# Dubrovnik Skein Theory and Power Sum Elements

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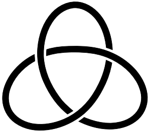
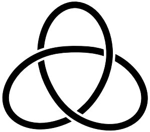
# INTRODUCTION

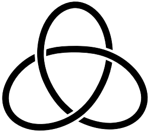
# (FRAMED) LINK INVARIANTS



$\stackrel{?}{=}$



# (FRAMED) LINK INVARIANTS



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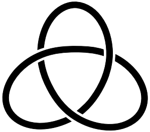
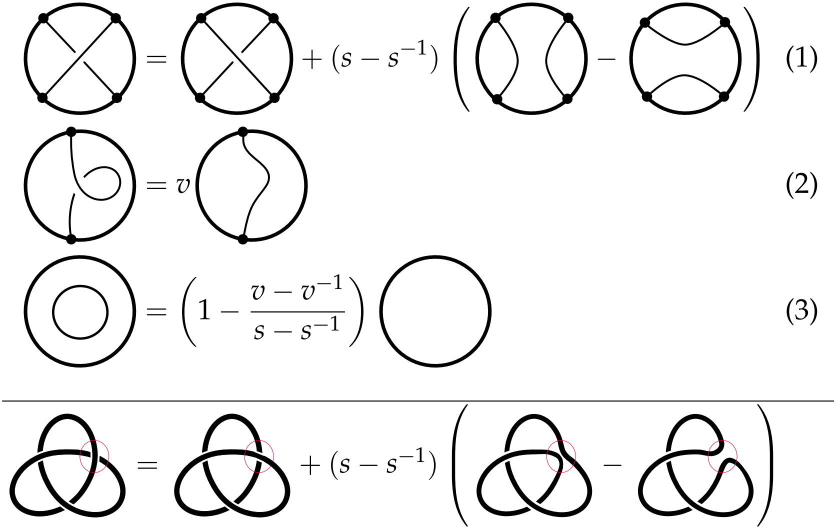


$$\downarrow \qquad \text{Dubrovnik Polynomial} \qquad \downarrow$$

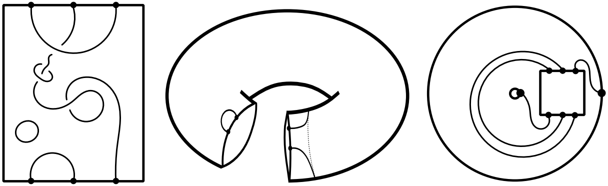
$$p_1(s,v) \neq p_2(s,v)$$

# Kauffman (Dubrovnik) Skein Relations



# Skein Modules

Observation: Skein relations are defined locally.
Consequence: May impose skein relations on tangles in arbitrary 3-dimensional manifolds.



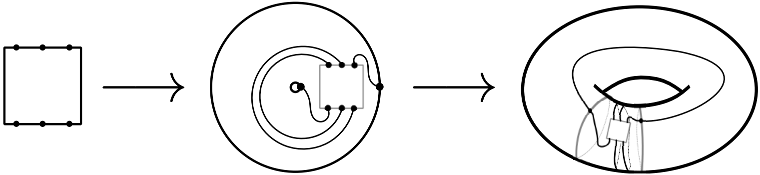
## Definition

Let $M$ be an oriented 3-manifold, $N \subset \partial M$ finite, and $R := \mathbb{Q}(s, v)$.

$$\mathcal{D}(M, N) := R\{\text{Tangles in } M \text{ relative to } N\}/\sim$$

# FUNCTORIALITY

A “nice" embedding $f : (M, N) \to (M', N')$
$+$ A wiring diagram in image complement
$=$ A linear transformation $\mathcal{D}(f) : \mathcal{D}(M, N) \to \mathcal{D}(M', N')$



Consequence: Dubrovnik skein theory is a type of algebraic topology for smooth, oriented, 3-manifolds.
Hope: Develop this theory by finding algebraic presentations of nice examples of modules D(M,N) and how they relate to each other.

# Special Case: Skein Algebras

If

- $M = \Sigma \times I$
- $N = (X \times \{0\}) \sqcup (X \times \{1\})$

Then $\mathcal{D}(M, N)$ is naturally an algebra.

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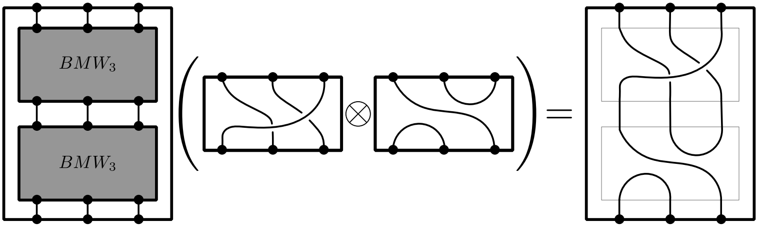
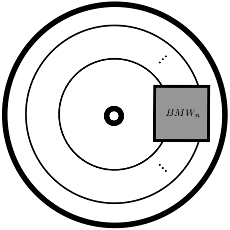
e.g.: $\Sigma =$ Square, $N = 2n$ points $\rightsquigarrow \mathcal{D}(\Sigma, N) \cong BMW_n$



# Idempotents in Birman-Murakami-Wenzl Algebras

**Theorem (Ram-Wenzl 1992, Beliakova-Blanchet 2001)**

*For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.*




$\rightsquigarrow \mathrm{cl} : BMW_n \to \mathcal{D}(A)$

**Theorem (Lu-Zhong 2002)**

*The elements $\widetilde{Q}_\lambda := \mathrm{cl}(\tilde{y}_\lambda)$ form a basis of $\mathcal{D}(A)$.*

# DUBROVNIK POWER SUM ELEMENTS

Define a family of elements $\widetilde{P}_k \in \mathcal{D}(A)$ for $k \in \mathbb{Z}_{\geq 1}$ via

$$\sum_{k\geq 1} \frac{\widetilde{P}_k}{k} t^k = \ln\left(1 + \sum_{n\geq 1} \widetilde{Q}_{(n)} t^n\right)$$

Idea:

$\widetilde{Q}_{(n)}$ are "like" complete homogeneous symmetric functions.

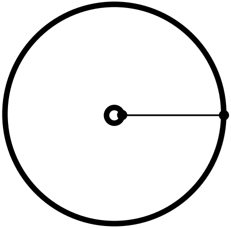
$\rightsquigarrow \widetilde{P}_k$ are "like" power sum symmetric functions.

(Will make more precise later)

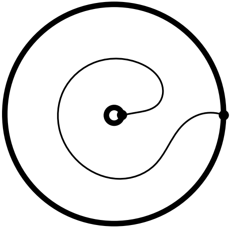
# A RELATIVE SKEIN ALGEBRA

Let $\mathcal{A} := \mathcal{D}(A, 1)$.

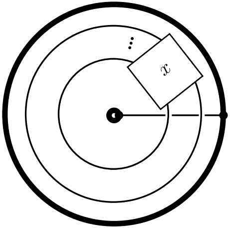
$e =$



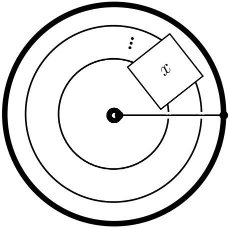
$a =$



$\mathrm{cl}(x) \cdot e =$




$e \cdot \mathrm{cl}(x) =$



# Commutation Relations

**Theorem (Morton-P.-Samuelson)**

$$e \cdot \widetilde{P}_k = \widetilde{P}_k \cdot e + (s^k - s^{-k})(a^k - a^{-k})$$

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$$e \cdot \widetilde{P}_k = \widetilde{P}_k \cdot e + (s^k - s^{-k})(a^k - a^{-k})$$

**Theorem (P.)**

$$e \cdot \widetilde{Q}_{(n)} = \widetilde{Q}_{(n)} \cdot e + \sum_{i=1}^{n} d_i \big(\widetilde{Q}_{(n-i)} \cdot e\big)$$

*where*

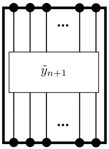
$$d_i = \sum_{l=0}^{i-1} (1 - s^{-2}) s^{i-2l} a^{i-2l} + (1 - s^2) s^{2l-i} a^{2l-i}$$

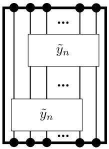
# IDEA OF PROOF

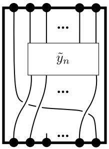
1) By power series manipulations, the statement is equivalent to

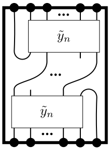
$$e \cdot \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) - \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) \cdot e$$
$$=$$
$$(sa + s^{-1}a^{-1})\left(e \cdot \widetilde{Q}_{(n+1)}\right) - (s^{-1}a + sa^{-1})\left(\widetilde{Q}_{(n+1)} \cdot e\right)$$

2) [Shelly, 2016] The $\widetilde{y}_{(n)}$ satisfy a skein-theoretic recurrence relation.

$[n+1]$ 

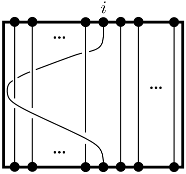
$= [n]s^{-1}$ 

$+$ 

$+ [n]s^{-1}\beta_n$ 

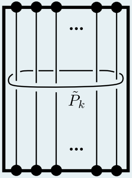
# Application: Central Elements of $BMW_n$

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of $BMW_n$.

$M_{n,i} =$ 

$2 \leq i \leq n$

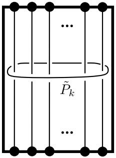
## Proposition (P.)

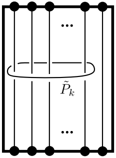
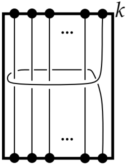
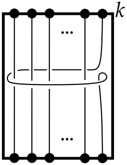


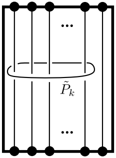
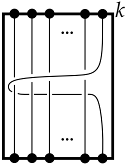
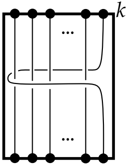

$$= \langle \widetilde{P}_k \rangle 1 + (s^k - s^{-k}) \sum_{i=2}^{n} v^{-k} M_{n,i}^{k} - v^{k} M_{n,i}^{-k}$$

# Application: Central Elements of $BMW_n$

<u>Sketch:</u> Apply commutation relation for $\widetilde{P}_k$.

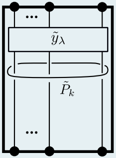


$=$  $+ (s^k - s^{-k})$  $- (s^k - s^{-k})$ 

$=$  $+ (s^k - s^{-k})v^{-k}$  $- (s^k - s^{-k})v^{k}$ 

# APPLICATION: MERIDIANS OF $\widetilde{y}_\lambda$

## Proposition (P.)

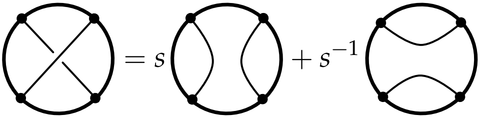



$$= \left( \langle \widetilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left( v^{-k} s^{2\mathrm{cn}(\square)} - v^k s^{-2\mathrm{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

Observation: For any fixed $k$, the eigenvalues of $\tilde{y}_\lambda$ are distinct.
Consequence: The basis $\{\widetilde{Q}_\lambda\}$ of $\mathcal{D}(A)$ is a simultaneous eigenbasis with 1-dimensional eigenspaces for all $\widetilde{P}_k$. Setting $k = 1$ recovers the result from [Lu-Zhong, 2002].

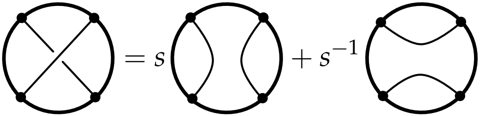
# Compatibility With Kauffman Bracket Skein Theory $\mathcal{K}$

The Kauffman bracket skein relation $\rightsquigarrow$ Jones polynomial



# COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY $\mathcal{K}$

The Kauffman bracket skein relation $\rightsquigarrow$ Jones polynomial




Fact: The Kauffman bracket skein relation implies the Dubrovnik skein relation.
Consequence: There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

**Theorem (Morton-P.-Samuelson)**

*The image of $\widetilde{P}_k \in \mathcal{D}(A)$ under $\eta_A$ is the Chebyshev polynomial $T_k \in \mathcal{K}(A)$.*

# SKEIN ALGEBRAS OF $T^2$

**Theorem (Frohman-Gelca 2000)**

*The algebra $\mathcal{K}(T^2)$ is presented by generators $T_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations*

$$T_{\mathbf{x}} T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}-\mathbf{y}}$$

**Theorem (Morton-Samuelson 2017)**

*The algebra $\mathcal{H}(T^2)$ is presented by generators $P_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$ subject to the relations*

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})}) P_{\mathbf{x}+\mathbf{y}}$$

# A PRESENTATION OF $\mathcal{D}(T^2)$

Let $\mathbf{x} = (a, b), k = \gcd(\mathbf{x})$.
Define $\widetilde{P}_{\mathbf{x}} \in \mathcal{D}(T^2)$ be the embedding of $\widetilde{P}_k$ along the closed curve of slope $a/b$.

**Theorem (Morton-P.-Samuelson)**

*The algebra $\mathcal{D}(T^2)$ is presented by generators $\widetilde{P}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x}\rangle$ subject to the relations*

$$[\widetilde{P}_{\mathbf{x}}, \widetilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\widetilde{P}_{\mathbf{x}+\mathbf{y}} - \widetilde{P}_{\mathbf{x}-\mathbf{y}})$$

# MAIN THEOREM: A PRESENTATION OF $\mathcal{D}(T^2)$

Proof:

1. The $\widetilde{P}_{\mathbf{x}}$ generate $\mathcal{D}(T^2)$.
    1.1 The $\widetilde{P}_k$ generate $\mathcal{D}(A)$.
    1.2 Each link in $\mathcal{D}(T^2)$ is a sum of products of annular knots.
2. $[\widetilde{P}_{\mathbf{x}}, \widetilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\widetilde{P}_{\mathbf{x}+\mathbf{y}} - \widetilde{P}_{\mathbf{x}-\mathbf{y}})$.
    2.1 $[\widetilde{P}_{1,0}, \widetilde{P}_{0,n}] = (s^n - s^{-n})(\widetilde{P}_{1,n} - \widetilde{P}_{1,-n})$
    2.2 $[\widetilde{P}_{1,0}, \widetilde{P}_{1,n}] = (s^n - s^{-n})(\widetilde{P}_{2,n} - \widetilde{P}_{0,n})$
    2.3 Use $SL_2(\mathbb{Z})$-action on $\mathcal{D}(T^2)$. Induct on $|\det(\mathbf{x}, \mathbf{y})|$.
3. These relations present the algebra.
    3.1 Diamond lemma type argument $\rightsquigarrow$ unordered words in $\widetilde{P}_{\mathbf{x}}$ form a basis.
    3.2 The above relations allow reordering of words.

# Relationship With $\mathcal{K}(T^2)$

**Corollary (Morton-P.-Samuelson)**

*There is a surjective algebra homomorphism $\mathcal{D}(T^2) \to \mathcal{K}(T^2)$ defined by*

$$\widetilde{P}_{\mathbf{x}} \mapsto T_{\mathbf{x}}.$$

Proof: Use the natural transformation $\eta : \mathcal{D} \Rightarrow \mathcal{K}$. Recall

$$\eta_A(\widetilde{P}_k) = T_k.$$

Note: $\mathcal{D}(T^2)$ is much bigger than $\mathcal{K}(T^2)$.

# Relationship with $\mathcal{H}(T^2)$

## Corollary (P.)

*The algebras $\mathcal{D}(T^2)$ and $\mathcal{H}(T^2)$ are universal enveloping algebras of some Lie algebras $\mathfrak{g}_{\mathcal{D}}$ and $\mathfrak{g}_{\mathcal{H}}$ spanned by the $\widetilde{P}_{\mathbf{x}}$ and $P_{\mathbf{x}}$, respectively. There is an injective Lie algebra homomorphism $\mathfrak{g}_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{H}}$ defined by*

$$\widetilde{P}_{\mathbf{x}} \mapsto P_{\mathbf{x}} + P_{-\mathbf{x}}.$$

<u>Proof:</u> $[\widetilde{P}_{\mathbf{x}}, \widetilde{P}_{\mathbf{y}}] - (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\widetilde{P}_{\mathbf{x}+\mathbf{y}} - \widetilde{P}_{\mathbf{x}-\mathbf{y}})$ is sent to 0.
<u>Note:</u> This restricts to an algebra homomorphism

$$\mathcal{D}(A) \to \mathcal{H}(A).$$

<u>Note:</u> $\mathcal{D}(A)$ and $\mathcal{H}(A)$ are related to characters of classical Lie groups.

# A $\mathcal{D}(T^2)$-ACTION ON $\mathcal{D}(D^2 \times S^1)$

## Lemma (P.)

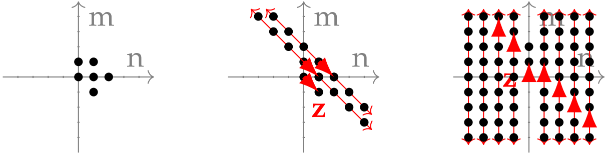
*$\mathcal{D}(T^2)$ is generated by the identity and the elements*

$$\widetilde{P}_{1,0}, \widetilde{P}_{0,1}, \widetilde{P}_{1,1}, \widetilde{P}_{2,0}$$

<u>Proof:</u> Can generate a "line" of $\widetilde{P}_{\mathbf{x}}$ by

$$\widetilde{P}_{\mathbf{y}+n\mathbf{z}} = (s^d - s^{-d})^{-1}\left([\widetilde{P}_{\mathbf{y}+(n-1)\mathbf{z}}, \widetilde{P}_{\mathbf{z}}] + \{d\}\widetilde{P}_{\mathbf{y}+(n-2)\mathbf{z}}\right)$$

as long as $d := \det(\mathbf{y} + n\mathbf{z}, \mathbf{z}) = \det(\mathbf{y}, \mathbf{z}) \neq 0$.

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# A $\mathcal{D}(T^2)$-ACTION ON $\mathcal{D}(D^2 \times S^1)$

## Proposition (P.)

*The $\mathcal{D}(T^2)$-action on $\mathcal{D}(D^2 \times S^1)$ is determined by the equations*

$$\widetilde{P}_{1,0} \cdot \widetilde{Q}_\lambda = \left( \langle \widetilde{P}_1 \rangle + \{1\} \left( v^{-1} \sum_{\square \in \lambda} s^{2\mathrm{cn}(\square)} - v \sum_{\square \in \lambda} s^{-2\mathrm{cn}(\square)} \right) \right) \widetilde{Q}_\lambda$$

$$\widetilde{P}_{2,0} \cdot \widetilde{Q}_\lambda = \left( \langle \widetilde{P}_2 \rangle + \{2\} \left( v^{-2} \sum_{\square \in \lambda} s^{4\mathrm{cn}(\square)} - v^2 \sum_{\square \in \lambda} s^{-4\mathrm{cn}(\square)} \right) \right) \widetilde{Q}_\lambda$$

$$\widetilde{P}_{0,1} \cdot \widetilde{Q}_\lambda = \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} \widetilde{Q}_\mu + \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} \widetilde{Q}_\nu$$

$$\widetilde{P}_{1,1} \cdot \widetilde{Q}_\lambda = v^{-1} \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} s^{2\mathrm{cn}(\square)} \widetilde{Q}_\mu + v \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} s^{-2\mathrm{cn}(\square)} \widetilde{Q}_\nu$$

# SYMMETRIC FUNCTIONS AND $\mathcal{H}(A)^+$

$\Lambda$ - The ring of symmetric functions
$s_\lambda$ - Schur function
$\mathcal{H}(A)^+$ - Annular HOMFLYPT links oriented counter-clockwise

**Theorem (Lukac 2005)**

*There is a basis of elements $Q_\lambda$ in $\mathcal{H}(A)^+$ indexed by partitions $\lambda$. The assignment $\Lambda \to \mathcal{H}(A)^+$ defined by*

$$s_\lambda \mapsto Q_\lambda$$

*is an algebra isomorphism.*

Hope: Generalize this to Dubrovnik case.

# Orthogonal & Symplectic Schur Functions

$$\text{Partition } \lambda \quad \longrightarrow \quad \text{Symmetric functions } sb_\lambda, sc_\lambda$$

**Theorem (Koike-Terada 1987, Koike 1989)**

*The sets $\{sb_\lambda\}$ and $\{sc_\lambda\}$ are bases of $\Lambda$.*
*The structure constants of $\Lambda$ with respect to each basis are identical natural numbers.*

**Conjecture**

*Both of the assignments $\Lambda \to \mathcal{D}(A)$ defined by*

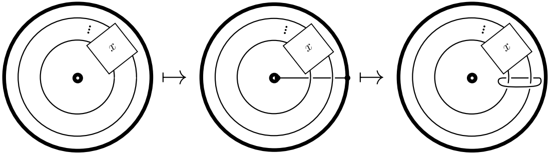
$$sb_{(n)} \mapsto \widetilde{Q}_{(n)} \qquad sc_{(n)} \mapsto \widetilde{Q}_{(n)}$$

*are algebra isomorphisms which send $sb_\lambda$ and $sc_\lambda$ to $\widetilde{Q}_\lambda$, respectively.*

# PARTIAL PROOF OF CONJECTURE

Idea: Let $S_\lambda$ be the image of $sc_\lambda$. Show $S_\lambda = \widetilde{Q}_\lambda$.

1. Show $S_\lambda$ is in the same eigenspace of $\widetilde{Q}_\lambda$ with respect to the meridian map $\Gamma$.
   1.1 Use a Jacobi-Trudi type identity to write $S_\lambda$ as a determinant whose entries are in terms of $S_{(n)} = \widetilde{Q}_{(n)}$.
   1.2 Factor $\Gamma$ as $\Gamma(x) = \mathrm{cl}_{\mathcal{A}}(x \cdot e)$:




   1.3 $\mathrm{cl}_{\mathcal{A}}(e \cdot x) = \delta x$. Try to translate left-action to right-action (works if $|\lambda| \leq 2$).
   1.4 Compute eigenvalue.
2. Eigenspaces are 1-dimensional $\Rightarrow S_\lambda = dQ_\lambda$. Show $d = 1$ with branching rules: $S^n_\square = \sum c_\mu S_\mu$ and $\widetilde{Q}^n_\square = \sum c_\mu \widetilde{Q}_\mu$

Thanks for listening!