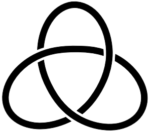
Dubrovnik Skein Theory and Power Sum Elements

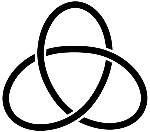
Alexander Pokorny
University of California, Riverside



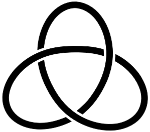
May 26, 2021

INTRODUCTION

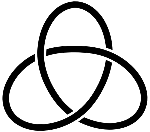
(FRAMED) LINK INVARIANTS



$$\stackrel{?}{=}$$



(Framed) Link Invariants



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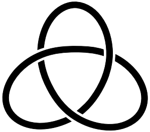
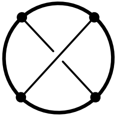
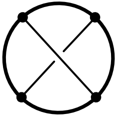
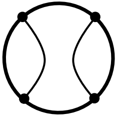
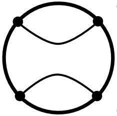
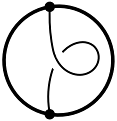
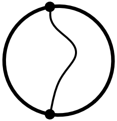


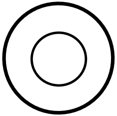
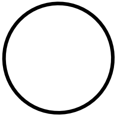
$\mapsto$ Dubrovnik Polynomial $\mapsto$

$$p_1(s, v) \neq p_2(s, v)$$

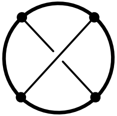
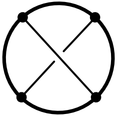
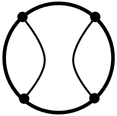
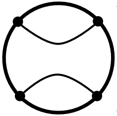
Dubrovnik Skein Relations

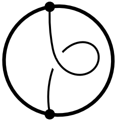
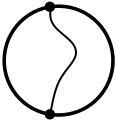
 $=$  $+ (s - s^{-1}) \Bigg($  $-$  $\Bigg)$ (1)

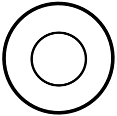
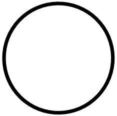
 $= v$  (2)

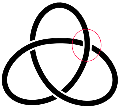
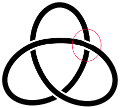
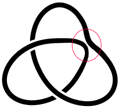
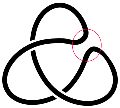
 $= \left(1 - \dfrac{v - v^{-1}}{s - s^{-1}}\right)$  (3)

DUBROVNIK SKEIN RELATIONS

 $=$  $+ (s - s^{-1}) \Bigg($  $-$  $\Bigg)$ (1)

 $= v$  (2)

 $= \left(1 - \dfrac{v - v^{-1}}{s - s^{-1}}\right)$  (3)

 $=$  $+ (s - s^{-1}) \Bigg($  $-$  $\Bigg)$

Skein Modules

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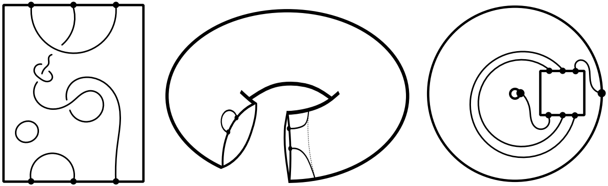
Consequence: May impose skein relations on tangles in arbitrary 3-dimensional manifolds.



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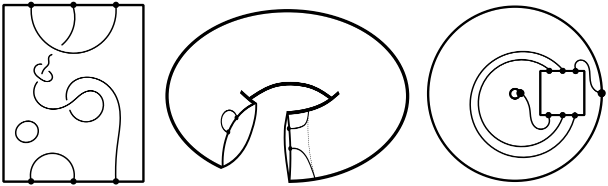


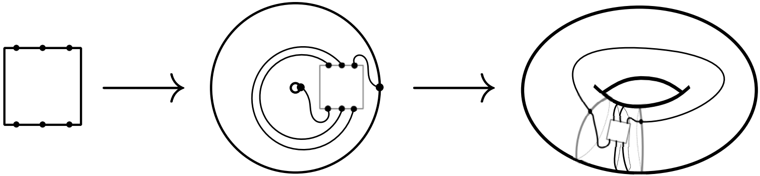
Definition

Let M be an oriented 3-manifold and $R := \mathbb{Q}(s, v)$.

$$\mathcal{D}(M, N) := R\{\text{Tangles in } M \text{ relative to } N\}/\sim$$

FUNCTORIALITY

$\quad$ A "nice" embedding $f : M \to M'$
$+$ A wiring diagram in image complement
$=$ A linear transformation $D(f) : \mathcal{D}(M, N) \to \mathcal{D}(M', N')$



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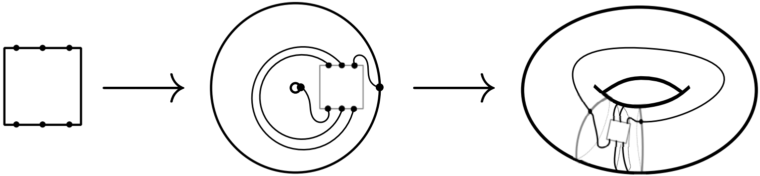

<u>Consequence:</u> Dubrovnik skein theory is a type of algebraic topology for smooth, oriented, 3-manifolds.

Special Case: Skein Algebras

If

- $M = \Sigma \times I$
- $N = (X \times \{0\}) \sqcup (X \times \{1\})$

Then $\mathcal{D}(M, N)$ is naturally an algebra.

SPECIAL CASE: SKEIN ALGEBRAS

If

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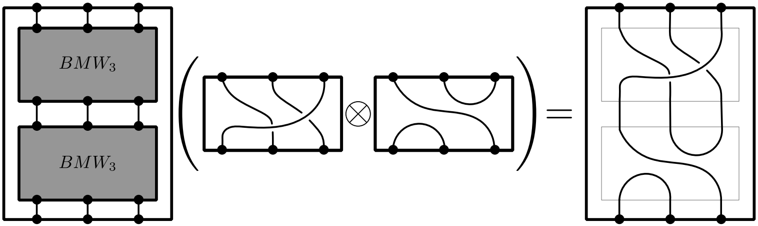
---

e.g.: $\Sigma =$ Square, $N = 2n$ points $\rightsquigarrow$ $\mathcal{D}(\Sigma, N) \cong BMW_n$



Idempotents in Birman-Murakami-Wenzl Algebras

Theorem (Ram-Wenzl 1992, Beliakova-Blanchet, 2001)

For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.

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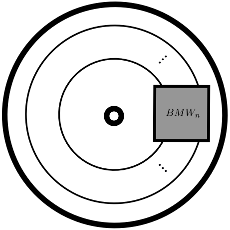

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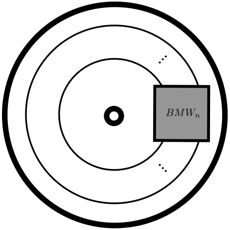


$\rightsquigarrow \mathrm{cl} : BMW_n \to \mathcal{D}(A)$

Theorem (Lu-Zhong 2002)

The elements $\widetilde{Q}_\lambda := \mathrm{cl}(\tilde{y}_\lambda)$ form a basis of $\mathcal{D}(A)$.

DUBROVNIK POWER SUM ELEMENTS

Define a family of elements $\widetilde{P}_k \in \mathcal{D}(A)$ for $k \in \mathbb{Z}_{\geq 1}$ via

$$\sum_{k \geq 1} \frac{\widetilde{P}_k}{k} t^k = \ln\left(1 + \sum_{n \geq 1} \widetilde{Q}_{(n)} t^n\right)$$

<u>Idea:</u>

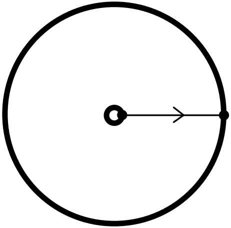
$\widetilde{Q}_{(n)}$ are "like" complete homogeneous symmetric functions.

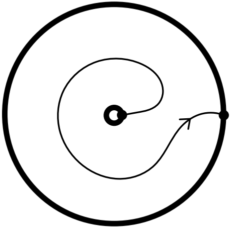
$\rightsquigarrow \widetilde{P}_k$ are "like" power sum symmetric functions.

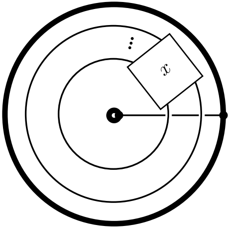
(Will make more precise later)

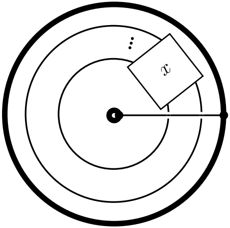
A Relative Skein Algebra

Let $\mathcal{A} := \mathcal{D}(A, 1)$.

$e =$ 

$a =$ 

$\mathrm{cl}(x) \cdot e =$ 

$e \cdot \mathrm{cl}(x) =$ 

Commutation Relations

Theorem (Morton-P.-Samuelson)

$$e \cdot \widetilde{P}_k - \widetilde{P}_k \cdot e = (s^k - s^{-k})(a^k - a^{-k})$$

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Theorem (P.)

$$e \cdot \widetilde{Q}_{(n)} - \widetilde{Q}_{(n)} \cdot e = \sum_{i=1}^{n} d_i \left(e \cdot \widetilde{Q}_{(n-i)}\right)$$

where

$$d_i = \sum_{l=0}^{i-1} (s^2 - 1)s^{2l-i}a^{i-2l} + (s^{-2} - 1)s^{i-2l}a^{2l-i}$$

IDEA OF PROOF

1) By power series manipulations, the statement is equivalent to

$$e \cdot \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) - \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) \cdot e$$

$$=$$

$$(sa + s^{-1}a^{-1})\left(e \cdot \widetilde{Q}_{(n+1)}\right) - (s^{-1}a + sa^{-1})\left(\widetilde{Q}_{(n+1)} \cdot e\right)$$

Idea of Proof

1) By power series manipulations, the statement is equivalent to

$$e \cdot \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) - \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) \cdot e$$
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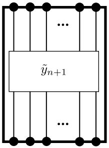
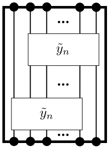
$$(sa + s^{-1}a^{-1})\left(e \cdot \widetilde{Q}_{(n+1)}\right) - (s^{-1}a + sa^{-1})\left(\widetilde{Q}_{(n+1)} \cdot e\right)$$

2) [Shelly, 2016] The $\widetilde{y}_{(n)}$ satisfy a skein-theoretic recurrence relation.

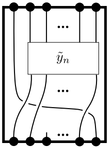
$[n+1]$



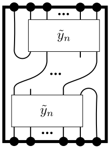
$= [n]s^{-1}$



$+$

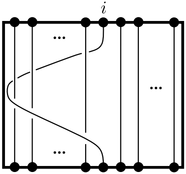


$+ [n]s^{-1}\beta_n$



Application: Central Elements of BMW_n

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of BMW_n.

$M_{n,i} =$  $2 \leq i \leq n$

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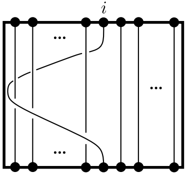

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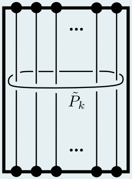


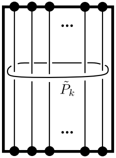
$$= \langle \widetilde{P}_k \rangle 1 + (s^k - s^{-k}) \sum_{i=2}^{n} v^{-k} M_{n,i}^{k} - v^{k} M_{n,i}^{-k}$$

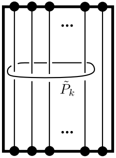
Application: Central Elements of BMW_n

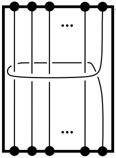
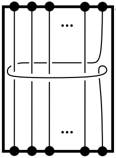
Sketch: Apply commutation relation for $\widetilde{P}_k$.

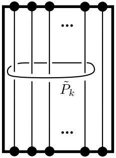
Application: Central Elements of BMW_n

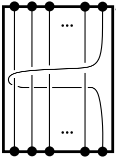
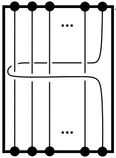
<u>Sketch:</u> Apply commutation relation for $\widetilde{P}_k$.



$=$ 

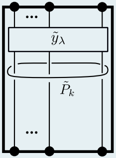
$+ (s^k - s^{-k})$  ^{k} $- (s^k - s^{-k})$  ^{k}

$=$ 

$+ (s^k - s^{-k})v^{-k}$  ^{k} $- (s^k - s^{-k})v^{k}$  ^{k}

Application: Meridians of $\widetilde{y}_\lambda$

Theorem (P.)



$$= \left(\langle \widetilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left(v^{-k} s^{2\mathrm{cn}(\square)} - v^k s^{-2\mathrm{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

APPLICATION: MERIDIANS OF $\widetilde{y}_\lambda$

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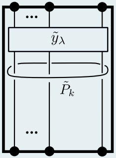

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<u>Observation:</u> For any fixed k, the eigenvalues of $\tilde{y}_\lambda$ are distinct.

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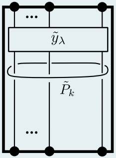

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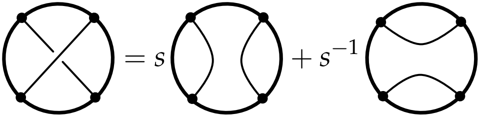
Observation: For any fixed k, the eigenvalues of $\tilde{y}_\lambda$ are distinct.
Consequence: The basis $\{\widetilde{Q}_\lambda\}$ of $\mathcal{D}(A)$ is an eigenbasis with 1-dimensional eigenspaces. Setting $k = 1$ recovers the result from [Lu-Zhong, 2002].

Compatibility With Kauffman Bracket Skein Theory $\mathcal{K}$

The Kauffman bracket skein relation $\rightsquigarrow$ Jones polynomial



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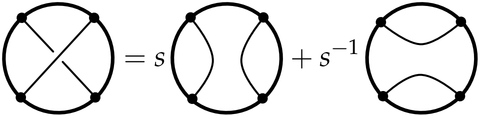

<u>Fact:</u> The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

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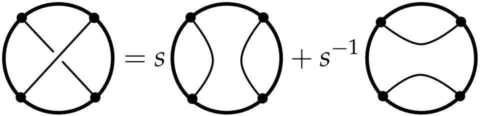

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.
Consequence: There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

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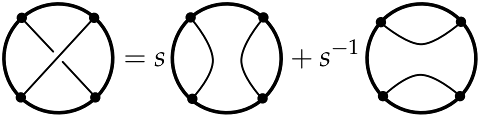

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Theorem (Morton-P.-Samuelson)

The image of $\widetilde{P}_k \in \mathcal{D}(A)$ under η_A is the Chebyshev polynomial $T_k \in \mathcal{K}(A)$.

SKEIN ALGEBRAS OF T^2

Theorem (Frohman-Gelca, 2000)

The algebra $\mathcal{K}(T^2)$ is presented by generators $T_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$T_{\mathbf{x}} T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}-\mathbf{y}}$$

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Theorem (Morton-Samuelson, 2017)

The algebra $\mathcal{H}(T^2)$ is presented by generators $P_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$ subject to the relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}+\mathbf{y}}$$

A Presentation of $\mathcal{D}(T^2)$

Let $\mathbf{x} = (a, b), k = \gcd(\mathbf{x})$.
Define $\widetilde{P}_{\mathbf{x}} \in \mathcal{D}(T^2)$ be the embedding of $\widetilde{P}_k$ along the closed curve of slope a/b.

Theorem (Morton-P.-Samuelson)

The algebra $\mathcal{D}(T^2)$ is presented by generators $\widetilde{\mathrm{P}}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x}\rangle$ subject to the relations

$$[\widetilde{P}_{\mathbf{x}}, \widetilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\widetilde{P}_{\mathbf{x}+\mathbf{y}} - \widetilde{P}_{\mathbf{x}-\mathbf{y}})$$

A PRESENTATION OF $\mathcal{D}(T^2)$

Proof:

1. $[\widetilde{P}_{\mathbf{x}}, \widetilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\widetilde{P}_{\mathbf{x}+\mathbf{y}} - \widetilde{P}_{\mathbf{x}-\mathbf{y}})$.
   1.1 $[\widetilde{P}_{1,0}, \widetilde{P}_{0,n}] = (s^n - s^{-n})(\widetilde{P}_{1,n} - \widetilde{P}_{1,-n})$
   1.2 $[\widetilde{P}_{1,0}, \widetilde{P}_{1,n}] = (s^n - s^{-n})(\widetilde{P}_{2,n} - \widetilde{P}_{0,n})$
   1.3 Use $SL_2(\mathbb{Z})$-action on $\mathcal{D}(T^2)$. Induct on $\gcd(\mathbf{x})$.
2. The $\widetilde{P}_{\mathbf{x}}$ generate $\mathcal{D}(T^2)$.
   2.1 Each link in $\mathcal{D}(T^2)$ is a sum of products of annular knots.
   2.2 The $\widetilde{P}_k$ generate $\mathcal{D}(A)$.
3. The $\widetilde{P}_{\mathbf{x}}$ are linearly independent.
   3.1 Diamond lemma type argument.

RELATIONSHIP WITH $\mathcal{K}(T^2)$

Corollary (Morton-P.-Samuelson)

There is a surjective algebra homomorphism $\mathcal{D}(T^2) \to \mathcal{K}(T^2)$ defined by

$$\widetilde{P}_{\mathbf{x}} \mapsto T_{\mathbf{x}}.$$

Proof: Use the natural transformation $\eta : \mathcal{D} \Rightarrow \mathcal{K}$. Recall

$$\eta_A(\widetilde{P}_k) = T_k.$$

Note: $\mathcal{D}(T^2)$ is much bigger than $\mathcal{K}(T^2)$.

Relationship with $\mathcal{H}(T^2)$

Corollary (P.)

The algebras $\mathcal{D}(T^2)$ and $\mathcal{H}(T^2)$ are universal enveloping algebras of some Lie algebras $\mathfrak{g}_{\mathcal{D}}$ and $\mathfrak{g}_{\mathcal{H}}$ generated by the $\widetilde{P}_{\mathbf{x}}$ and $T_{\mathbf{x}}$, respectively. There is an injective Lie algebra homomorphism $\mathfrak{g}_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{H}}$ defined by

$$\widetilde{P}_{\mathbf{x}} \mapsto P_{\mathbf{x}} + P_{-\mathbf{x}}.$$

Proof: $[\widetilde{P}_{\mathbf{x}}, \widetilde{P}_{\mathbf{y}}] - (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\widetilde{P}_{\mathbf{x}+\mathbf{y}} - \widetilde{P}_{\mathbf{x}-\mathbf{y}})$ is sent to 0.
Note: This restricts to an algebra homomorphism

$$\mathcal{D}(A) \to \mathcal{H}(A).$$

Note: $\mathcal{D}(A)$ and $\mathcal{H}(A)$ are related to characters of classical Lie groups.

A $\mathcal{D}(T^2)$-ACTION ON $\mathcal{D}(D^2 \times S^1)$

Lemma (P.)

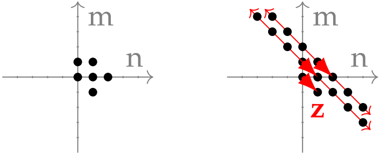
$\mathcal{D}(T^2)$ is generated by the identity and the elements

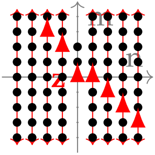
$$\widetilde{P}_{1,0}, \widetilde{P}_{0,1}, \widetilde{P}_{1,1}, \widetilde{P}_{2,0}$$

Proof: Can generate a "line" of $\widetilde{P}_{\mathbf{x}}$ by

$$\widetilde{P}_{\mathbf{y}+n\mathbf{z}} = (s^d - s^{-d})^{-1}\left([\widetilde{P}_{\mathbf{y}+(n-1)\mathbf{z}}, \widetilde{P}_{\mathbf{z}}] + \{d\}\widetilde{P}_{\mathbf{y}+(n-2)\mathbf{z}}\right)$$

as long as $d := \det(\mathbf{y}+n\mathbf{z}, \mathbf{z}) = \det(\mathbf{y}, \mathbf{z}) \neq 0$.





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A $\mathcal{D}(T^2)$-Action on $\mathcal{D}(D^2 \times S^1)$

Theorem (P.)

The $\mathcal{D}(T^2)$-action on $\mathcal{D}(D^2 \times S^1)$ is determined by the equations

$$\widetilde{P}_{1,0} \cdot \widetilde{Q}_\lambda = \left(\langle \widetilde{P}_1 \rangle + \{1\} \left(v^{-1} \sum_{\square \in \lambda} s^{2\mathrm{cn}(\square)} - v \sum_{\square \in \lambda} s^{-2\mathrm{cn}(\square)} \right) \right) \widetilde{Q}_\lambda$$

$$\widetilde{P}_{2,0} \cdot \widetilde{Q}_\lambda = \left(\langle \widetilde{P}_2 \rangle + \{2\} \left(v^{-2} \sum_{\square \in \lambda} s^{4\mathrm{cn}(\square)} - v^2 \sum_{\square \in \lambda} s^{-4\mathrm{cn}(\square)} \right) \right) \widetilde{Q}_\lambda$$

$$\widetilde{P}_{0,1} \cdot \widetilde{Q}_\lambda = \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} \widetilde{Q}_\mu + \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} \widetilde{Q}_\nu$$

$$\widetilde{P}_{1,1} \cdot \widetilde{Q}_\lambda = v^{-1} \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} s^{2\mathrm{cn}(\square)} \widetilde{Q}_\mu + v \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} s^{-2\mathrm{cn}(\square)} \widetilde{Q}_\nu$$

SYMMETRIC FUNCTIONS AND $\mathcal{H}(A)^+$

Frame 1

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