4CCS1ELA–Elementary Logic with Applications

4–Checking the Validity of Arguments

4.2–Forward Reasoning Systems

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Introduction to Formal Proofs

Introducing forward rules

We have already seen how to check logical consequence using the truth-tables. However, that potentially involves checking an exponential number of rows in the truth-table.

In addition, as we have seen in the previous part, it is often useful to *demonstrate* how the conclusion can be obtained from the premises, through a sequence of manipulation steps following well-understood principles.

Forward reasoning rules provide a way of doing just that!

Example

Consider the following argument

$$\frac{\begin{array}{c} P \\ R \vee S \\ T \\ P \to Q \end{array}}{Q}$$

If we were to use truth-tables to demonstrate its validity, we would need to check 2^5 rows (because of the number of propositional variables).

However, if we look at the argument, we can see that we only 2 premises (P and $P{\to}Q$) to ascertain that the conclusion must follow, hence rendering the check quite straightforward.

Formal Proofs

Formal proof

A *proof* (or deduction) of a formula B from a set of premises $A_1, \ldots, A_n$ is a finite sequence of formulas ending in B such that each formula in the sequence is either a premise, an "axiom", or can be obtained from previous formulas in the sequence using an inference rule.

We will denote that B can be derived from $A_1, \ldots, A_n$ by $A_1, \ldots, A_n \vdash B$.

The symbol $\vdash$ defines a *consequence relation* between a set of formulas and a formula.

Notice that this is a syntactical process. This being the reason why we use the special symbol $\vdash$, instead of its semantical counterpart $\models$.

Common presentation of an inference rule

An inference rule is a representation of a valid step in a formal proof, indicating the conditions that must be satisfied before a conclusion can be obtained in the proof.

A rule will be presented in the form: $$\frac{A_1, A_2, \ldots, A_n}{B} \quad \text{(R)} .$$

$A_1, A_2, \ldots, A_n$ are the premises or conditions of the rule, and B is the conclusion.

Informally, "if $A_1, A_2, \ldots, A_n$ are true, then B is also true" (via R).

Example

Modus ponens comes from the Latin "method of affirming (the antecedent)". We are already familiar with the inference pattern of this rule. In our presentation, we will sometimes refer to the rule as "elimination of the implication" ($\rightarrow$E):

$$\frac{A,\ A\rightarrow B}{B} \quad (\rightarrow\text{E}) \text{ or "modus ponens"}$$

In English.

"If Paul was born in Britain, he is British. Paul was born in Britain. Therefore, Paul is British."

Notation for proofs

A proof sequence $W_1, W_2, \ldots, W_n$ will be represented in the following way:

1.	W_1	Justification for W_1
2.	W_2	Justification for W_2
$\vdots$	$\vdots$	$\vdots$
n.	W_n	Justification for W_n

If the derivation of formula in a line depends on other formulas, then the line numbers of those formulas must be indicated as part of its *justification*.

If a proof contains premises, then those are justified by the words 'given', 'assumption' or 'data'.

Example

Our initial argument was $P, R \vee S, T, P \to Q \vdash Q$.

Using the sample rules we provided, a possible proof of its validity could be:

1.	P	data
2.	$P \to Q$	data
3.	Q	From 1. and 2. and ($\to$E)

We can use the premises for "free" (they are assumptions). Thus the justification for lines 1. and 2. are that they are givem or "data". Line 3. demonstrates that Q is a valid conclusion, because lines 1. and 2. fulfil the conditions of the modus ponens rule.

Deduction Theorem

If there is a proof of B that uses a formula A as a premise, then there is a proof of the implication $A \rightarrow B$ that does not use A as a premise. In other words,

$$\mathcal{S} \cup \{A\} \vdash B \text{ iff } \mathcal{S} \vdash A \rightarrow B$$

Let us recall, that in the special case when $\mathcal{S}$ is empty, this becomes

$$\{A\} \vdash B \text{ iff } \vdash A \rightarrow B$$

So we are actually saying that proving that B follows from A is equivalent to proving that $A \rightarrow B$ is a tautology (this should come as no surprise).

We will revisit this in Natural Deduction in our *introduction of the implication* rule.

Example

In the Example of Slide 4.1.6, we wanted to show that

“If n is an odd integer (N1), then n^2 is odd (N2)”

This means showing $\vdash N1 \rightarrow N2$.

The deduction theorem tells us that we can show $\vdash$ N1$\rightarrow$N2, if we assume N1, then show that N2 follows: N1 $\vdash$ N2. We could even describe our previous direct proof as follows:

1. n is odd	Assume
2. $n = 2k + 1$, for some k	Property of odd nums
3. $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$	Square of sums
4. $n^2 = 2(2k^2 + 2k) + 1$	Factorising
5. n^2 is odd	From 4. and prop. of odd nums

Notes about proofs I

- **Only premises are allowed as assumptions**
  - ⇒ In the previous example, we resorted to the deduction theorem, but in natural deduction, this is not (directly) allowed. There is an explicit rule dictating how to derive a conditional statement.
- **Only rules explicitly defined can be used**
  - ⇒ We will not allow any steps that are not formally or fully justified, e.g., in 2., 3., etc. Justifications for all items must be fully provided.
- The proof patterns seen before can still be used as a "strategy" to find a proof, but the proofs can only use the rules defined.

Notes about proofs II

- **Do not use unnecessary premises**
  - ⇒ The conjunction of the premises used to prove something is precisely the antecedent (the premise) of the "tautology" proved.

  To show $P, R \vee S, T, P \rightarrow Q \vdash Q$, we only need P and $P{\rightarrow}Q$. We could see this as a proof that $P{\rightarrow}((P{\rightarrow}Q){\rightarrow}Q)$ is a tautology, hence the other premises are not needed!

$$P, P{\rightarrow}Q \vdash Q \text{ iff } P \vdash (P{\rightarrow}Q){\rightarrow}Q \text{ iff } \vdash P{\rightarrow}((P{\rightarrow}Q){\rightarrow}Q)$$

Classical logic is a **monotonic** reasoning system: adding assumptions preserves conclusions. However, using only the premises we need make the proofs more compact and easier to read.

Notes about proofs III

- **The order of the premises is irrelevant.** Remember that premises are *sets* of formulas, which is an unordered structure. This is also a valid proof of the argument $P, R \vee S, T, P \to Q \vdash Q$:

  1. $P \to Q$ data
  2. P data
  3. Q From 1. and 2. and ($\to$E)

  $$P, P \to Q \vdash Q \text{ iff } P \vdash (P \to Q) \to Q \text{ iff } \vdash P \to ((P \to Q) \to Q)$$

  But also...

  $$P, P \to Q \vdash Q \text{ iff } P \to Q \vdash P \to Q \text{ iff } \vdash (P \to Q) \to (P \to Q)$$

  The conditions of a rule do not specify any order either!

Proof theory vs. Semantics

Soundness and Completeness

In general one wants a proof system to closely match the semantics of the logic it is used for. There are two important properties describing this relationship.

Soundness: An proof system is *sound* (or correct) if all of its rules are sound, i.e., whenever $A_1, \ldots, A_n \vdash B$, then $A_1, \ldots, A_n \models B$.

Completeness: A proof system is *complete* if whenever $A_1, \ldots, A_n \models B$, then B can be derived from $A_1, \ldots, A_n$ using its inference rules.

Some ramifications. . .

Suppose a proof system $\vdash$ is sound and complete. Then we know that

$$A_1, \ldots, A_n \vdash B \text{ iff } A_1, \ldots, A_n \models B$$

So if $A_1, \ldots, A_n \not\models B$, then we know that $A_1, \ldots, A_n \not\vdash B$.

This **only** means that we will not be able to find a proof for B from $A_1, \ldots, A_n$, but we may try in vain to find it. . .

If we are confident that $A_1, \ldots, A_n \not\vdash B$, this means that $A_1, \ldots, A_n \not\models B$, so there must be a model of $A_1, \ldots, A_n$ that is not a model of B.

NATURAL DEDUCTION PROOFS

Natural Deduction

Natural deduction is a type of forward reasoning proof system: the objective is to derive the conclusion of the argument, starting from its premises.

There are two ways of generating new conclusions:

- We either eliminate a connective from a formula in the proof to generate a new conclusion (via a connective **elimination rule**); or
- We produce a new conclusion with a connective, by combining formulas in the proof (via a connective **introduction rule**)

In general, coming up with a strategy to complete the proof is the hardest part.

Applying a rule

As we saw, a rule may have premises or *conditions*. For example, consider the ($\wedge$E) ("and-elimination") rule below:

$$\frac{A \wedge B}{A} \quad (\wedge\mathrm{E})$$

Since, this rule has $A \wedge B$ as a premise, before it can be applied you need to have $A \wedge B$ in the proof.

The result of the application of the rule is a new item in the proof sequence with the formula A and the justification "From x. and ($\wedge$E)", where x. is line in the proof where $A \wedge B$ appears.

Example

$$A \wedge B, A \to C \wedge D \vdash C$$

1.	$A \wedge B$	data
2.	$A \to C \wedge D$	data
3.	A	from 1. and ($\wedge$E)
4.	$C \wedge D$	from 2., 3., and ($\to$E)
5.	C	from 4. and ($\wedge$E)

Notice that every line in the proof is a logical consequence of the set of premises of the argument, and the proof is successful because it ends with the argument's conclusion.

To know more. . .

- There is a good introduction to formal reasoning in Section 6.3 of Hein's "Discrete Structures, Logic, and Computability", 4th edition.
- Sections 1.6, 1.7 and 1.8 of Rosen's "Discrete Mathematics and Its Applications", 8th edition also discuss proofs and strategies.
- Forward reasoning is introduced in Chapter 3 of Gabbay and Rodrigues' "Elementary Logic with Applications", 1st edition.
- A more comprehensive introduction to consequence relations, soundness and completeness can be found in Section 1.2.2 and Chapter 5 of Gabbay and Rodrigues' "Elementary Logic with Applications", 1st edition.