4CCS1ELA–Elementary Logic with Applications

5–Proving with Natural Deduction

5.3–Introducing Variant Rules

Dr. Odinaldo Rodrigues

odinaldo.rodrigues@kcl.ac.uk

Room BH(S) TBC, +44 (0)20 7848 2087

Department of Informatics

King's College London

Outline

1. Variant Rules

2. Examples with Variant Rules

Variant Rules

Motivation

We have seen that the basic rules are complete. However, many steps in some proofs could be avoided if we were to have additional rules to simplify valid derivations of conclusions from premises.

For example, consider the following argument $B \vdash A \rightarrow B$.

1.	B	data
2.	$A \rightarrow B$	From subcomputation box below

		$\underline{B}$
2.1	A	Assume
2.2	B	From 1.

We have to show this through a subcomputation box, because that is the only way to introduce implication using the basic set of rules.

Motivation (contd.)

Yet, *under the assumption that the consequent is true*, any implication immediately follows…

We can avoid the need for a subcomputation box, if we augment our basic set of rules with an extra implication-introduction rule of the form:

$$\frac{B}{A \to B} \quad (\to\text{I2})$$

This *variant* implication-introduction rule avoids the need for a subcomputation box in the cases where the consequent of the implication is already part of the current proof.

Motivation (contd.)

Using this variant form of implication introduction

$$\frac{B}{A \rightarrow B} \quad (\rightarrow\text{I2})$$

we can re-write the proof $B \vdash A{\rightarrow}B$ simply as as follows:

1. B data
2. $A{\rightarrow}B$ From 1. and ($\rightarrow$I2)

Just remember that all variants rules can be obtained from the basic set of rules, and hence they're not strictly needed!

Example two

1.	$P \vee Q$	data
2.	$\neg Q$	data
3.	$P \rightarrow R$	data
4.	$P \rightarrow P$	From subcomputation box
		$\underline{P}$
4.1	P	Assume
4.2	P	From 4.1
5.	$Q \rightarrow P$	From subcomputation box
		$\underline{P}$
5.1	Q	Assume
5.2	$\neg P \rightarrow Q$	From subcomputation
		$\underline{Q}$
5.2.1	$\neg P$	Assume
5.2.2	Q	From 5.1
5.3	$\neg P \rightarrow \neg Q$	From subcomputation
		$\underline{\neg Q}$
5.3.1	$\neg P$	Assume
5.3.2	$\neg Q$	From 2
5.4	P	From 5.2, 5.3, and ($\neg$E)
6.	P	From 1., 4., 5., and ($\vee$E)
7.	R	From 3., 6., and ($\rightarrow$E)

(without variant)

$$P \vee Q, \neg Q, P \rightarrow R \vdash R$$

$\Rightarrow$

1.	$P \vee Q$	data
2.	$\neg Q$	data
3.	$P \rightarrow R$	data
4.	P	From 1., 2., and ($\vee$E2)
5.	R	From 3., 4., and ($\rightarrow$E)

(with variant)

$$\frac{A \vee B, \neg B}{A} \quad (\vee E2)$$

Variant rules for disjunction and negation

Variant disjunction elimination rules

$$\frac{A \vee B, \neg A}{B} \quad (\vee E1) \qquad\qquad \frac{A \vee B, \neg B}{A} \quad (\vee E2)$$

Variant negation elimination rules

$$\frac{\neg\neg A}{A} \quad (\neg E1) \qquad\qquad \frac{\neg A \to B, A \to B}{B} \quad (\neg E2)$$

Variant rules for implication introduction

With ($\vee$I), we can get from $\neg A$ to $\neg A \vee B$ and from B to $\neg A \vee B$, which we know to be equivalent to $A \rightarrow B$.

Of course we can show that $\neg A \vee B$ leads to $A \rightarrow B$ (see next slide), but the next two rules allow us to this directly:

($\rightarrow$I1) allows us to derive an implication whose antecedent is false, and ($\rightarrow$I2) allows us to derive an implication whose consequent is true:

Variant introduction rules for implication

$$\frac{\neg A}{A \rightarrow B} \quad (\rightarrow\text{I1}) \qquad\qquad \frac{B}{A \rightarrow B} \quad (\rightarrow\text{I2})$$

Showing that $\neg A \vee B \vdash A \to B$

1.	$\neg A \vee B$	Data
2.	$A \to B$	From subcomputation box

> $\underline{B}$
>
> | | | |
> |---|---|---|
> | 2.1 | A | Assume |
> | 2.2 | $\neg A \to B$ | From subcomputation |
>
> > $\underline{B}$
> >
> > | | | |
> > |---|---|---|
> > | 2.2.1 | $\neg A$ | Assume |
> > | 2.2.2 | $\neg B \to A$ | From subcomputation |
> >
> > > $\underline{A}$
> > >
> > > | | | |
> > > |---|---|---|
> > > | 2.2.2.1 | $\neg B$ | Assume |
> > > | 2.2.2.2 | A | From 2.1 |
> >
> > | | | |
> > |---|---|---|
> > | 2.2.3 | $\neg B \to \neg A$ | From subcomputation |
> >
> > > $\underline{\neg A}$
> > >
> > > | | | |
> > > |---|---|---|
> > > | 2.2.3.1 | $\neg B$ | Assume |
> > > | 2.2.3.2 | $\neg A$ | From 2.2.1 |
> >
> > | | | |
> > |---|---|---|
> > | 2.2.4 | B | From 2.2.2, 2.2.3, ($\neg$E) |
>
> | | | |
> |---|---|---|
> | 2.3 | $B \to B$ | From subcomputation (omitted) |
> | 2.4 | B | From 1., 2.2., 2.3, and ($\vee$E) |

$$\neg A \vee B, A \vdash B$$

1.	$\neg A \vee B$	Data
2.	$A \to B$	copy steps from left
3.	A	Data
4.	B	2., 3. and ($\to$E)

Variant rules for implication elimination

($\to$E1) allows us to write an implication in terms of disjunction and negation:

Variant elimination rule for implication

$$\frac{A \to B}{\neg A \vee B} \quad (\to\text{E1})$$

Notice that we already knew that $A{\to}B \equiv \neg A \vee B$. We just did not have rules to show this correspondence in a more direct way.

Examples with Variant Rules

Sample proof 1

Show that $A \vee B, \neg B \vdash A$

1.	$A \vee B$	data
2.	$\neg B$	data
3.	A	From 1., 2., and ($\vee$E2)

Sample proof 2

Show that $A \wedge \neg A \vdash B$

1.	$A \wedge \neg A$	Data
2.	A	From 1. and ($\wedge$E)
3.	$\neg B \rightarrow A$	From 2. and ($\rightarrow$I2)
4.	$\neg A$	From 1. and ($\wedge$E)
5.	$\neg B \rightarrow \neg A$	From 4. and ($\rightarrow$I2)
6.	B	From 3. and 5. and ($\neg$E)

Sample proof 3

Show that $\neg A, B \to C \vdash A \to C$

1.	$\neg A$	Data
2.	$B \to C$	Data
3.	$A \to B$	From 1. and ($\to$I1)
4.	$A \to C$	From subcomputation

$\underline{C}$

4.1	A	Assume
4.2	B	From 4.1., 3., and ($\to$E)
4.3	C	From 4.2., 2., and ($\to$E)

1.	$\neg A$	Data
2.	$A \to C$	From 1. and ($\to$I1)

To know more. . .

- Natural deduction is explained in detail in Chapter 3 of Gabbay and Rodrigues' "Elementary Logic with Applications", 1st edition.
- An alternative presentation is given in Hein's "Discrete Structures, Logic, and Computability", 4th edition. The presentation may use different rules to the ones presented here and we will require adherence to "our" rules for consistency in the approach.
- The last exercise in tutorial list 5 suggests that you to prove all variant rules from the basic ones.