4CCS1ELA–Elementary Logic with Applications

3–Important Semantical Notions

3.3–Logical Consequence

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Logical consequence

Logical consequence and arguments

The Internet Encyclopedia of Philosophy defines an *argument* as a sequence of statements (*the premises, or the hypotheses*) which are intended to provide support, justification or evidence for the truth of another statement (the conclusion).

In the argument

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \\ \hline B \end{array}$$

$A_1, \ldots, A_n$ are the premises and B is the conclusion (you can read '$A_1, \ldots, A_n$, *therefore* B').

Logical (semantic) consequence

In a valid **argument**, we say informally that a conclusion B "follows" from a set of premises $A_1, \ldots, A_n$.

More formally, we say that the formula B is a *logical consequence* of the set of formulae $\{A_1, \ldots, A_n\}$, if the following implication holds for *every* interpretation v:

$$\text{If } v(A_i) = 1, \text{ for all } 1 \leq i \leq n, \text{ then } v(B) = 1.$$

We often drop the brackets in the set of premises above.

Alternative definition of $\models$

Let the symbol $\mathcal{I}$ denote the set of all interpretations.

Let $\mathcal{S} = \{A_1, \ldots, A_n\}$. We have that

$$\mathcal{S} \models B \text{ iff } \operatorname{mod}(\mathcal{S}) \subseteq \operatorname{mod}(B)$$

Notice that $\operatorname{mod}(\neg B) = \mathcal{I} - \operatorname{mod}(B)$ and hence

$$\operatorname{mod}(B) \cap \operatorname{mod}(\neg B) = \varnothing.$$

Therefore, if $\mathcal{S} \models B$, then $\operatorname{mod}(\mathcal{S}) \cap \operatorname{mod}(\neg B) = \emptyset$ and hence $\mathcal{S} \cup \{\neg B\}$ is unsatisfiable (see bullet point 3 in slide 3.3.7).

Example

Show that $P, P \rightarrow Q \models Q$.

Solution:

	P	Q	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$	Q	$\models$
v_0	0	0	1	0	0	✓ (premises false)
v_1	0	1	1	0	1	✓ (premises false)
v_2	1	0	0	0	0	✓ (premises false)
v_3	1	1	1	1	1	✓ (premises + conclusion true)

The statement follows because in every row in which the columns for P and $P \rightarrow Q$ contain 1 (the premises), so does the column for Q (the conclusion): $\operatorname{mod}(\{P, P \rightarrow Q\}) = \{v_3\} \subseteq \operatorname{mod}(\{Q\}) = \{v_1, v_3\}$.

In this example, the only required row to check is the one in red.

This type of derivation is called *modus ponens*.

Alternative terminologies

The following statements are equivalent.

- B is a logical consequence of $A_1, \ldots, A_n$.
- $A_1, \ldots, A_n \models B$.
- The argument $A_1, \ldots, A_n \models B$ is *valid*.
- B is *semantically entailed (or implied)* by $A_1, \ldots, A_n$
- B is a *valid consequence* of $A_1, \ldots, A_n$.

Relationship with other semantic concepts

$A_1, \ldots, A_n \models B$ if and only if

- $A_1 \wedge \ldots \wedge A_n \rightarrow B$ is a tautology (i.e., logically valid).
- $A_1 \wedge \ldots \wedge A_n \wedge \neg B$ is a contradiction
- The set $\{A_1, \ldots, A_n, \neg B\}$ is inconsistent (i.e., unsatisfiable)

Exercise: Check that the above is indeed the case for the example in Slide 3.3.5.

SPECIAL CASES OF LOGICAL CONSEQUENCE

Special case: unsatisfiable premises

If Jack takes a holiday, then Jill will be happy and she will not cry. Jack will take a holiday and if Jill is happy she will cry. **Therefore** *Jack will take a holiday*.

Let J stand for 'Jack will take a holiday'; H stand for 'Jill will be happy'; and C stand for 'Jill will cry'.

The argument $J \to (H \wedge \neg C), J \wedge (H \to C) \models J$ is valid!

$$\begin{array}{ll} \mathcal{S}: & \left\{ \begin{array}{l} J \to (H \wedge \neg C) \\ J \wedge (H \to C) \end{array} \right. \\ \hline B: & J \end{array}$$

This is because $\text{mod}(\mathcal{S}) = \varnothing \subseteq \text{mod}(B)$. In fact, any conclusion follows from an unsatisfiable set of premises!

Special case: tautological conclusions

Tautologies are always true, so if A is a tautology, then $\mathrm{mod}(A) = \mathcal{I}$.

This fact has two immediate effects:

1. A tautology is a logical consequence of any set of formulae.
2. A tautology also follows from “nothing”.

Notice that 2. is a special case of 1:

Any interpretation satisfies all of the formulae in the empty set (because there are none in it!). Therefore, $\mathrm{mod}(\varnothing) = \mathcal{I}$.

However, if A is a tautology, then $\mathrm{mod}(A) = \mathcal{I}$ and hence $\mathrm{mod}(\varnothing) = \mathcal{I} \subseteq \mathrm{mod}(A) = \mathcal{I}$.

INVALID ARGUMENTS

Invalid arguments

An argument that is not valid is said to be **invalid**.

By the definition of logical consequence, $A_1, \ldots, A_n \not\models B$ if there exists an interpretation v such that

$$v(A_i) = 1 \text{ for all } 1 \leq i \leq n, \text{ but } v(B) = 0.$$

Thus, in order to show that a conclusion does not follow from a set of premises, we must find an interpretation that makes all of the premises true, but under which the conclusion is false.

Notice the similarity with the truth-table for implication!

Examples

1. Show that $P \not\models Q$, where P, Q are atoms.

   Solution: Take the interpretation v with

   $$v(P) = 1 \text{ and } v(Q) = 0.$$

2. Show that $P \rightarrow Q \not\models Q$, where P, Q are atoms.

   Solution: Take the interpretation v with

   $$v(P) = 0 \text{ and } v(Q) = 0.$$

   These are not solutions:

   - The interpretation v_1 with $v_1(P) = 1$, $v_1(Q) = 0$, because v_1 does not satisfy $P \rightarrow Q$.
   - The interpretation v_2 with $v_1(P) = 1$, $v_1(Q) = 1$, because v_2 does satisfy Q.

To know more...

Logical consequence is explained in Sections 1.2 and 1.3 of Gabbay and Rodrigues' "Elementary Logic with Applications, 1st edition.