Instituto Tecnológico y de Estudios Superiores de Monterrey
Campus Monterrey

MA3014
Estancia de Investigación Control de Qbits para Cómputo Cuántico
(Grupo 577)

Actividad: Random Qbit

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Tecnológico de Monterrey

Activity

Qubit Control for Quantum Computing

Follow the instructions and generate the required code.

The expectation value of an operator (or quantum gate) A over a qubit $|\psi\rangle$ is defined as

$$\langle A\rangle = \langle\psi|A|\psi\rangle = (\langle\psi|)A|\psi\rangle$$

Consider the general state

$$|\psi\rangle = a|0\rangle + b|1\rangle, \text{ with } a, b \in \mathbb{C},$$

and define the map

$$|\psi\rangle \mapsto \hat{n} = (\langle X\rangle, \langle Y\rangle, \langle Z\rangle)$$

A) Show that the entries of the vector $\hat{n}$ fulfill

$$n_x = \langle X\rangle = 2\mathrm{Re}(\bar{a}b)$$
$$n_y = \langle Y\rangle = 2\mathrm{Im}(\bar{a}b)$$
$$n_z = \langle Z\rangle = |a|^2 - |b|^2$$

and its norm is equal to 1. The overline $\bar{x}$ stands for the complex conjugate of x. You might work in the conventional matrix representation.

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Random Qubit

A) Show that the entries of the vector $\hat{n}$ fulfill

$$n_x = \langle X\rangle = 2\mathrm{Re}(\bar{a}b)$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Expected value:

$$\langle X\rangle = \langle\psi|X|\psi\rangle = (\langle\psi|)\underbrace{X|\psi\rangle}_{①}$$

① * We need to apply X gate to Ket ψ

② Then we apply bra ψ

① $$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0\cdot a + b\cdot 1 \\ 1\cdot a + 0\cdot b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$|\psi\rangle = a|0\rangle + b|1\rangle = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle\psi| = |\psi\rangle^{+} = \begin{pmatrix} a \\ b \end{pmatrix}^{+} = (\bar{a} \ \bar{b})$$

dagger

② $$\langle\psi|X|\psi\rangle = (\bar{a} \ \bar{b})\begin{pmatrix} b \\ a \end{pmatrix} = \bar{a}b + \bar{b}a$$

$$\langle X\rangle = \bar{a}b + a\bar{b} = \bar{a}b + \overline{(\bar{a}b)}$$

$$\overline{(\bar{a}b)} = a\bar{b}$$

complejo conjugado del complejo conjugado es el numero original. real

Por lo tanto

$$\langle X\rangle = 2\mathrm{Re}(\bar{a}b)$$

$n_y = \langle Y \rangle = 2\,\mathrm{Im}(\bar{a}b)$

$y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\langle Y \rangle = \langle \psi | y | \psi \rangle = (\langle \psi |)\, Y | \psi \rangle$

$= (\langle \psi |) \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$\begin{matrix} 0 \cdot a + -i \cdot \beta \\ i \cdot a + 0 \cdot \beta \end{matrix} = \begin{bmatrix} -i\beta \\ i a \end{bmatrix}$

$= (\langle \psi |) \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix}$

$= (\bar{a} \;\; \bar{\beta}) \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix}$

$= -i\bar{\alpha}B + i\alpha\bar{B} = -i(\bar{\alpha}\beta - \alpha\bar{\beta})$

$z = \bar{\alpha}\beta$

$\bar{z} = \alpha\bar{b}$

$\bar{\alpha}B + (\alpha\bar{\beta}) = \bar{\alpha}B + \overline{\bar{\alpha}B}$

$= 2\,\mathrm{Im}(\bar{a}b)$

$n_z = \langle z \rangle = |a|^2 - |b|^2$

$z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\langle z \rangle = \langle \psi | z | \psi \rangle = (\langle \psi |)\, z | \psi \rangle$

$\langle z \rangle = (\langle \psi |) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{matrix} 1 \cdot a + 0 \cdot b \\ 0 \cdot a + -1 \cdot b \end{matrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$

$= (\bar{a} \cdot \bar{b}) \begin{pmatrix} a \\ -b \end{pmatrix}$

$= \bar{\alpha} \cdot a - b\bar{b}$

$= |a|^2 - |b|^2$

B) The qubit $|\psi\rangle$ can be parametrized in the following way

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle, \text{ with } \theta \in [0,\pi/2], \quad \varphi \in [0,2\pi),$$

Using the results obtained in part A prove that the components of $\hat{n}$ are the usual spherical coordinates

$n_x = \sin\theta\cos\varphi$

$n_y = \sin\theta\sin\varphi$

$n_z = \cos\theta$

This procedure justifies why an arbitrary qubit is identified with a point in the Bloch sphere, which is also called the qubit projective space.

Sabemos que $\cos\varphi = \dfrac{e^{i\varphi} + e^{-i\varphi}}{2}$ en su forma compleja

$n_x =$

Partiendo de $|\psi\rangle$ parametrized:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle = \begin{bmatrix}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix}$$ Matricial form

Para n_x usamos el resultado anterior:

sea $x = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$ y $\langle\psi| = \begin{bmatrix}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix}$

Tenemos

$$n_x = (\langle\psi|)\, x\, |\psi\rangle = (\langle\psi|)\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix}$$

$$= (\langle\psi|)\begin{bmatrix}\cancel{\cos\frac{\theta}{2}\cdot 0} + 1\cdot e^{i\varphi}\sin\frac{\theta}{2} \\ 1\cdot\cos\frac{\theta}{2} + \cancel{e^{i\varphi}\sin\frac{\theta}{2}\cdot 0}\end{bmatrix} = (\langle\psi|)\begin{pmatrix}e^{i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix}$$

$(\langle\psi|) \to$ We traspose and do complex conjugate

$$= \begin{bmatrix}\cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}e^{i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{bmatrix} =$$

$$= \left[\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\varphi}\right]$$

Usamos identidad $\cos(\alpha)\sin(\beta) = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$ $\to 0$

en este caso

$$\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{1}{2}\left[\sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) - \cancel{\sin\left(\frac{\theta}{2}-\frac{\theta}{2}\right)}\right] = \frac{1}{2}\sin\theta$$

Sustituyendo resultado

$$\frac{1}{2}\sin\theta e^{i\varphi} + \frac{1}{2}\sin\theta e^{-i\varphi}$$

factorizamos

$$= \sin\theta\left(\frac{e^{i\varphi} + e^{-i\varphi}}{2}\right) = \boxed{\sin\theta\cos\varphi}$$

$n_y =$

$|\psi\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{bmatrix}$ $y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\langle Y \rangle = \langle \psi | Y | \psi \rangle = (\langle\psi|)\ Y|\psi\rangle = (\langle\psi|) \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{bmatrix}$

$= \left(\cos\frac{\theta}{2} \quad e^{-i\varphi}\sin\frac{\theta}{2}\right) \begin{pmatrix} \cancel{0\cdot\cos\frac{\theta}{2}} - i\cdot e^{i\varphi}\sin\frac{\theta}{2} \\ i\cdot\cos\frac{\theta}{2} + \cancel{0\cdot e^{i\varphi}\sin\frac{\theta}{2}} \end{pmatrix}$

$= \left(\cos\frac{\theta}{2}, e^{-i\varphi}\sin\frac{\theta}{2}\right) \begin{pmatrix} -ie^{i\varphi}\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix}$

$\boxed{\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta}$

$= -ie^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + ie^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= ie^{-i\varphi} - ie^{i\varphi}\left(\frac{i}{2}\sin\theta\right)$

$\frac{ie^{-i\varphi} - ie^{i\varphi}}{2}\left(\sin\theta\right)$

En su forma compleja:

$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$

Sacamos $-i$ resultando

$= \frac{-i\left(e^{i\varphi} - e^{i\varphi}\right)}{2\ (1)}$

$= \frac{i\left(e^{i\varphi} - e^{-i\varphi}\right)}{2(-1)}$

$= \frac{\sqrt{-1}\left(e^{i\varphi} - e^{-i\varphi}\right)}{(-1\ (2)}\sin\theta$

Podemos visualizar un 1 y asignarle un negativo ya que $i = \sqrt{-1}$

$= \frac{e^{i\varphi} - e^{-i\varphi}}{\sqrt{-1}\ (2)}\sin\theta = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}\sin\theta = \boxed{\sin\theta\sin\theta}$

$n_z =$

$|\psi\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{bmatrix}$ $\qquad z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\langle z \rangle = \langle \psi | z | \psi \rangle = (\langle \psi |) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$= (\langle \psi |) \begin{pmatrix} 1\cdot\cos\frac{\theta}{2} + \cancel{0\cdot e^{i\varphi}\sin\frac{\theta}{2}} \\ \cancel{0\cdot\cos\frac{\theta}{2}} + -1\cdot e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ -e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \cos\frac{\theta}{2}\cos\frac{\theta}{2} - \cancel{e^{-i\varphi}}\sin\frac{\theta}{2}\sin\frac{\theta}{2}\cancel{e^{i\varphi}}$$

Tenemos $e^{-i\varphi}e^{i\varphi} = e^{0} = 1$ se cancela

Usaremos las identidades

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$$

$$\sin(\alpha)\sin(\beta) = \frac{\sin(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

Por lo tanto:

$$\cos\frac{\theta}{2}\cos\frac{\theta}{2} \rightarrow \frac{\cos\theta}{2}$$

$$\sin\frac{\theta}{2}\sin\frac{\theta}{2} \rightarrow -\frac{\cos\theta}{2}$$

Sustituyendo

$$\frac{\cos\theta}{2} - \left(-\frac{\cos\theta}{2}\right) = \underline{\cos\theta}$$

C) Figure out what a random qubit would mean in this representation.

Un qbit aleatorio es aquel que puede representar cualquier estado de superposición que reciba dentro de su conjunto $|\Psi>$ puntos cualquiera en la superficie de la esfera de Bloch, siempre y cuando tenga como limitantes de sus ángulos que:

$$\theta \in [0, \pi/2]$$

&

$$\varphi \in [0, 2\pi]$$

Así mismo, cuando se trata de más de un qbit aleatorio, la distribución de los estados se debe generar de manera uniforme, esto con el fin de evitar un sesgo o palarización de sus estados.

D) Create a computational algorithm (in Matlab, Python, Mathematica, etc) generating a random qubit and representing it in the Bloch sphere as a point.

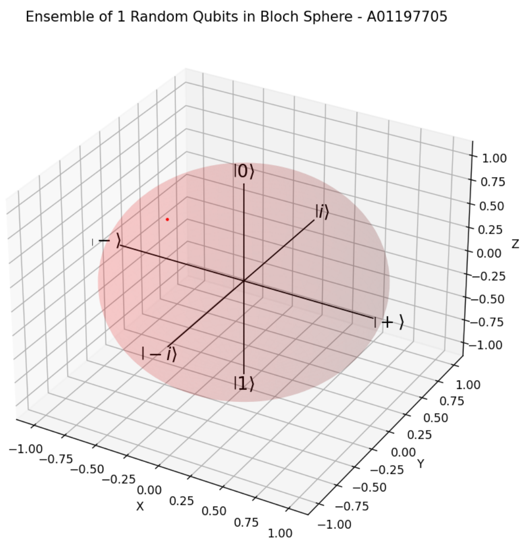
Código:

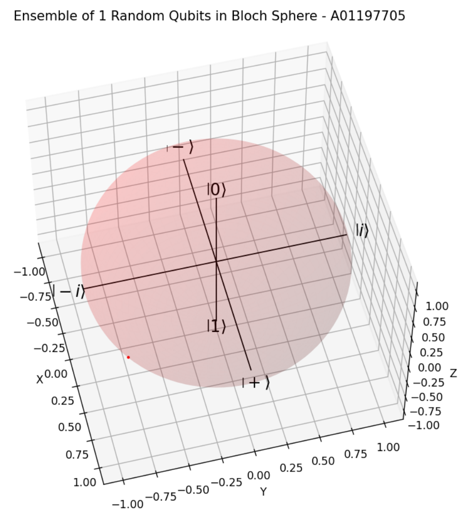
```
#Andrea Catalina Fernandez Mena  A01197705
#Actividad Qbits Aleatorios



#-------------------------------------------------------------------
#Inciso D
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

#Generar 1 punto que represente un estado random dentro de la esfera de Bloch
n_rand = 1

theta_rand = np.arccos(2 * np.random.rand(n_rand) - 1)
phi_rand = np.random.uniform(0, 2 * np.pi, n_rand)

state_0 = np.array([1, 0])
state_0 = state_0.reshape(-1, 1)

state_1 = np.array([0, 1])
state_1 = state_1.reshape(-1, 1)
```

```
psi_rand = np.cos(theta_rand / 2) * state_0 + np.sin(theta_rand / 2) *
np.exp(1j * phi_rand) * state_1

n_x_rand = np.sin(theta_rand) * np.cos(phi_rand)
n_y_rand = np.sin(theta_rand) * np.sin(phi_rand)
n_z_rand = np.cos(theta_rand)

fig = plt.figure(figsize=(8, 8))
ax = fig.add_subplot(111, projection='3d')

u = np.linspace(0, 2 * np.pi, 100)
v = np.linspace(0, np.pi, 100)
x = np.outer(np.cos(u), np.sin(v))
y = np.outer(np.sin(u), np.sin(v))
z = np.outer(np.ones(np.size(u)), np.cos(v))

ax.plot_surface(x, y, z, color='r', alpha=0.1)

ax.plot([-1, 1], [0, 0], [0, 0], linewidth=1, color='black')
ax.plot([0, 0], [-1, 1], [0, 0], linewidth=1, color='black')
ax.plot([0, 0], [0, 0], [-1, 1], linewidth=1, color='black')

ax.text(0, 0, 1.12, r'$\left| 0 \right>$', fontsize=15, ha='center',
va='center')
ax.text(1.12, 0, 0, r'$\left| + \right>$', fontsize=15, ha='center',
va='center')
ax.text(-1.12, 0, 0, r'$\left| - \right>$', fontsize=15, ha='center',
va='center')
ax.text(0, 0, -1.12, r'$\left| 1 \right>$', fontsize=15, ha='center',
va='center')
ax.text(0, 1.12, 0, r'$\left| i \right>$', fontsize=15, ha='center',
va='center')
ax.text(0, -1.12, 0, r'$\left| -i \right>$', fontsize=15, ha='center',
va='center')

inBS_rand = np.sqrt(n_x_rand**2 + n_y_rand**2 + n_z_rand**2) <= 1
ax.scatter(n_x_rand[inBS_rand], n_y_rand[inBS_rand],
n_z_rand[inBS_rand], c='red', marker='.', s=10)

ax.set_title('Ensemble of 1 Random Qubits in Bloch Sphere - A01197705')
ax.set_xlabel('X')
ax.set_ylabel('Y')
```

```
ax.set_zlabel('Z')

plt.show()
```

Resultado:





E) Generate an *ensemble* of 1000 random qubits and place them all in the qubit projective space. Do you think that they are uniformly distributed on the sphere?

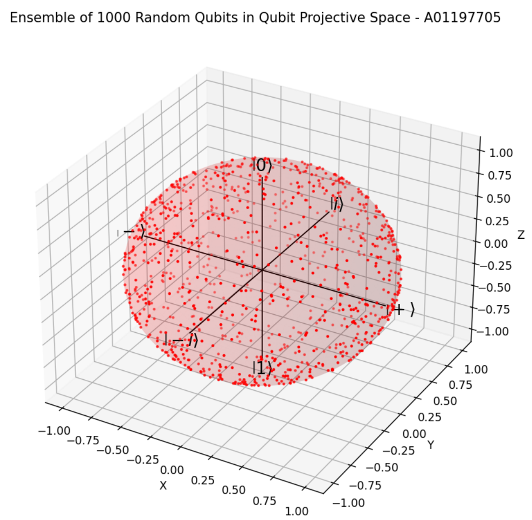
Código:

```
#Andrea Catalina Fernandez Mena  A01197705
#Actividad Qbits Aleatorios

#Inciso E)
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
#------------------------------------------------------------
n_rand = 1000

theta_rand = np.arccos(2 * np.random.rand(n_rand) - 1)
phi_rand = np.random.uniform(0, 2 * np.pi, n_rand)

state_0 = np.array([1, 0])
state_0 = state_0.reshape(-1, 1)

state_1 = np.array([0, 1])
```

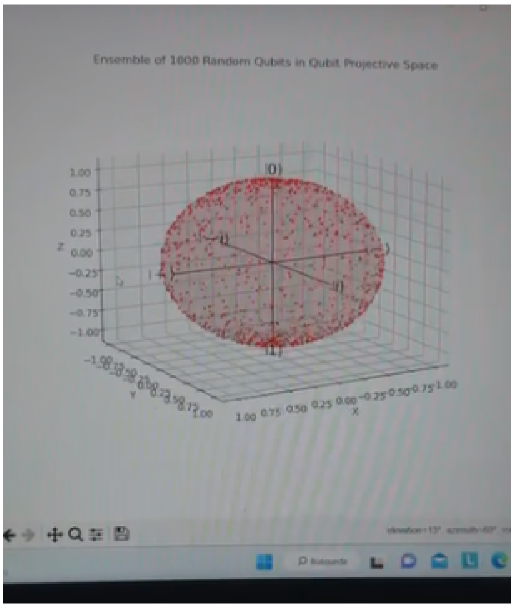
```
state_1 = state_1.reshape(-1, 1)

psi_rand = np.cos(theta_rand / 2) * state_0 + np.sin(theta_rand / 2) *
np.exp(1j * phi_rand) * state_1

n_x_rand = np.sin(theta_rand) * np.cos(phi_rand)
n_y_rand = np.sin(theta_rand) * np.sin(phi_rand)
n_z_rand = np.cos(theta_rand)

fig = plt.figure(figsize=(8, 8))
ax = fig.add_subplot(111, projection='3d')

u = np.linspace(0, 2 * np.pi, 100)
v = np.linspace(0, np.pi, 100)
x = np.outer(np.cos(u), np.sin(v))
y = np.outer(np.sin(u), np.sin(v))
z = np.outer(np.ones(np.size(u)), np.cos(v))

ax.plot_surface(x, y, z, color='r', alpha=0.1)

ax.plot([-1, 1], [0, 0], [0, 0], linewidth=1, color='black')
ax.plot([0, 0], [-1, 1], [0, 0], linewidth=1, color='black')
ax.plot([0, 0], [0, 0], [-1, 1], linewidth=1, color='black')

ax.text(0, 0, 1.12, r'$\left| 0 \right>$', fontsize=15, ha='center',
va='center')
ax.text(1.12, 0, 0, r'$\left| + \right>$', fontsize=15, ha='center',
va='center')
ax.text(-1.12, 0, 0, r'$\left| - \right>$', fontsize=15, ha='center',
va='center')
ax.text(0, 0, -1.12, r'$\left| 1 \right>$', fontsize=15, ha='center',
va='center')
ax.text(0, 1.12, 0, r'$\left| i \right>$', fontsize=15, ha='center',
va='center')
ax.text(0, -1.12, 0, r'$\left| -i \right>$', fontsize=15, ha='center',
va='center')

inBS_rand = np.sqrt(n_x_rand**2 + n_y_rand**2 + n_z_rand**2) <= 1
ax.scatter(n_x_rand[inBS_rand], n_y_rand[inBS_rand],
n_z_rand[inBS_rand], c='red', marker='.', s=10)

ax.set_title('Ensemble of 1000 Random Qubits in Qubit Projective Space
- A01197705')
```

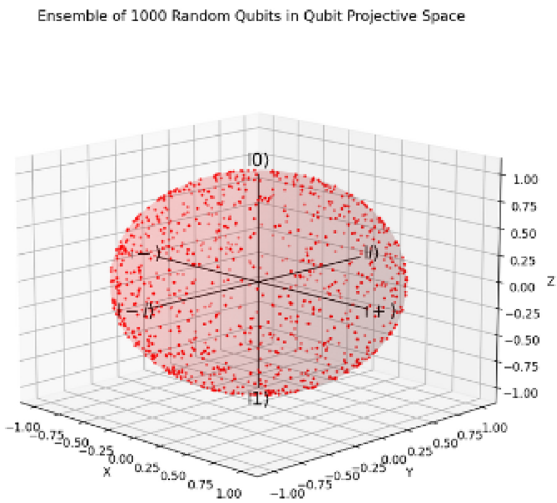
```python
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')

plt.show()
```

Resultado:



Considero que esta implementación ha logrado una distribución uniforme en la esfera, aunque con mediciones más precisas quizás no sea completamente uniforme en todos los aspectos.
Se obtuvo una mejora significativa en comparación con resultados anteriores que presentaban un marcado sesgo en los polos de la esfera de Bloch, especialmente en las coordenadas |0> y |1>. Este acercamiento hacia la uniformidad se ha conseguido mediante un ajuste en los ángulos correspondientes a los polos, utilizando la función del arcocoseno, por medio de la fórmula del elemento diferencial de área en coordenadas esféricas que es **dA = sin(theta) dtheta dphi** dentro del código . Esta función ha contribuido a dispersar de manera más equitativa los puntos tanto en el ecuador como en los polos de la esfera. Con esta función es posible generar una mayor uniformidad al aumentar el número de puntos dentro de la superficie.
Se adjunta evidencia de cambio de distribución que se logró gracias a la función de distribución de arcoseno.
Antes - Después


Ensemble of 1000 Random Qubits in Qubit Projective Space
|0⟩
|1⟩
X
Y
Z


Ensemble of 1000 Random Qubits in Qubit Projective Space
|0⟩
|1⟩
|−⟩
|+⟩
|i⟩
|−i⟩
X
Y
Z