4CCS1ELA–Elementary Logic with Applications

5–Proving with Natural Deduction

5.1–Natural Deduction Rules

Dr. Odinaldo Rodrigues

odinaldo.rodrigues@kcl.ac.uk
Room BH(S) TBC, +44 (0)20 7848 2087
Department of Informatics
King's College London

Outline

1. General Notions
2. Rules for Conjunction, Disjunction and Negation
3. Implication Introduction and Elimination
4. Subcomputations

General Notions

Basic set of rules

We start with a simple set of rules that is *complete* (the "basic rules").

Since our rules will either eliminate or introduce a connective, our *proof system* will have a pair rules for each connective of the language.

$\ominus$I will be used to designate the introduction rule of the connective $\ominus$, and $\ominus$E will be used for its elimination rule.

For instance, ($\wedge$E) will be used to denote the rule that eliminates the connective $\wedge$ and ($\wedge$I) for the rule that introduces it, and so forth.

Notes about proofs

We will follow the definition of a natural deduction proof seen in the last lecture. In particular,

- Only premises and derived conclusions are allowed.
- The rules have to be used exactly as they are given (equivalences cannot be used – although they can be proven!)
- Although the set of basic rules is complete, we will introduce some variant rules later, that offer "shortcuts".

Example

$$\neg(A \vee B), \neg C \to A \vdash C$$

1.	$\neg(A \vee B)$	Data
2.	$\neg A \wedge \neg B$	Since $\neg(A \vee B) \equiv \neg A \wedge \neg B$ ×
3.	$\neg A$	2. and ($\wedge$E)
4.	$\vdots$	

The step 2. above is not permitted unless we have a specific natural deduction rule whose premise we can instantiate with 1. to produce 2.

Spoiler alert: we do not! But 2. can be proven from 1. using the basic rules, since they are complete.

Rules for Conjunction, Disjunction and Negation

Rules for conjunction ($\wedge$)

Look at the truth-table for $\wedge$:

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

$A \wedge B$ is only true when both A and B are true. Similarly, whenever we know that $A \wedge B$ is true, we also know that A and B are both true.

This gives us the ($\wedge$I) and ($\wedge$E) rules below.

$$\frac{A, B}{A \wedge B}\ (\wedge\mathrm{I}) \qquad\qquad \frac{A \wedge B}{A} \ \text{ and } \ \frac{A \wedge B}{B}\ (\wedge\mathrm{E})$$

Introduction of the disjunction ($\vee$I)

Look at the truth-table for $\vee$:

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

We can obtain $A \vee B$ as long as at least one of A and B is true. This gives us the introduction rules ($\vee$I)below.

$$\frac{A}{A \vee B} \quad \text{and} \quad \frac{B}{A \vee B} \quad (\vee\text{I})$$

Example with the rules defined so far

Prove that $A \wedge B \vdash B \vee C$

1. $A \wedge B$ data
2. B from (1.) and ($\wedge$E)
3. $B \vee C$ from (2.) and ($\vee$I)

Elimination of disjunction ($\vee$E)

The first ($\vee$E) we present is an *indirect* rule.

We can eliminate a disjunction $A \vee B$ by showing that some formula C follows from both A and B, so whichever of A or B is the case, C will hold:

$$\frac{A \to C, B \to C, A \vee B}{C} \ (\vee\mathrm{E})$$

The rule above was called the "proof by cases"-rule in Hein's book (notice the similarity with the proof by case pattern from Lecture 4).

Example

$A,\ A{\to}(B \vee C),\ B{\to}D,\ C{\to}D \vdash D$

1. A data
2. $A{\to}(B \vee C)$ data
3. $B{\to}D$ data
4. $C{\to}D$ data
5. $B \vee C$ From 1., 2., and ($\to$E)
6. D From 3., 4., 5., and ($\vee$E)

Rules for negation ($\neg$)

The rules for negation are done indirectly and are somewhat tricky.

In order to conclude $\neg A$, we must show that if A were the case, then we would reach a contradiction (e.g., B and $\neg B$).

Thus,

$$\frac{A \rightarrow B, A \rightarrow \neg B}{\neg A} \ (\neg\text{I})$$

$\neg$ is eliminated through the same reasoning, except we start with the negated formula ($\neg A$).

$$\frac{\neg A \rightarrow B, \neg A \rightarrow \neg B}{A} \ (\neg\text{E})$$

Example

$A \rightarrow B,\ A \rightarrow \neg B, C \vdash \neg A \wedge C$

1. $A \rightarrow B$ data
2. $A \rightarrow \neg B$ data
3. C data
4. $\neg A$ From 1., 2., and ($\neg$I)
5. $\neg A \wedge C$ From 3., 4. and ($\wedge$I)

Implication Introduction and Elimination

Elimination of the Implication ($\to$E)

The elimination of the implication rule can be easily understood from the connective's truth-table:

A	B	$A \to B$
0	0	1
0	1	1
1	0	0
1	1	1

If we know that an implication and its antecedent are both true, then we must also have that the consequent of the implication is also true.

$$\frac{A, A \to B}{B} \ (\to\text{E})$$

Example

$A \wedge C, A \rightarrow B \vdash B$

1. $A \wedge C$ data
2. $A \rightarrow B$ data
3. A From 1. and ($\wedge$E)
4. B From 3., 2., and ($\rightarrow$E)

Introduction of the Implication ($\rightarrow$I)

The introduction of the implication is a little more elaborated.

If we want to derive $A \rightarrow B$ from some premises, then we must show that these premises together with A imply B.

$$\frac{\text{premises}}{A \rightarrow B}, \quad \text{if} \quad \frac{\text{premises}, A}{B} \qquad (\rightarrow \mathrm{I})$$

The proof that B indeed follows from the addition of A to premises is done in a separate subcomputation box (explained next).

SUBCOMPUTATIONS

Subcomputation boxes

A subcomputation box defines a sub-proof that is dependent on an extra assumption.

The box starts with the extra assumption in the antecedent of the implication and is used to define the scope in which this assumption can be used.

To introduce the implication, the box must constitute a valid sub-proof in its own right of the consequent of the implication, using as many previously reached conclusions as needed.

Subcomputation boxes

The box demonstrates that the consequent of the implication is a valid conclusion under the assumption that the antecedent is true. This justifies the derivation of the implication because:

- If the assumption is true, the box provides a valid proof of the consequent, therefore the implication holds.
- If the assumption is false, the implication will hold anyway because an implication is true when its antecedent is false (look at $\rightarrow$'s truth-table).

Therefore, if the box manages to show the conclusion, the implication will be true whether or not the extra assumption is true, and therefore true under the original assumptions.

Example

Consider the following proof that $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

1.	$A \rightarrow B$	data
2.	$B \rightarrow C$	data
3.	$A \rightarrow C$	($\rightarrow$I), from the subcomputation below

		$\underline{C}$
3.1	A	assumption
3.2	B	from (3.1), (1.) and ($\rightarrow$E)
3.3	C	from (3.2), (2.) and ($\rightarrow$E)

You can think of the box as a proof that $A \rightarrow C$, under the assumption that $A \rightarrow B$ and $B \rightarrow C$.

The above proof is a derivation of the “hypothetical syllogism” rule using the basic set of rules (see Example 16 in Hein’s book).

Using subcomputation boxes

In order to correctly perform a subcomputation, follow these rules

- The assumption at the beginning of a subcomputation box may only be used inside the box
- Any conclusions previously available, including premises and formulas derived can also be used inside the box
- Any conclusion obtained inside the box that depends on the box's initial assumption cannot be used on their own

  In the previous example, C cannot be used on its own, because its proof depended on A's assumption (line 3.1).

Nested subcomputation boxes

Prove that the following argument is valid: $A \to B \vdash \neg B \to \neg A$

1. $A \to B$ data
2. $\neg B \to \neg A$ $(\to\mathrm{I})$, subcomputation below

$\underline{\neg A}$

2.1 $\neg B$ assumption

2.2 $A \to \neg B$ $(\to\mathrm{I})$, subcomputation below

$\underline{\neg B}$

2.2.1 A Assumption

2.2.2 $\neg B$ From 2.1

2.3 $\neg A$ from 1., 2.2, and $(\neg\mathrm{I})$

Exercise: Show that the converse is also true: $\neg B \to \neg A \vdash A \to B$.
(These two proofs show that $A \to B \equiv \neg B \to \neg A$)

Rules for “if and only if”

Just for completion, we present the following two rules for $\leftrightarrow$:

$$\frac{A \to B,\ B \to A}{A \leftrightarrow B}\ (\leftrightarrow\mathrm{I}) \qquad\qquad \frac{A \leftrightarrow B}{(A \to B) \wedge (B \to A)}\ (\leftrightarrow\mathrm{E})$$

To know more. . .

- Natural deduction is explained in detail in Chapter 3 of Gabbay and Rodrigues’ “Elementary Logic with Applications”, 1st edition.
- An alternative version, using a different set of rules, is also avaliable in Section 6.3 of Hein’s “Discrete Structures, Logic, and Computability”, 4th edition.