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# Oddness of the number of Nash equilibria: The case of polynomial payoff functions

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ABSTRACT

In 1971, Wilson (1971) proved that "almost all" finite games have an odd number of mixed Nash equilibria. Since then, several other proofs have been given, but always for mixed extensions of finite games. In this paper, we present a new oddness theorem for large classes of polynomial payoff functions and semi-algebraic sets of strategies. Additionally, we provide some applications to recent models of games on networks such that Patacchini-Zenou's model about juvenile delinquency and conformism (Patacchini and Zenou, 2012), Calvó-Armengol-Patacchini-Zenou's model about social networks in education (Calvó-Armengol et al., 2009), Konig-Liu-Zenou's model about R&D networks (König et al., 2019), Helsley-Zenou's model about social networks and interactions in cities (Helsley and Zenou, 2014).

## 1. Introduction

In 1971, Wilson (1971) proved the so-called oddness theorem: generically, finite (strategic-form) games have an odd number of mixed Nash equilibria. Since then, several other proofs have been given, but always for mixed extensions of finite games (see for example Blume and Zame, 1994; Govindan and McLennan, 2001; Govindan and Wilson, 2001; Harsanyi, 1973; Herings and Peeters, 2001; Mas-Colell, 2008; Pimienta, 2009; Wilson, 1971). The purpose of this paper is to prove the oddness theorem for large classes of polynomial payoff functions and semi-algebraic sets of strategies (a subset $S$ of $\mathbb{R}^m$ is semi-algebraic if it is defined by finitely many polynomial inequalities). For instance, we prove that when payoff functions are polynomial and concave in players' own strategies, and when the sets of strategies are semi-algebraic, then the generic oddness theorem holds. Our result provides new insights on the set of Nash equilibria for many recent models: Patacchini-Zenou's model about juvenile delinquency and conformism (Patacchini and Zenou, 2012), Calvó-Armengol-Patacchini-Zenou's model about social networks in education (Calvó-Armengol et al., 2009), Konig-Liu-Zenou's model about R&D networks (König et al., 2019), Helsley-Zenou's model about social networks and interactions in cities (Helsley and Zenou, 2014), etc.

Our main *oddness theorem* (Theorem 3.1 in Section 3) states that for every subset $\mathcal{U}$ of profiles of polynomial payoff functions with a fixed maximal degree and own-strategy concave,[1] there are generically (in $\mathcal{U}$) an odd number of Nash equilibria, provided that the two following properties are satisfied:

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[1] Concavity (or at least quasiconcavity) of payoff functions $u_i(x_i, x_{-i})$ of each player $i$ with respect to $x_i$ is a standard assumption in the literature when considering conditions for the existence of a Nash equilibrium.

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1. **(Semi-algebraicity assumption).** The set of coefficients of $\mathcal{U}$ is semi-algebraic (we call this first requirement "*semi-algebraicity assumption*"); in particular, this gives meaning to the dimension of $\mathcal{U}$, defined as the (semi-algebraic) dimension of its set of coefficients[2];
2. **(Stability assumption).** Adding to $\mathcal{U}$ the set of profiles of linear payoff functions $(u_1, ..., u_n)$, where $u_i$ depends only on $i$'s strategy, does not modify the (semi-algebraic) dimension of $\mathcal{U}$ (we call this second requirement "*stability assumption*").

In short, the first assumption requires that the coefficients of the polynomial payoff functions in $\mathcal{U}$ are defined with finitely many inequalities involving other polynomials; the intuition of the second assumption is that the set of profiles of payoff functions in $\mathcal{U}$ should contain some linear part which is "large enough". We also provide examples that make us think that both of our assumptions are tight: indeed, if we remove one of the two above assumptions, then it is easy to find games with polynomial and own-strategy concave payoff functions for which oddness theorem fails to be true.

A first difficulty in the proof is that it requires some extension of Kohlberg-Mertens' structure theorem (Kohlberg and Mertens, 1986), which is known to imply Wilson's oddness theorem: considering a fix number $n$ of players and fixing the set of pure strategies of each player, if we denote by

$$\mathcal{N}' = \{(u, \sigma) \in F \times \Sigma : \sigma \text{ is a mixed Nash equilibrium of } u\}$$

the *graph of Nash equilibria* associated to the set $F$ of finite games (where $\Sigma$ is the set of mixed strategy profiles), and by $\pi : \mathcal{N}' \to F$ the projection on the first factor, then Kohlberg-Mertens' theorem states that $\pi$ is properly homotopic to some homeomorphism from $\mathcal{N}'$ to $F$ (which means, intuitively, that this homeomorphism can be continuously deformed into the projection mapping $\pi$), where $F$ is itself homeomorphic to some Euclidean space. In particular, applying topological degree to $\pi$, it can be obtained that the number of mixed Nash equilibria of any finite game $u$ is generically odd.[3] As a matter of fact, our *structure theorem* (Theorem 2.2 in Section 2) is a slight generalization of Predtetchinski's structure theorem (Predtetchinski, 2009, Theorem 2 and Theorem 3), itself an extension of Kohlberg-Mertens' theorem. Roughly, Predtetchinski's theorem states that the space $\mathcal{F}$ of games which are own-strategy concave and which satisfy some differentiability assumption is homeomorphic to its associated graph of Nash equilibria

$$\mathcal{N} = \{(u, x) \in \mathcal{F} \times X : x \text{ is a Nash equilibrium of } u\}$$

(were $X$ is the compact and convex set of strategy profiles), and that the provided homeomorphism $\eta : \mathcal{N} \to \mathcal{F}$ is properly homotopic to the projection mapping $\pi : \mathcal{N} \to \mathcal{F}$. We extend this theorem by showing that this result still holds for any subspace $\mathcal{U}$ of $\mathcal{F}$ which satisfies "stability under affine addition", a condition which requires that $\mathcal{U}$ is closed under addition, for each player $i$, of affine payoff functions which only depend on the strategy $x_i$ of player $i$.

A second difficulty in the proof of our oddness theorem is that, in general, the set $\mathcal{U}$ of profiles of polynomial payoff functions is not homeomorphic to some Euclidean space, and even worse, $\mathcal{U}$ is not always a topological manifold. This implies that there is no clear and tractable definition of topological degree anymore on $\mathcal{U}$, and that the standard proof summarized above in the case of finite games cannot be mimicked here. For example, considering one player and the subspace $\mathcal{U} = \{x \in [0, 1] \mapsto ax^2 + bx : a \leq 0, \text{ and } [b \leq a \text{ or } b \geq -a]\}$ of concave polynomial games, then we observe that $\mathcal{U}$ is homeomorphic to $\{(a, b) \in \mathbb{R}^2 : a \leq 0, \text{ and } [b \leq a \text{ or } b \geq -a]\}$ (its associated set of coefficients), which is not a topological manifold. To address this shortcoming, one of the main ingredients of our proof is to show that if $\mathcal{U}$ is semi-algebraic and satisfies "stability under affine addition", then it can be decomposed into a disjoint union of subspaces $\mathcal{V}^k$ satisfying these two properties, with the additional property that $\mathcal{V}^k$ is homeomorphic to some Euclidean space. Then, we can apply our structure theorem to each $\mathcal{V}^k$, and use techniques similar as the ones used in the case of finite games.[4] Here, semi-algebraicity of $\mathcal{U}$ plays a crucial role for obtaining our decomposition result.

Our paper is organized as follows: in Section 2, we first recall some basic definitions and notations, and state the structure theorem (Theorem 2.2), which is crucial for the next section; in Section 3, we state the oddness theorem (Theorem 3.1), and then provide several examples of applications; in Section 4, after some necessary reminders about real algebraic geometry and about topological degree, we prove the structure theorem and the oddness theorem.

## 2. Structure of the graph of Nash equilibria

First of all, Table 1 sums up some of the notations which are used in this paper (in particular, in the proof of our oddness theorem).

Also, let us recall some mathematical definitions or notations used in this paper. For every sets $A$ and $B$, both $\mathcal{F}(A, B)$ and $B^A$ denote the set of mappings from $A$ to $B$. For every sets $A$ and $B$ such that $A \subset B$, both $B \backslash A$ and $B - A$ denote the relative complement of $A$ in $B$. For every set $A$, both $\text{card}(A)$ and $|A|$ denotes the cardinal of $A$. A correspondence $\Phi$ from a set $A$ to another

[2] Throughout this paper, we simply write "$\mathcal{U}$ is semi-algebraic" instead of "$\mathcal{U}$ satisfies semi-algebraic assumption".

[3] Hereafter is a sketch of the proof: from homotopy invariance of topological degree, the topological degree of $\pi$ is equal to the topological degree of some proper homeomorphism from $\mathcal{N}'$ to $F$ ($F$ being homeomorphic to some Euclidean space), which is itself equal to 1 or $-1$ (from some homeomorphism property of topological degree). From some covering space property, there exists a generic subset $G$ of the set $F$ of finite games such that for every $u \in G$, $\pi^{-1}(u) = \{u\} \times \{\sigma : \sigma \text{ is a mixed Nash equilibrium of } u\}$ is a finite set $\{(u, \sigma^1), \dots, (u, \sigma^k)\}$ (for some integer $k$). The same covering space property ensures that $\pi$ is a local homeomorphism at each $(u, \sigma^p)$, $p \in \{1, \dots, k\}$, hence that the local degree of $\pi$ at $(u, \sigma^p)$ has to be equal to 1 or $-1$. Last, the sum of local degrees of $\pi$ is equal to the degree of $\pi$, which finally ends the proof.

[4] A similar generic oddness result in network formation theory, where Nash equilibrium is replaced by pairwise stability concept, can be found in Bich and Fixary (2022).

 

**Table 1**
Notations.

| | |
|---|---|
| $X_i$ | *Set of strategies of player $i$ ($X = \prod_{i\in N} X_i$: set of strategy profiles)* |
| $\mathrm{N}(u)$ | *Set of Nash equilibria of the game $u = (u_1,\dots,u_n)$* |
| $\mathcal{F}_i$ | *Set of continuous payoff functions of player $i$ which are concave in $i$'s strategy and such that $x \mapsto \nabla_{x_i} u_i(\cdot, x_{-i})$ is continuous ($\mathcal{F} = \prod_{i\in N} \mathcal{F}_i$)* |
| $\Phi$ | *Nash correspondence which associates to each game $u \in \mathcal{F}$ its set of Nash equilibria* |
| $\mathcal{N}$ | *Graph of Nash equilibria, i.e. $\mathcal{N} = \{(u,x) \in \mathcal{F}\times X : x \in \mathrm{N}(u)\}$* |
| $\mathcal{N}_{\mathcal{U}}$ | *Graph of Nash equilibria restricted to $\mathcal{U}\times X \subset \mathcal{F}\times X$* |
| $\pi$ | *Projection mapping from $\mathcal{N}$ to $\mathcal{F}$* |
| $\eta$ | *Predtetchinski's homeomorphism from $\mathcal{N}$ to $\mathcal{F}$* |
| $\mathcal{A}_i$ | *Set of payoff functions of player $i$ which are affine in $i$'s strategy and do not depend on other players' strategies ($\mathcal{A} = \prod_{i\in N} \mathcal{A}_i$)* |
| $\mathcal{L}_i$ | *Set of payoff functions of player $i$ which are linear in $i$'s strategy and do not depend on other players' strategies ($\mathcal{L} = \prod_{i\in N} \mathcal{L}_i$)* |
| $\mathcal{C}$ | *Set of constant payoff functions* |
| $\mathbb{R}_{\delta_i}[x]$ | *Set of payoff functions of player $i$ which are polynomial and whose degrees are less or equal to $\delta_i \in \mathbb{N}$* |
| $\varphi_i$ | *Vector space isomorphism which assigns to each payoff function in $\mathbb{R}_{\delta_i}[x]$ its coefficients in $\mathbb{R}^{m_i}$, $m_i \in \mathbb{N}$ ($m = \sum_{i\in N} m_i$, $\varphi = \times_{i\in N}\varphi_i : \prod_{i\in N}\mathbb{R}_{\delta_i}[x] \to \mathbb{R}^m$)* |
| $\mathbb{R}^0_{\delta_i}[x]$ | *Set of payoff functions of player $i$ which are polynomial, whose degrees are less or equal to $\delta_i \in \mathbb{N}$ and with null coefficient on the constant part* |
| $\hat{\pi}_{-\mathcal{C}^n}$ | *Linear projection from $\prod_{i\in N}\mathbb{R}_{\delta_i}[x]$ to $\prod_{i\in N}\mathbb{R}^0_{\delta_i}[x]$ (which removes the constant part)* |
| $\tilde{\mathbb{R}}_{\delta_i}[x]$ | *Set of payoff functions of player $i$ which are polynomial, whose degrees are less or equal to $\delta_i \in \mathbb{N}$ and with null coefficients on the "$\mathcal{A}_i$ part"* |
| $\hat{\pi}_{-\mathcal{A}}$ | *Linear projection from $\prod_{i\in N}\mathbb{R}_{\delta_i}[x]$ to $\prod_{i\in N}\tilde{\mathbb{R}}_{\delta_i}[x]$ (which removes the "$\mathcal{A}_i$ part")* |
| $\mathcal{U}^k$ | *Set of games of the semi-algebraic decomposition of $\hat{\pi}_{-\mathcal{A}}(\mathcal{U})$ ($\mathcal{U}$ being a set of polynomial games)* |
| $\mathcal{V}^k$ | *Minkowski sum of $\mathcal{U}^k$ and $\mathcal{A}$, i.e. $\mathcal{V}^k = \mathcal{U}^k + \mathcal{A}$* |

set $B$ is a mapping from $A$ to the set of all subsets of $B$, and we denote it by $\Phi : A \twoheadrightarrow B$. A mapping $f$ from a topological space $X$ to another topological space $Y$ is proper if for every compact subspace $K$ of $Y$, $f^{-1}(K)$ is compact in $X$. For every topological spaces $X$ and $Y$, a homotopy from a continuous mapping $f : X \to Y$ to another continuous mapping $g : X \to Y$ is a continuous mapping $H : [0,1]\times X \to Y$ such that $H(0,\cdot) = f$ and $H(1,\cdot) = g$. We say that: (i) $f$ and $g$ are homotopic if there exists a homotopy $H$ from $f$ to $g$; (ii) $f$ and $g$ are properly homotopic if there exists a proper homotopy $H$ from $f$ to $g$. Every cartesian product of any family of topological spaces is endowed with the product topology. For every positive integer $m$, $\mathbb{R}^m$ is endowed with its usual Euclidean topology; for every $x = (x_1,\dots,x_m) \in \mathbb{R}^m$, the Euclidean norm $\|x\|$ of $x$ is defined by $\|x\| = \sqrt{\sum_{i=1}^m x_i^2}$. For every positive integer $m$, a semi-algebraic subset of $\mathbb{R}^m$ can be written $\bigcup_{i=1}^s \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^m : f_{i,j} \star_{i,j} 0\}$, where $\star_{i,j}$ denotes either $<$ or $=$ and $f_{i,j} \in \mathbb{R}[X_1,\dots,X_m]$ for every $i = 1,\dots,s$ and every $j = 1,\dots,r_i$ (see Appendix 4.1 for more details about semi-algebraic sets).

### *2.1. The graph of Nash equilibria*

Throughout this paper, we fix some integer $n \geq 1$, and we define $N = [\![1,n]\!]$, called the *set of players*. Also, for every $i \in N$, let $X_i$ be a nonempty, convex and compact semi-algebraic subset of $\mathbb{R}^{\lambda_i}$ called the *set of strategies of player i*, where $\lambda_i$ is a positive integer, and let $X = \prod_{i\in N} X_i$ be the *set of strategy profiles*. Without any loss of generality, we can assume that 0 is interior to $X_i$ in $\mathbb{R}^{\lambda_i}$ (since we can always translate the initial strategy sets $X_i$ and define the payoff functions "identically" on these translated strategy sets, and then we can replace $\mathbb{R}^{\lambda_i}$, if necessary, by the vector space generated by the translation of $X_i$).

**Definition 2.1.** A *(strategic-form) game* is an element $u = (u_1,\dots,u_n) \in \mathcal{F}(X,\mathbb{R})^n$, where each mapping $u_i$ is called the *payoff function of player $i \in N$*.

**Notations.** For every player $i \in N$, an element $x_i \in X_i$ is also denoted $x_i = (x_{i,1},\dots,x_{i,\lambda_i})$. We denote $X_{-i} = \prod_{j\neq i} X_j$ and $x_{-i} = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \in X_{-i}$ the vector derived from $(x_1,\dots,x_n) \in X$ by deleting the $i$-th coordinate. We adopt the usual notation $u_i(x) = u_i(x_i,x_{-i})$ for every $x \in X$ and every payoff function $u_i$ of player $i$, where $x_i \in X_i$ is the strategy of player $i$ and $x_{-i} \in X_{-i}$ is the vector of strategies of all players except player $i$.

We now recall the seminal concept of *Nash equilibrium*.

**Definition 2.2.** A profile of strategies $x = (x_1,\dots,x_n) \in X$ is a *Nash equilibrium* of the game $u = (u_1,\dots,u_n)$ if for every $i \in N$ and every $d_i \in X_i$, $u_i(d_i,x_{-i}) \leq u_i(x)$. The *set of Nash equilibria of the game u* is denoted $\mathrm{N}(u)$.

The payoff functions considered in this paper satisfy some concavity and some differentiability assumption. More precisely, for every $i \in N$, let

$$\mathcal{F}_i = \left\{u_i \in \mathcal{C}^0(X,\mathbb{R}) : \forall x_{-i} \in X_{-i}, x_i \mapsto u_i(x_i,x_{-i}) \text{ is concave and } x \mapsto \nabla_{x_i} u_i(\cdot,x_{-i}) \text{ continuous}\right\}$$

and $\mathcal{F} = \prod_{i\in N} \mathcal{F}_i$. Here, $\nabla_{x_i} u_i(\cdot, x_{-i})$ denotes the gradient of $u_i(\cdot, x_{-i})$ at the point $x_i$, assumed to exist when $u_i$ belongs to $\mathcal{F}_i$.[5] For every $i \in N$, we consider on $\tilde{\mathcal{F}}_i := \left\{u_i \in C^0(X,\mathbb{R}) : x \mapsto \nabla_{x_i} u_i(\cdot, x_{-i}) \text{ continuous}\right\}$ the topology generated by all subsets which can be written $\{u_i \in \tilde{\mathcal{F}}_i : \forall x = (x_i, x_{-i}) \in K, u_i(x) \in U \text{ and } \nabla_{x_i} u_i(\cdot, x_{-i}) \in U'\}$, for some compact subspace $K$ of $X$, some open subset $U$ of $\mathbb{R}$ and some open subset $U'$ of $\mathbb{R}^{\lambda_i}$. The set $\mathcal{F}_i \subset \tilde{\mathcal{F}}_i$ is endowed with the induced topology. This topology is used by Predtetchinski (2009) to prove his structure theorem (see Theorem 2.1 in what follows).

**Definition 2.3.** The *Nash correspondence associated to* $\mathcal{F}$ is the correspondence

$$\begin{cases} \Phi : & \mathcal{F} \twoheadrightarrow X \\ & u \mapsto \mathrm{N}(u) \end{cases}$$

and the graph of the Nash correspondence, called the *graph of Nash equilibria associated to* $\mathcal{F}$, is denoted $\mathcal{N}$, i.e.,

$$\mathcal{N} = \mathrm{Gr}(\Phi) = \{(u,x) \in \mathcal{F} \times X : x \in \mathrm{N}(u)\}.$$

We denote the projection from $\mathcal{N}$ to $\mathcal{F}$ by $\pi$ (i.e. $\pi(u,x) = u$ for every $(u,x) \in \mathcal{N}$).

### *2.2. A slight extension of Predtetchinski's structure theorem*

First, let us recall (the first part of) Predtetchinski's structure theorem about the graph of Nash equilibria (Predtetchinski, 2009, Theorem 2 and Theorem 3).[6]

**Theorem 2.1.** *(Predtetchinski) The projection mapping $\pi : \mathcal{N} \to \mathcal{F}$ is properly homotopic to some homeomorphism $\eta : \mathcal{N} \to \mathcal{F}$.*

Recall that the homeomorphism $\eta : \mathcal{N} \to \mathcal{F}$ is defined as follows: for every $(u,x) \in \mathcal{N}$, $\eta(u,x) = (\eta_i(u,x))_{i\in N}$, where for every $i \in N$,

$$\eta_i(u,x) : y \mapsto u_i(y) + \langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(\cdot, x^0_{-i}), y_i - x_i\rangle + \langle x_i, y_i\rangle, \tag{1}$$

where $x^0 \in X$ is a fixed strategy profile and where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^{\lambda_i}$.

Now, our structure theorem (see Theorem 2.2 below) is a slight extension of Theorem 2.1, and states that this last theorem remains true if we replace $\mathcal{F}$ by any of its subset which satisfies some assumption of "stability under affine addition". For every $i \in N$, let

$$\mathcal{A}_i = \{x \in X \mapsto \langle \alpha, x_i\rangle + c : \alpha \in \mathbb{R}^{\lambda_i}, c \in \mathbb{R}\}$$

(that is to say, $a_i \in \mathcal{A}_i$ if $a_i$ is affine with respect to $x_i$ and does not depend on $x_{-i}$), and let $\mathcal{A} = \prod_{i\in N} \mathcal{A}_i$. Moreover, for every subset $\mathcal{U}$ of $\mathcal{F}$, we define

$$\mathcal{N}_{\mathcal{U}} = \{(u,x) \in \mathcal{N} : u \in \mathcal{U}\},$$

and we denote $\pi_{\mathcal{U}}$ (resp. $\eta_{\mathcal{U}}$) the restriction of the projection $\pi$ (resp. of Predtetchinski's homeomorphism $\eta$) from $\mathcal{N}_{\mathcal{U}}$ to $\mathcal{U}$.

**Theorem 2.2.** *(Structure theorem) For every subspace $\mathcal{U}$ of $\mathcal{F}$ such that $\mathcal{U} + \mathcal{A} = \mathcal{U}$, $\eta_{\mathcal{U}} : \mathcal{N}_{\mathcal{U}} \to \mathcal{U}$ is a homeomorphism which is properly homotopic to $\pi_{\mathcal{U}} : \mathcal{N}_{\mathcal{U}} \to \mathcal{U}$.*

The proof is given in Appendix 4.3. Briefly, the idea is to consider the homotopy $H : [0,1] \times \mathcal{N} \to \mathcal{F}$ between $\pi$ and $\eta$ defined in the proof of Theorem 2.1 (see Predtetchinski, 2009), and to remark that this homotopy is constructed by adding progressively an affine mapping in $\mathcal{A}_i$ to each payoff function in $\mathcal{F}_i$. Such addition preserves the concavity of the payoff functions, and the condition $\mathcal{U} + \mathcal{A} = \mathcal{U}$ (of "stability under affine addition") guarantees that the restriction $H_{\mathcal{U}}$ of $H$ from $[0,1] \times \mathcal{N}_{\mathcal{U}}$ to $\mathcal{U}$ is well-defined. We emphasize that the assumption of semi-algebraicity of the sets of strategies is not used in the proof of this theorem, but only in the proof of our oddness theorem (Theorem 3.1 in Section 3).

Remark that Theorem 2.2 plays a crucial role in all our applications, since our oddness theorem is obtained by an application of this result to particular "nice" subclasses of payoff functions.

[5] When $x_i$ belongs to the boundary of $X_i$, $\nabla_{x_i} u_i(\cdot, x_{-i})$ is simply defined to be equal to $\nabla_{x_i} v_i(\cdot, x_{-i})$ where $v_i(., x_{-i})$ is some extension of $u_i(., x_{-i})$ (assumed to exist) on some open neighborhood of $x_i$. This does not depend on the choice of the extension, because $X_i$ has a nonempty interior.

[6] It is important to highlight that Predtetchinski has also extended Kohlberg-Mertens' theorem (see Predtetchinski, 2004): in Predtetchinski's paper, Theorem 3 provides a structure theorem for continuous games such that for every player $i$, each payoff function of $i$ is concave in $i$'s strategy, and Theorem 5, derived from Theorem 3 and with additional assumptions on the sets of strategies, corresponds to Kohlberg-Mertens' theorem.

## 3. Generic oddness of the graph of Nash equilibria

### 3.1. Oddness theorem

In this section, we prove that for some large classes of polynomial payoff functions (which we call *regular*), there exists an odd number of Nash equilibria.

From now on, for every player $i \in N$, we fix some integer $\delta_i \in \mathbb{N}$, and we consider the vector space $\mathbb{R}_{\delta_i}[x]$ of polynomial payoff functions of player $i$ whose degree is less or equal to $\delta_i$.[7] In particular, any element $u_i \in \mathbb{R}_{\delta_i}[x]$ can be expressed as

$$u_i(x) = \sum_{k \in \mathbb{N}^\Lambda} \alpha_k^i x^k$$

where $\Lambda := \{(i,j) : i \in N,\ j \in \{1,\dots,\lambda_i\}\}$, $x \in X$, $k = (k_{i,j})_{(i,j)\in\Lambda} \in \mathbb{N}^\Lambda$ is a multi-index (with $\alpha_k^i = 0$ if $\deg(k) := \sum_{(i,j)\in\Lambda} k_{i,j} > \delta_i$), and where $x^k := \prod_{(i,j)\in\Lambda} x_{i,j}^{k_{i,j}}$ is called a monomial.

For every $i \in N$, we denote by $\varphi_i$ the vector space isomorphism which associates to every payoff function $u_i \in \mathbb{R}_{\delta_i}[x]$ its coefficients (with respect to some predefined order on $\mathbb{N}^\Lambda$) in $\mathbb{R}^{m_i}$, for some integer $m_i$. In the following, we define $m = \sum_{i\in N} m_i$, $\varphi = \times_{i\in N}\varphi_i : \prod_{i\in N} \mathbb{R}_{\delta_i}[x] \to \mathbb{R}^m$. By abuse of notation, for every subset $\mathcal{U}$ of $\prod_{i\in N}\mathbb{R}_{\delta_i}[x]$, both the restriction of $\varphi$ from $\mathcal{U}$ to $\mathbb{R}^m$ and the restriction of $\varphi$ from $\mathcal{U}$ to $\varphi(\mathcal{U})$ are also denoted $\varphi$ (in particular, notice that the restriction $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ is a homeomorphism).

On the other hand, if we consider the set

$$\mathcal{C} := \{x \in X \mapsto c : c \in \mathbb{R}\}$$

(of constant payoff functions), we can remark that for every game $u = (u_1,\dots,u_n)$ and every $c = (c_1,\dots,c_n) \in \mathcal{C}^n$, the set $\mathrm{N}(u)$ of Nash equilibria of $u$ and the set $\mathrm{N}(u+c)$ of Nash equilibria of $u+c$ are equal. Thus, we want to focus only on games without constant part. For every player $i$, let

$$\mathcal{L}_i := \{x \in X \mapsto \langle \alpha_i, x_i\rangle : \alpha_i \in \mathbb{R}^{\lambda_i}\}$$

(that is to say, $l_i \in \mathcal{L}_i$ if $l_i$ is linear with respect to $x_i$ and does not depend on $x_{-i}$), and $\mathcal{L} = \prod_{i\in N}\mathcal{L}_i$ (in particular, notice that $\mathcal{A}_i = \mathcal{L}_i + \mathcal{C}$ and that $\mathcal{A} = \mathcal{L} + \mathcal{C}^n$). Moreover, for every $i \in N$, let

$$\mathbb{R}_{\delta_i}^0[x] = \{u_i \in \mathbb{R}_{\delta_i}[x] : u_i(0) = 0\}$$

be the vector subspace of $\mathbb{R}_{\delta_i}[x]$ of polynomial payoff functions of player $i$ whose degree is less or equal to $\delta_i$ and with null coefficient on the constant part.

Last, before presenting our notion of regular set and our oddness theorem, let us introduce some simplifications that will be made throughout this paper in order to avoid too complex notations. Since we can identify a set $\mathcal{U} \subset \prod_{i\in N}\mathbb{R}_{\delta_i}[x]$ of games with its set of coefficients in $\mathbb{R}^m$ (using the mapping $\varphi$), we can "transport" all the definitions and results about semi-algebraic subsets of $\mathbb{R}^m$ to subsets of $\prod_{i\in N}\mathbb{R}_{\delta_i}[x]$. For example:

- A subset $\mathcal{U}$ of $\prod_{i\in N}\mathbb{R}_{\delta_i}[x]$ is *semi-algebraic* if its set of coefficients $\varphi(\mathcal{U})$ is semi-algebraic, and in that case, the *dimension of* $\mathcal{U}$ (denoted $\dim(\mathcal{U})$) is defined as the dimension of $\varphi(\mathcal{U})$.
- A mapping $f$ from $\mathcal{U} \subset \prod_{i\in N}\mathbb{R}_{\delta_i}[x]$ to $\mathcal{U}' \subset \prod_{i\in N}\mathbb{R}_{\delta_i}[x]$ is *semi-algebraic* ($\mathcal{U}$ and $\mathcal{U}'$ being semi-algebraic in the above sense) if the mapping $\varphi \circ f \circ \varphi^{-1}$, defined from $\varphi(\mathcal{U})$ to $\varphi(\mathcal{U}')$, is semi-algebraic.
- A semi-algebraic subset $\mathcal{G}$ of a semi-algebraic set $\mathcal{U} \subset \prod_{i\in N}\mathbb{R}_{\delta_i}[x]$ is *generic in* $\mathcal{U}$ if $\varphi(\mathcal{G})$ is generic (in the classical sense) in $\varphi(\mathcal{U})$, i.e. if: (i) $\varphi(\mathcal{G})$ is open in $\varphi(\mathcal{U})$; (ii) $\dim(\varphi(\mathcal{U})\backslash\varphi(\mathcal{G})) < \dim(\varphi(\mathcal{U}))$.
- Tarski-Seidenberg's theorem can be established in that framework (see Appendix 4.1 for more details).

Now, the following notion of regularity plays a central role in our paper.

**Definition 3.1.** (Regular set of polynomial games) A set $\mathcal{U} \subset \prod_{i\in N}\mathbb{R}_{\delta_i}^0[x]$ is said to be *regular* if the two following conditions are fulfilled:

1. **(Semi-algebraicity assumption).** $\mathcal{U}$ is a semi-algebraic set.
2. **(Stability assumption).** $\dim(\mathcal{U} + \mathcal{L}) = \dim(\mathcal{U})$.

[7] For every $i \in N$, the set $\mathbb{R}_{\delta_i}[x]$ is endowed with the topology induced by the topology of $\tilde{\mathcal{F}}_i$ (defined in Section 2). Endowed with this topology, $\mathbb{R}_{\delta_i}[x]$ is a finite-dimensional Hausdorff topological vector space, thus every linear mapping from $\mathbb{R}_{\delta_i}[x]$ to any other topological vector space is continuous.

Roughly, the first assumption requires that the coefficients of the polynomial payoff functions in $\mathcal{U}$ are defined with finitely many inequalities involving other polynomials; the second assumption requires that the set of profiles of payoff functions in $\mathcal{U}$ contains some linear part which is "large enough". For example, $\prod_{i\in N}(\mathcal{F}_i \cap \mathbb{R}^0_{\delta_i}[x])$ is regular (see Section 3.2 for more elaborated examples).

The following oddness theorem is in the spirit of Wilson's oddness theorem (Wilson, 1971), but for polynomial payoff functions satisfying the previous regularity condition.

**Theorem 3.1.** *(Oddness theorem) For every regular subset $\mathcal{U}$ of $\mathcal{F}$, there exists a generic subset $\mathcal{U}^*$ of $\mathcal{U}$ such that for every $u \in \mathcal{U}^*$, the game $u$ has an odd number of Nash equilibria.*

Remark that the regularity condition seems to be tight, since we can easily find games with polynomial and own-strategy concave payoff functions for which the theorem is false when we remove either stability assumption or semi-algebraicity assumption. For example, first, consider two players $i = 1, 2$ whose strategy spaces are $[0, 1]$, and a set of payoff functions $\mathcal{U} = \{(x_1, x_2) \in [0, 1]^2 \mapsto (ax_1, b(1 - x_1)x_2), (a, b) \in \mathbb{R} \times \mathbb{R}\}$. The set $\mathcal{U}$ is semi-algebraic (it is a vector space of dimension 2), but stability assumption does not hold: adding to $\mathcal{U}$ the set of profiles of linear payoff functions $(cx_1, dx_2)$ (where $(c, d) \in \mathbb{R} \times \mathbb{R}$), we get a vector space of dimension 3. As a matter of fact, Theorem 3.1 does not hold since $(1, x_2)$ is a Nash equilibrium for every $a \geq 0$, $b \in \mathbb{R}$ and for every $x_2 \in [0, 1]$, i.e. there is an infinite number of Nash equilibria for a full-dimensional set $\{(a, b) \in [0, +\infty) \times \mathbb{R}\}$ of games. Second, consider the following modification of the previous example $\mathcal{U} = \{(x_1, x_2) \in [0, 1]^2 \mapsto (ax_1, b(1 - x_1)x_2 + d(b, F)x_2), (a, b) \in \mathbb{R} \times \mathbb{R}\}$, where $F$ is any closed subset of $\mathbb{R}$ with a strictly positive Lebesgue measure, and where $d(b, F)$ denotes the Euclidean distance between $b$ and $F$. For every $a \geq 0$, $b \in F$, and every $x_2 \in [0, 1]$ $(1, x_2)$ is a Nash equilibrium, thus Theorem 3.1 does not hold. In this last example, a problem is that $\mathcal{U}$ is, in general, not a semi-algebraic set (when $F$ is not semi-algebraic), and, in that case, the dimension of $\mathcal{U}$ cannot be defined properly. This illustrates that semi-algebraicity assumption is important, if only to give a clear meaning to Stability assumption.

The proof of the theorem is given in Appendix 4.4, with some applications in the next section. Below, we provide a sketch of the proof:

**Sketch of the proof.** The detailed proof is given in the appendix. A non correct (but instructive) proof would be the following: from homotopy invariance of topological degree and from Theorem 2.2, the topological degree of $\pi_{\mathcal{U}}$ is equal to the topological degree of $\eta_{\mathcal{U}}$; but $\eta_{\mathcal{U}}$ is a proper homeomorphism from $\mathcal{N}_{\mathcal{U}}$ to $\mathcal{U}$, thus its topological degree is equal to 1 or $-1$; from some covering space property, there exists a generic subset $\mathcal{G}$ of the set $\mathcal{U}$ such that for every $u \in \mathcal{G}$, $\pi^{-1}_{\mathcal{U}}(u) = \{u\} \times \{x : x \text{ is a Nash equilibrium of } u\}$ is a finite set $\{(u, x^1), \ldots, (u, x^k)\}$ (for some integer $k$). The same covering space property ensures that $\pi_{\mathcal{U}}$ is a local homeomorphism at each $(u, x^p)$, hence (from homeomorphism property of topological degree) that the local degree of $\pi_{\mathcal{U}}$ at each $(u, x^p)$ is equal to 1 or $-1$; last, from additivity property of topological degree, the sum of local degrees of $\pi_{\mathcal{U}}$ at each $(u, x^p)$ is equal to the degree of $\pi_{\mathcal{U}}$, and this would finally prove that $k$ is odd. At least two things are not correct in this proof:

1. First, stability assumption in Theorem 2.2 (which is required in the above proof) may be false under the weaker stability assumption of Theorem 3.1. This can be solved easily as follows: we can define the following three stability assumptions:
   Strong stability assumption A1 (necessary in Theorem 2.2): $\mathcal{U} + \mathcal{A} = \mathcal{U}$ and $\mathcal{U} \subset \prod_{i\in N} \mathbb{R}_{\delta_i}[x] \cap \mathcal{F}$.
   Weak stability assumption A2: $\dim(\mathcal{U} + \mathcal{A}) = \dim(\mathcal{U})$ and $\mathcal{U} \subset \prod_{i\in N} \mathbb{R}_{\delta_i}[x] \cap \mathcal{F}$.
   Weak stability assumption A3 (necessary in Theorem 3.1): $\dim(\mathcal{U} + \mathcal{L}) = \dim(\mathcal{U})$ and $\mathcal{U} \subset \prod_{i\in N} \mathbb{R}^0_{\delta_i}[x] \cap \mathcal{F}$.
   First, if we assume that Theorem 3.1 is true under A1, then we can prove the theorem for $\mathcal{U}$ satisfying A2 (this is done in Step II): indeed, it is easy to see in that case that $\mathcal{U} + \mathcal{A}$ satisfies A1. Thus there exists a generic subset $\mathcal{G}$ of $\mathcal{U} + \mathcal{A}$ such that $u$ has an odd number of Nash equilibria for every $g \in \mathcal{G}$. Then, under A2, we can prove that $\mathcal{G}' = \mathcal{G} \cap \mathcal{U}$ is generic in $\mathcal{U}$, which proves Theorem 3.1 under A2.
   Second, if we assume Theorem 3.1 is true under A2, then we can prove it for $\mathcal{U}$ satisfying A3 (this is done in Step III): indeed, it is easy to see in that case that $\mathcal{U} + C^n$ satisfies A2. Thus there exists a generic subset $\mathcal{G}$ of $\mathcal{U} + C^n$ such that $u$ has an odd number of Nash equilibria for every $g \in \mathcal{G}$. Then, under A3, we can prove that $\mathcal{G}' = \mathcal{G} \cap \mathcal{U}$ is generic in $\mathcal{U}$, which proves Theorem 3.1 under A2.
   Thus, to prove Theorem 3.1, we only have to prove the theorem under the stronger stability assumption A1 (this is done in Step I).
2. A second issue, in the sketch of the proof given above, is that we cannot use the standard topological degree if $\mathcal{U}$ is not a topological manifold (which can be the case in general). Thus, the idea (Step 1-2) is to decompose $\mathcal{U}$ in a union of sets $\mathcal{V}^k$ which are topological manifolds (actually we can even choose $\mathcal{V}^k$ so that they are homeomorphic to some Euclidean spaces) and satisfies Stability assumption A1. Then we can apply the idea developed above to $\pi_{\mathcal{V}^k}$ and $\eta_{\mathcal{V}^k}$: in Step 3, the degree of $\pi_{\mathcal{V}^k}$ is proved to be equal to 1 (modulo 2), by applying Theorem 2.2 and Homotopy invariance of topological degree to these two mappings; in Step 4, $\pi^{-1}_{\mathcal{V}^k}(v)$ is proved to be finite for generic $v$ (here is used semi-algebraicity assumption); in Step 5 and 6, using topological degree, we prove that there are an odd number of Nash equilibria for every $v \in \mathcal{G}^k$, a generic subset of $\mathcal{V}^k$; the last step 7 consist in taking for $\mathcal{G}$ the union of the $\mathcal{G}^k$ which are open in $\mathcal{U}$, and it can be proved that it satisfies the conclusions of Theorem 3.1.

### 3.2. Some applications of oddness theorem

In this section, we suppose that for every $i \in N$, the set $X_i$ of strategies of player $i$ is a nonempty compact interval of $\mathbb{R}$. When $X_i \subset [0, +\infty)$, a strategy of player $i$ can be interpreted as an amount of time or effort to exert some activity.

#### 3.2.1. Linear perturbations

Hereafter, we prove that oddness theorem holds for generic linear perturbations of a given game $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n) \in \mathcal{F} \cap \mathbb{R}^0_{\delta_i}[x]$ (i.e., for every $i \in N$, $\bar{u}_i$ is a polynomial mapping concave with respect to player $i$'s strategy, whose degree is less or equal to $\delta_i$ and such that $\bar{u}_i(0) = 0$). Consider a semi-algebraic subset $C$ of $\mathbb{R}^n$ of parameters such that $\dim(C) = n$ (a typical case is when $C$ is a product of intervals of nonempty interiors, and naturally, we can take $C = \mathbb{R}^n$), and consider the following set $\mathcal{U}$ of perturbed games:

$$\mathcal{U} = \{ \left(x \mapsto u_i[\gamma](x) := \bar{u}_i(x) + \gamma_i x_i\right)_{i \in N} : \gamma = (\gamma_1, \ldots, \gamma_n) \in C \}.$$

**Proposition 3.1.** *There exists a generic subset $G$ of $C$ such that for every $\gamma \in G$, the game $u[\gamma] = (u_i[\gamma])_{i \in N}$ has an odd number of Nash equilibria.*

**Proof.** Formally, the set $\mathcal{S}_{\mathcal{U}} \subset \mathbb{R}^m$ of coefficients of polynomials in $\mathcal{U}$ can be written

$$\prod_{i \in N} E_i^1 \times \left( \prod_{i \in N}^{n} E_i^2 + C \right)$$

where for every player $i$, $E_i^1$ is a finite product of singletons (corresponding to the coefficients of $\bar{u}_i$ associated to the monomials other than $x_i$) and $E_i^2$ is a singleton (corresponding to the coefficient of $\bar{u}_i$ associated to the monomial $x_i$). We can check that $\mathcal{U}$ is regular:

1. First, $\mathcal{U} \subset (\prod_{i \in N} \mathbb{R}^0_{\delta_i}[x]) \cap \mathcal{F}$.
2. Second, $\mathcal{U}$ is semi-algebraic (in Definition 3.1) since $\mathcal{S}_{\mathcal{U}}$ is a finite product of semi-algebraic sets (the set $C$ being semi-algebraic by assumption).
3. Third, $\mathcal{U}$ satisfies stability assumption (in Definition 3.1) since

   $$\dim(\mathcal{U}) = \dim(\mathcal{S}_{\mathcal{U}}) = \dim(C) = n = \dim(\mathcal{U} + \mathcal{L})$$

   (indeed, the dimension of each singleton is equal to 0, $\dim(C) = n$, and $\mathcal{U} + \mathcal{L} = \{ \left(x \mapsto \bar{u}_i(x) + \gamma_i x_i\right)_{i \in N} : \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n \}$ is clearly $n$-dimensional).

Thus, from Theorem 3.1, there exists a generic subset $\mathcal{U}^*$ of $\mathcal{U}$ such that every $u \in \mathcal{U}^*$ has an odd number of Nash equilibria. We can reformulate the genericity with respect to the parameters $\gamma$ as follows: let $\mathcal{S}_{\mathcal{U}^*}$ ($\subset \mathcal{S}_{\mathcal{U}} \subset \mathbb{R}^m$) be the set of coefficients of profiles of polynomial payoff functions in $\mathcal{U}^*$. The mapping $f$ which associates to any $\gamma \in C \subset \mathbb{R}^n$ the vector of coefficients of $u[\gamma](x)$ (i.e., formally, $f(\gamma) = \varphi(u[\gamma])$) is a semi-algebraic homeomorphism. Thus, we get that for every $\gamma \in G := f^{-1}(\mathcal{S}_{\mathcal{U}^*})$, the game $u[\gamma]$ has an odd number of Nash equilibria, and that $G$ is a generic subset of $C$. $\square$

#### 3.2.2. Quadratic perturbations

Similarly to the previous section, we now prove that oddness theorem holds for generic quadratic perturbations of a given game $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n) \in \mathcal{F} \cap \mathbb{R}^0_{\delta_i}[x]$. Let $A$ be a semi-algebraic subset of $[0, +\infty)^n$, $B$ be a semi-algebraic subset of $\mathbb{R}^{n(n-1)}$ and $C$ be a semi-algebraic subset of $\mathbb{R}^n$ such that $\dim(C) = n$. Consider the set $A \times B \times C$ of parameters, and consider the following set $\mathcal{U}$ of perturbed games:

$$\mathcal{U} = \{ \left(x \mapsto u_i[\alpha, \beta, \gamma](x) = \bar{u}_i(x) - \alpha_i x_i^2 + \sum_{j \neq i} \beta_{i,j} x_i x_j + \gamma_i x_i\right)_{i \in N} : (\alpha, \beta, \gamma) = (\alpha_i, (\beta_{i,j})_{j \neq i}, \gamma_i)_{i \in N} \in A \times B \times C \},$$

where any game in $\mathcal{U}$, which depends on some parameters $\alpha, \beta, \gamma$, is denoted $u[\alpha, \beta, \gamma]$.

**Proposition 3.2.** *There exists a generic subset $G$ of $A \times B \times C$ such that for every $(\alpha, \beta, \gamma) \in G$, the game $u[\alpha, \beta, \gamma] = (u_i[\alpha, \beta, \gamma])_{i \in N}$ has an odd number of Nash equilibria.*

**Proof.** The proof is very similar to the above proof. We can check that $\mathcal{U}$ is regular:

1. First, $\mathcal{U} \subset (\prod_{i \in N} \mathbb{R}^0_{\delta_i}[x]) \cap \mathcal{F}$ (since, by assumption, the set $A$ is included in $[0, +\infty)^n$).

2. Second, $\mathcal{U}$ is semi-algebraic since $\mathcal{S}_{\mathcal{U}}$ is a finite product of semi-algebraic sets[8] (the sets $A, B, C$ being semi-algebraic by assumption).
3. Third, $\mathcal{U}$ satisfies stability assumption (in Definition 3.1) since

$$\dim(\mathcal{U}) = \dim(\mathcal{S}_{\mathcal{U}}) = \dim(A \times B \times C) = \dim(A) + \dim(B) + \dim(C) = \dim(A) + \dim(B) + n = \dim(\mathcal{U} + \mathcal{L})$$

(indeed, the dimension of each singleton is equal to 0, $\dim(C) = n$, and $\mathcal{U} + \mathcal{L} = \{ \big(x \mapsto u_i[\alpha,\beta,\gamma](x) = \bar{u}_i(x) - \alpha_i x_i^2 + \sum_{j\neq i} \beta_{i,j} x_i x_j + \gamma_i x_i\big)_{i\in N} : (\alpha,\beta,\gamma) = (\alpha_i, (\beta_{i,j})_{j\neq i}, \gamma_i)_{i\in N} \in A \times B \times \mathbf{R}^n \}$).

Thus, from Theorem 3.1, there exists a generic subset $\mathcal{U}^*$ of $\mathcal{U}$ such that every $u \in \mathcal{U}^*$ has an odd number of Nash equilibria, and we get the generic result with respect to the coefficients $(\alpha,\beta,\gamma)$ in the same manner as in the previous proof. □

#### 3.2.3. The benchmark quadratic model

As an application of the previous sections, we prove that there exists generically an odd number of Nash equilibria for several models introduced in network formation theory of games on networks: Patacchini-Zenou's model about juvenile delinquency and conformism (Patacchini and Zenou, 2012), Calvó-Armengol-Patacchini-Zenou's model about social networks in education (Calvó-Armengol et al., 2009), Konig-Liu-Zenou's model about R&D networks (König et al., 2019), Helsley-Zenou's model about social networks and interactions in cities (Helsley and Zenou, 2014), etc. These models are in fact particular cases of the *benchmark quadratic model* (see Young and Zamir, 2015). In the following, we consider the set $L := \{\{i,j\} \subset N : i \neq j\}$ of (undirected) links (on $N$), and the set $\mathbb{G} := \mathcal{F}(L,[0,1])$ of (weighted) networks (on $N$); the strength of any link $\{i,j\}$ in a network $g$, denoted $g_{ij}$, is measured by an element in $[0,1]$.

*The benchmark quadratic model with ex ante heterogeneity* Fix $\phi \in (0,+\infty)$, and suppose that for every $i \in N$, the payoff function of player $i$ is defined by

$$x \in X \mapsto -\frac{1}{2}x_i^2 + \phi \sum_{j\neq i} g_{ij} x_i x_j + \gamma_i x_i,$$

where $g = (g_{ij})_{\{i,j\}\in L} \in \mathbb{G}$, and where for every player $i$, $\gamma_i \in [0,+\infty)$. For every $i \in N$, we can rewrite the payoff function of player $i$ in the following way:

$$x \in X \mapsto -\frac{1}{2}x_i^2 + \sum_{j\neq i} \beta_{i,j} x_i x_j + \gamma_i x_i,$$

where for every player $i$ and every player $j \neq i$, $\beta_{i,j} = \beta_{j,i} = \phi g_{ij} \in [0,\phi]$.

We can get two different generic existence results, depending on which parameters of the model are fixed:

1. First, we can apply Section 3.2.1 where for every player $i$, and every $x \in X$, $\bar{u}_i(x) := -\frac{1}{2}x_i^2 + \phi\sum_{j\neq i} g_{ij}x_i x_j$, and where $C := [0,+\infty)^n$. This application corresponds to the case where the network $g$ is supposed to be fixed. From Proposition 3.1, we obtain a generic subset $G^{\text{lin}}$ of $C$ such that for every $\gamma \in G^{\text{lin}}$, the game $u[\gamma] := \big(x \mapsto -\frac{1}{2}x_i^2 + \phi\sum_{j\neq i} g_{ij}x_ix_j + \gamma_i x_i\big)_{i\in N}$ has an odd number of Nash equilibria.
2. Second, we can apply Section 3.2.2, where for every player $i$, and every $x \in X$, $\bar{u}_i(x) := 0$, and where $A := \{\frac{1}{2}\}^n$, $B := \{(\beta_{i,j})_{\substack{i\in N\\ j\neq i}} : \forall i \in N, \forall j \neq i, \beta_{i,j} = \beta_{j,i} \in [0,\phi]\}$ and $C := [0,+\infty)^n$. This application corresponds to the case where the network $g$ is not supposed to be fixed. From Proposition 3.2, we obtain a generic subset $G$ of $A\times B\times C$ such that for every $\gamma \in G$, the game $u[\alpha,\beta,\gamma] := \big(x \mapsto -\frac{1}{2}x_i^2 + \sum_{j\neq i}\beta_{i,j}x_ix_j + \gamma_i x_i\big)_{i\in N}$ has an odd number of Nash equilibria (and, obviously, we also get that the game $u[\alpha,g,\gamma] := \big(x \mapsto -\frac{1}{2}x_i^2 + \phi\sum_{j\neq i} g_{ij}x_ix_j + \gamma_i x_i\big)_{i\in N}$ has an odd number of Nash equilibria for a generic set $G^{\text{quad}}$ of parameters in $A\times\mathbb{G}\times C$). Remark that this result is not comparable to the previous one, since the generic set $G^{\text{lin}}$ depends on $g$.

*The model with global congestion and ex ante heterogeneity* Consider the benchmark quadratic model with ex ante heterogeneity with the following modification: for every $i \in N$, the payoff function of player $i$ is defined by

$$x \in X \mapsto -\frac{1}{2}x_i^2 + \phi\sum_{j\neq i} g_{ij}x_ix_j - \lambda\sum_{j\neq i} x_ix_j + \gamma_i x_i,$$

[8] Formally, the set $\mathcal{S}_{\mathcal{U}} \subset \mathbb{R}^m$ of coefficients associated to $\mathcal{U}$ can be written

$$\prod_{i\in N} E_i^1 \times \Big(\prod_{i\in N}^n E_i^2 - A\Big) \times \Big(\prod_{i\in N}^n E_i^3 + B\Big) \times \Big(\prod_{i\in N}^n E_i^4 + C\Big)$$

where for every player $i$, $E_i^1$ is a finite product of singletons (corresponding to the coefficients of $\bar{u}_i$ associated to the monomials other than $x_i^2$, $(x_ix_j)_{j\neq i}$, and $x_i$), $E_i^2$ is a singleton (corresponding to the coefficient of $\bar{u}_i$ associated to the monomial $x_i^2$), $E_i^3$ is a finite product of singletons (corresponding to the coefficients of $\bar{u}_i$ associated to the monomials $(x_ix_j)_{j\neq i}$), and $E_i^4$ is a singleton (corresponding to the coefficient of $\bar{u}_i$ associated to the monomial $x_i$).

where $\lambda \in [0, +\infty)$. For every $i \in N$, the payoff function of player $i$ can be rewritten in the following way:

$$x \in X \mapsto -\frac{1}{2}x_i^2 + \sum_{j \neq i} \beta'_{i,j} x_i x_j + \gamma_i x_i,$$

where for every player $i$ and every player $j \neq i$, $\beta'_{i,j} = \beta'_{j,i} = \phi g_{ij} - \lambda \in [-\lambda, \phi - \lambda]$. Similarly to the previous model (without global congestion):

1. If we apply Proposition 3.1 (i.e. linear perturbations), then we obtain for every $g \in \mathbb{G}$ the existence of a generic subset of parameters $\gamma \in [0, +\infty)^n$ for which oddness theorem is satisfied.
2. If we apply Proposition 3.2 (i.e. quadratic perturbations), then we obtain a generic subset of parameters $(g, \gamma) \in \mathbb{G} \times [0, +\infty)^n$ for which oddness theorem is satisfied.

## 4. Appendix

### 4.1. Reminders about real algebraic geometry

**Theorem 4.1.** *(Tarski-Seidenberg) If $S$ is a semi-algebraic subset of $\mathbb{R}^{m+p}$ and $\Pi$ is the canonical projection from $\mathbb{R}^{m+p}$ to $\mathbb{R}^m$, then $\Pi(S)$ is a semi-algebraic subset of $\mathbb{R}^m$.*

From this result, we can also define semi-algebraic sets using quantified variables (or $m$-tuples of variables) which range in $\mathbb{R}$, $\mathbb{R}^m$ or more generally in a semi-algebraic set (for example, see Bochnak et al., 1998, p. 28):

**Definition 4.1.** A first-order formula of the language of ordered fields with parameters in $\mathbb{R}$ is a formula written with a finite number of conjunctions, disjunctions, negations, and universal or existential quantifiers on variables in semi-algebraic sets, starting from atomic formulas which are formulas of the kind $f(x_1, \dots, x_m) = 0$ or $g(x_1, \dots, x_m) > 0$, where $f$ and $g$ are polynomials with coefficients in $\mathbb{R}$.

**Proposition 4.1.** *Let $\Phi(x_1, \dots, x_m)$ be a first-order formula of the language of ordered fields with parameters in $\mathbb{R}$. Then, $\{x \in \mathbb{R}^m : \Phi(x)\}$ is a semi-algebraic set.*

For example, if $f$ is a polynomial with four variables and if $\mathbb{S}^1$ denotes the unit circle of $\mathbb{R}^2$, then the set $\{(x, y) \in \mathbb{R}^2 : \forall (z, t) \in \mathbb{S}^1, f(x, y, z, t) \geq 0\}$ is semi-algebraic (as $\mathbb{S}^1$ is itself a semi-algebraic set).

**Definition 4.2.** Let $S$ be a semi-algebraic subset of $\mathbb{R}^m$ and $T$ be a semi-algebraic subset of $\mathbb{R}^p$. A mapping $f : S \to T$ is semi-algebraic if its graph

$$\mathrm{Gr}(f) = \{(x, f(x)) : x \in S\}$$

is a semi-algebraic subset of $\mathbb{R}^m \times \mathbb{R}^p$.

The following proposition is a consequence of Tarski-Seidenberg's theorem.

**Proposition 4.2.** *Let $S$ be a semi-algebraic subset of $\mathbb{R}^m$ and $f : S \to \mathbb{R}^p$ be a semi-algebraic mapping. Then, $f(S)$ is a semi-algebraic subset of $\mathbb{R}^p$.*

**Corollary 4.1.** *Let $S_1$ and $S_2$ be semi-algebraic subsets of $\mathbb{R}^m$. Then, the $S_1 + S_2$ is a semi-algebraic set.*

**Proof.** We can remark that $S_1 + S_2 = f(S_1 \times S_2)$, where $f : (x_1, x_2) \in S_1 \times S_2 \mapsto x_1 + x_2 \in \mathbb{R}^m$ is a polynomial function and where $S_1 \times S_2$ is a semi-algebraic set. □

A semi-algebraic homeomorphism is a homeomorphism which is semi-algebraic (in that case, $f^{-1}$ is also semi-algebraic).

We now define the dimension of a semi-algebraic set (see Bochnak et al., 1998, Theorem 2.3.6., p. 33 and Corollary 2.8.9., p. 53).

**Definition 4.3.** For every semi-algebraic subset $S$ of $\mathbb{R}^m$, there exists an increasing sequence of non-negative integers $d_0 \leq d_1 \leq \dots \leq d_k$ such that

$$S = \bigcup_{i=0}^{k} S_i,$$

the union being disjoint, where $S_i$ is semi-algebraically homeomorphic to $]0,1[^{d_i}$ for every $i = 0, \dots, k$ (where, by convention, $]0,1[^0$ is a point). The dimension of $S$ is defined as

$$\dim(S) := \max\{d_0, d_1, \dots, d_k\}$$

(and does not depend on the decomposition of $S$). Remark also that for every $S_i$ in this decomposition such that $\dim(S_i) = \dim(S)$, $S_i$ has to be open in $S$.

Then, we have the following properties.

**Proposition 4.3.** *Let $S$, $S_1$ and $S_2$ be semi-algebraic subsets of $\mathbb{R}^m$:*

1. *If $f : S \to \mathbb{R}^p$ is semi-algebraic and is a bijection from $S$ to $f(S)$, then* $\dim(S) = \dim(f(S))$.
2. *If $S_1 \subset S_2$, then* $\dim(S_1) \leq \dim(S_2)$.
3. $\dim(S_1 \times S_2) = \dim(S_1) + \dim(S_2)$.

Finally, we recall the notion of generic semi-algebraic set, and we state one particular theorem about generic semi-algebraic sets which is used in the proof of our oddness theorem (Theorem 3.1).

**Definition 4.4.** Let $S$ be a semi-algebraic set and $G$ be a semi-algebraic subset of $S$. We say that $G$ is a generic semi-algebraic subset of $S$ if $G$ is open in $S$ and if $\dim(S \backslash G) < \dim(S)$.

**Theorem 4.2.** *Let $S$ and $T$ be two semi-algebraic sets such that* $\dim(S) = \dim(T)$ *and $f : S \to T$ be a surjective continuous semi-algebraic mapping. Then, there exists a generic semi-algebraic subset $G$ of $T$ such that for every $t \in G$:*

- *$f^{-1}(t)$ is a (nonempty) finite set;*
- *there exists an open neighborhood $V_t$ of $t$ such that $f^{-1}(V_t)$ is a finite union of pairwise disjoint open sets $(V_t^k)_{k \in \mathcal{K}}$ (where $\mathcal{K}$ is a finite set) such that for every $k \in \mathcal{K}$, $f|_{V_t^k}$ is a homeomorphism between $V_t^k$ and $V_t$.*

**Proof.** To prove this theorem, we use the following theorem (see Bochnak et al., 1998, p. 224).

**Theorem 4.3.** *Let $S$ and $T$ be two semi-algebraic sets and $f : S \to T$ be a continuous semi-algebraic mapping. Then, there exists a generic semi-algebraic subset $G$ of $T$ such that $f$ has a semi-algebraic trivialization $\theta^C$ over each semi-algebraically connected component $C$ of $G$ with fiber $\mathcal{K}^C$.*[9,10]

Let us consider $t \in G$ and the connected component $C$ of $G$ which contains $t$. Applying the above theorem above to $C$, there exists a semi-algebraic trivialization $\theta^C$ of $f$ over $C$ with fiber $\mathcal{K}^C$. In particular, $C \times \mathcal{K}^C$ and $f^{-1}(C)$ are semi-algebraically homeomorphic, and we obtain that $\dim(f^{-1}(t)) = \dim(S) - \dim(T) = 0$. Then, as $f^{-1}(t)$ is a 0-dimensional semi-algebraic set, it is a finite (nonempty) set (see Bochnak et al., 1998, Theorem 2.3.6.) with the same cardinal as $\mathcal{K}^C$. Finally, defining $V_t$ any open neighborhood of $t$ included in $C$, and $V_t^k = \theta^C(V_t \times \{k\})$, for every $k \in \mathcal{K} := \mathcal{K}^C$, we get directly the result. □

### 4.2. Topological degree of a continuous mapping

In this subsection, we gather all the properties of topological degree which are used in this paper (see Dold, 1995, Proposition 4.5, p. 268).

Let $X, Y$ be two oriented topological $m$-manifolds (without boundaries), and suppose that $Y$ is connected. To every proper continuous mapping $f : X \to Y$, one can associate an integer $\deg(f) \in \mathbb{Z}$ called the (topological) degree of $f$, which satisfies the following properties:

(i) **(Homotopy invariance).** If $f : X \to Y$ and $g : X \to Y$ are homotopic, then $\deg(f) = \deg(g)$ (see Dold, 1995, Exercise 3, p. 271).
(ii) **(Surjectivity).** If $f : X \to Y$ is such that $\deg(f) \neq 0$, then $f$ is surjective (see Dold, 1995, after Proposition 4.5, p. 268).
(iii) **(Homeomorphism).** If $f : X \to Y$ is a homeomorphism, then $\deg(f) \in \{-1, 1\}$ (see Dold, 1995, after Proposition 4.5, p. 268).

[9] Recall that a semi-algebraic trivialization of $f$ over a semi-algebraically connected component $C$ of $G$ with fiber $\mathcal{K}^C$ (where $\mathcal{K}^C$ is a semi-algebraic set) is a semi-algebraic homeomorphism $\theta^C : C \times \mathcal{K}^C \to f^{-1}(C)$ such that $f(\theta^C(c,k)) = c$ for every $(c,k) \in C \times \mathcal{K}^C$. Remark that if $f^{-1}(C) = \emptyset$, then this condition is automatically verified by taking the empty mapping $\theta^C = \emptyset$.

[10] Let us recall that in $\mathbb{R}^m$, a semi-algebraic set is semi-algebraically connected if and only if it is connected (see Bochnak et al., 1998, Theorem 2.4.5., p. 35).

(iv) **(Additivity).** Let $f : X \to Y$ and $y \in Y$, if $f^{-1}(y) = \{x_1, \dots, x_k\}$ (for some integer $k > 0$), then

$$\deg(f) = \sum_{i=1}^{k} \deg(f_{|V_i})$$

where for every $i = 1, \dots, k$, $V_i$ is an open subset of $X$ such that $V_i \cap f^{-1}(y) = \{x_i\}$ (see Dold, 1995, Proposition 4.7, p. 269).

### *4.3. Proof of structure theorem (Theorem 2.2)*

First, we prove that $\eta(\mathcal{N}_{\mathcal{U}}) \subset \mathcal{U}$. From Equation (1), for every $(u, x) \in \mathcal{N}_{\mathcal{U}}$, we have $\eta(u, x) = (\eta_i(u, x))_{i \in N}$, where for every $i \in N$,

$$\eta_i(u, x) : y \mapsto u_i(y) + \langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(\cdot, x^0_{-i}), y_i - x_i \rangle + \langle x_i, y_i \rangle.$$

Now, since for every player $i$, $\eta_i(u, x)$ is equal to $u_i$ up to an element of $\mathcal{A}_i$, we obtain that $\eta(u, x) \in \mathcal{U}$ (since by assumption we have $\mathcal{U} + \mathcal{A} = \mathcal{U}$), i.e. finally $\eta(\mathcal{N}_{\mathcal{U}}) \subset \mathcal{U}$.

Second, we prove that $\eta^{-1}(\mathcal{U}) \subset \mathcal{N}_{\mathcal{U}}$, where we recall (see Predtetchinski, 2009) that for every $u \in \mathcal{U}$, we have $\eta^{-1}(u) = (\Phi(u), \tilde{x}) = (\phi_i(u), \tilde{x}_i)_{i \in N}$, where for every $i \in N$,

$$\phi_i(u) : y \mapsto u_i(y) - \langle \nabla_{\tilde{x}_i} u_i(\cdot, \tilde{x}_{-i}) - \nabla_{\tilde{x}_i} u_i(\cdot, x^0_{-i}), y_i - \tilde{x}_i \rangle - \langle \tilde{x}_i, y_i \rangle,$$

and where $\tilde{x}_i$ is the unique maximizer of the strictly concave mapping $x_i \in X_i \mapsto u_i(x_i, x^0_{-i}) - \frac{1}{2}\langle x_i, x_i \rangle$. For every $u \in \mathcal{F}$, the strategy profile $\tilde{x}$ is a Nash equilibrium of $\Phi(u)$ (see Predtetchinski, 2009), and since for every player $i$, $\phi_i(u)$ is equal to $u_i$ up to an element of $\mathcal{A}_i$, we obtain that $\Phi(u) \in \mathcal{U}$ (since $\mathcal{U} + \mathcal{A} = \mathcal{U}$), i.e. $\eta^{-1}(\mathcal{U}) \subset \mathcal{N}_{\mathcal{U}}$.

From the last two results, and from the continuity of $\eta$ and $\eta^{-1}$ (see Predtetchinski, 2009), we obtain that the mapping $\eta_{\mathcal{U}} : \mathcal{N}_{\mathcal{U}} \to \mathcal{U}, (u, x) \mapsto \eta(u, x)$ is a homeomorphism.

Now, consider the following mapping:

$$\begin{cases} H_{\mathcal{U}} : & [0,1] \times \mathcal{N}_{\mathcal{U}} & \to & \mathcal{U} \\ & (t, (u, x)) & \mapsto & t\eta_{\mathcal{U}}(u, x) + (1 - t)\pi_{\mathcal{U}}(u, x) \end{cases}.$$

Notice that it is well-defined since for every $(t, (u, x)) \in [0, 1] \times \mathcal{N}_{\mathcal{U}}$,

$$H_{\mathcal{U}}(t, (u, x)) = t\eta_{\mathcal{U}}(u, x) + (1 - t)\pi_{\mathcal{U}}(u, x) = t(u + a) + (1 - t)u = u + ta,$$

for some $a \in \mathcal{A}$ (from Equation (1) defining $\eta_i$). Thus $H_{\mathcal{U}}(t, (u, x)) \in \mathcal{U}$, since $\mathcal{U} + \mathcal{A} = \mathcal{U}$. We finally show that the mapping $H_{\mathcal{U}}$ is a proper homotopy between $\pi_{\mathcal{U}}$ and $\eta_{\mathcal{U}}$. Actually, the only difficulty is to prove that it is a proper mapping. Let

$$\begin{cases} G^+_{\mathcal{U}} : & [0,1] \times \mathcal{U} \times X & \to & \mathcal{U} + \mathcal{A} = \mathcal{U} \\ & (t, u, x) & \mapsto & \left(y \mapsto u_i(y) + t(\langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(\cdot, x^0_{-i}), y_i - x_i \rangle + \langle x_i, y_i \rangle)\right)_{i \in N} \end{cases}.$$

Remark that the restriction of $G^+_{\mathcal{U}}$ to $[0, 1] \times \mathcal{N}_{\mathcal{U}}$ is equal to $H_{\mathcal{U}}$. Moreover, let

$$\begin{cases} \psi : & [0,1] \times \mathcal{U} \times X & \to & [0,1] \times \mathcal{U} \times X \\ & (t, u, x) & \mapsto & (t, G^+_{\mathcal{U}}(t, u, x), x) \end{cases}.$$

This mapping is invertible: its inverse is the mapping

$$\begin{cases} \psi^{-1} : & [0,1] \times \mathcal{U} \times X & \to & [0,1] \times \mathcal{U} \times X \\ & (t, u, x) & \mapsto & (t, G^-_{\mathcal{U}}(t, u, x), x) \end{cases},$$

where

$$\begin{cases} G^-_{\mathcal{U}} : & [0,1] \times \mathcal{U} \times X & \to & \mathcal{U} + \mathcal{A} = \mathcal{U} \\ & (t, u, x) & \mapsto & \left(y \mapsto u_i(y) - t(\langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(\cdot, x^0_{-i}), y_i - x_i \rangle + \langle x_i, y_i \rangle)\right)_{i \in N} \end{cases}.$$

Also, both $\psi$ and $\psi^{-1}$ are continuous mappings (the proof is similar to the one of Theorem 2 in Predtetchinski (2009)), thus $\psi$ is a homeomorphism. To finish, let $K$ be a compact subspace of $\mathcal{U}$. By definition,

$$\begin{aligned} H^{-1}_{\mathcal{U}}(K) &= \{(t, (u, x)) \in [0, 1] \times \mathcal{N}_{\mathcal{U}} : H_{\mathcal{U}}(t, (u, x)) \in K\} \\ &= \psi^{-1}([0, 1] \times K \times X) \cap ([0, 1] \times \mathcal{N}_{\mathcal{U}}). \end{aligned}$$

Moreover, since $\mathcal{N}_{\mathcal{U}}$ is a closed subset of $\mathcal{U} \times X$,[11] we obtain that $[0,1] \times \mathcal{N}_{\mathcal{U}}$ is also closed in $[0,1] \times \mathcal{U} \times X$, thus that $H_{\mathcal{U}}^{-1}(K)$ is closed in $\psi^{-1}([0,1] \times K \times X)$, which is a compact subspace of $[0,1] \times \mathcal{U} \times X$ (since $K$ is compact, and since $\psi$ is a homeomorphism). This finally implies that $H_{\mathcal{U}}^{-1}(K)$ is a compact subspace of $[0,1] \times \mathcal{N}_{\mathcal{U}}$, thus that $H_{\mathcal{U}}$ is a proper mapping.

### 4.4. *Proof of oddness theorem (Theorem 3.1)*

*Step I. For every set $\mathcal{U} \subset \prod_{i \in N} \mathbb{R}_{\delta_i}[x] \cap \mathcal{F}$ such that $\mathcal{U}$ is semi-algebraic and $\mathcal{U} + \mathcal{A} = \mathcal{U}$, there exists a generic subset $\mathcal{G}$ of $\mathcal{U}$ such that for every $u \in \mathcal{G}$, the game $u$ has an odd number of Nash equilibria.*

In Step I, we prove a slightly different version of Theorem 3.1, with a stronger notion of regularity. Indeed, we suppose that: (i) $\mathcal{U} \subset \prod_{i \in N} \mathbb{R}_{\delta_i}[x] \cap \mathcal{F}$ (in particular, payoff functions can have a constant part); (ii) $\mathcal{U}$ is semi-algebraic; (iii) and $\mathcal{U} + \mathcal{A} = \mathcal{U}$ (i.e. the set $\mathcal{U}$ is stable under affine addition). The proof of Step I is divided into several parts: in Step 1, it is proved that $\mathcal{U}$ can be decomposed into a union of spaces $\mathcal{V}^k$ of polynomial games satisfying the three above properties (in particular it is semi-algebraic and satisfies the equality $\mathcal{V}^k + \mathcal{A} = \mathcal{V}^k$), with the additional property that each $\mathcal{V}^k$ is homeomorphic to some Euclidean space (thus $\mathcal{V}^k$ is a topological manifold: this is a crucial property to be able to apply topological degree later in the proof). In particular, we can apply our structure theorem (Theorem 2.2) on each set $\mathcal{V}^k$, and since this set homeomorphic to some Euclidean space, we can use techniques similar to those used for mixed extensions of finite games, to get the existence of a generic semi-algebraic subset $\mathcal{G}^k$ of $\mathcal{V}^k$ such that for every $u \in \mathcal{G}^k$, the game $u$ has an odd number of Nash equilibria (see Step 2 to Step 6). Then, in Step 7, we prove that the union $\mathcal{G}$ of all the $\mathcal{G}^k$ which are open in $\mathcal{U}$ is generic in $\mathcal{U}$, and satisfies oddness theorem.

*Step 1. A decomposition result.*

For every $i \in N$, we consider the linear subspace

$$\tilde{\mathbb{R}}_{\delta_i}[x] := \mathrm{Span}\big(\{x^k : k \in \mathbb{N}^\Lambda, \deg(k) \le \delta_i\} \backslash (\{x_{i,j} : j = 1, \dots, \lambda_i\} \cup \{1\})\big)$$

of $\mathbb{R}_{\delta_i}[x]$ generated by all the monomials in $\mathbb{R}_{\delta_i}[x]$, except the ones in $\mathcal{A}_i$.

By definition, $\mathbb{R}_{\delta_i}[x] = \tilde{\mathbb{R}}_{\delta_i}[x] \oplus \mathcal{A}_i$, for every $i \in N$, and $\prod_{i \in N} \mathbb{R}_{\delta_i}[x] = \prod_{i \in N} \tilde{\mathbb{R}}_{\delta_i}[x] \oplus \mathcal{A}$. Denote by $\hat{\pi}_{-\mathcal{A}}$ the linear projection from $\prod_{i \in N} \mathbb{R}_{\delta_i}[x] = \prod_{i \in N} \tilde{\mathbb{R}}_{\delta_i}[x] \oplus \mathcal{A}$ to $\prod_{i \in N} \tilde{\mathbb{R}}_{\delta_i}[x]$. Since $\mathcal{U}$ is semi-algebraic (from semi-algebraicity assumption in Definition 3.1) and since $\hat{\pi}_{-\mathcal{A}}$ is a semi-algebraic mapping, from Tarski-Seidenberg's theorem (see Proposition 4.2 in Section 4.1), the set $\hat{\pi}_{-\mathcal{A}}(\mathcal{U})$ is semi-algebraic. In particular, from the decomposition result for semi-algebraic sets (see Definition 4.3 in Section 4.1),

$$\hat{\pi}_{-\mathcal{A}}(\mathcal{U}) = \bigcup_{k=1}^{\kappa} \mathcal{U}^k,$$

the union being disjoint, where for every $k \in \{1, \dots, \kappa\}$, the set $\mathcal{U}^k$ is homeomorphic to $(0,1)^{d_k}$ (for some $d_k \in \mathbb{N}$). For every $k \in \{1, \dots, \kappa\}$, we now consider the set

$$\mathcal{V}^k := \mathcal{U}^k + \mathcal{A}.$$

We can notice that for every $k \in \{1, \dots, \kappa\}$:

1. $\mathcal{V}^k$ is included in $\mathcal{F}$. Indeed, $\mathcal{U}^k \subset \hat{\pi}_{-\mathcal{A}}(\mathcal{U}) \subset \mathcal{F}$, since $\mathcal{U} \subset \mathcal{F}$ and since substracting any affine mapping to a payoff function in $\mathcal{F}_i$ preserves its concavity property. Similarly, $\mathcal{V}^k = \mathcal{U}^k + \mathcal{A} \subset \mathcal{F}$.
2. $\mathcal{V}^k$ is semi-algebraic because $\mathcal{V}^k = \mathcal{U}^k + \mathcal{A}$ is the sum of two semi-algebraic sets.
3. $\mathcal{V}^k + \mathcal{A} = (\mathcal{U}^k + \mathcal{A}) + \mathcal{A} = \mathcal{U}^k + \mathcal{A} = \mathcal{V}^k$.
4. $\mathcal{V}^k$ is homeomorphic to $(0,1)^{e_k}$ (for some $e_k \in \mathbb{N}$).[12] In particular, $\mathcal{V}^k$ is homeomorphic to some Euclidean space.

Moreover, we can also notice that the family $(\mathcal{V}^k)_{k=1}^{\kappa}$ forms a covering of $\mathcal{U}$, since $\bigcup_{k=1}^{\kappa} \mathcal{V}^k = \hat{\pi}_{-\mathcal{A}}(\mathcal{U}) + \mathcal{A} = \mathcal{U}$ (from the assumption $\mathcal{U} + \mathcal{A} = \mathcal{U}$).

From now on, and until Step 7, we fix some element $k \in \{1, \dots, \kappa\}$.

*Step 2. The set $\mathcal{N}_{\mathcal{V}^k}$ (the graph of Nash equilibria associated to $\mathcal{V}^k$) is semi-algebraic, and the mapping $\eta_{\mathcal{V}^k}$ is a semi-algebraic homeomorphism from $\mathcal{N}_{\mathcal{V}^k}$ to $\mathcal{V}^k$. In particular, $\dim(\mathcal{N}_{\mathcal{V}^k}) = \dim(\mathcal{V}^k)$, and both $\mathcal{N}_{\mathcal{V}^k}$ and $\mathcal{V}^k$ are oriented connected topological manifolds of the same dimension.*

First, we show that $\mathcal{N}_{\mathcal{V}^k}$ is semi-algebraic. From the concavity assumption of payoff functions in each $\mathcal{U}_i$ and from the first-order necessary (and sufficient) condition at a maximum for a concave function, the condition $(u,x) \in \mathcal{N}_{\mathcal{V}^k}$, that is "$x$ is a Nash equilibrium of $u$", is equivalent to the following condition: for every $i \in N$, for every $y_i \in X_i$,

[11] Indeed, for every $(u,x) \in \mathcal{F} \times X$, $(u,x) \in \mathcal{N}$ if and only if $(u,x) \in \bigcap_{i \in N, d_i \in X_i} \Psi_{i,d_i}^{-1}((-\infty, 0])$, where for every $i \in N$ and every $d_i \in X_i$, $\Psi_{i,d_i} : \mathcal{F} \times X \to \mathbb{R}, (u,x) \mapsto u_i(d_i, x_{-i}) - u_i(x)$. Since for every $i \in N$ and every $d_i \in X_i$, $\Psi_{i,d_i}$ is continuous (see Preliminaries in Predtetchinski, 2004), we get that $\mathcal{N}$ is closed in $\mathcal{F} \times X$ as an intersection of closed subsets of $\mathcal{F} \times X$. Similarly, we get that $\mathcal{N}_{\mathcal{U}}$ is closed in $\mathcal{U} \times X$.

[12] Indeed, the mapping $f : \mathcal{U}^k \times \mathcal{A} \to \mathcal{U}^k + \mathcal{A}, (u,v) \mapsto u + v$ is such a homeomorphism, since the linear subspace of $\prod_{i \in N} \mathbb{R}_{\delta_i}[x]$ generated by $\mathcal{U}^k$ and the linear subspace $\mathcal{A}$ of $\prod_{i \in N} \mathbb{R}_{\delta_i}[x]$ are in direct sum. Then, we can use that $\mathcal{U}^k$ is homeomorphic to $(0,1)^{d_k}$ and $\mathcal{A}$ is homeomorphic to $(0,1)^{e'_k}$ (for some $e'_k \in \mathbb{N}$).

$$\langle \nabla_{x_i} u_i(\cdot, x_{-i}), y_i - x_i \rangle \le 0.$$

This condition involves semi-algebraic mappings and involves quantifiers defined on semi-algebraic sets, thus we get that $\mathcal{N}_{\mathcal{V}^k}$ is semi-algebraic (see Proposition 4.1 in Section 4.1).

Second, we show that the mapping $\eta_{\mathcal{V}^k}$ is semi-algebraic, which is equivalent to say that for every $i \in N$, the mapping

$$(u, x) \in \mathcal{N}_{\mathcal{V}^k} \mapsto \left(y \mapsto u_i(y) + \langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(., x^0_{-i}), y_i - x_i \rangle + \langle x_i, y_i \rangle\right)$$

is a semi-algebraic mapping. We can see directly that this is the case, since each coefficient of the polynomial function

$$y \in X \mapsto u_i(y) + \langle \nabla_{x_i} u_i(\cdot, x_{-i}) - \nabla_{x_i} u_i(., x^0_{-i}), y_i - x_i \rangle + \langle x_i, y_i \rangle$$

is a polynomial function of the coefficients of $u \in \mathcal{V}^k$ and of $x \in \mathrm{N}(u)$.

Now, since we know that $\eta_{\mathcal{V}^k}$ is a semi-algebraic homeomorphism from $\mathcal{N}_{\mathcal{V}^k}$ to $\mathcal{V}^k$ (from Theorem 2.2), this implies that $\dim(\mathcal{N}_{\mathcal{V}^k}) = \dim(\mathcal{V}^k)$ (see Bochnak et al., 1998, Theorem 2.8.8.). Finally, since $\mathcal{V}^k$ is homeomorphic to some Euclidean space (from Step 1), this ends Step 2.

*Step 3. The degree of $\pi_{\mathcal{V}^k} : \mathcal{N}_{\mathcal{V}^k} \to \mathcal{V}^k$ is equal to $-1$ or 1. In particular, $\pi_{\mathcal{V}^k}$ is surjective.*

From Step 2, we know that $\mathcal{N}_{\mathcal{V}^k}$ and $\mathcal{V}^k$ are oriented connected topological manifolds of the same dimension, and in particular, we can apply topological degree. Since $\eta_{\mathcal{V}^k}$ is a proper homeomorphism from $\mathcal{N}_{\mathcal{V}^k}$ to $\mathcal{V}^k$, we get that $\deg(\eta_{\mathcal{V}^k}) \in \{-1, +1\}$ (see Appendix 4.2, Property (iii)). Moreover, from Theorem 2.2, $\pi_{\mathcal{V}^k}$ and $\eta_{\mathcal{V}^k}$ are properly homotopic. Thus, from homotopy invariance of topological degree (see Appendix 4.2, Property (i)), we get that $\deg(\pi_{\mathcal{V}^k}) \in \{-1, +1\}$, which implies that $\pi_{\mathcal{V}^k}$ is a surjective mapping (see Appendix 4.2, Property (ii)).

*Step 4. There exists a generic semi-algebraic subset $\mathcal{G}^k$ of $\mathcal{V}^k$ such that for every $u \in \mathcal{G}^k$, the set of Nash equilibria of $u$ is nonempty and finite (its cardinal is denoted $K_u$).*

Indeed, $\pi_{\mathcal{V}^k} : \mathcal{N}_{\mathcal{V}^k} \to \mathcal{V}^k$ is a surjective, semi-algebraic continuous mapping (from Step 3) and $\dim(\mathcal{N}_{\mathcal{V}^k}) = \dim(\mathcal{V}^k)$ (from Step 1). Thus, from Theorem 4.2 (see Section 4.1), we obtain the existence of a generic semi-algebraic subset $\mathcal{G}^k$ of $\mathcal{V}^k$ (now fixed) such that for every $u \in \mathcal{G}^k$, $(\pi_{\mathcal{V}^k})^{-1}(u) = \{u\} \times \mathrm{N}(u)$ is nonempty and finite, which ends Step 4.

*Step 5. For every $u \in \mathcal{G}^k$, there exists an open subset $V_u$ of $\mathcal{V}^k$ containing $u$ such that $(\pi_{\mathcal{V}^k})^{-1}(V_u)$ is a union of pairwise disjoint open sets $(V_u^\ell)_{\ell \in \mathcal{K}}$ (where $\mathcal{K}$ is a finite set) and such that for every $\ell \in \mathcal{K}$, $\pi_{\mathcal{V}^k}|_{V_u^\ell}$ is a homeomorphism between $V_u^\ell$ and $V_u$.*

Let $u \in \mathcal{G}^k$, and let $C$ be the connected component of $\mathcal{G}^k$ containing $u$. From Step 4, we know that $(\pi_{\mathcal{V}^k})^{-1}(u)$ is nonempty and finite, of cardinal $K_u$. Moreover, from Theorem 4.2 (see Section 4.1), there exists an open neighborhood $V_u$ of $u$ included in $C$ such that $(\pi_{\mathcal{V}^k})^{-1}(V_u)$ is a union of pairwise disjoint open sets $(V_u^\ell)_{\ell \in \mathcal{K}}$ (where $\mathcal{K}$ is a finite set such that $\mathrm{card}(\mathcal{K}) = K_u$, which only depends on $C$) such that for every $\ell \in \mathcal{K}$, $\pi_{\mathcal{V}^k}|_{V_u^\ell}$ is a homeomorphism between $V_u^\ell$ and $V_u$.

*Step 6. Every game $u \in \mathcal{G}^k$ has an odd number of Nash equilibria.*

We want to prove that for every $u \in \mathcal{G}^k$, the integer $K_u$ (introduced in Step 4) is odd. From Step 4 and Step 5, since $(\pi_{\mathcal{V}^k})^{-1}(u)$ is a finite subset of $\bigcup_{\ell \in \mathcal{K}} V_u^\ell$ (with cardinal equal to $K_u = \mathrm{card}(\mathcal{K})$) and since for every $\ell \in \mathcal{K}$, $\pi_{\mathcal{V}^k}|_{V_u^\ell}$ is a homeomorphism between $V_u^\ell$ and $V_u$, we obtain that

$$(\pi_{\mathcal{V}^k})^{-1}(u) = \{x_u^\ell : \ell \in \mathcal{K}\},$$

for some $x_u^\ell \in V_u^\ell$, $\ell \in \mathcal{K}$. Also, since $\pi_{\mathcal{V}^k}|_{V_u^\ell} : V_u^\ell \to V_u$ is a homeomorphism, we obtain that $\deg(\pi_{\mathcal{V}^k}|_{V_u^\ell})$ is equal to 1 or $-1$, for every $\ell \in \mathcal{K}$ (see Appendix 4.2, Property (iii)). Now, from additivity of topological degree (see Appendix 4.2, Property (iv)) and from Step 3, we obtain (modulo 2), that

$$\deg(\pi_{\mathcal{V}^k}) = 1 = \sum_{\ell \in \mathcal{K}} \deg(\pi_{\mathcal{V}^k}|_{V_u^\ell}) = \sum_{\ell \in \mathcal{K}} 1 = \mathrm{card}(\mathcal{K}) = K_u \ [2],$$

i.e. that $K_u$ is odd.

*Step 7. There exists a generic semi-algebraic subset $\mathcal{G}$ of $\mathcal{U}$ such that for every $u \in \mathcal{G}$, $u$ has an odd number of Nash equilibria.*

For every $k \in \{1, \dots, \kappa\}$, applying Step 2 to Step 6 to the set $\mathcal{V}^k$ built in Step 1, there exists a generic semi-algebraic subset $\mathcal{G}^k$ of $\mathcal{V}^k = \mathcal{U}^k + \mathcal{A}$ such that for every $u \in \mathcal{G}^k$, the game $u$ has an odd number of Nash equilibria. This implies that for every $u$ in

$$\mathcal{G} := \bigcup_{\substack{k \in \{1, \dots, \kappa\} \\ \mathcal{U}^k \text{ is open in } \hat{\pi}_{-\mathcal{A}}(\mathcal{U})}} \mathcal{G}^k,$$

the game $u$ has an odd number of Nash equilibria. To finish, we prove that $\mathcal{G}$ is generic in $\mathcal{U}$. First, from Step 1, we get that the complement of $\mathcal{G}$ in $\mathcal{U}$ is included in

$$\bigcup_{k=1}^{\kappa}(\mathcal{V}^k\backslash\mathcal{G}^k) \cup \tilde{\mathcal{V}},$$

where

$$\tilde{\mathcal{V}} := \bigcup_{\substack{k\in\{1,\dots,\kappa\} \\ \mathcal{U}^k \text{ is not open in } \hat{\pi}_{-\mathcal{A}}(\mathcal{U})}} \mathcal{V}^k.$$

Indeed, suppose that $u \in \mathcal{U}$ and $u \notin \mathcal{G}$. From Step 1, the family $(\mathcal{V}^k)_{k=1}^{\kappa}$ forms a covering of $\mathcal{U}$, and as $u \in \mathcal{U}$, there exists $l \in \{1,\dots,\kappa\}$ such that $u \in \mathcal{V}^l$. Since $u \notin \mathcal{G}$, (1) either $\mathcal{U}^l$ is open in $\hat{\pi}_{-\mathcal{A}}(\mathcal{U})$, in which case $u \notin \mathcal{G}^l$ by the definition of $\mathcal{G}$, thus $u \in \mathcal{V}^l\backslash\mathcal{G}^l \subset \bigcup_{k=1}^{\kappa}(\mathcal{V}^k\backslash\mathcal{G}^k)$; (2) or $\mathcal{U}^l$ is not open in $\hat{\pi}_{-\mathcal{A}}(\mathcal{U})$, and we get $u \in \tilde{\mathcal{V}}$, which finally proves the inclusion.

Second, the dimension of $\bigcup_{k=1}^{\kappa}(\mathcal{V}^k\backslash\mathcal{G}^k)$ is strictly less than $\dim(\mathcal{U})$ because for every $k \in \{1,\dots,\kappa\}$, $\dim(\mathcal{V}^k\backslash\mathcal{G}^k) < \dim(\mathcal{V}^k) \le \dim(\mathcal{U})$ (the first strict inequality is a consequence of $\mathcal{G}^k$ being generic in $\mathcal{V}^k$). Also, the dimension of $\tilde{\mathcal{V}}$ is strictly less than $\dim(\mathcal{U})$, because for every $k \in \{1,\dots,\kappa\}$ such that $\mathcal{U}^k$ is not open in $\hat{\pi}_{-\mathcal{A}}(\mathcal{U})$, we get $\dim(\mathcal{U}^k) < \dim(\hat{\pi}_{-\mathcal{A}}(\mathcal{U}))$ (see Definition 4.3 in Section 4.1). This implies that

$$\begin{aligned}
\dim(\mathcal{V}^k) &= \dim(\mathcal{U}^k + \mathcal{A}) \\
&= \dim(\mathcal{U}^k) + \dim(\mathcal{A}) \text{ (since Span}(\mathcal{U}^k) \cap \mathcal{A} = \emptyset) \\
&< \dim(\hat{\pi}_{-\mathcal{A}}(\mathcal{U})) + \dim(\mathcal{A}) \\
&= \dim(\hat{\pi}_{-\mathcal{A}}(\mathcal{U}) + \mathcal{A}) \text{ (since Span}(\hat{\pi}_{-\mathcal{A}}(\mathcal{U})) \cap \mathcal{A} = \emptyset) \\
&= \dim(\mathcal{U}) \text{ (since from } \mathcal{U} + \mathcal{A} = \mathcal{U} \text{ we get } \hat{\pi}_{-\mathcal{A}}(\mathcal{U}) + \mathcal{A} = \mathcal{U})
\end{aligned}$$

Finally, this proves that dimension of the complement of $\mathcal{G}$ in $\mathcal{U}$ is strictly less than the dimension of $\mathcal{U}$.

Third, $\mathcal{G}$ is open in $\mathcal{U}$. Indeed, $\mathcal{G}^k$ is generic in $\mathcal{V}^k$, which implies that it is open in $\mathcal{V}^k$. Thus, for every $k \in \{1,\dots,\kappa\}$ such that $\mathcal{U}^k$ is open in $\hat{\pi}_{-\mathcal{A}}(\mathcal{U})$, we get that $\mathcal{V}^k = \mathcal{U}^k + \mathcal{A}$ is open in $\hat{\pi}_{-\mathcal{A}}(\mathcal{U}) + \mathcal{A} = \mathcal{U}$ (from the assumption $\mathcal{U} + \mathcal{A} = \mathcal{U}$).

Finally, by definition, the set $\mathcal{G}$ is generic in $\mathcal{U}$, and for every $u \in \mathcal{G}$, the game $u$ has an odd number of Nash equilibria, which ends Step I.

*Step II. For every subset $\mathcal{U}$ of $\mathcal{F}$ such that: (i) $\mathcal{U} \subset \prod_{i\in N}\mathbb{R}_{\delta_i}[x] \cap \mathcal{F}$; (ii) $\mathcal{U}$ is semi-algebraic, and (iii) $\dim(\mathcal{U} + \mathcal{A}) = \dim(\mathcal{U})$, there exists a generic subset $\mathcal{G}'$ of $\mathcal{U}$ such that for every $u \in \mathcal{G}'$, the game $u$ has an odd number of Nash equilibria.*

In this step, we replace the assumption $\mathcal{U} + \mathcal{A} = \mathcal{U}$ made in Step I by the weaker assumption $\dim(\mathcal{U} + \mathcal{A}) = \dim(\mathcal{U})$. To prove Step II, we simply apply Step I to the set $\mathcal{U} + \mathcal{A}$: we have $\mathcal{U} + \mathcal{A} \subset \mathcal{F}$ (because $\mathcal{U} \subset \mathcal{F}$), it is semi-algebraic (as the sum of two semi-algebraic sets), and satisfies the equality $(\mathcal{U} + \mathcal{A}) + \mathcal{A} = \mathcal{U} + \mathcal{A}$. Thus, from Step I, there exists a generic subset $\mathcal{G}$ of $\mathcal{U} + \mathcal{A}$ such that for every $u \in \mathcal{G}$, the game $u$ has an odd number of Nash equilibria. Now, let $\mathcal{G}' = \mathcal{U} \cap \mathcal{G}$. In particular, for every $u \in \mathcal{G}'$, the game $u$ has an odd number of Nash equilibria. Finally, remark that the set $\mathcal{G}'$ is generic in $\mathcal{U}$, from the following lemma applied to $S = \mathcal{U}$, $T = \mathcal{U} + \mathcal{A}$, and $G = \mathcal{G}$.

**Lemma 4.1.** *Let $S$ and $T$ be two semi-algebraic sets such that $S \subset T$ and $\dim(S) = \dim(T)$. If $G$ is a generic semi-algebraic subset of $T$, then $S \cap G$ is a generic semi-algebraic subset of $S$.*

**Proof.** We can remark that $S\backslash(S \cap G) \subset T\backslash G$, so that $\dim(S\backslash(S \cap G)) \le \dim(T\backslash G)$. Now, since $\dim(T\backslash G) < \dim(T)$ (from genericity of $G$ in $T$) and since $\dim(S) = \dim(T)$ (by assumption), we get that $\dim(S\backslash(S \cap G)) < \dim(S)$. Moreover, $S \cap G$ is open in $S$: indeed, $G$ is open in $T$ (from genericity of $G$ in $T$) and $S \subset T$, so by definition of the induced topology on $S$, we get directly the result. □

This ends Step II.

*Step III. Oddness theorem.*

In this last step, we treat the general case. Namely, we now suppose that: (i) $\mathcal{U} \subset \prod_{i\in N}\mathbb{R}^0_{\delta_i}[x] \cap \mathcal{F}$ (i.e. payoff functions have no more any constant part); (ii) $\mathcal{U}$ is semi-algebraic; (iii) and $\mathcal{U}$ satisfies stability assumption (i.e. $\dim(\mathcal{U} + \mathcal{L}) = \dim(\mathcal{U})$). The idea is to apply Step II to the set $\mathcal{U} + C^n$ (the set obtained from $\mathcal{U}$ by adding all possible constant functions). Thus, we have to prove that it satisfies the assumptions in Step II: first, we have $\mathcal{U} + C^n \subset \mathcal{F}$ (from $\mathcal{U} \subset \mathcal{F}$), second it is semi-algebraic (from Corollary 4.1 in Section 4.1, since both $\mathcal{U}$ and $C^n$ are semi-algebraic sets), and third we have:

$$\begin{aligned}
\dim((\mathcal{U} + C^n) + \mathcal{A}) &= \dim((\mathcal{U} + C^n) + (\mathcal{L} + C^n)) \\
&= \dim((\mathcal{U} + \mathcal{L}) + C^n) \\
&= \dim(\mathcal{U} + \mathcal{L}) + \dim(C^n) \text{ (since span}(\mathcal{U} + \mathcal{L}) \cap C^n = \{0\}) \\
&= \dim(\mathcal{U}) + \dim(C^n) \text{ (since by regularity, } \dim(\mathcal{U} + \mathcal{L}) = \dim(\mathcal{U})) \\
&= \dim(\mathcal{U} + C^n) \text{ (since span}(\mathcal{U}) \cap C^n = \{0\}).
\end{aligned}$$

Thus, from Step II, there exists a generic subset $\mathcal{G}'$ of $\mathcal{U} + C^n$ such that for every $u \in \mathcal{G}'$, the game $u$ has an odd number of Nash equilibria.

Now, remark that for every $i \in N$, $\mathbb{R}_{\delta_i}[x] = \mathbb{R}^0_{\delta_i}[x] \oplus C$, and that $\prod_{i\in N} \mathbb{R}_{\delta_i}[x] = \prod_{i\in N} \mathbb{R}^0_{\delta_i}[x] \oplus C^n$. Denote by $\hat{\pi}_{-C^n}$ the linear projection from $\prod_{i\in N} \mathbb{R}_{\delta_i}[x] = \prod_{i\in N} \mathbb{R}^0_{\delta_i}[x] \oplus C^n$ to $\prod_{i\in N} \mathbb{R}^0_{\delta_i}[x]$. To finish the proof, define $\mathcal{U}^* = \hat{\pi}_{-C^n}(\mathcal{G}')$. In particular, for every $u \in \mathcal{U}^*$, the game $u$ has an odd number of Nash equilibria (indeed, for every game $u = (u_1, \dots, u_n)$ and every $c = (c_1, \dots, c_n) \in C^n$, the set $\mathrm{N}(u)$ of Nash equilibria of $u$ and the set $\mathrm{N}(u + c)$ of Nash equilibria of $u + c$ are equal). Thus, we only have to prove that $\mathcal{U}^*$ is generic in $\mathcal{U}$. Since $\mathcal{G}'$ is generic in $\mathcal{U} + C^n$, the result follows from the following lemma applied to $E_1 = \prod_{i\in N} \mathbb{R}^0_{\delta_i}[x]$, $E_2 = S_2 = C^n$, $S_1 = \mathcal{U}$, and $T = \mathcal{G}'$.

**Lemma 4.2.** *Let $E_1$ and $E_2$ be two finite-dimensional vector subspaces of $\prod_{i\in N} \mathbb{R}[x]$ such that $E_1 \cap E_2 = \{0\}$. If $T$ is a generic semi-algebraic subset of $S_1 + S_2$, where $S_1$ is a semi-algebraic subset of $E_1$ and $S_2$ is a semi-algebraic subset of $E_2$, then $\pi_{E_1}(T)$ is a generic semi-algebraic subset of $S_1$.*

**Proof.** First,[13] notice that $\pi_{E_1}(T)$ is semi-algebraic, from Tarski-Seidenberg's theorem. Let us now prove that $\pi_{E_1}(T)$ is open in $S_1$. If $x_1 \in \pi_{E_1}(T)$, then there exists $t \in T$ such that $x_1 = \pi_{E_1}(t)$, and from the definition of $T$, there exists $x_2 \in S_2$ such that $t = x_1 + x_2$. Since $T$ is open in $S_1 + S_2$ (because it is generic in $S_1 + S_2$), there exists a neighborhood $V_t \subset T$ of $t$ in $S_1 + S_2$. Consider the mapping $f : (e_1, e_2) \in E_1 \times E_2 \mapsto e_1 + e_2$. Since it is continuous at $(x_1, x_2)$, and since $V_t$ is a neighborhood of $t = f(x_1, x_2)$, there exists a neighborhood $V$ of $(x_1, x_2)$ such that $f(V) \subset V_t$. But the set of all $V_{e_1} \times V_{e_2}$ (where $(e_1, e_2) \in E_1 \times E_2$, $V_{e_1}$ is a neighborhood of $e_1$ and $V_{e_2}$ is a neighborhood of $e_2$) forms a neighborhood basis of $(e_1, e_2)$ for the product topology on $E_1 \times E_2$, so there exists in particular some neighborhoods $V_{x_1}$ of $x_1$ and $V_{x_2}$ of $x_2$ such that $V_{x_1} \times V_{x_2} \subset V$, and we finally have $f(V_{x_1} \times V_{x_2}) \subset V_t$ (i.e. $V_{x_1} + V_{x_2} \subset V_t$). In particular, $V_{x_1}$ is a neighborhood of $x_1$ included in $\pi_{E_1}(T)$ (because for every $x_1' \in V_{x_1}$, $t' := x_1' + x_2 \in V_{x_1} + V_{x_2} \subset V_t \subset T$, thus $\pi_{E_1}(t') = x_1'$, which implies that $x_1' \in \pi_{E_1}(T)$), thus $\pi_{E_1}(T)$ is open in $S_1$. Now, suppose by contradiction that $\pi_{E_1}(T)$ is not generic in $S_1$. This means that $\dim(S_1 \backslash \pi_{E_1}(T)) = \dim(S_1)$, and hence that

$$\dim((S_1 \backslash \pi_{E_1}(T)) \times S_2) = \dim(S_1 \times S_2) = \dim(S_1 + S_2)$$

(since, by assumption, $E_1$ and $E_2$ are in direct sum). But $(S_1 \backslash \pi_{E_1}(T)) \times S_2 \subset (S_1 \times S_2) \backslash T$ (because if $(x_1, x_2) \in (S_1 \times S_2) \cap T$, we get that $x_1 \in \pi_{E_1}(T)$), which implies that

$$\dim(S_1 + S_2) = \dim((S_1 \backslash \pi_{E_1}(T)) \times S_2) \le \dim((S_1 \times S_2) \backslash T) \le \dim(S_1 \times S_2) = \dim(S_1 + S_2),$$

which finally contradicts the genericity of $T$ in $S_1 + S_2$. $\square$

This ends Step III, and ends the proof of Theorem 3.1.

## Declaration of competing interest

None.

## Data availability

No data was used for the research described in the article.

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