Sets: what is a set?

To communicate, we sometimes need to agree on the meaning of certain terms. If the same idea is mentioned several times in a discussion, we often replace it with some new terminology and shorthand notation.

A **set** is a collection of things, any things, called its **elements**.

If S is a set and x is an element in S, then we write

$$\boxed{x \in S}\,.$$

If x is **not** an element in S, then we write

$$\boxed{x \notin S}\,.$$

If **both** $x \in S$ **and** $y \in S$, we often denote this fact by the shorthand notation

$$\boxed{x, y \in S}\,.$$

Describing sets by listing

To describe a set we need to describe its elements in some way.

- The simplest way to describe a set is to **explicitly name its elements**.

  FOR EXAMPLE: We can form a set A by collecting three 'things':
  Cinderella, *Tasmania* and *Tuesday*.

  Any set defined this way is denoted by listing its elements, separated by commas, and surrounding the listing with braces:

  $$\boxed{A = \{\textit{Cinderella}, \textit{Tasmania}, \textit{Tuesday}\}}.$$

- **Anything** can be an element of a set.

  In particular, a set can be an element of another set.

  FOR EXAMPLE: The set $B = \{x, \{x, y\}\}$ has two elements:
  - one element is x ,
  - and the other element is the set $\{x, y\}$.

  (So we can write $x \in B$ and $\{x, y\} \in B$.)

Important features of sets

The only thing that matters about a set:

what is **IN** it and what is **NOT IN** it.

- **Repeated occurrences of elements don't matter:**

  $\{H, E, L, L, O\}$, $\{H, H, H, E, L, L, O\}$, and $\{H, E, L, O\}$ describe the **same** set (namely, the set having four elements: the letters H, E, L, and O).

  So it is best to be economical and list everything only once: $\{H, E, L, O\}$.

- **The order of listing doesn't matter either:**

  $\{H, E, L, O\}$, $\{E, H, L, O\}$, $\{E, H, O, L\}$, and $\{O, L, E, H\}$

  all describe the **same** set.

Special sets

- The set with no elements is called the **empty set**. The empty set is denoted by $\{\ \}$ or more often by the symbol

$$\boxed{\emptyset}\,.$$

The empty set has no elements. So no matter what thing x denotes, $x \notin \emptyset$.

On the other hand, the empty set can be an element of another set.

FOR EXAMPLE: $\emptyset \in \{a, \emptyset, \mathit{Joe}\}$ or $\emptyset \in \{\emptyset\}$.

- Any set with one element is called a **singleton**.

FOR EXAMPLE: $\{a\}$ and $\{\mathit{Monday}\}$ are singletons,
but $\{x, \emptyset\}$ is not.

Equality of sets

Two sets are **equal** if they have the same elements.

We denote the fact that two sets A and B are equal by writing

$$\boxed{A = B}.$$

If the sets A and B are **not equal,** we write

$$\boxed{A \neq B}.$$

FOR EXAMPLE: $\{u, g, h\} = \{h, u, g\}$

$\{a, b, c\} \neq \{a, c\}$

$\{\textit{Monday}\} \neq \emptyset$

$\{1, 2\} = \{1, 1, 1, 2, 2\}$

$\{2\} \neq \{\{2\}\}$

Finite and infinite sets

Suppose we start counting the elements of a nonempty set S, one element per second. If a point of time is reached when all the elements of S have been counted (we might need to have one of our descendants to finish up), then we say that S is **finite**.

If the counting never stops, then S is an **infinite** set.

EXAMPLES OF INFINITE SETS:

Z $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ — **integers**

N $= \{0, 1, 2, 3, \ldots\}$ — **natural numbers**

N$^+$ $= \{1, 2, 3, \ldots\}$ — **positive natural numbers** $=$ **positive integers**

odd natural numbers

Q — the set of **rational numbers:**

any number that can be expressed as $\frac{m}{n}$ for some integers m and $n \neq 0$

R — the set of **real numbers:**

they include all rational numbers, and

also **irrational numbers** like π, $\sqrt{2}$, $\ldots$

R$^+$ — the set of **positive real numbers**

Describing sets by properties: the set builder notation

Describing an infinite set by listing, using $\ldots$, might sometimes be possible (like we did with **N**), but even then a bit imprecise.

But listing, say, rational numbers can be quite tricky. Instead of listing elements, we can describe a property that the elements of the set satisfy.

If $\mathcal{P}$ is a property, then the set S whose elements have property $\mathcal{P}$ is denoted by

$$\boxed{S \;=\; \{x \mid x \text{ has property } \mathcal{P}\}}\,.$$

We read this as

"S **is the set of all** x **such that** x **has property** $\mathcal{P}$."

Or, if we also know that all the elements in S come from a larger set A, then we can write

$$\boxed{S = \{x \in A \mid x \text{ has property } \mathcal{P}\}}\,.$$

Describing sets by properties: an example

Let *Odd* be the set of all odd integers. Then we can describe *Odd* in several ways:

$$
\begin{aligned}
\textit{Odd} &= \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} \\
&= \{x \mid x \text{ is an odd integer}\} \\
&= \{x \in \mathbf{Z} \mid x \text{ is odd}\} \\
&= \{x \mid x = 2k + 1 \text{ for some integer } k\} \\
&= \{x \mid x = 2k + 1 \text{ for some } k \in \mathbf{Z}\}
\end{aligned}
$$

We can also use more complex expressions on the left hand side of $\mid$ in the set builder notation:

$$
\begin{aligned}
\textit{Odd} &= \{2k + 1 \mid k \text{ is an integer}\} \\
&= \{2k + 1 \mid k \in \mathbf{Z}\}
\end{aligned}
$$

Russell's paradox

Not every property is suitable for describing sets. Here is a tricky one:

Let T be the **set** of all sets that are not elements of themselves:

$$\boxed{T = \{A \mid A \text{ is a set and } A \notin A\}}$$

<u>QUESTION: Is T an element of T or not?</u>

Well, either $T \in T$ or $T \notin T$. Let us examine both cases:

- If $\underline{T \in T}$, then the property we used to describe T must hold for T. But then $\underline{T \notin T}$. So this case is impossible.
- If $\underline{T \notin T}$, then the property we used to describe T does not hold for T. So it is not the case that "T is a set and $T \notin T$." So, by de Morgan's law in logic, either T is not a set, or $T \in T$. But as T is a set, $\underline{T \in T}$ follows. So this case is impossible as well.

Describing sets by recursion

A recursive description of a set S consists of three steps:

(1) **Basis step:**
Specify one or more elements of S.

(2) **Recursive step:**
Give one or more rules to construct new elements of S from existing elements of S.

(3) **Exclusion rule:**
State that S consists **only** of the elements that are specified by the basis step, or generated by successive applications of the recursive step. Nothing else is in S.
(This step is usually assumed rather than stated explicitly.)

Describing sets by recursion: examples

- The set **N** of natural numbers can be defined recursively:

  Basis step: $0 \in$ **N**

  Recursive step: If $n \in$ **N** then $n+1 \in$ **N**.

- The set *Odd* of odd integers can be defined recursively:

  Basis step: $813 \in$ *Odd*

  Recursive step: If $n \in$ *Odd* then $n+2 \in$ *Odd* and $n-2 \in$ *Odd*.

- The set $A = \{3k+1 \mid k \in \mathbf{N}\}$ can be defined recursively:

  Basis step: $1 \in A$

  Recursive step: If $x \in A$ then $x+3 \in A$.

- The set F of all formulas of propositional logic can be defined recursively:

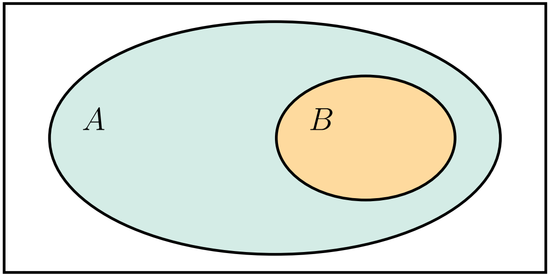
  Basis step: every propositional variable is in F

  Recursive step: If $p \in F$ and $q \in F$, then $(\neg p) \in F$,
  $(p \wedge q) \in F$, $(p \vee q) \in F$, $(p \to q) \in F$ and $(p \leftrightarrow q) \in F$.

Subsets

A set B is a **subset** of a set A, if every element of B is also an element of A.

Notation: $\boxed{B \subseteq A}$.



Venn diagram of $B \subseteq A$

FOR EXAMPLE: $\{a, c\} \subseteq \{a, b, c, d\}$ $\quad \{0, 1, 5\} \subseteq \mathbf{N}$ $\quad \mathbf{N} \subseteq \mathbf{Z}$ $\quad \mathbf{N} \subseteq \mathbf{N}$

- We **always** have, for every set S: $\boxed{S \subseteq S}$ $\quad \boxed{\emptyset \subseteq S}$
- If B is a subset of A, and there is some element in A that is **not** in B, then we say that B is a **proper subset** of A, and write $\boxed{B \subset A}$.
  In other words, $B \subset A$ if $B \subseteq A$ but $B \neq A$.

How to prove that $A \subseteq B$ or $A \subset B$?

Let $A = \{x \mid x$ is a prime number and $42 \leq x \leq 51\}$,

$B = \{x \mid x = 4k + 3$ and $k \in \mathbf{N}\}$.

<u>EXERCISE 1:</u> Show that $A \subseteq B$.

SOLUTION: We need to show that **every** element in A is also in B.

This is a statement of the form $\forall x \, \big(\text{“}x \in A\text{”} \to \text{“}x \in B\text{”}\big)$.

Take an **arbitrary** $x \in A$. Then x is a prime number and $42 \leq x \leq 51$.

So either $x = 43$ or $x = 47$.

- We can have $43 = 4 \cdot 10 + 3$. So the choice of $k = 10$ shows that $\underline{43 \in B}$.
- We can have $47 = 4 \cdot 11 + 3$. So the choice of $k = 11$ shows that $\underline{47 \in B}$ as well.

Therefore, we have shown that $A \subseteq B$.

<u>EXERCISE 2:</u> Show that $A \subset B$.

SOLUTION: We have just shown that $A \subseteq B$. So it remains **to find an element** $x \in B$ such that $x \notin A$. For example, $x = 3$ is such:

- As $0 \in \mathbf{N}$ and $3 = 4 \cdot 0 + 3$, we have $\underline{3 \in B}$.
- On the other hand, $42 \not\leq 3$, so we have $\underline{3 \notin A}$.

How to prove that $A \not\subseteq B$?

Let $A = \{3k + 1 \mid k \in \mathbf{N}\}$,

$B = \{4k + 1 \mid k \in \mathbf{N}\}$.

<u>EXERCISE:</u> Show that $A \not\subseteq B$, that is, A is **not** a subset of B.

SOLUTION: This is a statement of the form $\neg\forall x\,\big(\text{“}x \in A\text{”} \to \text{“}x \in B\text{”}\big)$.
So we need **to find a counterexample:** an element $x \in A$ such that $x \notin B$.
For example, $x = 4$ is such:

- $4 = 3 \cdot 1 + 1$. So $k = 1$ shows that $\underline{4 \in A}$.
- We need to show that $4 \notin B$, that is, there is **no** $k \in \mathbf{N}$ such that $4 = 4k + 1$.
  Let's see. If $k = 0$, then $4k + 1 = 4 \cdot 0 + 1 = 1 \neq 4$.
  And if $k \geq 1$, then $4k + 1 \geq 4 \cdot 1 + 1 = 5 > 4$.
  So it is not possible to find a $k \in \mathbf{N}$ such that $4 = 4k + 1$, and so $\underline{4 \notin B}$.

Therefore, we have shown $A \not\subseteq B$.

How to prove $A = B$ or $A \neq B$?

$\boxed{A = B}$ means that sets A and B have the **same elements.**

This is a statement of the form $\forall x\,\big(\text{“}x \in A\text{”} \leftrightarrow \text{“}x \in B\text{”}\big)$.

$$\boxed{A = B \qquad \text{iff} \qquad A \subseteq B \qquad \text{and} \qquad B \subseteq A}$$

$$\forall x\,\big(\text{“}x \in A\text{”} \to \text{“}x \in B\text{”}\big) \;\wedge\; \forall x\,\big(\text{“}x \in B\text{”} \to \text{“}x \in A\text{”}\big)$$

$$\boxed{A \neq B \qquad \text{iff} \qquad A \not\subseteq B \qquad \text{or} \qquad B \not\subseteq A}$$

$$\exists x\,\big(\text{“}x \in A\text{”} \wedge \text{“}x \notin B\text{”}\big) \;\vee\; \exists x\,\big(\text{“}x \in B\text{”} \wedge \text{“}x \notin A\text{”}\big)$$

So:

- If the task is to show that $\boxed{A = B}$,

  then we need to show **BOTH** $\boxed{A \subseteq B}$ **AND** $\boxed{B \subseteq A}$.

- If the task is to show that $\boxed{A \neq B}$,

  then we **EITHER** need **to find an element** in A that is not in B,

  OR to find an element in B that is not in A.

The power set of a set

The **set of all subsets** of a set S is called the **power set of S** which we denote by

$$\boxed{P(S)}\,.$$

So $P(S) = \{A \mid A \subseteq S\}$.

FOR EXAMPLE:

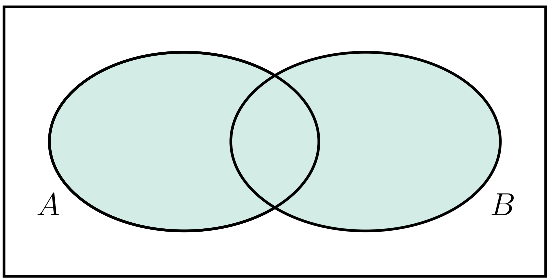
- As for EVERY set S, we always have $\emptyset \subseteq S$ and $S \subseteq S$,
  we always have $\boxed{\emptyset \in P(S)}$ and $\boxed{S \in P(S)}$.
- $P\big(\{u,v,w\}\big) = \big\{\emptyset, \{u\}, \{v\}, \{w\}, \{u,v\}, \{u,w\}, \{v,w\}, \{u,v,w\}\big\}$
  WATCH OUT! $\{\emptyset\} \notin P\big(\{u,v,w\}\big)$
- $P\big(\{\textit{Joe}, \textit{Tuesday}\}\big) = \big\{\emptyset, \{\textit{Joe}\}, \{\textit{Tuesday}\}, \{\textit{Joe}, \textit{Tuesday}\}\big\}$
- $P(\mathbf{N}) = \{X \mid X \subseteq \mathbf{N}\}$. So, say, $\{453, 11, 5\} \in P(\mathbf{N})$, $\{81, 2\} \in P(\mathbf{N})$, $\emptyset \in P(\mathbf{N})$,
  but $\{3, -1\} \notin P(\mathbf{N})$, $\{\frac{5}{2}, 23\} \notin P(\mathbf{N})$, $\{\emptyset\} \notin P(\mathbf{N})$.

Set operations: union

The **union** of sets A and B is the set

$$\boxed{A \cup B \ = \ \{x \mid x \in A \ \text{(inclusive) or} \ x \in B\}}.$$

$A \cup B$ consists of those elements that are either in A, or in B, or in both.



Venn diagram of $A \cup B$

FOR EXAMPLE: $A = \{4, 7, 8\}$ and $B = \{10, 4, 9\}$.

Then $A \cup B = \{4, 7, 8, 9, 10\}$.

Properties of union

- $A \cup \emptyset = A$ — **identity law**
- $A \cup B = B \cup A$ — **commutative law**
- $(A \cup B) \cup C = A \cup (B \cup C)$ — **associative law**

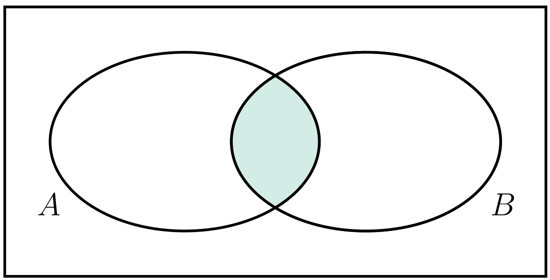
  So we can meaningfully write, say, $A \cup B \cup C \cup D \cup E$ or $\bigcup_{i=1}^{n} A_i$.
- $A \cup A = A$ — **idempotent law**
- $A \subseteq B$ iff $A \cup B = B$

Set operations: intersection

The **intersection** of sets A and B is the set

$$\boxed{A \cap B \;=\; \{x \mid x \in A \text{ and } x \in B\}}\,.$$

$A \cap B$ consists of those elements that are both in A and in B.



Venn diagram of $A \cap B$

FOR EXAMPLE: $A = \{4, 7, 8\}$ and $B = \{10, 4, 9\}$.

Then $A \cap B = \{4\}$.

If $A \cap B = \emptyset$, then A and B are called **disjoint**.

Properties of intersection and union

- $A \cap \emptyset = \emptyset$ — **domination law**
- $A \cap B = B \cap A$ — **commutative law**
- $(A \cap B) \cap C = A \cap (B \cap C)$ — **associative law**

  So we can meaningfully write, say, $A \cap B \cap C \cap D \cap E$ or $\bigcap_{i=1}^{n} A_i$.
- $A \cap A = A$ — **idempotent law**
- $A \subseteq B$ iff $A \cap B = A$
- **distributive laws:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- BUT WATCH OUT! often $A \cap (B \cup C) \neq (A \cap B) \cup C$

  So writing $A \cap B \cup C$ might NOT make sense!

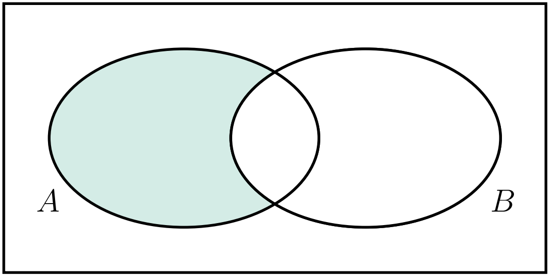
Set operations: difference

The **difference** of sets A and B is the set

$$\boxed{A - B \;=\; \{x \mid x \in A \text{ and } x \notin B\}}.$$

$A - B$ consists of those elements that are in A but not in B.

$A - B$ is also called the **complement of B with respect to A**.



Venn diagram of $A - B$

FOR EXAMPLE: $A = \{4, 7, 8\}$ and $B = \{10, 4, 9\}$.

Then $A - B = \{7, 8\}$ and $B - A = \{10, 9\}$.

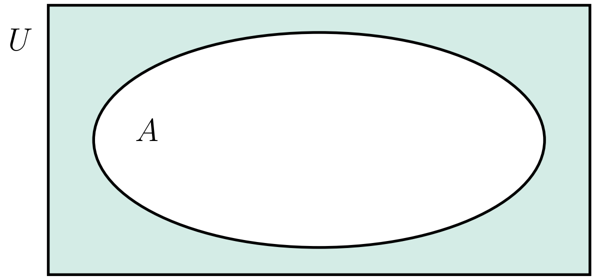
Set operations: (absolute) complement

In certain contexts we may consider all sets under consideration as being subsets of some given **universal set** U

(it is like the domain of a quantification).

Given a universal set U and $A \subseteq U$, the <u>**complement of** A</u> (w.r.t. U) is the set

$$\boxed{\bar{A} = U - A = \{x \in U \mid x \notin A\}}.$$



Venn diagram of $\bar{A}$

More properties of set operations

- **complementation laws:** $A \cup \bar{A} = U$

  $A \cap \bar{A} = \emptyset$

  $\bar{\bar{A}} = A$

  $\bar{U} = \emptyset$

  $\bar{\emptyset} = U$

- **De Morgan's laws:** $\overline{A \cup B} = \bar{A} \cap \bar{B}$

  $\overline{A \cap B} = \bar{A} \cup \bar{B}$

- $A - B = A \cap \bar{B}$

- $A \subseteq B$ iff $\bar{B} \subseteq \bar{A}$

Exercise 2.1 (examples of general direct proofs)

Prove that, for **any** X, Y and Z, we **always** have $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

<u>SOLUTION:</u> According to lecture slide 60, we need to show both

(1) $X \cap (Y \cup Z) \subseteq (X \cap Y) \cup (X \cap Z)$ and **(2)** $(X \cap Y) \cup (X \cap Z) \subseteq X \cap (Y \cup Z)$.

<u>Watch out!</u> **It is <u>not</u> enough to give (or draw) some example sets X, Y, Z for which (1) and (2) hold.**

We need to show (1) and (2) using **no** assumptions on the sets X, Y, Z and their elements, <u>using **only** the properties of set operations $\cup$ and $\cap$.</u>

<u>**For (1):**</u> We need to show that every element of $X \cap (Y \cup Z)$ is also an element of $(X \cap Y) \cup (X \cap Z)$. Take an arbitrary ☺ $\in X \cap (Y \cup Z)$. Then both ☺ $\in X$ and ☺ $\in Y \cup Z$. There are two cases: either ☺ $\in Y$ or ☺ $\in Z$. In the first case, ☺ $\in X \cap Y$. In the second case ☺ $\in X \cap Z$. So in either case, ☺ $\in (X \cap Y) \cup (X \cap Z)$.

<u>**For (2):**</u> Take an arbitrary ☹ $\in (X \cap Y) \cup (X \cap Z)$. Then there are two cases: either ☹ $\in X \cap Y$, or ☹ $\in X \cap Z$. In the first case, both ☹ $\in X$ and ☹ $\in Y$. In the second case, both ☹ $\in X$ and ☹ $\in Z$. So ☹ $\in X$ in both cases, and either ☹ $\in Y$ or ☹ $\in Z$, depending on the case. Therefore, ☹ $\in Y \cup Z$, and so ☹ $\in X \cap (Y \cup Z)$.

Exercise 2.2

Show that there are sets A and B such that $A \cup \bar{B} \neq \bar{A} \cup B$.

<u>SOLUTION:</u> We need to find and describe **some** sets A and B such that $A \cup \bar{B}$ and $\bar{A} \cup B$ are different sets. There can be many possible solutions, here is one:

Let the universal set be $U = \{1, 2, 3\}$. Let $A = \{1\}$ and $B = \{2\}$.

Then $\bar{A} = \{2, 3\}$ and $\bar{B} = \{1, 3\}$. So $A \cup \bar{B} = \{1, 3\}$ and $\bar{A} \cup B = \{2, 3\}$.

These are two different sets as, for example, $1 \in A \cup \bar{B}$, but $1 \notin \bar{A} \cup B$.

Sequences

A **sequence** is a list of things, taken in a certain order.

To distinguish sequences from sets, we use brackets (instead of braces) when we list the elements of a sequence:

$(1, 5, 8, -2, 3)$ $\qquad$ (*Cinderella*, *Tasmania*, *Tuesday*)

Important features of sequences:

- **The order of listing DOES matter:**
  $(1, 2, 3)$, $(1, 3, 2)$ and $(3, 1, 2)$ are different sequences.

- **Repeated occurrences DO matter:**
  (H, E, L, L, O), (H, H, H, E, L, L, O) and (H, E, L, O) are different sequences.

Tuples

Finite sequences are called **tuples**. A sequence with k elements is a **k-tuple**. The number k is called the **length** of the tuple.

Two tuples are **equal** if they have the same length and their corresponding elements are the same:

$\boxed{(x_1, x_2, \ldots, x_k) = (y_1, y_2, \ldots, y_n)}$ means that $k = n$ and

$x_1 = y_1$ and $x_2 = y_2$ $\ldots$ and $x_k = y_k$.

A 2-tuple is also called an **ordered pair**.

$\boxed{(a, b) = (c, d)}$ means that $a = c$ and $b = d$.

Cartesian product of sets

The **Cartesian product of sets A and B** is the set

$$\boxed{A \times B \;=\; \{(x,y) \mid x \in A \text{ and } y \in B\}}.$$

$A \times B$ consists of those ordered pairs (x,y) where $x \in A$ and $y \in B$.

FOR EXAMPLE: Let $A = \{1,2\}$ and $B = \{a,b,c\}$.
Then $A \times B \;=\; \{(1,a),(2,a),(1,b),(2,b),(1,c),(2,c)\}$.

The **Cartesian product of sets $A_1, A_2, \dots, A_k$** is the set

$$\boxed{A_1 \times A_2 \times \cdots \times A_k \;=\; \{(x_1, x_2, \dots, x_k) \mid x_i \in A_i \text{ for } i = 1, 2, \dots, k\}}.$$

$A_1 \times A_2 \times \cdots \times A_k$ consists of those k-tuples $(x_1, x_2, \dots, x_k)$
where $x_1 \in A_1,\; x_2 \in A_2,\; \dots,\; x_k \in A_k$.

FOR EXAMPLE: Let $A = \{1,2\}$, $B = \{a,b,c\}$ and $C = \{\diamond, \square\}$.

Then $A \times B \times C \;=\; \big\{(1,a,\diamond),(1,a,\square),(2,a,\diamond),(2,a,\square),(1,b,\diamond),(1,b,\square),(2,b,\diamond),$
$(2,b,\square),(1,c,\diamond),(1,c,\square),(2,c,\diamond),(2,c,\square)\big\}$.

Binary relations

For sets A and B, a **(binary) relation from A to B** is *any* subset R of the Cartesian product $A \times B$. (So a binary relation is a set consisting of some ordered pairs.)

We use notation $\boxed{aRb}$ to denote that $(a, b) \in R$, and say that **a is R-related to b**.
If $(a, b) \notin R$, then we write $\boxed{a \not R\, b}$.

(Other notations: $R(a, b)$ for aRb, $\neg R(a, b)$ for $a \not R\, b$)

FOR EXAMPLE:

Let P be the set of people, and C the set of football clubs in the English Premier League. Define a relation Fan_of by taking

$$\text{Fan\_of} = \{(x, y) \in P \times C \mid x \text{ is a fan of } y\}.$$

If I am a fan of MU, but can't stand Arsenal, then

$(\textit{Agi}, \textit{MU}) \in \text{Fan\_of}$ but $(\textit{Agi}, \textit{Arsenal}) \notin \text{Fan\_of}$

We can also write the same things as

Agi Fan_of *MU* and *Agi* $\not\text{Fan\_of}$ *Arsenal*

Relations on a set

A relation from a set A to A itself is called a **<u>relation on A</u>**.

In other words: a relation on a set A is a subset of $A \times A$

a relation on a set A is a set consisting *some* ordered pairs of elements from A

FOR EXAMPLE:

Let A be the set of all people in the class.

$$R_1 = \{(u, v) \in A \times A \mid u \text{ likes } v\}$$

$$R_2 = \{(u, v) \in A \times A \mid u \text{ is taller than } v\}$$

Raj is 180 cm tall and likes Jill who is 165 cm. Unfortunately, Jill doesn't like Raj.

Joe is 190 cm tall, and they don't like each other with Jill.

<u>Then:</u> $(\text{Raj}, \text{Jill}) \in R_1$ $(\text{Raj}, \text{Jill}) \in R_2$ $(\text{Jill}, \text{Raj}) \notin R_1$ $(\text{Jill}, \text{Raj}) \notin R_2$

$(\text{Joe}, \text{Jill}) \notin R_1$ $(\text{Joe}, \text{Jill}) \in R_2$ $(\text{Jill}, \text{Joe}) \notin R_1$ $(\text{Jill}, \text{Joe}) \notin R_2$

Relations on a set: some more examples

Here are some relations on the set **Z** of integers:

- **'smaller than':** $\boxed{< \;=\; \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is smaller than } y\}}$

  -1 is smaller than 0, so $(-1,0) \in <$ used more often: $-1 < 0$

  5 is smaller than 25, so $(5,25) \in <$ used more often: $5 < 25$

  -3 is smaller than -2, so $(-3,-2) \in <$ used more often: $-3 < -2$

  0 is not smaller than -1, so $(0,-1) \notin <$ used more often: $0 \not< -1$

  10 is not smaller than 2, so $(10,2) \notin <$ used more often: $10 \not< 2$

  -4 is not smaller than -4, so $(-4,-4) \notin <$ used more often: $-4 \not< -4$

- **'smaller than or equal to':** $\boxed{\leq \;=\; \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is smaller than or equal to } y\}}$

  $-1 \leq -1$, $5 \leq 6$, but $1 \not\leq 0$, $-2 \not\leq -10$

  $\boxed{< \subset \leq}$

- **'divisibility':** $\boxed{\text{div} \;=\; \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ divides } y\}}$

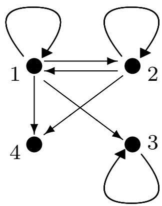
  $(1,25) \in \text{div}$, $(-3,12) \in \text{div}$, $(10,1000) \in \text{div}$, but $(0,4) \notin \text{div}$, $(2,-21) \notin \text{div}$, $(6,3) \notin \text{div}$

Representing relations

FOR EXAMPLE: Let $A = \{1,2,3,4\}$ and take the following relation R on A:

$$R = \big\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,4),(3,3)\big\}.$$

Then R can be represented by 'points and arrows':



directed graph

Or, by a 0-1 **matrix:**

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|cccc} & 2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{c|cccc} & 4 & 3 & 1 & 2 \\ 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \end{array}$$

we can choose **any** order of listing of the set A, but we must choose the **same** order **both** from left-to-right and from top-to-bottom

Properties of relations

- A relation R on a set A is called **reflexive**,

  if $(a, a) \in R$ for every element $a \in A$.

  FOR EXAMPLE:

  - reflexive: $\leq$, $\geq$, $=$, 'divisibility' on $\mathbf{N}^+$

    $R_1 = \{(1,1),(1,2),(3,1),(2,2),(3,3)\}$ on $\{1,2,3\}$
  - not reflexive: $R_2 = \{(1,1),(1,2),(3,1),(3,3)\}$ on $\{1,2,3\}$

- A relation R on a set A is called **irreflexive**,

  if $(a, a) \notin R$ for every element $a \in A$.

  FOR EXAMPLE: $<$, $>$, $\neq$ on $\mathbf{N}$ or on $\mathbf{Z}$

  $R_3 = \{(1,2),(1,3),(2,1),(3,2)\}$ on $\{1,2,3\}$

  Irreflexive is more than just 'not reflexive' !

Properties of relations (cont.)

- A relation R on a set A is called **symmetric**, if for all elements $a, b \in A$,

  $(b, a) \in R$ whenever $(a, b) \in R$.

  FOR EXAMPLE:

  - symmetric: $R_1 = \{(1,1),(1,2),(2,1),(2,3),(3,2),(3,3)\}$ on $\{1,2,3\}$

    $R_2 = \{(1,1),(2,2),(3,3)\}$ on $\{1,2,3\}$

    $\{(x,y) \in \mathbf{N} \times \mathbf{N} \mid x - y$ is divisible by $3\}$ on $\mathbf{N}$
  - not symmetric: $R_3 = \{(1,1),(1,2),(2,1),(3,1),(3,3)\}$ on $\{1,2,3\}$

- A relation R on a set A is called **antisymmetric**, if

  (a, b) and (b, a) cannot be both in R unless $a = b$.

  (It does NOT mean that (a, a) should be in R !)

  FOR EXAMPLE: $<$, $>$, $\leq$, $\geq$ on **N**, **Z**, **Q**, or **R**

  $R_4 = \{(1,1),(1,3),(2,3),(3,3)\}$ on $\{1,2,3\}$

  R_2 above

Symmetric and antisymmetric are not opposites !

Properties of relations (cont.)

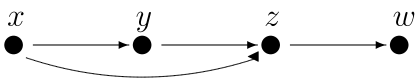
A relation R on a set A is called **transitive**, if the following hold, **for all elements** $a, b, c \in A$:

$$\text{if } \textbf{both } (a,b) \in R \textbf{ and } (b,c) \in R, \textbf{ then } (a,c) \in R.$$

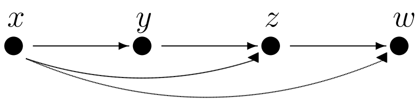
In the directed graph representing R:

R is transitive, if **every** two-step journey along arrows can be done in one step.

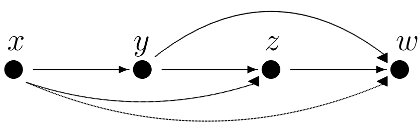
FOR EXAMPLE:



not transitive



still not transitive



transitive

Transitive relations: more examples and non-examples

- Transitive:

  $<$, $>$, $\leq$, $\geq$ on **N**, **Z**, **Q**, or **R**

  $R_1 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

  $R_2 = \{(3,4)\}$ (both R_1 and R_2 on $\{1,2,3,4\}$)

- Not transitive:

  $R_3 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

  $R_4 = \{(1,1),(1,2),(2,1)\}$

  $R_5 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

  (all R_3, R_4, R_5 on $\{1,2,3,4\}$)

Transitive closure

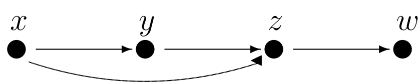
- Let R be a relation on a set A.

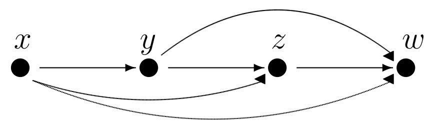
  The **transitive closure of R** is the smallest transitive relation on A containing R.

- We denote the transitive closure of R by $\boxed{R^*}$.

- (If R is already transitive, then $R^* = R$.)

  FOR EXAMPLE:

R : 

R^* : 

- Given R, we can define R^* recursively:

  Basis step: $R \subseteq R^*$

  (the pairs in R are all in R^*)

  Recursive step: If $(a, b) \in R^*$ and $(b, c) \in R^*$ then $(a, c) \in R^*$.

  (we add the missing pairs step-by-step)

Computing the transitive closure: Warshall's algorithm

An **algorithm** is a finite sequence of precise step-by-step instructions.

Warshall's algorithm computes the matrix of the transitive closure R^* of R.

- Given a relation R on a set A with n elements, we begin with its $n \times n$ matrix $\boxed{M_0}$.
  (**any** order of listing A is fine)
- There are n rounds.

  In each round, we turn the previous matrix to a new matrix:

  $$M_0 \overset{\text{round 1}}{\rightsquigarrow} M_1 \overset{\text{round 2}}{\rightsquigarrow} M_2 \overset{\text{round 3}}{\rightsquigarrow} \quad \dots \quad \overset{\text{round n}}{\rightsquigarrow} M_n$$

  $\boxed{M_n}$ is the matrix of the transitive closure R^* of R.

  - 1st rule. we never change a 1 to 0
  - 2nd rule. rule for changing **some** 0s to 1s:

$$\underbrace{\begin{array}{cc} & \begin{array}{ccc} j & & k\text{th column} \end{array} \\ \begin{array}{c} i \\ \\ k\text{th row} \\ \\ \end{array} & \begin{bmatrix} 0 & \to & 1 & \\ \downarrow & & & \\ 1 & & & \\ & & & \end{bmatrix} \end{array}}_{M_{k-1}} \quad \overset{\text{round k}}{\rightsquigarrow} \quad \underbrace{\begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ \\ k \\ \\ \end{array} & \begin{bmatrix} \underline{1} & 1 & \\ & & \\ 1 & & \\ & & \end{bmatrix} \end{array}}_{M_k}$$

Warshall's algorithm: an example

Let R be a relation on $\{a, b, c, d\}$: $\boxed{R = \{(a,d),(b,a),(b,c),(c,a),(c,d),(d,c)\}}$

As $\{a, b, c, d\}$ has 4 elements, $n = 4$. The matrix of R is the 4×4 matrix

$$M_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(here the chosen order of listing is a, b, c, d)

There will be 4 rounds.

Round 1.

$$\begin{array}{cc} & \text{1st column} \\ \text{1st row} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \underline{0} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ & M_0 \end{array} \quad \overset{\text{round 1}}{\rightsquigarrow} \quad \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \underline{\mathbf{1}} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ M_1 \end{array}$$

Warshall's algorithm: an example (cont.)

Round 2.

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \overset{\text{round 2}}{\leadsto} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = M_2 \quad \text{(no change)}$$

Round 3.

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \underline{0} & 0 & 1 & \underline{0} \end{bmatrix} \overset{\text{round 3}}{\leadsto} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \underline{\mathbf{1}} & 0 & 1 & \underline{\mathbf{1}} \end{bmatrix} = M_3$$

Round 4.

$$M_3 = \begin{bmatrix} \underline{0} & 0 & \underline{0} & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & \underline{0} & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \overset{\text{round 4}}{\leadsto} \begin{bmatrix} \underline{\mathbf{1}} & 0 & \underline{\mathbf{1}} & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & \underline{\mathbf{1}} & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = M_4$$

$$\boxed{R^* = \{(a,a),(a,c),(a,d),(b,a),(b,c),(b,d),(c,a),(c,c),(c,d),(d,a),(d,c),(d,d)\}}$$

Equivalence relations

A relation R on a set A is called an **equivalence relation** if it is

- reflexive,
- symmetric, and
- transitive.

FOR EXAMPLE:

- $=$ on any set,
- $\{(x, y) \in \mathbf{Z} \times \mathbf{Z} \mid x - y$ is divisible by $4\}$ on $\mathbf{Z}$

Partial orders

A relation R on a set A is called a **partial order** if it is

- reflexive,
- antisymmetric, and
- transitive.

FOR EXAMPLE: $\leq$, $\geq$, and 'divisibility' on $\mathbf{N}^+$

EXERCISE: Prove that $\subseteq$ is a partial order on the power set $P(S)$ of a set S.

SOLUTION:

- $\subseteq$ is reflexive: It is because $A \subseteq A$ for every set A.
- $\subseteq$ is antisymmetric: If $A \subseteq B$ and $B \subseteq A$, then $A = B$ holds.
- $\subseteq$ is transitive: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ holds.

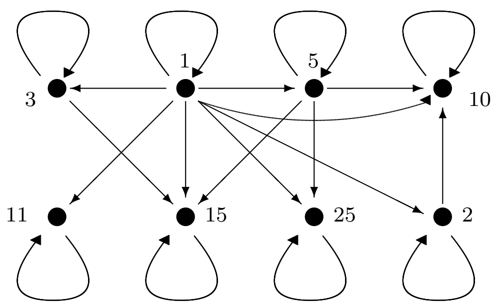
Representing partial orders: Hasse diagrams

If we **know that a relation is a partial order**, then there is a more 'economical' way of representing it than by a directed graph.

Say, take the 'divisibility' relation on the set $\{1, 2, 3, 5, 10, 11, 15, 25\}$:

$$\{(1,1),(1,2),(1,3),(1,5),(1,10),(1,11),(1,15),(1,25),(2,2),(2,10),(3,3),$$
$$(3,15),(5,5),(5,10),(5,15),(5,25),(10,10),(11,11),(15,15),(25,25)\}$$

As this is a relation, it can be represented by a directed graph:

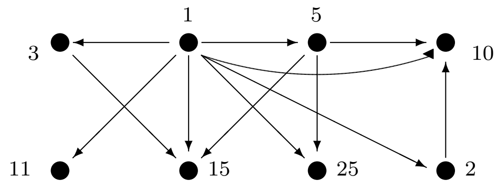


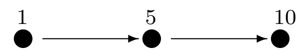
But we can do better, **using** our knowledge while drawing.

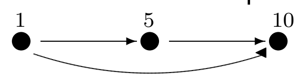
Constructing Hasse diagrams (cont.)

As partial orders are always **reflexive,** a loop is always present at every point.

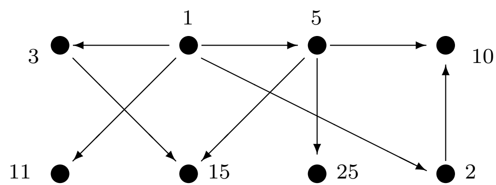
So by removing these loops we don't lose info:



Partial orders are always **transitive.** Say, if 

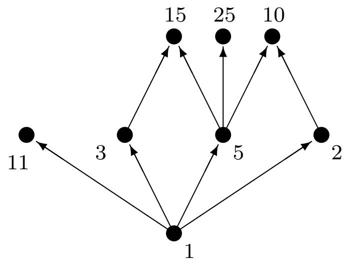
is part of our diagram, then we know that we must also have 

So we don't lose info by indicating only 'one-step' arrows, and removing the rest:

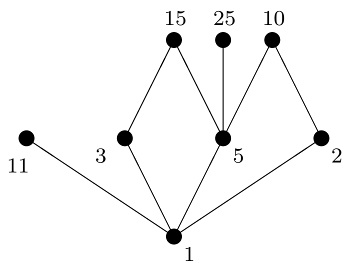


Constructing Hasse diagrams (cont.)

Partial orders are always **antisymmetric.** This means that between any two points there can be an arrow one way only, NOT both. So we can rearrange the points such that all the arrows 'point' from a lower position 'upwards':



So we don't lose info by removing the arrow-heads, and using lines instead:

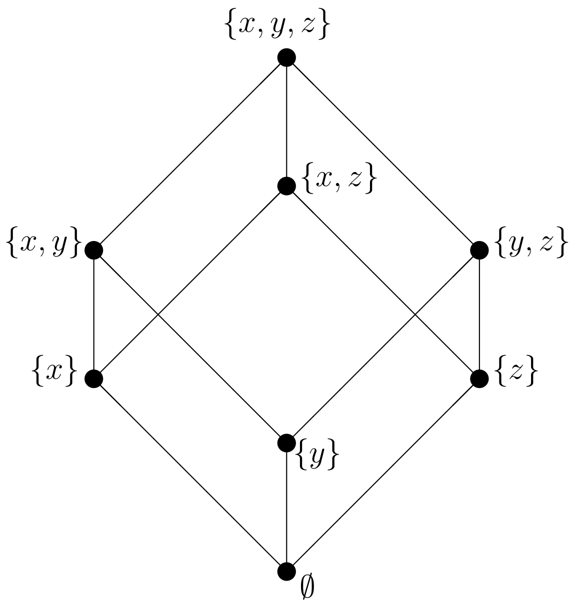


Hasse diagram
of the 'divisibility' relation
on $\{1, 2, 3, 5, 10, 11, 15, 25\}$

(Overall shape does not matter, but WATCH OUT: 'horizontal' lines are NO GOOD!)

Hasse diagrams: another example

The Hasse diagram of $\subseteq$ on the power set $P(\{x, y, z\})$ of $\{x, y, z\}$:



Linear orders

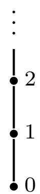
A relation R on a set A is called a **linear order** (or **total order**) if

- R is a partial order, and
- for all $a, b \in A$, either $(a, b) \in R$ or $(b, a) \in R$
  (that is, every pair of elements from A is in R this way or the other).

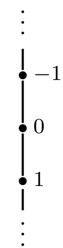
The name 'linear' comes from the fact that the Hasse diagram of a linear order is always a **line.**

FOR EXAMPLE:

- $\leq$ on **N**:



- $\geq$ on **Z**:



- BUT: 'divisibility' and $\subseteq$ are partial orders, but NOT linear orders
  (see previous two lecture slides)

Exercise 2.3

Let S be a set with more than one element.

Show that $\subseteq$ is **not** a linear order on $P(S)$.

SOLUTION:

If S has more than one element, then there are at least two different elements in S, let's call them x and y. Then:

- $x \in S$, so $\{x\} \subseteq S$, and so $\{x\} \in P(S)$.
- $y \in S$, so $\{y\} \subseteq S$, and so $\{y\} \in P(S)$.
- Neither $\{x\} \subseteq \{y\}$ nor $\{y\} \subseteq \{x\}$ holds, as x and y are different.

So $\{x\}$ and $\{y\}$ are two elements in $P(S)$ such that neither $\{x\} \subseteq \{y\}$ nor $\{y\} \subseteq \{x\}$, and so $\subseteq$ is not a linear order.

(Check the Hasse diagram on lecture slide 91. It is not a line.)