1. and ($\wedge$E)
		$\underline{\neg A}$
4.	B	From 2. and 3. and ($\neg$E)

How to practice natural deduction proofs

Now you have everything you need to provide a natural deduction proof for every valid argument in propositional logic.

These are some exercises you can do:

Practice makes perfect!

How to practice natural deduction proofs

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- Prove the validity of all arguments already shown to be valid via the truth-tables

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Now you have everything you need to provide a natural deduction proof for every valid argument in propositional logic.

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- Prove the validity of all arguments already shown to be valid via the truth-tables
- Show both directions of every logical equivalence we have seen

That is, if $A \equiv B$, then you can show that $A \vdash B$ and $B \vdash A$

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- That all tautologies are valid

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How to practice natural deduction proofs

Now you have everything you need to provide a natural deduction proof for every valid argument in propositional logic.

These are some exercises you can do:

- Prove the validity of all arguments already shown to be valid via the truth-tables
- Show both directions of every logical equivalence we have seen

  That is, if $A \equiv B$, then you can show that $A \vdash B$ and $B \vdash A$
- That all tautologies are valid
- That all variant rules can be derived from the basic ones alone (variant rules are presented in the next part)

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To know more. . .

- Natural deduction is explained in detail in Chapter 3 of Gabbay and Rodrigues' "Elementary Logic with Applications", 1st edition.
- Tutorial list 5 contains several natural deduction examples.
- Proofs of all variant rules from the basic ones will be available as a supplemental material.