

Dubrovnik Skein Theory and Power Sum Elements

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May 24, 2021

INTRODUCTION

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BACKGROUND

POWER SUM ELEMENTS

THE ALGEBRA $\mathcal{D}(T^2)$

(FRAMED) LINK INVARIANTS

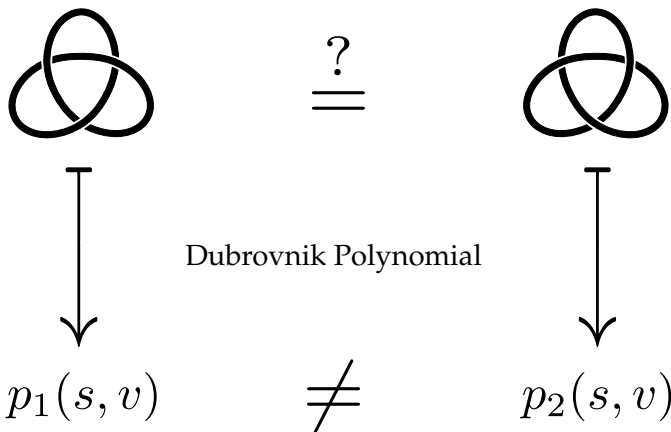


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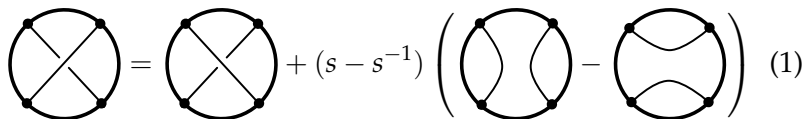
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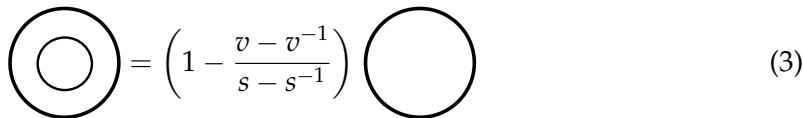
(FRAMED) LINK INVARIANTS



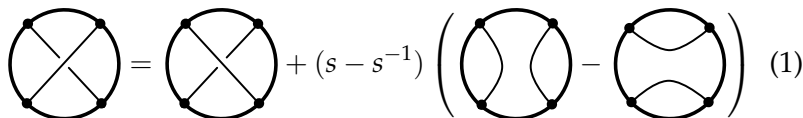
DUBROVNIK SKEIN RELATIONS


$$\text{Diagram 1} = \text{Diagram 2} + (s - s^{-1}) \left(\text{Diagram 3} - \text{Diagram 4} \right) \quad (1)$$

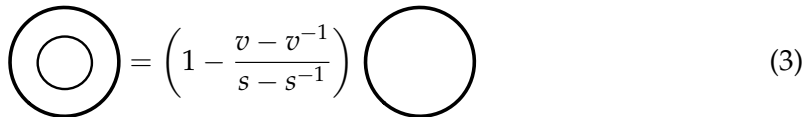

$$\text{Diagram 1} = v \text{Diagram 2} \quad (2)$$

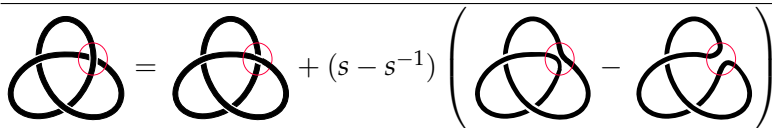

$$\text{Diagram 1} = \left(1 - \frac{v - v^{-1}}{s - s^{-1}} \right) \text{Diagram 2} \quad (3)$$

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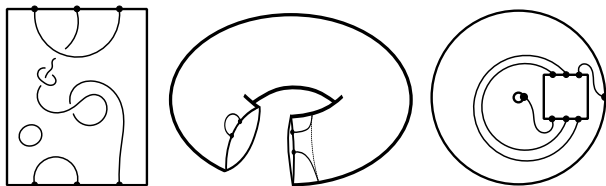
SKEIN MODULES

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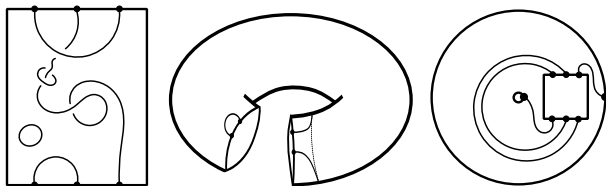
Consequence: May impose skein relations on tangles in arbitrary 3-dimensional manifolds.



SKEIN MODULES

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Definition

Let M be an oriented 3-manifold and $R := \mathbb{Q}(s, v)$.

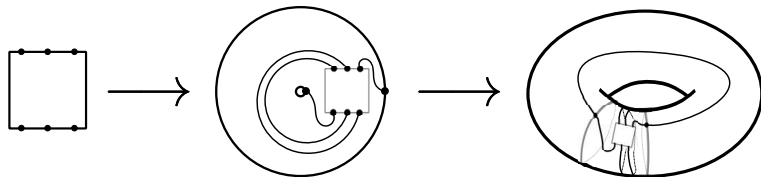
$$\mathcal{D}(M, N) := R\{\text{Tangles in } M \text{ relative to } N\} / \sim$$

FUNCTORIALITY

A “nice” embedding $f : M \rightarrow M'$

+ A wiring diagram in image complement

= A linear transformation $D(f) : \mathcal{D}(M, N) \rightarrow \mathcal{D}(M', N')$

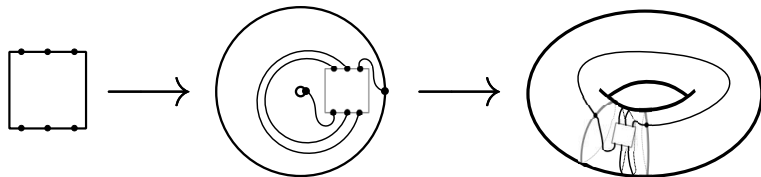


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Consequence: Dubrovnik skein theory is a type of algebraic topology for smooth, oriented, 3-manifolds.

SPECIAL CASE: SKEIN ALGEBRAS

If

- ▶ $M = \Sigma \times I$
- ▶ $N = (X \times \{0\}) \sqcup (X \times \{1\})$

Then $\mathcal{D}(M, N)$ is naturally an algebra.

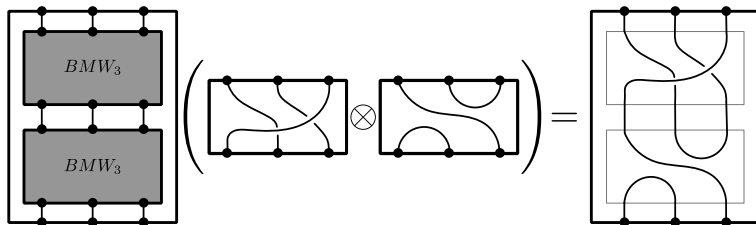
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If

- ▶ $M = \Sigma \times I$
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Then $\mathcal{D}(M, N)$ is naturally an algebra.

e.g.: $\Sigma = \text{Square}$, $N = 2n$ points $\rightsquigarrow \mathcal{D}(\Sigma, N) \cong BMW_n$



IDEMPOTENTS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

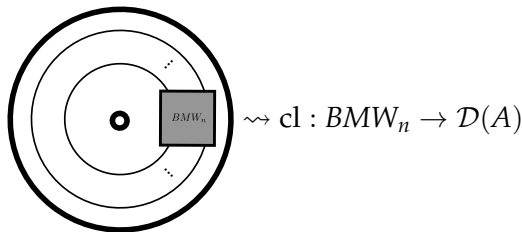
Theorem (Ram-Wenzl 1992, Beliakova-Blanchet, 2001)

For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.

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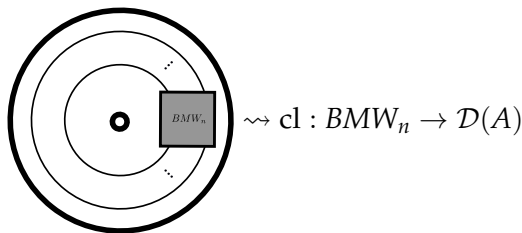
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For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.



Theorem (Lu-Zhong 2002)

The elements $\tilde{Q}_\lambda := \text{cl}(\tilde{y}_\lambda)$ form a basis of $\mathcal{D}(A)$.

DUBROVNIK POWER SUM ELEMENTS

Define a family of elements $\tilde{P}_k \in \mathcal{D}(A)$ for $k \in \mathbb{Z}_{\geq 1}$ via

$$\sum_{k \geq 1} \frac{\tilde{P}_k}{k} t^k = \ln \left(1 + \sum_{n \geq 1} \tilde{Q}_{(n)} t^n \right)$$

Idea:

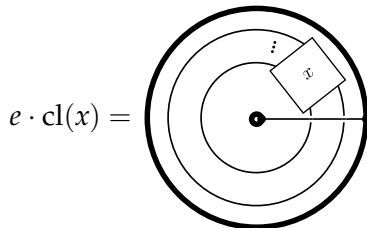
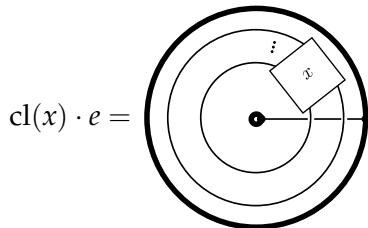
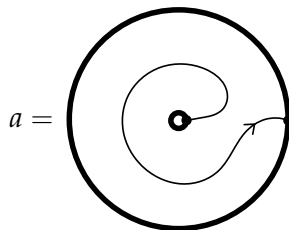
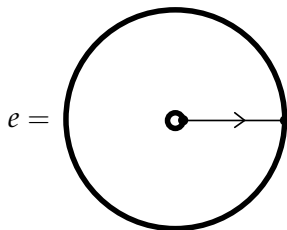
$\tilde{Q}_{(n)}$ are “like” complete homogeneous symmetric functions.

$\rightsquigarrow \tilde{P}_k$ are “like” power sum symmetric functions.

(Will make more precise later)

A RELATIVE SKEIN ALGEBRA

Let $\mathcal{A} := \mathcal{D}(A, 1)$.



COMMUTATION RELATIONS

Theorem (Morton-P.-Samuelson)

$$e \cdot \tilde{P}_k - \tilde{P}_k = (s^k - s^{-k})(a^k - a^{-k})$$

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Theorem (P.)

$$e \cdot \tilde{Q}_{(n)} - \tilde{Q}_{(n)} \cdot e = \sum_{i=1}^n d_i (e \cdot \tilde{Q}_{(n-i)})$$

where

$$d_i = \sum_{l=0}^{i-1} (s^2 - 1) s^{2l-i} a^{i-2l} + (s^{-2} - 1) s^{i-2l} a^{2l-i}$$

IDEA OF PROOF

1) By power series manipulations, the statement is equivalent to

$$\begin{aligned} e \cdot (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) - (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) \cdot e \\ = \\ (sa + s^{-1}a^{-1})(e \cdot \tilde{Q}_{(n+1)}) - (s^{-1}a + sa^{-1})(\tilde{Q}_{(n+1)} \cdot e) \end{aligned}$$

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 \end{aligned}$$

2) [Shelly, 2016] The $\tilde{y}_{(n)}$ satisfy a skein-theoretic recurrence relation.

$$[n+1] \begin{array}{|c|} \hline \tilde{y}_{n+1} \\ \hline \end{array} = [n]s^{-1} \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array} + \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array} + [n]s^{-1}\beta_n \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array}$$

APPLICATION: CENTRAL ELEMENTS OF BMW_n

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of BMW_n .

$$M_{n,i} = \begin{array}{c} \text{Diagram of a braid with } n \text{ strands. The } i\text{-th strand from the left crosses over the } (i-1)\text{-th strand. Ellipses indicate other strands.} \end{array} \quad 2 \leq i \leq n$$

APPLICATION: CENTRAL ELEMENTS OF BMW_n

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of BMW_n .

$$M_{n,i} = \text{Diagram} \quad 2 \leq i \leq n$$

Theorem (P.)

$$\text{Diagram} = \langle \tilde{P}_k \rangle 1 + (s^k - s^{-k}) \sum_{i=2}^n v^{-k} M_{n,i}^k - v^k M_{n,i}^{-k}$$

APPLICATION: CENTRAL ELEMENTS OF BMW_n

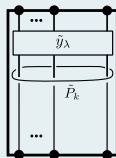
Sketch: Apply commutation relation for \tilde{P}_k .

APPLICATION: CENTRAL ELEMENTS OF BMW_n Sketch: Apply commutation relation for \tilde{P}_k .

$$\begin{aligned}
 & \text{Diagram 1: A rectangle with } k \text{ vertical strands. A horizontal oval labeled } \tilde{P}_k \text{ encircles the strands. Ellipses indicate more strands above and below the oval.} \\
 &= \text{Diagram 2: Same as Diagram 1.} + (s^k - s^{-k}) \text{Diagram 3: } k \text{ vertical strands. A horizontal oval encircles the strands. The rightmost strand crosses over the oval.} - (s^k - s^{-k}) \text{Diagram 4: } k \text{ vertical strands. A horizontal oval encircles the strands. The rightmost strand crosses under the oval.} \\
 &= \text{Diagram 5: Same as Diagram 2.} + (s^k - s^{-k})v^{-k} \text{Diagram 6: } k \text{ vertical strands. A horizontal oval encircles the strands. The rightmost strand crosses over the oval.} - (s^k - s^{-k})v^k \text{Diagram 7: } k \text{ vertical strands. A horizontal oval encircles the strands. The rightmost strand crosses under the oval.}
 \end{aligned}$$

APPLICATION: MERIDIANS OF \tilde{y}_λ

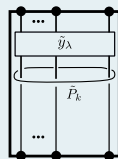
Theorem (P.)



$$= \left(\langle \tilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left(v^{-k} s^{2\text{cn}(\square)} - v^k s^{-2\text{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

APPLICATION: MERIDIANS OF \tilde{y}_λ

Theorem (P.)

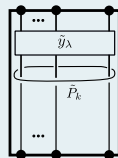


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Observation: For any fixed k , the eigenvalues of \tilde{y}_λ are distinct.

APPLICATION: MERIDIANS OF \tilde{y}_λ

Theorem (P.)



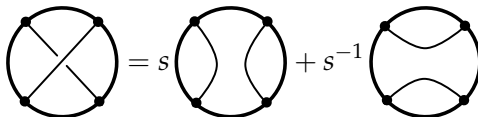
$$= \left(\langle \tilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left(v^{-k} s^{2\text{cn}(\square)} - v^k s^{-2\text{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

Observation: For any fixed k , the eigenvalues of \tilde{y}_λ are distinct.

Consequence: The basis $\{\tilde{Q}_\lambda\}$ of $\mathcal{D}(A)$ is an eigenbasis with 1-dimensional eigenspaces. Setting $k = 1$ recovers the result from [Lu-Zhong, 2002].

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

The Kauffman bracket skein relation \rightsquigarrow Jones polynomial

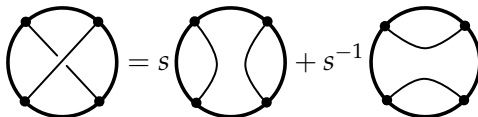


The diagram illustrates the Kauffman bracket skein relation. It shows three circular diagrams with four marked points on the boundary. The first diagram on the left has two straight lines crossing each other. This is equal to s times the second diagram, which has two vertical arcs, plus s^{-1} times the third diagram, which has two horizontal arcs.

$$\text{Crossing} = s \cdot \text{Two Vertical Arcs} + s^{-1} \cdot \text{Two Horizontal Arcs}$$

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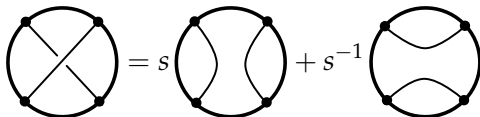
The diagram illustrates the Kauffman bracket skein relation. It shows a circle with four marked points (top, bottom, left, right). On the left, the circle is divided into four quadrants by two straight lines crossing at the center. This is followed by an equals sign, then the variable s , then a circle with two vertical arcs (left and right) connecting the top and bottom points. This is followed by a plus sign, then s^{-1} , then a circle with two horizontal arcs (top and bottom) connecting the left and right points.

$$\text{Crossing} = s \text{ (Vertical Arcs)} + s^{-1} \text{ (Horizontal Arcs)}$$

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

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The Kauffman bracket skein relation \rightsquigarrow Jones polynomial



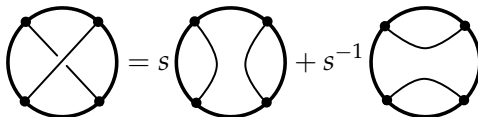
The diagram illustrates the Kauffman bracket skein relation. It shows a circle with four marked points on its boundary. On the left, the circle is divided into four quadrants by two diagonal lines connecting opposite points. This is followed by an equals sign, then the term s multiplied by a circle where the two diagonal lines are replaced by two vertical arcs, each connecting the two points on the same side of the circle. This is followed by a plus sign, then the term s^{-1} multiplied by a circle where the two diagonal lines are replaced by two horizontal arcs, each connecting the two points on the same side of the circle.

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

Consequence: There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

The Kauffman bracket skein relation \rightsquigarrow Jones polynomial



The diagram illustrates the Kauffman bracket skein relation. On the left is a circle with four black dots at the top, bottom, left, and right positions. Two straight lines connect the dots diagonally, forming an 'X' shape. This is followed by an equals sign, then the variable 's'. To the right of 's' is a circle with the same four dots, but with two curved lines connecting the top and bottom dots, forming two separate loops. This is followed by a plus sign, then 's^{-1}', and finally another circle with the same four dots and two curved lines connecting the top and bottom dots, but with the curves facing the opposite direction (concave outwards).

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

Consequence: There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

Theorem (P.)

The image of $P_k \in \mathcal{D}(A)$ under η_A is the Chebyshev polynomial $T_k \in \mathcal{K}(A)$.

OTHER SKEIN ALGEBRAS OF T^2

Theorem (Frohman-Gelca, 2000)

The algebra $\mathcal{K}(T^2)$ is presented by generators $T_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2 / \langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$T_{\mathbf{x}}T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}-\mathbf{y}}$$

Theorem (Morton-Samuelson, 2017)

The algebra $\mathcal{H}(T^2)$ is presented by generators $P_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$ subject to the relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}+\mathbf{y}}$$

A PRESENTATION OF $\mathcal{D}(T^2)$

Let $\mathbf{x} = (a, b)$, $k = \gcd(\mathbf{x})$.

Define $\tilde{P}_{\mathbf{x}} \in \mathcal{D}(T^2)$ be the embedding of \tilde{P}_k along the closed curve of slope a/b .

Theorem (Morton-P.-Samuelson)

The algebra $\mathcal{D}(T^2)$ is presented by generators $\tilde{P}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2 / \langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$$

FRAME 1

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