Name: Solutions

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Practice Midterm

Problem 1

Let $X_1, ..., X_n$ be a random sample from the following distribution with PDF

$$f(x;\theta) = \theta x^{\theta-1} \qquad 0 \le x \le 1$$
$$0 < \theta < \infty$$

a. Evaluate $\hat{\theta}_{\text{MLE}}$, the maximum likelihood estimator of θ.

b. Evaluate the Fisher information for this model.

c. Determine the asymptotic distribution of $\hat{\theta}_{\text{MLE}}$.

d. Find a sufficient statistic for θ. Can we argue that $\hat{\theta}_{\text{MLE}}$ is sufficient as well? Why or why not?

a. $f(\underline{x}|\theta) = \prod_{i=1}^{n} (\theta x_i^{\theta-1}) = \theta^n (\prod_i x_i)^{\theta-1} = \mathcal{L}(\theta)$

$\ell(\theta) = \log \mathcal{L}(\theta) = n \log\theta + (\theta-1)\log(\prod_i x_i)$

$= n\log\theta + (\theta-1)\sum_i \log X_i$

Extrema occur when $\ell'(\theta) = 0$.

$$\ell'(\theta) = \frac{n}{\theta} + \sum_i \log X_i \overset{!}{=} 0$$

$$\Rightarrow \hat{\theta}_{\text{MLE}} = \frac{-n}{\sum_i \log X_i}$$

Check max:

$$\ell''(\hat{\theta}) = \frac{-n}{\left(\frac{-n}{\sum_i \log X_i}\right)^2} = -\frac{(\sum_i \log X_i)^2}{n} < 0$$

b. Single obs:

$$I(\theta) = -E\left(\frac{\partial^2 \log f(x_i|\theta)}{\partial\theta^2}\right)$$

$$= -E\left(\frac{\partial^2}{\partial\theta^2}\left(\log\theta + (\theta-1)x\right)\right)$$

$$= -E\left(\frac{\partial}{\partial\theta}\left(\frac{1}{\theta} + x\right)\right)$$

$$= -E\left(\frac{-1}{\theta^2} + 0\right) = E\left(\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}.$$

Sample:

$$I_n(\theta) = nI(\theta) = \frac{n}{\theta^2}$$

c. MLEs have asymptotic distributions in the form:

$$\sqrt{n}(\hat\theta - \theta) \dot\sim N\left(0, \frac{1}{I_n(\theta)}\right)$$

So:

$$\sqrt{n}(\hat\theta_{MLE} - \theta) \dot\sim N\left(0, \frac{\theta^2}{n}\right)$$

OR $\hat\theta_{MLE} \dot\sim N(\theta, \theta^2)$

d. Factor $f(\underline{x}|\theta) = \theta^n (\Pi_i x_i)^{\theta-1}$ into $u(\underline{x}) = 1$

$v(T, \theta) = \theta^n (\Pi_i x_i)^{\theta-1}$

The only way the data comes in is through $T(\underline{X}) = \Pi_i X_i$

$\therefore$ $T(\underline{X}) = \Pi_i X_i$ is sufficient by factorization theorem.

Notice the MLE

$$\hat\theta_{MLE} = \frac{-n}{\Sigma_i \log X_i} = \frac{-n}{\log(\Pi_i X_i)} = \frac{-n}{\log(T)} = g(T)$$

Where $g(t) = \frac{-n}{\log(t)}$ is 1:1.

A 1:1 function of a sufficient statistic is still sufficient, so yes.

Problem 2

Let $X_1, ..., X_n$ be a random sample from a Gamma(α,β) distribution.

a. Find method of moments estimators for α and β.

b. If β is known to be β_0, find a sufficient statistic for α.

c. If α is known to be α_0, find a sufficient statistic for β.

d. If α and β are both unknown, find jointly sufficient statistics for the pair of them.

a. 2 unknowns $\Rightarrow$ need 2 equations.

Set $E(X) \overset{!}{=} \bar{X}$ (1st moments)

& $E(X^2) = \frac{1}{n}\sum_{i=1}^n X_i^2$ (2nd moments)

We know (table) that:

$E(X) = \alpha/\beta \Rightarrow$ $\therefore$ Set: $\alpha/\beta \overset{!}{=} \bar{X} \Rightarrow \alpha = \beta\bar{X}$ ①

Also:

$$E(X^2) = Var(X) + E^2(X) = \frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2} = \frac{\alpha+\alpha^2}{\beta^2}$$

$\therefore$ Set

$$\frac{\alpha+\alpha^2}{\beta^2} = \frac{1}{n}\sum_i X_i^2 \quad ②$$

Solve ① & ② together:

$$\frac{\alpha+\alpha^2}{\beta^2} = \frac{(\beta\bar{X}) + (\beta\bar{X})^2}{\beta^2} = \frac{\bar{X}}{\beta} + \bar{X}^2 = \frac{1}{n}\sum_i X_i^2$$

$$\Rightarrow \bar{X} = \beta\left(\frac{1}{n}\sum_i X_i^2 - \bar{X}^2\right)$$

$$\rightarrow \boxed{\hat{\beta} = \bar{X}\Big/\left(\frac{1}{n}\sum_i X_i^2 - \bar{X}^2\right)}$$

Also $\alpha = \beta\bar{X} \Rightarrow \hat{\alpha} = \hat{\beta}\bar{X} = \boxed{\bar{X}^2\Big/\left(\frac{1}{n}\sum_i X_i^2 - \bar{X}^2\right) = \hat{\alpha}}$

b. $f(\underset{\sim}{X}|\alpha) = \frac{\beta_0^{\alpha}}{\Gamma(\alpha)} \cdot (\Pi_i X_i)^{\alpha-1} e^{-\beta_0 \Sigma_i X_i}$

Factor: $u(\underset{\sim}{X}) = e^{-\beta_0 \Sigma_i X_i}$

$v(T, \alpha) = \frac{\beta_0^{\alpha}}{\Gamma(\alpha)} (\underline{\Pi_i X_i})^{\alpha-1}$

$\hookrightarrow T(\underset{\sim}{X}) = \Pi_i X_i$ is sufficient by factorization.

c. $f(\underset{\sim}{X}|\beta) = \frac{\beta^{\alpha_0}}{\Gamma(\alpha_0)} \cdot (\Pi_i X_i)^{\alpha_0 - 1} e^{-\beta \Sigma_i X_i}$

Factor: $u(\underset{\sim}{X}) = \frac{(\Pi_i X_i)^{\alpha_0 - 1}}{\Gamma(\alpha_0)}$

$v(T, \beta) = \beta^{\alpha_0} \cdot e^{-\beta \Sigma_i X_i} \Rightarrow T(\underset{\sim}{X}) = \Sigma_i X_i$ is suff. by facto.

d. $f(\underset{\sim}{X}|\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} (\Pi_i X_i)^{\alpha-1} e^{-\beta \Sigma_i X_i}$

Factor: $u(\underset{\sim}{X}) = \frac{1}{\Pi_i X_i}$

$v(T, \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} (\Pi_i X_i)^{\alpha-1} e^{-\beta \Sigma_i X_i}$

$\begin{cases} T_1(\underset{\sim}{X}) = \Pi_i X_i \\ T_2(\underset{\sim}{X}) = \Sigma_i X_i \end{cases}$ are jointly suff. by facto.

Problem 3

Let $X_1, ..., X_n$ be a random sample from a Poisson(λ) distribution.

a. Find a method of moment estimator of λ.

b. Find a method of moment estimator of λ based off of the second moment. (Hint: equations of the form $ax^2 + bx + c = 0$ can be solved with the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.)

c. Assume now that we take a Bayesian perspective and suppose that the prior distribution of λ is Gamma(α,β). Find the posterior distribution of λ given $X_1, ..., X_n$.

d. Calculate the posterior mean.

e. Set up (but do not solve) an equation for the posterior median.

a. $E(X) \overset{!}{=} \bar{X}$

For Poisson... $E(X) = \lambda \Rightarrow \hat{\lambda}_{MoM} = \bar{X}$

call this

b. $E(X^2) \overset{!}{=} \frac{1}{n}\sum_i X_i = m_2$

$E(X^2) = Var(X) + E^2(X) = \lambda + \lambda^2$

$\Rightarrow E(X^2) = \lambda^2 + \lambda \overset{!}{=} m_2$

$\Rightarrow \hat{\lambda}^2 + \hat{\lambda} - m_2 = 0 \Rightarrow$ Quad. Eq: $\hat{\lambda} = \frac{-1 \pm \sqrt{1 - 4(1)(m_2)}}{2}$

$\lambda > 0 \Rightarrow$ positive root $\Rightarrow \hat{\lambda} = \frac{-1 + \sqrt{4m_2 + 1}}{2}$

(only positive if $m_2 > 0$)

c. $\xi(\lambda) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$

$f(\underset{\sim}{x}|\lambda) = \prod_{i=1}^{n} \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \dfrac{e^{-n\lambda}\lambda^{\sum_i x_i}}{(\prod_i x_i!)}$

Posterior pdf:

$$\xi(\lambda|\underset{\sim}{x}) = \frac{\xi(\lambda) f(\underset{\sim}{x}|\lambda)}{f(\underset{\sim}{x})} = \frac{\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}\left(\frac{e^{-n\lambda}\lambda^{\sum_i x_i}}{(\prod_i x_i!)}\right)}{f(\underset{\sim}{x})}$$

$$= k \underbrace{\lambda^{\alpha+\sum_i x_i - 1} e^{-(\beta+n)\lambda}}_{\Gamma \text{ kernel w/ } \tilde{\alpha} = \alpha + \sum_i x_i,\ \tilde{\beta} = \beta + n}$$

$\Rightarrow \lambda|\underset{\sim}{x} \sim \Gamma(\tilde{\alpha} = \alpha + \sum_i x_i, \tilde{\beta} = \beta + n)$

d. $E(\lambda|\underset{\sim}{x}) \overset{\text{(table)}}{=} \dfrac{\tilde{\alpha}}{\tilde{\beta}} = \dfrac{\alpha + \sum_i x_i}{\beta + n}$

e. Posterior median is M such that:

$$\int_0^M \frac{(\beta+n)^{\alpha+\sum_i x_i}}{\Gamma(\alpha+\sum_i x_i)} \lambda^{\alpha+\sum_i x_i - 1} e^{-(\beta+n)\lambda}\, d\lambda = 1/2.$$

Problem 4

Suppose that $X_1, ..., X_n$ is a random sample from the Bernoulli(p) distribution.

a. Find the maximum likelihood estimator for p and call it $\hat{p}$. Show that it is unbiased and calculate its variance.

b. Compute the Fisher Information for p. Does the MLE attain the Cramèr-Rao lower bound?

c. Consider the adjusted estimator for the proportion:

$$\tilde{p} = \frac{\sum_{i=1}^{n} X_i + 1}{n+2}.$$

Calculate the bias of this estimator and show that $\tilde{p}$ is unbiased if and only if $p = 0.5$.

d. Calculate MSE($\hat{p}$) and MSE($\tilde{p}$).

e. Suppose that the true value of p is 0.1. Which estimator performs better in terms of MSE? What about if the true value of p is 0.5? (Hint: you can choose a specific value for n and compare numerically.)

a. $f(\underline{x}|p) = p^{\sum_i x_i}(1-p)^{n-\sum_i x_i} = \mathcal{L}(p)$

$\ell(p) = \sum_i x_i \log p + (n - \sum_i x_i)\log(1-p)$

$\ell'(p) = \frac{\sum_i x_i}{p} + \frac{(n-\sum_i x_i)}{1-p}(-1) \overset{\text{set}}{=} 0$

$\Rightarrow \frac{\sum_i x_i}{p} = \frac{n-\sum_i x_i}{1-p} \Rightarrow \sum_i x_i - \cancel{p\sum_i x_i} = np - \cancel{p\sum_i x_i}$

$\Rightarrow \sum_i x_i = np \Rightarrow \boxed{\hat{p} = \frac{\sum_i x_i}{n}}$

Check: $\ell''(\hat{p}) = \frac{-\sum_i x_i}{\hat{p}^2} - \frac{n-\sum_i x_i}{(1-\hat{p})^2} < 0$

$\text{bias}(\hat{p}) = E(\hat{p}) - p$

$E(\hat{p}) = E(\frac{1}{n}\sum_i x_i) = \frac{1}{n}\sum_i E(X_i) = \frac{1}{n}\sum_i p = \frac{np}{n} = p.$

$\Rightarrow \text{bias}(\hat{p}) = p - p = 0. \Rightarrow$ unbiased.

$\text{Var}(\hat{p}) = \text{Var}(\frac{1}{n}\sum_i x_i) = \frac{1}{n^2}\sum_i \text{Var}(X_i) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$

b. $I(p) = -E\left(\frac{\partial^2 \log f(x_i|p)}{\partial p^2}\right)$

$$\log f(x_i|p) = \ell(p) = x_i \log p + (1-x_i)\log(1-p)$$

$$\ell'(p) = x_i/p - \frac{1-x_i}{1-p}$$

$$\ell''(p) = \frac{-x_i}{p^2} - \frac{(1-x_i)}{(1-p)^2}$$

$$I(p) = -E(\ell''(p)) = E\left(\frac{x_i}{p^2} + \frac{1-x_i}{(1-p)^2}\right)$$

$$= \frac{E(X_i)}{p^2} + \frac{1-E(X_i)}{(1-p)^2} = \frac{p}{p^2} + \frac{1-p}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p}$$

$$= \frac{1}{p(1-p)}$$

$$I_n(p) = nI(p) = \frac{n}{p(1-p)}.$$

CRLB:

$$Var(\hat{p}) \stackrel{?}{=} \frac{m'(p)}{I_n(p)};$$

LHS:

$Var(\hat{p}) = \frac{p(1-p)}{n}$ from part a.

RHS: $m(p) = E(\hat{p}) = p$
$m'(p) = 1$
$\Rightarrow \frac{m'(p)}{I_n(p)} = \frac{1}{(n/p(1-p))} = \frac{p(1-p)}{n}.$

LHS = RHS $\Rightarrow$ $Var(\hat{p})$ attains the CRLB & is efficient.
(left hand side) (right " ")

Problem 5

Suppose that $X_1, ..., X_n$ is a random sample from the following

$$f(x|\theta) = \frac{3}{\theta^3}x^2 \quad \text{for } 0 < x < \theta$$

a. Show that the expected value of X_1 is $\frac{3}{4}\theta$.

b. Find a method of moments estimator of θ.

c. Is it possible for us to calculate Fisher Information for this sample? If not, which of the the assumptions are violated and how?

a. $E(X_1) = \int_0^\theta x \cdot f(x)dx = \int_0^\theta x \cdot \frac{3}{\theta^3} x^2 dx = \frac{3}{\theta^3} \left. \frac{x^4}{4} \right|_0^\theta = \frac{3\theta}{4}$

b. $E(X) = \frac{3}{4}\theta \overset{!}{=} \bar{X}$

$\Rightarrow \hat{\theta} = \frac{4}{3}\bar{X}$

c. Support depends on θ!

That violates a Fisher Info. assumption $\Rightarrow$ cannot calculate $I(\theta)$.