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Network formation and pairwise stability: A new oddness theorem

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ABSTRACT

We prove that for large classes of polynomial payoff functions, there exist generically an odd number of pairwise stable networks, as a consequence of the topological structure of the *graph of pairwise stable weighted networks*, which we characterize. This improves recent results in Bich and Morhaim (2020) or in Herings and Zhan (2022), and can be applied to many existing models, as for example to the public good provision model of Bramoullé and Kranton (2007), the information transmission model of Calvó-Armengol and İlkılıç (2009) or the two-way flow model of Bala and Goyal (2000).

1. Introduction

Pairwise stability concept, proposed by Jackson and Wolinsky (1996), plays a central role in strategic network formation models. Roughly, a network is pairwise stable if "*no two agents could gain from linking and no single agent could gain by severing one of his or her link*" (Jackson and Wolinsky, 1996). First introduced for unweighted networks (i.e. networks for which relationships are measured by 0 or 1), it was extended for weighted networks (i.e. networks for which relationships are measured by real numbers between 0 and 1), and recently, Bich and Morhaim proved that a pairwise stable weighted network always exists, for large classes of payoff functions (Bich and Morhaim, 2017, 2020).

The aim of this paper is to provide new important properties of the set of pairwise stable networks. In game theory, similar properties have already been proved for the set of Nash equilibria: in 1971, Wilson (1971) has established his famous oddness theorem (*generically*, each finite strategic-form game has an odd number of mixed Nash equilibria); then, several authors have proposed alternative proofs (e.g. Blume and Zame, 1994; Govindan and McLennan, 2001; Govindan and Wilson, 2001; Harsanyi, 1973; Herings and Peeters, 2001; Mas-Colell, 2008; Pimienta, 2009), some of them being based on Kohlberg–Mertens' structure theorem (Kohlberg and Mertens, 1986). In this paper, we prove similar results for the set of pairwise stable networks, by allowing large classes of payoff functions:

1. The first main result, called *structure theorem* (see Theorem 2.1), deals with the topological structure of the *graph of pairwise stable networks*, i.e. the space $\mathcal{P}$ of all pairs (v, g), where $v = (v_1, \ldots, v_n)$ is a *society* (i.e. a profile of payoff functions of n agents involved in a network formation process) satisfying some mild differentiability and concavity assumptions, and where g is a pairwise stable network associated to v. More precisely, if $\mathcal{F}$ denotes the space of societies satisfying the above assumptions, then we prove that $\mathcal{P}$ is homeomorphic to $\mathcal{F}$ (i.e., intuitively, $\mathcal{P}$ can be deformed continuously into the simpler space $\mathcal{F}$). Our structure theorem is similar to Kohlberg–Mertens' theorem (Kohlberg and Mertens, 1986) and to Predtetchinski's structure theorem (Predtetchinski, 2009), but for pairwise stable networks instead of Nash equilibria.[1]
2. The second main result, called *oddness theorem* (Theorem 3.1), states that for large classes of polynomial payoff functions, there exist generically an odd number of pairwise stable networks.[2] We show that our oddness theorem is actually a consequence of our structure theorem. In particular, as a byproduct, we encompass a recent work of

[1] Kohlberg–Mertens' theorem states that the graph of mixed Nash equilibria, when defined on the space of all finite strategic-form games (of fixed dimension), is homeomorphic to the space of all finite strategic-form games (of the same dimension). Predtetchinski generalizes this result for more general classes of payoff functions than those involved in mixed extensions of finite strategic-form games.

[2] We also have developed a similar (but incomparable) result for Nash equilibria in Bich and Fixary (2021).

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Herings and Zhan (2022),[3] which states the same result, but for multilinear payoff functions.

We would like to emphasize that our theorems are not applications of existing results in game theory. First, by nature, a pairwise stable network is not a Nash equilibrium: two agents who want to create a link together have to decide it simultaneously (i.e. deviations have to be bilateral in some cases), whereas deviations are always unilateral in Nash equilibrium concept. More precisely, assessing that there exist generically an odd number of Nash equilibria is equivalent to say that there are generically an odd number of fixed-points of the best-reply correspondence, but there is no natural and analogue formulation for pairwise stability concept. Second, a notion of "mixed pairwise stability", comparable to the notion of mixed Nash equilibrium in game theory, seems less meaningful in network formation theory. This explains also – apart from its mathematical interest – why we consider general classes of polynomial payoff functions, going beyond the case of multilinear payoff functions. However, this also creates new technical difficulties: to prove our oddness theorem, we have to enter more deeply in the theory of semi-algebraic sets (in particular, we will provide some new decomposition result for semi-algebraic sets). Third, we have to "extend" Kohlberg–Mertens' theorem to the framework of network formation theory and pairwise stability. Our extension (i.e., our structure theorem) is not only a rewriting of existing proofs in game theory, since, in essence, we do not deal with standard non-cooperative strategic-form games.[4]

To conclude this paper, we describe several standard models in network formation theoretical literature to which our oddness theorem can be applied. A first example is the public good provision model of Bramoullé and Kranton (2007): n agents are characterized by some level of effort $e_i \in [0, +\infty)$, $i = 1, \ldots, n$, (e.g. it could be the amount of time a consumer spends researching a new product) and interact in some network g (the idea being that agents could benefit from the other agents' efforts, thanks to network externalities). Oddness theorem implies that, in this standard model, there exist an odd number of pairwise stable networks, generically with respect to some parameters of the model (i.e., in short, there exist an odd number of pairwise stable networks for "most" parameters of the model). We prove similar results in the information transmission model of Calvó-Armengol and İlkılıç (2009) and the two-way flow model of Bala and Goyal (2000).

The paper is organized as follows: in Section 2, we first recall some basic definitions and notations about strategic network formation theory and pairwise stability, and we define the graph of pairwise stable networks (Section 2.1), then we state our structure theorem (Theorem 2.1, Section 2.2) and a corollary of this theorem (Corollary 2.1) which will be important for the next section; in Section 3, we first state our oddness theorem (Theorem 3.1, Section 3.1), then we provide several examples of applications, including in particular the ones mentioned before (Section 3.2); in (Appendix), we provide first the necessary reminders about real algebraic geometry (Appendix A.1) and about specific notions of general topology (Appendix A.2), we provide the proofs of Theorem 2.1 (Appendix A.3), of Corollary 2.1 (Appendix A.4) and of Theorem 3.1 (Appendix A.5). We also provide sketches of proofs at the beginning of Appendix A.3 and of Appendix A.5 to help the reader. Finally, we provide a table of the main notations which are used in this paper (Appendix A.6).

[3] As a matter of fact, Herings and Zhan's paper also treats the problem of computation of pairwise stable networks, a problem that we do not consider here.

[4] However, several important ingredients of our proof comes from the proof of Predtetchinski's structure theorem (Predtetchinski, 2009).

2. Structure of the graph of pairwise stable networks

First of all, let us recall some mathematical definitions or notations used in this paper. For every sets X and Y, $\mathcal{F}(X, Y)$ denotes the set of mappings from X to Y. For every finite set X, $\text{card}(X)$ (or sometimes $|X|$) denotes the cardinal of X. A correspondence Ψ from a set X to another set Y is a mapping from X to the set of all subsets of Y, and we denote it by $\Psi : X \twoheadrightarrow Y$. A mapping f from a topological space X to another topological space Y is proper if for every compact subspace K of Y, $f^{-1}(K)$ is compact in X. For every topological spaces X and Y, a homotopy from a continuous mapping $f : X \to Y$ to another continuous mapping $g : X \to Y$ is a continuous mapping $H : [0, 1] \times X \to Y$ such that $H(0, \cdot) = f$ and $H(1, \cdot) = g$. We say that: (i) f and g are homotopic if there exists a homotopy H from f to g; (ii) f and g are properly homotopic if there exists a proper homotopy H from f to g.

2.1. The graph of pairwise stable networks

Throughout this paper, we fix some integer $n \geq 2$, and we define $N = [\![1, n]\!]$, called the *set of agents*. The set $L = \{\{i, j\} : (i, j) \in N^2, i \neq j\}$ denotes the *set of links on* N and $\mathbb{G} = [0, 1]^L$ the *set of (weighted) networks on* N. We now define a *society*, which is a n-tuple of payoff functions (one for each agent) defined on the set of all possible networks on N.

Definition 2.1. A *society* is an element $v = (v_1, \ldots, v_n) \in \mathcal{F}(\mathbb{G}, \mathbb{R})^n$, where each mapping v_i is called the *payoff function of agent* $i \in N$.

Notations. Every link $\{i, j\} \in L$ is denoted ij. For every network g and every $ij \in L$, the element $g(\{i, j\})$ is simply denoted g_{ij}. For every $ij \in L$, we denote $L_{-ij} = L - ij$ (i.e. $L_{-ij} = \{kl \in L : kl \neq ij\}$) and $\mathbb{G}_{-ij} = [0, 1]^{L_{-ij}}$. For every $ij \in L$, every $g_{-ij} = (g_{kl})_{kl \neq ij} \in \mathbb{G}_{-ij}$ and every $w \in [0, 1]$, $g' = (w, g_{-ij}) \in \mathbb{G}$ is the network defined by $g'_{kl} = g_{kl}$ for every $kl \neq ij$ and $g'_{ij} = w$. For every network $g = (g_{ij})_{ij \in L} \in \mathbb{G}$ and every $ij \in L$, $g_{-ij} := (g_{kl})_{kl \neq ij} \in \mathbb{G}_{-ij}$.

We now recall the seminal concept of *pairwise stability*, introduced by Jackson and Wolinsky (1996) for unweighted societies,[5] and extended by Bich and Morhaim to weighted societies (see Bich and Morhaim (2017, 2020)).

Definition 2.2. A network $g \in \mathbb{G}$ is *pairwise stable* with respect to a society $v = (v_1, \ldots, v_n)$ if for every $ij \in L$, the two following conditions hold:

1. For every $w \in [0, g_{ij})$, $v_i(w, g_{-ij}) \leq v_i(g)$ and $v_j(w, g_{-ij}) \leq v_j(g)$.
2. For every $w \in (g_{ij}, 1]$, $v_i(w, g_{-ij}) \leq v_i(g)$ or $v_j(w, g_{-ij}) \leq v_j(g)$.

The payoff functions considered in this paper have to satisfy some differentiability and some concavity assumption. More

[5] The *set of unweighted networks on* N is defined by $\mathbb{G}_u = \{0, 1\}^L$, and an *unweighted society* is a n-tuple $v = (v_1, \ldots, v_n) \in \mathcal{F}(\mathbb{G}_u, \mathbb{R})^n$.

precisely, for every $i \in N$, let

$$\mathcal{F}_i = \Big\{ v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \forall g_{-ij} \in \mathbb{G}_{-ij},\ g_{ij} \mapsto v_i(g_{ij}, g_{-ij}) \text{ is concave and } \partial_{ij} v_i \text{ is } \mathcal{C}^0 \text{ on } \mathbb{G} \Big\},$$

where $\partial_{ij} v_i$ denotes the function defined by

$$\partial_{ij} v_i : g \in \mathbb{G} \mapsto \frac{\partial v_i}{\partial g_{ij}}(g_{ij}, g_{-ij}),$$

i.e. $\partial_{ij} v_i$ assigns to each network $g \in \mathbb{G}$ the first-order derivative of v_i at g with respect to the ijth variable. For every $i \in N$, the set $\mathcal{F}_i$ is endowed with the norm[6]

$$\| \cdot \|_i : v_i \in \mathcal{F}_i \mapsto \max\{\max\{\|v_i\|_\infty, \|\partial_{ij} v_i\|_\infty\} : j \neq i\}.$$

The set $\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$ is endowed with the product norm $v = (v_1, \ldots, v_n) \in \mathcal{F} \mapsto \max\{\|v_i\|_i : i = 1, \ldots, n\}$.

Definition 2.3. The pairwise stable networks correspondence (with respect to $\mathcal{F}$), or PSN correspondence (with respect to $\mathcal{F}$), is the correspondence

$$\begin{cases} \Psi : & \mathcal{F} \rightrightarrows \mathbb{G} \\ & v \mapsto \{g \in \mathbb{G} : g \text{ is pairwise stable with respect to } v\} \end{cases}$$

and the graph of the PSN correspondence, called the *graph of pairwise stable networks*, is denoted $\mathcal{P}$, i.e.,

$$\mathcal{P} = \mathrm{Gr}(\Psi) = \{(v, g) \in \mathcal{F} \times \mathbb{G} : g \text{ is pairwise stable with respect to } v\}.$$

We endow $\mathcal{P}$ with the following product norm:

$$(v, g) \in \mathcal{P} \mapsto \max\{\max\{\|v_i\|_i : i = 1, \ldots, n\}, \|g\|\},$$

where $\|g\|$ denotes the Euclidean norm of g,[7] and we denote the projection from $\mathcal{P}$ to $\mathcal{F}$ by π, i.e. $\pi(v, g) = v$, for every $(v, g) \in \mathcal{P}$.

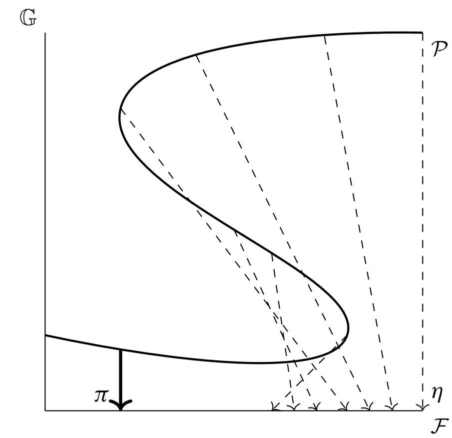
2.2. Structure theorem

In this section, we characterize the topological structure of the graph of pairwise stable networks, which corresponds to our first main result.

Theorem 2.1 (*Structure Theorem*)**.** *The projection mapping $\pi : \mathcal{P} \to \mathcal{F}$ is properly homotopic to some homeomorphism $\eta : \mathcal{P} \to \mathcal{F}$.*

This theorem gives two important informations: (1) thanks to η, the graph $\mathcal{P}$ of pairwise stable networks is homeomorphic to $\mathcal{F}$, which corresponds to the intuition that $\mathcal{P}$ can be continuously deformed into the very simple space $\mathcal{F}$; (2) the homeomorphism η itself can be continuously deformed into a very simple mapping from $\mathcal{P}$ to $\mathcal{F}$, the projection π. This result will be a key ingredient in our second main result (Theorem 3.1): indeed, this theorem relies partly on two important properties of topological degree (see Appendix A.2.2), homotopy invariance and homeomorphism properties, which can be derived from Theorem 2.1. Last, the fact that the homotopy is proper means intuitively that it maps points "close to infinity" to points "close to infinity", and it will play an important role in the proof of Theorem 3.1.

Here is a (simple) representation of the graph $\mathcal{P} \subset \mathcal{F} \times \mathbb{G}$ of pairwise stable networks, of the projection mapping π (in thick line) and of the homeomorphism η (in dashed line):



The proof of Theorem 2.1 is given in Appendix A.3. For a similar result applied to mixed Nash equilibria of strategic-form games, see Kohlberg–Mertens' structure theorem (Kohlberg and Mertens, 1986), and also a recent generalization due to Predtetchinski (2009).[8]

Now, for every $i \in N$, let

$$\mathcal{A}_i = \{g \in \mathbb{G} \mapsto \sum_{j \neq i} \alpha_{ij} g_{ij} + c : (\alpha_{ij})_{j \neq i} \in \mathbb{R}^{n-1}, c \in \mathbb{R}\},$$

that is to say, $a_i \in \mathcal{A}_i$ if a_i is affine in $(g_{ij})_{j \neq i}$ and only depends on those weights. In particular, $\mathcal{A}_i \subset \mathcal{F}_i$.

Moreover, for every subset $\mathcal{U}$ of $\mathcal{F}$, we define

$$\mathcal{P}_{\mathcal{U}} = \{(v, g) \in \mathcal{P} : v \in \mathcal{U}\},$$

and we consider the restriction $\pi_{|\mathcal{P}_{\mathcal{U}}} : \mathcal{P}_{\mathcal{U}} \to \mathcal{U}$ of π to $\mathcal{P}_{\mathcal{U}}$ and the restriction $\eta_{|\mathcal{P}_{\mathcal{U}}} : \mathcal{P}_{\mathcal{U}} \to \mathcal{U}$ of η to $\mathcal{P}_{\mathcal{U}}$ (η being the mapping defined in Theorem 2.1).

The following corollary states that Theorem 2.1 remains true if we replace $\mathcal{F}$ by a subset of $\mathcal{F}$ satisfying some additional stability assumption:

Corollary 2.1. *For every $i \in N$, consider $\mathcal{U}_i \subset \mathcal{F}_i$ such that for every $v_i \in \mathcal{U}_i$, and every $a_i \in \mathcal{A}_i$, $v_i + a_i \in \mathcal{U}_i$, and let $\mathcal{U} = \prod_{i \in N} \mathcal{U}_i$. Then, $\eta_{|\mathcal{P}_{\mathcal{U}}} : \mathcal{P}_{\mathcal{U}} \to \mathcal{U}$ is a homeomorphism which is properly homotopic to $\pi_{|\mathcal{P}_{\mathcal{U}}} : \mathcal{P}_{\mathcal{U}} \to \mathcal{U}$.*

The proof is given in Appendix A.4. Corollary 2.1 is very important for applications, since our main oddness theorem (Theorem 3.1) is obtained by an application of this result to particular "nice" subclasses of payoff functions.

3. Generic oddness of the graph of pairwise stable networks

3.1. Oddness theorem

In this section, we prove that for some large classes of polynomial payoff functions (which we call *regular*), there exist generically an odd number of pairwise stable networks.

From now on, for every agent $i \in N$, we consider a fixed integer $\delta_i \in \mathbb{N}$, and we consider the vector space $\mathbb{R}_{\delta_i}[g]$ of

[6] This norm is defined on the vector space $\{v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \partial_{ij} v_i \text{ is } \mathcal{C}^0 \text{ on } \mathbb{G}\}$.

[7] For every $g \in \mathbb{G}$, the Euclidean norm $\|g\|$ of g is defined by $\|g\| = \sqrt{\sum_{ij \in L} g_{ij}^2}$.

[8] We can also mention the interesting works of Demichelis and Germano which provide sharper results on the topological structure of the graph of Nash equilibria (Demichelis and Germano, 2000a, 2002), and similarly for the graph of Walrasian equilibria (Demichelis and Germano, 2000b).

polynomial functions[9] whose degree is less or equal to δ_i. For every $i \in N$, $\mathbb{R}_{\delta_i}[g]$ is endowed with the topology induced by the norm[10] defined in Section 2.1. In particular, any element $v_i \in \mathbb{R}_{\delta_i}[g]$ can be expressed as

$$v_i(g) = \sum_{k \in \mathbb{N}^L} \alpha^i_k g^k$$

where $g \in \mathbb{G}$, $k = (k_{ij})_{ij \in L} \in \mathbb{N}^L$ is a multi-index (with $\alpha^i_k = 0$ if $\deg(k) := \sum_{ij \in L} k_{ij} > \delta_i$) and where $g^k := \prod_{ij \in L} g_{ij}^{k_{ij}}$ is called a monomial. Recall that for every such monomial $g^k \neq 0$, the degree of g^k is by definition equal to $\deg(k)$, and that the degree of v_i is equal to $\max\{\deg(k) : \alpha^i_k \neq 0\}$ (which is equal, by convention, to $-\infty$ when $v_i = 0$).

For example, if we consider three agents and if $v_1(g_{12}, g_{13}, g_{23}) = -g_{12}^2 g_{23} + 3 g_{12} g_{13}^5 g_{23}^4 - g_{13}^2 g_{23}$, then we can write $v_1(g) = \alpha^1_{k_1} g^{k_1} + \alpha^1_{k_2} g^{k_2} + \alpha^1_{k_3} g^{k_3}$ with $k_1 = (2, 0, 1)$, $k_2 = (1, 5, 4)$ and $k_3 = (0, 2, 1)$ and with $\alpha^1_{k_1} = -1$, $\alpha^1_{k_2} = 3$ and $\alpha^1_{k_3} = -1$ (in particular, $\deg(v_i) = 10$).

The mapping which associates to every payoff function $v_i \in \mathbb{R}_{\delta_i}[g]$ its coefficients (with respect to some predefined order on $\mathbb{N}^L$) is an isomorphism from $\mathbb{R}_{\delta_i}[g]$ to $\mathbb{R}^{m_i}$, for some integer m_i. In the following, we define $m = \sum_{i \in N} m_i$ and $\varphi = \times_{i \in N} \varphi_i : \prod_{i \in N} \mathbb{R}_{\delta_i}[g] \to \mathbb{R}^m$.

Moreover, for every agent $i \in N$ and every subset $\mathcal{U}_i$ of $\mathbb{R}_{\delta_i}[g]$, we define $\mathcal{S}_i = \varphi_i(\mathcal{U}_i)$, and $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$. Also, in order to avoid unnecessary complexity, both the restriction of φ_i from $\mathcal{U}_i$ to $\mathbb{R}^{m_i}$ and the restriction of φ_i from $\mathcal{U}_i$ to $\mathcal{S}_i$ will be denoted φ_i, by abuse of notation (however, notice that $\varphi_i : \mathcal{U}_i \to \mathcal{S}_i$ is a homeomorphism).

We now introduce the notion of *regular set* (recall that for every $i \in N$, $\mathcal{A}_i$ is the set of payoff functions of agent i which are affine in $(g_{ij})_{j \neq i}$ and only depend on those weights):

Definition 3.1. For every $i \in N$, let $\mathcal{U}_i$ be a subset of $\mathbb{R}_{\delta_i}[g]$. The set $\mathcal{U} := \prod_{i \in N} \mathcal{U}_i$ is said to be *regular* if for every $i \in N$, the two following conditions are fulfilled:

1. **(Stability assumption).** For every $v_i \in \mathcal{U}_i$ and every $a_i \in \mathcal{A}_i$, $v_i + a_i \in \mathcal{U}_i$.
2. **(Semi-algebraicity assumption).** $\mathcal{S}_i = \varphi_i(\mathcal{U}_i)$ is a semi-algebraic set.

Stability assumption simply means that adding an affine mapping in $\mathcal{A}_i$ to any element in $\mathcal{U}_i$ gives an element in $\mathcal{U}_i$. Semi-algebraicity assumption means that the set of coefficients of polynomial payoff functions in $\mathcal{U}_i$ can be defined by using a finite number of polynomial equalities or inequalities (see Appendix A.1 for some reminders about real algebraic geometry).

Before introducing our second main theorem, we quickly recall some details about semi-algebraic sets (again, see Appendix A.1 for more details). First, it can be proved that every semi-algebraic subset S of $\mathbb{R}^m$ can be written $S = \bigcup_{i=0}^k S_i$, where each S_i is semi-algebraically homeomorphic to $]0, 1[^{d_i}$ for some integers d_i ($i = 0, \dots, k$). The dimension of S is then defined as $\max\{d_0, d_1, \dots, d_k\}$ and is denoted $\dim(S)$. Actually, it can be proved that this notion of "dimension" corresponds to the standard one in the particular case where S is an affine subspace of $\mathbb{R}^m$. Second, we say that a semi-algebraic subset G of a semi-algebraic set S is generic (in S) if G is open in S and if $\dim(S - G) < \dim(S)$. The intuition behind this definition is that a generic subset G in S fills "completely" S.

Now, we introduce the second main result of this paper. Briefly, this theorem states that "most" societies in network formation theory should have an odd number of pairwise stable networks, when the payoff functions are polynomial, with the additional "standard" concavity assumption.

Theorem 3.1 (*Oddness Theorem*). *Consider a regular set $\mathcal{U} = \prod_{i \in N} \mathcal{U}_i$ such that $\mathcal{U} \subset \mathcal{F}$. Then, there exists a generic semi-algebraic subset G of $\mathcal{S}$ such that for every $x \in G$, the society $v^x := \varphi^{-1}(x)$ has an odd number of pairwise stable networks.*

The proof is given in Appendix A.5.

3.2. Some applications of Theorem 3.1

3.2.1. Polynomial payoff functions concave with respect to agents' weights

For every $i \in N$, consider the particular case where $\mathcal{U}_i = \mathbb{R}_{\delta_i}[g] \cap \mathcal{F}_i$ is the set of polynomial functions whose degree is less or equal to $\delta_i \in \mathbb{N}$ and which are concave in g_{ij} on $[0, 1]$, for every $j \neq i$, and define $\mathcal{U} = \prod_{i \in N} \mathcal{U}_i$. The following corollary of Theorem 3.1 states that societies in $\mathcal{U}$ have generically an odd number of pairwise stable networks.

Corollary 3.1. *The set $\mathcal{U} \subset \mathcal{F}$ is regular, that is, it satisfies stability assumption and semi-algebraicity assumption. Thus, there exists a generic semi-algebraic subset G of $\mathcal{S} = \varphi(\mathcal{U})$ such that for every $x \in G$, the society $v^x = \varphi^{-1}(x)$ has an odd number of pairwise stable networks.*

Proof. We show that $\mathcal{U}$ is a regular set:

1. Stability assumption is satisfied, since the sum of a concave function and an affine function is concave.
2. Let us prove that semi-algebraicity assumption is also satisfied. For every $i \in N$ and every $g \in \mathbb{G}$, any payoff function $v_i \in \mathcal{U}_i$ can be written $v_i(g) = \sum_{k \in \mathbb{N}^L} \alpha^i_k g^k$ (by definition of $\mathbb{R}_{\delta_i}[g]$). Then, we can consider the polynomial function

$$P_i : ((\alpha^i_k)_{k \in \mathbb{N}^L}, g) \in \mathbb{R}^{m_i} \times \mathbb{G} \mapsto \sum_{k \in \mathbb{N}^L} \alpha^i_k g^k,$$

so that we finally get

$$\mathcal{S}_i = \varphi(\mathcal{U}_i) = \{\alpha^i = (\alpha^i_k)_{k \in \mathbb{N}^L} \in \mathbb{R}^{m_i} : \forall g \in \mathbb{G}, \\ \forall j \neq i, \frac{\partial^2 P_i(\alpha^i, g_{ij}, g_{-ij})}{\partial g_{ij}^2} \leq 0\}.$$

This set is semi-algebraic, from $\mathbb{G}$ being semi-algebraic and from Proposition A.1 (see Appendix A.1).

Finally, from Theorem 3.1, we obtain directly the existence of a generic semi-algebraic subset G of $\mathcal{S}$ such that for every $x \in G$, v^x has an odd number of pairwise stable networks. □

3.2.2. Network formation models with linear costs

For every $i \in N$, let $\bar{v}_i$ be a polynomial function in $\mathbb{R}_{\delta_i}[g] \cap \mathcal{F}_i$ (i.e. $\bar{v}_i$ is concave in g_{ij} for each $j \neq i$), and let $L' = \{(i,j) \in N^2 : i \neq j\}$ denote the *set of directed links on* N. Let $c = (c_{i,j})_{(i,j) \in L'} \in \mathbb{R}^{L'}$, and consider the society parameterized by c, $v^c = (v^c_i)_{i \in N}$ where for every $i \in N$, $v^c_i(g) := \bar{v}_i(g) - \sum_{j \neq i} c_{i,j} g_{ij}$ for every $g \in \mathbb{G}$. Here, $c_{i,j} \in \mathbb{R}$ can be interpreted as a marginal cost of maintaining the weight g_{ij} of link ij (at least when $c_{i,j} \geq 0$). The following corollary states that societies with such payoff functions have generically an odd number of pairwise stable networks.

Corollary 3.2. *There exists a generic semi-algebraic subset $\mathcal{G}$ of $\mathbb{R}^{L'}$ such that for every $c \in \mathcal{G}$, v^c has an odd number of pairwise stable networks.*

[9] Here, a mapping $f : \mathbb{G} \to \mathbb{R}$ is said to be polynomial if f is a mapping polynomial in the weights g_{ij} of g, where $ij \in L$.

[10] Actually, from equivalence of norms in finite-dimensional spaces, any norm would give the same topology.

Proof. To apply Theorem 3.1, we first add constant terms to the payoff functions, to ensure that stability assumption is satisfied: for every $c = (c_{i,j})_{(i,j)\in L'} \in \mathbb{R}^{L'}$ and every $k = (k_1, \dots, k_n) \in \mathbb{R}^n$, consider the society

$$v^{c,k} := (g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j\neq i} c_{i,j} g_{ij} + k_i)_{i\in N},$$

and let $\mathcal{U} = \{v^{c,k} : (c, k) \in \mathbb{R}^{L'} \times \mathbb{R}^n\}$. Stability assumption is now clearly satisfied, and to prove that $\mathcal{U}$ is a regular set, we only have to prove that semi-algebraicity assumption is satisfied: this is the case since for every $i \in N$, the set $\mathcal{S}_i$ of coefficients of polynomial payoff functions in $\mathcal{U}_i$ (the ith factor of $\mathcal{U}$) is a finite product of copies of $\mathbb{R}$ (due to the coefficients $c_{i,j}$ and k_i) and of singletons (corresponding to the coefficients in $\bar{v}_i$ associated to the non constant monomials other than $g_{ij}, j \neq i$).

Thus, from Theorem 3.1, there exists a generic semi-algebraic subset G of $\mathcal{S}$ such that the society $\varphi^{-1}(x)$ has an odd number of pairwise stable networks for every $x \in G$. Now, since the mapping $f : (c, k) \mapsto \varphi(v^{c,k})$ from $\mathbb{R}^{L'} \times \mathbb{R}^n$ to $\mathcal{S}$ is a semi-algebraic homeomorphism, we get that for every $(c, k) \in \mathcal{G} := f^{-1}(G)$ (which is a generic subset of $\mathbb{R}^{L'} \times \mathbb{R}^n$), the society $v^{c,k}$ has an odd number of pairwise stable networks. Then, denoting $\Pi : \mathbb{R}^{L'} \times \mathbb{R}^n \to \mathbb{R}^{L'}$ the projection on the first factor, this is straightforward to see that for every $c \in \Pi(\mathcal{G})$ (which is a generic[11] subset of $\mathbb{R}^{L'}$), the society v^c has an odd number of pairwise stable networks. Indeed, the sets of pairwise stable networks of v^c and of $v^{c,k}$ are identical for every k. □

3.2.3. Network formation models with quadratic costs

Consider the same example as above, but where the payoff functions are parameterized by $c = (c_{i,j})_{(i,j)\in L'} \in (0, +\infty)^{L'}$ and $d = (d_{i,j})_{(i,j)\in L'} \in \mathbb{R}^{L'}$, where $L' = \{(i, j) \in N^2 : i \neq j\}$ denotes the set of directed links on N, as follows: $v_i^{c,d}(g) := \bar{v}_i(g) - \sum_{j\neq i} c_{i,j} g_{ij}^2 - \sum_{j\neq i} d_{i,j} g_{ij}$ for every $i \in N$ and for every $g \in \mathbb{G}$. Here, the cost of maintaining the weight g_{ij} of link ij is assumed to be quadratic. Again, we get that societies with such payoff functions have generically an odd number of pairwise stable networks.

Corollary 3.3. *There exists a generic semi-algebraic subset $\mathcal{G}$ of $(0, +\infty)^{L'} \times \mathbb{R}^{L'}$ such that for every $(c, d) \in \mathcal{G}$, $v^{c,d}$ has an odd number of pairwise stable networks.*

Proof. It is easy to check that $\mathcal{U} = \prod_{i\in N} \mathcal{U}_i$ is a regular[12] set, where $\mathcal{U}_i = \{g \in \mathbb{G} \mapsto \bar{v}_i(g) - \sum_{j\neq i} c_{i,j} g_{ij}^2 - \sum_{j\neq i} d_{i,j} g_{ij} : \forall j \neq i, c_{i,j} \in (0, +\infty), d_{i,j} \in \mathbb{R}\}$. □

3.2.4. Bramoullé–Kranton's public good provision model

We are interested by the following modification of a model introduced by Bramoullé and Kranton (2007). Consider n agents, each of them being characterized by some level of effort $e_i \in [0, +\infty)$ (it could be the amount of time a consumer spends researching a new product). The marginal cost of effort of each individual is $z \in \mathbb{R}$. Agents interact in some network g, and g_{ij} measures the benefit for agent i (resp. j) of agent's j (resp. i) efforts. For every $c = (c_{i,j})_{(i,j)\in L'} \in \mathbb{R}^{L'}$, consider a society $v^c = (v_i^c)_{i\in N}$ parameterized by c, and defined in the following way: for every $i \in N$, $v_i^c(g) := b(e_i + \sum_{j\neq i} g_{ij} e_j) - z e_i - \sum_{j\neq i} c_{i,j} g_{ij}$, where $b : \mathbb{R} \to \mathbb{R}$ is a concave polynomial benefit function and for every $j \neq i$, $c_{i,j} \in \mathbb{R}$ is the marginal cost of maintaining the weight g_{ij} of the link ij (at least when $c_{i,j} \geq 0$).

In their paper, Bramoullé and Kranton are interested by the optimal levels of efforts e_i $(i = 1, \dots, n)$, which are here exogenous, and they consider exogenous values of the weights $g_{ij} \in \{0, 1\}$ $(ij \in L)$, which here are endogenous and belong to $[0, 1]$.

Corollary 3.4. *There exists a generic semi-algebraic subset $\mathcal{G}$ of $\mathbb{R}^{L'}$ such that for every $c \in \mathcal{G}$, the society v^c has an odd number of pairwise stable networks.*

Proof. The proof is similar to the proof of Corollary 3.2, remarking that for every $i \in N$, $g \in \mathbb{G} \mapsto b(e_i + \sum_{j\neq i} g_{ij} e_j) - z e_i$ is a polynomial function in $\mathcal{F}_i$ (i.e. concave in g_{ij} for each $j \neq i$). □

3.2.5. Calvó–Armengol–İlkılıç's information transmission model

We now apply Theorem 3.1 to a weighted version of an information transmission model due to Calvó-Armengol and İlkılıç (Calvó-Armengol and İlkılıç, 2009, see also Bich and Morhaim, 2020). We consider n agents: if agent i and agent j are in a full relationship ($g_{ij} = 1$), some information can be transmitted between the two agents with some probability p_{ij}. We assume that if the relationship is weighted ($g_{ij} \in [0, 1]$), the probability of transmission is $p_{ij} g_{ij}$. The payoff function of agent $i \in N$ is defined by

$$v_i^c(g) = 1 - \prod_{j\neq i}(1 - p_{ij} g_{ij}) - \sum_{j\neq i} c_{i,j} g_{ij},$$

for every $c = (c_{i,j})_{(i,j)\in L'} \in \mathbb{R}^{L'}$ and for every weighted network g on N. The first term corresponds to the probability that the message is transmitted to agent i, and the second term to the cost of maintaining his links. Denoting $v^c = (v_i^c)_{i\in N}$, another application of the linear cost example treated above is the following corollary:

Corollary 3.5. *There exists a generic semi-algebraic subset $\mathcal{G}$ of $\mathbb{R}^{L'}$ such that for every $c \in \mathcal{G}$, the society v^c has an odd number of pairwise stable networks.*

3.2.6. Bala–Goyal's two-way flow model

We now adapt a model of Bala and Goyal (2000) to weighted networks: we consider n agents, and for every agent i and agent $j \neq i$, a path from i to j is a finite sequence $x_0 = i, x_1, \dots, x_k = j$ of distinct elements of N. Let $\mathbf{P}_{i\to j}$ be the (finite) set of all paths from i to j, and $n_i(g)$ be the sum on all paths in $\mathbf{P}_{i\to j}$ $(j \neq i)$ of the product of weights along these paths. We can interpret $n_i(g)$ as the benefit that agent i receives from his links. For every agent $i \in N$, define $v^c = (v_i^c)_{i\in N}$ and

$$v_i^c(g) = n_i(g) - \sum_{j\neq i} c_{i,j} g_{ij},$$

where $g \in \mathbb{G}$ and where $c_{i,j} \in \mathbb{R}$ can be interpreted as the marginal cost of maintaining the weight g_{ij} of the link ij between agent i and agent j (at least when $c_{i,j} \geq 0$). Denoting $c = (c_{i,j})_{(i,j)\in L'} \in \mathbb{R}^{L'}$, we get from the linear cost example:

Corollary 3.6. *There exists a generic semi-algebraic subset $\mathcal{G}$ of $\mathbb{R}^{L'}$ such that for every $c \in \mathcal{G}$, the society v^c has an odd number of pairwise stable networks.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

[11] This set is semi-algebraic from Tarski–Seidenberg's theorem (see Theorem A.1 in Appendix A.1).

[12] Actually, we have to add a constant part to payoff functions to ensure stability assumption. However, we have seen in the previous example that considering a constant part or not is innocuous, so we now skip this difficulty for simplification.

Appendix

A.1. Reminders about real algebraic geometry

Let us first recall the definition of a semi-algebraic subset of $\mathbb{R}^m$ and of a semi-algebraic mapping (see Bochnak et al. (1998)).

Definition A.1. A semi-algebraic subset of $\mathbb{R}^m$ is a set of the form

$$\bigcup_{i=1}^{s}\bigcap_{j=1}^{r_i}\{x \in \mathbb{R}^m : f_{i,j}(x) \star_{i,j} 0\},$$

where $\star_{i,j}$ denotes either $<$ or $=$ and $f_{i,j} \in \mathbb{R}[X_1, \ldots, X_m]$, for every $i = 1, \ldots, s$ and every $j = 1, \ldots, r_i$.

Thus, by definition, semi-algebraic sets are closed under finite unions and finite intersections. Actually, these sets are also closed under projections:

Theorem A.1 (*Tarski–Seidenberg*)**.** *If S is a semi-algebraic subset of $\mathbb{R}^{m+p}$ and Π is the canonical projection from $\mathbb{R}^{m+p}$ to $\mathbb{R}^m$, then $\Pi(S)$ is a semi-algebraic subset of $\mathbb{R}^m$.*

From this result, we can also define semi-algebraic sets using quantified variables (or m-tuples of variables) which range in $\mathbb{R}$, $\mathbb{R}^m$ or more generally in a semi-algebraic set (for example, see Bochnak et al. (1998), p. 28).

Definition A.2. A first-order formula of the language of ordered fields with parameters in $\mathbb{R}$ is a formula written with a finite number of conjunctions, disjunctions, negations, and universal or existential quantifiers on variables in semi-algebraic sets, starting from atomic formulas which are formulas of the kind $f(x_1, \ldots, x_m) = 0$ or $g(x_1, \ldots, x_m) > 0$, where f and g are polynomials with coefficients in $\mathbb{R}$.

Proposition A.1. *Let $\Phi(x_1, \ldots, x_m)$ be a first-order formula of the language of ordered fields with parameters in $\mathbb{R}$. Then, $\{x \in \mathbb{R}^m : \Phi(x)\}$ is a semi-algebraic set.*

For example, if f is a polynomial with four variables and if $\mathbb{S}^1$ denotes the unit circle of $\mathbb{R}^2$, then the set $\{(x, y) \in \mathbb{R}^2 : \forall(z, t) \in \mathbb{S}^1, f(x, y, z, t) \geq 0\}$ is semi-algebraic (as $\mathbb{S}^1$ is itself a semi-algebraic set).

Definition A.3. Let S be a semi-algebraic subset of $\mathbb{R}^m$ and T be a semi-algebraic subset of $\mathbb{R}^p$. A mapping $f : S \to T$ is semi-algebraic if its graph

$$\mathrm{Gr}(f) = \{(x, f(x)) : x \in S\}$$

is a semi-algebraic subset of $\mathbb{R}^m \times \mathbb{R}^p$.

The following proposition is a consequence of Tarski-Seidenberg's theorem:

Proposition A.2. *Let S be a semi-algebraic subset of $\mathbb{R}^m$ and $f : S \to \mathbb{R}^p$ be a semi-algebraic mapping. Then, $f(S)$ is a semi-algebraic subset of $\mathbb{R}^p$.*

Corollary A.1. *Let S_1 and S_2 be semi-algebraic subsets of $\mathbb{R}^m$. Then, $S_1 + S_2$ is a semi-algebraic set.*

A semi-algebraic homeomorphism is a homeomorphism which is semi-algebraic (in that case, f^{-1} is also semi-algebraic).

We now define the dimension of a semi-algebraic set (see Bochnak et al. (1998), Theorem 2.3.6., p. 33 and Corollary 2.8.9., p. 53).

Definition A.4. For every semi-algebraic subset S of $\mathbb{R}^m$, there exists an increasing sequence of non-negative integers $d_0 \leq d_1 \leq \cdots \leq d_k$ such that

$$S = \bigcup_{i=0}^{k} S_i,$$

the union being disjoint, where S_i is semi-algebraically homeomorphic to $]0, 1[^{d_i}$ for every $i = 0, \ldots, k$ (where, by convention, $]0, 1[^0$ is a point). The dimension of S is defined by

$$\dim(S) := \max\{d_0, d_1, \ldots, d_k\}$$

(and does not depend on the decomposition of S).

Then, we have the following properties:

Proposition A.3. *Let S, S_1 and S_2 be semi-algebraic subsets of $\mathbb{R}^m$:*

1. *If $f : S \to \mathbb{R}^p$ is semi-algebraic and is a bijection from S to $f(S)$, then $\dim(S) = \dim(f(S))$.*
2. *If $S_1 \subset S_2$, then $\dim(S_1) \leq \dim(S_2)$.*
3. *$\dim(S_1 \times S_2) = \dim(S_1) + \dim(S_2)$.*

Finally, we recall the notion of generic semi-algebraic set, together with a theorem which plays an important role in the proof of our oddness theorem (Theorem 3.1):

Definition A.5. Let S be a semi-algebraic set and G be a semi-algebraic subset of S. We say that G is a generic semi-algebraic subset of S if G is open in S and if $\dim(S - G) < \dim(S)$.

Theorem A.2. *Let S and T be two semi-algebraic sets such that $\dim(S) = \dim(T)$ and $f : S \to T$ be a surjective continuous semi-algebraic mapping. Then, there exists a generic semi-algebraic subset G of T such that for every $t \in G$:*

- *$f^{-1}(t)$ is a (nonempty) finite set;*
- *there exists an open neighborhood V_t of t such that $f^{-1}(V_t)$ is a finite union of pairwise disjoint open sets $(V_t^k)_{k \in \mathcal{K}}$ (where $\mathcal{K}$ is a finite set), and such that for every $k \in \mathcal{K}$, $f_{|_{V_t^k}}$ is a homeomorphism between V_t^k and V_t.*

Proof. To prove this theorem, we use the following theorem (see Bochnak et al. (1998), p. 224):

Theorem A.3. *Let S and T be two semi-algebraic sets and $f : S \to T$ be a continuous semi-algebraic mapping. Then, there exists a generic semi-algebraic subset G of T such that f has a semi-algebraic trivialization over each semi-algebraically connected component C of G.*[13,14]

Now, let us consider $t \in G$ and the connected component C of G which contains t. We apply the theorem above to C: thus, f has semi-algebraic trivialization θ^C with fiber $\mathcal{K}^C$ over C. In particular, $C \times \mathcal{K}^C$ and $f^{-1}(C)$ are semi-algebraically homeomorphic, thus we obtain that

$$\dim(C \times \mathcal{K}^C) = \dim(C) + \dim(\mathcal{K}^C) = \dim(f^{-1}(C))$$

with $\dim(f^{-1}(C)) \leq \dim(S)$ and $\dim(C) = \dim(T)$, since $f^{-1}(C) \subset S$ and since G is open in T (thus C is also open in T). Now,

13 Recall that a semi-algebraic trivialization of f with fiber $\mathcal{K}$ (a semi-algebraic set) over a semi-algebraically connected component C of G is a semi-algebraic homeomorphism $\theta : C \times \mathcal{K} \to f^{-1}(C)$ such that $f(\theta(c, k)) = c$ for every $(c, k) \in C \times \mathcal{K}$. Remark that if $f^{-1}(C) = \emptyset$, then this condition is automatically verified by taking the empty mapping $\theta = \emptyset$.

14 Let us recall that in $\mathbb{R}^m$, a semi-algebraic set is semi-algebraically connected if and only if it is connected (see Bochnak et al. (1998), Theorem 2.4.5., p. 35).

$f^{-1}(t)$ is also semi-algebraically homeomorphic to $\mathcal{K}^C$ (indeed, $\mathcal{K}^C$ is semi-algebraically homeomorphic to $\{t\} \times \mathcal{K}^C$ and $\theta^C_{|\{t\}\times\mathcal{K}^C}$ is a semi-algebraic homeomorphism from $\{t\} \times \mathcal{K}^C$ to $f^{-1}(t)$). In particular, $\dim(f^{-1}(t)) = \dim(\mathcal{K}^C)$, thus from the above equality, we obtain that

$$\dim(f^{-1}(t)) = \dim(f^{-1}(C)) - \dim(C) \leq \dim(S) - \dim(T).$$

However, by assumption, $\dim(S) = \dim(T)$, thus we obtain that $\dim(f^{-1}(t)) = 0$. Finally, as $f^{-1}(t)$ is a 0-dimensional semi-algebraic set, it is a finite nonempty set (see Bochnak et al. (1998), Theorem 2.3.6.) with the same cardinal as $\mathcal{K}^C$ (nonemptiness of $f^{-1}(t)$ comes from the fact that f is a surjective mapping).

Then, defining V_t as any open neighborhood of t included in C, and $V_t^k = \theta^C(V_t \times \{k\})$ for every $k \in \mathcal{K} := \mathcal{K}^C$, we easily get that:

1. The sets V_t^k, $k \in \mathcal{K}$, are open (because $V_t \times \{k\}$ is open in $C \times \mathcal{K}$ and θ^C is a homeomorphism) and disjoint.
2. For every $k \in \mathcal{K}$, $f_{|V_t^k}$ is a homeomorphism between V_t^k and V_t, because $f_{|V_t^k} = \Pi \circ (\theta^C_{|V_t^k})^{-1}$ (where Π is the projection from $V_t \times \{k\}$ to V_t) and because both Π and $\theta^C_{|V_t^k}$ are homeomorphisms. □

A.2. Reminders about topology

A.2.1. Alexandroff one-point compactification of a topological space

First, we recall that every noncompact locally compact Hausdorff space X admits a (Alexandroff one-point) compactification X^*.

Definition A.6. Let (X, τ) be a noncompact locally compact Hausdorff topological space, and let $X^* = X \cup \{\infty\}$ (where $\infty \notin X$). The set

$$\tau^* = \tau \cup \{X^* - K : K \subset X \text{ is compact}\}$$

is a topology[15] on X^*, and the space (X^*, τ^*) is called a compactification of X.

Then, we have the following property:

Proposition A.4. *Let X and Y be two noncompact locally compact Hausdorff topological spaces and $f : X \to Y$ be a continuous proper mapping. Then, f can be (uniquely) extended to a continuous mapping $f^* : X^* \to Y^*$ for which $f(\infty) = \infty$.*

A.2.2. Topological degree of a continuous mapping from $\mathbb{S}^m$ to $\mathbb{S}^m$

In this subsection, we gather all the properties of topological degree which are used in this paper.

To every continuous mapping $f : \mathbb{S}^m \to \mathbb{S}^m$, one can associate an integer $\deg(f) \in \mathbb{Z}$, called the degree of f (or the topological degree of f, or the Brouwer degree of f), which satisfies the following properties:

(i) **(Degree of identity).** $\deg(\mathrm{id}_{\mathbb{S}^m}) = 1$ (see Dold, 1995, Proposition 4.2., (i), p. 62).
(ii) **(Homotopy invariance).** If $f : \mathbb{S}^m \to \mathbb{S}^m$ and $g : \mathbb{S}^m \to \mathbb{S}^m$ are homotopic, then $\deg(f) = \deg(g)$ (see Dold, 1995, Proposition 4.2., (iii), p. 62, or Hatcher, 2002, p. 134).
(iii) **(Surjectivity).** If $f : \mathbb{S}^m \to \mathbb{S}^m$ is such that $\deg(f) \neq 0$, then f is surjective (see Hatcher, 2002, p. 134).
(iv) **(Homeomorphism).** If $f : \mathbb{S}^m \to \mathbb{S}^m$ is a homeomorphism, then $\deg(f) \in \{-1, 1\}$ (see Hatcher, 2002, p. 135, in the proof of Proposition 2.29.).

To every continuous mapping $f : \mathbb{S}^m \to \mathbb{S}^m$ and every $y \in \mathbb{S}^m$, one can also associate an integer $\deg(f)_{|y} \in \mathbb{Z}$ called the local degree of f at y, which satisfies the following properties:

(v) **(Local homeomorphism).** If $f : \mathbb{S}^m \to \mathbb{S}^m$ is a homeomorphism from some neighborhood of y to some neighborhood of $f(y)$, then $\deg(f)_{|y} \in \{-1, 1\}$ (see Hatcher, 2002, p. 135–136, or Dold, 1995, Example 5.4., p. 67).
(vi) **(Additivity of degree modulo 2).** For every $z \in \mathbb{S}^m$, if $f : \mathbb{S}^m \to \mathbb{S}^m$ is such that $f^{-1}(z) = \{y_1, \ldots, y_k\}$, then

$$\deg(f) = \sum_{i=1}^{k} \deg(f)_{|y_i} \text{ modulo } 2$$

(see Hatcher, 2002, Proposition 2.30., p. 135–136, or Dold, 1995, Proposition 5.8., p. 68). In particular, from the above formula and from property (v), if for every $i = 1, \ldots, k$, f is a homeomorphism from some neighborhood of y_i to some neighborhood of $f(y_i)$ and if $\deg(f) \in \{-1, 1\}$, then k is odd.

A.3. Proof of Theorem 2.1

A.3.1. Sketch of the proof

To construct a homomorphism η from $\mathcal{P}$ to $\mathcal{F}$ (Step 1), we have to associate to a pair (v, g), where v is a profile of payoff functions in $\mathcal{F}$ and g is a pairwise stable network associated to it, another profile of payoff functions $\eta(v, g)$. Moreover, $\eta(v, g)$ has to contain all the information conveyed by v and g, since we want to be able to define an inverse mapping $\eta^{-1} : \mathcal{F} \to \mathcal{P}$ (Step 2–4).

The idea is to define $\eta(v, g)$ by adding to v a profile of affine payoff functions, whose coefficients contain some coefficients g_{ij} as well as first-order derivatives of each v_i at g and at (g_{ij}, g^0_{-ij}) (where g^0 will be a fixed reference network). These coefficients are chosen so that $\eta^{-1}(v)$ has a simple form $(\tilde{v}, g^v)$ where:

1. g^v_{ij} is defined from v taking the minimum of the two (unique) solutions of
$$\max_{w\in[0,1]} v_k(w, g^0_{-ij}) - \frac{w^2}{2}$$
for $k \in \{i, j\}$. Here, uniqueness is guaranteed by concavity assumption on v_k. The intuition is that from the first-order necessary condition, the solution of the above maximization problem should depend on some derivatives of $v_k(w, g^0_{-ij})$: we precisely fix the coefficients in the affine part added to v in $\eta(v, g)$ (see the discussion above) in accordance to the coefficients given by this first-order condition, so that it cancels when considering $g^{\eta(v,g)}_{ij}$, which guarantees $g^{\eta(v,g)}_{ij} = g_{ij}$.
2. $\tilde{v}$ is obtained from v using a very similar formula defining $\eta(v, g)$ (simply reversing some signs), i.e. $\tilde{v}$ is equal to v up to some profile of affine functions.

In Step 5 and Step 6, we prove that η and η^{-1} are continuous, and finally in Step 7, we prove that the straight line homotopy $H : (t, (v, g)) \mapsto (1 - t)\pi(v, g) + t\eta(v, g)$ from $[0, 1] \times \mathcal{P}$ to $\mathcal{F}$ is a proper homotopy between the projection $\pi : \mathcal{P} \to \mathcal{F}$ and $\eta : \mathcal{P} \to \mathcal{F}$.

A.3.2. The proof

In the proof, we will use the following first-order necessary and sufficient condition for a maximum:

Lemma A.1. *A $\mathcal{C}^1$ concave function $f : [0, 1] \to \mathbb{R}$ has a maximum at $x \in [0, 1]$ if and only if one of the three following conditions is verified:*

[15] That is, $O \subset X^*$ is open in X^* if (i) either $\infty \notin O$ and O is open in X, (ii) or $\infty \in O$ and the complement of O is compact for the topology induced by τ.

1. $f'(x) = 0$ *and* $x \in (0, 1)$.
2. $f'(x) \geq 0$ *and* $x = 1$.
3. $f'(x) \leq 0$ *and* $x = 0$.

From now on and until the end of this proof, we consider a fixed network $g^0 \in \mathbb{G}$.

Step 1: Construction of η.

Let

$$\begin{cases} \eta : & \mathcal{P} \to \mathcal{F} \\ & (v, g) \mapsto v^g \end{cases}$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$v_i^g(\gamma) = v_i(\gamma) + \sum_{j \neq i} \big(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\big)(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij} \quad (1)$$

(for every $(v, g) \in \mathcal{P}$ and every $i \in N$, v_i^g and v_i are equal up to an affine mapping). Moreover, to define *the inverse of* η, consider the mapping

$$\begin{cases} \rho : & \mathcal{F} \to \mathcal{F} \times \mathbb{G} \\ & v \mapsto (\tilde{v}, g^v) \end{cases}$$

where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\tilde{v}_i(\gamma) = v_i(\gamma) - \sum_{j \neq i} \big(\partial_{ij} v_i(g^v_{ij}, g^v_{-ij}) - \partial_{ij} v_i(g^v_{ij}, g^0_{-ij})\big)(\gamma_{ij} - g^v_{ij}) - \sum_{j \neq i} g^v_{ij}\gamma_{ij} \quad (2)$$

and where for every link $ij \in L$, $g^v_{ij} = \min\{w^v_{i,j}, w^v_{j,i}\}$, with $w^v_{i,j} \in [0, 1]$ being the unique solution of the strictly concave maximization problem

$$\max_{w \in [0,1]} v_i(w, g^0_{-ij}) - \frac{w^2}{2}$$

and $w^v_{j,i} \in [0, 1]$ being the unique solution of the strictly concave maximization problem

$$\max_{w \in [0,1]} v_j(w, g^0_{-ij}) - \frac{w^2}{2}.$$

Step 2: For every $v \in \mathcal{F}$, $\rho(v) \in \mathcal{P}$ *(i.e.* $\rho(\mathcal{F}) \subset \mathcal{P}$*).*

Let $v \in \mathcal{F}$ and consider $\rho(v) = (\tilde{v}, g^v)$. We have to prove that the network g^v is pairwise stable with respect to $\tilde{v}$. To do so, let $ij \in L$ be a fixed link. Since $g^v_{ij} = \min\{w^v_{i,j}, w^v_{j,i}\}$, we can assume, without any loss of generality (permuting $w^v_{i,j}$ and $w^v_{j,i}$ if necessary), that $g^v_{ij} = w^v_{i,j}$, which corresponds to the case where $w^v_{i,j} \leq w^v_{j,i}$.

First, we prove that $g^v_{ij} = w^v_{i,j}$ maximizes the payoff function $\tilde{v}_i(\cdot, g^v_{-ij})$ of agent i; from Lemma A.1, we have to prove that $\tilde{v}_i(\cdot, g^v_{-ij})$ satisfies one of the three conditions of the lemma at $g^v_{ij} = w^v_{i,j}$. An easy computation gives at every $w \in [0, 1]$:

$$\partial_{ij}\tilde{v}_i(w, g^v_{-ij}) = \partial_{ij} v_i(w, g^v_{-ij}) - (\partial_{ij} v_i(g^v_{ij}, g^v_{-ij}) - \partial_{ij} v_i(g^v_{ij}, g^0_{-ij})) - g^v_{ij}$$

and in particular:

$$\partial_{ij}\tilde{v}_i(g^v_{ij}, g^v_{-ij}) = \partial_{ij} v_i(g^v_{ij}, g^0_{-ij}) - g^v_{ij}. \quad (3)$$

However, this last quantity is also the derivative with respect to w of the concave function $w \in [0, 1] \mapsto v_i(w, g^0_{-ij}) - \frac{w^2}{2}$ at $g^v_{ij} = w^v_{i,j}$. By definition, $g^v_{ij} = w^v_{i,j}$ is the maximum of $w \in [0, 1] \mapsto v_i(w, g^0_{-ij}) - \frac{w^2}{2}$, and writing the first-order necessary and sufficient condition given by Lemma A.1 applied to this function, we also get (still from Lemma A.1) that $g^v_{ij} = w^v_{i,j}$ maximizes $\tilde{v}_i(\cdot, g^v_{-ij})$. In particular, this implies that Condition 1 of pairwise stability is fulfilled for agent i at g^v_{ij}, and that Condition 2 of pairwise stability is also fulfilled.

It remains to show that Condition 1 of pairwise stability is satisfied for agent j at g^v_{ij}. It is clearly the case if $g^v_{ij} = 0$, thus we can assume $g^v_{ij} > 0$, and finally the proof of Step 1 will be finished if we prove that $\tilde{v}_j(\cdot, g^v_{-ij})$ is nondecreasing on $[0, g^v_{ij})$. Because $\tilde{v}_j(\cdot, g^v_{-ij})$ is concave, this is equivalent to show that $\partial_{ij}\tilde{v}_j(g^v_{ij}, g^v_{-ij}) \geq 0$, or, after a computation similar to the one in Eq. (3), it is also equivalent to show that $\partial_{ij} v_j(g^v_{ij}, g^0_{-ij}) - g^v_{ij} \geq 0$. However, by definition, the concave function $w \mapsto v_j(w, g^0_{-ij}) - \frac{w^2}{2}$ reaches its maximum at $w_{j,i}$. Hence, from Lemma A.1 and from concavity of this function, its derivative at $g^v_{ij} \leq w_{j,i}$ (which is precisely $\partial_{ij} v_j(g^v_{ij}, g^0_{-ij}) - g^v_{ij}$) has to be nonnegative. This ends the proof that g^v is pairwise stable with respect to $\tilde{v}$.

Finally, we have proved that $\rho(\mathcal{F}) \subset \mathcal{P}$, and we now consider, by abuse of notation, that ρ is a mapping from $\mathcal{F}$ to $\mathcal{P}$.

Step 3: For every $v \in \mathcal{F}$, $(\eta \circ \rho)(v) = v$ *(i.e.* $\eta \circ \rho = \mathrm{id}_{\mathcal{F}}$*).*

For every $v \in \mathcal{F}$, let us define $\check{v} = (\eta \circ \rho)(v) = \eta(\tilde{v}, g^v)$. We have to prove that $\check{v} = v$.

By definition of η, for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$\check{v}_i(\gamma) = \tilde{v}_i(\gamma) + \sum_{j \neq i} (\partial_{ij}\tilde{v}_i(g^v_{ij}, g^v_{-ij}) - \partial_{ij}\tilde{v}_i(g^v_{ij}, g^0_{-ij})(\gamma_{ij} - g^v_{ij})) + \sum_{j \neq i} g^v_{ij}\gamma_{ij} \quad (4)$$

and summing Eqs. (2) and (4), we get that

$$\begin{aligned} \check{v}_i(\gamma) - v_i(\gamma) &= \sum_{j \neq i} (\partial_{ij}\tilde{v}_i(g^v_{ij}, g^v_{-ij}) - \partial_{ij}\tilde{v}_i(g^v_{ij}, g^0_{-ij})(\gamma_{ij} - g^v_{ij})) \\ &\quad - \sum_{j \neq i} (\partial_{ij} v_i(g^v_{ij}, g^v_{-ij}) - \partial_{ij} v_i(g^v_{ij}, g^0_{-ij})(\gamma_{ij} - g^v_{ij})) \\ &= \sum_{j \neq i} (\partial_{ij}(\tilde{v}_i - v_i)(g^v_{ij}, g^v_{-ij}) - \partial_{ij}(\tilde{v}_i - v_i)(g^v_{ij}, g^0_{-ij})) \\ &\quad \times (\gamma_{ij} - g^v_{ij}). \end{aligned}$$

However, $g \in \mathbb{G} \mapsto (\tilde{v}_i - v_i)(g)$ is an affine mapping, and in particular, for every $j \neq i$, $\partial_{ij}(\tilde{v}_i - v_i)(g_{ij}, g_{-ij})$ does not depend on g_{-ij}. Thus, each term in the previous sum is null, which implies that $\check{v}_i(\gamma) = v_i(\gamma)$, and which ends Step 3.

Step 4: For every $(v, g) \in \mathcal{P}$, $(\rho \circ \eta)(v, g) = (v, g)$ *(i.e.* $\rho \circ \eta = \mathrm{id}_{\mathcal{P}}$*).*

For every $(v, g) \in \mathcal{P}$, let us define $(\tilde{v^g}, g') = (\rho \circ \eta)(v, g) = \rho(v^g)$, where $g' = g^{v^g}$, i.e. for every $ij \in L$, $g'_{ij} = \min\{w'_{i,j}, w'_{j,i}\}$, with $w'_{i,j} \in [0, 1]$ being the unique solution of the strictly concave maximization problem

$$\max_{w \in [0,1]} v^g_i(w, g^0_{-ij}) - \frac{w^2}{2},$$

and $w'_{j,i} \in [0, 1]$ being the unique solution of the strictly concave maximization problem

$$\max_{w \in [0,1]} v^g_j(w, g^0_{-ij}) - \frac{w^2}{2}.$$

We have to show that $\tilde{v^g} = v$ and that $g = g'$.

Step 4.1: $g = g'$.

Let us consider an arbitrary link $ij \in L$. Recall that since $(v, g) \in \mathcal{P}$, g is pairwise stable with respect to v. The following lemma (cf. Bich and Morhaim, 2017, 2020) establishes the relationship between the weight of a link ij at some pairwise stable network and the optimal weights for agent i and agent j:

Lemma A.2. *Let $g = (g_{ij})_{ij \in L}$ be some pairwise stable network with respect to a continuous society $v = (v_i)_{i \in N}$ such that for every $i \in N$ and every $j \neq i$, the function $w \in [0,1] \mapsto v_i(w, g_{-ij})$ is concave. Moreover, for every $ij \in L$, let $I_i = [a_i, b_i] = \arg\max_{w \in [0,1]} v_i(w, g_{-ij})$ and $I_j = [a_j, b_j] = \arg\max_{w \in [0,1]} v_j(w, g_{-ij})$.*[16] *Then,*

$$g_{ij} \in [\min\{a_i, a_j\}, \min\{b_i, b_j\}].$$

To prove that $g_{ij} = g'_{ij}$, we can assume, without any loss of generality (up to a permutation of i and j if necessary), that $a_i \leq a_j$. In particular, from Lemma A.2, we get that $g_{ij} \in [a_i, b_i]$ in all cases. Thus, $g_{ij} \in I_i$, and as a maximizer of $v_i(\cdot, g_{-ij})$, it has to satisfy the necessary (and sufficient) condition of Lemma A.1 for this function.

Now, the derivative of the function $q_i : w \in [0,1] \mapsto v_i^g(w, g^0_{-ij}) - \frac{w^2}{2}$ at g_{ij} is equal to $\partial_{ij} v_i^g(g_{ij}, g^0_{-ij}) - g_{ij}$, which is also equal to $\partial_{ij} v_i(g_{ij}, g_{-ij})$ (computing $\partial_{ij} v_i^g(g_{ij}, g^0_{-ij})$, thanks to Eq. (1)). Thus, q_i has also to verify the necessary (and sufficient) condition of Lemma A.1 at g_{ij}, and by strict concavity of q_i, g_{ij} has to be the unique maximum of this function, i.e. $g_{ij} = w'_{i,j}$.

To finish, we only have to prove that $g'_{ij} = w'_{i,j}$, or equivalently that $w'_{i,j} \leq w'_{j,i}$. Since this inequality is verified when $w'_{i,j} = 0$ or when $w'_{j,i} = 1$, we can assume $w'_{i,j} > 0$ and $w'_{j,i} < 1$. The same computation as above gives that the derivative of $q_j : w \in [0,1] \mapsto v_j^g(w, g^0_{-ij}) - \frac{w^2}{2}$ at g_{ij} is equal to $\partial_{ij} v_j(g_{ij}, g_{-ij})$. Thus, since $w'_{j,i} < 1$ is the unique maximizer of q_j, we get (from Lemma A.1) that

$$q'_j(w'_{j,i}) \leq 0.$$

Also, since $v_j(\cdot, g_{-ij})$ is concave and since $w'_{i,j} = g_{ij} \leq b_j$ (because $g_{ij} \leq \min\{b_i, b_j\}$), we get that $\partial_{ij} v_j(w'_{i,j}, g_{-ij}) \geq 0$, and in particular, that

$$q'_j(w'_{i,j}) \geq 0.$$

Finally, $q'_j(w'_{j,i}) \leq q'_j(w'_{i,j})$, thus in particular $w'_{i,j} \leq w'_{j,i}$ because q'_j is strictly decreasing from strict concavity of q_j, which proves that $g = g'$.

Step 4.2: $\tilde{v^g} = v$.

The proof is very similar to the one in Step 2 (i.e. the proof that $\eta \circ \rho = \mathrm{id}_{\mathcal{F}}$). Indeed, for every $i \in N$ and every $\gamma \in \mathbb{G}$, by definition of ρ and from Step 3.1 (where we have proved $g^{v^g} = g$ and $\rho(v^g) = (\tilde{v^g}, g)$), we get that

$$\tilde{v}_i^g(\gamma) = v_i^g(\gamma) - \sum_{j \neq i} \big(\partial_{ij} v_i^g(g_{ij}, g_{-ij}) - \partial_{ij} v_i^g(g_{ij}, g^0_{-ij})\big)(\gamma_{ij} - g_{ij}) - \sum_{j \neq i} g_{ij}\gamma_{ij} \tag{5}$$

and by definition of η, we get that

$$v_i^g(\gamma) = v_i(\gamma) + \sum_{j \neq i} \big(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\big)(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij}. \tag{6}$$

Summing the two above equations, we get that

$$\tilde{v}_i^g(\gamma) - v_i(\gamma) = \sum_{j \neq i} (\partial_{ij}(v_i - v_i^g)(g_{ij}, g_{-ij}) - \partial_{ij}(v_i - v_i^g)(g_{ij}, g^0_{-ij})) \times (\gamma_{ij} - g_{ij}).$$

[16] These sets are nonempty closed intervals, since the functions $w \in [0,1] \mapsto v_i(w, g_{-ij})$ and $w \in [0,1] \mapsto v_j(w, g_{-ij})$ are continuous and concave on a compact set.

However, $g \in \mathbb{G} \mapsto (v_i - v_i^g)(g)$ is an affine mapping, and in particular, for every $j \neq i$, $\partial_{ij}(v_i - v_i^g)(g_{ij}, g_{-ij})$ does not depend on g_{-ij}. Thus, each term in the previous sum is null, which implies that $\tilde{v}_i^g(\gamma) = v_i(\gamma)$, and which proves that $\tilde{v^g} = v$ and ends Step 4.

Step 5: η is a continuous mapping.

Recall that for every $(v, g) \in \mathcal{P}$, $\eta(v, g) = v^g$, where for every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$v_i^g(\gamma) = v_i(\gamma) + \sum_{j \neq i} \big(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\big)(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij}$$

(see Eq. (1)).

To prove the continuity of η, notice that:

1. The mapping $(v, g) \in \mathcal{P} \mapsto v_i \in \mathcal{F}_i$ is continuous, as a projection, since $\mathcal{F}$ is endowed with the product norm (see Section 2.1).
2. The mapping

$$(v, g) \in \mathcal{P} \mapsto \Big(\gamma \mapsto \sum_{j \neq i} \big(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\big) \times (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij} \Big) \in \mathcal{F}_i$$

is continuous. Indeed, let us first prove the continuity of the mapping

$$F_{ij} : (v, g) \in \mathcal{P} \mapsto \big(\gamma \mapsto \partial_{ij} v_i(g_{ij}, g_{-ij})\gamma_{ij}\big) \in \mathcal{F}_i,$$

for every $j \neq i$. Consider $(\bar{v}, \bar{g}) \in \mathcal{P}$, and for every $(v, g) \in \mathcal{P}$, let us compute

$$\|F_{ij}(v, g) - F_{ij}(\bar{v}, \bar{g})\|_i = \max\{\max\{\|F_{ij}(v, g) - F_{ij}(\bar{v}, \bar{g})\|_\infty, \|\partial_{ik} F_{ij}(v, g) - \partial_{ik} F_{ij}(\bar{v}, \bar{g})\|_\infty\} : k \neq i\}.$$

For every $\gamma \in \mathbb{G}$,

$$\begin{aligned} F_{ij}(v, g)(\gamma) - F_{ij}(\bar{v}, \bar{g})(\gamma) &= \partial_{ij} v_i(g_{ij}, g_{-ij})\gamma_{ij} - \partial_{ij}\bar{v}_i(\bar{g}_{ij}, \bar{g}_{-ij})\gamma_{ij} \\ &= \partial_{ij}(v_i - \bar{v}_i)(g_{ij}, g_{-ij})\gamma_{ij} \\ &\quad - (\partial_{ij}\bar{v}_i(\bar{g}_{ij}, \bar{g}_{-ij}) \\ &\quad - \partial_{ij}\bar{v}_i(g_{ij}, g_{-ij}))\gamma_{ij}, \end{aligned}$$

which implies that

$$\begin{aligned} \|F_{ij}(v, g) - F_{ij}(\bar{v}, \bar{g})\|_\infty &\leq |\partial_{ij}(v_i - \bar{v}_i)(g_{ij}, g_{-ij})| \\ &\quad + |(\partial_{ij}\bar{v}_i(\bar{g}_{ij}, \bar{g}_{-ij}) - \partial_{ij}\bar{v}_i(g_{ij}, g_{-ij}))| \\ &\leq \|v_i - \bar{v}_i\|_i + |(\partial_{ij}\bar{v}_i(\bar{g}_{ij}, \bar{g}_{-ij}) \\ &\quad - \partial_{ij}\bar{v}_i(g_{ij}, g_{-ij}))|, \end{aligned}$$

which proves that

$$\lim_{(v,g) \to (\bar{v}, \bar{g})} \|F_{ij}(v, g) - F_{ij}(\bar{v}, \bar{g})\|_\infty = 0.$$

Moreover, for every $\gamma \in \mathbb{G}$,

$$\partial_{ij} F_{ij}(v, g)(\gamma) - \partial_{ij} F_{ij}(\bar{v}, \bar{g})(\gamma) = \partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij}\bar{v}_i(\bar{g}_{ij}, \bar{g}_{-ij}),$$

which gives by a same argument as above:

$$\begin{aligned} &\|\partial_{ij} F_{ij}(v, g)(\gamma) - \partial_{ij} F_{ij}(\bar{v}, \bar{g})(\gamma)\|_\infty \\ &\quad \leq \|v_i - \bar{v}_i\|_i + |(\partial_{ij}\bar{v}_i(\bar{g}_{ij}, \bar{g}_{-ij}) - \partial_{ij}\bar{v}_i(g_{ij}, g_{-ij}))|, \end{aligned}$$

which proves that

$$\lim_{(v,g) \to (\bar{v}, \bar{g})} \|\partial_{ij} F_{ij}(v, g)(\gamma) - \partial_{ij} F_{ij}(\bar{v}, \bar{g})(\gamma)\|_\infty = 0,$$

and which finally proves the continuity of F_{ij}. Similarly, we can prove the continuity of the mappings $(v, g) \in \mathcal{P} \mapsto$

$\left(\gamma \mapsto \left(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\right) g_{ij}\right)$ and of the mappings $(v, g) \in \mathcal{P} \mapsto \left(\gamma \mapsto g_{ij}\gamma_{ij}\right)$, for every $j \neq i$.

Summing all these mappings, we finally get the continuity of η.

Step 6: ρ is a continuous mapping.

Recall that for every $v \in \mathcal{F}$, $\rho(v) = (\tilde{v}, g^v)$, where for every $i \in N$ and for every $\gamma \in \mathbb{G}$,

$$\tilde{v}_i(\gamma) = v_i(\gamma) - \sum_{j \neq i} \left(\partial_{ij} v_i(g^v_{ij}, g^v_{-ij}) - \partial_{ij} v_i(g^v_{ij}, g^0_{-ij})\right)(\gamma_{ij} - g^v_{ij}) - \sum_{j \neq i} g^v_{ij}\gamma_{ij}$$

(see Eq. (2)), and where for every $ij \in L$, $g^v_{ij} = \min\{w^v_{i,j}, w^v_{j,i}\}$, with $w^v_{i,j}$ being the unique maximum of the strictly concave function $q_i : w \in [0, 1] \mapsto v_i(w, g^0_{-ij}) - \frac{w^2}{2}$ and $w^v_{j,i}$ being the unique maximum of the strictly concave function $q_j : w \in [0, 1] \mapsto v_j(w, g^0_{-ij}) - \frac{w^2}{2}$ on $[0, 1]$.

Actually, the continuity of ρ can be proved in the same way as the continuity of η. The only additional point to be proved is that the mapping $v \in \mathcal{F} \mapsto g^v$ is continuous, which is a consequence of the continuity of the mappings $v \in \mathcal{F} \mapsto w^v_{i,j}$ and $v \in \mathcal{F} \mapsto w^v_{j,i}$ (from Berge's theorem).[17]

Step 7: The mapping $H : [0, 1] \times \mathcal{P} \to \mathcal{F}, (t, (v, g)) \mapsto H(t, (v, g)) = (1 - t)\pi(v, g) + t\eta(v, g)$ is a proper homotopy between π and η.

For every $i \in N$, define the mapping $\Phi_i : [0, 1] \times \mathcal{F} \times \mathbb{G} \to \mathcal{F}_i$ by

$$\Phi_i(t, (v, g))(\gamma) = v_i(\gamma) + t \left(\sum_{j \neq i} \left(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\right) \times (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij} \right),$$

for every $(t, (v, g)) \in [0, 1] \times \mathcal{F} \times \mathbb{G}$ and for every $\gamma \in \mathbb{G}$. Moreover, define

$$\Phi : [0, 1] \times \mathcal{F} \times \mathbb{G} \to \mathcal{F}, (t, (v, g)) \mapsto (\Phi_i)_{i \in N}.$$

It is easy to see that the restriction of Φ to $[0, 1] \times \mathcal{P}$ is equal to H.

Now, define the mapping $\psi : [0, 1] \times \mathcal{F} \times \mathbb{G} \to [0, 1] \times \mathcal{F} \times \mathbb{G}$ by

$$\psi(t, (v, g)) = (t, \Phi(t, (v, g)), g)$$

for every $(t, (v, g)) \in [0, 1] \times \mathcal{F} \times \mathbb{G}$. The mapping ψ is invertible: for every $t \in [0, 1]$, $\psi^{-1}(t, (v, g)) = (t, \tilde{\Phi}(t, (v, g)), g)$, where $\tilde{\Phi} = (\tilde{\Phi}_i)_{i \in N}$ and where for every $i \in N$, $\tilde{\Phi}_i : [0, 1] \times \mathcal{F} \times \mathbb{G} \to \mathcal{F}_i$ is defined by

$$\tilde{\Phi}_i(t, (v, g))(\gamma) = v_i(\gamma) - t \left(\sum_{j \neq i} \left(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\right) \times (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij} \right)$$

for every $(t, (v, g)) \in [0, 1] \times \mathcal{F} \times \mathbb{G}$ and for every $\gamma \in \mathbb{G}$ (the proof is similar to those of Step 3 and Step 4). Moreover, ψ and ψ^{-1} are continuous (the proof is similar to those of Step 5 and Step 6). Now, for every compact subspace K of $\mathcal{F}$,

$$\begin{aligned} H^{-1}(K) &= \{(t, (v, g)) \in [0, 1] \times \mathcal{P} : H(t, (v, g)) \in K\} \\ &= \{(t, (v, g)) \in [0, 1] \times \mathcal{P} : \psi(t, (v, g)) \in [0, 1] \times K \times \mathbb{G}\} \\ &= \psi^{-1}([0, 1] \times K \times \mathbb{G}) \cap ([0, 1] \times \mathcal{P}). \end{aligned}$$

However, ψ^{-1} is a continuous mapping and $[0, 1] \times K \times \mathbb{G}$ is a compact subspace of $[0, 1] \times \mathcal{F} \times \mathbb{G}$, thus $\psi^{-1}([0, 1] \times K \times \mathbb{G})$ is a compact subspace of $[0, 1] \times \mathcal{F} \times \mathbb{G}$. Moreover, $[0, 1] \times \mathcal{P}$ is closed in $[0, 1] \times \mathcal{F} \times \mathbb{G}$ (see Lemma A.3). Thus, finally $H^{-1}(K) = \psi^{-1}([0, 1] \times K \times \mathbb{G}) \cap ([0, 1] \times \mathcal{P})$ is compact as a closed subset of a compact space, which proves that H is a proper mapping.

Lemma A.3. *The set $\mathcal{P}$ is a closed subset of $\mathcal{F} \times \mathbb{G}$.*

Proof. Consider a sequence (v^k, g^k) in $\mathcal{P}$ which converges to $(v, g) \in \mathcal{F} \times \mathbb{G}$. To prove that $(v, g) \in \mathcal{P}$, suppose by contradiction that $(v, g) \notin \mathcal{P}$:

1. Suppose first that there exists some $ij \in L$ and some $w < g_{ij}$ such that $v_i(w, g_{-ij}) > v_i(g)$. Since v_i is continuous, this implies that there exists $k \geq 0$ such that $v^k_i(w, g^k_{-ij}) > v^k_i(g^k)$. Indeed, from the topology defined on $\mathcal{F}$, $v^k_i(g^k)$ converges to $v_i(g)$ since
$$\begin{aligned} |v^k_i(g^k) - v_i(g)| &\leq |v^k_i(g^k) - v_i(g^k)| + |v_i(g^k) - v_i(g)| \\ &\leq \|v^k_i - v_i\|_\infty + |v_i(g^k) - v_i(g)| \\ &\leq \|v^k_i - v_i\|_i + |v_i(g^k) - v_i(g)|, \end{aligned}$$
and similarly $v^k_i(w, g^k_{-ij})$ converges to $v_i(w, g_{-ij})$. However, this contradicts the fact that g^k is pairwise stable with respect to v^k.
2. Now, suppose that there exists some $ij \in L$ and some $w > g_{ij}$ such that $v_i(w, g_{-ij}) > v_i(g)$ and $v_j(w, g_{-ij}) > v_j(g)$. Since v_i and v_j are continuous, this implies that there exists $k \geq 0$ such that $v^k_i(w, g^k_{-ij}) > v_i(g^k)$ and $v^k_j(w, g^k_{-ij}) > v_j(g^k)$ (the proof is the same as the one above), which again contradicts the fact that g^k is pairwise stable with respect to v^k. $\square$

A.4. Proof of Corollary 2.1

The idea is to follow the proof of Theorem 2.1 by restricting η to $\mathcal{P}_{\mathcal{U}}$. The only important point is to remark that $\eta(\mathcal{P}_{\mathcal{U}}) \subset \mathcal{U}$. Indeed, by definition, for every $(v, g) \in \mathcal{P}_{\mathcal{U}}$, every $i \in N$ and every $\gamma \in \mathbb{G}$,

$$(\eta_{|\mathcal{P}_{\mathcal{U}}})_i(v, g)(\gamma) = v_i(\gamma) + \sum_{j \neq i} \left(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\right) \times (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij}$$

(where $\eta_{|\mathcal{P}_{\mathcal{U}}} = ((\eta_{|\mathcal{P}_{\mathcal{U}}})_i)_{i \in N}$), with $v_i \in \mathcal{U}_i$. Now, notice that the mapping

$$a_i : \gamma \in \mathbb{G} \mapsto \sum_{j \neq i} \left(\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g^0_{-ij})\right)(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij}$$

belongs to $\mathcal{A}_i$. Thus, we can use the assumption of the corollary (stability assumption) to get that $(\eta_{|\mathcal{P}_{\mathcal{U}}})_i(v, g) = v_i + a_i \in \mathcal{U}_i$, which implies that $\eta_{|\mathcal{P}_{\mathcal{U}}}(v, g) \in \mathcal{U} = \prod_{i \in N} \mathcal{U}_i$.

[17] Indeed, Berge's theorem states that if we consider a continuous correspondence $\Psi : X \twoheadrightarrow Y$ with nonempty compact values and if we consider a continuous mapping $f : X \times Y \to \mathbb{R}$, then the argmax correspondence of the maximization problem $\max_{y \in \Psi(x)} f(x, y)$ has a closed graph and nonempty values. In particular, if we take $Y = [0, 1]$, $X = \mathcal{F}$, $\Psi(x) = [0, 1]$ for every $x \in X$ and $f : (v, w) \in \mathcal{F} \times [0, 1] \mapsto v_i(w, g^0_{-ij}) - w^2/2$ (which is continuous), then we get that the (single-valued) correspondence $v \in \mathcal{F} \mapsto w^v_{i,j}$ has a closed graph, or equivalently, that the mapping $v \in \mathcal{F} \to w^v_{i,j}$ is continuous.

A.5. Proof of Theorem 3.1

A.5.1. Sketch of the proof

We want to prove that for every regular set $\mathcal{U} \subset \mathcal{F}$ of polynomial societies, there is (generically in $\mathcal{U}$) an odd number of pairwise stable networks.

From Theorem 2.1, there exists a homeomorphism η from $\mathcal{P}$, the graph of pairwise stable networks, to $\mathcal{F}$, the set of profiles of $\mathcal{C}^1$ payoff functions which are concave in each agent's links.

Since $\mathcal{U}$ is regular, is satisfies stability assumption (see Definition 3.1), thus Corollary 2.1 implies that $\eta_{|\mathcal{P}_\mathcal{U}}$, the restriction of η to $\mathcal{P}_\mathcal{U}$ (the graph of pairwise stable networks restricted to $\mathcal{U}$) is a homeomorphism from $\mathcal{P}_\mathcal{U}$ to $\mathcal{U}$. Moreover, still from Corollary 2.1, we know that $\eta_{|\mathcal{P}_\mathcal{U}} : \mathcal{P}_\mathcal{U} \to \mathcal{U}$ is properly homotopic to the projection $\pi_{|\mathcal{P}_\mathcal{U}} : \mathcal{P}_\mathcal{U} \to \mathcal{U}$ (where $\pi : \mathcal{P} \to \mathcal{F}$ denotes the projection from $\mathcal{P}$ to $\mathcal{F}$).

Following a tradition of existence proofs in game theory or in general equilibrium, one could try to apply topological degree to the projection mapping $\pi_{|\mathcal{P}_\mathcal{U}} : \mathcal{P}_\mathcal{U} \to \mathcal{U}$, to get that generically, $\pi^{-1}_{|\mathcal{P}_\mathcal{U}}(v)$ (which "counts" the number of pairwise stable networks associated to the society v) has an odd number of elements. The idea would be, first, to prove that the topological degree of $\eta_{|\mathcal{P}_\mathcal{U}}$ is equal to 1, using homeomorphism property of topological degree. This would imply that the degree of $\pi_{|\mathcal{P}_\mathcal{U}}$ is also equal to 1 (by homotopy invariance of topological degree). Also, from some covering property (see Theorem A.2), we could get that $\pi^{-1}_{|\mathcal{P}_\mathcal{U}}(v) = \{(v, g_1), \dots, (v, g_k)\}$ is generically finite, where each (v, g_ℓ) has some neighborhood which is mapped by $\pi_{|\mathcal{P}_\mathcal{U}}$ homeomorphically onto its image. Still from homeomorphism property of topological degree, this would imply that the local degree of $\pi_{|\mathcal{P}_\mathcal{U}}$ at each (v, g_ℓ) (which is by definition the degree of the restriction of $\pi_{|\mathcal{P}_\mathcal{U}}$ to some neighborhood of (v, g_ℓ)) is equal, modulo 2, to 1. Then, from additivity property of degree, the sum of local degrees, equal (modulo 2) to k, has also to be equal to the degree of $\pi_{|\mathcal{P}_\mathcal{U}}$ (which is equal to 1), i.e. there are generically an odd number of pairwise stable networks.

The main problem with this approach is that to be able to apply topological degree to $\pi_{|\mathcal{P}_\mathcal{U}}$, $\mathcal{P}_\mathcal{U}$ and $\mathcal{U}$ have to be topological manifolds, which is false in general. In Step 1, we first skip this difficulty by assuming that $\mathcal{S} := \varphi(\mathcal{U})$, the set of coefficients of profiles of polynomial payoff functions in $\mathcal{U}$, is homeomorphic to the Euclidean space $\mathbb{R}^m$. Under this assumption, φ is a global chart which allows to identify $\mathcal{U}$ (now a topological manifold) with $\mathbb{R}^m$. Using this chart, the two mappings $\eta_{|\mathcal{P}_\mathcal{U}}$ and $\pi_{|\mathcal{P}_\mathcal{U}}$, from $\mathcal{P}_\mathcal{U}$ to $\mathcal{U}$, can be "red" as mappings $\tilde{\eta}$ and $\tilde{\pi}$, now defined from $\mathcal{M}$ to $\mathbb{R}^m$ (where $\mathcal{M} := \{(\varphi(v), g) \in \mathbb{R}^m \times \mathbb{G} : (v, g) \in \mathcal{P}_\mathcal{U}\}$), these two sets being homeomorphic.

A second problem (see Step 2 and Step 3), which could prevent from using additivity property of degree, is the possibility to have pairwise stable networks "at infinity". Formally, we avoid that by proving that the homotopy between η and π is proper (and similarly for the homotopy between $\tilde{\pi}$ and $\tilde{\eta}$, which we also construct using the chart φ). Property allows to use compactification: that is, we can extend $\tilde{\pi}$ and $\tilde{\eta}$ from $\mathcal{M}_\mathcal{U} \cup \{\infty\}$ to $\mathcal{S} \cup \{\infty\}$, by defining $\tilde{\pi}(\infty) = \infty$ and $\tilde{\eta}(\infty) = \infty$ (this last ∞ represents intuitively profiles of payoff functions with some "infinite" coefficients). It is easy to see that the compactifications $\mathcal{M} \cup \{\infty\}$ and $\mathcal{S} \cup \{\infty\} = \mathbb{R}^m \cup \{\infty\}$ are also homeomorphic to $\mathbb{S}^m$ (the m-sphere). These constructions are summarized by the following picture, where $\eta_{|\mathcal{P}_\mathcal{U}}$ and $\pi_{|\mathcal{P}_\mathcal{U}}$ are the initial mappings we consider, $\tilde{\eta}$ and $\tilde{\pi}$ are the "equivalent" mappings red in the chart defined by φ, and $\hat{\eta}$ and $\hat{\pi}$ are the mappings obtained by compactification. In particular, these three pairs of mappings should be thought as very similar, but one great advantage of this construction is that we can apply topological degree (see Appendix A.2.2) to the last pair $\hat{\eta}$ and $\hat{\pi}$ (which will also give informations about $\pi_{|\mathcal{P}_\mathcal{U}}$).

$$\begin{tikzcd}
\mathcal{P} \arrow[d, shift right, "\pi"'] \arrow[d, shift left, "\eta"] & \mathcal{P}_\mathcal{U} \arrow[l, hook'] \arrow[r] \arrow[d, shift right, "\pi_{|\mathcal{P}_\mathcal{U}}"'] \arrow[d, shift left, "\eta_{|\mathcal{P}_\mathcal{U}}"] & \mathcal{M} \arrow[r] \arrow[d, shift right, "\tilde{\pi}"'] \arrow[d, shift left, "\tilde{\eta}"] & \mathbb{S}^m \arrow[d, shift right, "\hat{\pi}"'] \arrow[d, shift left, "\hat{\eta}"] \\
\mathcal{F} & \mathcal{U} \arrow[l, hook'] \arrow[r] & \mathcal{S} \arrow[r] & \mathbb{S}^m
\end{tikzcd}$$

In Step 4, we get that the topological degree of $\hat{\pi} : \mathbb{S}^m \to \mathbb{S}^m$ is equal to 1 (because $\hat{\pi}$ is homotopic to the homeomorphism $\hat{\eta} : \mathbb{S}^m \to \mathbb{S}^m$), which implies the same for $\tilde{\pi}$, from its definition. In Step 5, we use semi-algebraicity properties (also derived from Step 1) to get that for generic $x \in \mathbb{R}^m$, $\tilde{\pi}^{-1}(x)$ is finite (here, the regularity assumption made on $\mathcal{U}$ plays an important role). We get the same (from their definitions) for $\hat{\pi}$ (Step 7) and for π (Step 6). Finally, in Step 8, we use additivity property of degree, as discussed above, to get the theorem when $\varphi(\mathcal{U}) = \mathbb{R}^m$.

In the last step (Step 9), perhaps the most important in terms of providing new proof techniques, we want to get rid of the assumption that $\mathcal{S} = \varphi(\mathcal{U})$ is equal to $\mathbb{R}^m$. The idea is to decompose $\mathcal{U}$ into a union of sets $\mathcal{V}^\lambda$ to which we can apply the previous steps (Step 1 to Step 8). This requires that each set $\mathcal{V}^\lambda$ is included in $\mathcal{F}$, satisfies regularity assumption (thus in particular stability assumption) and is homeomorphic to some Euclidean space. To construct the $\mathcal{V}^\lambda$, we could try, using semi-algebraicity of $\mathcal{S}$, to decompose $\mathcal{S}$ in a finite disjoint union of sets $\mathcal{S}^\lambda$, each one being homeomorphic to some Euclidean space. The problem is that there is no guarantee that the sets of profiles of polynomial payoff functions $\varphi^{-1}(\mathcal{S}^\lambda)$ satisfy stability assumption. To obtain this property, the idea is: (i) to remove some affine part in each payoff function of $\mathcal{U}$ (Step 9.1) using some projection of $\mathcal{U}$ (call $\mathcal{U}'$ the outcome); (ii) to use a standard decomposition result of $\varphi(\mathcal{U}')$ (the set of coefficients of profiles of polynomial payoff functions in $\mathcal{U}'$), which can be proved to be semi-algebraic by Tarski–Seidenberg's theorem (Step 9.2); (iii) to re-introduce the affine parts in each set of profiles of payoff functions obtained from the previous decomposition (Step 9.3). In Step 9.4, it is proved that the sets $\mathcal{V}^\lambda$ obtained in this decomposition are regular. The end of the proof consists in applying Steps 1–8 to each set $\mathcal{V}^\lambda$: this provides a generic set G^λ in $\varphi(\mathcal{V}^\lambda)$ such that every society $v \in G^\lambda$ has an odd number of pairwise stable networks. The final generic set G (in $\mathcal{S}$) is simply obtained taking the union of the sets G^λ, retaining only the λ for which $\mathcal{V}^\lambda$ is "thick enough", to be sure that G is open in $\mathcal{S}$ (Steps 9.5–9.8).

A.5.2. The proof

From Step 1 to Step 8, we assume that the set $\mathcal{S} = \varphi(\mathcal{U})$ of coefficients of polynomial societies in $\mathcal{U}$ is equal to[18] $\mathbb{R}^m$, and the general case (where $\mathcal{S}$ is simply semi-algebraic) is treated in Step 9. Similarly to the proof of Theorem 2.1 (see the definition of η in Eq. (1)), from now on and until the end of this proof, we consider a fixed network $g^0 \in \mathbb{G}$.

Step 1: The set $\mathcal{M} := \{(\varphi(v), g) \in \mathbb{R}^m \times \mathbb{G} : (v, g) \in \mathcal{P}_\mathcal{U}\}$ *is semi-algebraic and the mapping* $\tilde{\eta} : \mathcal{M} \to \mathbb{R}^m, (x, g) \mapsto \varphi(\eta_{|\mathcal{P}_\mathcal{U}}(\varphi^{-1}(x), g))$ *is a semi-algebraic homeomorphism. In particular,* $\dim(\mathcal{M}) = m$.

First, we show that $\mathcal{M}$ is semi-algebraic. From the concavity assumption of payoff functions in $\mathcal{U}$, we can remark that the condition $(x, g) \in \mathcal{M}$ (i.e. the condition "g is a pairwise stable network of $\varphi^{-1}(x)$") is equivalent to the following conditions (where we recall that $v^x = (v_i^x)_{i \in N} = \varphi^{-1}(x)$): for every $ij \in L$,

[18] Actually, the proof is similar if we only assume that $\mathcal{S}$ is homeomorphic to $\mathbb{R}^p$, for some integer $p \leq m$.

1. either $g_{ij} \in (0, 1)$, $\partial_{ij} v_i^x(g_{ij}, g_{-ij}) = 0$ and $\partial_{ij} v_j^x(g_{ij}, g_{-ij}) \geq 0$,
2. or $g_{ij} \in (0, 1)$, $\partial_{ij} v_j^x(g_{ij}, g_{-ij}) = 0$ and $\partial_{ij} v_i^x(g_{ij}, g_{-ij}) \geq 0$,
3. or $g_{ij} = 0$ and $[\partial_{ij} v_i^x(g_{ij}, g_{-ij}) \leq 0$ or $\partial_{ij} v_j^x(g_{ij}, g_{-ij}) \leq 0]$,
4. or $g_{ij} = 1$ and $[\partial_{ij} v_i^x(g_{ij}, g_{-ij}) \geq 0$ and $\partial_{ij} v_j^x(g_{ij}, g_{-ij}) \geq 0]$.

These conditions involve a finite number of equalities and of inequalities with semi-algebraic mappings, thus $\mathcal{M}$ is a semi-algebraic set, as announced.

Second, we show that the mapping $\tilde{\eta}$ is semi-algebraic. From its definition and the definition of η (in Eq. (1)), this is equivalent to say that for every $i \in N$, the mapping

$$(x, g) \in \mathcal{M} \mapsto \varphi\ \Big(\ \gamma \mapsto v_i^x(\gamma) + \sum_{j \neq i} \Big(\ \partial_{ij} v_i^x(g_{ij}, g_{-ij}) - \partial_{ij} v_i^x(g_{ij}, g_{-ij}^0)\ \Big)\ (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij}\ \Big)$$

is semi-algebraic. We can see directly that this is the case, since each coefficient of the polynomial function

$$\gamma \in \mathbb{G} \mapsto v_i^x(\gamma) + \sum_{j \neq i} \big(\partial_{ij} v_i^x(g_{ij}, g_{-ij}) - \partial_{ij} v_i^x(g_{ij}, g_{-ij}^0)\big)(\gamma_{ij} - g_{ij}) + \sum_{j \neq i} g_{ij}\gamma_{ij}$$

is a polynomial function of (x, g), for every $(x, g) \in \mathcal{M}$. Now, the following diagram summarizes the situation:

$$\begin{tikzcd}
\mathcal{P} \arrow[d, shift right, "\pi"'] \arrow[d, shift left, "\eta"] & \mathcal{P}_{\mathcal{U}} \arrow[l, hook'] \arrow[r, "\tilde{\varphi}"] \arrow[d, shift right, "\pi_{|\mathcal{P}_{\mathcal{U}}}"'] \arrow[d, shift left, "\eta_{|\mathcal{P}_{\mathcal{U}}}"] & \mathcal{M} \arrow[d, shift right, "\tilde{\pi}"'] \arrow[d, shift left, "\tilde{\eta}"] \\
\mathcal{F} & \mathcal{U} \arrow[l, hook'] \arrow[r, "\varphi"'] & \mathbb{R}^m
\end{tikzcd}$$

where $\tilde{\pi}$ and $\tilde{\eta}$ are the "transportations" of $\pi_{|\mathcal{P}_{\mathcal{U}}}$ and $\eta_{|\mathcal{P}_{\mathcal{U}}}$ from $\mathcal{M}$ to $\mathbb{R}^m$ by the homeomorphisms φ and $\tilde{\varphi} : \mathcal{P}_{\mathcal{U}} \to \mathcal{M}, (v, g) \mapsto (\varphi(v), g)$. With this last notation, we can see that $\tilde{\pi} = \varphi \circ \pi_{|\mathcal{P}_{\mathcal{U}}} \circ \tilde{\varphi}^{-1}$ (which is the usual projection from $\mathcal{M}$ to $\mathbb{R}^m$) and $\tilde{\eta} = \varphi \circ \eta_{|\mathcal{P}_{\mathcal{U}}} \circ \tilde{\varphi}^{-1}$. We can remark that $\tilde{\eta}$ is a semi-algebraic homeomorphism from $\mathcal{M}$ to $\mathbb{R}^m$, since both $\eta_{|\mathcal{P}_{\mathcal{U}}}$ and $\tilde{\varphi}$ are homeomorphisms. This implies (see Bochnak et al. (1998), Theorem 2.8.8.) that the dimensions of $\mathcal{M}$ and of $\mathbb{R}^m$ coincide, i.e. that $\dim(\mathcal{M}) = m$, which ends the proof of Step 1.

Step 2: Extensions of $\tilde{\pi}$ and $\tilde{\eta}$ to the compactifications of $\mathcal{M}$ and $\mathbb{R}^m$.

For every $(x, g) \in \mathcal{M}$, the network g is pairwise stable in the society v^x if and only if $(x, g) \in \tilde{\pi}^{-1}(x)$. Thus, to finish the proof of Theorem 3.1, we need informations on the cardinal of $\tilde{\pi}^{-1}(x)$, which can be done by using topological degree. The definition of topological degree that we use (see Appendix A.2) is applied to mappings defined on spheres (which avoids boundary issues). However, $\tilde{\pi}$ is not defined on a sphere (but on $\mathcal{M}$), that is why we try to extend $\tilde{\pi}$ on the compactification of $\mathcal{M}$, which will be proved to be homeomorphic to the m-dimensional sphere.

Since $\mathcal{M}$ is homeomorphic to $\mathbb{R}^m$ (the homeomorphism being $\tilde{\eta}$), $\mathcal{M}$ is a noncompact locally compact Hausdorff space and thus, it admits a compactification $\mathcal{M}^*$ (see Appendix A.2 for some reminders about the notion of Alexandroff one-point compactification of a topological space). From Corollary 2.1 and from stability assumption in Definition 3.1, $\pi_{|\mathcal{P}_{\mathcal{U}}}$ and $\eta_{|\mathcal{P}_{\mathcal{U}}}$ are proper mappings. In particular, since φ and $\tilde{\varphi}$ are homeomorphisms, $\tilde{\pi}$ and $\tilde{\eta}$ are also proper mappings. Thus, from Proposition A.4 (see Appendix A.2), these mappings admit (unique) continuous extensions $\tilde{\pi}^* : \mathcal{M}^* \to (\mathbb{R}^m)^*$ and $\tilde{\eta}^* : \mathcal{M}^* \to (\mathbb{R}^m)^*$ (we can notice that $\tilde{\eta}^*$ is also a homeomorphism). Moreover, we recall that $(\mathbb{R}^m)^*$ is also homeomorphic to the m-sphere $\mathbb{S}^m$ (where the inverse of the stereographic projection, denoted $s : (\mathbb{R}^m)^* \to \mathbb{S}^m$, is such a homeomorphism).

The following diagram summarizes the situation:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, hook] \arrow[d, shift right, "\tilde{\pi}"'] \arrow[d, shift left, "\tilde{\eta}"] & \mathcal{M}^* \arrow[r, "s \circ \tilde{\eta}^*"] \arrow[d, shift right, "\tilde{\pi}^*"'] \arrow[d, shift left, "\tilde{\eta}^*"] & \mathbb{S}^m \arrow[d, shift right, "\hat{\pi}"'] \arrow[d, shift left, "\hat{\eta}"] \\
\mathbb{R}^m \arrow[r, hook] & (\mathbb{R}^m)^* \arrow[r, "s"'] & \mathbb{S}^m
\end{tikzcd}$$

where

$$\hat{\pi} = s \circ \tilde{\pi}^* \circ (s \circ \tilde{\eta}^*)^{-1} = s \circ \tilde{\pi}^* \circ (\tilde{\eta}^*)^{-1} \circ s^{-1}$$

$$\hat{\eta} = s \circ \tilde{\eta}^* \circ (s \circ \tilde{\eta}^*)^{-1} = s \circ \mathrm{id}_{(\mathbb{R}^m)^*} \circ s^{-1} = \mathrm{id}_{\mathbb{S}^m}$$

and where $s \circ \tilde{\eta}^*$ is a homeomorphism from $\mathcal{M}^*$ to $\mathbb{S}^m$.

Step 3: $\tilde{\pi}^*$ (resp. $\hat{\pi}$) and $\tilde{\eta}^*$ (resp. $\hat{\eta}$) are homotopic.

We recall that from Corollary 2.1, there exists some proper homotopy $H_{|\mathcal{P}_{\mathcal{U}}}$ between $\pi_{|\mathcal{P}_{\mathcal{U}}}$ and $\eta_{|\mathcal{P}_{\mathcal{U}}}$. Using this homotopy, we can deduce an homotopy between $\tilde{\pi}$ and $\tilde{\eta}$. Indeed, the map

$$\begin{cases} \tilde{H} : & [0, 1] \times \mathcal{M} & \to & \mathbb{R}^m \\ & (t, (x, g)) & \mapsto & \varphi(H_{|\mathcal{P}_{\mathcal{U}}}(t, (\tilde{\varphi}^{-1}(x, g)))) \end{cases}$$

is continuous (by composition) and we obtain that

$$\begin{aligned} \tilde{H}(0, x, g) &= \tilde{\varphi}(H_{|\mathcal{P}_{\mathcal{U}}}(0, \tilde{\varphi}^{-1}(x, g))) = \tilde{\varphi}(\eta|_{\mathcal{P}_{\mathcal{U}}}(\tilde{\varphi}^{-1}(x, g))) \\ &= \tilde{\eta}(x, g), \end{aligned}$$

by definition of $\tilde{\eta}$, and similarly that

$$\begin{aligned} \tilde{H}(1, x, g) &= \tilde{\varphi}(H_{|\mathcal{P}_{\mathcal{U}}}(1, \tilde{\varphi}^{-1}(x, g))) = \tilde{\varphi}(\pi|_{\mathcal{P}_{\mathcal{U}}}(\tilde{\varphi}^{-1}(x, g))) \\ &= \tilde{\pi}(x, g), \end{aligned}$$

by definition of $\tilde{\pi}$. We can notice that, as $H_{|\mathcal{P}_{\mathcal{U}}}$ is a proper map, $\tilde{H}$ is also proper.

Now, let us define the following map:

$$\begin{cases} \tilde{H}^* : & [0, 1] \times \mathcal{M}^* & \to & (\mathbb{R}^m)^* \\ & (t, (x, g)) & \mapsto & \begin{cases} \tilde{H}(t, (x, g)) & \text{if} & (x, g) \neq \infty \\ \infty & \text{otherwise} & \end{cases} \end{cases}$$

and let us show that $\tilde{H}^*$ is an homotopy between $\tilde{\pi}^*$ and $\tilde{\eta}^*$. First, we can remark that

$$\tilde{H}^*(0, x, g) = \begin{cases} \tilde{H}(0, x, g) = \tilde{\eta}(x, g) \\ \quad = \tilde{\eta}^*(x, g) & \text{if} \quad (x, g) \neq \infty \\ \infty = \tilde{\eta}^*(\infty) & \text{otherwise} \end{cases}$$

and similarly that

$$\tilde{H}^*(1, x, g) = \begin{cases} \tilde{H}(1, x, g) = \tilde{\pi}(x, g) \\ \quad = \tilde{\pi}^*(x, g) & \text{if} \quad (x, g) \neq \infty\ . \\ \infty = \tilde{\pi}^*(\infty) & \text{otherwise} \end{cases}$$

This homotopy can be "transported", using the homeomorphisms $s \circ \tilde{\eta}^*$ and s, to an homotopy $\hat{H}$ between $\hat{\pi}$ and $\hat{\eta}$, which is defined in the following way:

$$\begin{cases} \hat{H} : & [0, 1] \times \mathbb{S}^m & \to & \mathbb{S}^m \\ & (t, x) & \mapsto & s(\tilde{H}^*(t, (s \circ \tilde{\eta}^*)^{-1}(x))). \end{cases}$$

Now, we only have to prove that $\tilde{H}^*$ is a continuous map: let $O \subset (\mathbb{R}^m)^*$ be an open subset in $(\mathbb{R}^m)^*$ and let us prove that $(\tilde{H}^*)^{-1}(O)$ is open in $[0, 1] \times \mathcal{M}^*$ (again, see Appendix A.2 for some reminders about compactifications of topological spaces).

First, if $\infty \notin O$ and O is open in $\mathbb{R}^m$, then $(\tilde{H}^*)^{-1}(O) = \tilde{H}^{-1}(O)$ is open in $[0, 1] \times \mathcal{M}$ (by continuity of $\tilde{H}$), thus open in $[0, 1] \times \mathcal{M}^*$.

Second, if $O = \{\infty\} \cup K^c$ (where K is compact for the induced topology in $\mathbb{R}^m$), then

$$(\tilde{H}^*)^{-1}(O) = ([0, 1] \times \{\infty\}) \cup \tilde{H}^{-1}(K)^c.$$

To prove that this set is open in $[0,1]\times\mathcal{M}^*$, let us consider an element $(t,(x,g))\in(\tilde{H}^*)^{-1}(O)$ and let us find an open neighborhood of $(t,(x,g))$ which is included in $(\tilde{H}^*)^{-1}(O)$. We prove that there exists $\epsilon>0$ such that

$$V_\epsilon=(]t-\epsilon,t+\epsilon[\cap[0,1])\times(L_\epsilon^c\cup\{\infty\})$$

is the neighborhood we are looking for, where

$$\begin{aligned}L_\epsilon &:= \{(x',g')\in\mathcal{M}:\exists t'\in[0,1]\cap[t-\epsilon,t+\epsilon]:\\ &\quad (t',(x',g'))\in\tilde{H}^{-1}(K)\}\\ &=\{(x',g')\in\mathcal{M}:\exists t'\in[0,1]\cap[t-\epsilon,t+\epsilon]:\\ &\quad \tilde{H}(t',(x',g'))\in K\}.\end{aligned}$$

We proceed in three steps:

- First, we show that $L_\epsilon\subset\mathcal{M}$ is compact. If (x^k,g^k) is sequence in L_ϵ, by definition, for every integer k, there exists $t^k\in[0,1]\cap[t-\epsilon,t+\epsilon]$ such that $(t^k,(x^k,g^k))\in\tilde{H}^{-1}(K)$, which is compact since K is compact and $\tilde{H}$ is a proper. Thus, $(t^k,(x^k,g^k))$ admits some subsequence converging to some $(\bar{t},(\bar{x},\bar{g}))\in\tilde{H}^{-1}(K)$, and finally $(\bar{x},\bar{g})\in L_\epsilon$ since $\bar{t}\in[0,1]\cap[t-\epsilon,t+\epsilon]$.
- Second, let us prove there exists $\epsilon>0$ such that $V_\epsilon=(]t-\epsilon,t+\epsilon[\cap[0,1])\times(L_\epsilon^c\cup\{\infty\})$ contains (t,x,g). This is directly verified if $(x,g)=\infty$. By contradiction, if it is not the case when $(x,g)\neq\infty$, then, in particular (choosing $\epsilon=\frac{1}{k}$ for every integer $k\geq1$), we get that $(x,g)\in L_{\frac{1}{k}}$, for every $k\geq1$.

  This means that there exists a sequence t^k converging to t such that for every $k\geq1$, $\tilde{H}(t^k,(x,g))\in K$. Passing to the limit, from the continuity of $\tilde{H}$ and the compactness of K (which is closed for the considered topology), we get that $\tilde{H}(t,(x,g))\in K$, which contradicts the fact that $(t,(x,g))\in(\tilde{H}^*)^{-1}(O)=([0,1]\times\{\infty\})\cup\tilde{H}^{-1}(K)^c$ (since this implies in particular that $(t,(x,g))\in\tilde{H}^{-1}(K)^c$, i.e. that $\tilde{H}(t,(x,g))\notin K$).
- Third, let us prove that $V_\epsilon\subset(\tilde{H}^*)^{-1}(O)$. If $(t',(x',g'))\in V_\epsilon$ and $(x',g')=\infty$, then we obtain directly that $(t',(x',g'))\in(\tilde{H}^*)^{-1}(O)$. Moreover, if $(x',g')\neq\infty$ and $(t',(x',g'))\in(]t-\epsilon,t+\epsilon[\cap[0,1])\times L_\epsilon^c$, then by definition of L_ϵ, for every $t''\in[0,1]\cap[t-\epsilon,t+\epsilon]$, we get that $(t'',(x',g'))\notin\tilde{H}^{-1}(K)$, and in particular, that $\tilde{H}(t',(x',g'))\notin K$, which implies that $(t',(x',g'))\in(\tilde{H}^*)^{-1}(O)$ and ends the proof.

Step 4: The topological degree of $\hat{\pi}$ is equal to 1, and $\tilde{\pi}$ is surjective (which implies that every society $v\in\mathcal{U}$ admits a pairwise stable network).

From Step 3, $\hat{\pi}:\mathbb{S}^m\to\mathbb{S}^m$ and $\hat{\eta}:\mathbb{S}^m\to\mathbb{S}^m$ are homotopic. Thus from homotopy invariance of topological degree (see Appendix A.2, property (ii)), we get that $\deg(\hat{\pi})=\deg(\hat{\eta})$. However, from Step 2, we know that $\hat{\eta}=\mathrm{id}_{\mathbb{S}^m}$, thus $\deg(\hat{\eta})=1$ (see Appendix A.2, property (i)). In particular, $\deg(\hat{\pi})=1$, which implies that $\hat{\pi}$ is a surjective mapping (see Appendix A.2, property (iii)). This also implies that $\tilde{\pi}$ is surjective: indeed, for every $x\in\mathbb{R}^m$, $s(x)\in\mathbb{S}^m$, thus there exists $y\in\mathbb{S}^m$ such that $\hat{\pi}(y)=s(x)$, which gives $x=(s^{-1}\circ\hat{\pi})(y)$, thus $x=\tilde{\pi}((s\circ\tilde{\eta})^{-1}(y))$ (by definition of $\hat{\pi}$). Remark that for every $v\in\mathcal{U}$, we also have that there exists $y\in\mathbb{S}^m$ such that $v=(\pi_{|\mathcal{P}_\mathcal{U}}\circ\tilde{\varphi}^{-1}\circ(s\circ\tilde{\eta})^{-1})(y)$, which proves that $\pi_{|\mathcal{P}_\mathcal{U}}$ is also surjective; equivalently this means that every society $v\in\mathcal{U}$ admits a pairwise stable network.

In particular, we recover (but for a smaller class of payoff functions) the result of Bich and Morhaim (2020), which proves the existence of a pairwise stable network for every continuous society $v=(v_1,\dots,v_n)$ such that v_i is quasiconcave with respect to g_{ij}, for every link ij.

Step 5: There exists a generic semi-algebraic subset G of $\mathbb{R}^m$ such that for every $x\in G$, $\tilde{\pi}^{-1}(x)$ is nonempty and finite.

From Step 4, we know that $\tilde{\pi}:\mathcal{M}\to\mathbb{R}^m$ is a surjective semi-algebraic continuous mapping, and from Step 1, we know that $\dim(\mathcal{M})=m$. Thus, from Theorem A.2 (see Appendix A.1), we obtain the existence of a generic semi-algebraic subset G (now fixed) of $\mathbb{R}^m$ such that for every $x\in G$, $\tilde{\pi}^{-1}(x)$ is nonempty and finite.

Step 6: For every $x\in G$, the set of pairwise stable networks of $v^x=\varphi^{-1}(x)$ is nonempty and finite (we denote K_x its cardinal).

From Step 5, we know that for every $x\in G$, $\tilde{\pi}^{-1}(x)$ is nonempty and finite. However, since $\tilde{\varphi}$ is a homeomorphism, we obtain that $\tilde{\varphi}^{-1}(\tilde{\pi}^{-1}(x))=\pi_{|\mathcal{P}_\mathcal{U}}^{-1}(\varphi^{-1}(x))=\pi_{|\mathcal{P}_\mathcal{U}}^{-1}(v^x)$ is also nonempty and finite (with the same cardinal as $\tilde{\pi}^{-1}(x)$). It remains to notice that, by definition, $\pi_{|\mathcal{P}_\mathcal{U}}^{-1}(v^x)=\{(v^x,g)\in\mathcal{P}_\mathcal{U}: g \text{ is a pairwise stable network of } v^x\}$.

Step 7: For every $x\in G$, there exists an open neighborhood $\hat{V}_x$ of $s(x)$ such that $\hat{\pi}^{-1}(\hat{V}_x)$ is a union of pairwise disjoint open sets $(\hat{V}_x^k)_{k\in\mathcal{K}}$ (where $\mathcal{K}$ is a finite set of cardinal K_x) such that for every $k\in\mathcal{K}$, $\hat{\pi}_{|\hat{V}_x^k}$ is a homeomorphism between $\hat{V}_x^k$ and $\hat{V}_x$.

Let $x\in G$, and let C be the connected component of G containing x. From Step 6, we know that $\tilde{\pi}^{-1}(x)$ is nonempty and finite, so we obtain that $(\tilde{\pi}^*)^{-1}(x)=(\tilde{\pi})^{-1}(x)$ and $\hat{\pi}^{-1}(s(x))$ are also non empty and finite, with the same cardinal as $\tilde{\pi}^{-1}(x)$ (since s is a homeomorphism), i.e. of cardinal K_x. Moreover, from Theorem A.2 (see Appendix A.1), there exists an open neighborhood V_x of x such that $\tilde{\pi}^{-1}(V_x)$ is a union of pairwise disjoint open sets $(V_x^k)_{k\in\mathcal{K}}$ (where $\mathcal{K}$ is a finite set) such that for every $k\in\mathcal{K}$, $\tilde{\pi}_{|V_x^k}$ is a homeomorphism between V_x^k and V_x. Denote by $\hat{V}_x$ the image by the homeomorphism s of V_x and for every $k\in\mathcal{K}$, denote by $\hat{V}_x^k$ the image by the homeomorphism $s\circ\tilde{\eta}^*$ of V_x^k. Since s is a homeomorphism, $\hat{V}_x$ is an open neighborhood of $s(x)$. Also, since $s\circ\tilde{\eta}^*$ is a homeomorphism, $\hat{\pi}^{-1}(\hat{V}_x)=\bigcup_{k\in\mathcal{K}}\hat{V}_x^k$, and $(\hat{V}_x^k)_{k\in\mathcal{K}}$ is a family of pairwise disjoint open sets such that for every $k\in\mathcal{K}$, $\hat{\pi}_{|\hat{V}_x^k}$ is a homeomorphism between $\hat{V}_x^k$ and $\hat{V}_x$.

Step 8: For every $x\in G$, the society $v^x=\varphi^{-1}(x)$ admits an odd number of pairwise stable networks.

We want to prove that for every $x\in G$, the integer K_x (introduced in Step 6) is odd. From Step 7, since $\hat{\pi}^{-1}(s(x))$ is a finite subset of $\bigcup_{k\in\mathcal{K}}\hat{V}_x^k$ (with cardinal equal to $K_x=\mathrm{card}(\mathcal{K})$) and since for every $k\in\mathcal{K}$, $\hat{\pi}_{|\hat{V}_x^k}$ is a homeomorphism between $\hat{V}_x^k$ and $\hat{V}_x$, we obtain that

$$\hat{\pi}^{-1}(s(x))=\{y_x^k:k\in\mathcal{K}\}$$

for some $y_x^k\in\hat{V}_x^k$, $k\in\mathcal{K}$. Also, from property (v) of local degree (see Appendix A.2), we obtain that $\deg(\hat{\pi})_{|_{y_x^k}}$ is equal to 1 or -1, for every $k\in\mathcal{K}$. Now, from additivity of the topological degree with respect to local degrees (see Appendix A.2, property (vi)), we obtain (modulo 2) that

$$1=\deg(\hat{\pi})=\sum_{k\in\mathcal{K}}\deg(\hat{\pi})_{|_{y_x^k}}=\sum_{k\in\mathcal{K}}1=\mathrm{card}(\mathcal{K})=K_x\ [2],$$

i.e. that K_x is odd.

Step 9: The general case.

We now return to the general case. Recall that for every $i\in N$, $\varphi_i:\mathcal{U}_i\to\mathbb{R}^{m_i}$ is the topological embedding which associates to any polynomial payoff function in $\mathcal{U}_i$ its coefficients in $\mathbb{R}^{m_i}$, that $\mathbb{R}_{\delta_i}[g]$ is the vector space of polynomial functions whose degree is less or equal to $\delta_i \in \mathbb{N}$, and that $\varphi = \times_{i\in N}\varphi_i : \mathcal{U} \to \mathbb{R}^m$ (with $m = \sum_{i\in N} m_i$).

Step 9.1: Direct sum decomposition of $\mathbb{R}^{m_i}$ and cancellation of the "$\mathcal{A}_i$ part" of $\mathcal{S}_i$ ($i \in N$) — application of the first part of the proof.

For every $i \in N$, we consider the linear subspace $\widetilde{\mathbb{R}}_{\delta_i}[g]$ of $\mathbb{R}_{\delta_i}[g]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[g]$, except the ones in $\mathcal{A}_i$, i.e.

$$\widetilde{\mathbb{R}}_{\delta_i}[g] := \mathrm{Span}\big(\{g^k : k \in \mathbb{N}^L, \deg(k) \le \delta_i\} - (\{g_{ij} : j \neq i\} \cup \{1\})\big).$$

By definition, notice that $\mathbb{R}_{\delta_i}[g] = \widetilde{\mathbb{R}}_{\delta_i}[g] \oplus \mathcal{A}_i$, thus that

$$\varphi_i(\mathbb{R}_{\delta_i}[g]) = \mathbb{R}^{m_i} = \varphi_i(\widetilde{\mathbb{R}}_{\delta_i}[g]) \oplus \varphi_i(\mathcal{A}_i).$$

Moreover, denote by $\pi_{-\mathcal{A}_i}$ the linear projection from the set $\mathbb{R}^{m_i} = \varphi_i(\widetilde{\mathbb{R}}_{\delta_i}[g]) \oplus \varphi_i(\mathcal{A}_i)$ (of coefficients of polynomial functions in $\mathbb{R}_{\delta_i}[g]$) to the set $\varphi_i(\widetilde{\mathbb{R}}_{\delta_i}[g])$ (of coefficients of polynomial functions in $\widetilde{\mathbb{R}}_{\delta_i}[g]$), and consider the set $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$.

Step 9.2: Semi-algebraic decomposition of $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$ ($i \in N$) — application of the first part of the proof.

For every $i \in N$, since $\mathcal{S}_i = \varphi(\mathcal{U}_i)$ is semi-algebraic (from semi-algebraicity assumption in Definition 3.1) and since $\pi_{-\mathcal{A}_i}$ is a semi-algebraic mapping, from Tarski–Seidenberg's theorem (see Proposition A.2 in Appendix A.1), the set $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$ is also semi-algebraic. In particular, from the decomposition result for semi-algebraic sets (see Definition A.4 in Appendix A.1),

$$\pi_{-\mathcal{A}_i}(\mathcal{S}_i) = \bigcup_{k=1}^{\kappa_i} T_k^i$$

(the union being disjoint), where $\kappa_i \in \mathbb{N}$ and where for every $k \in \{1, \dots, \kappa_i\}$, $T_k^i \subset \pi_{-\mathcal{A}_i}(\mathcal{S}_i)$ is semi-algebraic and homeomorphic to $]0,1[^{d_k^i}$ (for some $d_k^i \in \mathbb{N}$).

Thus,

$$\begin{aligned}\pi_{-\mathcal{A}_1}(\mathcal{S}_1) \times \cdots \times \pi_{-\mathcal{A}_n}(\mathcal{S}_n) &= \bigcup_{k=1}^{\kappa_1} T_k^1 \times \cdots \times \bigcup_{k=1}^{\kappa_n} T_k^n \\ &= \bigcup_{(k_1,\dots,k_n)\in\Lambda} (T_{k_1}^1 \times \cdots \times T_{k_n}^n), \end{aligned} \tag{7}$$

where $\Lambda := \prod_{i\in N}\{1, \dots, \kappa_i\}$. In particular, for every $\lambda = (k_1, \dots, k_n) \in \Lambda$, the set

$$\varphi_1^{-1}(T_{k_1}^1) \times \cdots \times \varphi_n^{-1}(T_{k_n}^n)$$

is homeomorphic to $]0,1[^{d_\lambda}$, for some $d_\lambda \in \mathbb{N}$.

Step 9.3: Addition of $\mathcal{A}_i$ to $\varphi_i^{-1}(T_{k_i}^i)$ ($i \in N$) — application of the first part of the proof.

For every $\lambda = (k_1, \dots, k_n) \in \Lambda$, we now consider the set

$$\mathcal{V}^\lambda := \big(\varphi_1^{-1}(T_{k_1}^1) \times \cdots \times \varphi_n^{-1}(T_{k_n}^n)\big) + \prod_{i=1}^n \mathcal{A}_i = \prod_{i=1}^n (\varphi_i^{-1}(T_{k_i}^i) + \mathcal{A}_i)$$

and we define $\mathcal{V}_i^{k_i} = \varphi_i^{-1}(T_{k_i}^i) + \mathcal{A}_i$, for every $i \in N$.

Step 9.4: For every $\lambda \in \Lambda$, $\mathcal{V}^\lambda$ is regular and homeomorphic to some Euclidean space — application of the first part of the proof.

We can notice that for every $\lambda = (k_1, \dots, k_n) \in \Lambda$ and every $i \in N$:

1. Stability assumption (in Definition 3.1) is easily seen to be satisfied for $\mathcal{V}_i^{k_i}$.
2. Semi-algebraicity assumption (in Definition 3.1) is satisfied for $\mathcal{V}_i^{k_i}$ because, from Corollary A.1 (see Appendix A.1),

   $$\varphi_i(\mathcal{V}_i^{k_i}) = T_{k_i}^i + \varphi_i(\mathcal{A}_i)$$

   is a semi-algebraic set, since $T_{k_i}^i$ and $\varphi_i(\mathcal{A}_i)$ are both semi-algebraic sets.
3. $\mathcal{V}_i^{k_i}$ is included in $\mathcal{F}_i$ because $T_{k_i}^i \subset \pi_{-\mathcal{A}_i}(\mathcal{S}_i) = \pi_{-\mathcal{A}_i}(\varphi_i(\mathcal{U}_i))$, which implies that $\varphi_i^{-1}(T_{k_i}^i) \subset (\varphi_i^{-1} \circ \pi_{-\mathcal{A}_i} \circ \varphi_i)(\mathcal{U}_i) \subset \mathcal{U}_i$.
4. $\mathcal{V}_i^{k_i}$ is homeomorphic to $\varphi_i^{-1}(T_{k_i}^i) \times \mathcal{A}_i$,[19] thus homeomorphic to $(0,1)^{d_{k_i}^i}$, for some $d_{k_i}^i \in \mathbb{N}$, which implies that $\mathcal{V}^\lambda$ is itself homeomorphic to some Euclidean space.[20]

Thus, from the first part of the proof (Steps 1–8), we know that for every $\lambda = (k_1, \dots, k_n) \in \Lambda$, there exists a generic semi-algebraic subset G^λ of $\mathcal{S}^\lambda := \varphi(\mathcal{V}^\lambda)$ such that for every $x \in G^\lambda$, the society $v^x = \varphi^{-1}(x)$ has an odd number of pairwise stable networks. This implies that for every x in

$$G := \bigcup_{\substack{\lambda=(k_1,\dots,k_n)\in\Lambda \\ \forall i\in N, T_{k_i}^i \text{ is open in } \pi_{-\mathcal{A}_i}(\mathcal{S}_i)}} G^\lambda,$$

the society $v^x = \varphi^{-1}(x)$ has an odd number of pairwise stable networks.

To finish, we need to show that G is indeed generic in $\mathcal{S} = \varphi(\mathcal{U})$.

Step 9.5: The family $(\mathcal{S}^\lambda)_{\lambda=(k_1,\dots,k_n)\in\Lambda}$ forms a covering of $\mathcal{S}$ — genericity of G in $\mathcal{S}$.

We get that

$$\begin{aligned}\bigcup_{\lambda\in\Lambda}\mathcal{S}^\lambda &= \bigcup_{\lambda\in\Lambda}\varphi(\mathcal{V}^\lambda) = \bigcup_{\lambda\in\Lambda}\varphi(\prod_{i=1}^n(\varphi_i^{-1}(T_{k_i}^i) + \mathcal{A}_i)) \\ &= \bigcup_{k\in\Lambda}\prod_{i=1}^n(\varphi_i(\varphi_i^{-1}(T_{k_i}^i) + \mathcal{A}_i)) \\ &= \bigcup_{k\in\Lambda}(\prod_{i=1}^n(T_{k_i}^i) + \varphi_i(\mathcal{A}_i)) = \bigcup_{k\in\Lambda}(\prod_{i=1}^n T_{k_i}^i) + (\prod_{i=1}^n \varphi_i(\mathcal{A}_i)) \\ &= \pi_{-\mathcal{A}_1}(\mathcal{S}_1) \times \cdots \times \pi_{-\mathcal{A}_n}(\mathcal{S}_n) + (\prod_{i=1}^n \varphi_i(\mathcal{A}_i)) \text{ (from Eq. (7))} \\ &= (\pi_{-\mathcal{A}_1}(\mathcal{S}_1) + \varphi_1(\mathcal{A}_1)) \times \cdots \times (\pi_{-\mathcal{A}_n}(\mathcal{S}_n) + \varphi_n(\mathcal{A}_n)) \\ &= (\pi_{-\mathcal{A}_1}(\varphi_1(\mathcal{U}_1)) + \varphi_1(\mathcal{A}_1)) \times \cdots \\ &\quad \times (\pi_{-\mathcal{A}_n}(\varphi_n(\mathcal{U}_n)) + \varphi_n(\mathcal{A}_n)) \\ &= \varphi_1(\mathcal{U}_1) \times \cdots \times \varphi_n(\mathcal{U}_n) = \varphi(\mathcal{U}).\end{aligned}$$

Step 9.6: Inclusion of $\mathcal{S} - G$ in some particular set — genericity of G in $\mathcal{S}$.

From Step 9.5, we get that the complement of G in $\mathcal{S}$ is included in

$$\bigcup_{\lambda\in\Lambda}(\mathcal{S}^\lambda - G^\lambda) \cup \widetilde{\mathbb{S}},$$

where

$$\widetilde{\mathbb{S}} := \bigcup_{\substack{\lambda=(k_1,\dots,k_n)\in\Lambda \\ \exists i\in N, T_{k_i}^i \text{ is not open in } \pi_{-\mathcal{A}_i}(\mathcal{S}_i)}} \mathcal{S}^\lambda.$$

Indeed, let $x \in \mathcal{S}$ such that $x \notin G$. Since $(\mathcal{S}^\lambda)_{\lambda\in\Lambda}$ forms a covering of $\mathcal{S}$, there exists $\ell \in \Lambda$ such that $x \in \mathcal{S}^\ell$. Moreover, since $x \notin G$, this means that: (i) either there exists $i \in N$ such that $T_{k_i}^i$ is

[19] Indeed, the mapping $f : \varphi_i^{-1}(T_{k_i}^i) \times \mathcal{A}_i \to \varphi_i^{-1}(T_{k_i}^i) + \mathcal{A}_i, (v_i, a_i) \mapsto v_i + a_i$ is such a homeomorphism, since the vector subspace of $\mathbb{R}_{\delta_i}[g]$ generated by $\varphi_i^{-1}(T_{k_i}^i)$ and the vector subspace $\mathcal{A}_i$ of $\mathbb{R}_{\delta_i}[g]$ are in direct sum.

[20] This condition is important in order to be able to use what has been done from Step 1 to Step 8.

not open in $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$, which implies that $x \in \widetilde{\mathbb{S}}$; (ii) or for every $i \in N$, $T^i_{k_i}$ is open in $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$, which implies that $x \notin G^\ell$, thus that $x \in \mathcal{S}^\ell - G^\ell \subset \bigcup_{\lambda\in\Lambda}(\mathcal{S}^\lambda - G^\lambda)$. Thus,

$$\mathcal{S} - G \subset \bigcup_{\lambda\in\Lambda}(\mathcal{S}^\lambda - G^\lambda) \cup \widetilde{\mathbb{S}}$$

Step 9.7: The dimension of $\mathcal{S} - G$ is strictly less than the dimension of $\mathcal{S}$ – genericity of G in $\mathcal{S}$.

From Step 9.6, notice that the dimension of $\bigcup_{\lambda\in\Lambda}(\mathcal{S}^\lambda - G^\lambda)$ is strictly less than $\dim(\mathcal{S})$, because for every $\lambda \in \Lambda$, $\dim(\mathcal{S}^\lambda - G^\lambda) < \dim(\mathcal{S}^\lambda) \le \dim(\mathcal{S})$ (the first strict inequality being a consequence of G^λ generic in $\mathcal{S}^\lambda$).

Also, the dimension of $\widetilde{\mathbb{S}}$ is strictly less than $\dim(\mathcal{S})$. Indeed, for every $\lambda = (k_1, \ldots, k_n) \in \lambda$ such that there exists $j \in N$ such that $T^j_{k_j}$ is not open in $\pi_{-\mathcal{A}_j}(\mathcal{S}_j)$, we get $\dim(T^j_{k_j}) < \dim(\pi_{-\mathcal{A}_j}(\mathcal{S}_j))$ (see the reminders about semi-algebraic sets and in particular Definition A.4). This implies that for every such $\lambda \in \Lambda$:

$$\begin{aligned}
\dim(\mathcal{S}^\lambda) &= \dim(\varphi(\mathcal{V}^\lambda)) = \dim\Big(\prod_{i=1}^{n}(T^i_{k_i}) + \varphi_i(\mathcal{A}_i)\Big) \\
&= \sum_{i=1}^{n}\dim(T^i_{k_i} + \varphi_i(\mathcal{A}_i)) \\
&= \sum_{i=1}^{n}\dim(T^i_{k_i}) + \sum_{i=1}^{n}\dim(\varphi_i(\mathcal{A}_i)) \text{ (since } \forall i \in N, \\
&\quad \text{Span}(T^i_{k_i}) \text{ and } \varphi(\mathcal{A}_i) \text{ are in direct sum)} \\
&< \sum_{i\neq j}\dim(\pi_{-\mathcal{A}_i}(\mathcal{S}_i)) + \dim(\pi_{-\mathcal{A}_j}(\mathcal{S}_j)) + \sum_{i=1}^{n}\dim(\varphi_i(\mathcal{A}_i)) \\
&= \sum_{i=1}^{n}\dim(\pi_{-\mathcal{A}_i}(\mathcal{S}_i) + \varphi_i(\mathcal{A}_i)) \text{ (since } \forall i \in N, \\
&\quad \text{Span}(\pi_{-\mathcal{A}_i}(\mathcal{S}_i)) \text{ and } \varphi_i(\mathcal{A}_i) \text{ are in direct sum)} \\
&= \dim\Big(\prod_{i=1}^{n}(\pi_{-\mathcal{A}_i}(\mathcal{S}_i) + \varphi_i(\mathcal{A}_i))\Big) = \dim\Big(\prod_{i=1}^{n}\mathcal{S}_i\Big) = \dim(\mathcal{S}).
\end{aligned}$$

Finally, this proves that dimension of the complement of G in $\mathcal{S}$ is strictly less than the dimension of $\mathcal{S}$.

Step 9.8: G is open in $\mathcal{S}$ – genericity of G in $\mathcal{S}$.

Since for every $\lambda = (k_1, \ldots, k_n) \in \Lambda$ such that for every $i \in N$, $T^i_{k_i}$ is open in $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$, G^λ is open in $\mathcal{S}$ because G^λ is open in $\mathcal{S}^\lambda$ (from G^λ generic in $\mathcal{S}^\lambda$) and because $\mathcal{S}^\lambda$ is open in $\mathcal{S}$.

Indeed, to understand this last point, recall that $\mathcal{S}^\lambda = \prod_{i\in N}(T^i_{k_i} + \varphi_i(\mathcal{A}_i))$, where for every $i \in N$, $T^i_{k_i}$ is open in $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$ (by definition of the semi-algebraic decomposition of $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$). Thus $T^i_{k_i} + \varphi_i(\mathcal{A}_i)$ is open in $\pi_{-\mathcal{A}_i}(\mathcal{S}_i) + \varphi_i(\mathcal{A}_i) = \mathcal{S}_i + \varphi_i(\mathcal{A}_i)$. Finally, since $\mathcal{U}_i = \mathcal{U}_i + \mathcal{A}_i$ (from stability assumption in Definition 3.1), we get that $\mathcal{S}_i = \mathcal{S}_i + \varphi_i(\mathcal{A}_i)$, thus that $T^i_{k_i} + \varphi_i(\mathcal{A}_i)$ is open in $\mathcal{S}_i$, which implies that $\mathcal{S}^\lambda$ is open in $\mathcal{S} = \prod_{i\in N}\mathcal{S}_i$.

Finally, we get G is open in $\mathcal{S}$ as an arbitrary union of open subsets of $\mathcal{S}$ (by definition).

This ends the proof of Theorem 3.1.

A.6. Table of notations

The following table sums up some of the notations which are used in this paper (in particular, in the different proofs):

$\mathbb{G}$	*Set of weighted networks*
$\mathcal{F}_i$	*Set of payoff functions of agent i which are concave and C^1 in g_{ij}, for every $j \neq i$ ($\mathcal{F} = \prod_{i\in N}\mathcal{F}_i$)*
$\mathcal{P}$	*Graph of pairwise stable networks, i.e. $\mathcal{P} = \{(v, g) \in \mathcal{F} \times \mathbb{G} : g$ is pairwise stable with respect to $v\}$*
π	*Projection mapping from $\mathcal{P}$ to $\mathcal{F}$*
η	*Bich–Fixary's homeomorphism from $\mathcal{P}$ to $\mathcal{F}$*
$\mathcal{A}_i$	*Set of payoff functions of agent i which are affine in $(g_{ij})_{j\neq i}$ and only depend on those weights*
$\mathbb{R}_{\delta_i}[g]$	*Set of payoff functions of agent i which are polynomial in $g = (g_{ij})_{ij\in L}$ and whose degree is less or equal to $\delta_i \in \mathbb{N}$*
φ_i	*Vector space isomorphism which assigns to each payoff function in $\mathbb{R}_{\delta_i}[g]$ its coefficients in $\mathbb{R}^{m_i}$, $m_i \in \mathbb{N}$ ($m = \sum_{i\in N} m_i$, $\varphi = \times_{i\in N}\varphi_i : \prod_{i\in N}\mathbb{R}_{\delta_i}[g] \to \mathbb{R}^m$)*
$\mathcal{S}_i$	*Set of coefficients of polynomial payoff functions in a subset $\mathcal{U}_i$ of $\mathbb{R}_{\delta_i}[g]$, i.e. $\mathcal{S}_i = \varphi_i(\mathcal{U}_i)$ ($\mathcal{S} = \prod_{i\in N}\mathcal{S}_i$)*
$\widetilde{\mathbb{R}}_{\delta_i}[g]$	*Set of payoff functions of agent i which are polynomial in $g = (g_{ij})_{ij\in L}$, whose degree is less or equal to $\delta_i \in \mathbb{N}$ and with null coefficients on the "$\mathcal{A}_i$ part"*
$\pi_{-\mathcal{A}_i}$	*Linear projection from $\mathbb{R}^{m_i} = \varphi_i(\widetilde{\mathbb{R}}_{\delta_i}[g]) \oplus \varphi_i(\mathcal{A}_i)$ to $\varphi_i(\mathcal{A}_i)$*
$(T^i_k)_{k=1}^{\kappa_i}$	*Semi-algebraic decomposition of $\pi_{-\mathcal{A}_i}(\mathcal{S}_i)$, for some $\kappa_i \in \mathbb{N}$*
$\mathcal{V}_i^{k_i}$	*Minkowski sum of $\varphi_i^{-1}(T^i_{k_i})$ and $\mathcal{A}_i$, i.e. $\mathcal{V}_i^{k_i} = \varphi_i^{-1}(T^i_{k_i}) + \mathcal{A}_i$ ($\mathcal{V}^\lambda = \prod_{i\in N}\mathcal{V}_i^{k_i}$, for every $\lambda = (k_1, \ldots, k_n) \in \Lambda = \prod_{i\in N}\{1, \ldots, \kappa_i\}$)*

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