Summary of natural deduction rules

Basic rules

$$\frac{A,B}{A \wedge B}$$
 Conjunction
$$\frac{A,B}{B \wedge A}$$
 (\wedge I)

$$(\land \mathsf{E}) \qquad \qquad \frac{A \land B}{A} \qquad \qquad (\land \mathsf{E})$$

$$\frac{A}{A \vee B} \qquad \qquad \textbf{Disjunction} \qquad \frac{B}{A \vee B} \qquad \qquad (\forall \mathsf{I})$$

$$(\vee \mathsf{E}) \qquad \qquad \frac{A \to C, B \to C, A \vee B}{C}$$

$$(\rightarrow \text{I}) \quad \text{If} \quad \frac{\text{assumptions, } A}{B} \quad \text{then} \quad \frac{\text{assumptions}}{A \rightarrow B} \qquad \qquad \text{Implication} \qquad \frac{A, A \rightarrow B}{B} \qquad (\rightarrow \text{E})$$
 (also known as modus ponens)

$$\frac{A \rightarrow B, A \rightarrow \neg B}{\neg A} \qquad \qquad \text{Negation} \qquad \frac{\neg A \rightarrow B, \neg A \rightarrow \neg B}{A} \qquad (\neg \mathsf{E})$$

$$(\leftrightarrow \mathsf{I}) \qquad \qquad \frac{(A \to B) \land (B \to A)}{A \leftrightarrow B} \qquad \qquad \mathsf{Equivalence} \qquad \frac{A \leftrightarrow B}{(A \to B) \land (B \to A)} \qquad \qquad (\leftrightarrow \mathsf{E})$$

Variant rules

$$\frac{A \vee B, \neg A}{B}$$
 Disjunction
$$\frac{A \vee B, \neg B}{A}$$
 (VE2) (disjunctive syllogism)

$$\frac{A \to C, B \to D, A \vee B}{C \vee D} \\ \text{(constructive dilemma)} \qquad \frac{A \to C, B \to D, \neg C \vee \neg D}{\neg A \vee \neg B} \\ \text{(destructive dilemma)}$$

$$(\rightarrow 11) \qquad \frac{\neg A}{A \rightarrow B} \qquad \qquad \text{Implication} \qquad \frac{B}{A \rightarrow B} \qquad (\rightarrow 12)$$

$$(\rightarrow E1) \qquad \qquad \frac{A \rightarrow B}{\neg A \lor B}$$

$$(\neg E1) \qquad \frac{\neg \neg A}{A} \qquad \text{Negation} \qquad \frac{\neg A \rightarrow B, A \rightarrow B}{B} \qquad (\neg E2)$$

$$\frac{A}{\neg \neg A} \qquad \frac{A \rightarrow B, \neg A \rightarrow B}{B} \qquad (\neg 2)$$

Proofs of the Derived Rules from the Basic Ones

$$\text{(}{\vee}\text{E1)}\ \frac{A\vee B, \neg A}{B}$$

- 1. $A \vee B$
- 2. $\neg A$ data
- 3. $A \rightarrow B$ (\rightarrow I) (subcomputation box below)

data

<u>B</u> 3.1 A assume $3.2 \neg B \rightarrow \neg A$ $(\rightarrow I)$ (subcomputation box below) $\underline{\neg A}$ $3.3.1 \neg B$ assume 3.3.2 $\neg A$ from 1. 3.3 $\neg B \rightarrow A$ (subcomputation box below) \underline{A} $3.3.1 \neg B$ assume 3.3.2 Afrom 3.1 3.4 B 3.2, 3.3 and $(\neg E)$

- 4. $B \rightarrow B$ (\rightarrow I) (subcomputation box below) \underline{B} 4.1 B assume 4.2 B from 4.1
- 5. B from 1., 3., 4., and (\vee E)

(CD) $\frac{A{ o}C,B{ o}D,A{ ext{\vee}}B}{C{ ext{\vee}}D}$

- 1. $A \rightarrow C$ data
- 2. $B \rightarrow D$ data
- 3. $A \lor B$ data
- 4. $A \rightarrow (C \lor D)$ (subcomputation box below)

 $\begin{array}{ccc} & & & & & & \\ 4.1 \ A & & \text{assume} & & \\ 4.2 \ C & & & 1., \ 4.1, \ \text{and} \ (\rightarrow \text{E}) \\ 4.3 \ C \lor D & & 4.2, \ \text{and} \ (\lor \text{I}) \\ \end{array}$

5. $B \rightarrow (C \lor D)$ (subcomputation box below)

 $\begin{array}{ccc} & & & & & & \\ \hline 5.1 & B & & & \text{assume} \\ \hline 5.2 & D & & & 2., \ 5.1, \ \text{and} \ (\rightarrow \text{E}) \\ \hline 5.3 & C \lor D & & 5.2, \ \text{and} \ (\lor \text{I}) \\ \end{array}$

6. $C \lor D$ from 3., 4., 5., and $(\lor E)$

(DD)
$$\frac{A{ o}C,B{ o}D,\neg C\vee \neg D}{\neg A\vee \neg B}$$

- 1. $A \rightarrow C$
- data
- 2. $B\rightarrow D$
- data
- 3. $\neg C \lor \neg D$
- data
- 4. $\neg C \rightarrow (\neg A \lor \neg B)$ (\rightarrow I) (subcomputation box below)

		$\neg A \lor \neg B$
$4.1 \neg C$		assume
$4.2 A \rightarrow \neg C$		(ightarrow I) (subcomputation box below)
		$\neg C$
	4.2.1 A	assume
	4.2.2 <i>¬C</i>	from 4.1
$4.3 \neg A$		1, 4.2 and (¬I)
4.4 $\neg A \lor \neg B$		4.3 and $(\lor I)$

5. $\neg D \rightarrow (\neg A \lor \neg B)$ (\rightarrow I) (subcomputation box below)

6. $\neg A \lor \neg B$ from 3., 4., 5., and $(\lor E)$

$$(\rightarrow$$
l1) $\frac{\neg A}{A \rightarrow B}$

- 1. $\neg A$
- data
- 2. $A \rightarrow B$
- $(\rightarrow I)$ (subcomputation box below)

(
$$\rightarrow$$
l2) $\frac{B}{A \rightarrow B}$

1. *B*

data

- 2. $\underline{A \rightarrow B}$
- $(\rightarrow I)$ (subcomputation box below)

2.1 A

assume

2.2 *B*

from 1.

$$(\rightarrow E1) \; \frac{A \rightarrow B}{\neg A \lor B}$$

1. $A \rightarrow B$

data

2. $\neg(\neg A \lor B) \rightarrow (\neg A \lor B)$ (subcomputation box below)

 \underline{B}

		$\neg A \lor B$
$2.1 \neg (\neg A \lor B)$		assume
$2.2 A \rightarrow (\neg A \lor B)$		(ightarrow I) (subcomputation box below)
		$\neg A \lor B$
	2.2.1 <i>A</i>	assume
	2.2.2 <i>B</i>	from 2.2.1, 1., and $(ightarrowE)$
	$2.2.3 \neg A \lor B$	from 2.2.2 and (\lor I)
$2.3 A \rightarrow \neg(\neg A \lor B)$		(ightarrow I) (subcomputation box below)
		$\neg(\neg A \lor B)$
	2.3.1 <i>A</i>	assume
	$2.3.2 \neg (\neg A \lor B)$	from 2.1
2.4 ¬	\overline{A}	2.2, 2.3, (¬I)
$2.5 \neg A \lor B$		2.5 and (∨I)

3. $\neg(\neg A \lor B) \rightarrow \neg(\neg A \lor B)$ (\rightarrow I) (subcomputation box below)

- 4. $\neg A \lor B$
- from 2., 3., and $(\neg E)$

$$(\neg E1) \frac{\neg \neg A}{A}$$

- 1. $\neg \neg A$ data
- 2. $\neg A \rightarrow \neg A$ (\rightarrow I) (subcomputation box below)

$$\begin{array}{ccc} & & \underline{\neg A} \\ 2.1 \ \neg A & \text{assume} \\ 2.2 \ \neg A & \text{from } 2.1 \end{array}$$

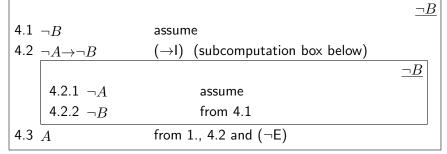
3. $\neg A \rightarrow \neg \neg A$ (\rightarrow I) (subcomputation box below)

4. A from 2., 3., and $(\neg E)$

$$(\neg E2) \ \frac{\neg A \rightarrow B, A \rightarrow B}{B}$$

- 1. $\neg A \rightarrow B$ data
- 2. $A \rightarrow B$ data
- 3. $\neg B \rightarrow \neg A$ (\rightarrow I) (subcomputation box below)

4. $\neg B \rightarrow A$ (\rightarrow I) (subcomputation box below)



5. B from 3., 4., and $(\neg E)$

$$(\neg 11) \frac{A}{\neg \neg A}$$

- 1. A data
- 2. $\neg A \rightarrow \neg A$ (\rightarrow I) (subcomputation box below)

$$\begin{array}{ccc} & & \underline{\neg A} \\ 2.1 \ \neg A & \text{assume} \\ 2.2 \ \neg A & \text{from } 2.1 \end{array}$$

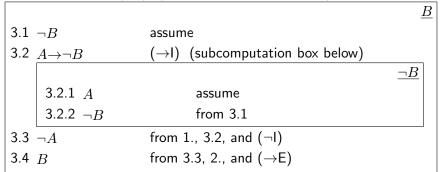
3. $\neg A \rightarrow A$ (\rightarrow I) (subcomputation box below)

$$\begin{array}{ccc} \underline{A} \\ 3.1 \ \neg A & \text{assume} \\ 3.2 \ A & \text{from 1} \end{array}$$

4. $\neg \neg A$ from 2., 3., and $(\neg I)$

$$(\neg 2) \ \frac{A {\rightarrow} B, \neg A {\rightarrow} B}{B}$$

- 1. $A{
 ightarrow}B$ data
- 2. $\neg A \rightarrow B$ data
- 3. $\neg B \rightarrow B$ (\rightarrow I) (subcomputation box below)



4. $\neg B \rightarrow \neg B$ (\rightarrow I) (subcomputation box below)

$$\begin{array}{ccc} & & \underline{\neg B} \\ 4.1 \ \neg B & \text{assume} \\ 4.2 \ \neg B & \text{from 4.1} \end{array}$$

5. B from 3., 4., and $(\neg E)$