Nakamura Labo. Seminar No.1

Limit of Sequence of Real Numbers and Vector Spaces

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Contents

Definition (convergence of sequence of real numbers)

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then,

$$(a_n)_{n=1}^{\infty} \textbf{ converges} \text{ to } \alpha \in \mathbb{R}$$
$$\overset{\text{def}}{\Leftrightarrow} \forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} \text{ s.t. } \forall n \geq N(\varepsilon) \Rightarrow |a_n - \alpha| < \varepsilon.$$

One writes

$$\lim_{n \to \infty} a_n = \alpha \text{ or } a_n \to \alpha \ (n \to \infty).$$

Then, α is a **limit** of $(a_n)_{n=1}^{\infty}$. If $(a_n)_{n=1}^{\infty}$ does not converge, we say $(a_n)_{n=1}^{\infty}$ **diverges**.

Property

(a) (convergence of monotone bounded sequences)
Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then,

$(a_n)_{n=1}^{\infty}$ is a **monotonically increasing (decreaing) sequence**
$\overset{\text{def}}{\Leftrightarrow} \forall n, a_n \leq a_{n+1}$ $(a_n \geq a_{n+1})$.

For all monotone sequences of real numbers $(a_n)_{n=1}^{\infty}$,

$$(a_n)_{n=1}^{\infty} \text{ converges } \Leftrightarrow (a_n)_{n=1}^{\infty} \text{ is bounded.}$$

Especially, if $(a_n)_{n=1}^{\infty}$ is bounded monotonically increasing (decreaing) sequence,

$$\lim_{n\to\infty} a_n = \sup a_n \ (\lim_{n\to\infty} a_n = \inf a_n).$$

Property

(b) (Cauchy(1789-1857))
Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then,

$$
\begin{aligned}
&(a_n)_{n=1}^{\infty}, \exists \alpha \in \mathbb{R}, (a_n)_{n=1}^{\infty} \text{ converges } \alpha \\
&\Leftrightarrow (a_n)_{n=1}^{\infty}, \forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, N(\varepsilon) \leq n, m \Rightarrow |a_n - a_m| < \varepsilon.
\end{aligned}
$$

Definition (limit superior and limit inferior)

For all sequences of real numbers $(a_n)_{n=1}^{\infty}$,

(a) We define **limit superior** $\limsup\limits_{n\to\infty} a_n$ as follows:

(i) If $(a_n)_{n=1}^{\infty}$ is not upper bounded,

$$\limsup_{n\to\infty} a_n := +\infty.$$

(ii) If $(a_n)_{n=1}^{\infty}$ is upper bounded, we define a new sequence $(\check{a}_p)_{p=1}^{\infty}$ as follows:

$$(\check{a}_p)_{p=1}^{\infty} := \sup\{a_n \mid n \geq p\} = \sup\{a_p, a_{p+1}, \ldots\},$$

and define $\limsup\limits_{n\to\infty} a_n$ as follows:

$$\limsup_{n\to\infty} a_n = \begin{cases} \lim_{p\to\infty} \check{a}_p & ((\check{a}_p)_{p=1}^{\infty} \text{ is lower bounded}) \\ -\infty & ((\check{a}_p)_{p=1}^{\infty} \text{ is not lower bounded}). \end{cases}$$

Definition (limit superior and limit inferior)

(b) Similarly, we define **limit inferior** $\liminf_{n\to\infty} a_n$ as follows:

(i) If $(a_n)_{n=1}^{\infty}$ is not lower bounded,

$$\liminf_{n\to\infty} a_n := -\infty.$$

(ii) If $(a_n)_{n=1}^{\infty}$ is lower bounded, we define a new sequence $(\check{a}_p)_{p=1}^{\infty}$ as follows:

$$(\check{a}_p)_{p=1}^{\infty} := \sup\{a_n \mid n \geq p\} = \inf\{a_p, a_{p+1}, \ldots\},$$

and define $\liminf_{n\to\infty} a_n$ as follows:

$$\liminf_{n\to\infty} a_n = \begin{cases} \lim_{p\to\infty} \check{a}_p & ((\check{a}_p)_{p=1}^{\infty} \text{ is upper bounded}) \\ +\infty & ((\check{a}_p)_{p=1}^{\infty} \text{ is not upper bounded}). \end{cases}$$

Definition (vector spaces over real number field)

Let X be a set with 2 **linear operators** (addition "$+$" and scalar multiplication "$\cdot$"). We call that set X is a **vector space over** $\mathbb{R}$ or **linear space over** $\mathbb{R}$ if set X satisfies the following conditions:

(a) Axiom of commutative group and the following conditions are satisfied. $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$,

  (i) $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} = \boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ (associativity 1),

  (ii) $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{x}$ (commutativity),

  (iii) $\forall \boldsymbol{x} \in X, \ \exists \boldsymbol{0} \in X$ s.t. $\boldsymbol{0}+\boldsymbol{x} = \boldsymbol{x}+\boldsymbol{0} = \boldsymbol{x}$ (existence of identity element of addition),

  (iv) $\forall \boldsymbol{x} \in X, \ \exists \boldsymbol{x}^{-1}$ s.t. $\boldsymbol{x}+\boldsymbol{x}^{-1} = \boldsymbol{x}^{-1}+\boldsymbol{x} = \boldsymbol{0}$ (existence of inverse elements of addition). We denote $\boldsymbol{x}^{-1}$ by $-\boldsymbol{x}$.

Definition (vector spaces over real number field)

(b) Axiom of scalar multiplication and the following conditions are satisfied. $\forall \alpha, \beta \in \mathbb{R}, \forall \boldsymbol{x}, \boldsymbol{y} \in X$,

(i) $\alpha \cdot (\boldsymbol{x} + \boldsymbol{y}) = \alpha \cdot \boldsymbol{x} + \alpha \cdot \boldsymbol{y}$ (distributivity 1),

(ii) $(\alpha + \beta) \cdot \boldsymbol{x} = \alpha \cdot \boldsymbol{x} + \beta \cdot \boldsymbol{x}$ (distributivity 2),

(iii) $\alpha \cdot (\beta \cdot \boldsymbol{x}) = (\alpha \cdot \beta) \cdot \boldsymbol{x}$ (associativity 2),

(iv) $\forall \boldsymbol{x} \in X,\ \exists \mathbf{1}$ s.t. $\mathbf{1} \cdot \boldsymbol{x} = \boldsymbol{x} \cdot \mathbf{1} = \boldsymbol{x}$ (existence of identity element of scalar multiplication).

Definition (linear mapping, linear independence)

(a) For 2 vector spaces X, Y over $\mathbb{R}$,

$$\text{mapping } \phi : X \to Y \text{ is a } \textbf{linear mapping}$$
$$\overset{\text{def}}{\Leftrightarrow} \forall \boldsymbol{x}, \boldsymbol{y} \in X, \forall \alpha, \beta \in \mathbb{R}, \ \phi(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) = \alpha\phi(\boldsymbol{x}) + \beta\phi(\boldsymbol{y}).$$

We denote the set of all linear mappings from X to Y by $\mathcal{L}(X, Y)$. And,

$$\text{linear mapping} \phi : X \to Y \text{ is } \textbf{isomorphism}$$
$$\overset{\text{def}}{\Leftrightarrow} \ \phi : X \to Y \text{ is bijection.}$$

Then, we call that X and Y are **isomorphic** (as vector spaces).

Definition (linear mapping, linear independence)

(b) For an infinite vector system $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$,

$$\textbf{linear combination} \text{ of } \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$$

$$\overset{\text{def}}{\Leftrightarrow} \forall \alpha_i \in \mathbb{R}, \ \sum_{i=1}^{m} \alpha_i \boldsymbol{x}_i = \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \cdots + \alpha_m \boldsymbol{x}_m.$$

Especially,

$$\text{vector system } \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\} \text{ is } \textbf{linear independence}$$

$$\overset{\text{def}}{\Leftrightarrow} \sum_{i=1}^{m} \alpha_i \boldsymbol{x}_i = 0 \Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0.$$

Otherwise, if a vector system is not linear independence, we call that the vector system is **linear dependence**.

Definition (linear mapping, linear independence)

(c) (Expand (b) for finite vector systems.)

Property

(a) If all $m+1$ vectors $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m, \boldsymbol{y}_{m+1}$ in vector space X over $\mathbb{R}$ are linear combinations of m vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m \in X$, the vector system $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m, \boldsymbol{y}_{m+1}\}$ is linear dependence.

Property

(b) **maximal system**

Definition (subspace, basis, and dimention)

(a) **subspace**

Definition (subspace, basis, and dimention)

(b) **spanned subspace**

Definition (subspace, basis, and dimention)

(c) **basis**

Definition (subspace, basis, and dimention)

(d) **dimention**