4CCS1ELA: Tutorial list — Week 7

1. Consider the following predicate symbols with which we associate the following meanings:

- $S(x)$ represents "x is a student"
- $L(x)$ represents "x is a lecture"
- $A(x, y)$ represents "x attends y"

Provide a first-order formula encoding the following sentence:

"*At least one student attended every lecture*"

(be careful, this sentence is ambiguous. . . , provide a formula for each of the two different meanings)

<u>*SOLUTION*</u>

The sentence can have two different meanings:

(i) A first meaning could be "There exists one student that attend every lecture". In this case, a possible encoding in FOL could be:

$$\exists x(S(x) \wedge \forall y(L(y) \rightarrow A(x, y)))$$

The formula reads: there exists an object x such that x is a student and, for all objects y, if y is a lecture, then x attends y.

(ii) A second meaning could be "For every lecture there is at least a student attending the lecture". In this case, a possible encoding could be:

$$\forall x(L(x) \rightarrow \exists y(S(y) \wedge A(y, x)))$$

The formula reads: for all objects x, if x is a lecture, then there exists a object y such that y is a student and y attends x.

Notice that (ii) is a logical consequence of (i).

2. Consider the following predicate symbols with which we associate the following meanings:

- $B(x)$ means "x is a bird"
- $W(x)$ means "x is a worm"
- $E(x, y)$ means "x eats y"

Using these predicates, represent in first-order logic each of the following statements:

(i) *Every bird eats every worm*
(ii) *Some birds do not eat some worms*
(iii) *No bird is eaten by a worm*
(iv) *Some worms do not get eaten by birds*

<u>*SOLUTION*</u>

(i) *Every bird eats every worm:*
$\forall x(B(x) \rightarrow \forall y(W(y) \rightarrow E(x, y)))$.
Equivalently,
$\forall x \forall y(B(x) \wedge W(y) \rightarrow E(x, y))$.

(ii) *Some birds do not eat some worms:*
$\exists x(B(x) \wedge \exists y(W(y) \wedge \neg E(x, y)))$.
Equivalently,
$\exists x \exists y(B(x) \wedge W(y) \wedge \neg E(x, y))$.

(iii) *No bird is eaten by a worm:*
$\forall x(B(x) \to \forall y(W(y) \to \neg E(y,x)))$.
Equivalently,
$\neg\exists x \exists y(B(x) \wedge W(y) \wedge E(y,x))$.

(iv) *Some worms do not get eaten by birds:*
$\exists x(W(x) \wedge \forall y(B(y) \to \neg E(y,x)))$.
Equivalently,
$\exists x(W(x) \wedge \neg\exists y(B(y) \wedge E(y,x)))$.

3. Identify which occurrences of variables in the formulas below are free and which occurrences are bound. Justify you answers.

(i) $y \geq 0 \;\; \wedge \forall x(N(x) \to x \geq y)$
(ii) $x \geq 0 \;\; \wedge \forall x(N(x) \to x \geq y)$
(i) $\forall x(N(x) \to \exists y(N(y) \wedge x \geq y))$

Here N is a unary predicate symbol, $\geq$ is a binary predicate symbol in infix notation, and $x \geq y$ is an atom in infix notation (infix notation means that the predicate symbol appears in between the terms).

<u>*SOLUTION*</u>

(i) All occurrences of x are bound (therefore x is a bound variable in this formula). Both occurrences of y are free (therefore y is a free variable in this formula).

The free occurrences are boxed:
$\boxed{y} \geq 0 \;\; \wedge \forall x(N(x) \to x \geq \boxed{y}\;)$

(ii) The variable x is free and bound (i.e., there are free occurrences of x and there are bound occurrences of x). The variable y is free, i.e., all occurrences of y are free.

The free occurrences are boxed:
$\boxed{x} \geq 0 \;\; \wedge \forall x(N(x) \to x \geq \boxed{y})$

(iii) Both x and y are bound (i.e., all occurrences of x and y are bound).

4. Let ϕ be a well-formed formula (wff), i.e. a σ-formula belonging to $\mathcal{L}[\sigma]$, interpreted over the domain set D and $d \in D$. Then $\phi(x/d)$ denotes the wff obtained from ϕ by replacing all *free* occurrences of X in ϕ by d.

Compute the following substitutions and determine the meaning (the truth-values) of the resulting sentences over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Here $N(x)$ denotes "x is a natural number", predicates $\geq$ and $>$ have their usual interpretation and are expressed with their usual infix notation.

(i) $\big(y \geq 0 \;\; \wedge \forall x(N(x) \to x \geq y)\big)(y/3)$
(ii) $\big(x \geq 0 \;\; \wedge \exists y(N(y) \wedge x \geq y)\big)(x/3)$
(iii) $\big(\forall x(N(x) \to \exists y(N(y) \wedge x > y))\big)(x/3)$
(iv) $\big(\forall x(N(x) \to \exists y(N(y) \wedge y > x))\big)(y/3)$

<u>*SOLUTION*</u>

(i) $(3 \geq 0 \;\; \wedge \forall x(N(x) \to x \geq 3)$ is false (witness $x = 1$).

(ii) $(3 \geq 0 \;\; \wedge \exists y(N(y) \wedge 3 \geq y)$ is true (witness $y = 3$).

(iii) $(\forall x(N(x) \to \exists y(N(y) \wedge x > y)))$ is false (witness $x = 0$).

(iv) $(\forall x(N(x) \to \exists y(N(y) \wedge y > x)))$ is true (witness $y = x + 1$).

Formula (iii) does not contain free occurrences of x. Similarly, all occurrences of y in formula (iv) are bound.