# 4CCS1ELA–Elementary Logic with Applications

## 5–Proving with Natural Deduction

## 5.2–Examples Using the Basic Rules

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# Worked Examples

# Sample proof 1

Show that $(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$

1. $(A \vee B) \to C$ — Data
2. $A \to C$ — From subcomputation box below

| | | | |
|---|---|---|---|
| | | | $\underline{C}$ |
| 2.1 | $A$ | Assume | |
| 2.2 | $A \vee B$ | From 2.1 and ($\vee$I) | |
| 2.3 | $C$ | From 2.2, 1. and ($\to$E) | |

3. $B \to C$ — From subcomputation box below

| | | | |
|---|---|---|---|
| | | | $\underline{C}$ |
| 3.1 | $B$ | Assume | |
| 3.2 | $A \vee B$ | From 3.1 and ($\vee$I) | |
| 3.3 | $C$ | From 3.2, 1. and ($\to$E) | |

4. $(A \to C) \wedge (B \to C)$ — From 2. and 3. and ($\wedge$I)





# Sample proof 2

Show that $A \vee B, \neg B \vdash A$

1. $A \vee B$ — data
2. $\neg B$ — data
3. $A \to A$ — From subcomputation box below

| | | | |
|---|---|---|---|
| | | | $\underline{A}$ |
| 3.1 | $A$ | Assume | |
| 3.2 | $A$ | From 3.1 | |

4. $B \to A$ — From subcomputation box below

| | | | |
|---|---|---|---|
| | | | $\underline{A}$ |
| 4.1 | $B$ | Assume | |
| 4.2 | $\neg A \to B$ | From subcomputation below | |
| | | | $\underline{B}$ |
| 4.2.1 | $\neg A$ | Assume | |
| 4.2.2 | $B$ | From 4.1 | |
| 4.3 | $\neg A \to \neg B$ | From subcomputation below | |
| | | | $\underline{\neg B}$ |
| 4.3.1 | $\neg A$ | Assume | |
| 4.3.2 | $\neg B$ | From 2 | |
| 4.4 | $A$ | From 4.2, 4.3, and ($\neg$E) | |

5. $A$ — From 1., 3., 4., and ($\vee$E)

$$\frac{A \to C, B \to C, A \vee B}{C} \ (\vee\text{E})$$

$$\frac{A \vee B, \neg B}{A} \ (\vee\text{E2})$$

will be introduced as a variant rule!

# Sample proof 3

Show that $\vdash \neg(A \wedge \neg A)$ (This shows that $\neg(A \wedge \neg A)$ is a tautology)

1. $A \wedge \neg A \rightarrow A$ — From subcomputation box below

| | | $\underline{A}$ |
|---|---|---|
| 1.1 $A \wedge \neg A$ | Assume | |
| 1.2 $A$ | From 1.1 and ($\wedge$E) | |

2. $A \wedge \neg A \rightarrow \neg A$ — From subcomputation box below

| | | $\underline{\neg A}$ |
|---|---|---|
| 2.1 $A \wedge \neg A$ | Assume | |
| 2.2 $\neg A$ | From 2.1 and ($\wedge$E) | |

3. $\neg(A \wedge \neg A)$ — From 1. and 2. and ($\neg$I)





# Sample proof 4

Show that $A \wedge \neg A \vdash B$

(This shows you how to derive any conclusion from an inconsistent set)

1. $A \wedge \neg A$ — Data

2. $\neg B \rightarrow A$ — From subcomputation box below

| | | $\underline{A}$ |
|---|---|---|
| 2.1 $\neg B$ | Assume | |
| 2.2 $A$ | From 1. and ($\wedge$E) | |

3. $\neg B \rightarrow \neg A$ — From subcomputation box below

| | | $\underline{\neg A}$ |
|---|---|---|
| 3.1 $\neg B$ | Assume | |
| 3.2 $\neg A$ | From 1. and ($\wedge$E) | |

4. $B$ — From 2. and 3. and ($\neg$E)

# How to practice natural deduction proofs

Now you have everything you need to provide a natural deduction proof for every valid argument in propositional logic.

These are some exercises you can do:

- Prove the validity of all arguments already shown to be valid via the truth-tables
- Show both directions of every logical equivalence we have seen

  That is, if $A \equiv B$, then you can show that $A \vdash B$ and $B \vdash A$
- That all tautologies are valid
- That all variant rules can be derived from the basic ones alone (variant rules are presented in the next part)

Practice makes perfect!





# To know more. . .

- Natural deduction is explained in detail in Chapter 3 of Gabbay and Rodrigues' "Elementary Logic with Applications", 1st edition.
- Tutorial list 5 contains several natural deduction examples.
- Proofs of all variant rules from the basic ones will be available as a supplemental material.