4CCS1ELA–Elementary Logic with Applications

4–Checking the Validity of Arguments

4.1–Introduction to Informal Proofs

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Outline

1. Introduction to Informal Proofs
2. Common Proof Patterns
3. Fallacies

Introduction to Informal Proofs

Formal vs. Informal Proofs

Our ultimate objective in this part of the course is to present a *formal* deductive system which:

- shows how a conclusion follows from a set of premises via a sequence of well-defined proof steps
- fully agrees with the semantical concept of logical consequence
- is purely syntactical

Before we do that, let us do a little digression into *informal* proofs.

Informal proofs

The term “informal” here is used in the sense that we may *omit* some steps and/or use some “accepted” results (statements) in the proof: axioms, theorems, etc.

For instance, “Two points lie on one and only one line” is an axiom for a reasoning system for geometry.

Our interim objective is to relate reasoning patterns in the proofs with known tautologies.

We will then formalise the notion of proof, to completely forbid any assumptions outside the set of premises or any steps which are not fully defined at the outset.

COMMON PROOF PATTERNS

Direct proofs

A **direct proof**, as the name implies, involves the successive application of valid inference steps to demonstrate that a conclusion follows from a set of assumptions.

Assume: n is even if and only if $n = 2k$, for some integer k; and n is odd if and only if $n = 2k + 1$, for some integer k.

Example. If n is an odd integer, then n^2 is odd.

Proof. Assume that n is any odd integer, and then show that n^2 is odd. If n is odd, then $n = 2k + 1$ for some integer k. Therefore, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. If k is integer, then $2k^2 + 2k$ is also an integer and hence n^2 is also odd.

Proof by contradiction

To prove $A \to B$, construct a proof of $A \land \neg B \to \mathbf{0}$ (using $A \land \neg B \to \mathbf{0} \equiv \neg(A \land \neg B) \equiv (\neg A \lor B) \equiv (A \to B)$).

Example. If $3n + 2$ is odd, then n is odd.

Proof. Suppose $3n + 2$ is odd, but n is not odd (i.e., n is even), and let us show that this leads to a contradiction.

Since n is even, we know that $n = 2k$ for some integer k. Thus, we can expand $3n + 2$ as $3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Since $3k + 1$ is integer and $3n + 2$ can be expressed as $2(3k + 1)$, then $3n + 2$ must be even, and this contradicts our orginal assumption.

Therefore, if $3n + 2$ is odd, then n is odd.

Proof by contraposition

To prove $A \to B$, you can show that $\neg B \to \neg A$ (using $\neg B \to \neg A \equiv A \to B$).

Example. If $3n + 2$ is odd, then n is odd.

Proof. Suppose that n is not odd (i.e., even), we show that $3n + 2$ is also not odd (i.e., it is also even).

If n is even, then $n = 2k$, for some integer k. Then $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$, which is an even number.

Proof by cases

To prove $(A_1 \vee A_2 \vee \ldots \vee A_2) \to B$, you can show that $(A_1 \to B) \wedge (A_2 \wedge B) \wedge \ldots \wedge (A_n \to B)$
(using $(A_1 \vee A_2 \vee \ldots \vee A_2) \to B \equiv (A_1 \to B) \wedge (A_2 \to B) \wedge \ldots \wedge (A_n \to B)$).

Example. If n is an integer, then $n^2 \geq n$.

Proof. Let us divide the set of integers into three classes: 0, positive numbers and negative numbers, and show that in each case $n^2 \geq n$.

a) If $n = 0$, then $n^2 = 0^2 = 0 \geq 0$. ***b)*** If $n \geq 1$, then $(n \cdot n) \geq (1 \cdot n)$, and therefore $n^2 \geq n$. ***c)*** if $n \leq -1$, then again we know that $n^2 > 0$. Therefore, $n^2 \geq n$. *a)–c)* show that for all integers n, $n^2 \geq n$.

FALLACIES

Fallacies

We want every step in our proof to be correct, and by this we mean that if we apply a step of the form "from P and $P \rightarrow Q$, we can conclude Q", in symbols,

$$\frac{P, \ P \rightarrow Q}{Q},$$

then indeed the formula $(P \wedge (P \rightarrow Q)) \rightarrow Q$ is a tautology.

A fallacy *resembles* a correct inference, but it is not valid, leading to incorrect reasoning.

Fallacy of affirming the conclusion

Suppose we have the following argument (Example 10 in Rosen's "Discrete Mathematics and Its Applications", 8th edition):

"If you do every problem in this book (P), then you will learn discrete mathematics (M). You learned discrete mathematics (M). Therefore, you did every problem in this book. (P)"

$$P \rightarrow M, M \models P \qquad\qquad \textbf{not}\text{ valid!}$$

Affirming the consequent of the implication $P \rightarrow M$ does not guarantee the truth of its antecedent P. Notice that valuation $v(P) = 0,\ v(M) = 1$ makes all assumptions of the argument true, but its conclusion false.

Fallacy of affirming the conclusion

The incorrect reasoning pattern is based on the formula

$$((P \rightarrow M) \land M) \rightarrow P$$

which is **not** a tautology, but merely a contingency.

Example of this fallacy in practice.

"If Odinaldo was born in Britain, then he is a British citizen. Odinaldo is a British citizen. Therefore, he was born in Britain."

This argument fails because an implication being true does not guarantee its antecedent also being true (it may be true or false).

In fact, when the consequent of an implication is true, the implication will be true even when its antecedent is false.

Fallacy of denying the hypothesis

Using the same language of the example in Slide 4.1.11, consider the following argument:

“If you do every problem in this book (P), then you will learn discrete mathematics (M). You did not do every problem in this book ($\neg P$). Therefore, you did not learn discrete mathematics ($\neg M$).”

$$P \to M, \neg P \models \neg M \qquad\qquad \textbf{not valid!}$$

Denying the antecedent of the implication $P \to M$ does not guarantee the falsity of its consequent M.

Notice that the valuation $v(P) = 0$, $v(M) = 1$ makes all assumptions of the argument true, but its conclusion false.

Fallacy of denying the hypothesis

This incorrect reasoning pattern is based on the formula

$$((P \to M) \wedge \neg P) \to \neg M$$

which is **not** a tautology, but merely a contingency.

Example of this fallacy in practice.

“If Odinaldo was born in Britain, then he is a British citizen.
Odinaldo was not born in Britain. Therefore, he is not a British citizen.”

This argument fails, because the premises do not state that being born in Britain is the only way to be a British citizen – someone may become a British citizen by other means.

Where from now?

These “informal” proofs used a number of external assumptions and did not justify all the steps in terms of well-understood principles.

They are intended for humans who have the privilege of understanding the “proof” despite of its missing steps, untold assumptions, etc.

In the next part, we will tighten the definition of a proof to provide a *formal* well-defined sequence of syntactical manipulations from premises to conclusions that fully agrees with the concept of logical consequence.

To know more. . .

- There is a good introduction to formal reasoning in Section 6.3 of Hein’s “Discrete Structures, Logic, and Computability”, 4th edition.
- Sections 1.6, 1.7 and 1.8 of Rosen’s “Discrete Mathematics and Its Applications”, 8th edition also discuss proofs and strategies.
- A more comprehensive introduction to consequence relations, soundness and completeness can be found in Section 1.2.2 and Chapter 5 of Gabbay and Rodrigues’ “Elementary Logic with Applications”, 1st edition.