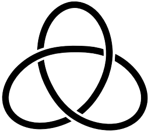
Dubrovnik Skein Theory and Power Sum Elements

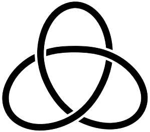
Alexander Pokorny
University of California, Riverside



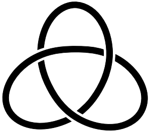
May 24, 2021

INTRODUCTION

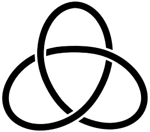
(FRAMED) LINK INVARIANTS



$$\stackrel{?}{=}$$



(Framed) Link Invariants



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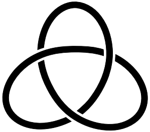
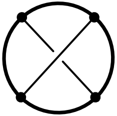
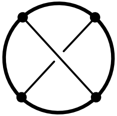
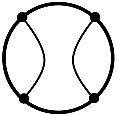
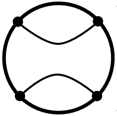
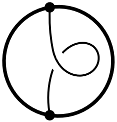
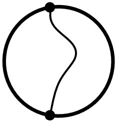


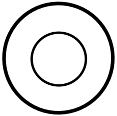
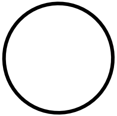
Dubrovnik Polynomial

$$p_1(s,v) \neq p_2(s,v)$$

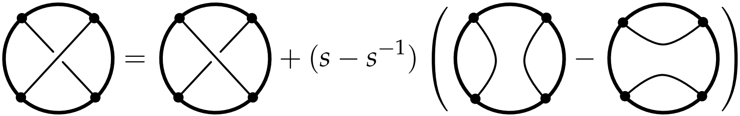
DUBROVNIK SKEIN RELATIONS

 $= $  $+ (s - s^{-1}) \Bigg($  $-$  $\Bigg)$ (1)

 $= v$  (2)

 $= \left(1 - \dfrac{v - v^{-1}}{s - s^{-1}}\right)$  (3)

Dubrovnik Skein Relations



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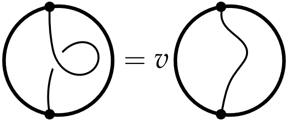

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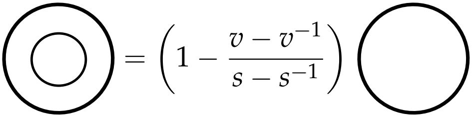

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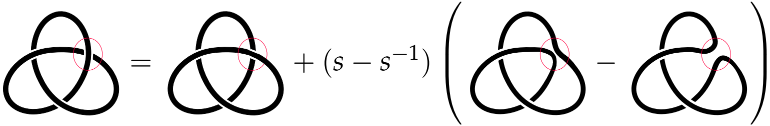

SKEIN MODULES

Observation: Skein relations are defined locally.

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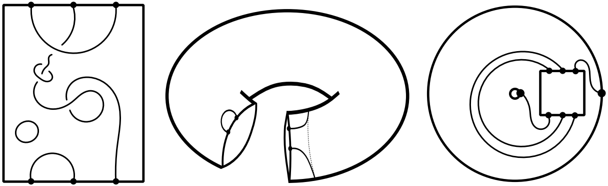
Consequence: May impose skein relations on tangles in arbitrary 3-dimensional manifolds.



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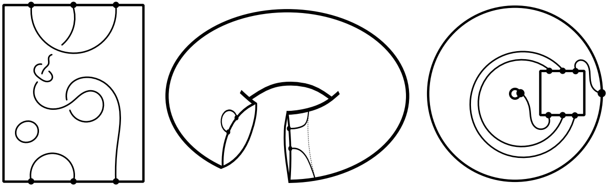


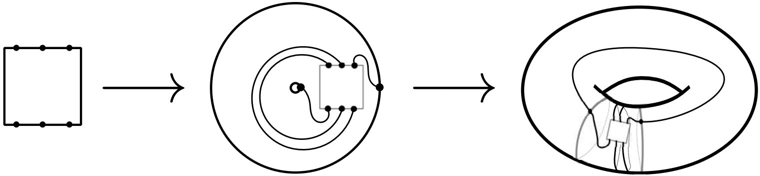
Definition

Let M be an oriented 3-manifold and $R := \mathbb{Q}(s, v)$.

$$\mathcal{D}(M, N) := R\{\text{Tangles in } M \text{ relative to } N\} / \sim$$

FUNCTORIALITY

A "nice" embedding $f : M \to M'$
$+$ A wiring diagram in image complement
$=$ A linear transformation $D(f) : \mathcal{D}(M, N) \to \mathcal{D}(M', N')$



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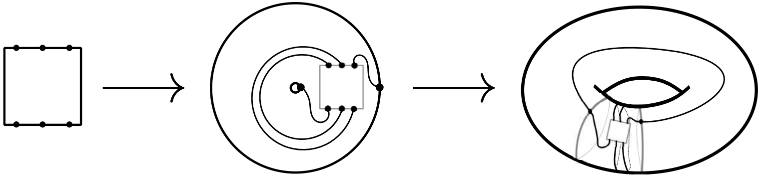

Consequence: Dubrovnik skein theory is a type of algebraic topology for smooth, oriented, 3-manifolds.

SPECIAL CASE: SKEIN ALGEBRAS

If

- $M = \Sigma \times I$
- $N = (X \times \{0\}) \sqcup (X \times \{1\})$

Then $\mathcal{D}(M, N)$ is naturally an algebra.

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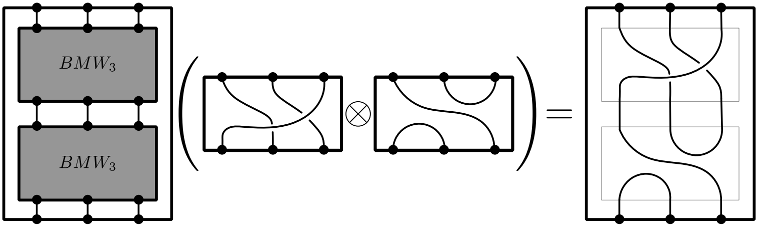
---

e.g.: $\Sigma =$ Square, $N = 2n$ points $\rightsquigarrow$ $\mathcal{D}(\Sigma, N) \cong BMW_n$



Idempotents in Birman-Murakami-Wenzl Algebras

Theorem (Ram-Wenzl 1992, Beliakova-Blanchet, 2001)

For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.

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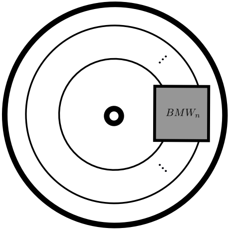

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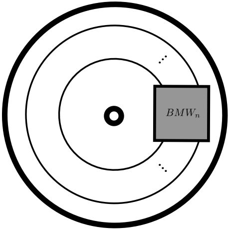


$$\rightsquigarrow \mathrm{cl} : BMW_n \to \mathcal{D}(A)$$

Theorem (Lu-Zhong 2002)

The elements $\widetilde{Q}_\lambda := \mathrm{cl}(\tilde{y}_\lambda)$ form a basis of $\mathcal{D}(A)$.

DUBROVNIK POWER SUM ELEMENTS

Define a family of elements $\widetilde{P}_k \in \mathcal{D}(A)$ for $k \in \mathbb{Z}_{\geq 1}$ via

$$\sum_{k\geq 1} \frac{\widetilde{P}_k}{k} t^k = \ln \left(1 + \sum_{n \geq 1} \widetilde{Q}_{(n)} t^n \right)$$

<u>Idea:</u>

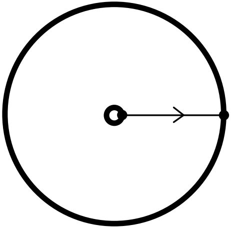
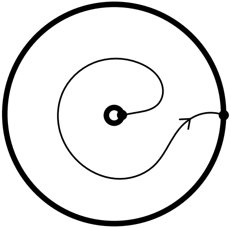
$\widetilde{Q}_{(n)}$ are "like" complete homogeneous symmetric functions.

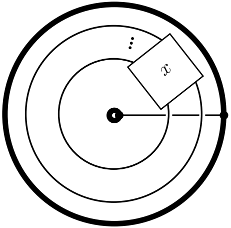
$\rightsquigarrow \widetilde{P}_k$ are "like" power sum symmetric functions.

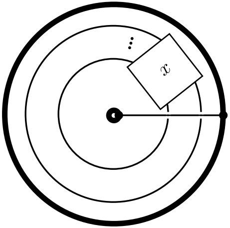
(Will make more precise later)

A RELATIVE SKEIN ALGEBRA

Let $\mathcal{A} := \mathcal{D}(A, 1)$.

$e =$  $a =$ 

$\mathrm{cl}(x) \cdot e =$ 

$e \cdot \mathrm{cl}(x) =$ 

COMMUTATION RELATIONS

Theorem (Morton-P.-Samuelson)

$$e \cdot \widetilde{P}_k - \widetilde{P}_k = (s^k - s^{-k})(a^k - a^{-k})$$

Commutation Relations

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Theorem (P.)

$$e \cdot \widetilde{Q}_{(n)} - \widetilde{Q}_{(n)} \cdot e = \sum_{i=1}^{n} d_i \big(e \cdot \widetilde{Q}_{(n-i)}\big)$$

where

$$d_i = \sum_{l=0}^{i-1} (s^2 - 1) s^{2l-i} a^{i-2l} + (s^{-2} - 1) s^{i-2l} a^{2l-i}$$

IDEA OF PROOF

1) By power series manipulations, the statement is equivalent to

$$e \cdot \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) - \left(\widetilde{Q}_{(n+2)} + \widetilde{Q}_{(n)}\right) \cdot e$$

$$=$$

$$(sa + s^{-1}a^{-1})\left(e \cdot \widetilde{Q}_{(n+1)}\right) - (s^{-1}a + sa^{-1})\left(\widetilde{Q}_{(n+1)} \cdot e\right)$$

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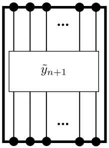
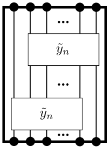
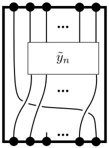
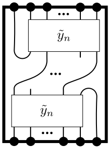
2) [Shelly, 2016] The $\widetilde{y}_{(n)}$ satisfy a skein-theoretic recurrence relation.

$[n+1]$ 

$= [n]s^{-1}$ 

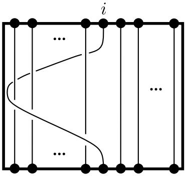
$+$ 

$+ [n]s^{-1}\beta_n$ 

APPLICATION: CENTRAL ELEMENTS OF BMW_n

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of BMW_n.

$M_{n,i} =$



$2 \leq i \leq n$

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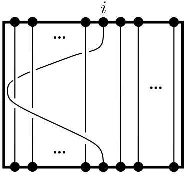
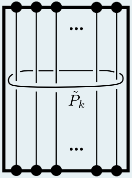
$M_{n,i} =$  $2 \leq i \leq n$

Theorem (P.)



$$= \langle \widetilde{P}_k \rangle 1 + (s^k - s^{-k}) \sum_{i=2}^{n} v^{-k} M_{n,i}^{k} - v^{k} M_{n,i}^{-k}$$

Application: Central Elements of BMW_n

Sketch: Apply commutation relation for $\widetilde{P}_k$.

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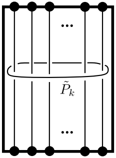
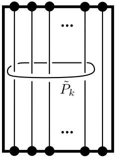
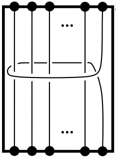
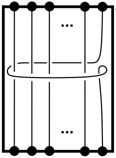
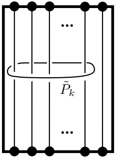
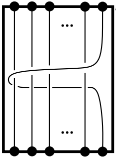
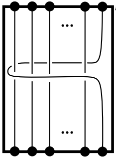
<u>Sketch:</u> Apply commutation relation for $\widetilde{P}_k$.



$=$ 

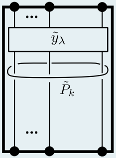
$+ (s^k - s^{-k})$  ^{k} $- (s^k - s^{-k})$  ^{k}

$=$ 

$+ (s^k - s^{-k})v^{-k}$  ^{k} $- (s^k - s^{-k})v^k$  ^{k}

APPLICATION: MERIDIANS OF $\widetilde{y}_\lambda$

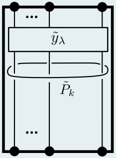
Theorem (P.)



$$= \left(\langle \widetilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left(v^{-k} s^{2\mathrm{cn}(\square)} - v^k s^{-2\mathrm{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

Application: Meridians of $\widetilde{y}_\lambda$

Theorem (P.)



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Observation: For any fixed k, the eigenvalues of $\tilde{y}_\lambda$ are distinct.

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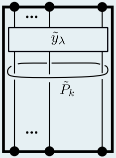

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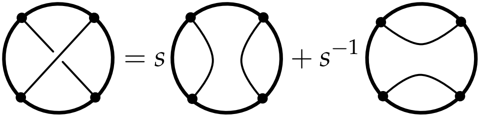
<u>Observation:</u> For any fixed k, the eigenvalues of $\tilde{y}_\lambda$ are distinct.

<u>Consequence:</u> The basis $\{\widetilde{Q}_\lambda\}$ of $\mathcal{D}(A)$ is an eigenbasis with 1-dimensional eigenspaces. Setting $k = 1$ recovers the result from [Lu-Zhong, 2002].

Compatibility With Kauffman Bracket Skein Theory $\mathcal{K}$

The Kauffman bracket skein relation $\rightsquigarrow$ Jones polynomial



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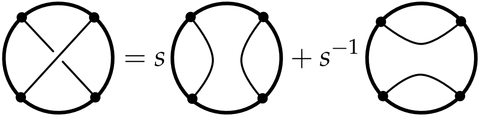

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

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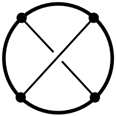
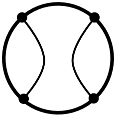
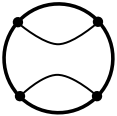
 $= s$  $+ s^{-1}$ 

<u>Fact:</u> The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.
<u>Consequence:</u> There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

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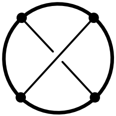
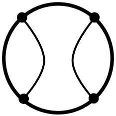
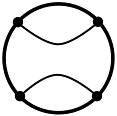


$= s$



$+ s^{-1}$



<u>Fact:</u> The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.
<u>Consequence:</u> There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

Theorem (P.)

The image of $P_k \in \mathcal{D}(A)$ under η_A is the Chebyshev polynomial $T_k \in \mathcal{K}(A)$.

OTHER SKEIN ALGEBRAS OF T^2

Theorem (Frohman-Gelca, 2000)

The algebra $\mathcal{K}(T^2)$ is presented by generators $T_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$T_{\mathbf{x}} T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}-\mathbf{y}}$$

Theorem (Morton-Samuelson, 2017)

The algebra $\mathcal{H}(T^2)$ is presented by generators $P_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$ subject to the relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})}) P_{\mathbf{x}+\mathbf{y}}$$

A PRESENTATION OF $\mathcal{D}(T^2)$

Let $\mathbf{x} = (a, b), k = \gcd(\mathbf{x})$.
Define $\widetilde{P}_{\mathbf{x}} \in \mathcal{D}(T^2)$ be the embedding of $\widetilde{P}_k$ along the closed curve of slope a/b.

Theorem (Morton-P.-Samuelson)

The algebra $\mathcal{D}(T^2)$ is presented by generators $\widetilde{\mathrm{P}}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x}\rangle$ subject to the relations

$$[\widetilde{P}_{\mathbf{x}}, \widetilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\widetilde{P}_{\mathbf{x}+\mathbf{y}} - \widetilde{P}_{\mathbf{x}-\mathbf{y}})$$

Frame 1

FRAME 1

Frame 1

Frame 1

FRAME 1