

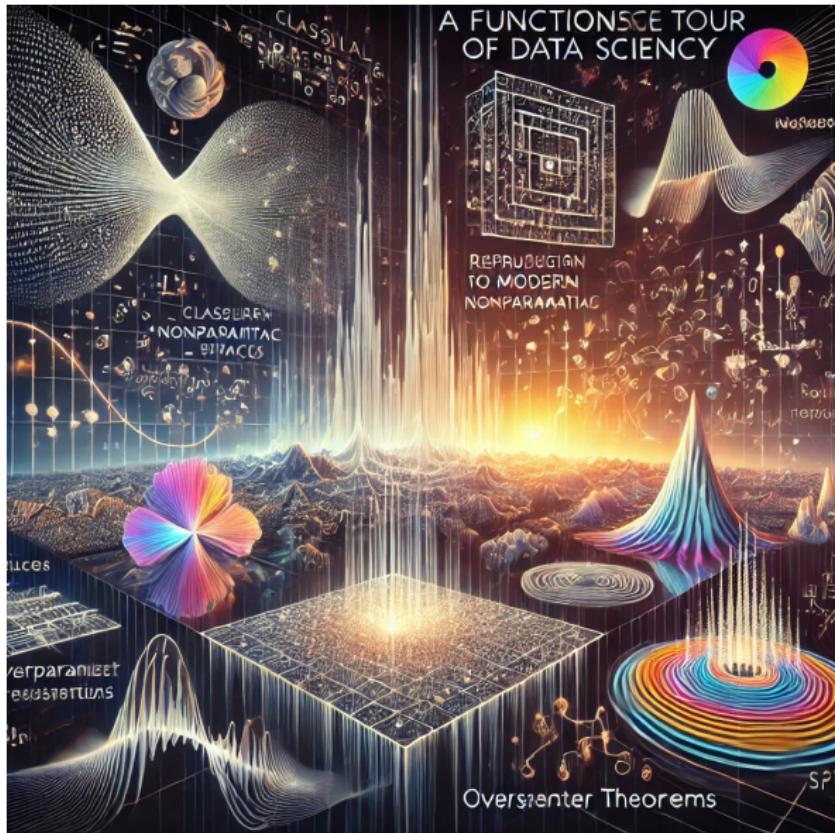
A Function-Space Tour of Data Science

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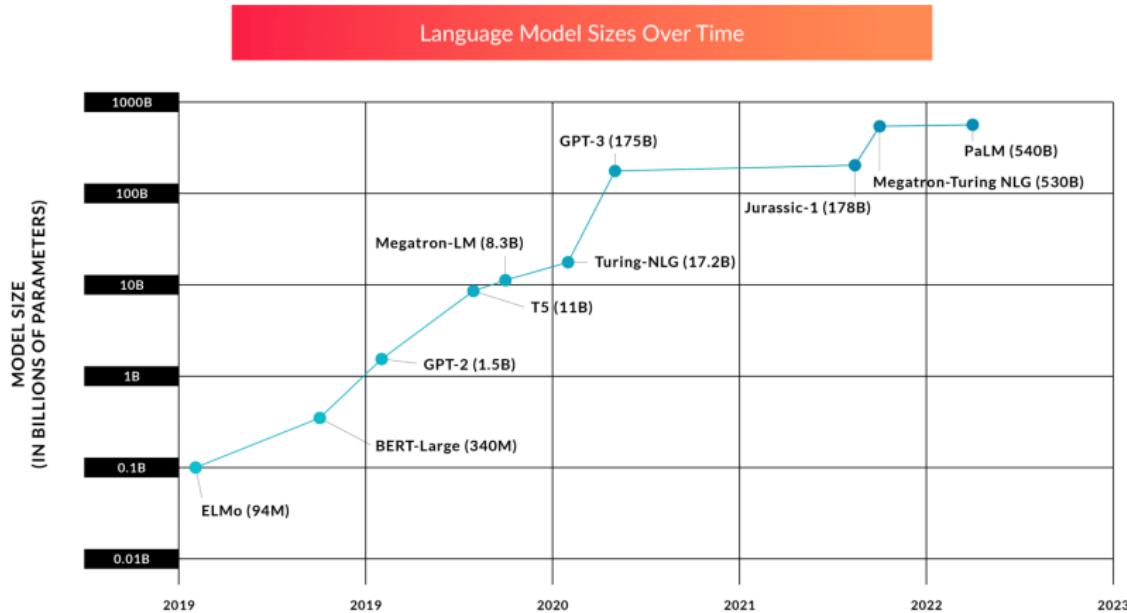
(made in conjunction with Greg Ongie)

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Large Models Have Taken The World By Storm



SOTA Models are Getting Bigger and Bigger



Modern Machine Learning is Overparameterized

Let $\{(\mathbf{x}_i, y_i)\}_{i=1}^n \subset \mathbb{R}^d \times \mathbb{R}$ be a training dataset and let \mathcal{F} be a space of real-valued functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Consider the learning problem

$$\min_{f \in \mathcal{F}} \sum_{i=1}^n (y_i - f(\mathbf{x}_i))^2 + \lambda R(f), \quad \lambda > 0.$$

If \mathcal{F} is an infinite-dimensional space, then this problem is an “overparameterized” learning problem.

Example: Suppose $\{\varphi_k\}_{k \in \mathbb{Z}}$ is a basis for \mathcal{F} . Then, each $f \in \mathcal{F}$ can be represented as a model with an **infinite** parameters $\boldsymbol{\theta} = \{\theta_k\}_{k \in \mathbb{Z}}$ such that

$$f_{\boldsymbol{\theta}} = \sum_{k \in \mathbb{Z}} \theta_k \varphi_k.$$

The norm $R(f)$ reflects the “size” of the parameters $\boldsymbol{\theta}$.

The function-space view is a powerful tool to study the **infinite-parameter** limit of overparameterization.

Nonparametric methods as opposed to **parametric** methods.

Regularization is Necessary

Without regularization (either **implicit** from the optimization algorithm or **explicit** in the optimization problem), the overparameterized learning problem

$$\min_{f \in \mathcal{F}} \sum_{i=1}^n (y_i - f(\mathbf{x}_i))^2$$

is **ill-posed** since there are many interpolating (zero-loss solutions).

Which interpolating function (of the many possible) will be selected?

How does this choice affect performance/generalization?

Without regularization, it becomes challenging to answer these questions.

Plan: A tour through (nonparametric) methods in data science through the unifying lens of **explicit regularization** in **function space**.

Goal: To provide sharp characterizations of the **inductive bias** of various data-fitting methods.

From Parametric to Nonparametric

Let Θ denote the **parameter space**. The associated **parametric model space** is

$$\mathcal{F}_\Theta = \{f_\theta : \theta \in \Theta\}$$

Let $C : \Theta \rightarrow \mathbb{R}_{\geq 0}$ denote a **parameter cost function**. Given any $f \in \mathcal{F}_\Theta$, its **parametric representation cost** is defined by

$$\mathring{R}(f) = \inf\{C(\theta) : f = f_\theta, \theta \in \Theta\}$$

For an arbitrary (measurable) function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we can define its **nonparametric representation cost** as

$$R(f) = \begin{cases} \liminf_{f_k \rightarrow f} \mathring{R}(f), & \exists (f_k)_{k \in \mathbb{N}} \subset \mathcal{F}_\Theta \text{ that converges}^2 \text{ to } f \\ +\infty, & \text{else.} \end{cases}$$

The **native space** is given by

$$\mathcal{F} = \{f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ measurable} : R(f) < +\infty\}.$$

⁴In an appropriate topology.

Function-Space Inductive Bias

Suppose that we have a **parametric method**.

- A parameter space Θ .
- A parametric model space \mathcal{F}_Θ .
 - ⇒ A subset of measurable functions $\mathbb{R}^d \rightarrow \mathbb{R}$.
- A parametric cost $C : \Theta \rightarrow \mathbb{R}_{\geq 0}$.
 - ⇒ $C(\mathbf{0}) = 0$.
 - ⇒ $\|\boldsymbol{\theta}\|_2 \leq \|\boldsymbol{\theta}'\|_2 \Leftrightarrow C(\boldsymbol{\theta}) \leq C(\boldsymbol{\theta}')$.

A parametric method **induces** a native space \mathcal{F} and a corresponding nonparametric representation cost $R : \mathcal{F} \rightarrow \mathbb{R}_{\geq 0}$.

$$\min_{\boldsymbol{\theta} \in \Theta} \sum_{i=1}^n \mathcal{L}(y_i, f_{\boldsymbol{\theta}}(\mathbf{x}_i)) + \lambda C(\boldsymbol{\theta}) \Leftrightarrow \min_{f \in \mathcal{F}} \sum_{i=1}^n \mathcal{L}(y_i, f(\mathbf{x}_i)) + \lambda R(f)$$

This is characterizes **function-space inductive bias** of \mathcal{F}_Θ .

How Do Different Methods Look in Function Space?

- Can we characterize the nonparametric representation cost R ?
- What are the properties of the native space \mathcal{F} ?
 - ⇒ Is it a vector space?
 - ⇒ Is it a metric space?
 - ⇒ Is it a Banach space?
 - ⇒ Is it a Hilbert space?
 - ⇒ Does it have some other (topological) structure?
 - ⇒ How is \mathcal{F} related to classical function spaces?
- How well does the parametric model space \mathcal{F}_Θ approximate \mathcal{F} ?
- How well can we learn functions in \mathcal{F} from data?

This is the **function-space view** of studying data-fitting methods.

The function-space view **unifies** classical and modern data-fitting methods.

Three Remarkable Ideas in Data Science

① Kernel Methods

- ⇒ ℓ^2 -regularization of parameters
- ⇒ Reproducing Kernel Hilbert Spaces
- ⇒ Linear methods = not adaptive

② Wavelet and Sparse Methods

- ⇒ ℓ^1 -regularization of parameters
- ⇒ Besov Spaces and Bounded Variation (BV) Spaces
- ⇒ Nonlinear methods = adaptive

③ Neural Networks

- ⇒ ℓ^2 -regularization of parameters
- ⇒ Barron Spaces, Variation Spaces, and Radon BV Spaces
- ⇒ Nonlinear methods = adaptive
- ⇒ **Shallow vs. deep**

Classical methods were studied function space first.

Can we understand modern methods by characterizing their function spaces?

Kernel Methods

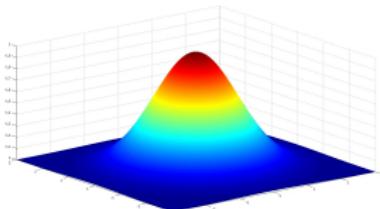
$$\left\{ f_{\mathbf{a}} = \sum_{i=1}^n a_i k(\cdot, \mathbf{x}_i) : \begin{array}{l} n \in \mathbb{N} \\ a_i \in \mathbb{R} \\ \mathbf{x}_i \in \mathbb{R}^d \end{array} \right\}$$

$$C(\mathbf{a}) = \mathbf{a}^\top \mathbf{K} \mathbf{a}$$

\mathcal{F} : RKHS induced by k

$R(f) = \|f\|_{\mathcal{F}}^2$
squared RKHS norm

$$\min_{\mathbf{a} \in \mathbb{R}^n} \sum_{i=1}^n \mathcal{L}(y_i, [\mathbf{K}\mathbf{a}]_i) + \lambda \mathbf{a}^\top \mathbf{K} \mathbf{a} \quad \Leftrightarrow \quad \min_{f \in \mathcal{F}} \sum_{i=1}^n \mathcal{L}(y_i, f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{F}}^2$$



- The equivalence \Leftrightarrow is understood via the **representer theorem**.
- There always exists a solution to the problem over \mathcal{F} that lies in the span of shifted kernels.

Wavelet and Sparse Methods

$$\left\{ f_{\boldsymbol{\theta}} = \sum_{j,k} \theta_{j,k} \psi_{j,k} : \begin{aligned} \psi_{j,k}(x) &= \\ 2^{-j/2} \psi(2^j x - k) \end{aligned} \right\}$$

$$C(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_{\ell^1}$$

\mathcal{F} : Besov space $B_{1,1}^1$

$R(f) = \|f\|_{B_{1,1}^1}$
Besov norm

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^n \mathcal{L}(y_i, f_{\boldsymbol{\theta}}(x_i)) + \lambda \|\boldsymbol{\theta}\|_{\ell^1} \quad \Leftrightarrow \quad \min_{f \in B_{1,1}^1} \sum_{i=1}^n \mathcal{L}(y_i, f(x_i)) + \lambda \|f\|_{B_{1,1}^1}$$



- The equivalence \Leftrightarrow is understood via the **wavelet shrinkage algorithm**.
- There always exists a solution to the problem over $B_{1,1}^1$ that is a sparse combination of wavelets.

Shallow Neural Networks

$$\left\{ f_{\boldsymbol{\theta}}(\mathbf{x}) = \sum_{k=1}^K v_k \sigma(\mathbf{w}_k^\top \mathbf{x}) : \begin{array}{l} v_k \in \mathbb{R} \\ \mathbf{w}_k \in \mathbb{R}^d \end{array} \right\}$$

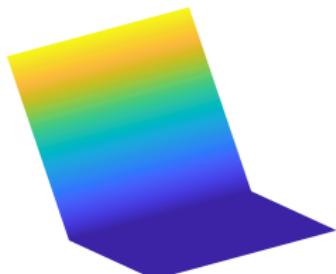
$$C(\boldsymbol{\theta}) = \frac{1}{2} \sum_{k=1}^K |v_k|^2 + \|\mathbf{w}_k\|_2^2$$

\mathcal{F} : Variation Space
(Barron Space,
Radon-BV Space)

$$R(f) = \|f\|_{\mathcal{F}}$$

variation norm

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^n \mathcal{L}(y_i, f_{\boldsymbol{\theta}}(\mathbf{x}_i)) + \frac{\lambda}{2} \sum_{k=1}^K |v_k|^2 + \|\mathbf{w}_k\|_2^2 \Leftrightarrow \min_{f \in \mathcal{F}} \sum_{i=1}^n \mathcal{L}(y_i, f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{F}}$$



- The equivalence \Leftrightarrow is understood via **Banach-space representer theorems**.
- There always exists a solution to the problem over \mathcal{F} that is a **sparse** combination of neurons.

Deep Neural Networks

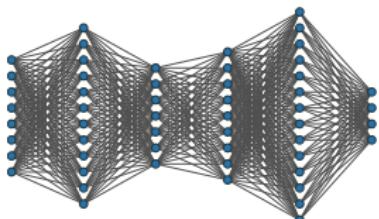
$$\{f_{\theta}(x) = \sigma(W_L \sigma(W_{L-1} \sigma(\dots W_1 x)))\}$$

$$C(\theta) = \frac{1}{L} \sum_{\ell=1}^L \|W_\ell\|_F^2$$

\mathcal{F} : exists

$R(f)$: exists

$$\min_{\theta} \sum_{i=1}^n \mathcal{L}(y_i, f_{\theta}(x_i)) + \frac{\lambda}{L} \sum_{\ell=1}^L \|W_\ell\|_F^2 \quad \Leftrightarrow \quad \min_{f \in \mathcal{F}} \sum_{i=1}^n \mathcal{L}(y_i, f(x_i)) + \lambda R(f)$$



- The equivalence \Leftrightarrow is by construction.
- We currently do not know how to characterize \mathcal{F} or $R(f)$.
➡ For $L > 2$, is it even a linear space?

A Note on Universal Approximation

A common heuristic to explain the success of deep learning is that neural networks are **universal approximators**. This heuristic is **meaningless** since any reasonable parametric model space is a universal approximator.

- polynomials
- kernel machines
- Fourier series
- wavelets
- shallow and deep neural networks

Theorem (Stone–Weierstraß)

Let $\Omega \subset \mathbb{R}^d$ be compact and $A \subset C(\Omega)$ be a subalgebra (vector subspace closed under multiplication) that contains a nonzero constant function. Then, A is dense in $C(\Omega)$ if and only if it separates points (for every $x, x' \in \Omega$ such that $x \neq x'$, there exists $p \in A$ such that $p(x) \neq p(x')$).

Pop Quiz: Does the closing procedure mean $\mathcal{F} = C(\Omega)$?

When $R(f) < \infty$, it is (typically) the case that $\mathcal{F} \subsetneq C(\Omega)$.

Outline

- ① Hilbert Spaces \Leftrightarrow Linear/Kernel Methods
- ② Banach Spaces \Leftrightarrow Nonlinear/Sparse Methods
- ③ Banach Spaces \Leftrightarrow Shallow Neural Networks
- ④ Beyond(?) Banach Spaces \Leftrightarrow Deep Neural Networks

Outline

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Hilbert Spaces: Basic Definition

Assume \mathcal{F} is a **vector space of functions** defined on a domain $\Omega \subset \mathbb{R}^d$.

Will focus on functions with real outputs (scalar- or vector-valued).

We say $\langle \cdot, \cdot \rangle_{\mathcal{F}} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ is an **inner product** on \mathcal{F} if it is:

- *bilinear*: $\langle \alpha f + \beta g, h \rangle_{\mathcal{F}} = \alpha \langle f, h \rangle_{\mathcal{F}} + \beta \langle g, h \rangle_{\mathcal{F}}$ and
 $\langle f, \alpha g + \beta h \rangle_{\mathcal{F}} = \alpha \langle f, g \rangle_{\mathcal{F}} + \beta \langle f, h \rangle_{\mathcal{F}}$
- *symmetric*: $\langle f, g \rangle_{\mathcal{F}} = \langle g, f \rangle_{\mathcal{F}}$
- *positive definite*: $\langle f, f \rangle_{\mathcal{F}} \geq 0$ and $\langle f, f \rangle_{\mathcal{F}} = 0$ iff $f = 0$.

Any inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ on \mathcal{F} defines a **norm** on \mathcal{F} by:

$$\|f\|_{\mathcal{F}} := \sqrt{\langle f, f \rangle_{\mathcal{F}}}$$

Definition

A **Hilbert space** is a vector space \mathcal{F} equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ that is *complete* with respect to the norm $\|\cdot\|_{\mathcal{F}}$

Example: Finite Collection of Basis Functions

Suppose \mathcal{F} is the span of K linearly independent basis functions $\varphi_1, \dots, \varphi_K$:

$$f_{\boldsymbol{\theta}} = \sum_{k=1}^K \theta_k \varphi_k, \quad \theta_k \in \mathbb{R}, k = 1, \dots, K$$

equipped with the inner product and norm

$$\langle f_{\boldsymbol{\theta}}, f_{\boldsymbol{\beta}} \rangle_{\mathcal{F}} = \boldsymbol{\theta}^T \boldsymbol{\beta} \implies \|f_{\boldsymbol{\theta}}\|_{\mathcal{F}}^2 = \|\boldsymbol{\theta}\|_2^2.$$

Then \mathcal{F} is a finite-dimensional Hilbert space, and we have the equivalence

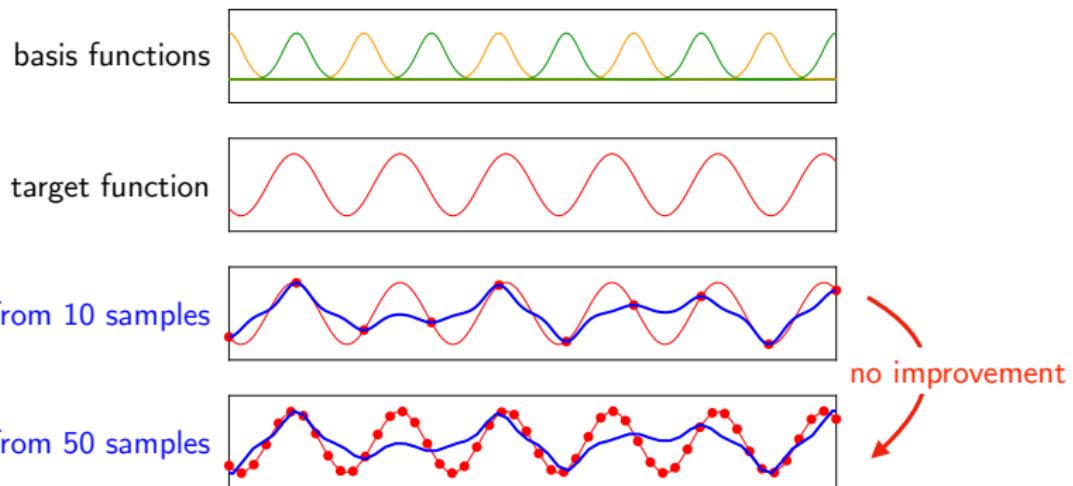
$$\min_{f \in \mathcal{F}} \sum_{i=1}^n (f(\mathbf{x}_i) - y_i)^2 + \lambda \|f\|_{\mathcal{F}}^2 \iff \min_{\boldsymbol{\theta} \in \mathbb{R}^K} \mathcal{L}(y_i, [\mathbf{V}\boldsymbol{\theta}]_i) + \lambda \|\boldsymbol{\theta}\|_2^2$$

where $\mathbf{V} \in \mathbb{R}^{n \times K}$ is such that $[\mathbf{V}]_{ik} = \varphi_k(\mathbf{x}_i)$.

Learning over \mathcal{F} is a simple finite dimensional ℓ^2 -regularized problem!

Limitations to Finite-Dimensional Hilbert Spaces

Finite-dimensional Hilbert spaces have **limited approximation capability**.



Could improve by adding basis functions (but how many? what type?)

Can we solve this issue with an **infinite-dimensional** Hilbert space?

L^2 -space

One of the most fundamental infinite-dimensional Hilbert spaces is

$$L^2(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} : \int_{\Omega} |f(\mathbf{x})|^2 d\mathbf{x} < +\infty \right\},$$

$$\langle f, g \rangle_{L^2(\Omega)} = \int_{\Omega} f(\mathbf{x})g(\mathbf{x}) d\mathbf{x} \Rightarrow \|f\|_{L^2(\Omega)} = \sqrt{\int_{\Omega} |f(\mathbf{x})|^2 d\mathbf{x}}$$

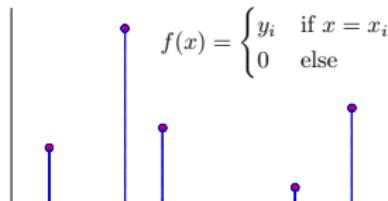
However, $L^2(\Omega)$ is “too big” of a space to be useful for learning.

Example: learning with L^2 -norm regularization

$$\min_{f \in L^2(\Omega)} \sum_{i=1}^n (f(\mathbf{x}_i) - y_i)^2 + \lambda \|f\|_{L^2(\Omega)}^2$$

Pop Quiz: What functions minimize this loss?

Obtain **zero loss** by putting “spikes” at the datapoints:



Reproducing Kernel Hilbert Spaces

For learning to be possible in an infinite-dimensional Hilbert space, we need some additional regularity assumptions.

- Continuity seems to be a necessary requirement.
- But it is not sufficient – we also need to ensure that any sequence of functions approaching a “spike” cannot vanish in norm.

The largest class of Hilbert Spaces having precisely this property are known as **Reproducing Kernel Hilbert Spaces**

Definition

A Hilbert space of functions \mathcal{F} is called a **Reproducing Kernel Hilbert Space (RKHS)** if for all $x \in \Omega$ there exists a constant C_x such that

$$|f(x)| \leq C_x \|f\|_{\mathcal{F}} \text{ for all } f \in \mathcal{F}.$$

Interpretation: if f is non-zero at any point, its norm is also non-zero.

Kernel Functions

In the language of functional analysis, an RKHS \mathcal{F} is a Hilbert space where the evaluation functionals $f \mapsto f(\mathbf{x})$ are continuous.

By a result known as the **Riesz Representation Theorem**, this implies for all $\mathbf{x} \in \Omega$ there exists a function $K_{\mathbf{x}} \in \mathcal{F}$ such that

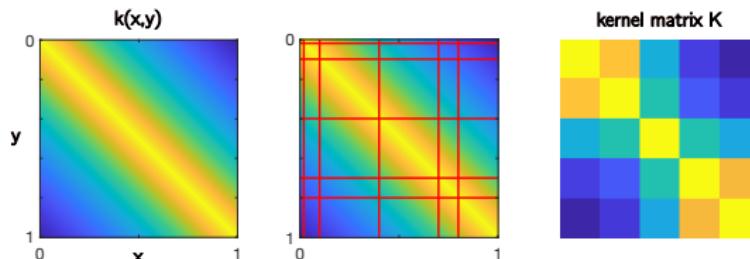
$$\langle K_{\mathbf{x}}, f \rangle_{\mathcal{F}} = f(\mathbf{x}) \text{ for all } f \in \mathcal{F}.$$

Define the associated **kernel** $k : \Omega \times \Omega \rightarrow \mathbb{R}$ by

$$k(\mathbf{x}, \mathbf{x}') = \langle K_{\mathbf{x}}, K_{\mathbf{x}'} \rangle = K_{\mathbf{x}}(\mathbf{x}') = K_{\mathbf{x}'}(\mathbf{x}).$$

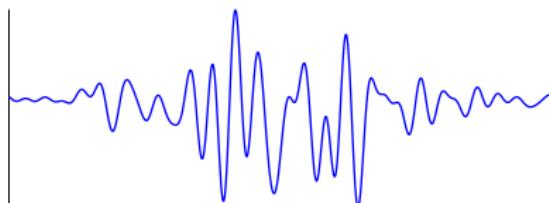
Two important properties: kernels are

- ① **symmetric**: $k(\mathbf{x}, \mathbf{x}') = k(\mathbf{x}', \mathbf{x})$, for all $\mathbf{x}, \mathbf{x}' \in \Omega$ and
- ② **positive semidefinite**: for any finite set of points $\{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \Omega$, the *kernel matrix* $\mathbf{K} \in \mathbb{R}^{n \times n}$ with $K_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)$ is a PSD matrix.



Example: Bandlimited Functions

Let $\mathcal{B} \subset L^2(\mathbb{R})$ be the space of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ whose Fourier transform $\widehat{f}(\xi)$ vanishes for all frequencies $|\xi| > B$.



Pop Quiz: Why aren't "spike" functions allowed in this space?

Define the *sinc function* $s(x) := \mathcal{F}^{-1}(1_{[-B,B]})(x) = \sin(Bx)/\pi x$

Key property: $s * f = f$ for all $f \in \mathcal{B}$, since $\widehat{s * f} = \widehat{s} \cdot \widehat{f} = \widehat{f}$.

Put another way, we have

$$f(x) = (s * f)(x) = \int_{\mathbb{R}} f(x') s(x' - x) dx' = \langle f, s(\cdot - x) \rangle$$

Therefore, \mathcal{B} is an RKHS with kernel function $k(x, x') = s(x - x')$.

Example: the Sobolev space $H^s(\Omega)$

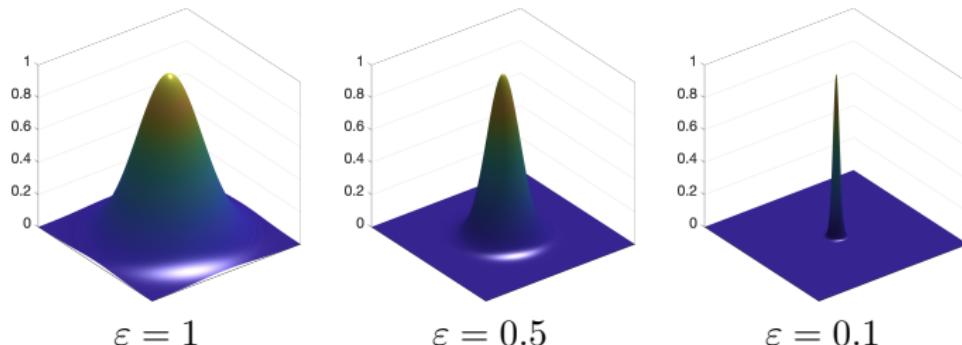
Let $H^s(\Omega)$ denote the space of functions $f : \Omega \rightarrow \mathbb{R}$ whose partial derivatives up to order s belong to $L^2(\Omega)$, equipped with the inner product

$$\langle f, g \rangle_{H^s(\Omega)} = \sum_{|\alpha| \leq s} \int_{\Omega} D^\alpha f(x) D^\alpha g(x) dx \Rightarrow \|f\|_{H^s(\Omega)}^2 = \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2(\Omega)}^2$$

Fact: $H^s(\Omega)$ is an RKHS iff $s > d/2$

Smoothness $s > d/2$ is necessary, since otherwise arbitrarily thin “spike” functions would have vanishing norm:

$$f_\varepsilon(x) := f(x/\varepsilon) \Rightarrow \|\partial^s f_\varepsilon\|_{L^2} = \varepsilon^{d/2-s} \|\partial^s f\|_{L^2}$$



Building RKHSs from Kernels

Every RKHS induces a kernel $k(\cdot, \cdot)$ that is symmetric and positive semidefinite.

On the flipside, given any function $k : \Omega \times \Omega \rightarrow \mathbb{R}$ that is symmetric and positive semidefinite, we can **construct** a **RKHS** having k as its kernel.

- ① Take the span of all kernel translates $k(\cdot, \boldsymbol{x})$:

$$\text{span}\{k(\cdot, \boldsymbol{x}) : \boldsymbol{x} \in \Omega\} = \left\{ f_{\boldsymbol{a}} = \sum_{i=1}^n a_i k(\cdot, \boldsymbol{x}_i) : \begin{array}{l} n \in \mathbb{N} \\ a_i \in \mathbb{R} \\ \boldsymbol{x}_i \in \mathbb{R}^d \end{array} \right\}$$

- ② Equip this space with the inner product:

$$f = \sum_{i=1}^n a_i k(\cdot, \boldsymbol{x}_i), \quad g = \sum_{j=1}^m a'_j k(\cdot, \boldsymbol{x}'_j) \implies \langle f, g \rangle = \sum_{i=1}^n \sum_{j=1}^m a_i a'_j k(\boldsymbol{x}_i, \boldsymbol{x}'_j)$$

- ③ Take the **closure** of the space (in the induced norm) to get the RKHS.

Theorem (Moore-Aronszajn)

Every SPD function $k(\cdot, \cdot)$ defines a **unique** RKHS with k as its kernel.

Examples: Common Kernels

Linear Kernel

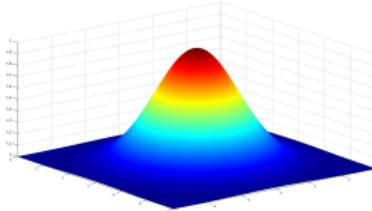
$$k(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{y},$$

Polynomial Kernel

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^\top \mathbf{y} + 1)^d, \quad \text{for some } k \in \mathbb{N}$$

Gaussian Kernel

$$k(\mathbf{x}, \mathbf{y}) = \exp\left(-\frac{1}{2\sigma^2}\|\mathbf{x} - \mathbf{y}\|_2^2\right), \quad \text{for some } \sigma > 0$$



Another common way to create kernels is with a **feature map** $\varphi : \Omega \rightarrow \mathcal{H}$ where \mathcal{H} is a Hilbert space (typically \mathbb{R}^D):

$$k(\mathbf{x}, \mathbf{y}) = \langle \varphi(\mathbf{x}), \varphi(\mathbf{y}) \rangle_{\mathcal{H}}$$

Example: $\varphi : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $\varphi(x) = [1, \sqrt{2}x, x^2]^\top$ gives the polynomial kernel $k(x, y) = 1 + 2xy + x^2y^2 = (xy + 1)^2$

Example: Tangent Kernels

Given a parametric model $f_{\theta}(x) = f(\theta; x)$, linearizing about $\theta = \theta_0$ gives

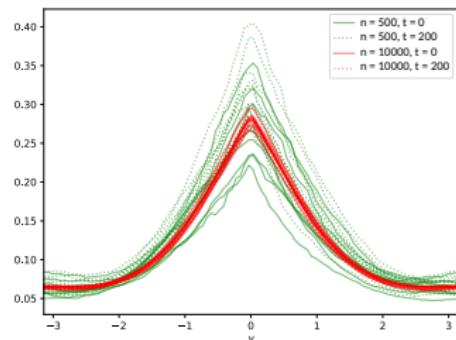
$$f(\theta; x) \approx f(\theta_0; x) + \nabla_{\theta} f(\theta_0; x)^T (\theta - \theta_0).$$

For any fixed θ define the **tangent kernel** k_{θ}

$$k_{\theta}(x, x') := \nabla_{\theta} f(\theta; x)^T \nabla_{\theta} f(\theta; x'), \quad \text{for all } x, x' \in \Omega$$

which is the kernel arising from the feature map $\phi(x) = \nabla_{\theta} f(\theta, x)$.

Jacot et al. 2018 showed that when f_{θ} is a **randomly initialized neural network architecture**, in the limit of the hidden-layer widths approaching infinity, $k_{\theta}(x, x')$ converges to an explicit kernel that stays constant during training—the **neural tangent kernel**.



Ininitely-wide neural network architectures define kernels

Does this RKHS perspective explain the astounding success of neural networks?

RKHS Representer Theorem

Theorem

Let \mathcal{F} be an RKHS with kernel $k : \Omega \times \Omega \rightarrow \mathbb{R}$. Fix $\lambda > 0$. Then

$$f^* \in \arg \min_{f \in \mathcal{F}} \sum_{i=1}^n \mathcal{L}(f(\mathbf{x}_i), y_i) + \lambda \|f\|_{\mathcal{F}}^2 \Rightarrow f^*(\mathbf{x}) = \sum_{i=1}^n a_i k(\mathbf{x}, \mathbf{x}_i).$$

for some $a_i \in \mathbb{R}$.

Proof sketch for square-loss:

$$\min_{f \in \mathcal{F}} \sum_{i=1}^n (f(\mathbf{x}_i) - y_i)^2 + \lambda \|f\|_{\mathcal{F}}^2 = \min_{f \in \mathcal{F}} \sum_{i=1}^n (\langle K_{\mathbf{x}_i}, f \rangle_{\mathcal{F}} - y_i)^2 + \lambda \langle f, f \rangle_{\mathcal{F}}$$

Set the “derivative” $\partial/\partial f$ of the loss to zero:

$$\sum_{i=1}^n 2(\langle K_{\mathbf{x}_i}, f^* \rangle_{\mathcal{F}} - y_i) K_{\mathbf{x}_i} + 2\lambda f^* = 0 \implies f^* = \sum_{i=1}^n a_i K_{\mathbf{x}_i}.$$

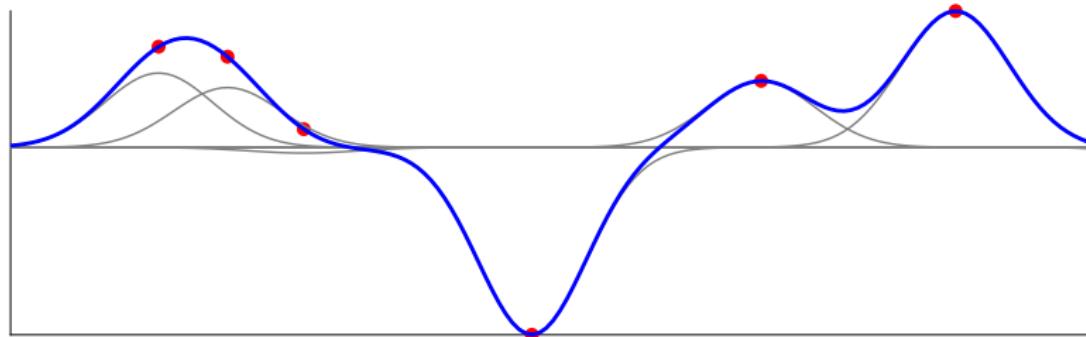
The “Kernel Trick”

Restricting to only functions of the form $f = \sum_{i=1}^n a_i k(\cdot, \mathbf{x}_i)$ we have

$$\min_{f \in \mathcal{F}} \sum_{i=1}^n \mathcal{L}(y_i, f(\mathbf{x}_i)) + \lambda \|f\|_{\mathcal{F}}^2 \quad \Leftrightarrow \quad \min_{\mathbf{a} \in \mathbb{R}^n} \sum_{i=1}^n \mathcal{L}(y_i, [\mathbf{K}\mathbf{a}]_i) + \lambda \mathbf{a}^\top \mathbf{K} \mathbf{a}$$

where $\mathbf{K} \in \mathbb{R}^{n \times n}$ is the kernel matrix $K_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)$. This is now a finite dimensional optimization problem we can solve easily.

Example: Gaussian RBF kernel $f(x) = \sum_{i=1}^n a_i \exp(-\frac{1}{2\sigma^2}(x - x_i)^2)$



Example: Smoothing Splines

The solution to

$$\min_f \sum_{i=1}^n (y_i - f(x_i))^2 + \lambda \int_0^1 |D^2 f(x)|^2 dx$$

is a cubic (smoothing) spline,

$$\|D^2 f\|_{L^2}^2$$

$$f_{\text{spline}}(x) = \sum_{i=1}^n a_i^* k(x, x_i),$$

where $a^* = \arg \min_{a \in \mathbb{R}^n} \|y - \mathbf{K}a\|_2^2 + \lambda a^\top \mathbf{K}a$.

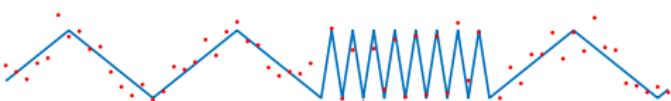
quadratic regularizer \Rightarrow
solution linear in data y

If $y_i = f^*(x_i) + \varepsilon_i$ with $f^* \in H^2$, then

$$\mathbf{E} \|f^* - f_{\text{spline}}\|_{L^2}^2 = O(n^{-\frac{4}{5}}).$$

minimax rate

Limitations of Linear/Kernel Methods

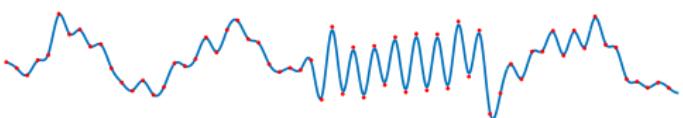


True function and noisy data



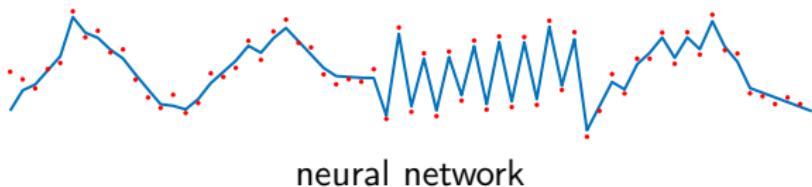
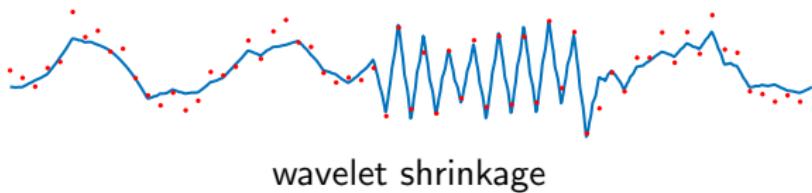
small λ :
overfits low variation
portion of the data

large λ :
oversmooths high variation
portion of the data



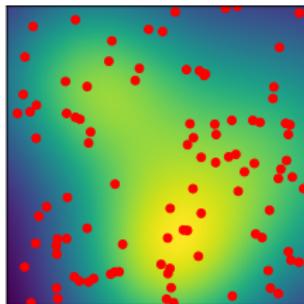
Linear methods cannot adapt to spatially varying smoothness.

Limitations of Linear/Kernel Methods

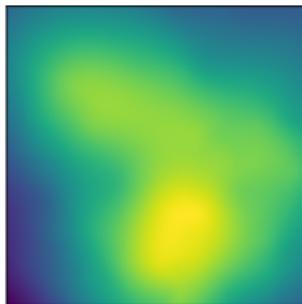


Limitations of Linear/Kernel Methods

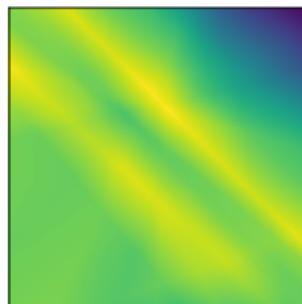
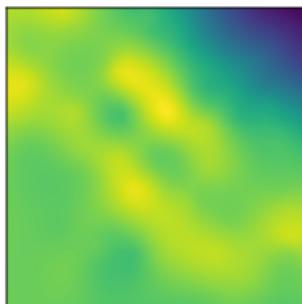
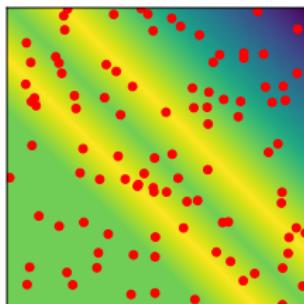
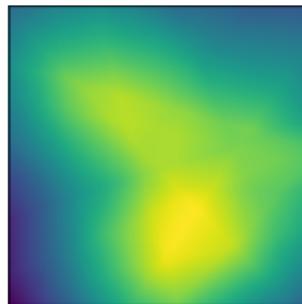
True function
and noisy data



Thin-plate spline
(kernel method)



Neural network
(nonlinear method)



Neural networks can adapt to **low-dimensional structure**.

Summary

- Kernel methods are well-understood from the function-space view.
 - ⇒ Essentially by construction.
 - ⇒ There is a one-to-one correspondance between a kernel $k(\cdot, \cdot)$ and their associated RKHS \mathcal{F}_k .
- We know when kernel methods work and how well they work.
 - ⇒ Kernel methods are “optimal” for learning functions in their associated RKHS.
- We know that there are situations where they do not work.
 - ⇒ There are fundamental drawbacks to linear methods.

Outline

- ① Hilbert Spaces \Leftrightarrow Linear/Kernel Methods
- ② Banach Spaces \Leftrightarrow Nonlinear/Sparse Methods
- ③ Banach Spaces \Leftrightarrow Shallow Neural Networks
- ④ Beyond(?) Banach Spaces \Leftrightarrow Deep Neural Networks

Banach Spaces - Basic Definition

Assume \mathcal{F} is a **vector space of functions** defined on a domain $\Omega \subset \mathbb{R}^d$.

We will focus on functions with real outputs (scalar- or vector-valued).

We say that $\|\cdot\|_{\mathcal{F}} : \mathcal{F} \rightarrow \mathbb{R}_{\geq 0}$ is a **norm** if it is:

- *subadditive*: $\|f + g\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}} + \|g\|_{\mathcal{F}}$
- *homogeneous*: $\|\alpha f\|_{\mathcal{F}} = |\alpha| \|f\|_{\mathcal{F}}$
- *positive definite*: $\|f\|_{\mathcal{F}} = 0$ if and only if $f \equiv 0$

Remark: Every inner product $\langle \cdot, \cdot \rangle$ induces a valid norm: $\|f\|^2 := \langle f, f \rangle$.

Definition

A **Banach space** is a vector space \mathcal{F} equipped with a norm $\|\cdot\|_{\mathcal{F}}$ that is *complete* with respect to the norm $\|\cdot\|_{\mathcal{F}}$.

Reproducing Kernel Banach Spaces

Definition

A Banach space of functions \mathcal{F} is called a **Reproducing Kernel Banach Space (RKBS)** if its norm $\|\cdot\|_{\mathcal{F}}$ is strictly convex, if its dual norm $\|\cdot\|_{\mathcal{F}'}$ is strictly convex, and for all $\mathbf{x} \in \Omega$ there exists a constant $C_{\mathbf{x}}$ such that

$$|f(\mathbf{x})| \leq C_{\mathbf{x}} \|f\|_{\mathcal{F}} \text{ for all } f \in \mathcal{F}.$$

- Strict convexity of the norm and dual norm ensures the existence of a **unique** reproducing kernel $k : \Omega \times \Omega \rightarrow \mathbb{R}$ with $k(\mathbf{x}, \cdot) \in \mathcal{F}$ and $k(\cdot, \mathbf{x}) \in \mathcal{F}'$ and

$$\langle k(\cdot, \mathbf{x}), f \rangle = f(\mathbf{x}), \quad \text{for all } f \in \mathcal{F}.$$

$$\langle f', k(\mathbf{x}, \cdot) \rangle = f'(\mathbf{x}), \quad \text{for all } f' \in \mathcal{F}'.$$

- The RKBS shares many similarities to the RKHS framework, but reflexivity is too strong of a condition to capture important spaces related to **sparsity**.

Sparsity = Feature Learning?

In Hilbert-space methods, the learned models are **linear in parameters**.

Linear methods cannot adapt to spatially varying smoothness.

Linear methods do not **learn features**.

Early approaches to circumvent this issue were based on sparsity:

- lasso (Tibshirani 1996)
- sparse approximation (DeVore 1998)
- wavelet shrinkage/thresholding (Donoho and Johnstone 1998)
- compressed sensing (Candès et al. 2006; Donoho 2006)

Sparse methods are **nonlinear in parameters**.

Is sparsity key to understand **feature learning**?

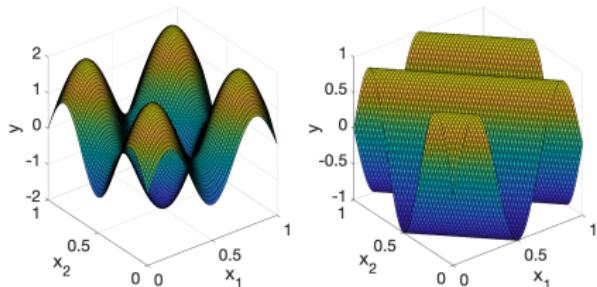
Sparsity and the Quest for Adaptivity

Latent Variable Perspective: The target function depends only on an r -dimensional projection ($r \ll d$) of the input.

Manifold Hypothesis: d -dimensional data sets that occur in the real world actually lie along r -dimensional latent manifolds ($r \ll d$).

Structured Smoothness: The target function has some unknown structured smoothness.

Can we design methods that **adapt** to the **unknown** structure?



Sparsity \Rightarrow adaptation

Adaptation = feature learning?

Sparse Models: Finite-Dimensional Case

Returning to our simplest parametric model, where \mathcal{F} is the linear span of a **dictionary** of finitely many functions $\varphi_1, \dots, \varphi_K$.

$$f_{\boldsymbol{\theta}} = \sum_{k=1}^K \theta_k \varphi_k, \quad C(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_1$$

$$\mathcal{F}_{\Theta} = \mathcal{F} = \text{span}\{\varphi_k\}_{k=1}^K$$

$$R(f) = \inf_{\boldsymbol{\theta}: f=f_{\boldsymbol{\theta}}} \|\boldsymbol{\theta}\|_1$$

The learning problem is

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^K} \sum_{i=1}^n \mathcal{L}(y_i, [\mathbf{V}\boldsymbol{\theta}]_i) + \lambda \|\boldsymbol{\theta}\|_1,$$

where the i th row of $\mathbf{V} \in \mathbb{R}^{n \times K}$ is

$$\mathbf{V} = [\varphi_1(x_i), \varphi_2(x_i), \dots, \varphi_K(x_i)]$$

- Data-fitting over \mathcal{F} is equivalent to a **finite-dimensional convex** optimization problem.
- There always exists a solution with at most K dictionary functions.

Sparse Models: Infinite-Dimensional Case

What if we had an infinitely large dictionary? $\{\varphi_k\}_{k \in \mathbb{Z}}$

$$f_{\boldsymbol{\theta}} = \sum_{k \in \mathbb{Z}} \theta_k \varphi_k, \quad C(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_{\ell^1}$$

$$\mathcal{F}_{\Theta} \subset \mathcal{F} = \overline{\text{span}}\{\varphi_k\}_{k \in \mathbb{Z}}$$

$$R(f) = \inf_{\boldsymbol{\theta}: f=f_{\boldsymbol{\theta}}} \|\boldsymbol{\theta}\|_1$$

The learning problem is

$$\min_{\boldsymbol{\theta} \in \ell^1(\mathbb{Z})} \sum_{i=1}^n \mathcal{L}(y_i, V\{\boldsymbol{\theta}\}) + \lambda \|\boldsymbol{\theta}\|_{\ell^1},$$

where $V : \ell^1(\mathbb{Z}) \rightarrow \mathbb{R}^n$.

- Data-fitting over \mathcal{F} is equivalent to a **infinite-dimensional convex** optimization problem.
- Solutions have infinitely many dictionary functions? Are we screwed?

Luckily, we are not.

Intuition: Soft Thresholding

Consider the “denoising” problem

$$\min_{\boldsymbol{\theta} \in \ell^1(\mathbb{Z})} \|\mathbf{y} - \boldsymbol{\theta}\|_{\ell^2}^2 + \lambda \|\boldsymbol{\theta}\|_{\ell^1} = \min_{\boldsymbol{\theta} \in \ell^1(\mathbb{Z})} \sum_{k \in \mathbb{Z}} [(y_k - \theta_k)^2 + \lambda |\theta_k|]$$

Pop Quiz: What is the solution to this problem?

This problem can be “decoupled”.

$$\begin{aligned} \min_{\theta_k \in \mathbb{R}} (y_k - \theta_k)^2 + \lambda |\theta_k| &\Rightarrow \text{soft thresholding of } y_k \\ &\Rightarrow \hat{\theta}_k = \text{sgn}(y_k) \max\{0, |y| - \lambda/2\} \end{aligned}$$

Since $\boldsymbol{\theta} \in \ell^1(\mathbb{Z})$, the sorted coefficients $|\theta_{(1)}| \geq |\theta_{(2)}| \geq \dots$ must decay strictly faster than $1/k$.

For every $\lambda > 0$, only a finite number of coefficients will be nonzero.

Soft thresholding is **nonlinear in parameters**.

Representer Theorems for ℓ^1 -Norm Regularization

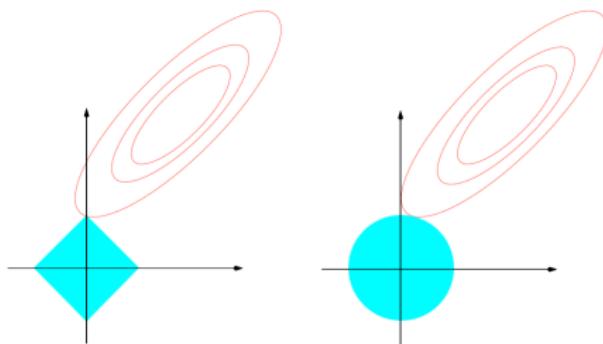
The **sparsity** of solutions is related to the **convex geometry** of the optimization problem.

$$\begin{aligned} \min_{\boldsymbol{\theta} \in \ell^1(\mathbb{Z})} \sum_{i=1}^n \mathcal{L}(y_i, V\{\boldsymbol{\theta}\}) + \lambda \|\boldsymbol{\theta}\|_{\ell^1} &\Leftrightarrow \min_{\boldsymbol{\theta} \in \ell^1(\mathbb{Z})} \|\boldsymbol{\theta}\|_{\ell^1} \\ &\text{s.t. } \sum_{i=1}^n \mathcal{L}(y_i, V\{\boldsymbol{\theta}\}) \leq B \\ &\Leftrightarrow \min_{\boldsymbol{\theta} \in \ell^1(\mathbb{Z})} \|\boldsymbol{\theta}\|_{\ell^1} \text{ s.t. } V\{\boldsymbol{\theta}\} \in \mathcal{C} \subset \mathbb{R}^n \end{aligned}$$

There exists a solution $\widehat{\boldsymbol{\theta}}$ that is n -sparse.

- Tightly linked to Carathéodory's theorem for convex hulls.
- This is an example of a **Banach-space representer theorem**.
 $\implies \ell^1(\mathbb{Z})$ is a non-Hilbertian Banach space.

Geometry of ℓ^1 -Norm Regularization



The **extreme points** of the ℓ^1 -ball are 1-sparse vectors.

1-sparse vectors are **Kronecker deltas**:

$$e_k[n] = \delta_k[n] = \begin{cases} 1, & \text{if } n = k \\ 0, & \text{else.} \end{cases}$$

Pop Quiz: What are the extreme points of the ℓ^2 -ball?

Convex Optimization in Infinite Dimensions?

$$\min_{\boldsymbol{\theta} \in \ell^1(\mathbb{Z})} \sum_{i=1}^n \mathcal{L}(y_i, V\{\boldsymbol{\theta}\}) + \lambda \|\boldsymbol{\theta}\|_{\ell^1}$$

Convex problem, but infinite-dimensional.

BUT, we know there exists a solution $\hat{\boldsymbol{\theta}}$ that is n -sparse:

$$f_{\hat{\boldsymbol{\theta}}} = \sum_{j=1}^n \theta_{[j]} \varphi_{[j]}$$

Pop Quiz: Can we just optimize over n -sparse sequences?

Yes, but the problem becomes **nonconvex**.

Wavelet Shrinkage

$$f_{\theta}(x) = \sum_{j,k} \theta_{j,k} 2^{-j/2} \psi(2^j x - k)$$

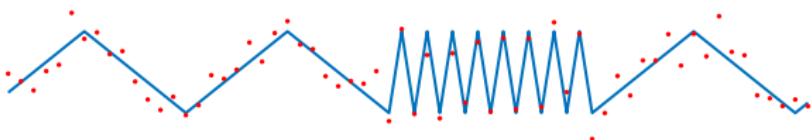
$$C(\theta) = \|\theta\|_{\ell^1}$$

$$\min_{\theta} \sum_{i=1}^n \mathcal{L}(y_i, f_{\theta}(x_i)) + \lambda \|\theta\|_{\ell^1}$$



- We can **efficiently** solve this optimization problem by thresholding the empirical wavelet coefficients.
- The resulting solution is able to **adapt** to intrinsic structure in the data-generating function.

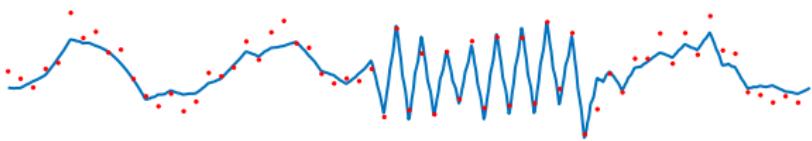
Spatially Inhomogeneous Functions



Spatially inhomogeneous functions have different kinds of **local regularity**.

Designing **locally adaptive** estimators was of great interest in the 1990s.

- Real-world signals (**low-dimensional** objects) are spatially inhomogeneous.
- Linear methods cannot adapt to spatial inhomogeneities while sparse/nonlinear methods can.
⇒ Mathematical foundations for the success and popularity of wavelets.



What kinds of function spaces capture spatial inhomogeneities?

Besov Spaces and Bounded Variation (BV) Spaces

Spatially inhomogeneous functions are well-captured by **Besov** and **Bounded Variation** (BV) spaces.

- The Besov space $B_{p,q}^s$ is the space of functions with s derivatives in L^p , where q allows for finer control of the regularity:

$$B_{p,q}^s \subset B_{p,q'}^s \quad \Leftrightarrow \quad q < q'$$

When $p < 2$, $B_{p,q}^s$ contains functions that are spatially inhomogeneous.

- The (total) variation of a function is

$$\text{TV}(f) = \sup \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \quad \left(\approx \int |f'(x)| dx \right).$$

where the sup is over all partitions. BV is the space of functions with bounded (total) variation $\text{TV}(f) < \infty$.

$\Rightarrow f \in \text{BV}^k \Leftrightarrow f^{(k-1)} \in \text{BV} \Leftrightarrow \text{TV}^k(f) = \text{TV}(f^{(k-1)})$.

- BV^k is morally a Besov space since we have the sandwich

$$B_{1,1}^k \subset \text{BV}^k \subset B_{1,\infty}^k.$$

Native Space of Wavelet Soft Thresholding

$$\left\{ f_{\theta} = \sum_{j,k} \theta_{j,k} \psi_{j,k} : \begin{aligned} \psi_{j,k}(x) &= \\ 2^{-j/2} \psi(2^j x - k) \end{aligned} \right\}$$

$$C(\theta) = \|\theta\|_{\ell^1}$$

\mathcal{F} : Besov space $B_{1,1}^1$

$$R(f) = \|f\|_{B_{1,1}^1}$$

Besov norm

$$\min_{\theta} \sum_{i=1}^n \mathcal{L}(y_i, f_{\theta}(x_i)) + \lambda \|\theta\|_{\ell^1} \quad \Leftrightarrow \quad \min_{f \in B_{1,1}^1} \sum_{i=1}^n \mathcal{L}(y_i, f(x_i)) + \lambda \|f\|_{B_{1,1}^1}$$

- ℓ^1 -limits of wavelet coefficients converge to $B_{1,1}^1$ functions
⇒ Other choices of sequence space norms give rise to **all** other Besov spaces $B_{p,q}^s$.
- There always exists a solution to the problem over $B_{1,1}^1$ that is a **sparse and finite** combination of wavelets.
⇒ Soft-thresholding is the algorithm of choice.



Integral Representations of Functions

For smooth functions on $[0, 1]$, by the fundamental theorem of calculus, we have that

$$f(x) = f(0) + \int_0^x f'(t) dt \quad \Rightarrow \quad f(x) = f(0) + \int_0^1 \text{ReLU}^0(x-t) f'(t) dt.$$

If we iterate this process...

$$f(x) = \sum_{j=0}^{k-1} f^{(j)}(0)x^j + \int_0^1 \text{ReLU}^{k-1}(x-t) f^{(k)}(t) dt$$

Pop Quiz: When is this quantity well-defined?

- $\|f^{(k)}\|_{L^1} < \infty$.
- Suppose $k = 1$ and $f(x) = \text{ReLU}^0(x - 0.5)$.
 $\implies f'(x) = \delta_{0.5}$.
 $\implies f(x) = \int_0^1 \text{ReLU}^0(x-t) d\delta_{0.5}$.
- Identify $f^{(k)}$ with a measure $\nu \Rightarrow \|\nu\|_{\mathcal{M}} < \infty$.

BV Spaces and Continuously-Indexed Dictionaries

For smooth functions, we have that

$$\text{TV}(f) = \sup \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| = \int_0^1 |f'(x)| \, dx.$$

For nonsmooth functions,

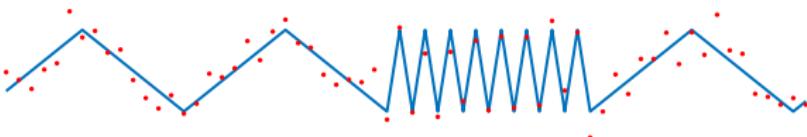
$$\text{TV}(f) = \|f'\|_{\mathcal{M}} \quad \Rightarrow \quad \text{TV}^k(f) = \|\mathbf{D}^k f\|_{\mathcal{M}}.$$

If $f \in \text{BV}^k$ and $\nu = \mathbf{D}^k f$, then,

$$f(x) = \sum_{j=1}^{k-1} c_j x^j + \int_0^1 \text{ReLU}^{k-1}(x-t) \, d\nu(t)$$

- Peano kernel formula
- Infinite-width neural network?
- $f \in \text{BV}^k \Rightarrow f$ can be build from the **continuously-indexed dictionary** $\{\text{ReLU}^{k-1}(\cdot - t)\}_{t \in [0,1]}$.

Minimax Optimality of Nonlinear Methods



Pop Quiz: What is the regularity of this function?

- This is a BV^2 function.
- The **minimax rate** for BV^2 is

$$\inf_{\hat{f}} \sup_{\substack{f \in BV^2 \\ TV^2(f) \leq C}} \mathbf{E} \|f - \hat{f}\|_{L^2}^2 \asymp n^{-4/5}$$

smoothing spline		wavelet shrinkage
$n^{-3/4}$		$n^{-4/5}$

- $n^{-3/4}$ is the **linear minimax rate**

Donoho and Johnstone 1998; Mammen and Geer 1997

Sparsity in the Continuum

Discrete sparse model:

$$\sum_{k \in \mathbb{Z}} \theta_k \varphi_k, C(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_{\ell^1}$$

Continuous sparse model:

$$\int_{\Xi} \varphi_{\xi} d\nu(\xi), C(\nu) = \|\nu\|_{\mathcal{M}}$$

The continuous model is “backwards compatible” with the discrete model.

$$\left\| \sum_{k \in \mathbb{Z}} \theta_k \delta_{\xi_k} \right\|_{\mathcal{M}} = \sum_{k \in \mathbb{Z}} |\theta_k| = \|\boldsymbol{\theta}\|_{\ell^1}$$

Pop Quiz: Why don't we use the L^1 -norm for continuous sparsity?

- The extreme points of the unit ball of $\ell^1(\mathbb{Z})$ are the **Kronecker deltas**: $\{e_k\}_{k \in \mathbb{Z}}$.
- The extreme points of the unit ball of $\mathcal{M}(\Xi)$ are the **Dirac deltas/measures**: $\{\delta_{\xi}\}_{\xi \in \Xi}$.
- The extreme points of the unit ball of $L^1(\Xi)$...**do not exist**.

Variation Spaces

Given a continuously-indexed dictionary $\mathcal{D} = \{\varphi_\xi\}_{\xi \in \Xi}$, the **variation space** of \mathcal{D} is the space

$$\mathcal{V} = \mathcal{V}(\mathcal{D}) = \left\{ f_\nu = \int_{\Xi} \varphi_\xi \, d\nu(\xi) : \nu \in \mathcal{M}(\Xi) \right\}.$$

This forms a Banach space when equipped with the norm/representation cost

$$R(f) = \|f\|_{\mathcal{V}} = \inf_{\substack{\nu \in \mathcal{M}(\Xi) \\ f=f_\nu}} \|\nu\|_{\mathcal{M}}$$

Example: BV^k (modulo polynomials) is a variation space with respect to the $\{\text{ReLU}^{k-1}(\cdot - t)\}_{t \in [0,1]}$ dictionary.

The variation norm is the continuous counterpart of the **atomic norm**.

Variation Spaces as a Closure

$$f_{\boldsymbol{\theta}} = \sum_{k=1}^K \vartheta_k \varphi_{\xi_k}, \quad \boldsymbol{\theta} = (\boldsymbol{\vartheta}, \boldsymbol{\xi})$$
$$C(\boldsymbol{\theta}) = \|\boldsymbol{\vartheta}\|_{\ell^1}$$

$$\mathcal{F} = \overline{\text{span}}\{\varphi_{\xi}\}_{\xi \in \Xi}$$

If the dictionary $\mathcal{D} = \{\varphi_{\xi}\}_{\xi \in \Xi}$ is sufficiently regular, then

$$\mathcal{V}(\mathcal{D}) = \mathcal{F}$$

Observation: Variation spaces are ℓ^1 -limits of combinations from continuous dictionaries.

Meyer's Bump Algebra

Let K_x denote the Gaussian kernel centered at x . Consider the dictionary $\mathcal{G} = \{K_x\}_{x \in \Omega}$, where $\Omega \subset \mathbb{R}^d$.

The variation space $\mathcal{V}(\mathcal{G})$ is called **Meyer's Bump Algebra**.

$$\mathcal{V}(\mathcal{G}) = B_{1,1}^d(\Omega)$$

On the other hand, the RKHS \mathcal{F} generated by \mathcal{G} is **fundamentally different**.

$$\mathcal{F} \subset H^s(\Omega) \text{ for any } s \geq 0$$

- \mathcal{F} is an extremely small space.
- $\mathcal{V}(\mathcal{G})$ is reasonably large.
 $\implies \mathcal{F} \subsetneq \mathcal{V}(\mathcal{G})$

Pop Quiz: Why did we get different spaces from the same parametric model space \mathcal{F}_Θ ?

This is alluding to a gap between Hilbert- and Banach-space methods.

Representer Theorems for \mathcal{M} -Norm Regularization

Theorem

Fix a dictionary $\mathcal{D} = \{\varphi_\xi\}_{\xi \in \Xi}$ of continuously-parameterized functions. For any data set $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ and lower semicontinuous, bounded from below, and proper loss function $\mathcal{L}(\cdot, \cdot)$, there exists a solution $\widehat{\nu}$ to

$$\min_{\nu \in \mathcal{M}(\Xi)} \sum_{i=1}^n \mathcal{L}(y_i, f_\nu(\mathbf{x}_i)) + \lambda \|\nu\|_{\mathcal{M}}, \quad \lambda > 0,$$

that leads to a representation of the form

$$f_{\widehat{\nu}}(\mathbf{x}) = \sum_{k=1}^K v_k \varphi_{\xi_k}(\mathbf{x}), \quad K \leq n.$$

There always exists a solution that is a K -sparse sum of dictionary functions.

Convex or Nonconvex?

$$\min_{\nu \in \mathcal{M}(\Xi)} \sum_{i=1}^n \mathcal{L}(y_i, f_\nu(\mathbf{x}_i)) + \lambda \|\nu\|_{\mathcal{M}}$$

Convex problem, but infinite-dimensional.

BUT, we know there exists a solution $\hat{\nu}$ that is K -sparse:

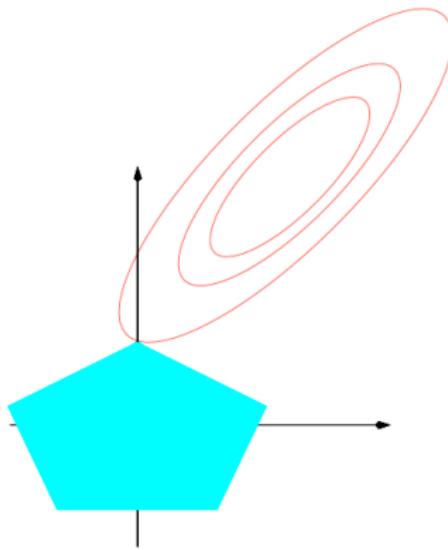
$$f_{\hat{\nu}}(\mathbf{x}) = \sum_{k=1}^K v_k \varphi_{\xi_k}(\mathbf{x})$$

Pop Quiz: Can we just optimize over K -sparse sums?

Yes, but the problem becomes **nonconvex**.

$$\min_{\substack{v_1, \dots, v_K \\ \xi_1, \dots, \xi_K}} \sum_{i=1}^n \mathcal{L}\left(y_i, \sum_{k=1}^K v_k \varphi_{\xi_k}(\mathbf{x}_i)\right) + \lambda \|\mathbf{v}\|_1$$

Geometry of Convex Regularization



The “atoms” of the solution are the **extreme points** of the **regularization ball**.

Abstract Representer Theorems

Theorem

Consider the learning problem over the function space \mathcal{F}

$$\inf_{f \in \mathcal{F}} \sum_{i=1}^n \mathcal{L}(y_i, f(\mathbf{x}_i)) + \lambda R(f).$$

Under appropriate hypotheses on \mathcal{F} and R , there always exists a **sparse** solution of the form

$$\hat{f} = \sum_{k=1}^K v_k \mathbf{e}_k, \quad K \leq n,$$

where $\mathbf{e}_k \in \text{Ext}(\{f \in \mathcal{F} : R(f) \leq 1\})$.

Infinite-dimensional optimization problems with **finite data constraints** always admit **sparse solutions**.

Summary

- Sparsity allows for methods to **adapt** to structure.
 - ⇒ Linear methods (including kernel methods) cannot.
 - ⇒ Quantification via minimax and linear minimax rates.
- Infinite-dimensional sparse models correspond to **convex problems**.
 - ⇒ These problems can be recast as **finite-dimensional non-convex problems**.
 - ⇒ This will play a key role in understanding neural networks from the function-space view.
- The infinite-dimensional perspective reveals interesting aspects about the **geometry of convex regularization**.
 - ⇒ Abstract representer theorems and extreme points.

Outline

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- ④ Beyond(?) Banach Spaces \Leftrightarrow Deep Neural Networks

What is the Inductive Bias Shallow Neural Networks?

What kinds of functions do neural networks prefer?

930

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Universal Approximation Bounds for Superpositions of a Sigmoidal Function

Andrew R. Barron, *Member, IEEE*



Andrew Barron

Barron (1993) introduced a class of d -dimensional functions that can be approximated **extremely well** by neural networks.

- Such functions can be approximated by a neural network with K neurons at a rate $K^{-\frac{1}{2}}$.
- Rates for classical function classes behave as $K^{-\frac{s}{d}}$ the curse
⇒ Andrew Barron broke the curse of dimensionality!

Spectral Barron Spaces

Let $\mathcal{B}^s \subset L^1(\mathbb{R}^d)$ be the space of functions for which

$$\|f\|_{\mathcal{B}^s} := \int_{\mathbb{R}^d} |\widehat{f}(\omega)| (1 + \|\omega\|)^s d\omega < +\infty$$

Then \mathcal{B}^s is a Banach space and is now referred to the s th order **spectral Barron space**.

- Barron 1993 proved that functions in \mathcal{B}^1 can be approximated by shallow **sigmoid** neural networks at a rate that does not grow with input dimension!
 - ⇒ Klusowski and Barron 2018 extended this result and proved that functions in \mathcal{B}^2 can be approximated by shallow **ReLU** neural networks at a rate that does not grow with input dimension.
- Spectral Barron spaces are **variation spaces** for the dictionary

$$\{(1 + \|\omega\|)^{-s} e^{i\omega^\top x}\}_{\omega \in \mathbb{R}^d}$$

Maurey–Barron–Jones Lemma

Theorem

Fix a dictionary $\mathcal{D} = \{\varphi_\xi\}_{\xi \in \Xi}$ of bounded functions and let $\mathcal{V} = \mathcal{V}(\mathcal{D})$ be the associated variation space. Given $f \in \mathcal{V}$, there exists

$$f_K = \sum_{k=1}^K v_k \varphi_{\xi_k}$$

such that

$$\|f - f_K\|_{L^2(\Omega)} \lesssim_{\Omega} \|f\|_{\mathcal{V}} K^{-1/2}.$$

Variation spaces admit dimension-free approximation rates.

Limitations to Spectral Barron Spaces

- Spectral Barron spaces offer an incomplete description.
 - ⇒ \mathcal{B}^2 provides a **sufficient** condition for approximation by shallow ReLU networks with dimension-free rates.
 - ⇒ This kind of regularity is not necessary.

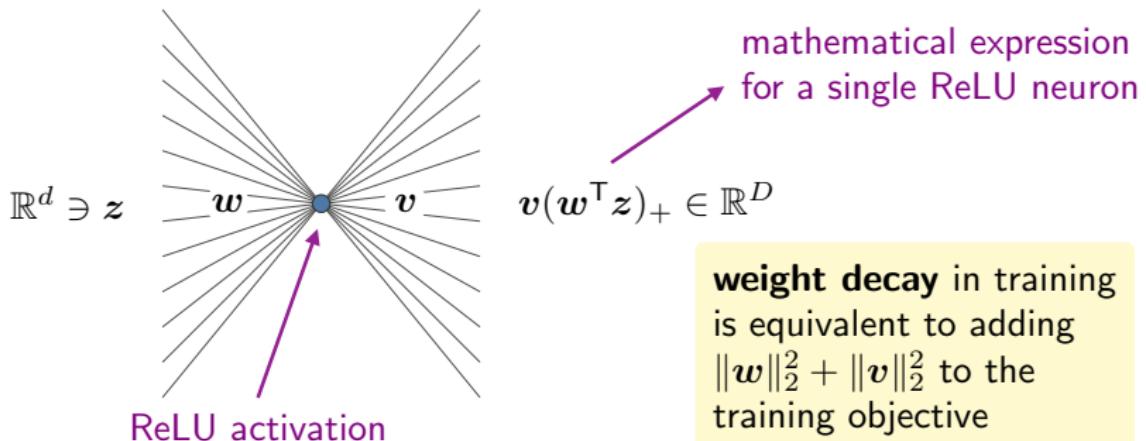
Example: A single ReLU neuron does not lie in \mathcal{B}^2 .

Question: What is the **largest space** of functions approximable by a shallow network (with a given activation function) at a particular error rate?

This is a fundamental problem in approximation theory.

- Currently, there is not a complete characterization of the **approximation spaces** of shallow neural networks, but sufficient conditions are known. (DeVore et al. 2025)
 - ⇒ In 1D, these spaces are known and coincide with Besov spaces. (Petrushev 1988)

Neural Balance and Weight Decay



Neural Balance Theorem

If a DNN is trained with weight decay, then the 2-norms of the input and output weights to each ReLU neuron must be **balanced**.

$$\|w\|_2 = \|v\|_2$$

Neural Balance

The ReLU activation is **homogeneous**

$$\mathbf{v}(\mathbf{w}^\top \mathbf{z})_+ = \gamma^{-1} \mathbf{v}(\gamma \mathbf{w}^\top \mathbf{z})_+, \quad \text{for any } \gamma > 0.$$

At a global minimizer of the weight decay objective, $\|\mathbf{v}\|_2 = \|\mathbf{w}\|_2$.

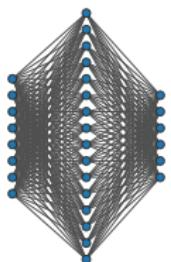
Proof. The solution to

$$\min_{\gamma > 0} \|\gamma^{-1} \mathbf{v}\|_2 + \|\gamma \mathbf{w}\|_2$$

is $\gamma = \sqrt{\|\mathbf{v}\|_2 / \|\mathbf{w}\|_2}$. □

$$\text{At a global minimizer, } \frac{\|\mathbf{v}\|_2^2 + \|\mathbf{w}\|_2^2}{2} = \|\mathbf{v}\|_2 \|\mathbf{w}\|_2.$$

Secret Sparsity of Weight Decay



$$f_{\theta}(x) = \sum_{k=1}^K v_k (\mathbf{w}_k^\top x)_+$$

$$\theta = \{(\mathbf{w}_k, v_k)\}_{k=1}^K$$

weight decay

$$\min_{\theta=\{(\mathbf{w}_k, v_k)\}_{k=1}^K} \sum_{i=1}^n \mathcal{L}(\mathbf{y}_i, f_{\theta}(\mathbf{x}_i)) + \frac{\lambda}{2} \sum_{k=1}^K \|v_k\|_2^2 + \|\mathbf{w}_k\|_2^2$$

path-norm

$$\min_{\theta=\{(\mathbf{w}_k, v_k)\}_{k=1}^K} \sum_{i=1}^n \mathcal{L}(\mathbf{y}_i, f_{\theta}(\mathbf{x}_i)) + \lambda \sum_{k=1}^K \|v_k\|_2 \|\mathbf{w}_k\|_2$$

multitask lasso

$$\min_{\theta=\{(\mathbf{w}_k, v_k)\}_{k=1}^K, \|\mathbf{w}_k\|_2=1} \sum_{i=1}^n \mathcal{L}(\mathbf{y}_i, f_{\theta}(\mathbf{x}_i)) + \lambda \sum_{k=1}^K \|v_k\|_2$$

Rebalancing

Secret Sparsity of Weight Decay

$$\text{weight decay} \iff \min_{\theta = \{(\mathbf{w}_k, \mathbf{v}_k)\}_{k=1}^K \atop \|\mathbf{w}_k\|_2 = 1} \sum_{i=1}^n \mathcal{L}(\mathbf{y}_i, f_\theta(\mathbf{x}_i)) + \lambda \sum_{k=1}^K \|\mathbf{v}_k\|_2$$

- Weight decay is equivalent to a **non-convex** multitask lasso.

⇒ Convex reformulations of
neural network training problems.

Ergen and Pilanci (2021, JMLR)
Sahiner et al. (2021, ICLR)

What Kinds of Functions Do Neural Networks Learn?

Why Do Neural Networks Work Well in High-Dimensional Problems?

Path-Norm and Representation Costs

$$\mathcal{F}_\Theta = \left\{ f(\mathbf{x}) = \sum_{k=1}^K v_k (\mathbf{w}_k^\top \mathbf{x})_+ : v_k \in \mathbb{R}, \mathbf{w}_k \in \mathbb{R}^d, K \in \mathbb{N} \right\}$$

finite-width
networks

The path-norm is a **valid norm** on \mathcal{F}_Θ :

$$\|f\|_{\mathcal{F}} = \sqrt{\sum_{k=1}^K |v_k| \|\mathbf{w}_k\|_2^2}$$

The “completion” of \mathcal{F}_Θ (in an appropriate sense) is a Banach space.
It is the Banach space of all functions of the form

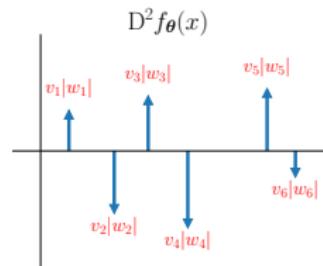
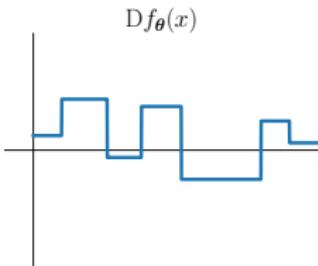
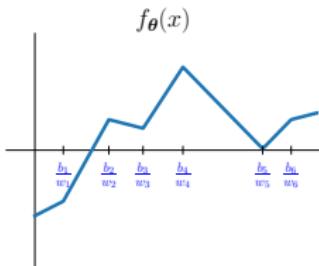
$$f(\mathbf{x}) = \int_{\mathbb{S}^{d-1}} (\mathbf{w}^\top \mathbf{x})_+ d\nu(\mathbf{w}).$$

Barron (1993, IEEE Transactions on Information Theory)
Bach (2017, Journal of Machine Learning Research)
Siegel and Xu (2023, Constructive Approximation)

“output weights”

Path-Norm and Derivatives

$$f_{\theta}(x) = \sum_{k=1}^K v_k(w_k x - b_k)_+$$

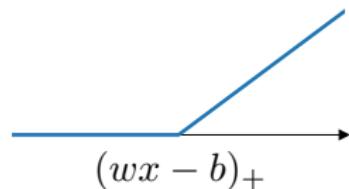


$$\text{path-norm}(f_{\theta}) = \sum_{k=1}^K |v_k| |w_k| = \int_{-\infty}^{\infty} |D^2 f_{\theta}(x)| dx$$

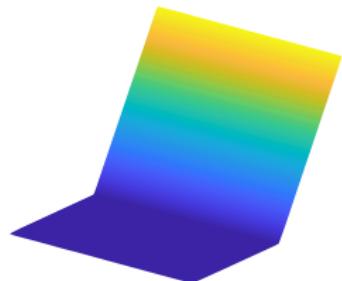
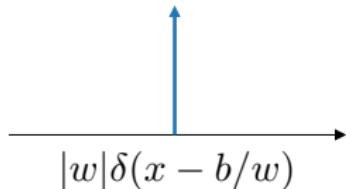
More rigorously:
total variation of Df_{θ}

BV^2 is the native space for univariate shallow neural networks.

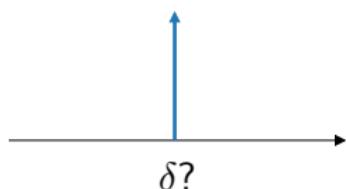
What About the Multivariate Case?



$$\xrightarrow{D^2}$$

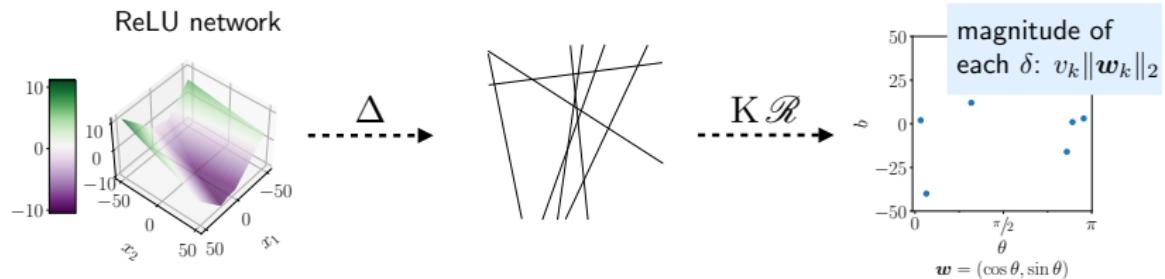
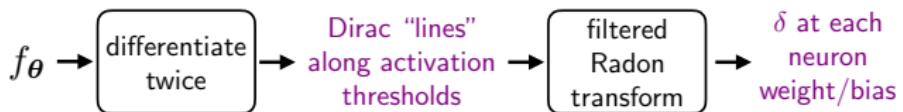


$$\xrightarrow{???$$



$$(w^\top x - b)_+$$

Multivariate Extension: The Radon Transform



$$\text{path-norm}(f_{\theta}) = \sum_{k=1}^K |v_k| \|\mathbf{w}_k\|_2 = \|K\mathcal{R} \Delta f_{\theta}\|_{\mathcal{M}}$$

Functions of Radon Bounded Variation

Radon-domain TV^2 : $\mathcal{R}\text{TV}^2(f) := \|\mathbf{K}\mathcal{R}\Delta f\|_{\mathcal{M}}$

total variation
of the measure
 $\mathbf{K}\mathcal{R}\Delta f$

$\mathbf{K}\mathcal{R}$ = filtered Radon transform

$\widehat{\mathbf{K}g}(\omega) \propto |\omega|^{d-1} \widehat{g}(\omega)$

$\Delta = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ = Laplacian operator

Average measure of **sparsity** of second derivatives along each **direction** in \mathbb{R}^d .

$\mathcal{R}\text{BV}^2$ is the space of all functions on \mathbb{R}^d with $\mathcal{R}\text{TV}^2(f) < \infty$.

Banach, not Hilbert!

Representer Theorem

Neural Network Representer Theorem

For any data set $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ and lower semicontinuous $\mathcal{L}(\cdot, \cdot)$, there exists a solution to

$$\min_{f \in \mathcal{R} \text{ BV}^2} \sum_{i=1}^n \mathcal{L}(y_i, f(\mathbf{x}_i)) + \lambda \mathcal{R} \text{TV}^2(f), \quad \lambda > 0,$$

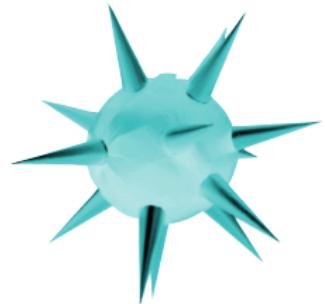
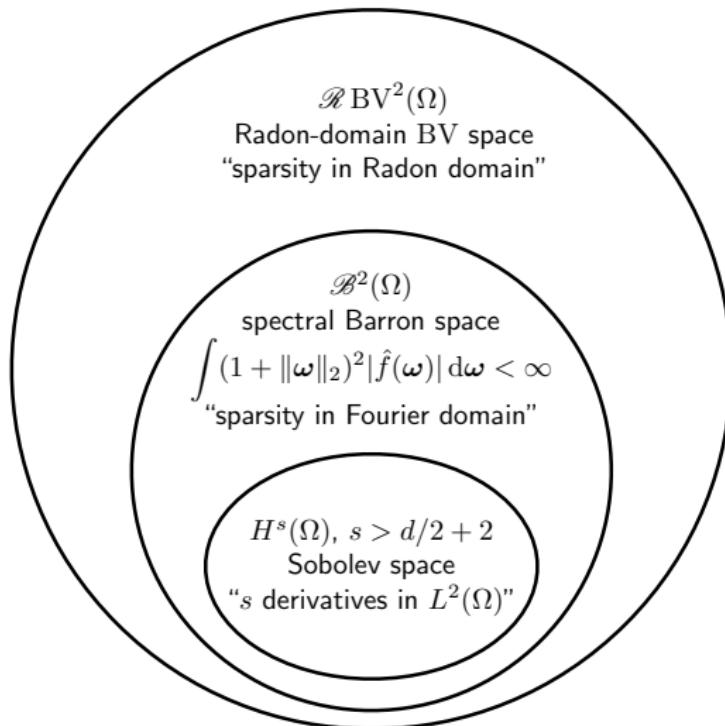
that admits a representation of the form

$$f_{\text{ReLU}}(\mathbf{x}) = \sum_{k=1}^K v_k (\mathbf{w}_k^\top \mathbf{x} - b_k)_+ + \mathbf{w}_0^\top \mathbf{x} + b_0, \quad K < n.$$

Training a **sufficiently parameterized** neural network ($K \geq N$) with weight decay (to a global minimizer) is a solution to the Banach space problem.

Neural networks learn $\mathcal{R} \text{BV}^2$ -functions.

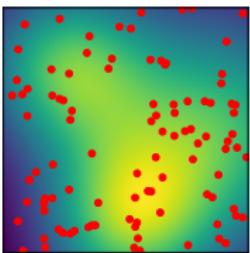
Neural Spaces



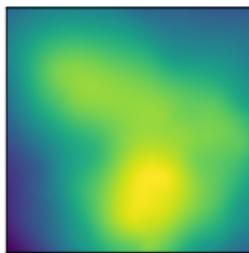
cartoon diagram
of unit $\mathcal{R} \text{BV}^2$ -ball

Adaptation to Directional Smoothness

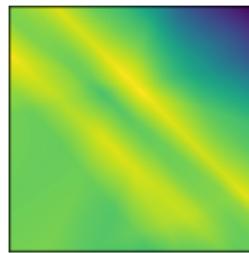
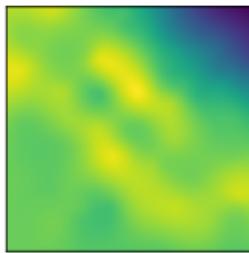
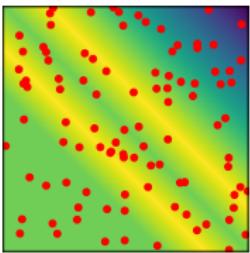
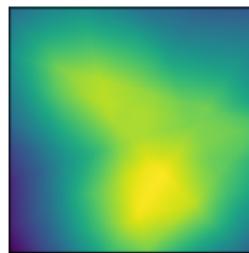
True function
and noisy data



Thin-plate spline
(kernel method)



Neural network
(nonlinear method)



Variation in only a **few directions** is a defining characteristic of $\mathcal{R}BV^2$.

Breaking the Curse of Dimensionality?

Given $f \in \mathcal{R}BV^2$, there exists a finite-width ReLU network f_K with K neurons such that

$$\|f - f_K\|_{L^\infty(\Omega)} = O(K^{-\frac{1}{2} - \frac{3}{2d}}) = O(K^{-\frac{1}{2}}).$$

$\nearrow^{-\alpha}$

Barron (1993)
Matoušek (1996)
Bach (2017)
Siegel (2023)

By the inequality of Carl (1981), this implies

$$\log \mathcal{N}(\delta, U(\mathcal{R}BV^2), \|\cdot\|_{L^\infty(\Omega)}) = \tilde{O}(\delta^{-\frac{2d}{d+3}}) = \tilde{O}(\delta^{-2}).$$

$\nearrow^{-\frac{1}{\alpha}}$
unit ball

Approximation rates and metric entropies
do not grow with the input dimension d .

Minimax Optimality of Shallow Neural Networks

Suppose that $\{\mathbf{x}_i\}_{i=1}^n$ are i.i.d. uniform on a bounded domain $\Omega \subset \mathbb{R}^d$. If $y_i = f^*(\mathbf{x}_i) + \varepsilon_i$ with $\mathcal{R}\text{TV}^2(f^*) < \infty$, then any solution to

$$f_{\text{ReLU}} \in \arg \min_{\theta} \sum_{i=1}^n \mathcal{L}(y_i, f_{\theta}(\mathbf{x}_i)) + \frac{\lambda}{2} \sum_{k=1}^K |v_k|^2 + \|\mathbf{w}_k\|_2^2 \quad \begin{matrix} \text{weight decay} \\ \text{objective} \end{matrix}$$

satisfies

$$\mathbf{E} \|f^* - f_{\text{ReLU}}\|_{L^2(\Omega)}^2 = \tilde{O}(n^{-\frac{d+3}{2d+3}}) = \tilde{O}(n^{-\frac{1}{2}}). \quad \begin{matrix} \text{no curse} \\ \text{minimax rate} \end{matrix}$$

Linear methods (thin-plate splines, kernel methods, neural tangent kernels, etc.) **necessarily** suffer the curse of dimensionality.

Linear minimax lower bound: $n^{-\frac{3}{d+3}}$ the curse

Depth Separation Result

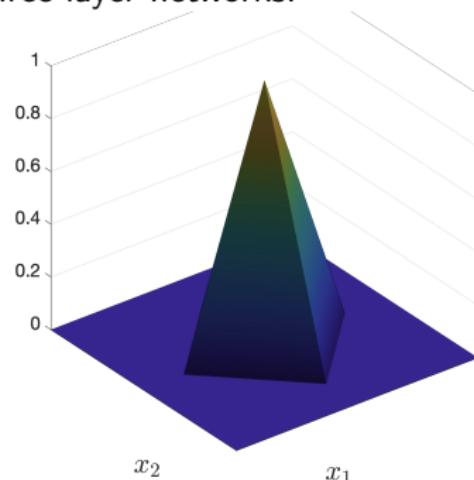
Are there fundamental differences in the native spaces described by (infinite-width) shallow ReLU networks versus deeper ReLU networks?

Put another way are there functions that have small three-layer representation costs but large, or even infinite, two-layer cost?

The answer is yes: [Ongie et al. 2020](#) showed there exists a class of piecewise linear functions $f^* : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $R(f^*)$ is infinite, yet they can be represented exactly by small three-layer networks.

Example: “pyramid function”

$$f^*(\mathbf{x}) = (1 - \|\mathbf{x}\|_1)_+$$



Depth Separation, Cont.

Recent work (McCarty 2023) proved that if $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous piecewise linear function with finite pieces, then $R(f)$ is finite iff f is realizable as a finite-width shallow ReLU network.

Implication: If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ for $d \geq 2$ is any continuous piecewise linear function with compact support, then $R(f) = +\infty$ (b/c a finite sum of ReLUs is never compactly supported).

What can be said about deep network representation costs?

Outline

- ① Hilbert Spaces \Leftrightarrow Linear/Kernel Methods
- ② Banach Spaces \Leftrightarrow Nonlinear/Sparse Methods
- ③ Banach Spaces \Leftrightarrow Shallow Neural Networks
- ④ Beyond(?) Banach Spaces \Leftrightarrow Deep Neural Networks

Deep Networks - Overview

Previously we saw that shallow NN trained with parameter ℓ^2 cost are naturally associated with Banach spaces known as variation spaces.

The associated Banach space norm promotes functions that are a sparse linear combination of neurons.

Question: What are the representation costs associated with deep NNs?
And what function space properties do they promote?

Question: What are the function spaces \mathcal{F} associated with deep NNs?
Are they fundamentally different than variation spaces?

These are still mostly open questions!



Deep Newtorks - Overview, Cont.

Goal: Highlight recent efforts to characterize the **representation costs** and **function spaces** associated with deep NNs.

- ① Deep representation costs and non-linear notions of function rank.
 - ⇒ Warm-up: Deep Linear Networks
 - ⇒ Shallow ReLU nets w/multiple linear layers (Parkinson et al. 2023)
 - ⇒ Deep ReLU networks (Jacot 2023b; Jacot 2023a)
- ② Deep compositions/hiearchies of function spaces, and representer theorems
 - ⇒ Compositions of Variation Spaces (Parhi and Nowak 2022)
 - ⇒ Compositions of RKBSs (Bartolucci et al. 2024)
 - ⇒ Deep Kernel Compositions (Chen 2024; Heeringa et al. 2025)

Warm-up: Deep Linear Networks

Let \mathcal{F}_{lin} be the space of *linear functions* from \mathbb{R}^d to \mathbb{R}^k . Then $f \in \mathcal{F}_{\text{lin}}$ iff

$$f(\mathbf{x}) = \mathbf{W}\mathbf{x}$$

for some matrix $\mathbf{W} \in \mathbb{R}^{k \times d}$.

Suppose we parameterize elements of \mathcal{F}_{lin} as **two-layer linear networks**:

$$f_{\theta}(\mathbf{x}) = \mathbf{U}\mathbf{V}\mathbf{x}$$

where \mathbf{U} and \mathbf{V} have inner dimension $r \geq \min\{k, d\}$, with associated parameter cost $C(\theta) = \frac{1}{2}(\|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2)$.

Lemma (Burer and Monteiro 2003; Srebro et al. 2004)

$$\|\mathbf{W}\|_* = \min_{\mathbf{W}=\mathbf{U}\mathbf{V}} \frac{1}{2}(\|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2).$$

where $\|\mathbf{W}\|_*$ is the **nuclear norm** (the sum of all singular values of \mathbf{W}).

This shows the representation cost of a linear function $f(\mathbf{x}) = \mathbf{W}\mathbf{x}$ parametrized as a two-layer linear network is $\|\mathbf{W}\|_*$.

Warm-up: Deep Linear Networks

Now, suppose we parameterize elements of \mathcal{F}_{lin} as L -layer linear networks:

$$f_{\theta}(x) = \mathbf{W}_L \cdots \mathbf{W}_2 \mathbf{W}_1 x$$

with associated parameter cost:

$$C_L(\theta) = \frac{1}{L} \sum_{\ell=1}^L \|\mathbf{W}_\ell\|_F^2.$$

Then if $f(x) = \mathbf{W}x$, one can show the L -layer representation cost is the **Schatten- $2/L$ quasi-norm** of \mathbf{W} (Dai et al. 2021; Wang et al. 2023):

$$R_L(f) = \|\mathbf{W}\|_{S^{2/L}}^{2/L} = \sum_{i=1}^{\text{rank}(\mathbf{W})} \sigma_i(\mathbf{W})^{2/L}.$$

This is a *non-convex* penalty when $L > 2$.

Generalizations to deep linear convolution networks and other structured matrix classes are considered in (Gunasekar et al. 2018; Dai et al. 2021).

Warm-up: Deep Linear Networks

For a linear function $f(\mathbf{x}) = \mathbf{W}\mathbf{x}$, define $\text{rank}(f) = \text{rank}(\mathbf{W})$. Then, in the limit as the number of linear layers L tends to infinity

$$\lim_{L \rightarrow \infty} R_L(f) = \lim_{L \rightarrow \infty} \|\mathbf{W}\|_{S^2/L}^{2/L} = \text{rank}(\mathbf{W}) = \text{rank}(f)$$

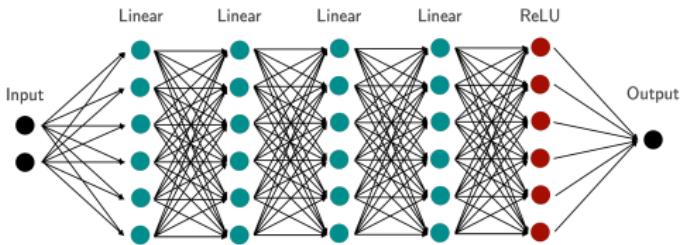
Therefore, (informally) we have:

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^n \mathcal{L}(y_i, f_{\boldsymbol{\theta}}(\mathbf{x}_i)) + \frac{\lambda}{L} \sum_{\ell=1}^L \|\mathbf{W}_{\ell}\|_F^2 \xrightarrow{L \rightarrow \infty} \min_{f \in \mathcal{F}_{\text{lin}}} \sum_{i=1}^n \mathcal{L}(y_i, f(\mathbf{x}_i)) + \lambda \text{rank}(f)$$

Deep linear networks trained with ℓ^2 -regularization
are biased toward **low-rank linear functions**.

Can we understand representation costs of deep *nonlinear* neural networks by alternative notions of function “rank”?

Shallow ReLU Nets with Added Linear Layers



Parameteric Model: L -layer network, first $L - 1$ layers have linear activation, final layer has ReLU activation, scalar outputs.

$$f_{\theta}(x) = \mathbf{a}^T \sigma(\mathbf{W}_{L-1} \cdots \mathbf{W}_1 x + \mathbf{b}) + c$$

This is a **re-parameterization** of shallow ReLU networks.

Parameter Cost: $C_L(\theta) = \frac{1}{L} \left(\|\mathbf{a}\|^2 + \sum_{\ell=1}^{L-1} \|\mathbf{W}_\ell\|_F^2 \right).$

Call the associated **representation cost** $R_L^{\text{lin}}(f)$.

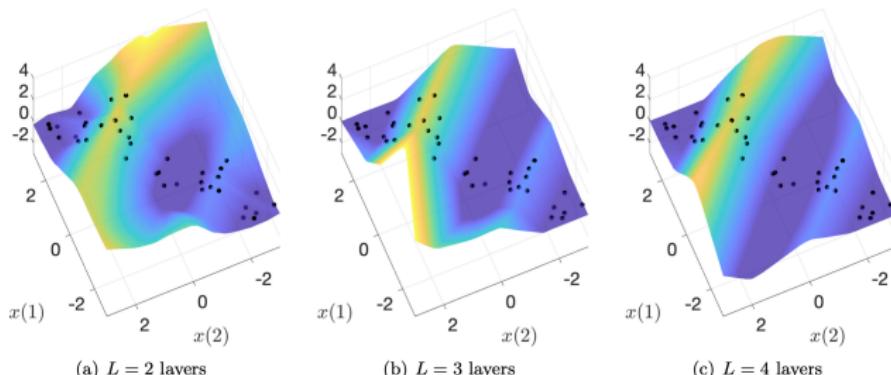
How does the representation cost $R_L^{\text{lin}}(f)$ change (if at all) as the number of linear layers L increases?

Unit Alignment Effect of Linear Layers

The addition of linear layers is equivalent to penalizing a non-convex Schatten quasi-norm on a single “virtual” inner-layer weight matrix:

$$R_L^{\text{lin}}(f) = \min_{\theta} \frac{1}{L} \|\boldsymbol{a}\|^2 + \frac{L-1}{L} \|\boldsymbol{W}\|_{S^{2/(L-1)}}^{2/(L-1)} \text{ s.t. } f(\boldsymbol{x}) = \boldsymbol{a}^\top \sigma(\boldsymbol{W}\boldsymbol{x} + \boldsymbol{b}) + c.$$

This implies $R_L^{\text{lin}}(f)$ promotes functions that are realizable as a shallow ReLU network with **low-rank inner-layer weight matrix**. This can be thought of as a unit alignment effect:



Can this effect be described in function space terms?

Relation to Single- and Multi-Index Models

If $f(\mathbf{x}) = \mathbf{a}^\top \sigma(\mathbf{W}\mathbf{x} + \mathbf{b}) + c$ where \mathbf{W} is rank- r , then there exists a matrix $\mathbf{V} \in \mathbb{R}^{d \times r}$ and function $g : \mathbb{R}^r \rightarrow \mathbb{R}$ such that

$$f(\mathbf{x}) = g(\mathbf{V}^\top \mathbf{x})$$

This is known as a **multi-index model** in the statistics literature.

The column space of \mathbf{V} is often called the **index space**.

Estimating the index space from samples of f (and/or gradients of f) is a classical problem (Li 1991). The **expected gradient outer product (EGOP) matrix** is commonly used tool for this.

Definition

Given any weakly differentiable $f : \Omega \rightarrow \mathbb{R}$ and probability density function ρ defined over $\Omega \subset \mathbb{R}^d$, define its **EGOP matrix** $\mathbf{C}_f \in \mathbb{R}^{d \times d}$ by

$$\mathbf{C}_f := \mathbb{E}_{\mathbf{X}}[\nabla f(\mathbf{X}) \nabla f(\mathbf{X})^\top] = \int_{\Omega} \nabla f(\mathbf{x}) \nabla f(\mathbf{x})^\top \rho(\mathbf{x}) \, d\mathbf{x}$$

Index Rank of a Function

For a multi-index model $f(\mathbf{x}) = g(\mathbf{V}^\top \mathbf{x})$, the EGOP factors as

$$\mathbf{C}_f = \underbrace{\mathbf{V}}_{d \times r} \underbrace{\mathbf{E}_{\mathbf{X}} [\nabla g(\mathbf{V}^\top \mathbf{X}) \nabla g(\mathbf{V}^\top \mathbf{X})^\top]}_{r \times r} \underbrace{\mathbf{V}^\top}_{r \times r}$$

Under general conditions on g , can show that $\text{col}(\mathbf{C}_f) = \text{col}(\mathbf{V})$.
This motivates the following definition:

Definition (Parkinson et al. 2023)

Define the **index rank** of a function f , $\text{rank}_I(f)$, to be the rank of its EGOP matrix:

$$\text{rank}_I(f) = \text{rank}(\mathbf{C}_f).$$

In particular, for the multi-index model $f(\mathbf{x}) = g(\mathbf{V}^\top \mathbf{x})$

$$\text{rank}_I(f) = \text{rank}(\mathbf{V}).$$

under general conditions on g .

Bounds on the L -Linear-Layer Representation Cost

Theorem (Parkinson et al. 2023)

For all $f : \Omega \rightarrow \mathbb{R}$ realizable as a finite width two-layer ReLU network we have the bounds

$$\max\left\{R_2(f)^{2/L}, \|\mathbf{C}_f^{1/2}\|_{S^{2/L}}^{2/L}\right\} \leq R_L^{\text{lin}}(f) \leq \text{rank}_I(f)^{\frac{L-2}{L}} R_2(f)^{2/L}$$

Note: $\lim_{L \rightarrow \infty} \|\mathbf{C}_f^{1/2}\|_{S^{2/L}}^{2/L} = \text{rank}(\mathbf{C}_f^{1/2}) = \text{rank}(\mathbf{C}_f) = \text{rank}_I(f)$.

Also: $\lim_{L \rightarrow \infty} \text{rank}_I(f)^{\frac{L-2}{L}} R_2(f)^{2/L} = \text{rank}_I(f)$

Therefore, as a corollary, we have

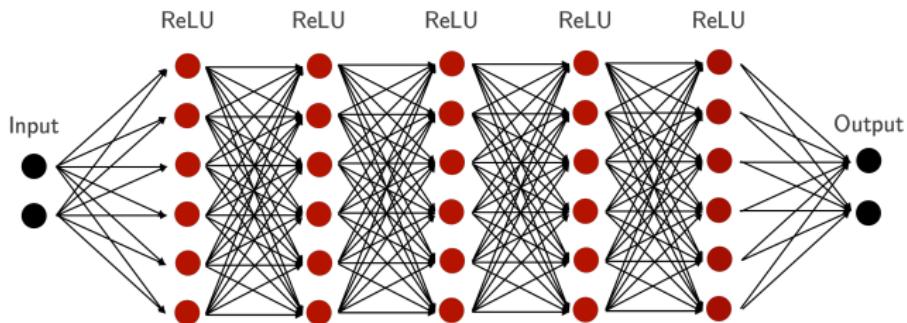
$$\lim_{L \rightarrow \infty} R_L^{\text{lin}}(f) = \text{rank}_I(f).$$

The R_L^{lin} -cost favors functions with **low index rank**, i.e., functions well-approximated by multi-index models.

Related Work

- Bach 2017 gives generalization bounds for infinite-width shallow networks having bounded variation norm assuming the target function is a multi-index model.
- Recent line of work studies ability neural networks to provably learn low-index rank structure when trained via gradient methods:
 - ⇒ Shallow networks (Damian et al. 2022; Bietti et al. 2022; Mousavi-Hosseini et al. 2022)
 - ⇒ Three-layer networks (Nichani et al. 2023)
- EGOP analysis is central to the recently proposed “deep neural feature ansatz” (Radhakrishnan et al. 2024) as a means to explain feature learning in deep networks.

Representation Costs of Deep ReLU Networks



Model class: L -layer fully connected ReLU network, unbounded widths

$$f_{\theta}(x) = W_L(W_{L-1} \cdots \sigma(W_2 \sigma(W_1 x + b_1)_+ + b_2)_+ \cdots + b_{L-1})_+ + b_L.$$

Focus on networks with *vector outputs*: $f_{\theta} : \mathbb{R}^{d_{\text{in}}} \rightarrow \mathbb{R}^{d_{\text{out}}}$.

Parameter Cost: $C_L(\theta) = \frac{1}{L} \|\theta\|_2^2$ (sum-of-squares of all weights/biases)

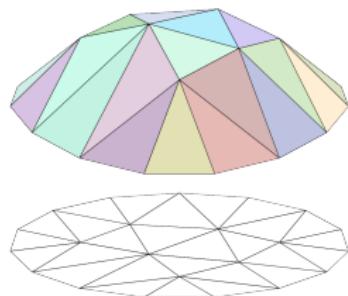
Call the associated **representation cost** $R_L(f)$.

CPWL Functions

Every ReLU net realizes a **continuous piecewise linear (CPWL) function**, in the following sense:

Definition

We say $f : \Omega \rightarrow \mathbb{R}^D$ is **CPWL** if f is continuous and there is a polyhedral decomposition of Ω such that f is affine on each polyhedra in the decomposition.



Conversely, every CPWL function over \mathbb{R}^d can be represented by a ReLU NN with at most $\lceil \log_2(d+1) \rceil$ hidden layers (Arora et al. 2018).

The parametric model space of unbounded width L -layer ReLU nets coincides with all CPWL functions when $L \geq \lceil \log_2(d+1) \rceil$

The space of CPWL functions is a vector space, and is closed under compositions: if g and h are CPWL, then so is $f = h \circ g$.

However, it is not a closed space under any “reasonable” topology.

Infinite-Depth Representation Cost

Define the “**infinite-depth**” **representation cost** of a CPWL function f :

$$R_\infty(f) = \lim_{L \rightarrow \infty} R_L(f).$$

R_∞ has the properties we would expect a “rank” on CPWL functions to have (Jacot 2023b):

- $R_\infty(f \circ g) \leq \min\{R_\infty(f), R_\infty(g)\}$
- $R_\infty(f + g) \leq R_\infty(f) + R_\infty(g)$
- if $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ then $R_\infty(f) = \text{rank}(\mathbf{A})$.

Is there a function space description of R_∞ ?

A suggestive bound:

Lemma

Let f be CPWL, and suppose $\mathbf{x} \in \Omega$ is a point where f is differentiable. Then

$$\|Jf(\mathbf{x})\|_{S^{2/L}}^{2/L} \leq R_L(f)$$

where Jf is the Jacobian of f .

Jacobian Rank and Bottleneck Rank

Definition:

The **Jacobian rank** of a CPWL function $f : \Omega \rightarrow \mathbb{R}^{d_{\text{out}}}$ is

$$\text{rank}_J(f) = \sup_{\mathbf{x}} \text{rank}(Jf(\mathbf{x})),$$

taking the sup over points $\mathbf{x} \in \Omega$ where f is differentiable.

Definition:

The **bottleneck rank** of a CPWL function $f : \Omega \rightarrow \mathbb{R}^{d_{\text{out}}}$, denoted $\text{rank}_{\text{BN}}(f)$, is the smallest integer $r \in \mathbb{N}$ such that $f|_{\Omega} = (g \circ h)|_{\Omega}$ where g and h are CPWL functions with inner dimension r .

If $f = h \circ g$ with inner dimension r , then by the chain rule

$$Jf(\mathbf{x}) = \underbrace{Jh(g(\mathbf{x}))}_{d_{\text{out}} \times r} \underbrace{Jg(\mathbf{x})}_{r \times d_{\text{in}}} \implies \text{rank}_J(f) \leq r.$$

This shows $\text{rank}_J(f) \leq \text{rank}_{\text{BN}}(f)$ for any CPWL f .

But there are CPWL functions where strict inequality holds.

Bounds on the “Infinite-Depth” Representation Cost

Theorem (Jacot 2023b)

For all CPWL functions $f : \Omega \rightarrow \mathbb{R}$

$$\text{rank}_J(f) \leq R_\infty(f) \leq \text{rank}_{\text{BN}}(f).$$

Further, it is conjectured that for all CPWL functions f

$$R_\infty(f) = \text{rank}_{\text{BN}}(f).$$

Related Work

Recent work by Jacot *et al.* explores the implications of the bottleneck rank for learning:

- Emergent bottleneck rank structure CNNs (Wen and Jacot 2024)
- Neural collapse phenomenon (Jacot et al. 2024)
- Feature learning in Leaky ResNets (Jacot and Kaiser 2025)

Related nonlinear notions of function rank have been proposed to characterize **implicit regularization** in deep networks:

- Deep matrix factorization (Arora et al. 2019; Razin and Cohen 2020)
- Deep tensor factorization (Razin et al. 2021; Razin et al. 2022)
- Graph Neural Networks and Separation Rank (Razin et al. 2023).
- Rank minimization in deep ReLU networks (Timor et al. 2023)

Summary and Outlook

Function spaces give a unified perspective to learning with kernel methods, sparse methods, shallow NN, and (to some extent) deep NN.

Powerful tool for characterizing **approximation**, **estimation**, and **generalization** capabilities of neural networks. (Bach 2017; E and Wojtowytsch 2020; Siegel and Xu 2020; Schmidt-Hieber 2020; Zhang and Wang 2023; E et al. 2022; Siegel and Xu 2023; DeVore et al. 2025)

Practical implications for **efficient optimization** and **compression** of neural network models. (Ergen and Pilancı 2021; Yang et al. 2022; Varshney and Pilancı 2024; Shenouda et al. 2024)

Open Problems

Several Key Open Problems Remain

- Is there a function space characterization of R_L , the representation cost associated w/ L -layer ReLU networks, for $L > 2$?
- What is the native space associated with R_L -cost? How does it change with input dimension d ? How does it change with depth L ?
- Does the depth- L native space “saturate” after some depth L_0 ?
- Prove or disprove the conjecture $R_\infty(f) = \text{rank}_{\text{BN}}(f)$. ([Jacot 2023b](#))
- What connections can be drawn between function space perspectives of explicit regularization versus *implicit regularization* arising from practical training with gradient methods?

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