

# Dubrovnik Skein Theory and Power Sum Elements

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# INTRODUCTION

INTRODUCTION

BACKGROUND

POWER SUM ELEMENTS

THE ALGEBRA  $\mathcal{D}(T^2)$

TYPE B/C/D SCHUR FUNCTIONS

# (FRAMED) LINK INVARIANTS



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=



# (FRAMED) LINK INVARIANTS

 $\stackrel{?}{=}$ 

Dubrovnik Polynomial

 $p_1(s, v)$  $\neq$  $p_2(s, v)$

## DUBROVNIK SKEIN RELATIONS

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \bigcirc = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \bigcirc + (s - s^{-1}) \left( \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \bigcirc - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \bigcirc \right) \quad (1)$$

$$\begin{array}{c} \text{twist} \end{array} \bigcirc = v \begin{array}{c} \text{crossing} \end{array} \bigcirc \quad (2)$$

$$\bigcirc \bigcirc = \left( 1 - \frac{v - v^{-1}}{s - s^{-1}} \right) \bigcirc \quad (3)$$

# DUBROVNIK SKEIN RELATIONS

$$\text{Diagram 1} = \text{Diagram 2} + (s - s^{-1}) \left( \text{Diagram 3} - \text{Diagram 4} \right) \quad (1)$$

$$\text{Diagram 5} = v \text{Diagram 6} \quad (2)$$

$$\text{Diagram 7} = \left( 1 - \frac{v - v^{-1}}{s - s^{-1}} \right) \text{Diagram 8} \quad (3)$$

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$$\text{Diagram 9} = \text{Diagram 10} + (s - s^{-1}) \left( \text{Diagram 11} - \text{Diagram 12} \right)$$

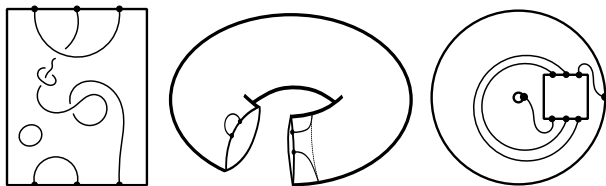
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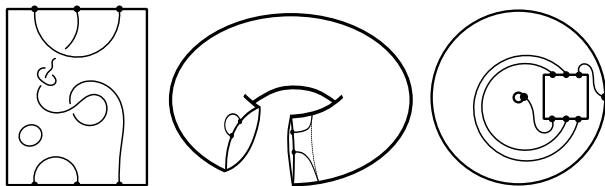




# SKEIN MODULES

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## Definition

Let  $M$  be an oriented 3-manifold and  $R := \mathbb{Q}(s, v)$ .

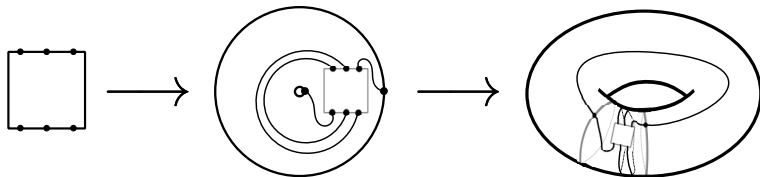
$$\mathcal{D}(M, N) := R\{\text{Tangles in } M \text{ relative to } N\} / \sim$$

# FUNCTORIALITY

A “nice” embedding  $f : (M, N) \rightarrow (M', N')$

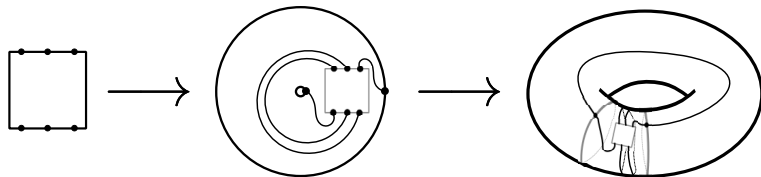
+ A wiring diagram in image complement

= A linear transformation  $D(f) : \mathcal{D}(M, N) \rightarrow \mathcal{D}(M', N')$



# FUNCTORIALITY

A “nice” embedding  $f : (M, N) \rightarrow (M', N')$   
 + A wiring diagram in image complement  
 = A linear transformation  $D(f) : \mathcal{D}(M, N) \rightarrow \mathcal{D}(M', N')$



Consequence: Dubrovnik skein theory is a type of algebraic topology for smooth, oriented, 3-manifolds.

# SPECIAL CASE: SKEIN ALGEBRAS

If

- ▶  $M = \Sigma \times I$
- ▶  $N = (X \times \{0\}) \sqcup (X \times \{1\})$

Then  $\mathcal{D}(M, N)$  is naturally an algebra.

# SPECIAL CASE: SKEIN ALGEBRAS

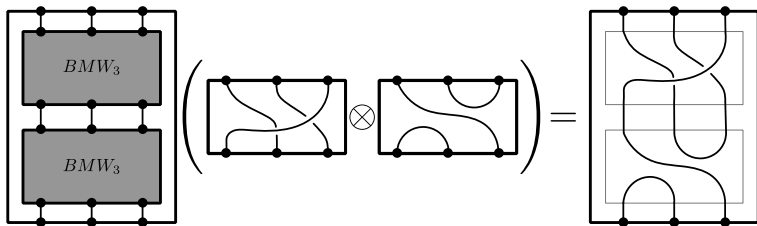
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---

e.g.:  $\Sigma = \text{Square}$ ,  $N = 2n$  points  $\rightsquigarrow \mathcal{D}(\Sigma, N) \cong \text{BMW}_n$



# IDEMPOTENTS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

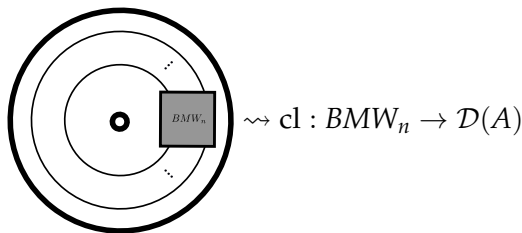
Theorem (Ram-Wenzl 1992, Beliakova-Blanchet 2001)

*For each partition  $\lambda \vdash n$ , there is a minimal idempotent  $\tilde{y}_\lambda \in BMW_n$ .*

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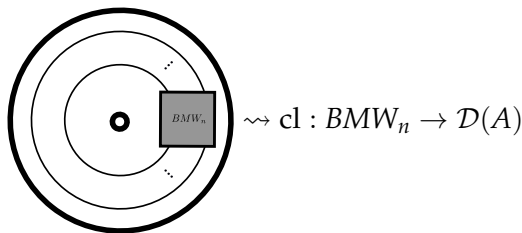
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*For each partition  $\lambda \vdash n$ , there is a minimal idempotent  $\tilde{y}_\lambda \in BMW_n$ .*



Theorem (Lu-Zhong 2002)

*The elements  $\tilde{Q}_\lambda := \text{cl}(\tilde{y}_\lambda)$  form a basis of  $\mathcal{D}(A)$ .*



# DUBROVNIK POWER SUM ELEMENTS

Define a family of elements  $\tilde{P}_k \in \mathcal{D}(A)$  for  $k \in \mathbb{Z}_{\geq 1}$  via

$$\sum_{k \geq 1} \frac{\tilde{P}_k}{k} t^k = \ln \left( 1 + \sum_{n \geq 1} \tilde{Q}_{(n)} t^n \right)$$

Idea:

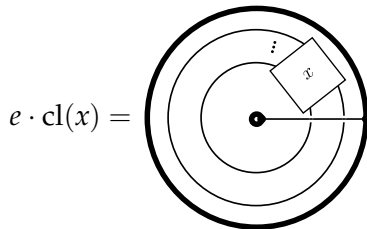
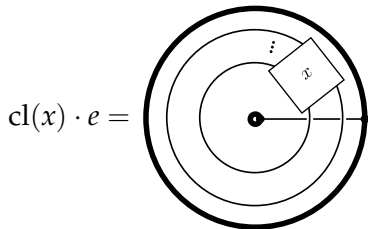
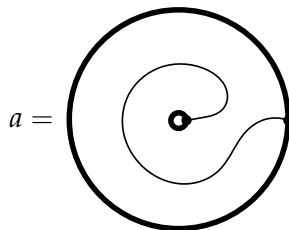
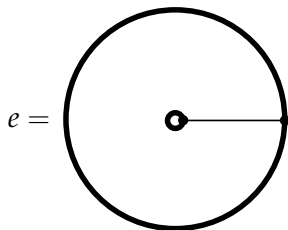
$\tilde{Q}_{(n)}$  are “like” complete homogeneous symmetric functions.

$\rightsquigarrow \tilde{P}_k$  are “like” power sum symmetric functions.

(Will make more precise later)

# A RELATIVE SKEIN ALGEBRA

Let  $\mathcal{A} := \mathcal{D}(A, 1)$ .



# COMMUTATION RELATIONS

## Theorem (Morton-P.-Samuelson)

$$e \cdot \tilde{P}_k - \tilde{P}_k \cdot e = (s^k - s^{-k})(a^k - a^{-k})$$

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## Theorem (P.)

$$e \cdot \tilde{Q}_{(n)} - \tilde{Q}_{(n)} \cdot e = \sum_{i=1}^n d_i (e \cdot \tilde{Q}_{(n-i)})$$

where

$$d_i = \sum_{l=0}^{i-1} (s^{2l-i} - 1)s^{2l-i}a^{i-2l} + (s^{-2l-i} - 1)s^{i-2l}a^{2l-i}$$

# IDEA OF PROOF

1) By power series manipulations, the statement is equivalent to

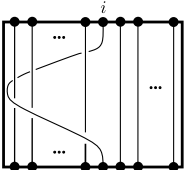
$$\begin{aligned}
 e \cdot (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) - (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) \cdot e \\
 = \\
 (sa + s^{-1}a^{-1})(e \cdot \tilde{Q}_{(n+1)}) - (s^{-1}a + sa^{-1})(\tilde{Q}_{(n+1)} \cdot e)
 \end{aligned}$$

2) [Shelly, 2016] The  $\tilde{y}_{(n)}$  satisfy a skein-theoretic recurrence relation.

$$[n+1] \begin{array}{|c|} \hline \tilde{y}_{n+1} \\ \hline \end{array} = [n]s^{-1} \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array} + \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array} + [n]s^{-1}\beta_n \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array}$$

APPLICATION: CENTRAL ELEMENTS OF  $BMW_n$ 

The Jucys-Murphy elements  $M_{n,i}$  generate a commutative subalgebra of  $BMW_n$ .

$$M_{n,i} =$$


$$2 \leq i \leq n$$

# APPLICATION: CENTRAL ELEMENTS OF $BMW_n$

The Jucys-Murphy elements  $M_{n,i}$  generate a commutative subalgebra of  $BMW_n$ .

$$M_{n,i} = \text{Diagram} \quad 2 \leq i \leq n$$

## Proposition (P.)

$$\text{Diagram} = \langle \tilde{P}_k \rangle 1 + (s^k - s^{-k}) \sum_{i=2}^n v^{-k} M_{n,i}^k - v^k M_{n,i}^{-k}$$

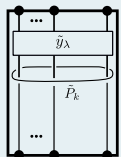
APPLICATION: CENTRAL ELEMENTS OF  $BMW_n$ Sketch: Apply commutation relation for  $\tilde{P}_k$ .

$$\begin{aligned}
 & \text{Diagram 1: A rectangle with } k \text{ vertical lines. A horizontal oval labeled } \tilde{P}_k \text{ encircles the middle of the lines.} \\
 &= \text{Diagram 2: Same as Diagram 1.} + (s^k - s^{-k}) \text{Diagram 3: A rectangle with } k \text{ vertical lines. A horizontal oval encircles the middle of the lines, with a crossing on the right side.} - (s^k - s^{-k}) \text{Diagram 4: A rectangle with } k \text{ vertical lines. A horizontal oval encircles the middle of the lines, with a crossing on the left side.} \\
 &= \text{Diagram 5: Same as Diagram 2.} + (s^k - s^{-k})v^{-k} \text{Diagram 6: Same as Diagram 3, but with a crossing on the left side.} - (s^k - s^{-k})v^k \text{Diagram 7: Same as Diagram 4, but with a crossing on the right side.}
 \end{aligned}$$



APPLICATION: MERIDIANS OF  $\tilde{y}_\lambda$ 

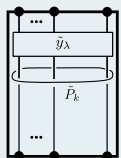
## Proposition (P.)



$$= \left( \langle \tilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left( v^{-k} s^{2\text{cn}(\square)} - v^k s^{-2\text{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

APPLICATION: MERIDIANS OF  $\tilde{y}_\lambda$ 

## Proposition (P.)

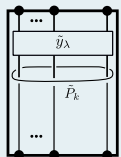


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Observation: For any fixed  $k$ , the eigenvalues of  $\tilde{y}_\lambda$  are distinct.

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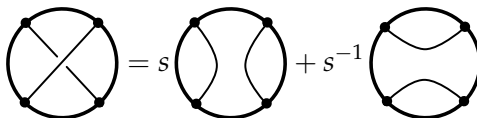
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Observation: For any fixed  $k$ , the eigenvalues of  $\tilde{y}_\lambda$  are distinct.

Consequence: The basis  $\{\tilde{Q}_\lambda\}$  of  $\mathcal{D}(A)$  is an eigenbasis with 1-dimensional eigenspaces. Setting  $k = 1$  recovers the result from [Lu-Zhong, 2002].

# COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY $\mathcal{K}$

The Kauffman bracket skein relation  $\rightsquigarrow$  Jones polynomial

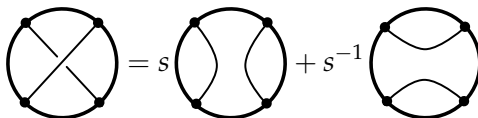


The diagram illustrates the Kauffman bracket skein relation. It shows three circular diagrams with four marked points on the boundary. The first diagram on the left has two straight lines crossing each other. This is equal to  $s$  times the second diagram, which has two vertical arcs, plus  $s^{-1}$  times the third diagram, which has two horizontal arcs.

$$\text{Crossing} = s \cdot \text{Two Vertical Arcs} + s^{-1} \cdot \text{Two Horizontal Arcs}$$

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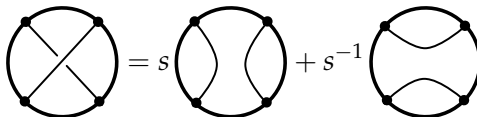
The diagram illustrates the Kauffman bracket skein relation. It shows a circle with four marked points (top, bottom, left, right). On the left, the circle is divided into four quadrants by two straight lines crossing at the center. This is followed by an equals sign, then the variable  $s$ , then a circle with two vertical arcs (left and right) connecting the top and bottom points. This is followed by a plus sign, then  $s^{-1}$ , then a circle with two horizontal arcs (top and bottom) connecting the left and right points.

$$\text{Crossing} = s \text{ (Vertical Arcs)} + s^{-1} \text{ (Horizontal Arcs)}$$

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

# COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY $\mathcal{K}$

The Kauffman bracket skein relation  $\rightsquigarrow$  Jones polynomial



The diagram illustrates the Kauffman bracket skein relation. On the left is a circle with four black dots at the top, bottom, left, and right positions. Two straight lines connect the dots: one from top-left to bottom-right, and another from top-right to bottom-left, forming an 'X' shape. This is followed by an equals sign, then the term  $s$  multiplied by a circle with four dots where the top and bottom arcs are thickened and the left and right arcs are thin. This is followed by a plus sign, then the term  $s^{-1}$  multiplied by a circle with four dots where the left and right arcs are thickened and the top and bottom arcs are thin.

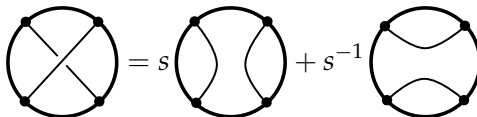
$$\text{Crossing} = s \cdot \text{Thickened Top/Bottom} + s^{-1} \cdot \text{Thickened Left/Right}$$

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

Consequence: There is a natural transformation of skein theories  $\eta : \mathcal{D} \Rightarrow \mathcal{K}$ .

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## Theorem (Morton-P.-Samuelson)

*The image of  $\tilde{P}_k \in \mathcal{D}(A)$  under  $\eta_A$  is the Chebyshev polynomial  $T_k \in \mathcal{K}(A)$ .*

# SKEIN ALGEBRAS OF $T^2$

## Theorem (Frohman-Gelca 2000)

*The algebra  $\mathcal{K}(T^2)$  is presented by generators  $T_{\mathbf{x}}$  for  $\mathbf{x} \in \mathbb{Z}^2 / \langle \mathbf{x} = -\mathbf{x} \rangle$  subject to the relations*

$$T_{\mathbf{x}}T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}-\mathbf{y}}$$



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## Theorem (Morton-Samuelson 2017)

*The algebra  $\mathcal{H}(T^2)$  is presented by generators  $P_{\mathbf{x}}$  for  $\mathbf{x} \in \mathbb{Z}^2$  subject to the relations*

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}+\mathbf{y}}$$

# A PRESENTATION OF $\mathcal{D}(T^2)$

Let  $\mathbf{x} = (a, b)$ ,  $k = \gcd(\mathbf{x})$ .

Define  $\tilde{P}_{\mathbf{x}} \in \mathcal{D}(T^2)$  be the embedding of  $\tilde{P}_k$  along the closed curve of slope  $a/b$ .

## Theorem (Morton-P.-Samuelson)

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$$[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$$

# A PRESENTATION OF $\mathcal{D}(T^2)$

## Proof:

1.  $[\tilde{P}_x, \tilde{P}_y] = (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}}).$ 
  - 1.1  $[\tilde{P}_{1,0}, \tilde{P}_{0,n}] = (s^n - s^{-n})(\tilde{P}_{1,n} - \tilde{P}_{1,-n})$
  - 1.2  $[\tilde{P}_{1,0}, \tilde{P}_{1,n}] = (s^n - s^{-n})(\tilde{P}_{2,n} - \tilde{P}_{0,n})$
  - 1.3 Use  $SL_2(\mathbb{Z})$ -action on  $\mathcal{D}(T^2)$ . Induct on  $|\det(\mathbf{x}, \mathbf{y})|$ .
2. The  $\tilde{P}_x$  generate  $\mathcal{D}(T^2)$ .
  - 2.1 Each link in  $\mathcal{D}(T^2)$  is a sum of products of annular knots.
  - 2.2 The  $\tilde{P}_k$  generate  $\mathcal{D}(A)$ .
3. The  $\tilde{P}_x$  are linearly independent.
  - 3.1 Diamond lemma type argument.

RELATIONSHIP WITH  $\mathcal{K}(T^2)$ 

## Corollary (Morton-P.-Samuelson)

*There is a surjective algebra homomorphism  $\mathcal{D}(T^2) \rightarrow \mathcal{K}(T^2)$  defined by*

$$\tilde{P}_x \mapsto T_x.$$

Proof: Use the natural transformation  $\eta : \mathcal{D} \Rightarrow \mathcal{K}$ . Recall

$$\eta_A(\tilde{P}_k) = T_k.$$

Note:  $\mathcal{D}(T^2)$  is much bigger than  $\mathcal{K}(T^2)$ .

# RELATIONSHIP WITH $\mathcal{H}(T^2)$

## Corollary (P.)

*The algebras  $\mathcal{D}(T^2)$  and  $\mathcal{H}(T^2)$  are universal enveloping algebras of some Lie algebras  $\mathfrak{g}_{\mathcal{D}}$  and  $\mathfrak{g}_{\mathcal{H}}$  generated by the  $\tilde{P}_{\mathbf{x}}$  and  $T_{\mathbf{x}}$ , respectively. There is an injective Lie algebra homomorphism  $\mathfrak{g}_{\mathcal{D}} \rightarrow \mathfrak{g}_{\mathcal{H}}$  defined by*

$$\tilde{P}_{\mathbf{x}} \mapsto P_{\mathbf{x}} + P_{-\mathbf{x}}.$$

Proof:  $[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] - (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$  is sent to 0.

Note: This restricts to an algebra homomorphism

$$\mathcal{D}(A) \rightarrow \mathcal{H}(A).$$

Note:  $\mathcal{D}(A)$  and  $\mathcal{H}(A)$  are related to characters of classical Lie groups.

# A $\mathcal{D}(T^2)$ -ACTION ON $\mathcal{D}(D^2 \times S^1)$

## Lemma (P.)

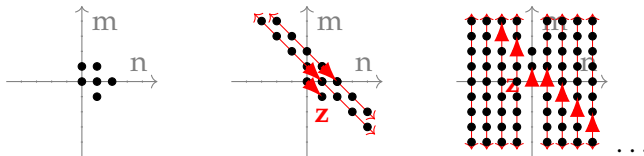
$\mathcal{D}(T^2)$  is generated by the identity and the elements

$$\tilde{P}_{1,0}, \tilde{P}_{0,1}, \tilde{P}_{1,1}, \tilde{P}_{2,0}$$

Proof: Can generate a “line” of  $\tilde{P}_x$  by

$$\tilde{P}_{\mathbf{y}+n\mathbf{z}} = (s^d - s^{-d})^{-1} \left( [\tilde{P}_{\mathbf{y}+(n-1)\mathbf{z}}, \tilde{P}_{\mathbf{z}}] + \{d\} \tilde{P}_{\mathbf{y}+(n-2)\mathbf{z}} \right)$$

as long as  $d := \det(\mathbf{y} + n\mathbf{z}, \mathbf{z}) = \det(\mathbf{y}, \mathbf{z}) \neq 0$ .



A  $\mathcal{D}(T^2)$ -ACTION ON  $\mathcal{D}(D^2 \times S^1)$ 

## Proposition (P.)

The  $\mathcal{D}(T^2)$ -action on  $\mathcal{D}(D^2 \times S^1)$  is determined by the equations

$$\tilde{P}_{1,0} \cdot \tilde{Q}_\lambda = \left( \langle \tilde{P}_1 \rangle + \{1\} \left( v^{-1} \sum_{\square \in \lambda} s^{2\text{cn}(\square)} - v \sum_{\square \in \lambda} s^{-2\text{cn}(\square)} \right) \right) \tilde{Q}_\lambda$$

$$\tilde{P}_{2,0} \cdot \tilde{Q}_\lambda = \left( \langle \tilde{P}_2 \rangle + \{2\} \left( v^{-2} \sum_{\square \in \lambda} s^{4\text{cn}(\square)} - v^2 \sum_{\square \in \lambda} s^{-4\text{cn}(\square)} \right) \right) \tilde{Q}_\lambda$$

$$\tilde{P}_{0,1} \cdot \tilde{Q}_\lambda = \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} \tilde{Q}_\mu + \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} \tilde{Q}_\nu$$

$$\tilde{P}_{1,1} \cdot \tilde{Q}_\lambda = v^{-1} \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} s^{2\text{cn}(\square)} \tilde{Q}_\mu + v \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} s^{-2\text{cn}(\square)} \tilde{Q}_\nu$$

# SYMMETRIC FUNCTIONS AND $\mathcal{H}(A)^+$

$\Lambda$  - The ring of symmetric functions

$s_\lambda$  - Schur function

$\mathcal{H}(A)^+$  - Annular HOMFLYPT links oriented counter-clockwise

## Theorem (Lukac 2005)

*There is a basis of elements  $Q_\lambda$  in  $\mathcal{H}(A)^+$  indexed by partitions  $\lambda$ .  
The assignment  $\Lambda \rightarrow \mathcal{H}(A)^+$  defined by*

$$s_\lambda \mapsto Q_\lambda$$

*is an algebra isomorphism.*

Hope: Generalize this to Dubrovnik case.



# ORTHOGONAL & SYMPLECTIC SCHUR FUNCTIONS

Partition  $\lambda$   $\longrightarrow$  Symmetric functions  $sb_\lambda, sc_\lambda$

# ORTHOGONAL & SYMPLECTIC SCHUR FUNCTIONS

Partition  $\lambda$   $\longrightarrow$  Symmetric functions  $sb_\lambda, sc_\lambda$

Theorem (Koike-Terada 1987, Koike 1989)

*The sets  $\{sb_\lambda\}$  and  $\{sc_\lambda\}$  are bases of  $\Lambda$ .*

*The structure constants of  $\Lambda$  with respect to each basis are identical natural numbers.*

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## Conjecture

*Both of the assignments  $\Lambda \rightarrow \mathcal{D}(A)$  defined by*

$$sb_{(n)} \mapsto \tilde{Q}_{(n)}$$

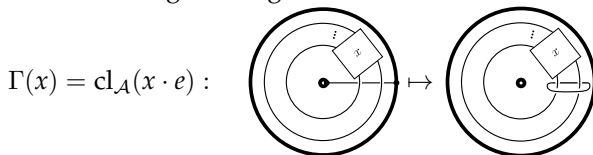
$$sc_{(n)} \mapsto \tilde{Q}_{(n)}$$

*are algebra isomorphisms.*

# PARTIAL PROOF OF CONJECTURE

Idea: Let  $S_\lambda$  be the image of  $sc_\lambda$ . Show  $S_\lambda = \tilde{Q}_\lambda$ .

1. Show  $S_\lambda$  is in the same eigenspace of  $\tilde{Q}_\lambda$  with respect to the meridian map  $\Gamma$ .
  - 1.1 Use a Jacobi-Trudi type identity to write  $S_\lambda$  as a determinant whose entries are in terms of  $S_{(n)} = \tilde{Q}_{(n)}$ .
  - 1.2 Factor  $\Gamma$  through the algebra  $\mathcal{A}$  so that



- 1.3 ???  $\text{cl}_{\mathcal{A}}(e \cdot x) = \delta x$ . Try to translate left-action to right-action. (Works if  $|\lambda| \leq 2$ )
  - 1.4 Compute eigenvalue.
2. Eigenspaces are 1-dimensional  $\Rightarrow S_\lambda = dQ_\lambda$ . Show  $d = 1$  with branching rules:  $S_{\square}^n = \sum c_\mu S_\mu$  and  $\tilde{Q}_{\square}^n = \sum c_\mu \tilde{Q}_\mu$

INTRODUCTION  
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BACKGROUND  
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POWER SUM ELEMENTS  
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THE ALGEBRA  $\mathcal{D}(T^2)$   
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TYPE B/C/D SCHUR FUNCTIONS  
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# FRAME 1