

Dubrovnik Skein Theory and Power Sum Elements

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INTRODUCTION

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BACKGROUND

POWER SUM ELEMENTS

THE ALGEBRA $\mathcal{D}(T^2)$

TYPE B/C/D SCHUR FUNCTIONS

(FRAMED) LINK INVARIANTS



?

=



(FRAMED) LINK INVARIANTS



$\stackrel{?}{=}$



Dubrovnik Polynomial

$p_1(s, v)$

\neq

$p_2(s, v)$

KAUFFMAN (DUBROVNIK) SKEIN RELATIONS

$$\text{Diagram 1} = \text{Diagram 2} + (s - s^{-1}) \left(\text{Diagram 3} - \text{Diagram 4} \right) \quad (1)$$

$$\text{Diagram 5} = v \text{Diagram 6} \quad (2)$$

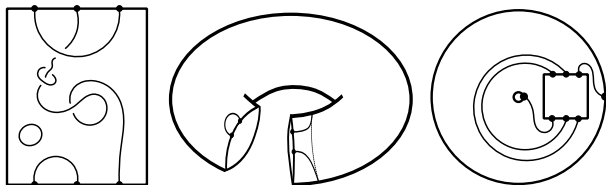
$$\text{Diagram 7} = \left(1 - \frac{v - v^{-1}}{s - s^{-1}} \right) \text{Diagram 8} \quad (3)$$

$$\text{Diagram 9} = \text{Diagram 10} + (s - s^{-1}) \left(\text{Diagram 11} - \text{Diagram 12} \right)$$

SKEIN MODULES

Observation: Skein relations are defined locally.

Consequence: May impose skein relations on tangles in arbitrary 3-dimensional manifolds.



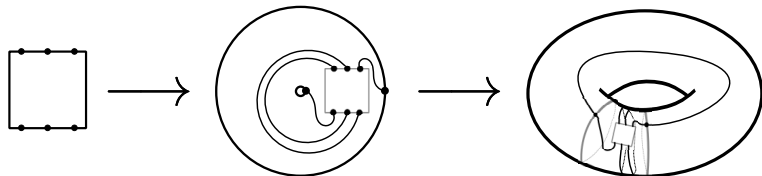
Definition

Let M be an oriented 3-manifold, $N \subset \partial M$ finite, and $R := \mathbb{Q}(s, v)$.

$$\mathcal{D}(M, N) := R\{\text{Tangles in } M \text{ relative to } N\} / \sim$$

FUNCTORIALITY

A “nice” embedding $f : (M, N) \rightarrow (M', N')$
+ A wiring diagram in image complement
= A linear transformation $\mathcal{D}(f) : \mathcal{D}(M, N) \rightarrow \mathcal{D}(M', N')$



Consequence: Dubrovnik skein theory is a type of algebraic topology for smooth, oriented, 3-manifolds.

Hope: Develop this theory by finding algebraic presentations of nice examples of modules $\mathcal{D}(M, N)$ and how they relate to each other.

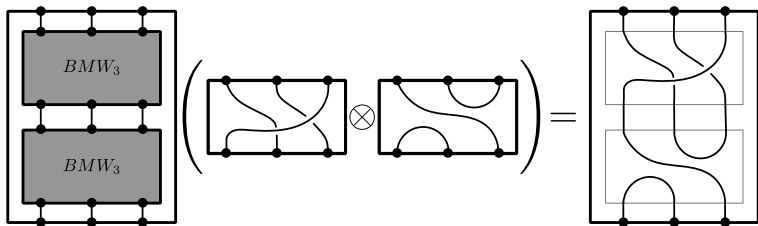
SPECIAL CASE: SKEIN ALGEBRAS

If

- ▶ $M = \Sigma \times I$
- ▶ $N = (X \times \{0\}) \sqcup (X \times \{1\})$

Then $\mathcal{D}(M, N)$ is naturally an algebra.

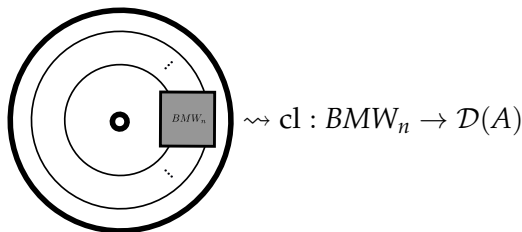
e.g.: $\Sigma = \text{Square}$, $N = 2n$ points $\rightsquigarrow \mathcal{D}(\Sigma, N) \cong BMW_n$



IDEMPOTENTS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

Theorem (Ram-Wenzl 1992, Beliakova-Blanchet 2001)

For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.



Theorem (Lu-Zhong 2002)

The elements $\tilde{Q}_\lambda := \text{cl}(\tilde{y}_\lambda)$ form a basis of $\mathcal{D}(A)$.

DUBROVNIK POWER SUM ELEMENTS

Define a family of elements $\tilde{P}_k \in \mathcal{D}(A)$ for $k \in \mathbb{Z}_{\geq 1}$ via

$$\sum_{k \geq 1} \frac{\tilde{P}_k}{k} t^k = \ln \left(1 + \sum_{n \geq 1} \tilde{Q}_{(n)} t^n \right)$$

Idea:

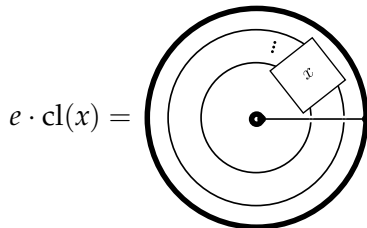
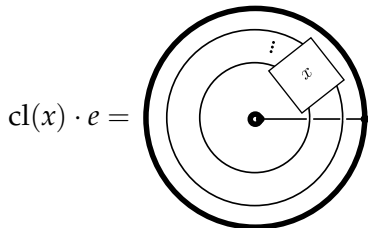
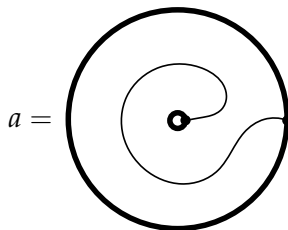
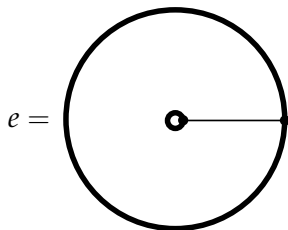
$\tilde{Q}_{(n)}$ are “like” complete homogeneous symmetric functions.

$\rightsquigarrow \tilde{P}_k$ are “like” power sum symmetric functions.

(Will make more precise later)

A RELATIVE SKEIN ALGEBRA

Let $\mathcal{A} := \mathcal{D}(A, 1)$.



COMMUTATION RELATIONS

Theorem (Morton-P.-Samuelson)

$$e \cdot \tilde{P}_k = \tilde{P}_k \cdot e + (s^k - s^{-k})(a^k - a^{-k})$$

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Theorem (P.)

$$e \cdot \tilde{Q}_{(n)} = \tilde{Q}_{(n)} \cdot e + \sum_{i=1}^n d_i (\tilde{Q}_{(n-i)} \cdot e)$$

where

$$d_i = \sum_{l=0}^{i-1} (1 - s^{-2}) s^{i-2l} a^{i-2l} + (1 - s^2) s^{2l-i} a^{2l-i}$$

IDEA OF PROOF

1) By power series manipulations, the statement is equivalent to

$$\begin{aligned}
 e \cdot (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) - (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) \cdot e \\
 = \\
 (sa + s^{-1}a^{-1})(e \cdot \tilde{Q}_{(n+1)}) - (s^{-1}a + sa^{-1})(\tilde{Q}_{(n+1)} \cdot e)
 \end{aligned}$$

2) [Shelly, 2016] The $\tilde{y}_{(n)}$ satisfy a skein-theoretic recurrence relation.

The diagram illustrates a skein-theoretic recurrence relation for the elements \tilde{y}_n . It shows that the element $[n+1]\tilde{y}_{n+1}$ is equal to the sum of three terms:

- $[n]s^{-1}$ times a diagram where two \tilde{y}_n boxes are stacked vertically, connected by a straight vertical line.
- plus a diagram where two \tilde{y}_n boxes are stacked vertically, connected by a crossing (a braid).
- plus $[n]s^{-1}\beta_n$ times a diagram where two \tilde{y}_n boxes are stacked vertically, connected by a loop (a cap and a cup).

Each diagram consists of a rectangular frame with dots on the top and bottom edges, representing strands. Ellipses (...) indicate additional strands or components.

APPLICATION: CENTRAL ELEMENTS OF BMW_n

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of BMW_n .

$$M_{n,i} = \text{Diagram} \quad 2 \leq i \leq n$$

Proposition (P.)

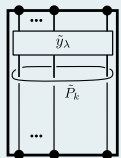
$$\text{Diagram} = \langle \tilde{P}_k \rangle 1 + (s^k - s^{-k}) \sum_{i=2}^n v^{-k} M_{n,i}^k - v^k M_{n,i}^{-k}$$

APPLICATION: CENTRAL ELEMENTS OF BMW_n Sketch: Apply commutation relation for \tilde{P}_k .

$$\begin{aligned}
 & \text{Diagram 1: A rectangular box with 4 vertical lines and dots at the top and bottom. A horizontal oval labeled \tilde{P}_k encircles the middle of the box. Ellipses (...) are placed above and below the oval.} \\
 &= \text{Diagram 2: Same as Diagram 1, but with a crossing on the right side. The crossing is labeled k at the top right.} + (s^k - s^{-k}) \text{Diagram 3: Same as Diagram 2, but with a crossing on the left side. The crossing is labeled k at the top left.} - (s^k - s^{-k}) \text{Diagram 4: Same as Diagram 3, but with a crossing on the right side. The crossing is labeled k at the top right.} \\
 &= \text{Diagram 5: Same as Diagram 1, but with a crossing on the right side. The crossing is labeled k at the top right.} + (s^k - s^{-k})v^{-k} \text{Diagram 6: Same as Diagram 5, but with a crossing on the left side. The crossing is labeled k at the top left.} - (s^k - s^{-k})v^k \text{Diagram 7: Same as Diagram 6, but with a crossing on the right side. The crossing is labeled k at the top right.}
 \end{aligned}$$

APPLICATION: MERIDIANS OF \tilde{y}_λ

Proposition (P.)



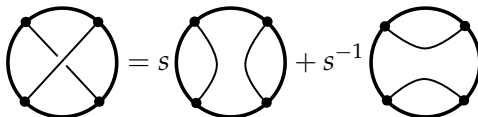
$$= \left(\langle \tilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left(v^{-k} s^{2\text{cn}(\square)} - v^k s^{-2\text{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

Observation: For any fixed k , the eigenvalues of \tilde{y}_λ are distinct.

Consequence: The basis $\{\tilde{Q}_\lambda\}$ of $\mathcal{D}(A)$ is a simultaneous eigenbasis with 1-dimensional eigenspaces for all \tilde{P}_k . Setting $k = 1$ recovers the result from [Lu-Zhong, 2002].

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

The Kauffman bracket skein relation \rightsquigarrow Jones polynomial

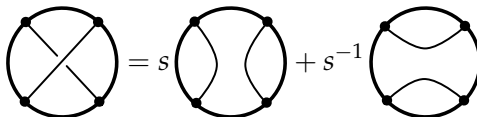


The diagram illustrates the Kauffman bracket skein relation. It shows three circular diagrams with four marked points on the boundary. The first diagram on the left has two straight lines crossing each other. This is equal to s times the second diagram, which has two vertical arcs, plus s^{-1} times the third diagram, which has two horizontal arcs.

$$\text{Crossing} = s \cdot \text{Vertical Arcs} + s^{-1} \cdot \text{Horizontal Arcs}$$

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

The Kauffman bracket skein relation \rightsquigarrow Jones polynomial



Fact: The Kauffman bracket skein relation implies the Dubrovnik skein relation.

Consequence: There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

Theorem (Morton-P.-Samuelson)

The image of $\tilde{P}_k \in \mathcal{D}(A)$ under η_A is the Chebyshev polynomial $T_k \in \mathcal{K}(A)$.

SKEIN ALGEBRAS OF T^2

Theorem (Frohman-Gelca 2000)

The algebra $\mathcal{K}(T^2)$ is presented by generators $T_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2 / \langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$T_{\mathbf{x}}T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}-\mathbf{y}}$$

Theorem (Morton-Samuelson 2017)

The algebra $\mathcal{H}(T^2)$ is presented by generators $P_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$ subject to the relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}+\mathbf{y}}$$

A PRESENTATION OF $\mathcal{D}(T^2)$

Let $\mathbf{x} = (a, b)$, $k = \gcd(\mathbf{x})$.

Define $\tilde{P}_{\mathbf{x}} \in \mathcal{D}(T^2)$ be the embedding of \tilde{P}_k along the closed curve of slope a/b .

Theorem (Morton-P.-Samuelson)

The algebra $\mathcal{D}(T^2)$ is presented by generators $\tilde{P}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2 / \langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$$

MAIN THEOREM: A PRESENTATION OF $\mathcal{D}(T^2)$

Proof:

1. The $\tilde{P}_{\mathbf{x}}$ generate $\mathcal{D}(T^2)$.
 - 1.1 The \tilde{P}_k generate $\mathcal{D}(A)$.
 - 1.2 Each link in $\mathcal{D}(T^2)$ is a sum of products of annular knots.
2. $[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$.
 - 2.1 $[\tilde{P}_{1,0}, \tilde{P}_{0,n}] = (s^n - s^{-n})(\tilde{P}_{1,n} - \tilde{P}_{1,-n})$
 - 2.2 $[\tilde{P}_{1,0}, \tilde{P}_{1,n}] = (s^n - s^{-n})(\tilde{P}_{2,n} - \tilde{P}_{0,n})$
 - 2.3 Use $SL_2(\mathbb{Z})$ -action on $\mathcal{D}(T^2)$. Induct on $|\det(\mathbf{x}, \mathbf{y})|$.
3. These relations present the algebra.
 - 3.1 Diamond lemma type argument \rightsquigarrow unordered words in $\tilde{P}_{\mathbf{x}}$ form a basis.
 - 3.2 The above relations allow reordering of words.

RELATIONSHIP WITH $\mathcal{K}(T^2)$

Corollary (Morton-P.-Samuelson)

There is a surjective algebra homomorphism $\mathcal{D}(T^2) \rightarrow \mathcal{K}(T^2)$ defined by

$$\tilde{P}_x \mapsto T_x.$$

Proof: Use the natural transformation $\eta : \mathcal{D} \Rightarrow \mathcal{K}$. Recall

$$\eta_A(\tilde{P}_k) = T_k.$$

Note: $\mathcal{D}(T^2)$ is much bigger than $\mathcal{K}(T^2)$.

RELATIONSHIP WITH $\mathcal{H}(T^2)$

Corollary (P.)

The algebras $\mathcal{D}(T^2)$ and $\mathcal{H}(T^2)$ are universal enveloping algebras of some Lie algebras $\mathfrak{g}_{\mathcal{D}}$ and $\mathfrak{g}_{\mathcal{H}}$ spanned by the $\tilde{P}_{\mathbf{x}}$ and $P_{\mathbf{x}}$, respectively. There is an injective Lie algebra homomorphism $\mathfrak{g}_{\mathcal{D}} \rightarrow \mathfrak{g}_{\mathcal{H}}$ defined by

$$\tilde{P}_{\mathbf{x}} \mapsto P_{\mathbf{x}} + P_{-\mathbf{x}}.$$

Proof: $[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] - (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$ is sent to 0.

Note: This restricts to an algebra homomorphism

$$\mathcal{D}(A) \rightarrow \mathcal{H}(A).$$

Note: $\mathcal{D}(A)$ and $\mathcal{H}(A)$ are related to characters of classical Lie groups.

A $\mathcal{D}(T^2)$ -ACTION ON $\mathcal{D}(D^2 \times S^1)$

Lemma (P.)

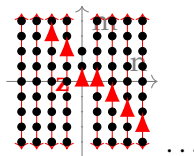
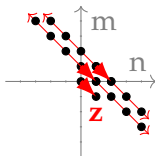
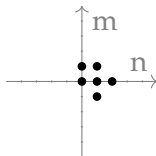
$\mathcal{D}(T^2)$ is generated by the identity and the elements

$$\tilde{P}_{1,0}, \tilde{P}_{0,1}, \tilde{P}_{1,1}, \tilde{P}_{2,0}$$

Proof: Can generate a “line” of \tilde{P}_x by

$$\tilde{P}_{\mathbf{y}+n\mathbf{z}} = (s^d - s^{-d})^{-1} \left([\tilde{P}_{\mathbf{y}+(n-1)\mathbf{z}}, \tilde{P}_{\mathbf{z}}] + \{d\} \tilde{P}_{\mathbf{y}+(n-2)\mathbf{z}} \right)$$

as long as $d := \det(\mathbf{y} + n\mathbf{z}, \mathbf{z}) = \det(\mathbf{y}, \mathbf{z}) \neq 0$.



A $\mathcal{D}(T^2)$ -ACTION ON $\mathcal{D}(D^2 \times S^1)$

Proposition (P.)

The $\mathcal{D}(T^2)$ -action on $\mathcal{D}(D^2 \times S^1)$ is determined by the equations

$$\tilde{P}_{1,0} \cdot \tilde{Q}_\lambda = \left(\langle \tilde{P}_1 \rangle + \{1\} \left(v^{-1} \sum_{\square \in \lambda} s^{2\text{cn}(\square)} - v \sum_{\square \in \lambda} s^{-2\text{cn}(\square)} \right) \right) \tilde{Q}_\lambda$$

$$\tilde{P}_{2,0} \cdot \tilde{Q}_\lambda = \left(\langle \tilde{P}_2 \rangle + \{2\} \left(v^{-2} \sum_{\square \in \lambda} s^{4\text{cn}(\square)} - v^2 \sum_{\square \in \lambda} s^{-4\text{cn}(\square)} \right) \right) \tilde{Q}_\lambda$$

$$\tilde{P}_{0,1} \cdot \tilde{Q}_\lambda = \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} \tilde{Q}_\mu + \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} \tilde{Q}_\nu$$

$$\tilde{P}_{1,1} \cdot \tilde{Q}_\lambda = v^{-1} \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} s^{2\text{cn}(\square)} \tilde{Q}_\mu + v \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} s^{-2\text{cn}(\square)} \tilde{Q}_\nu$$

SYMMETRIC FUNCTIONS AND $\mathcal{H}(A)^+$

Λ - The ring of symmetric functions

s_λ - Schur function

$\mathcal{H}(A)^+$ - Annular HOMFLYPT links oriented counter-clockwise

Theorem (Lukac 2005)

*There is a basis of elements Q_λ in $\mathcal{H}(A)^+$ indexed by partitions λ .
The assignment $\Lambda \rightarrow \mathcal{H}(A)^+$ defined by*

$$s_\lambda \mapsto Q_\lambda$$

is an algebra isomorphism.

Hope: Generalize this to Dubrovnik case.

ORTHOGONAL & SYMPLECTIC SCHUR FUNCTIONS

Partition λ \longrightarrow Symmetric functions sb_λ, sc_λ

Theorem (Koike-Terada 1987, Koike 1989)

The sets $\{sb_\lambda\}$ and $\{sc_\lambda\}$ are bases of Λ .

The structure constants of Λ with respect to each basis are identical natural numbers.

Conjecture

Both of the assignments $\Lambda \rightarrow \mathcal{D}(A)$ defined by

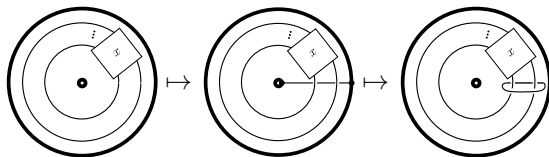
$$sb_{(n)} \mapsto \tilde{Q}_{(n)} \quad sc_{(n)} \mapsto \tilde{Q}_{(n)}$$

are algebra isomorphisms which send sb_λ and sc_λ to \tilde{Q}_λ , respectively.

PARTIAL PROOF OF CONJECTURE

Idea: Let S_λ be the image of sc_λ . Show $S_\lambda = \tilde{Q}_\lambda$.

1. Show S_λ is in the same eigenspace of \tilde{Q}_λ with respect to the meridian map Γ .
 - 1.1 Use a Jacobi-Trudi type identity to write S_λ as a determinant whose entries are in terms of $S_{(n)} = \tilde{Q}_{(n)}$.
 - 1.2 Factor Γ as $\Gamma(x) = \text{cl}_A(x \cdot e)$:



- 1.3 $\text{cl}_A(e \cdot x) = \delta x$. Try to translate left-action to right-action (works if $|\lambda| \leq 2$).
 - 1.4 Compute eigenvalue.
2. Eigenspaces are 1-dimensional $\Rightarrow S_\lambda = dQ_\lambda$. Show $d = 1$ with branching rules: $S_\square^n = \sum c_\mu S_\mu$ and $\tilde{Q}_\square^n = \sum c_\mu \tilde{Q}_\mu$

Thanks for listening!

