CHAPTER – 4

Linear Programming: II

Transportation Model

4.1. INTRODUCTION

In operations Research Linear programming is one of the model in mathematical programming, which is very broad and vast. Mathematical programming includes many more optimization models known as Non - linear Programming, Stochastic programming, Integer Programming and Dynamic Programming - each one of them is an efficient optimization technique to solve the problem with a specific structure, which depends on the assumptions made in formulating the model. We can remember that the general linear programming model is based on the assumptions:

(*a*) Certainty

The resources available and the requirement of resources by competing candidates, the profit coefficients of each variable are assumed to remain unchanged and they are certain in nature.

(*b*) Linearity

The objective function and structural constraints are assumed to be linear.

(*c*) Divisibility

All variables are assumed to be continuous; hence they can assume integer or fractional values.

(*d*) Single stage

The model is static and constrained to one decision only. And planning period is assumed to be fixed.

(*e*) Non-negativity

A non-negativity constraint exists in the problem, so that the values of all variables are to be ≥ 0, *i.e*. the lower limit is zero and the upper limit may be any positive number.

(*f*) Fixed technology

Production requirements are assumed to be fixed during the planning period.

(*g*) Constant profit or cost per unit

Regardless of the production schedules profit or cost remain constant.

Now let us examine the applicability of linear programming model for **transportation** and **assignment models**.

4.2. TRANSPORTATION MODEL

The transportation model deals with a special class of linear programming problem in which the objective is to **transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost.**

To understand the problem more clearly, let us take an example and discuss the rationale of transportation problem. Three factories *A*, *B* and *C* manufactures sugar and are located in different regions. Factory *A* manufactures, b_1 tons of sugar per year and *B* manufactures b_2 tons of sugar per year and *C* manufactures b_3 tons of sugar. The sugar is required by four markets *W*, *X*, *Y* and *Z*. The requirement of the four markets is as follows: Demand for sugar in Markets *W*, *X*, Yand *Z* is d_1, d_2, d_3 and d_4 tons respectively. The transportation cost of one ton of sugar from each factory to market is given in the matrix below. The objective is to transport sugar from factories to the markets at a minimum total transportation cost.

	Markets	*Transportation cost per ton in Rs.*				*Availability in tons*
		W	*X*	*Y*	*Z*	
Factories	*A*	c_{11}	c_{12}	c_{13}	c_{14}	b_1
	B	c_{21}	c_{22}	c_{23}	c_{24}	b_2
	C	c_{31}	c_{32}	c_{33}	c_{34}	b_3
Demand in Tons.		d_1	d_2	d_3	d_4	$\Sigma b_j / \Sigma d_j$

For the data given above, the mathematical model will be:

Minimize $Z = c_{11}\, x_{11} + c_{12}\, x_{12} + c_{13}\, x_{13} + c_{14}\, x_{14} + c_{21}\, x_{21} + c_{22}\, x_{22} + c_{23}\, x_{23} + c_{24}\, x_{24} + c_{31}\, x_{31} + c_{32}\, x_{32} + c_{33}\, x_{33} + c_{34}\, x_{34}$ subject to a condition: $\longrightarrow$ **OBJECTIVE FUNCTION.**

$a_{11}\, x_{11} + a_{12}\, x_{12} + a_{13}\, x_{13} + a_{14}\, x_{14} \leq b_1$ (because the sum must be less than or equal to the available capacity)

$a_{21}\, x_{21} + a_{22}\, x_{22} + a_{23}\, x_{23} + a_{24}\, x_{24} \leq b_2$

$a_{31}\, x_{31} + a_{32}\, x_{32} + a_{33}\, x_{33} + a_{34}\, x_{34} \leq b_3$ $\longrightarrow$ **MIXED STRUCTURAL CONSTRAINTS.**

$a_{11}\, x_{11} + a_{21}\, x_{21} + a_{31}\, x_{31} \geq d_1$

$a_{12}\, x_{12} + a_{22}\, x_{22} + a_{32}\, x_{32} \geq d_2$ (because the sum must be greater than or equal to the demand of the market. We cannot send less than what is required)

$a_{13}\, x_{13} + a_{23}\, x_{23} + a_{33}\, x_{33} \geq d_3$

$a_{14}\, x_{14} + a_{24}\, x_{24} + a_{34}\, x_{34} \geq d_4$ and

All x_{ij} and x_{ji} are ≥ 0 where $i = 1,2,3$ and $j = 1,2,3,4$. (This is because we cannot supply negative elements). $\longrightarrow$ **NON-NEGATIVITY CONSTRAINT.**

The above problem has got the following properties:

1. It has an objective function.
2. It has structural constraints.
3. It has a non-negativity constraint.
4. The relationship between the variables and the constraints are linear.

We know very well that these are the properties of **a linear programming problem**. Hence the transportation model is also a **linear programming problem**. But a **special type of linear programming problem.**

Once we say that the problem has got the characteristics of linear programming model, and then we can solve it by simplex method. Hence we can solve the transportation problem by using the simplex method. As we see in the above given transportation model, the structural constraints are of mixed type. That is some of them are of ≤ type and some of them are of ≥ type. When we start solving the transportation problem by simplex method, it takes more time and laborious. Hence we use **transportation algorithm or transportation method** to solve the problem. Before we discuss the transportation algorithm, let us see how a general model for transportation problem appears. The general problem will have '*m*' rows and '*n*' columns *i.e.*, $m \times n$ matrix.

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}\, x_j \quad \text{s.t. where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n.$$

$$\sum_{i=1}^{m} a_{ij}\, x_{ij} \le b_i \quad \text{where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

$$\sum_{j=1}^{n} a_{ij}\, x_{ji} \ge d_j \quad \text{where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

4.3. COMPARISON BETWEEN TRANSPORTATION MODEL AND GENERAL LINEAR PROGRAMMING MODEL

Similarities

1. Both have objective function.
2. Both have linear objective function.
3. Both have non - negativity constraints.
4. Both can be solved by simplex method. In transportation model it is laborious.
5. A general linear programming problem can be reduced to a transportation problem if (*a*) the a_{ij}*'s* (coefficients of the structural variables in the constraints) are restricted to the values 0 and/or 1 and (*b*) There exists homogeneity of units among the constraints.

Differences

1. Transportation model is basically a minimization model; where as general linear programming model may be of maximization type or minimization type.

2. The resources, for which, the structural constraints are built up is homogeneous in transportation model; where as in general linear programming model they are different. That is one of the constraint may relate to machine hours and next one may relate to man-hours etc. In transportation problem, all the constraints are related to one particular resource or commodity, which is manufactured by the factories and demanded by the market points.
3. The transportation problem is solved by transportation algorithm; where as the general linear programming problem is solved by simplex method.
4. The values of structural coefficients (*i.e.* x_{ij}) are not restricted to any value in general linear programming model, where as it is restricted to values either 0 or 1 in transportation problem. Say for example:

   Let one of the constraints in general linear programming model is: $2x - 3y + 10z \leq 20$. Here the coefficients of structural variables x, y and z may negative numbers or positive numbers of zeros. Where as in transportation model, say for example $x_{11} + x_{12} + x_{13} + x_{14} = b_i = 20$. Suppose the value of variables x_{11}, and x_{14} are 10 each, then $10 + 0.\ x_{12} + 0.\ x_{13} + 10 = 20$. Hence the coefficients of x_{11} and x_{14} are 1 and that of x_{12} and x_{13} are zero.

4.4. APPROACH TO SOLUTION TO A TRANSPORTATION PROBLEM BY USING TRANSPORTATION ALGORITHM

The steps used in getting a solution to a transportation problem is given below:

4.4.1. Initial Basic Feasible Solution

Step 1. Balancing the given problem. Balancing means check whether sum of availability constraints must be equals to sum of requirement constraints. That is $\Sigma b_i = \Sigma d_j$. Once they are equal, go to step two. If not by opening a *Dummy row* or *Dummy column* balance the problem. The *cost coefficients of dummy cells are zero*. If Σb_i is greater than Σd_j, then open a dummy column, whose requirement constraint is equals to $\Sigma b_i - \Sigma d_j$ and the cost coefficient of the cells are zeros. In case if Σd_j is greater than Σb_i, then open a dummy row, whose availability constraint will be equals to $\Sigma d_j - \Sigma b_i$ and the cost coefficient of the cells are zeros. Once the balancing is over, then go to second step. Remember while solving general linear programming problem to convert an inequality into an equation, we add (for maximization problem) a slack variable. In transportation problem, the dummy row or dummy column, exactly similar to a slack variable.

Step II. A .Basic feasible solution can be obtained by three methods, they are

(*a*) North - west corner method.

(*b*) Least - cost cell method. (Or Inspection method Or Matrix minimum - row minimum - column minimum method)

(*c*) Vogel's Approximation Method, generally known as VAM.

After getting the basic feasible solution (*b.f.s.*) give **optimality test** to check whether the solution is optimal or not.

There are two methods of giving optimality test:

(*a*) Stepping Stone Method.

(*b*) Modified Distribution Method, generally known as **MODI** method.

4.4.2.Properties of a Basic feasible Solution

1. The allocation made must satisfy the rim requirements, *i.e.*, it must satisfy availability constraints and requirement constraints.
2. It should satisfy non negativity constraint.
3. Total number of allocations must be equal to ($m + n - 1$), where 'm' is the number of rows and 'n' is the number of columns. Consider a value of $m = 4$ and $n = 3$, *i.e.* 4×3 matrix. This will have four constraints of $\leq$ type and three constraints of $\geq$ type. Totally it will have 4 + 3 (*i.e* $m + n$) inequalities. If we consider them as equations, for solution purpose, we will have 7 equations. In case, if we use simplex method to solve the problem, only six rather than seen structural constraints need to be specified. In view of the fact that the sum of the origin capacities (availability constraint) equals to the destination requirements (requirement constraint) *i.e.*, $\Sigma b_i = \Sigma d_j$, any solution satisfying six of the seven constraints will automatically satisfy the last constraint. In general, therefore, if there are 'm' rows and 'n' columns, in a given transportation problem, we can state the problem completely with $m + n - 1$ equations. This means that one of the rows of the simplex tableau represents a redundant constraint and, hence, can be deleted. This also means that a basic feasible solution of a transportation problem has only $m + n - 1$ positive components. If $\Sigma b_i = \Sigma d_j$, it is always possible to get a basic feasible solution by North-west corner method, Least Cost cell method or by VAM.

4.4.3. Basic Feasible Solution by North - West corner Method

Let us take a numerical example and discuss the process of getting basic feasible solution by various methods.

Example 4.1. Four factories, *A*, *B*, *C* and *D* produce sugar and the capacity of each factory is given below: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the transportation cost of 1 ton of sugar from each factory to the destinations. Find the Optimal Solution for least cost transportation cost.

Factories.	*Cost in Rs. per ton (× 100) Markets.*			*Availability in tons.*
	X	*Y*	*Z*	
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\Sigma\ b = 29$, $\Sigma\ d = 23$

Here Σb is greater than Σd hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by three different methods and see the advantages and disadvantages of these methods. After this let us give optimality test for the obtained basic feasible solutions.

a) North- west corner method

(*i*) Balance the problem. That is see whether $\Sigma b_i = \Sigma d_j$. If not open a dummy column or dummy row as the case may be and balance the problem.

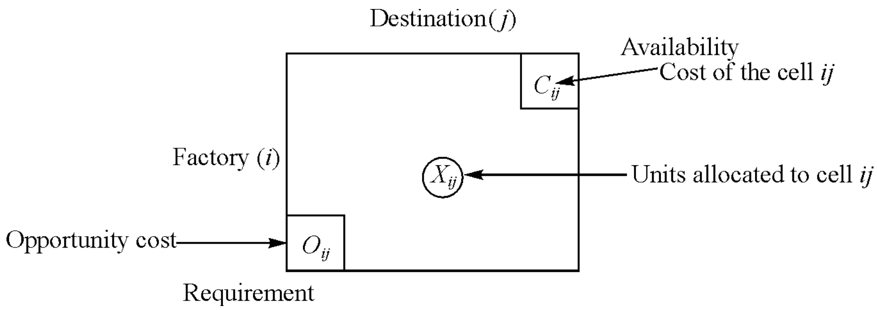
(*ii*) Start from the left hand side top corner or cell and make allocations depending on the availability and requirement constraint. If the availability constraint is less than the requirement constraint, then for that cell make allocation in units which is equal to the availability constraint. In general, verify which is the smallest among the availability and requirement and allocate the smallest one to the cell under question. Then proceed allocating either sidewise or downward to satisfy the rim requirement. Continue this until all the allocations are over.

(*iii*) Once all the allocations are over, *i.e.*, both rim requirement (column and row *i.e.*, availability and requirement constraints) are satisfied, write allocations and calculate the cost of transportation.

Solution by North-west corner method:

	X	Y	Z	Dummy	Availability
A	(7) 4	(3) 3	2	0	10
B	5	(8) 6	1	0	8
C	6	(1) 4	(4) 3	0	5
D	3	5	(1) 4	(5) 0	6
Requirement.	7	12	5	5	29

For cell *AX* the availability constraint is 10 and the requirement constraint is 7. Hence 7 is smaller than 10, allocate 7 to cell *AX*. Next 10 – 7 = 3, this is allocated to cell *AY* to satisfy availability requirement. Proceed in the same way to complete the allocations. Then count the allocations, if it is equals to $m + n - 1$, then the solution is ***basic feasible solution***. The solution, we got have **7** allocations which is = 4 + 4 – 1 = 7. Hence the solution is basic feasible solution.



Now allocations are:

From	To	Units in tons	Cost in Rs.
A	*X*	7	7 × 4 = 28
A	*Y*	3	3 × 3 = 09
B	*Y*	8	8 × 6 = 48
C	*Y*	1	1 × 4 = 04
C	*Z*	4	4 × 3 = 12
D	*Z*	1	1 × 4 = 04
D	DUMMY	5	5 × 0 = 00
	Total in Rs.		105

4.4.4. Solution by Least cost cell (or inspection) Method: (Matrix Minimum method)

(*i*) Identify the lowest cost cell in the given matrix. In this particular example it is = 0. Four cells of dummy column are having zero. When more than one cell has the same cost, then both the cells are competing for allocation. This situation in transportation problem is known as **tie**. To break the tie, select any one cell of your choice for allocation. Make allocations to this cell either to satisfy availability constraint or requirement constraint. Once one of these is satisfied, then mark crosses (×) in all the cells in the row or column which ever has completely allocated. Next search for lowest cost cell. In the given problem it is cell BZ which is having cost of Re.1/- Make allocations for this cell in similar manner and mark crosses to the cells in row or column which has allocated completely. Proceed this way until all allocations are made. Then write allocations and find the cost of transportation. As the total number of allocations are **7** which is equals to 4 + 4 – 1 = 7, the solution is basic feasible solution.

	X	Y	Z	Dummy	Availability
A	$\boxed{4}$ ×	$\boxed{3}$ 8	$\boxed{2}$ 2	$\boxed{0}$ ×	10(8) (0) iv
B	$\boxed{5}$ ×	$\boxed{6}$ ×	$\boxed{1}$ 3	$\boxed{0}$ 5	8(3) (0) ii
C	$\boxed{6}$ 1	$\boxed{4}$ 4	$\boxed{3}$ ×	$\boxed{0}$ ×	5(4) (0) viii
D	$\boxed{3}$ 6	$\boxed{5}$ ×	$\boxed{4}$ ×	$\boxed{0}$ ×	6 (0) v
Requirement.	7 (1) (0) vi	12 (4) (0) vii	5 (2) (0) iii	5 (0) i	29

(**Note:** The numbers under and side of rim requirements shows the sequence of allocation and the units remaining after allocation)

Allocations are:

From	*To*	*Units in tons*	*Cost in Rs.*
A	Y	8	8 × 3 = 24
A	*Z*	2	2 × 2 = 04
B	*Z*	3	3 × 1 = 03
B	DUMMY	5	5 × 0 = 00
C	*X*	1	1 × 6 = 06
C	*Y*	4	4 × 4 = 16
D	*X*	6	6 × 3 = 18
		Total in Rs.	**71**

4.4.5. Solution by Vogel's Approximation Method: (Opportunity cost method)

(*i*) In this method, we use concept of **opportunity cost**. Opportunity cost is the penalty for not taking correct decision. ***To find the row opportunity cost in the given matrix deduct the smallest element in the row from the next highest element. Similarly to calculate the column opportunity cost, deduct smallest element in the column from the next highest element. Write row opportunity costs of each row just by the side of availability constraint and similarly write the column opportunity cost of each column just below the requirement constraints. These are known as penalty column and penalty row.***

The rationale in deducting the smallest element form the next highest element is:

Let us say the smallest element is 3 and the next highest element is 6. If we transport one unit through the cell having cost Rs.3/-, the cost of transportation per unit will be Rs. 3/-. Instead we transport through the cell having cost of Rs.6/-, then the cost of transportation will be Rs.6/- per unit. That is for not taking correct decision; we are spending Rs.3/- more (Rs.6 – Rs.3 = Rs.3/-). ***This is the penalty for not taking correct decision and hence the opportunity cost. This is the lowest opportunity cost in that particular row or column as we are deducting the smallest element form the next highest element.***

Note: If the smallest element is three and the row or column having one more three, then we have to take next highest element as three and not any other element. Then the opportunity cost will be zero. In general, if the row has two elements of the same magnitude as the smallest element then the opportunity cost of that row or column is zero.

(*ii*) Write row opportunity costs and column opportunity costs as described above.

(*iii*) Identify the highest opportunity cost among all the opportunity costs and write a tick (√) mark at that element.

(*iv*) If there are two or more of the opportunity costs which of same magnitude, then select any one of them, to break the tie. While doing so, see that both availability constraint and requirement constraint are simultaneously satisfied. If this happens, we may not get basic feasible solution *i.e* solution with $m + n - 1$ allocations. As far as possible see that both are not satisfied simultaneously. In case if inevitable, proceed with allocations. We may not get a solution with, $m + n - 1$ allocations. For this we can allocate a small element epsilon (∈) to any one of the empty cells. This situation in transportation problem is known as degeneracy. (This will be discussed once again when we discuss about optimal solution).

In transportation matrix, all the cells, which have allocation, are known as **loaded cells** and those, which have no allocation, are known as **empty cells**.

(Note: All the allocations shown in matrix 1 to 6 are tabulated in the matrix given below:**)**

	X	Y	Z	Dummy	Availability
A	3 $\boxed{4}$	7 $\boxed{3}$	$\boxed{2}$	$\boxed{0}$	10
B	3 $\boxed{5}$	$\boxed{6}$	5 $\boxed{1}$	$\boxed{0}$	8
C	$\boxed{6}$	5 $\boxed{4}$	$\boxed{3}$	$\boxed{0}$	5
D	1 $\boxed{3}$	$\boxed{5}$	$\boxed{4}$	5 $\boxed{0}$	6
Requirement.	7	12	5	5	29

(1)

	X	Y	Z	DMY	
A	4	3	2	0	10 (2)
B	5	6	1	0	8 (1)
C	6	4	3	0	5 **(3)**
D	3	5	4	0 **5**	6 **(3)**
	7 (1)	12 (1)	5 (1)	5 (0)	29

(4)

	X	Y	
A	4	3	10 (1)
B	5	6	3 (1)
C	6	4 **5**	5 **(2)**
	6 (1)	12 (1)	18

(2)

	X	Y	Z	
A	4	3	2	10
B	5	6	1 **5**	8
C	6	4	3	5
D	3	5	4	1
	5 (1)	12 (1)	5 (1)	24

(5)

	X	Y		
A	4	**7**	3	10(1)
B	5		6	3(1)
	6 (1)		7 **(3)**	13

(3)

	X	Y		
A	4	3	10	(1)
B	5	6	3	(1)
C	6	4	5	**(2)**
D	**1** 3	5	1	**(2)**
	7 (1)	12 (1)	19	

(6)

		X	
(1)	A	4 **3**	3
(4)			
(1)	B	**3** 5	3
(1)		6	6

Consider matrix (1), showing cost of transportation and availability and requirement constraints. In the first row of the matrix, the lowest cost element is 0, for the cell A-Dummy and next highest element is 2, for the cell AZ. The difference is 2 – 0 = 2. The meaning of this is, if we transport the load through the cell A-Dummy, whose cost element is 0, the cost of transportation will be = Rs.0/- for each unit transported. Instead, if we transport the load through the cell, AZ whose cost element is Rs. 2/- the transportation cost is = Rs.2/- for each unit we transport. This means to say if we take decision to send the goods through the cell AZ, whose cost element is Rs.2/- then the management is going to loose Rs. 2/- for every unit it transport through *AZ*. Suppose, if the management decide to send load through the cell *AX*, Whose cost element is Rs.4/-, then the penalty or the opportunity cost is *Rs*.4/-. We write the minimum opportunity cost of the row outside the matrix. Here it is shown in brackets. Similarly, we find the column opportunity costs for each column and write at the bottom of each corresponding row (in brackets). After writing all the opportunity costs, then we select the highest among them. In the given matrix it is Rs.3/- for the rows *D* and *C*. **This situation is known as tie.** When tie exists, select any of the rows of your choice. At present, let us select the row *D*. Now in that row select the lowest cost cell for allocation. This is because; our objective is to minimize the transportation cost. For the problem, it is *D*-dummy, whose cost is zero. For this cell examine what is available and what is required? Availability is 6 tons and requirement is 5 tons. Hence allocate 5 tons to this cell and cancel the dummy row from the problem. Now the matrix is reduced to 3 × 4. Continue the above procedure and for every allocation the matrix goes on reducing, finally we get all allocations are over. Once the allocations are over, count them, if there are $m + n - 1$ allocations, then the solution is basic feasible solution. Otherwise, the **degeneracy** occurs in the problem. To solve degeneracy, we have to **add epsilon** ($\in$), a small element to one of the empty cells. This we shall discuss, when we come to discuss optimal solution. Now for the problem the allocations are:

From	*To*	*Load*	*Cost in Rs.*
A	*X*	3	3 × 4 = 12
A	*Y*	7	7 × 3 = 21
B	*X*	3	3 × 5 = 15
B	*Z*	5	5 × 1 = 05
C	*Y*	5	5 × 4 = 20
D	*X*	1	1 × 3 = 03
D	DUMMY	5	5 × 0 = 00
		Total Rs.	**76**

Now let us compare the three methods of getting basic feasible solution:

North – west corner method.	*Inspection or least cost cell method*	*Vogel's Approximation Method.*
1. The allocation is made from the left hand side top corner irrespective of the cost of the cell.	The allocations are made depending on the cost of the cell. Lowest cost is first selected and then next highest etc.	The allocations are made depending on the opportunity cost of the cell.
2. As no consideration is given to the cost of the cell, naturally the total transportation cost will be higher than the other methods.	As the cost of the cell is considered while making allocations, the total cost of transportation will be comparatively less.	As the allocations are made depending on the opportunity cost of the cell, the basic feasible solution obtained will be very nearer to optimal solution.
3. It takes less time. This method is suitable to get basic feasible solution quickly.	The basic feasible solution, we get will be very nearer to optimal solution. It takes more time than northwest coroner method.	It takes more time for getting basic Feasible solution. But the solution we get will be very nearer to Optimal solution.
4. When basic feasible solution alone is asked, it is better to go for northwest corner method.	When optimal solution is asked, better to go for inspection method for basic feasible solution and MODI for optimal solution.	VAM and MODI is the best option to get optimal solution.

In the problem given, the total cost of transportation for Northwest corner method is Rs. 101/-. The total cost of transportation for Inspection method is Rs. 71/- and that of VAM is Rs. 76/-. The total cost got by inspection method appears to be less. That of Northwest coroner method is highest. The cost got by VAM is in between.

Now let us discuss the method of getting optimal solution or methods of giving optimality test for basic feasible solution.

4.4.6. Optimality Test: (Approach to Optimal Solution)

Once, we get the basic feasible solution for a transportation problem, the next duty is to test whether the solution got is an optimal one or not? This can be done by two methods. (*a*) By Stepping Stone Method, and (*b*) By Modified Distribution Method, or MODI method.

(*a*) Stepping stone method of optimality test

To give an optimality test to the solution obtained, we have to find the opportunity cost of empty cells. As the transportation problem involves decision making under certainty, we know that an optimal solution must not incur any positive opportunity cost. Thus, we have to determine whether any positive opportunity cost is associated with a given progarmme, *i.e.*, for empty cells. **Once the opportunity cost of all empty cells are negative, the solution is said to be optimal.** In case any one cell has got positive opportunity cost, then the solution is to be modified. The Stepping stone method is used for finding the opportunity costs of empty cells. Every empty cell is to be evaluated for its opportunity cost. To do this the methodology is:

1. Put a small '+' mark in the empty cell.

2. Starting from that cell draw a loop moving horizontally and vertically from loaded cell to loaded cell. Remember, there should not be any diagonal movement. We have to take turn only at loaded cells and move to vertically downward or upward or horizontally to reach another loaded cell. In between, if we have a loaded cell, where we cannot take a turn, ignore that and proceed to next loaded cell in that row or column.
3. After completing the loop, mark minus (–) and plus (+) signs alternatively.
4. Identify the lowest load in the cells marked with negative sign.
5. This number is to be added to the cells where plus sign is marked and subtract from the load of the cell where negative sign is marked.
6. Do not alter the loaded cells, which are not in the loop.
7. The process of adding and subtracting at each turn or corner is necessary to see that rim requirements are satisfied.
8. Construct a table of empty cells and work out the cost change for a shift of load from loaded cell to loaded cell.
9. If the cost change is positive, it means that if we include the evaluated cell in the programme, the cost will increase. If the cost change is negative, the total cost will decrease, by including the evaluated cell in the programme.
10. The negative of cost change is the opportunity cost. Hence, in the optimal solution of transportation problem empty cells should not have positive opportunity cost.
11. Once all the empty cells have negative opportunity cost, the solution is said to be optimal.

One of the drawbacks of stepping stone method is that we have to write a loop for every empty cell. Hence it is tedious and time consuming. Hence, for optimality test we use MODI method rather than the stepping stone method.

Let us take the basic feasible solution we got by Vogel's Approximation method and give optimality test to it by stepping stone method.

Basic Feasible Solution obtained by VAM:

	X	Y	Z	Dummy	Availability
A	4 3 –	3 7 +	2 –2	0 +1	10
B	5 3	6 –2	1 5	0 +2	8
C	6 –1	4 5 –	3 –1	0 + 0	5
D	3 1 + 3	5 –3 5	4 –5 4	0 5 – 0	6
Requirement.	7	12	5	5	29

Table showing the cost change and opportunity costs of empty cells:

Table I.

S.No.	*Empty Cell*	*Evalution Loop formation*	*Cost change in Rs.*	*Opportunity cost -(Cost change)*
1.	AZ	+AZ – AX + BX – BZ	+2 – 4 + 5 – 1 = + 2	–2
2	A Dummy	+ A DUMMY – AX + BX – B DUMMY	+0 – 4 + 3 – 0 = – 1	+1
3	BY	+ BY – AY + AX – BX	+6 – 3 + 4 – 5 = +2	-2
4	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+0 – 5 +3 – 0 = –2	+2
5	CX	+CX – CY + AX – AY	6 – 4 + 3 – 4 = +1	– 1
6	CZ	+CZ – BZ + BX –AX + AY – CY	+2 –1 +5 – 4 +5 – 4 =+1	– 1
7	C DUMMY	+ C DUMMY – D DUMMY + DX – AX + AY – CY	+ 0 – 0 +3 – 4 +3 – 4 = –2	+2
8	DY	+DY – DX + AX – AY	+5 – 3 +4 – 3 = +3	– 3
9	DZ	+DZ – DX +BX – BZ	+4 – 3 + 5 – 1 = +5	– 5

In the table 1 cells A DUMMY, B DUMMY, C DUMMY are the cells which are having positive opportunity cost. Between these two cells B DUMMY and C DUMMY are the cells, which are having higher opportunity cost i.e Rs. 2/ - each. Let us select any one of them to include in the improvement of the present programme. Let us select C DUMMY.

	X	Y	Z	Dummy	Availability
A	4 –8	3 10	2 –4	0 +1	10
B	5 3 –	6 0	1 5	0 +2 +	8
C	6 –3	4 2	3 –3	0 3	5
D	3 4 +	5 –1	4 –5	0 2	6
Requirement.	7	12	5	5	29

Table II.

S.No.	Empty Cell	Evalution Loop formation	Cost change in Rs.	Opportunity Cost
1	AX	+AX –DX + D DUMMY – C DUMMY + CY – AY	+ 4 – 3 + 0 – 0 + 4 – 3 = + 2	–2
2	AX	AZ – AY + CY – C DUMMY + D DUMMY – DX+ BX – BZ	+ 2 – 3 + 4 – 0 + 0 – 3 + 3 – 0 = + 4	–4
3	ADUMMY	+ A DUMMY – AY + DX – D DUMMY	+ 0 – 4 + 3 – 0 = – 1	+1
4	BY	+BY – BX + DX – D DUMMY + C DUMMY – CY	+ 6 –5 + 3 –0 + 0 – 4 = 0	0
5	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+ 0 – 5 + 3 – 0 = –2	+2
6	CX	+ CX – DX + D DUMMY – C DUMMY	+ 6 –3 + 0 – 0 = +3	–3
7	CZ	+ CZ – C DUMMY + D DUMMY – DX + BX – BZ	+ 2 – 0 + 0 – 3 + 5 – 1 = + 3	–3
8	DY	DY – CY + C DUMMY – D DUMMY	+ 5 – 4 + 0 – 0 = 1	–1
9	DZ	+ DZ – DX + BX – BZ	+ 4 – 3 +5 – 1 = + 5	–5

Cells A DUMMY and B DUMMY are having positive opportunity costs. The cell B DUMMY is having higher opportunity cost. Hence let us include this cell in the next programme to improve the solution.

Table III.

S.No.	Empty Cell	Evaluation Loop formation	Cost change in Rs.	Opportunity Cost
1	AX	+AX – AY + CY – C DUMMY + B DUMMY – BX	+4 – 3 + 4 – 0 + 0 – 5 = 0	0
2	AZ	+ AZ – BZ + B DUMMY – C DUMMY + CX – AX	+2 –1 + 0 – 0 + 4 – 3 =+ 2	–2
3	A DUMMY	+ A DUMMY – C DUMMY + CY – AY	+0 – 0 + 4 – 3 = +1	–1
4	BY	+ BY – B DUMMY + C DUMMY – CY	+6 – 0 + 0 – 4 = + 2	–2
5	CX	+ CX – BX + B DUMMY – C DUMMY	+6 – 5 + 0 – 0 = +1	–1
6	CZ	+ CZ – BZ + B DUMMY – C DUMMY	+2 – 1 + 0 – 0 = +1	–1
7	DY	+DY – CY + C DUMMY – B DUMMY + BX – DX	+5 – 4 + 0 – 0 + 5 – 3 = +3	–3
8	DZ	+ DZ – BZ + BX – DX	+4 – 1 + 5 – 3 = +5	–5
9	D DUMMY	+ D DUMMY – DX + BX – B DUMMY	+ 0 – 3 + 5 – 0 = +2	–2

All the empty cells have negative opportunity cost hence the solution is optimal. The **allocations are:**

S.No	*Loaded cell*	*Load*	*Cost in Rs.*
1	AY	10	10 × 3 = 30
2	BX	01	01 × 5 = 05
3	BZ	05	05 × 1 = 05
4	B DUMMY	02	02 × 0 = 00
5	CY	02	02 × 4 = 08
6	C DUMMY	03	03 × 0 = 00
7	DX	06	06 × 3 = 18
	Total in Rs.		66

Total minimum transportation cost is Rs. 66/-

Optimal allocation.

	X	Y	Z	Dummy	Availability
A	$\boxed{4}$ 0	$\boxed{3}$ 10	$\boxed{2}$ 0	$\boxed{0}$ –1	10
B	$\boxed{5}$ 1	$\boxed{6}$ –2	$\boxed{1}$ 5	$\boxed{0}$ 2	8
C	$\boxed{6}$ –1	$\boxed{4}$ 2	$\boxed{3}$ –1	$\boxed{0}$ 3	5
D	$\boxed{3}$ 6	$\boxed{5}$ –2	$\boxed{4}$ –5	$\boxed{0}$ –2	6
Requirement.	7	12	5	5	29

(*b*) Modified Distribution Method of Optimality test

In stepping stone method, we have seen that to get the opportunity cost of empty cells, for every cell we have to write a loop and evaluate the cell, which is a laborious process. In MODI (Modified DIstribution method, we can get the opportunity costs of empty cells without writing the loop. After getting the opportunity cost of all the cells, we have to select the cell with highest positive opportunity cost for including it in the modified solution.

Steps in MODI method:

1. Select row element (u_i) and Column element (v_j) for each row and column, such that $u_i + v_j$ = the actual cost of loaded cell. In MODI method we can evaluate empty cells simultaneously and get the opportunity cost of the cell by using the formula $(u_i + v_j) – C_{ij}$, where C_{ij} is the actual cost of the cell.
2. In resource allocation problem (maximization or minimization method), we have seen that once any variable becomes basis variable, *i.e.*, the variable enters the programme; its opportunity cost or net evaluation will be zero. Here, in transportation problem also, once any cell is loaded, its opportunity cost will be zero. Now the opportunity cost is given by $(u_i + v_j) – C_{ij}$, which is, equals to zero for a loaded cell.

   i.e. $(u_i + v_j) – C_{ij} = 0$ which means, $(u_i + v_j) = C_{ij}$. Here $(u_i + v_j)$ is known as **implied cost of the cell. For any loaded cell the implied cost is equals to actual cost of the cell as its opportunity cost is zero. For any empty cell, (implied cost – actual cost) will give opportunity cost.**
3. How to select u_i and v_j? The answer is:

   (*a*) Write arbitrarily any one of them against a row or against a column. The written u_i or vj may be any whole number *i.e* u_i or v_j may be ≤ or ≥ to zero. By using the formula $(u_i + v_j) = C_{ij}$ for a loaded cell, we can write the other row or column element. For example, if the actual cost of the cell $C_{ij} = 5$ and arbitrarily we have selected $u_i = 0$, then v_j is given by $u_i + v_j = 0 + v_j = 5$. Hence $v_j = –5$. Like this, we can go from loaded cell to loaded cell and complete entering of all u_i *s* and v_j *s*.

   (*b*) Once we get all u_i *s* and v_j *s*, we can evaluate empty cells by using the formula $(u_i + v_j)$ – Actual cost of the cell = opportunity cost of the cell, and write the opportunity cost of each empty cell at left hand bottom corner.

   (*c*) Once the opportunity costs of all empty cells are negative, the solution is said to be optimal. In case any cell is having the positive opportunity cost, the programme is to be modified.

   Remember the formula that IMPLIED COST OF A CELL = $u_i + v_j$

   Opportunity cost of loaded cell is zero *i.e* $(u_i + v_j)$ = Actual cost of the cell.

   Opportunity cost of an empty cell = implied cost – actual cost of the cell = $(u_i + v_j) – C_{ij}$

   (*d*) In case of degeneracy, *i.e.* in a basic feasible solution, if the number of loaded cells are not equals to $m + n – 1$, then we have to add a small element *epsilon* (∈), to any empty cell to make the number of loaded cells equals to $m + n – 1$. While adding '∈' we must be careful enough to see that this ∈ should not form a closed loop when we draw horizontal and vertical lines from loaded cell to loaded cell. In case the cell to which we have added ∈ forms a closed loop, then if we cannot write all u_i s and v_j *s*.

   ∈ is such a small element such that $a + \in = a$ or $a – \in = a$ and $\in – \in = 0$.

Implied cost	Actual cost	Action
$u_i + v_j >$	C_{ij}	A better programme can be designed by including this cell in the solution.
$u_i + v_j =$	C_{ij}	Indifferent; however, an alternative programme with same total cost can be written by including this cell in the programme.
$u_i + v_j <$	C_{ij}	Do not include this cell in the programme.

Now let us take the basic feasible solution obtained by VAM method and apply MODI method of optimality test.

Basic feasible solution got by VAM method.

	X	Y	Z	Dummy	Availability	Row No. u_i
A	4 (3) −	3 (7) +	2	0	10	4
B	5 (3)	6	1 (5)	0	8	5
C	6	4 (5)	3	0 +	5	5
D	3 (1) +	5	4	0 (5) −	6	3
Requirement.	7	12	5	5	29	
Column No v_j	0	−1	−4	−3		

The cell C DUMMY is having a positive opportunity cost. Hence we have to include this cell in the programme. The solution has $m + n - 1$ allocations.

	X	Y	Z	DUMMY	Availability	Row No. u_i
A	4 −2	3 (10)	2 0	0 −1	10	0
B	5 (3) −	6 0	1 (5)	0 +2 +	8	3
C	6 −3	4 (2)	3 −2	0 (3)	5	1
D	3 (4) +	5 −1	4 −5	0 (2) −	6	1
Requirement .	7	12	5	5	29	
Column Number: v_j	2	3	−2	−1		

The cell B DUMMY is having a positive opportunity cost. Thïs is to be included in the modified programme.

	X	Y	Z	DUMMY	Availability	Row No. u_i
A	[4] 0	[3] (10)	[2] –2	[0] –1	10	–1
B	[5] (1)	[6] –2	[1] (5)	[0] (2)	8	0
C	[6] –1	[4] (2)	[3] –2	[0] (3)	5	0
D	[3] (6)	[5] –1	[4] –3	[0] 0	6	0
Requirement .	7	12	5	5	29	
Column Number: v_j	5	4	1	0		

As the opportunity cost of all empty cells are negative, the solution is optimal. The solution has $m + n - 1$ allocations.

The allocations are:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1	AY	10	10 × 3 = 30
2.	BX	01	01 × 5 = 05
3.	BZ	05	05 × 1 = 05
4.	B DUMMY	02	02 × 0 = 00
5.	CY	02	02 × 4 = 08
6.	C DUMMY	03	03 × 0 = 00
7.	CX	06	06 × 3 = 18
	Total Cost in Rs.		66

Readers can verify the optimal solution got by Stepping stone method and the MODI method they are same. And they can also verify the opportunity costs of empty cells they are also same. This is the advantage of using MODI method to give optimality test. Hence the combination of VAM and MODI can be conveniently used to solve the transportation problem when optimal solution is asked.

4.4.7. Alternate Solutions

By principle, we know that the opportunity cost of a loaded cell or a problem variable is always equals to zero. In case any empty cell of the optimal solution of a transportation problem got zero as the opportunity cost, it should be understood that it is equivalent to a loaded cell. Hence by including that cell, we can derive another solution, which will have same total opportunity cost, but different allocations. Once one alternate solution exists, we can write any number of alternate solutions. The methodology is:

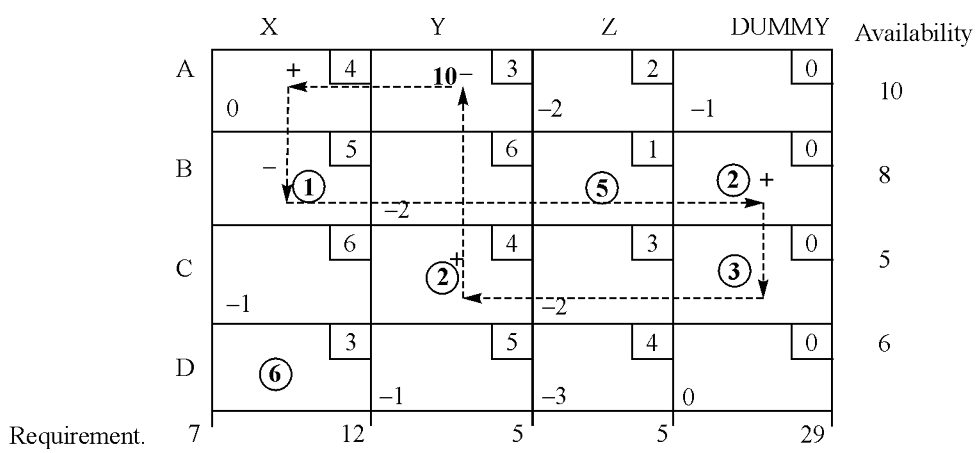
1. Let the Optimal solution is matrix A with one or more empty cells having zero as the opportunity cost.

2. By including the cell having zero as the opportunity cost, derive one more optimal solution, let it be the matrix B.
3. The new matrix C is obtained by the formula: $\boldsymbol{C = d\,A + (1 - d)\,B}$**,** where '$d$' is a positive fraction less than 1.

It is better to take always $d = 1/2$, so that $\boldsymbol{C = 1/2\,A + 1/2\,B}$**.**

Now we shall take the optimal solution of the problem above and write the alternate optimal solutions.

Matrix A (First optimal Solution).



The cell AX, having zero opportunity cost is included in revised solution. The loop is:
+ AX – BX + B DUMMY – C DUMMY + CY – AY = + 4 – 5 + 0 – 0 + 4 – 3 = 0

Allocation:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1.	AX	01	01 × 4 = 04
2.	AY	09	09 × 3 = 18
3.	BZ	05	05 × 1 = 05
4.	B Dummy	03	03 × 0 = 00
5.	CY	03	03 × 4 = 12
6.	C Dummy	02	02 × 0 = 00
7.	DX	06	06 × 3 = 18
	Total cost in Rs.		66

Matrix B (First alternative solution):

	X	Y	Z	Dummy	Availability
A	4 (1) 0	3 (9)	2 –2	0 –1	10
B	5	6 –2	1 (5)	0 (3)	8
C	6 –1	4 (3)	3 –2	0 (2)	5
D	3 (6)	5 –1	4 –3	0 0	6
Requirement	7	12	5	5	29

Matrix C (Second alternate solution)

	X	Y	Z	Dummy	Availability
A	4 (0.5)	3 (9.5)	2	0	10
B	5 (0.5)	6	1 (5)	0 (2.5)	8
C	6	4 (2.5)	3	0 (2.5)	5
D	3 (6)	5	4	0	6
Requirement	7	12	5	5	29

The total cost is $0.5 \times 4 + 9.5 \times 3 + 0.5 \times 5 + 5 \times 1 + 2.5 \times 0 + 2.5 \times 0 + 2.5 \times 0 + 6 \times 3 =$ Rs. 66/-

Once we get one alternate solution we can go on writing any number of alternate solutions until we get the first optimal solution.

4.5. MAXIMIZATION CASE OF TRANSPORTATION PROBLEM

Basically, the transportation problem is a minimization problem, as the objective function is to minimize the total cost of transportation. Hence, when we would like to maximize the objective function. There are two methods.

maximization

(*i*) The given matrix is to be multiplied by –1, so that the problem becomes maximization problem. Or ii) Subtract all the elements in the matrix from the highest element in the matrix. Then the problem becomes maximization problem. Then onwards follow all the steps of maximization problem to get the solution. Let us consider the same problem solved above.

Problem 4.2. Four factories, *A*, *B*, *C* and *D* produce sugar and the capacity of each factory is given below: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the returns the factory can get, by selling the sugar in each market. Formulate a transportation problem and solve for maximizing the returns.

	Profit in Rs. per ton (× 100)			*Availability in tons.*
		Markets.		
	X	*Y*	*Z*	
Factories.				
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\sum b = 29$, $\sum d = 23$

Here $\sum b$ is greater than $\sum d$ hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by VAM and then give optimality test by MODI method. The balanced matrix of the transportation problem is:

Profit per ton in Rs.

	X	Y	Z	Dummy	Availability Row No.	u_i
A	2 $\boxed{-4}$	3 $\boxed{-3}$	$\boxed{-2}$ 0	5 $\boxed{-0}$	10	0
B	$\boxed{-5}$ 2	$\boxed{-6}$ 8	$\boxed{-1}$ 4	$\boxed{-0}$ 3	8	3
C	$\boxed{-6}$ 5	$\boxed{-4}$ 1	$\boxed{-3}$ 2	$\boxed{-0}$ 2	5	2
D	$\boxed{-3}$ 3	$\boxed{-5}$ 1	$\boxed{-4}$ 5	$\boxed{-0}$ 2	6	2
Requirement	7	12	5	5	29	
column. no. v_j	4	3	2	0		

By multiplying the matrix by –1, we can convert it into a maximisation problem. Now in VAM we have to find the row opportunity cost and column opportunity costs. In minimisation problem, we use to subtract the smallest element in the row from next highest element in that row for finding row opportunity cost. Similarly, we use to subtract smallest element in the column by next highest element in that column to get column opportunity cost. Here as we have multiplied the matrix by –1 the highest element will become lowest element. Hence subtract the lowest element from the next highest element as usual. Otherwise, instead of multiplying by –1 simply find the difference between highest element and the next lowest element and take it as opportunity cost of that row or column. For example in the given problem in the row A, the highest element is 4 and the next lowest element is 3 and hence the opportunity cost is 4 – 3 = 1. (Or smallest element is –4 and the next highest element is –3 and the opportunity cost is –3 – (– 4) = –3 + 4 = 1). Similarly, we can write all opportunity costs. Once we find the opportunity costs, rest of the procedure is same. That is, we have to select highest opportunity cost and select the highest profit element in that row or column for allocation. Obtain the basic feasible solution. As usual the basic feasible solution must have $m + n - 1$ allocations. If the allocations are not equal to $m + n - 1$, the problem degenerate. In that case, add $\in$ to an empty cell, which do not form loop with other loaded cells. Once we have basic feasible solution, the optimality test by MODI method, is followed. Here, once the opportunity costs of all the cells are positive, (as we have converted the maximistion problem into minimisation problem) the solution is said to be optimal.

In the given problem as the opportunity costs of all empty cells are positive, the solution is optimal. And the optimal return to the company is Rs. 125/-.

Allocations:

S.No	Loaded Cell	*Load*	*Cost in Rs.*
1.	*AX*	02	02 × 4 = 08
2.	*AY*	03	03 × 3 = 09
3.	*A Dmy*	05	05 × 0 = 00
4.	*BY*	08	08 × 6 = 48
5.	*CX*	05	05 × 6 = 30
6.	*DY*	01	01 × 5 = 05
7.	DZ	05	05 × 4 = 20
	Total returns in Rs.		125

(1)

	X	Y	Z	DMY		
A	4	3	2	0	10	(1)
B	5	6	1	0	8	(1) (2)
C	**5** 6	4	2	0	5	(2)
D	3	5	4	0	6	(1)
	7	12	5	5	29	
	1	1	2	0		

(2)

	X	Y	Z	DMY		
A	4	3	2	0	10	1
B	5	6	1	0	8	1
D	3	5	4 **5**	0	6	1
	7	12	5	5	24	
	1	1	2	0		

(3)

	X	Y	DMY		
A	4	3	0	10	1
B	5	6	0	8	1
D	3	5 **1**	0	1	2
	2	12	5	19	
	1	1	0		

(4)

	X	Y	DMY		
A	4	3	0	10	1
B	5	6 **8**	0	8	1
	2	11	5	18	
	1	3	0		

(5)

	X	Y	DMY	
A	4 **2**	3 **3**	0 **5**	10
	2	3	5	10

4.6. DEGENERACY IN TRANSPORTATION PROBLEM

Earlier, it is mentioned that the basic feasible solution of a transportation problem must have $(m + n - 1)$ basis variables or allocations. This means to say that the number of occupied cells or loaded cells in a given transportation problem is 1 less than the sum of number of rows and columns in the transportation matrix. Whenever the number of occupied cells is less than $(m + n - 1)$, the transportation problem is said to be degenerate.

Degeneracy in transportation problem can develop in two ways. First, the problem becomes degenerate when the initial programme is designed by northwest corner or inspection or VAM, *i.e.* at the stage of initial allocation only.

To solve degeneracy at this stage, we can allocate extremely small amount of goods (very close to zero) to one or more of the empty cells depending on the shortage, so that the total occupied cells becomes $m + n - 1$. The cell to which small element (load) is allocated is considered to be an occupied cell. In transportation problems, Greek letter '$\in$' represents the small amount. One must be careful enough to see that the smallest element epsilon is added to such an empty cell, which will enable us to write row number 'u_i' and column number 'v_j' without any difficulty while giving optimality test to the basic feasible solution by MODI method. That is care must be taken to see that the epsilon is added to such a cell, which will not make a **closed loop**, when we move horizontally and vertically from loaded cell to loaded cell.

(Note: Epsilon is so small so that if it is added or subtracted from any number, it does not change the numerical value of the number for which it added or from which it is subtracted.).

Secondly, the transportation problem may become degenerate during the solution stages. This happens when the inclusion of a most favorable empty cell *i.e.* cell having highest opportunity cost results in simultaneous vacating of two or more of the currently occupied cells. Here also, to solve degeneracy, add epsilon to one or more of the empty cells to make the number of occupied cells equals to $(m + n - 1)$.

To understand the procedure let us solve one or two problems.

Problem. 4.3. Solve the transportation problem given below

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	2	1	2	20	
Y	3	4	1	40	
Requirement	20	15	25	60	
Column element v_j					

Solution by Northwest corner method:

Initial allocation show that the solution is not having (m+n–1) allocations. Hence degeneracy occurs.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	2 (20)	1	2	20	
Y	3	4 (15)	1 (25)	40	
Requirement	20	15	25	60	

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	2 (20)	1 (ε)	2	20	
Y	3	4 (15)	1 (25)	40	
Requirement	20	15	25	60	
Column element v_j					

The smallest load ∈ is added to cell *XB* which does not make loop with other loaded cells.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	2 (20) –	1 (ε) +	2	20	
Y	3 +	4 (15) –	1 (25)	40	
Requirement	20	15	25	60	
Column element v_j					

Shifting of load by drawing loops to cell YA.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	$\boxed{2}$ 5	$\boxed{1}$ 15	$\boxed{2}$	20	
Y	$\boxed{3}$ 15	$\boxed{4}$	$\boxed{1}$ 25	40	
Requirement	20	15	25	60	
Column element v_j					

The basic feasible solution is having four loaded cells. As the number of columns is 3 and number of rows is 2 the total number of allocations must be 2 + 3 – 1 = 4. The solution got has four allocations. Hence the basic feasible solution. Now let us give optimality test by MODI method.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	$\boxed{2}$ 5	$\boxed{1}$ 15	$\boxed{2}$ –2	20	0
Y	$\boxed{3}$ 15	$\boxed{4}$ –2	$\boxed{1}$ 25	40	1
Requirement	20	15	25	60	
Column element v_j	2	1	0		

Row numbers u_i s and column numbers v_j s are written in the matrix and opportunity cost of empty cells are evaluated. As the opportunity cost of all empty cells are negative, the solution is optimal. The allocations and the total cost of transportation is:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1.	*XA*	05	05 × 2 = 50
2.	*XB*	15	15 × 1 = 15
3.	*YA*	15	15 × 3 = 45
4.	*YC*	25	25 × 1 = 25
	Total cost in Rs.		135

Problem. 4.4. Solve the transportation problem given below:

Cost in Rs. per unit

To → / From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 (30)	3 (10)	1, 2	2	6	40	0
O_2	5	2 (20)	3 (10)	4	5	30	–1
O_3	3	5	6 (5)	3 (15)	2	20	2
O_4	2	4	4	5 (5)	3 (5)	10	4
Requirement	30	30	15	20	5	100	
v_j	4	3	4	1	–1		

Let us make initial assignment by using Northwest corner method. To modify the solution we include the cell O_1D_3 in the programme, as it is having highest opportunity cost.

Improved solution:

Cost in Rs. per unit

To → / From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 (20)	3 (ε)	1 (10), 2	2	6	40	0
O_2	5	2 (30)	3	4	5	30	0
O_3	3	5	6 (5)	3 (15)	2	20	5
O_4	2, **9**	4	4	5 (5)	3 (5)	10	7
Requirement	30	30	15	20	5	100	
v_j	4	2	1	–2	–4		

Total number of allocations are less than $m + n - 1$. Hence we have to add one epsilon to an empty cell. Remember, in transportation problem, which has minimization of cost as its objective function a, we have to add epsilon to recently vacated cell, which is having lowest shipping cost. We have a tie between two cells, *i.e.* O_1D_2 and O_2D_3. Let us select O_1D_2 to add epsilon. To improve the solution, let us take empty cell O_4D_1 in the programme.

Improved Programme: The solution is not having $m + n - 1$ allocations. We have to add epsilon; in the programme epsilon is added to cell 0_4D_4

Revised Programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 (25) +	3 (ε)	1 (15)	2 + 5	6	40	0
O_2	5	2 (30)	3	4	5	30	–1
O_3	3	5	6	3 (20)	2	20	–4
O_4	2 (5) –	4	4	5 ∈ +	3 (5)	10	–2
Requirement	30	30	15	20	5	100	
v_j	4	3	1	7	5		

The epsilon is shifted to an empty cell. The improved solution is having 8 allocations. Hence a feasible solution.

As the cell O_1D_4 having positive opportunity cost, let us include and revise the programme.

Revised programme. Cell 0_3D_5 having positive opportunity cost is included in revised programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 (25) –	3 (ε)	1 (15)	2 (ε) +	6	40	0
O_2	5	2 (30)	3	4	5	30	–1
O_3	3	5	6	3 (20) –	2 + 4	20	1
O_4	2 (5) +	4	4	5	3 (5) –	10	–2
Requirement	30	30	15	20	5	100	
v_j	4	3	1	2	5		

Revised programme: Cell O_3D_1 having positive opportunity cost is included in the revised programme.

Cost in Rs. per unit

To → / From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	$\boxed{4}$ – 20	$\boxed{3}$ ε	$\boxed{1}$ 15	$\boxed{2}$ + 5	$\boxed{6}$	40	0
O_2	$\boxed{5}$	$\boxed{2}$ 30	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	30	+–1
O_3	$\boxed{3}$ + 2	$\boxed{5}$	$\boxed{6}$	$\boxed{3}$ – 5	$\boxed{2}$ 2	20	1
O_4	$\boxed{2}$ 10	$\boxed{4}$	$\boxed{4}$	$\boxed{5}$	$\boxed{3}$	10	–2
Requirement	30	30	15	20	5	100	
v_j	4	2	1	2	1		

Revised Programme.

Cost in Rs. per unit

To → / From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	$\boxed{4}$ 5	$\boxed{3}$ ε	$\boxed{1}$ 15	$\boxed{2}$ 20	$\boxed{6}$ –3	40	0
O_2	$\boxed{5}$ –2	$\boxed{2}$ 30	$\boxed{3}$ –3	$\boxed{4}$ –3	$\boxed{5}$ –3	30	–1
O_3	$\boxed{3}$ 15	$\boxed{5}$ –3	$\boxed{6}$ –6	$\boxed{3}$ –2	$\boxed{2}$ 5	20	–1
O_4	$\boxed{2}$ 10	$\boxed{4}$ –4	$\boxed{4}$ –6	$\boxed{5}$ –6	$\boxed{3}$ –3	10	–3
Requirement	30	30	15	20	5	100	
v_j	4	3	1	2	3		

As the opportunity costs of all empty cells are negative, the solution is optimal. The allocations and the total cost of transportation is:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1.	O_1D_1	5	5 × 4 = 20
2.	O_1D_2	ε	----
3.	O_1D_3	15	15 × 1 = 15
4.	O_1D_4	20	20 × 2 = 40
5.	O_2D_2	30	30 × 2 = 60
6.	O_3D_1	15	15 × 3 = 45
7.	O_3D_5	5	5 × 2 = 10
8.	O_4D_1	10	10 × 2 = 20
	Total Cost in Rs.		210/-

The same problem, if we solve by VAM, the very first allocation will be feasible and optimality test shows that the solution is optimal.

Roc: Row opportunity cost, COC= Column opportunity cost, Avail: Availability, Req: Requirement.

	D_1	D_2	D_3	D_4	D_5	*Avail*	*ROC*
O_1	4	3	1 **15**	2	6	40	1
O_2	5	2	3	4	5	30	1
O_3	3	5	6	3	2	20	1
O_4	2	4	4	5	3	10	1
REQ	30	30	15	20	5	100	
COC	1	1	2	1	1		

	D_1	D_2	D_4	D_5	*Avail*	*ROC*
O_1	4	3	2	6	25	1
O_2	5	2 **30**	4	5	30	2
O_3	3	5	3		20	1
O_4	2	4	5	3	10	1
REQ	30	30	20	5	85	
COC	1	1	1	1		

	D_1	D_4	D_5	*Avail*	*ROC*
O_1	4	2 **20**	6	25	2
O_3	3	3	2	20	1
O_4	2	5	3	10	1
REQ	30	20	5	55	
ROC	1	1	1		

	D_1	D_5	Avail	ROC
~~O_1~~	~~4~~ **5**	~~6~~	~~5~~	~~2~~
O_3	3	2	20	1
O_4	2	3	10	1
REQ	30	5	35	
COC	1	1		

	D_1	D_5	Avail	ROC
O_3	3	2	20	1
~~O_4~~	~~2~~ **10**	~~3~~	~~10~~	~~1~~
REQ	25	5	30	
COC	1	1		

	D_1	D_5	AVAIL
O_3	3 **15**	2 **5**	20
REQ	15	5	20

Allocation by VAM:

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	[4] (5)	[3] (ε)	[1] (15)	[2] (20)	[6] −1	40	0
O_2	[5] −2	[2] (30)	[3] −3	[4] −3	[5] −1	30	−1
O_3	[3] (15)	[5] −3	[6] −6	[3] −2	[2] (5)	20	−1
O_4	[2] (10) 9	[4] −3	[4] −5	[5] −5	[3] 0	10	−2
Requirement	30	30	15	20	5	100	
v_j	4	3	1	2	5		

Allocations are same as in the optimal solution got by northwest corner method. All opportunity costs of empty cells are negative. Hence the total transportation cost is Rs. 210/-

4.7. TIME MINIMISATION MODEL OR LEAST TIME MODEL OF TRANSPORTATION TIME.

It is well known fact that the transportation problem is cost minimization model, *i.e* we have to find the least cost transportation schedule for the given problem. Some times the cost will become secondary factor when the time required for transportation is considered. This type of situation we see in military operation. When the army want to send weapons or food packets or medicine to the war front, then the time is important than the money. They have to think of what is the least time required to transport the goods than the least cost of transportation. Here the given matrix gives the time elements, *i.e*. time required to reach from one origin to a destination than the cost of transportation of one unit from one origin to a destination. A usual, we can get the basic feasible solution by Northwest corner method or by least time method or by VAM. To optimize the basic feasible solution, we have to identify the highest time element in the allocated cells, and try to eliminate it from the schedule by drawing loops and encouraging to take the cell, which is having the time element less than the highest one. Let us take a problem and work out the solution. Many a time, when we use VAM for basic feasible solution, the chance of getting an optimal solution is more. Hence, the basic feasible solution is obtained by Northwest corner method.

Problem 4.5. The matrix given below shows the time required to shift a load from origins to destinations. Formulate a least time schedule. Time given in hours.

Roc: Row opportunity cost, Coc: Column opportunity cost, Avail: Availability, Req: Requirement.

Destinations (Time in hours)

Origins

	D_1	D_2	D_3	D_4	Avail
O_1	7	8	4	5	5
O_2	8	10	2	3	7
O_3	7	6	17	8	8
O_4	19	10	11	3	10
Req	10	5	10	5	

1. Initial assignment by Northwest corner method: The Maximum time of allocated cell is 17 hours. Any cell having time element greater than 17 hours is cancelled, so that it will not in the programme.

Destinations (Time in hours)

Origins

	D_1	D_2	D_3	D_4	Avail
O_1	7, 5	8	4	5	5
O_2	8, 5	− 10, 2	+ 2	3	7
O_3	7	6, + 3	17, − 5	8	8
O_4	~~19~~	10	11, 5	3, 5	10
Req	10	5	10	5	

By drawing loops, let us try to avoid 17 hours cell and include a cell, which is having time element less than 17 hours. The basic feasible solution is having $m + n - 1$ allocations.

Destinations (Time in hours)

Origins		D1	D2	D3	D4	Avail	Roc
	O1	7 (5)	8	4	5	5	
	O2	8 – (5)	10	2 + (2)	3	7	
	O3	7 +	6 (5)	17 – (3)	8	8	
	O4	19	10	11 (5)	3 (5)	10	
	Req	10	5	10	5		
	Coc.						

Here also the maximum time of transport is 17 hours.

Destinations (Time in hours)

Origins		D1	D2	D3	D4	Avail	Roc
	O1	7 (5)	8	4	5	5	
	O2	8 – (2)	10	2 + (5)	3	7	
	O3	7 + (3)	6 – (5)	~~17~~	8	8	
	O4	~~19~~	10 +	11 – (5)	3 (5)	10	
	Req	10	5	10	5		
	Coc.						

In this allocation highest time element is 11 hours. Let us try to reduce the same.

Destinations (Time in hours)

Origins		D1	D2	D3	D4	Avail	Roc
	O1	7 – (5)	8	4 +	5	5	
	O2	8	10	2 (7)	3	7	
	O3	7 + (5)	6 – (3)	~~17~~	8	8	
	O4	~~19~~	10 + (2)	11 – (3)	3 (5)	10	
	Req	10	5	10	5		
	Coc.						

In this allocation also the maximum time element is 11 hours. Let us try to avoid this cell.

Destinations (Time in hours)

Origins

	D_1	D_2	D_3	D_4	Avail
O_1	7 (2)	8	4 (3)	5	5
O_2	8	10	2 (7)	3	7
O_3	7 (8)	6	~~17~~	8	8
O_4	~~19~~	10 (5)	~~11~~	3 (5)	10
Req	10	5	10	5	

No more reduction of time is possible. Hence the solution is optimal and the time required for completing the transportation is 10 Hours. T_{max} = 10 hours.

4.8. PURCHASE AND SELL PROBLEM: (TRADER PROBLEM)

Problem. 4.7 M/S Epsilon traders purchase a certain type of product from three manufacturing units in different places and sell the same to five market segments. The cost of purchasing and the cost of transport from the traders place to market centers in Rs. per 100 units is given below:

			Market Segments. (Transportation cost in Rs.per 100 units).				
Place of Manufacture.	*Availability In units x 10000.*	*Manufacturing cost in Rs. per unit*	*1*	*2*	*3*	*4*	*5*
Bangalore (*B*)	10	40	40	30	20	25	35
Chennai (*C*)	15	50	30	50	70	25	40
Hyderabad (*H*)	5	30	50	30	60	55	40
	Requirement in units × 10000		6	6	8	8	4

The trader wants to decide which manufacturer should be asked to supply how many to which market segment so that the total cost of transportation and purchase is minimized.

Solution

Here availability is 300000 units and the total requirement is 320000 units. Hence a dummy row (*D*) is to be opened. The following matrix shows the cost of transportation and purchase per unit in Rs. from manufacturer to the market centers directly.

	1	2	3	4	5	*Availability*
B	4040	4030	4020	4025	4035	10
C	5030	5050	5070	5025	5040	15
H	3050	3030	3060	3055	3040	5
D	0	0	0	0	0	2
Requirement.	6	6	8	8	4	32

Let us multiply the matrix by 100 to avoid decimal numbers and get the basic feasible solution by VAM.

Table. Avail: Availability. Req: Requirement, Roc: Row opportunity cost, Coc: Column opportunity cost.

Tableau. I Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030	4020	4025	4035	10	
C	5030	5050	5070	5025	5040	15	
H	3050	3030	3060	3055	3040	5	
D	0	0	0	0	0	2	
Req.	6	6	8	8	4	32	
Coc.							

Tableau. II Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail	Roc
B	4040	4030	4020	4025	4035	10	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030	3060	3055	3040	5	10
D	0	0	2 0	0	0	2	0
Req	6	6	8	8	4		
Coc	3050	3030	3060	3055	3040		

Tableau. II Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030	4020	4025	4035	10	5
C	5030	5050	5070	5025	5040	15	5
H	3050	5 3030	3060	3055	3040	5	10
D	0	0	2 0	0	0	2	
Req.	6	6	6	8	4	30	
Coc.	990	1000	960	970	995		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030	4020 5	4025	4035	10	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030 5	3060	3055	3040	5	5
D	0	0	0 2	0	0	2	0
Req.	6	1	6	8	4	27	
Coc.	990	1020	1050	1000	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030 1	4020 6	4025	4035	4	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030 5	3060	3055	3040		
D	0	0	0 2	0	0		
Req.	6	1		8	4	19	
Coc.	990	1020		1000	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030 1	4020 6	4025	4035 **3**	3	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030 5	3060	3055	3040		5
D	0	0	0 **2**	0	0		0
Req.	6			8	4	18	
Coc.	990			1000	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.
B	4040	1 4030	6 4020	4025	3 4035	10
C	6 5030	5050	5070	8 5025	1 5040	15
H	3050	5 3030	3060	3055	3040	5
D	0	0	2 0	0	0	2
Req.	6	6	8	8	4	32

Final Allocation by MODI method.

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.
B	4040	1 4030	8 4020	4025	1 4035	10
C	6 5030	5050	5070	8 5025	1 5040	15
H	3050	5 3030	3060	3055	3040	5
D	0	0	0	0	2 0	2
Req.	6	6	8	8	4	32

Allocation:

From	*To*	*Load*	*Cost in Rs.*
Bangalore	2	10,000	4,03,000
Bangalore	3	80,000	32, 16,000
Bangalore	5	10,000	4, 03,000
Chennai	1	60,000	30, 18,000
Chennai	4	80,000	40, 20,000
Chennai	5	10,000	5, 04,000
Hyderabad	2	50,000	15, 15,000
Total cost in Rs.			1,30, 79,000

4.9. MAXIMISATION PROBLEM: (PRODUCTION AND TRANSPORTATION SCHEDULE FOR MAXIMIZATION)

This type of problems will arise when a company having many units manufacturing the same product and wants to satisfy the needs of various market centers. The production manager has to work out for transport of goods to various market centers to cater the needs. Depending on the production schedules and transportation costs, he can arrange for transport of goods from manufacturing units to the market centers, so that his costs will be kept at minimum. At the same time, this problem also helps him to prepare schedules to aim at maximizing his returns.

Problem.4.8. A company has three manufacturing units at *X*, *Y* and *Z* which are manufacturing certain product and the company supplies warehouses at *A*, *B*, *C*, *D*, and *E*. Monthly regular capacities for regular production are 300, 400 and 600 units respectively for *X*, *Y* and *Z* units. The cost of production per unit being Rs.40, Rs.30 and Rs. 40 respectively at units *X*, *Y* and *Z*. By working overtime it is possible to have additional production of 100, 150 and 200 units, with incremental cost of Rs.5, Rs.9 and Rs.8 respectively. If the cost of transportation per unit in rupees as given in table below, find the allocation for the total minimum production cum transportation cost. Under what circumstances one factory may have to work overtime while another may work at under capacity?

Transportation cost in Rs. To

From	*A*	*B*	*C*	*D*	*E*
X	12	14	18	13	16
Y	11	16	15	11	12
Z	16	17	19	16	14
REQ	400	400	200	200	300

(*a*) If the sales price per unit at all warehouses is Rs. 70/- what would be the allocation for maximum profit? Is it necessary to obtain a new solution or the solution obtained above holds valid?

(*b*) If the sales prices are Rs.70/-, Rs. 80/-, Rs. 72/-, Rs. 68/- and Rs. 65/- at *A*, *B*, *C*, *D* and *E* respectively what should be the allocation for maximum profit?

Solution: Total production including the overtime production is 1750 units and the total requirement by warehouses is 1500 units. Hence the problem is unbalanced. This can be balance by opening a Dummy Row (*DR*), with cost coefficients equal to zero and the requirement of units is 250. The cost coefficients of all other cells are got by adding production and transportation costs. The production cum transportation matrix is given below:

	A	*B*	*C*	*D*	*E*	*DC*	*Availability*
X	52	54	58	53	56	0	300
Y	41	46	45	41	42	0	400
Z	56	57	59	56	54	0	600
XOT	57	59	63	58	61	0	100
YOT	50	55	54	50	51	0	150
ZOT	64	65	67	64	62	0	200
Requirement:	400	400	200	200	300	250	1750

Initial Basic feasible solution by VAM:

	A	B	C	D	E	DC	Avail.	u_i
X	$\boxed{52}$ 300	$\boxed{54}$ 0	$\boxed{58}$ –2	$\boxed{53}$ 0	$\boxed{56}$ –4	$\boxed{0}$ –7	300	52
Y	$\boxed{41}$ –1	$\boxed{46}$ –4	$\boxed{45}$ –1	$\boxed{41}$ 100	$\boxed{42}$ 300	$\boxed{0}$ –17	400	40
Z	$\boxed{56}$ –1	$\boxed{57}$ 400	$\boxed{59}$ 100	$\boxed{56}$ 100	$\boxed{54}$ –1	$\boxed{0}$ –2	600	55
XOT	$\boxed{57}$ 50	$\boxed{59}$ 0	$\boxed{63}$ –2	$\boxed{58}$ 0	$\boxed{61}$ –4	$\boxed{0}$ **50**	100	57
YOT	$\boxed{50}$ 50	$\boxed{55}$ –3	$\boxed{54}$ 100	$\boxed{50}$ –1	$\boxed{51}$ –1	$\boxed{0}$ –7	150	50
ZOT	$\boxed{64}$ –7	$\boxed{65}$ –6	$\boxed{67}$ –6	$\boxed{64}$ –6	$\boxed{62}$ –5	$\boxed{0}$ 200	200	57
REQ.	400	400	200	200	300	250	1750	
v_j	0	2	4	1	0	–57		

As we have $m + n - 1$ (= 11) allocations, the solution is feasible and all the opportunity costs of empty cells are negative, the solution is optimal.

Allocations:

Cell	Load	Cost in Rs.
XA	300	300 × 52 = 15,600
YD	100	100 × 41 = 4,100
YE	300	300 × 40 = 12,000
ZB	400	400 × 54 = 21,000
ZC	100	100 × 59 = 5,900
ZD	100	100 × 56 = 5,600
XOT A	50	50 × 57 = 2,850
XOT DR	50	50 × 0 = 0
YOT A	50	50 × 50 = 5,500
YOT C	100	100 × 54 = 5,400
ZOT DR	50	50 × 0 = 0
Total Cost in Rs.		75, 550

Allocation by VAM:

(1)

	A	*B*	*C*	*D*	*E*	*DC*	*AVAIL*	*ROC*
X	52	54	58	53	56	0	300	52
Y	41	46	45	41	42	0	400	41
Z	56	57	59	56	54	0	600	54
XOT	57	59	63	58	61	0	100	50
YOT	50	55	54	50	51	0	150	50
ZOT	64	65	67	64	62	0 **(200)**	2 00	62 ←
REQ	400	400	2 00	2 00	300	2 50	1750	
COC	9	8	9	9	9	0		

As for one allocation a row and column are getting eliminated. Hence, the degeneracy occurs.

(2)

	A	*B*	*C*	*D*	*E*	*DC*	*AVAIL*	*ROC*
X	52	54	58	53	56	0	300	52
Y	41	46	45	41	42	0	400	41
Z	56	57	59	56	54	0	600	54
XOT	57	59	63	58	61	0 **(50)**	100	57 ←
YOT	50	55	54	50	51	0	150	50
REQ	400	400	2 00	2 00	300	2 50	15 50	
COC	9	8	9	9	9	0		

(3)

	A	*B*	*C*	*D*	*E*	*AVAIL*	*ROC*
X	52	54	58	53	56	300	1
Y	41	46	45	41	42 **(300)**	400	0
Z	56	57	59	56	54	600	2
XOT	57	59	63	58	61	50	2
YOT	50	55	54	50	51	150	0
REQ	400	400	2 00	2 00	300	1500	
COC	9	8	9	9	9 ↑		

Here also for one allocation, a row and a column are getting eliminated. Degeneracy will occur. In all we may have to allocate two ∈ s to two empty cells.

(4)

	A	*B*	*C*	*D*	*AVAIL*	*ROC*
X	52	54	58	53	300	1
Y	41	46	45	41 **(100)**	100	0
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
YOT	50	55	54	50	150	0
REQ	400	400	2 00	2 00	1200	
COC	9	8	9	9		

(5)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	52	54	58	53	300	1
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
YOT	50	55	54 **(150)**	50	150	0
Req	400	400	200	100	1100	
Coc	2	1	4	3		

(6)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	52 **(300)**	54	58	53	300	1
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
Req	400	400	50	100	950	
Coc	4	3	1	3		

(6)

	A	*B*	*C*	*D*	*Avail*	*Roc*
Z	56	57	59 **(50)**	56	550	0
XOT	57	59	63	58	50	1
Req	100	400	50	100	600	
Coc	1	2	4	2		

(7)

	A	*B*	*D*		*Avail*	*Roc*
Z	56	57	56	**(100)**	550	0
XOT	57	59	58		50	1
Req	100	400	100		600	
Coc	1	2	2			

(8)

	A	*B*		*Avail*	*Roc*
Z	56	57	**(400)**	450	1
XOT	57	59		50	2
Req	100	400		500	
Coc	1	2			

(9)

	A	*Avail*
Z	56 **(50)**	50
XOT	57 **(50)**	50
	100	

In the table showing optimal solution, we can understand that the company *X* has to work 50% of its over time capacity, and company *Y* has to work 100% of its overtime capacity and company Z will not utilize its overtime capacity.

(*a*) Here the total profit or return that the trading company gets is equals to Sales revenue – total expenses, which include manufacturing cost and transportation cost. Hence,

Profit = (Total Sales Revenue) – (Manufacturing cost + transportation cost).

In the question given the sales price is same in all market segments, hence, the profit calculated is independent of sales price. Hence the programme, which minimizes the total cost will, maximizes the total profit. Hence the same solution will hold good. We need not work a separate schedule for maximization of profit.

(*b*) Here sales price in market segments will differ. Hence we have to calculate the total profit by the formula given above for all the markets and work for solution to maximise the profit.

The matrix showing the total profit earned by the company:

	A	B	C	D	E	DC	Avail.	u_j
X	18 3	26 300	14 5	15 3	9 8	0 6	900	6
Y	29 300	34 14	27 ε	27 1	25 0	0 14	400	14
Z	14 1	23 ∈	13 200	12 50	11 300	0 50	600	0
XOT	13 3	21 100	9 5	10 3	4 8	0 1	100	1
YOT	20 1	25 1	18 1	18 150	14 3	0 6	150	6
ZOT	6 9	15 5	5 8	4 8	3 8	0 200	200	0
Req.	400	400	200	200	300	250	1750	
Coc.	15	23	13	12	11	0		

As all the opportunity cost of empty cells are positive (maximization problem), the solution is optimal.

The allocations are:

Cell	*Load*	*Cost in Rs.*
XB	300	300 × 26 = 7, 800
YA	400	400 × 29 = 11, 600
ZC	200	200 × 13 = 2, 600
ZD	50	50 × 12 = 600
ZE	300	300 × 11 = 3, 300
Z DR	50	50 × 0 = 0
XOT B	100	100 × 21 = 2, 100
YOT D	150	150 × 18 = 2, 700
ZOT DR	200	200 × 0 = 0
Profit in Rs.		= 30, 700

	A	B	C	D	E	DC	Avail	Coc
X	18	26	14	15	9	0	300	8
Y	29 **400**	34	27	27	25	0	400	5
Z	14	23	13	12	11	0	600	9
XOT	13	21	9	10	4	0	100	8
YOT	20	25	18	18	14	0	150	5
ZOT	6	15	5	4	3	0	200	9
Req	400	400	200	200	300	250	1750	
Coc	11	8	9	9	9	0		

As for one allocation a row and column are getting eliminated. Hence, the degeneracy occurs.

(2)

	B	C	D	E	DC	Avail	Coc
X	26 **300**	14	15	9	0	300	11
Z	23	13	12	11	0	600	10
XOT	21	9	10	4	0	100	11
YOT	25	18	18	14	0	150	7
ZOT	15	5	4	3	0	200	10
Req	400	200	200	300	250	1350	
Coc	1	4	3	5	0		

(3)

	B	C	D	E	DC	Avail	Coc
Z	23	13	12	11	0	600	10
XOT	21 **100**	9	10	4	0	100	11
YOT	25	18	18	14	0	150	7
ZOT	15	5	4	3	0	200	10
Req	100	200	200	300	250	1050	
Coc	2	5	6	3	0		

Here also for one allocation, a row and a column are getting eliminated. Degeneracy will occur. In all we may have to allocate two ∈ s to two empty cells.

(4)

	C	*D*	*E*	*DC*	*Avail*	*Coc*
Z	13	12	11	0	600	1
YOT	18	18 **150**	14	0	150	0
ZOT	5	4	3	0	200	1
Req	200	200	300	250	950	
Coc	5	6	3	0		

↑

(5)

	C	*D*	*E*	*DC*	*Avail*	*Coc*
Z	13 **200**	12	11	0	600	1
ZOT	5	4	3	0	200	1
Req	200	50	300	250	800	
Coc	8	8	8	0		

(6)

	D	*E*	*DC*	*Avail*	*Coc*
Z	12	11 **300**	0	400	1
ZOT	4	3	0	200	1
Req	50	300	250	600	
Coc	8	8	0		

(7)

	D	DC		
Z	12 **50**	0	100	12 ←
ZOT	4	0	200	4
	50	250	300	
	8	0		

(8)

	DC	
Z	**50** 0	50
ZOT	0 **200**	200
	250	250

Problem. 4.9. A company has booked the orders for its consignment for the months of April, May, June and July as given below:

April: 900 units, May: 800 units, June: 900 units and July: 600 units. The company can produce 750 units per month in regular shift, at a cost of Rs. 80/- per unit and can produce 300 units per month by overtime production at a cost of Rs. 100/- per unit. Decide how much the company has to produce in which shift for minimizing the cost of production. It is given that there is no holding cost of inventory.

Solution: Remember here the production of April is available to meet the orders of April and subsequent months. But the production of May cannot be available to meet the demand of April. Similarly, the production of June is not available to meet the demand of April, May, but it can meet the demand of June and subsequent months and so on. Hence very high cost of production is allocated to the cells (Infinity or any highest number greater than the costs given in the problem), which cannot meet the demands of previous months (*i.e.* back ordering is not allowed). Here total availability is 4200 units and the total demand is for 3200 units. Hence we have to open a dummy column (DC), with cost coefficients equal to zero. The balanced matrix is shown below. Let us find the initial basic feasible solution by Northwest corner method and apply optimality test by MODI method.

A: April, M: May, J: June, Jl: July, AT: April Over time, MT: May overtime, JT: June overtime, JLT: July over time. DC : Dummy column.

Tableau 1.

Month of Demand (Cost in Rs)

	A	M	J	JI	DC	Avail.	u_i
A	80 / 750	80 / 0	80 / 0	80 / 0	0 / 0	750	0
AT	100 / 150	100 / 150	100 / 0	100 / 0	0 / 20	300	20
M	X	80 / 650	80 / 100	80 / 0	0 / 0	750	0
MT	X	100	100 / 300	100 / 0	0 / 20	300	20
J	X	X	80 / 500	80 / 250	0 / 0	750	0
JT	X	X	100 / 0	100 / – 300	0 / 20 +	300	20
JL	X	X	X	80 / 50 +	0 / – 700	750	0
JLT	X	X	X	100 / –20	0 / 300	300	0
Req.	900	800	900	600	1000	4200	
v_j	80	80	80	80	0		

Here the cell JT DC is having highest opportunity cost. Hence let us include the cell in the revised programme. To find the opportunity costs of empty cells, the row number u_i and column number v_j are shown. The cells marked with (*X*) are avoided from the programme. We can also allocate very high cost for these cells, so that they will not enter into the programme.

Tableau II. Revised programme.

Month of Demand (Cost in Rs)

	A	M	J	JI	DC	Avail.	u_i
A	80 (750)	80 0	80 0	80 0	0 0	750	0
AT	100 (150)	100 (150)	100 0	100 0	0 0	300	20
M	X	80 (650)	80 (100)	80 0	0 0	750	0
MT	X	100	100 (300)	100 0	0 0	300	20
J	X	X	80 (500)	80 (250)	0 0	750	0
JT	X	X	100 –20	100 **–20**	0 (300)	300	20
JL	X	X	X	80 (350)	0 (400)	750	0
JLT	X	X	X	100 –20	0 (300)	300	0
Req.	900	800	900	600	1000	4200	
v_j	80	80	80	80	0		

As the opportunity costs of all empty cells are either zeros or negative elements, the solution is optimal. As many empty cells are having zero as the opportunity cost, they can be included in the solution and get alternate solution.

Allocations:

Demand month.	*Production of the month*	*Load*	*Cost in Rs.*
April	April regular	750	$750 \times 80 = 60,000$
April	April over time	150	$150 \times 100 = 15,000$
May	April over time	150	$150 \times 100 = 15,000$
May	May regular	650	$650 \times 80 = 52,000$
June	May regular	100	$100 \times 80 = 8,000$
June	May over time	300	$300 \times 100 = 30,000$
June	June Regular	500	$500 \times 80 = 40,000$
July	June regular	250	$250 \times 80 = 20,000$
July	July regular	350	$350 \times 80 = 28,000$
Dummy column	June over time	300	300×0
Dummy Column	July regular	300	300×0
Dummy column	July over time	300	300×0
	Total cost in Rs.:		2,68,000

Problem: 4.10. Let us slightly change the details given in the problem 4.9. It is given that production of a month could be stored and delivered in next month without extra costs. Let us now consider that there is a cost associated with inventory holding or inventory carrying cost. Let the inventory carrying cost is Rs. 20 per month decide the new allocation.

Solution: In the cost matrix, for regular production, the cost is Rs. 80/-, for overtime production, the cost is Rs. 100 and for the stock held the inventory carrying cost is Rs. 20/ per month. If the stock is held for two months the inventory carrying cost is Rs. 40/-. That is if the production of April is supplied in June the cost will be Rs. 80/- + Rs. 40/- =

Rs. 120/- and do on. The initial basic feasible solution is obtained by Northwest corner method.

Production cost in Rs.

	April	May	June	July	DC	Avail.	u_i	
April	80 750	100 0	120 0	140 –20	0 40	750	120	
AOT	100 150	100 150 –	140 0	160 –20	0 + 60	300	140	
May	X	80 650 +	100 100 –	120 –20	0 20	750	100	
MOT	X	100	120 50 +	140 –20	0 250 –	300	120	
June	X	X	80 750	100 –20	0 0	750	80	
JOT	X	X	100	120 –40	0 300	300	80	
July	X	X	X	80 300	0 150	750	80	
JLOT	X	X	X	100 –20	0 300	300	80	
Req.	900	800	900	600	1000	4200		
v_j	–40	–20	0	0	–80			

Cell AOT DC is having highest positive opportunity cost. Hence we have to include this in the revised programme.

Production cost in Rs.

	April	May	June	July	DC	Avail.	u_i	
April	80 750	100 0	120 –20	140 –80	0 –20	750	–20	
AOT	100 150	120 50	140 –20	160 –80	0 **150**	300	0	
May	X	80 750	100 **–20**	120 –20	0 –40	750	–40	
MOT	X	100 20	120 150	140 –60	0 **150**	300	0	
June	X	X	80 750	100 –60	0 –40	750	–40	
JOT	X	X	100 20	120 –40	0 300	300	0	
July	X	X	X	80 **600**	0 150	750	0	
JLOT	X	X	X	100 –20	0 300	300	0	
Req.	900	800	900	600	1000	4200		
v_j	100	120	120	80	0			

In the above matrix, two cells, MO M and JO J are having positive opportunity costs = 20. Hence, they may be included in the revised programme. If we include them in the programme, the final optimal solution will be as follows:

Production cost in Rs.

	April	May	June	July	DC	Avail.	u_i	
April	80; 750	100; –20	120; –40	140; –80	0; –20	750	–20	
AOT	100; 150	120; –20	140; –40	160; –80	0; 150	300	0	
May	X	80; 750	100; –20	120; –80	0; –20	750	–20	
MOT	X	100; 50	120; –20	140; –80	0; 250	300	0	
June	X	X	80; 750	100; –40	0; –20	750	–20	
JOT	X	X	100; 150	120; –40	0; 150	300	0	
July	X	X	X	80; 600	0; 150	750	0	
JLOT	X	X	X	100; –20	0; **300**	300	0	
Req.	900	800	900	600	1000	4200		
v_j	100	100	100	80	0			

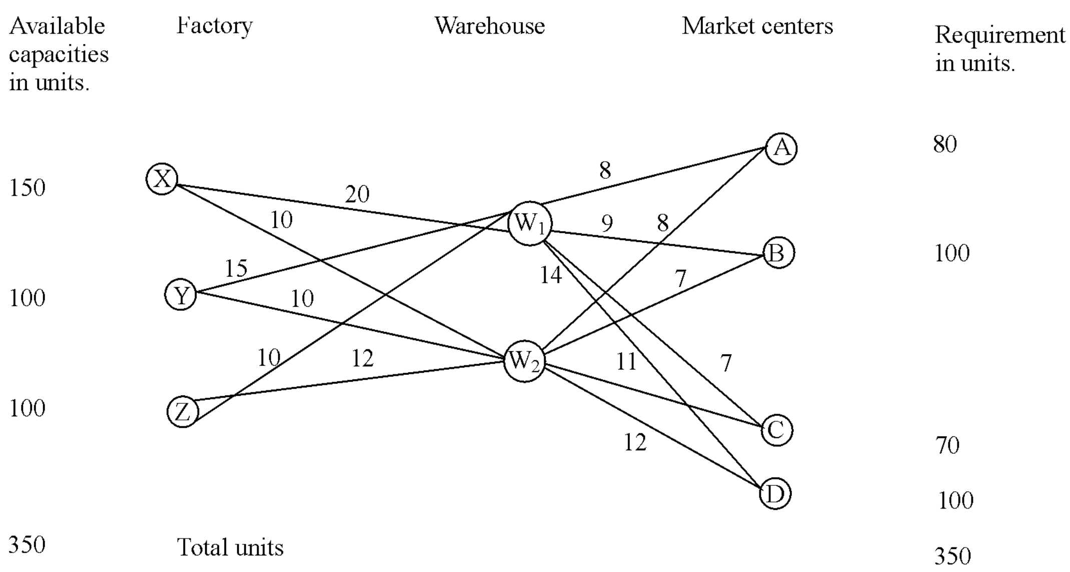
As all the opportunity costs of empty cells are negative, the solution is optimal. The optimal allocations are:

Month of demand	*Month of production*	*load*	*cost in Rs.*		*Rs.*
April	April regular	750	750 × 80	=	60,000
April	April over time	150	150 × 100	=	15,000
Dummy Col	April over time	150	150 × 0	=	0
May	May regular	750	750 × 80	=	60,000
May	May over time	50	50 × 100	=	5,000
Dummy Column	May over time	250	250 × 0	=	0
June	June regular	750	750 × 80	=	60,000
June	June over time	150	150 × 100	=	15,000
Dummy column	June over time	150	150 × 0	=	0
July	July regular	600	600 × 80	=	48,000
Dummy column	July regular	150	150 × 0	=	0
Dummy column	July overtime	300	300 × 0	=	0
	Total cost in Rs.				2,63,000

4.10. TRANSSHIPMENT PROBLEM

We may come across a certain situation, that a company (or companies) may be producing the product to their capacity, but the demand arises to these products during certain period in the year or the demand may reach the peak point in a certain period of the year. This is particularly true that products like Cool drinks, Textbooks, Notebooks and Crackers, etc. The normal demand for such products will exist, throughout the year, but the demand may reach peak points during certain months in the year. It may not possible for all the companies put together to satisfy the demand during peak months. It is not possible to produce beyond the capacity of the plant. Hence many companies have their regular production throughout the year, and after satisfying the existing demand, they stock the excess production in a warehouse and satisfy the peak demand during the peak period by releasing the stock from the warehouse. This is quite common in the business world. Only thing that we have to observe the inventory carrying charges of the goods for the months for which it is stocked is to be charged to the consumer. Take for example crackers; though their production cost is very much less, they are sold at very high prices, because of inventory carrying charges. When a company stocks its goods in warehouse and then sends the goods from warehouse to the market, the problem is known as **Transshipment problem**. Let us work one problem and see the methodology of solving the Transshipment Problem.

Problem. 4.11. A company has three factories *X*, *Y* and *Z* producing product *P* and two warehouses to stock the goods and the goods are to be sent to four market centers *A*, *B*, *C* and *D* when the demand arises. The figure given below shows the cost of transportation from factories to warehouses and from warehouses to the market centers, the capacities of the factories, and the demands of the market centers. Formulate a transportation matrix and solve the problem for minimizing the total transportation cost.



Solution:

To formulate a transportation problem for three factories and four market centers, we have to find out the cost coefficients of cells. For this, if we want the cost of the cell *XA*, the cost of transportation from *X* to warehouse W_1 + Cost transportation from W_1 to market center *A* are calculated and as our objective is to minimize the cost, the least of the above should be entered as the cost coefficient of cell *XA*. Similarly, we have to workout the costs and enter in the respective cells.

Cell *XA*: Route X-W_1-A and X- W_2- A minimum of these two (28 and 18) i.*e* 18

Cell *XB* Route X - W_1 - B and X - W_2 - B Minimum of the two is (29, 17) *i.e* 17

Cell *XC* Route X - W_1 - C and X- W_2 - C Minimum of the two is (27, 11) *i.e* 11

Cell *XD* Route X- W_1 - D and X- W_2 - D Minimum of the two is (34, 22) *i.e* 22

Similarly we can calculate for other cells and enter in the matrix. The required transportation problem is:

	A	*B*	*C*	*D*	*Available*
X	18	17	21	22	150
Y	18	17	21	22	100
Z	18	19	17	24	100
Required.	80	100	70	100	350

Basic Feasible Solution by VAM:

Market centers (Cost in Rs.)

	A	B	C	D	Available	u_i
X	18 (80)	17 + (70)	21	22	150	0
Y	18	17 − (30)	21	22 + (70)	100	0
Z	18	19	17 (70)	24 − (30)	100	2
REQUIRED	80	199	70	100	350	
v_j	18	17	15	22		

Market centers (Cost in Rs.)

	A	B	C	D	Available	u_i
X	18 (50)	17 (100)	21 –4	22 0	150	0
Y	18 0	17 (ε)	21 –4	22 (100)	100	0
Z	18 (30)	19 –2	17 (70)	24 –2	100	0
REQUIRED	80	100	70	100	350	
v_j	18	17	17	22		

As the opportunity costs of all empty cells are negative, the solution is optimal. The optimal allocation is:

Cell	*Route*	*Load*	*Cost in Rs.*		*Rs.*
XA	X-W_2-A	50	50×18	=	900
XB	X - W_2 - B	100	100×17	=	1700
YB	Y- W_2 - B	ε	---	=	----
YD	Y - W_2- D	100	100×22	=	2200
ZA	Z- W_1- A	30	30×18	=	540
ZC	Z - W_1 - C	70	70×17	=	1190
			Total Cost in Rs.		6530

(The answer shows that the capacity of W_2 is 250 units and capacity of W_1 is100 units).

(1)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	18	17	21	22	150	0
Y	18	17	21	22	100	0
Z	18	19	17 **70**	24	100	2
Req.	80	100	70	100	350	
Coc	18	17	15	22		

(2)

		A	*B*	*D*	*Avail*	*Roc*
X		18	17	22	150	1
Y	**80**	18	17	22	100	1
Z		18	19	24	30	1
Req.		80	100	100	280	
Coc		0	0	0		

(3)

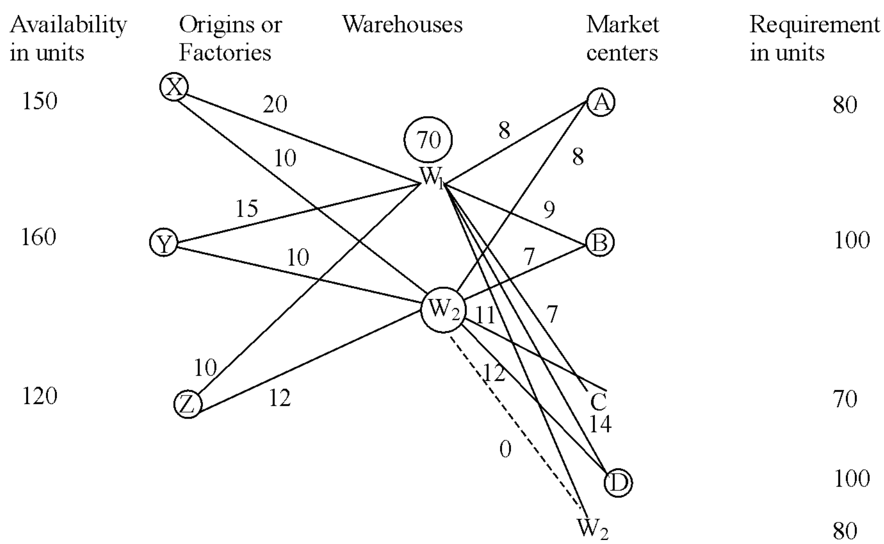
	B	*D*	*Avail*	*Roc*
X	17 **70**	22	70	5
Y	17	22	100	5
Z	19	24	30	5
Req.	100	100	200	
Coc	0	0		

(4)

	B	*D*	*Avail*	*Roc*
Y	17 **30**	22	100	5
Z	19	24	30	5
Req.	30	100	200	
Coc	2	2		

	D	*Avail*	*Roc*
Y	22 **70**	70	
Z	24 **30**	30	
Req.	100	100	
Coc			

Problem 4.12.



In the above some restrictions are imposed. The restrictions are:

Let warehouse W_1 be pure transshipment warehouse and W_2 is transshipment as well as distribution point.

(*i*) The capacity limitation on W_1 = 70 units.

(*ii*) The warehouse W_2 also deals with direct distribution of 80 units.

As per the given conditions, the following discussion will hold good.

Solution:

1. As a source and intermediate transshipment node, W_1 has the capacity limitations of 70 units. Hence, availability of W_1 and requirement of destination W_1 is 70 units.

2. W_2 has no capacity limitation. However, it deals partial direct distribution of 80 units. Therefore, as a source its availability should be the difference between the total availability from all factories *i.e* X, Y and Z less its own direct distribution. 430 – 80 = 350.
3. As an intermediate destination, it should have the capacity to route entire production *i.e.* 430 units.
4. Unit cost of transportation from X, Y, and Z to destinations A, B, C and D, through W_1 and W_2 can be had from figure given, this can be entered in the table- 1 showing the initial transportation matrix.
5. There is no direct transportation from X,Y, and Z to destinations A, B, C and D. To avoid this direct routes we can allocate very high cost of transportation costs for these cells or we can avoid these cells by crossing them, *i.e.* eliminating them from the programme.
6. W_1 as source giving to W_1 as warehouse or sink, and W_2 as a source giving to W_2 as warehouse or sink will have zero cost.

	W_1	W_2	A	B	C	D	Avail.	u_i
X	$\boxed{20}$ –12	$\boxed{10}$ 150	X	X	X	X	150	10
Y	$\boxed{15}$ –7	$\boxed{10}$ 160	X	X	X	X	160	10
Z	$\boxed{10}$ 70	$\boxed{12}$ 50	X	X	X	X	120	12
W_1	$\boxed{0}$ –2	X	$\boxed{8}$ ε	$\boxed{9}$ –2	$\boxed{7}$ 70	$\boxed{14}$ –2	70	0
W_2	X	$\boxed{0}$ 70	$\boxed{8}$ 80	$\boxed{7}$ 100	$\boxed{11}$ 0	$\boxed{12}$ 100	350	0
Req.	70	430	80	100	70	100	850	
v_j	–2	0	8	7	11	12		

As the total number of allocations are $m + n - 1$ after allocating $\in$ to cell W_1A, the solution is a basic feasible solution. By giving the optimality test by MODI method, we see that all the opportunity costs of empty cells are negative and hence the solution is optimal.

The allocation:

Cell	*Load*	*Cost in Rs.*		*Rs.*
XW_2	150	150 × 19	=	1500
YW_2	160	160 × 10	=	1600
ZW_1	70	70 × 10	=	700
ZW_2	50	50 × 12	=	600
W_1A	ε	-----	--	-----
W_1C	70	70 × 7	=	140
W_2W_2	70	70 × 0	=	0
W_2A	80	80 × 8	=	160
W_2B	100	100 × 7	=	700
W_2D	100	100 × 12	=	1200
		Total Cost in Rs.		6,600

VAM:

(1)

	W_1	W_2	A	B	C	D	$Avail$	ROC
X	20	10	X	X	X	X	150	10
Y	15	10	X	X	X	X	160	5
Z	10 **70**	12	X	X	X	X	120	2
W_1	0	X	8	9	7	14	70	1
W_2	X	0	8	7	11	12	350	1
Req.	70	430	80	100	70	100	850	
COC	10	10	0	2	4	2		

(2)

	W_2	A	B	C	D	$Avail$	ROC
X	10	X	X	X	X	150	10
Y	10	X	X	X	X	160	10
Z	12	X	X	X	X	50	12
	50						
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	430	80	100	70	100	780	
COC	10	0	2	4	2		

(3)

	W_2	A	B	C	D	$Avail$	ROC
X	10	X	X	X	X	150	10
	150						
Y	10	X	X	X	X	160	10
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	380	80	100	70	100	730	
COC	10	0	2	4	2		

(4)

	W_2	*A*	*B*	*C*	*D*	*Avail*	*ROC*
Y	10 **160**	X	X	X	X	160	10
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	230	80	100	70	100	580	
COC	10	0	2	4	2		

(5)

	W_2	*A*	*B*	*C*	*D*	*Avail*	*ROC*
W_1	X	8	9	7	14	70	1
W_2	0 **70**	8	7	11	12	350	7
Req.	70	80	100	70	100	420	
COC	INF	0	2	4	2		

(6)

	A	*B*	*C*	*D*	*Avail*	*ROC*
W_1	8	9	7	14	70	1
			70			
W_2	8	7	11	12	280	1
Req.	80	100	70	100	350	
COC	0	2	4	2		

(7)

	A	*B*	*D*	*Avail*	*ROC*
W_2	8 **80**	7 **100**	12 **100**	280	
Req.	80	100	100	280	
COC					

4.12. REDUNDANCY IN TRANSPORTATION PROBLEMS

Some times, it may very rarely happen or while writing the alternate solution it may happen or during modifying the basic feasible solution it may happen that the number of occupied cells of basic feasible solution or some times the optimal solution may be greater than $m + n - 1$. This is called **redundancy in transportation problem**. This type of situation is very helpful to the manager who is looking about shipping of available loads to various destinations. This is as good as having more number of independent simultaneous equations than the number of unknowns. It may fail to give unique values of unknowns as far as mathematical principles are concerned. But for a transportation manager, it enables him to plan for more than one orthogonal path for an or several cells to evaluate penalty costs, which obviously will be different for different paths.

4.13. SENSITIVITY ANALYSIS

(a) Non - basic variables

While discussing MODI method for getting optimal solution, we have discussed significance of **implied cost**, which fixes the upper limit of cost of the empty cell to entertain the cell in the next programme. Now let us discuss the influence of variations in present parameters on the optimum solution i.e **sensitivity of optimal solution for the variations in the costs of empty cells and loaded cells**. If unit cost of transportation of a particular non-basic variable changes, at what value of the cost of present optimum will no longer remain optimum? To answer this question, in the first instance, it is obvious that as the empty cell is not in the solution, any increase in its unit transportation cost will to qualify it for entering variable. But if the unit cost of empty cells is reduced the chances of changing the optimum value may be examined. Let us take an optimum solution and examine the above statement.

	A	B	C	D	E	Avail.	u_i
X	$\boxed{10}$ –3	$\boxed{15}$ –3	$\boxed{17}$ –1	$\boxed{19}$ –27	$\boxed{16}$ 50	50	0
Y	$\boxed{20}$ –23	$\boxed{12}$ 40	$\boxed{16}$ 30	$\boxed{18}$ –26	$\boxed{20}$ –4	70	0
Z	$\boxed{9}$ 40	$\boxed{14}$ 10	X	$\boxed{10}$ **30**	$\boxed{18}$ 0	80	2
DR	$\boxed{0}$ –9	$\boxed{0}$ –4	$\boxed{0}$ 30	$\boxed{0}$ –24	$\boxed{0}$ 20	50	–16
Req.	40	50	60	30	70	250	
v_j	7	12	16	–8	16		

In the solution shown above as all the opportunity costs of empty cells are negative. Consider empty cell *XA*. Its opportunity cost is Rs. -3/- This means to say that the units cost of transportation of cell *XA* decreases by Rs.3/- or more *i.e* Rs.10/- the unit cost of transportation of the empty cell *XA* minus 3 = 7, or less than 7 the optimal solution changes, *i.e*. the cell *XA* will become eligible for entering into solution. Hence this cost, which shows the limit of the unit cost of empty cell, is known as implied cost in transportation problems. We can see that the opportunity cost of empty cell *ZE* is zero. This shows that the cell ZE is as good as a loaded cell and hence we can write alternate solutions by taking the cell *ZE* into consideration. **(Note: No unit cost of transportation is given for the cell *ZC*. Hence that cell should not be included in the programme. For this purpose, we can cross the cell or allocate very high unit cost of transportation for the cell. In case zero or any negative element is given as the unit cost of transportation for a cell, the value can be taken for further treatment.)**

(*b*) Basic variables

If unit cost of loaded cell *i.e.* basic variable is changed, it affects the opportunity costs of several cells. Now let us take the same solution shown above for our discussion. In case the unit cost of transportation for the cell *XE* is θ instead of 16, and other values remaining unchanged. Now let us workout the opportunity costs of other cells.

	A	B	C	D	E	Avail.	u_i
X	[10] θ – 19	[15] θ – 19	[17] θ – 17	[19] θ – 27	[θ] (50)	50	0
Y	[20] –13	[12] (40)	[16] (30)	[18] –10	[20]	70	θ + 16
Z	[9] (40)	[14] **10**	X	[10] (30)	[18] 0	80	θ + 18
DR	[0] –9	[0] –4	[0] (30)	[0] –8	[0] (20)	50	–θ
Req.	40	50	60	30	70	250	
v_j	θ.9	θ.4	θ	θ–8	θ		

Cells *XA* and *XB* is positive when θ is > than 19. Cell *XC* is positive when θ is > 17 and cell *XD* is positive when θ is > 27. Other cells are not influenced by θ.

If unit cost of transportation increase and becomes 17, the present optimum may change. In case the unit cost of transportation of the cell *XA* is reduced, the solution will still remain optimum, as our objective is to minimize the total transportation cost.

A point to note here is we have used Northwest corner method and Vogel's approximation method to get basic feasible solution. Also we have discussed the least cost method and there are some methods such as row minimum and column minimum methods. These methods attempt to optimize the sub-system and do not consider marginal trade-offs. Therefore, such methods have no merit to serve useful purpose.

4.14. SUMMARY

1. Read the statement of the problem. Confirm whether you have to maximize the objective function or minimize the objective function.
2. Construct the transportation matrix.
3. Check whether the given problem is balanced or not.
4. If balanced proceed further. If not balanced, balance the problem by opening a dummy row or a dummy column depending on the need. Let the unit cost of transportation of cells of dummy row or column be zero.

5. If the problem is maximization one convert that into a minimization problem by multiplying the matrix by –1 or by subtracting all the elements of the matrix form the highest element in the matrix.
6. Find the basic feasible solution. The characteristics of the basic feasible solution are it must have $(m + n – 1)$ allocations, where **m** is the number of rows and **n** is the number of columns.
7. The basic feasible solution may be obtained by (*a*) Northwest corner method, (*b*) Least Cost method or Matrix minimum method, or (*c*) Vogel's approximation method or Opportunity cost method.
8. If initial allocations are equal to $(m + m – 1)$ proceed to next step. If it is not equal to $(m + n$ **-** $1)$ it is known as degenerate solution.
9. To solve degeneracy, add a small and negligible element ∈ to empty cells. Take care to see that the ∈ loaded cell do not make closed loop with other loaded cells when lines are drawn from epsilon loaded cells to other loaded cells by travelling vertically and horizontally by taking turns at loaded cells.
10. Write allocations and calculate the total cost of transportation.
11. Give optimality test to the basic feasible solution. Optimality test can be given by (*a*) Stepping stone method or (*b*) Modified distributing method or **MODI** method.
13. The characteristic of optimal solution is the opportunity costs of all empty cells are either negatives or zeros.
14. Remember if any empty cell has zero as its opportunity cost, then we can write alternate optimal solutions.
15. Write the allocations and calculate total transportation cost.
16. In case, the unit cost of transportation of any cell is zero or negative elements, take the same into considerations for further calculations. Suppose nothing is given in the cell as the unit cost of transportation, then presume that the route connecting the origin and the destination through that cell is not existing and cancel that cell and do not consider it at all while solving the problem, or else allocate very high cost of unit cost of transportation (infinity or any number which is greater than all the elements in the matrix), so that that cell will not enter into programme. (In maximization problem allocate a negative profit or return to the cell).

Problem 4.13. A company has three factories *X*, *Y*, and *Z* and four warehouses *A*, *B*, *C*, and *D*. It is required to schedule factory production and shipments from factories to warehouses in such a manner so as to minimize total cost of shipment and production. Unit variable manufacturing costs (UVMC) and factory capacities and warehouse requirements are given below:

From	*UVMC*	*To warehouses*				*Capacity in units per month.*
Factories.	Rs.	Unit shipping costs in Rs.				
		A	B	C	D	
X	10	0	1	1	2	75
Y	11	1	2	3	1	32
Z	12	4	3	3	6	67
Requirement:		65	24	16	15	

Find the optimal production and transportation schedule.

Solution: We have to optimize production and shipment cost. Hence the transportation matrix elements are the total of manufacturing cost plus transportation cost. For example, the manufacturing cost of factory *X* is Rs. 10. Hence the transportation and shipment cost will be equal to 10 + 0, 10 +1, 10 +1 and 10 +2 respectively for warehouses *A*, *B*, *C* and *D* respectively. As the total available is 174 units and the total demand is 120 units we have to open a dummy column with requirement of 54 units. The production cum transportation matrix is given below:

Production cum transportation cost per unit in Rs.

	A	B	C	D	DC	Avail	u_i
X	$\boxed{10}$ + 35	$\boxed{11}$ 24	$\boxed{11}$ − 16	$\boxed{11}$ −2	$\boxed{0}$ –6	75	0
Y	$\boxed{12}$ 17	$\boxed{13}$ 0	$\boxed{14}$ −1	$\boxed{12}$ 15	$\boxed{0}$ –4	32	2
Z	$\boxed{16}$ 13	$\boxed{15}$ 2	$\boxed{15}$ + 2	$\boxed{18}$ −2	$\boxed{0}$ 54	67	6
Req.	65	24	16	15	54	174	
v_j	10	10	11	10	–6		

(1) Initial basic feasible solution by VAM :

	A	*B*	*C*	*D*	*DC*	*Avail*	*Roc*
X	10	11	11	11	0	75	10
Y	12	13	14	12	0	32	12
Z	16	15	15	18	0 **54**	67	15
Req	65	24	16	15	54	174	
Coc	2	2	3	1	0		

(2)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	10	11	11 **16**	11	75	0
Y	12	13	14	12	32	0
Z	16	15	15	18	13	0
Req	65	24	16	15	124	
Coc	2	2	3	0		

(3)

	A	*B*	*D*	*Avail*	*Roc*
X	10	11 **24**	11	75	0
Y	12	13	12	32	1
Z	16	15	18	13	0
Req	65	24	15	104	
Coc	2	2	0		

(4)

	A	*D*	*Avail*	*Roc*
X	10	11	35	2
	35			
Y	12	12	32	0
Z	16	18	13	2
Req	65	15	80	
Coc	2	0		

(5)

	A	*D*	*Avail*	*Roc*
Y	12	12 **15**	32	0
Z	16	18	13	2
Req	30	15	45	
Coc	4	6		

(6)

	A	*Avail*	*Roc*
Y	12 **17**	17	0
Z	16 **13**	13	2
Req	30	30	
Coc	4		

Production cum transportation cost per unit in Rs.

	A	B	C	D	DC	Avail	u_i
X	10 48	11 24	11 29	11 –6	0 –4	75	0
Y	12 **17**	13 0	14 –1	12 15	0 –2	32	2
Z	16 –2	15 0	15 13 2	18 –4	0 54	67	4
Req.	65	24	16	15	54	174	
v_j	10	11	11	10	–4		

As there are $m + n - 1$ allocations and all the opportunity costs of empty cells are negative, the solution is optimal.

The optimal allocations are:

Cell	Load	Cost in Rs.		Rs.
XA	48	48×10	=	480
XB	24	24×11	=	264
XC	29	29×11	=	319
YA	17	17×12	=	204
YB	15	15×12	=	180
ZC	13	13×15	=	195
Z DC	54	54×0	=	0
	Total cost in RS.		=	1642

QUESTIONS

1. Explain the process of solving a transportation problem.
2. List out the differences and similarities between Resource allocation model and Transportation model in linear programming.
3. Explain the procedure of getting basic feasible solution by using VAM.
4. Explain what are degeneracy and redundancy in transportation problem. How do you solve degeneracy in transportation problem? Distinguish between tie and degeneracy in linear programming problem.
5. Is transportation problem is of maximization type or minimization type problem? If it is one of the two, how do you solve the other version of the transportation model?

6. How do you say that a transportation model has an alternate solution? In case it has an alternate optimal solution, how do you arrive at alternate solution?
7. What is transshipment problem? In what way it differs from general transportation problem?
8. Explain the terms: (*a*) Opportunity cost, (*b*) Implied cost, (*c*) Row opportunity cost, (*d*) Column opportunity cost.
9. The DREAM - DRINK Company has to work out a minimum cost transportation schedule to distribute crates of drinks from three of its factories *X*, *Y*, and *Z* to its three warehouses *A*, *B*, and *C*. The required particulars are given below. Find the least cost transportation schedule.

Transportation cost in Rs per crate.

From / To	*A*	*B*	*C*	*Crates Available.*
X	75	50	50	1040
Y	50	25	75	975
Z	25	125	25	715
Crates required.	1300	910	520	2730

10. The demand pattern for a product at for consumer centers, *A*, *B*, *C* and *D* are 5000 units, 7000 units, 4000 units and 2000 units respectively. The supply for these centers is from three factories *X*, *Y* and *Z*. The capacities for the factories are 3000 units, 6000 units and 9000 units respectively. The unit transportation cost in rupees from a factory to consumer center is given below in the matrix. Develop an optimal transportation schedule and find the optimal cost.

From:	*To*			
	A	*B*	*C*	*D*
X	8	9	12	8
Y	3	4	3	2
Z	5	3	7	4

11. From three warehouses, *A*, *B*, and *C* orders for certain commodities are to be supplied to demand points *X*, *Y*, and *Z*. Find the least cost transportation schedule with relevant information given below:

From Warehouses	*To demand points (Transportation cost in Rs. per units).*			*Availability in units.*
	X	*Y*	*Z*	
A	5	10	2	100
B	3	7	5	25
C	6	8	4	75
Units demand:	105	30	90	

12. From three warehouses *A*, *B*, and *C* orders for certain commodities are to be supplied to demand points 1, 2, 3, 4 and 5 monthly. The relevant information is given below:

Warehouses	*Demand points (Transportation cost in Rs per unit.*					*Availability in units.*
	1	2	3	4	5	
A	4	1	2	6	9	100
B	6	4	3	5	7	120
C	5	2	6	4	8	120
Units demand:	40	50	70	90	90	

During certain month a bridge on the road-connecting warehouse *B* to demand point 3 is closed for traffic. Modify the problem suitably and find the least cost transportation schedule. (The demand must be complied with).

13. A tin box company has four factories that supply to 5 warehouses. The variable cost of manufacturing and shipment of one ton of product from each factory to each warehouse are shown in the matrix given below, Factory capacities and warehouse requirements are shown in the margin. After several iterations the solution obtained is also shown.

Warehouses (Cost in Rs. per unit)

Factories	A	B	C	D	E	DMY	Capacity
W	17	9	14 **25**	10 **30**	14	0 **20**	75
X	13 **10**	6 **20**	11 **15**	11	12	0	45
Y	6 **30**	17	9	12	12	0	30
Z	15	20	11 **10**	14	6 **40**	0	50
Req	40	20	50	30	40	20	200

(*a*) Is this an optimal solution? How do you know?

(*b*) Is there an alternate solution? If so find it.

(*c*) Suppose some new equipment was installed that reduces the variable operation cost by Rs. 2/- per ton in factory *X*, is the shipping schedule remain optimum? If not what is the new optimum?

(*d*) Suppose the freight charges from *W* to A were reduced by Rs.2/- would this change the shipping schedule? If so what is the new optimum?

(*e*) How much would the manufacturing cost have to be reduced in W before production would be increased beyond 55 tons?

14. A company has a current shipping schedule, which is being questioned by the management as to whether or not it is optimal. The firm has three factories and five warehouses. The necessary data in terms of transportation costs in Rs. per unit from a factory to a destination and factory capacities and warehouse requirements are as follows:

Factories. (Transportation costs in Rs. per unit.)

Warehouses.	*X*	*Y*	*Z*	*Requirement of warehouses in units.*
A	5	4	8	400
B	8	7	4	400
C	6	7	6	500
D	6	6	6	400
E	3	5	4	800
Factory capacities.	800	600	1100	

Solve for an optimal shipping schedule in terms of lowest possible shipping costs.

15. Solve the following transportation problem.

Destination

Source	*A*	*B*	*C*	*D*	*E*	*Supply*
W	20	19	14	21	16	40
X	15	20	13	19	16	60
Y	18	15	18	20		70
Z	0	0	0	0	0	50
Demand.	30	40	50	40	60	

(Note: Nothing is given in cell *YE*. So you have to ignore it).

16. A manufacturing organization has 3 factories located at *X*, *Y* and *Z*. The centralized planning cell has to decide on allocation of 4 orders over the 3 factories with a view to minimizing the total cost to the organization, Demand and capacity and cost details are given as under:

Customer	*Demand per month in units.*
A	960
B	380
C	420
D	240

Capacities and Costs (Rs.).

Factories	*Capacity units per month*	*Overhead costs in Rs, per month*	*Direct cost in Rs. per unit.*
X	400	400	2.50
Y	900	720	3.00
Z	640	320	3.50

Shipping cost in Paise per unit dispatch.

To

From	*A*	*B*	*C*	*D*
X	50	70	40	35
Y	45	75	40	55
Z	70	65	60	75

It is also possible to produce 25% higher than the capacity in each factory by working overtime at 50% higher in direct costs.

(*a*) Build a transportation model so that the total demand is met with.

(*b*) Do the allocation of factory capacity by minimum cost allocation and check the solution for optimality.

16. In a transportation problem the distribution given in the table below was suggested as an optimal solution. The capacities and requirement are given. The number in bold are allocations. The transportation costs given in Rs, per unit from a source to a destination.

(*a*) Test whether the given distribution is optimal?

(*b*) If not optimal obtain all basic optimal solution.

Destination

Source	A	B	C	D	Capacity
X	9 **12**	8 **14**	12	10 **10**	36
Y	10	10 **16**	12 **28**	14	44
Z	8	9	11 **32**	12	32
Demand	12	30	60	10	

17. A department stores wishes to purchase 7,500 purses of which 2,500 are of style *X*, 2,500 are of style *Y* and 2,500 are of style *Z*. Four manufacturers *A*, *B*, *C* and *D* bid to supply not more than the following quantities, all styles combined. *A* = 1,000, *B* = 3,000, *C* = 2, 100 and *D* = 1,900. The following table gives the cost per purse of each style of the bidders in Rs. per purse.

MANUFACTURER.

Style	*A*	*B*	*C*	*D*
X	10	4	9	5
Y	6	7	8	7
Z	3	8	6	9

(*a*) How should orders to be placed by the department store to minimize the total cost?
(*b*) If the store were to introduce a new style W, which manufacturer can supply it? How many of W can he supply?

MULTIPLE CHOICE QUESTIONS

1. Transportation problem is basically a
   (*a*) Maximization model
   (*b*) Minimization model
   (*c*) Transshipment problem
   (*d*) Iconic model ( )
2. The column, which is introduced in the matrix to balance the rim requirements, is known as:
   (*a*) Key column
   (*b*) Idle column
   (*c*) Slack column
   (*d*) Dummy Column ( )
3. The row, which is introduced in the matrix to balance the rim requirement, is known as:
   (*a*) Key row
   (*b*) Idle row
   (*c*) Dummy row
   (*d*) Slack row ( )
4. One of the differences between the Resource allocation model and Transportation Model is:
   (*a*) The coefficients of problem variables in Resource allocation model may be any number and in transportation model it must be either zeros or ones.
   (*b*) The coefficients of problem variable in Resource allocation model must be either zeros or ones and in Transportation model they may be any number.
   (*c*) In both models they must be either zeros or ones only.
   (*d*) In both models they may be any number. ( )
5. To convert the transportation problem into a maximization model we have to
   (*a*) To write the inverse of the matrix
   (*b*) To multiply the rim requirements by –1
   (*c*) To multiply the matrix by –1
   (*d*) We cannot convert the transportation problem in to a maximization problem, as it is basically a minimization problem. ( )
6. In a transportation problem where the demand or requirement is equals to the available resource is known as
   (*a*) Balanced transportation problem,
   (*b*) Regular transportation problem,
   (*c*) Resource allocation transportation problem
   (*d*) Simple transportation model. ( )

7. The total number of allocation in a basic feasible solution of transportation problem of $m \times n$ size is equal to:
   (*a*) $m \times n$ (*b*) $(m / n) - 1$
   (*c*) $m + n + 1$ (*d*) $m + n - 1$ ( )

8. When the total allocations in a transportation model of $m \times n$ size is not equals to $m + n - 1$ the situation is known as:
   (*a*) Unbalanced situation (*b*) Tie situation
   (*c*) Degeneracy (*d*) None of the above ( )

9. The opportunity cost of a row in a transportation problem is obtained by:
   (*a*) Deducting the smallest element in the row from all other elements of the row,
   (*b*) Adding the smallest element in the row to all other elements of the row,
   (*c*) Deducting the smallest element in the row from the next highest element of the row
   (*d*) Deducting the smallest element in the row from the highest element in that row. ( )

10. In Northwest corner method the allocations are made
   (*a*) Starting from the left hand side top corner,
   (*b*) Starting from the right hand side top corner
   (*c*) Starting from the lowest cost cell
   (*d*) Starting from the lowest requirement and satisfying first. ( )

11. VAM stands for:
   (*a*) Value added method (*b*) Value assessment method
   (*c*) Vogel Adam method, (*d*) Vogel's approximation method. ( )

12. MODI stands for
   (*a*) Modern distribution, (*b*) Mendel's distribution method
   (*c*) Modified distribution method (*d*) Model index method ( )

13. In the optimal solution, more than one empty cell have their opportunity cost as zero, it indicates
   (*a*) The solution is not optimal (*b*) The problem has alternate solution
   (*c*) Something wrong in the solution (*d*) The problem will cycle. ( )

14. In case the cost elements of one or two cells are not given in the problem, it means:
   (*a*) The given problem is wrong
   (*b*) We can allocate zeros to those cells
   (*c*) Allocate very high cost element to those cells
   (*d*) To assume that the route connected by those cells are not available. ( )

15. To solve degeneracy in the transportation problem we have to:
   (*a*) Put allocation in one of the empty cell as zero
   (*b*) Put a small element epsilon in any one of the empty cell

(*c*) Allocate the smallest element epsilon in such a cell, which will not form a closed loop with other loaded cells.

(*d*) Allocate the smallest element epsilon in such a cell, which will form a closed loop with other loaded cells. ( )

16. A problem where the produce of a factory is stored in warehouses and then they are transported to various demand point as and when the demand arises is known as:
    (*a*) Transshipment problem
    (*b*) Warehouse problem
    (*c*) Storing and transport problem
    (*d*) None of the above ( )

17. Implied Cost in transportation problem sets (in the existing program):
    (*a*) The lowest limit for the empty cell beyond which it is not advisable to include in the programme,
    (*b*) The highest limit for the empty cell beyond which it is not advisable to include in the programme,
    (*c*) The opportunity cost of the empty cell,
    (*d*) None of the above. ( )

18. In transportation model, the opportunity cost is given by
    (*a*) Implied cost + Actual cost of the cell
    (*b*) Actual cost of the cell – Implied cost,
    (*c*) Implied cost – Actual cost of the cell
    (*d*) Implied cost × Actual cost of the cell ( )

19. If ui and v_j are row and column numbers respectively, then the implied cost is given by:
    (*a*) $u_i + v_j$ (*b*) $u_i - v_j$
    (*c*) $u_i \times v_j$ (*d*) u_i / v_j ( )

20. If a transportation problem has an alternate solution, then the other alternate solutions are derived by:
    (Given that the two matricides of alternate solutions are A and B, and d is any positive fraction number)
    (*a*) $A + (1 - d) \times B$ (*b*) $A\ (1 - d) + B$
    (*c*) $dA + dB$ (*d*) $dA + (1 - d) \times B$ ( )

ANSWERS

1. (*b*)	2. (*d*)	3. (*d*)	4. (*c*)
5. (*a*)	6. (*c*)	7. (*a*)	8. (*d*)
9. (*c*)	10. (*a*)	11. (*d*)	12. (*a*)
13. (*b*)	14. (*d*)	15. (*c*)	16. (*a*)
17. (*b*)	18. (*b*)	19. (*a*)	20. (*a*)

CHAPTER **– 5**

Linear Programming : III
Assignment Model

5.1. INTRODUCTION

In earlier discussion in chapter 3 and 4, we have dealt with two types of linear programming problems, *i.e*. Resource allocation method and Transportation model. We have seen that though we can use simplex method for solving transportation model, we go for transportation algorithm for simplicity. We have also discussed that how a resource allocation model differ from transportation model and similarities between them. Now we have another model comes under the class of linear programming model, which looks alike with transportation model with an objective function of minimizing the time or cost of manufacturing the products by allocating one job to one machine or one machine to one job or one destination to one origin or one origin to one destination only. This type of problem is given the name **ASSIGNMENT MODEL**. Basically assignment model is a minimization model. If we want to maximize the objective function, then there are two methods. One is to subtract all the elements of the matrix from the highest element in the matrix or to multiply the entire matrix by –1 and continue with the procedure. For solving the assignment problem we use Assignment technique or Hungarian method or Flood's technique. All are one and the same. Above, it is mentioned that one origin is to be assigned to one destination. This feature implies the existence of two specific characteristics in linear programming problems, which when present, give rise to an assignment problem. The first one being **the pay of matrix for a given problem is a square matrix** and the second is the optimum solution (or any solution with given constraints) for the problem is such that there can be **one and only one assignment in a given row or column of the given payoff matrix.** The transportation model is a special case of linear programming model (Resource allocation model) and assignment problem is a special case of transportation model, therefore it is also a special case of linear programming model. Hence it must have all the properties of linear programming model. That is it must have: (*i*) an objective function, (*ii*) it must have structural constraints, (*iii*) It must have non-negativity constraint and (*iv*) The relationship between variables and constraints must have linear relationship. In our future discussion, we will see that the assignment problem has all the above properties.

5.2. The Problem

There are some types in assignment problem. They are: