

Dubrovnik Skein Theory and Power Sum Elements

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INTRODUCTION

INTRODUCTION

BACKGROUND

POWER SUM ELEMENTS

THE ALGEBRA $\mathcal{D}(T^2)$

TYPE B/C/D SCHUR FUNCTIONS

(FRAMED) LINK INVARIANTS

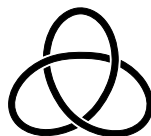


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(FRAMED) LINK INVARIANTS

 $\stackrel{?}{=}$ 

Dubrovnik Polynomial

 $p_1(s, v)$ \neq $p_2(s, v)$

DUBROVNIK SKEIN RELATIONS

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \bigcirc = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \bigcirc + (s - s^{-1}) \left(\begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \bigcirc - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \bigcirc \right) \quad (1)$$

$$\begin{array}{c} \text{twist} \end{array} \bigcirc = v \begin{array}{c} \text{crossing} \end{array} \bigcirc \quad (2)$$

$$\bigcirc \bigcirc = \left(1 - \frac{v - v^{-1}}{s - s^{-1}} \right) \bigcirc \quad (3)$$

DUBROVNIK SKEIN RELATIONS

$$\text{Diagram 1} = \text{Diagram 2} + (s - s^{-1}) \left(\text{Diagram 3} - \text{Diagram 4} \right) \quad (1)$$

$$\text{Diagram 5} = v \text{Diagram 6} \quad (2)$$

$$\text{Diagram 7} = \left(1 - \frac{v - v^{-1}}{s - s^{-1}} \right) \text{Diagram 8} \quad (3)$$

$$\text{Diagram 9} = \text{Diagram 10} + (s - s^{-1}) \left(\text{Diagram 11} - \text{Diagram 12} \right)$$

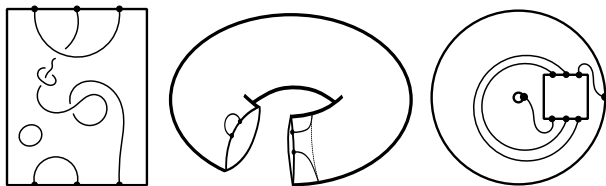
SKEIN MODULES

Observation: Skein relations are defined locally.

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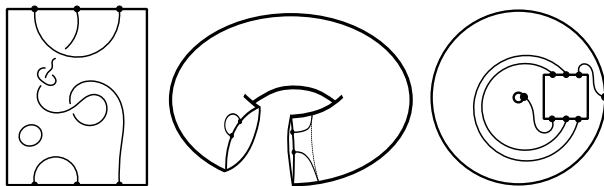
Consequence: May impose skein relations on tangles in arbitrary 3-dimensional manifolds.



SKEIN MODULES

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Consequence: May impose skein relations on tangles in arbitrary 3-dimensional manifolds.



Definition

Let M be an oriented 3-manifold and $R := \mathbb{Q}(s, v)$.

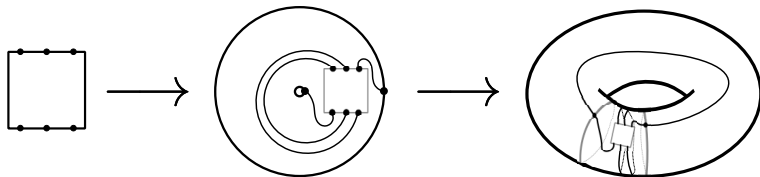
$$\mathcal{D}(M, N) := R\{\text{Tangles in } M \text{ relative to } N\} / \sim$$

FUNCTORIALITY

A “nice” embedding $f : M \rightarrow M'$

+ A wiring diagram in image complement

= A linear transformation $D(f) : \mathcal{D}(M, N) \rightarrow \mathcal{D}(M', N')$

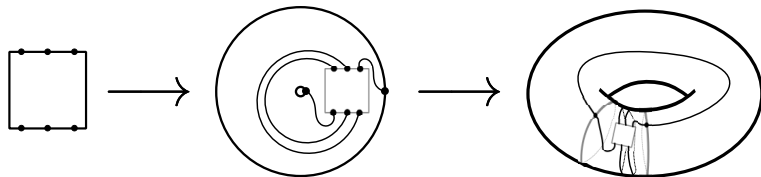


FUNCTORIALITY

A “nice” embedding $f : M \rightarrow M'$

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= A linear transformation $D(f) : \mathcal{D}(M, N) \rightarrow \mathcal{D}(M', N')$



Consequence: Dubrovnik skein theory is a type of algebraic topology for smooth, oriented, 3-manifolds.

SPECIAL CASE: SKEIN ALGEBRAS

If

- ▶ $M = \Sigma \times I$
- ▶ $N = (X \times \{0\}) \sqcup (X \times \{1\})$

Then $\mathcal{D}(M, N)$ is naturally an algebra.

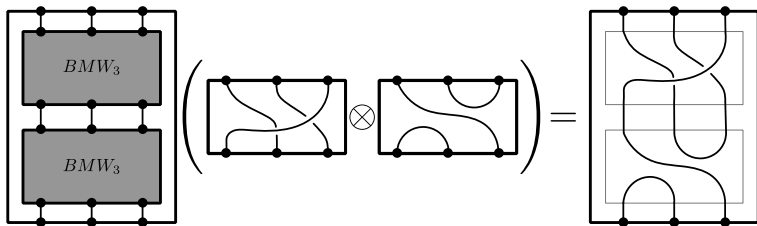
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If

- ▶ $M = \Sigma \times I$
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Then $\mathcal{D}(M, N)$ is naturally an algebra.

e.g.: $\Sigma = \text{Square}$, $N = 2n$ points $\rightsquigarrow \mathcal{D}(\Sigma, N) \cong BMW_n$



IDEMPOTENTS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

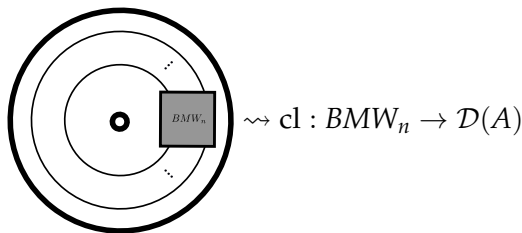
Theorem (Ram-Wenzl 1992, Beliakova-Blanchet, 2001)

For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.

IDEMPOTENTS IN BIRMAN-MURAKAMI-WENZL ALGEBRAS

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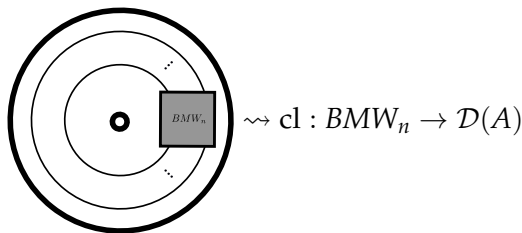
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For each partition $\lambda \vdash n$, there is a minimal idempotent $\tilde{y}_\lambda \in BMW_n$.



Theorem (Lu-Zhong 2002)

The elements $\tilde{Q}_\lambda := \text{cl}(\tilde{y}_\lambda)$ form a basis of $\mathcal{D}(A)$.

DUBROVNIK POWER SUM ELEMENTS

Define a family of elements $\tilde{P}_k \in \mathcal{D}(A)$ for $k \in \mathbb{Z}_{\geq 1}$ via

$$\sum_{k \geq 1} \frac{\tilde{P}_k}{k} t^k = \ln \left(1 + \sum_{n \geq 1} \tilde{Q}_{(n)} t^n \right)$$

Idea:

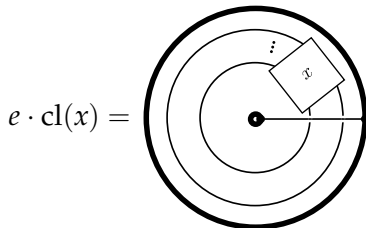
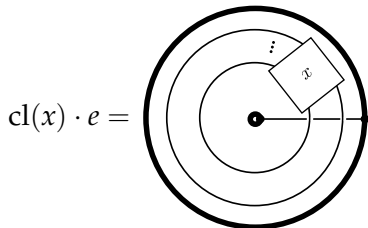
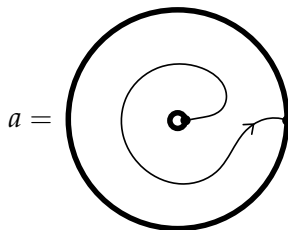
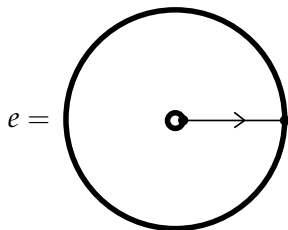
$\tilde{Q}_{(n)}$ are “like” complete homogeneous symmetric functions.

$\rightsquigarrow \tilde{P}_k$ are “like” power sum symmetric functions.

(Will make more precise later)

A RELATIVE SKEIN ALGEBRA

Let $\mathcal{A} := \mathcal{D}(A, 1)$.



COMMUTATION RELATIONS

Theorem (Morton-P.-Samuelson)

$$e \cdot \tilde{P}_k - \tilde{P}_k \cdot e = (s^k - s^{-k})(a^k - a^{-k})$$

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Theorem (P.)

$$e \cdot \tilde{Q}_{(n)} - \tilde{Q}_{(n)} \cdot e = \sum_{i=1}^n d_i (e \cdot \tilde{Q}_{(n-i)})$$

where

$$d_i = \sum_{l=0}^{i-1} (s^{2l-i} - 1)s^{2l-i}a^{i-2l} + (s^{-2l-i} - 1)s^{i-2l}a^{2l-i}$$

IDEA OF PROOF

1) By power series manipulations, the statement is equivalent to

$$\begin{aligned}
 & e \cdot (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) - (\tilde{Q}_{(n+2)} + \tilde{Q}_{(n)}) \cdot e \\
 & \quad = \\
 & (sa + s^{-1}a^{-1})(e \cdot \tilde{Q}_{(n+1)}) - (s^{-1}a + sa^{-1})(\tilde{Q}_{(n+1)} \cdot e)
 \end{aligned}$$

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 \end{aligned}$$

2) [Shelly, 2016] The $\tilde{y}_{(n)}$ satisfy a skein-theoretic recurrence relation.

$$[n+1] \begin{array}{|c|} \hline \tilde{y}_{n+1} \\ \hline \end{array} = [n]s^{-1} \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array} + \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array} + [n]s^{-1}\beta_n \begin{array}{|c|} \hline \tilde{y}_n \\ \hline \end{array}$$

APPLICATION: CENTRAL ELEMENTS OF BMW_n

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of BMW_n .

$$M_{n,i} = \begin{array}{c} \text{Diagram of } M_{n,i} \end{array} \quad 2 \leq i \leq n$$

APPLICATION: CENTRAL ELEMENTS OF BMW_n

The Jucys-Murphy elements $M_{n,i}$ generate a commutative subalgebra of BMW_n .

$$M_{n,i} = \text{Diagram} \quad 2 \leq i \leq n$$

Theorem (P.)

$$\text{Diagram} = \langle \tilde{P}_k \rangle 1 + (s^k - s^{-k}) \sum_{i=2}^n v^{-k} M_{n,i}^k - v^k M_{n,i}^{-k}$$

APPLICATION: CENTRAL ELEMENTS OF BMW_n

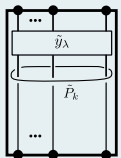
Sketch: Apply commutation relation for \tilde{P}_k .

APPLICATION: CENTRAL ELEMENTS OF BMW_n Sketch: Apply commutation relation for \tilde{P}_k .

$$\begin{aligned}
 & \text{Diagram 1: A rectangular box with vertical lines and dots at the top and bottom. A horizontal oval labeled \tilde{P}_k is in the center. Ellipses (...) are above and below the oval.} \\
 &= \text{Diagram 2: Same as Diagram 1.} + (s^k - s^{-k}) \text{Diagram 3: Same as Diagram 1, but with a crossing on the right side labeled } k. \\
 & \quad - (s^k - s^{-k}) \text{Diagram 4: Same as Diagram 1, but with a crossing on the right side labeled } k. \\
 &= \text{Diagram 5: Same as Diagram 1.} + (s^k - s^{-k}) v^{-k} \text{Diagram 6: Same as Diagram 3, but with a crossing on the left side labeled } k. \\
 & \quad - (s^k - s^{-k}) v^k \text{Diagram 7: Same as Diagram 4, but with a crossing on the left side labeled } k.
 \end{aligned}$$

APPLICATION: MERIDIANS OF \tilde{y}_λ

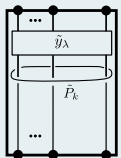
Theorem (P.)



$$= \left(\langle \tilde{P}_k \rangle + (s^k - s^{-k}) \sum_{\square \in \lambda} \left(v^{-k} s^{2\text{cn}(\square)} - v^k s^{-2\text{cn}(\square)} \right) \right) \tilde{y}_\lambda$$

APPLICATION: MERIDIANS OF \tilde{y}_λ

Theorem (P.)

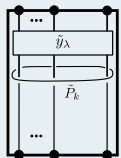


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Observation: For any fixed k , the eigenvalues of \tilde{y}_λ are distinct.

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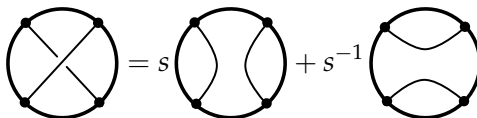
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Observation: For any fixed k , the eigenvalues of \tilde{y}_λ are distinct.

Consequence: The basis $\{\tilde{Q}_\lambda\}$ of $\mathcal{D}(A)$ is an eigenbasis with 1-dimensional eigenspaces. Setting $k = 1$ recovers the result from [Lu-Zhong, 2002].

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

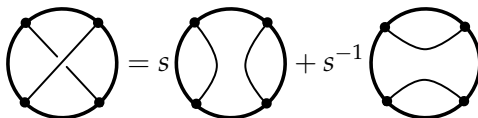
The Kauffman bracket skein relation \rightsquigarrow Jones polynomial



The diagram illustrates the Kauffman bracket skein relation. On the left is a circle with four black dots at the top, bottom, left, and right positions. Two straight lines connect the dots diagonally, forming an 'X' shape. This is followed by an equals sign, then the variable s , then a circle with four black dots at the top, bottom, left, and right positions. Two curved lines connect the dots, forming two separate loops (one on the left and one on the right). This is followed by a plus sign, then s^{-1} , then another circle with four black dots at the top, bottom, left, and right positions. Two curved lines connect the dots, forming two separate loops (one on the top and one on the bottom).

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

The Kauffman bracket skein relation \rightsquigarrow Jones polynomial



The diagram illustrates the Kauffman bracket skein relation. It shows a circle with four marked points (top, bottom, left, right). On the left, the circle is divided into four quadrants by two straight lines crossing at the center. This is followed by an equals sign, then the variable s , then a circle with two vertical arcs (left and right) connecting the top and bottom points. This is followed by a plus sign, then s^{-1} , then a circle with two horizontal arcs (top and bottom) connecting the left and right points.

$$\text{Crossing} = s \text{ (Vertical Arcs)} + s^{-1} \text{ (Horizontal Arcs)}$$

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

The Kauffman bracket skein relation \rightsquigarrow Jones polynomial

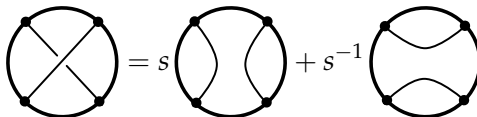
The diagram illustrates the Kauffman bracket skein relation. On the left is a circle with four black dots at the top, bottom, left, and right positions. Two straight lines connect the dots diagonally: one from the top-left to the bottom-right, and another from the top-right to the bottom-left. This is followed by an equals sign, then the variable s , then a circle with four black dots at the same positions. Two curved lines connect the dots: one from the top-left to the top-right, and another from the bottom-left to the bottom-right. This is followed by a plus sign, then s^{-1} , then a third circle with four black dots at the same positions. Two curved lines connect the dots: one from the top-left to the bottom-right, and another from the top-right to the bottom-left.

Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

Consequence: There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

COMPATIBILITY WITH KAUFFMAN BRACKET SKEIN THEORY \mathcal{K}

The Kauffman bracket skein relation \rightsquigarrow Jones polynomial



Fact: The Dubrovnik skein relation satisfies the Kauffman bracket skein relation.

Consequence: There is a natural transformation of skein theories $\eta : \mathcal{D} \Rightarrow \mathcal{K}$.

Theorem (Morton-P.-Samuelson)

The image of $\tilde{P}_k \in \mathcal{D}(A)$ under η_A is the Chebyshev polynomial $T_k \in \mathcal{K}(A)$.

SKEIN ALGEBRAS OF T^2

Theorem (Frohman-Gelca, 2000)

The algebra $\mathcal{K}(T^2)$ is presented by generators $T_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2 / \langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$T_{\mathbf{x}}T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}-\mathbf{y}}$$

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Theorem (Morton-Samuelson, 2017)

The algebra $\mathcal{H}(T^2)$ is presented by generators $P_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2$ subject to the relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}+\mathbf{y}}$$

A PRESENTATION OF $\mathcal{D}(T^2)$

Let $\mathbf{x} = (a, b)$, $k = \gcd(\mathbf{x})$.

Define $\tilde{P}_{\mathbf{x}} \in \mathcal{D}(T^2)$ be the embedding of \tilde{P}_k along the closed curve of slope a/b .

Theorem (Morton-P.-Samuelson)

The algebra $\mathcal{D}(T^2)$ is presented by generators $\tilde{P}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2 / \langle \mathbf{x} = -\mathbf{x} \rangle$ subject to the relations

$$[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$$

A PRESENTATION OF $\mathcal{D}(T^2)$

Proof:

1. $[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}}).$
 - 1.1 $[\tilde{P}_{1,0}, \tilde{P}_{0,n}] = (s^n - s^{-n})(\tilde{P}_{1,n} - \tilde{P}_{1,-n})$
 - 1.2 $[\tilde{P}_{1,0}, \tilde{P}_{1,n}] = (s^n - s^{-n})(\tilde{P}_{2,n} - \tilde{P}_{0,n})$
 - 1.3 Use $SL_2(\mathbb{Z})$ -action on $\mathcal{D}(T^2)$. Induct on $\gcd(\mathbf{x})$.
2. The $\tilde{P}_{\mathbf{x}}$ generate $\mathcal{D}(T^2)$.
 - 2.1 Each link in $\mathcal{D}(T^2)$ is a sum of products of annular knots.
 - 2.2 The \tilde{P}_k generate $\mathcal{D}(A)$.
3. The $\tilde{P}_{\mathbf{x}}$ are linearly independent.
 - 3.1 Diamond lemma type argument.

RELATIONSHIP WITH $\mathcal{K}(T^2)$

Corollary (Morton-P.-Samuelson)

There is a surjective algebra homomorphism $\mathcal{D}(T^2) \rightarrow \mathcal{K}(T^2)$ defined by

$$\tilde{P}_x \mapsto T_x.$$

Proof: Use the natural transformation $\eta : \mathcal{D} \Rightarrow \mathcal{K}$. Recall

$$\eta_A(\tilde{P}_k) = T_k.$$

Note: $\mathcal{D}(T^2)$ is much bigger than $\mathcal{K}(T^2)$.

RELATIONSHIP WITH $\mathcal{H}(T^2)$

Corollary (P.)

The algebras $\mathcal{D}(T^2)$ and $\mathcal{H}(T^2)$ are universal enveloping algebras of some Lie algebras $\mathfrak{g}_{\mathcal{D}}$ and $\mathfrak{g}_{\mathcal{H}}$ generated by the $\tilde{P}_{\mathbf{x}}$ and $T_{\mathbf{x}}$, respectively. There is an injective Lie algebra homomorphism $\mathfrak{g}_{\mathcal{D}} \rightarrow \mathfrak{g}_{\mathcal{H}}$ defined by

$$\tilde{P}_{\mathbf{x}} \mapsto P_{\mathbf{x}} + P_{-\mathbf{x}}.$$

Proof: $[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] - (s^{\det(\mathbf{x}, \mathbf{y})} - s^{-\det(\mathbf{x}, \mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})$ is sent to 0.

Note: This restricts to an algebra homomorphism

$$\mathcal{D}(A) \rightarrow \mathcal{H}(A).$$

Note: $\mathcal{D}(A)$ and $\mathcal{H}(A)$ are related to characters of classical Lie groups.

A $\mathcal{D}(T^2)$ -ACTION ON $\mathcal{D}(D^2 \times S^1)$

Lemma (P.)

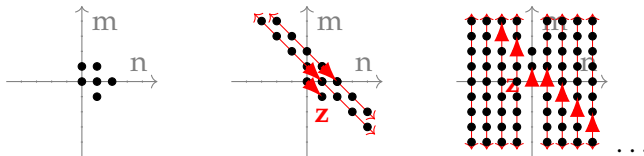
$\mathcal{D}(T^2)$ is generated by the identity and the elements

$$\tilde{P}_{1,0}, \tilde{P}_{0,1}, \tilde{P}_{1,1}, \tilde{P}_{2,0}$$

Proof: Can generate a “line” of \tilde{P}_x by

$$\tilde{P}_{\mathbf{y}+n\mathbf{z}} = (s^d - s^{-d})^{-1} \left([\tilde{P}_{\mathbf{y}+(n-1)\mathbf{z}}, \tilde{P}_{\mathbf{z}}] + \{d\} \tilde{P}_{\mathbf{y}+(n-2)\mathbf{z}} \right)$$

as long as $d := \det(\mathbf{y} + n\mathbf{z}, \mathbf{z}) = \det(\mathbf{y}, \mathbf{z}) \neq 0$.



A $\mathcal{D}(T^2)$ -ACTION ON $\mathcal{D}(D^2 \times S^1)$

Theorem (P.)

The $\mathcal{D}(T^2)$ -action on $\mathcal{D}(D^2 \times S^1)$ is determined by the equations

$$\tilde{P}_{1,0} \cdot \tilde{Q}_\lambda = \left(\langle \tilde{P}_1 \rangle + \{1\} \left(v^{-1} \sum_{\square \in \lambda} s^{2\text{cn}(\square)} - v \sum_{\square \in \lambda} s^{-2\text{cn}(\square)} \right) \right) \tilde{Q}_\lambda$$

$$\tilde{P}_{2,0} \cdot \tilde{Q}_\lambda = \left(\langle \tilde{P}_2 \rangle + \{2\} \left(v^{-2} \sum_{\square \in \lambda} s^{4\text{cn}(\square)} - v^2 \sum_{\square \in \lambda} s^{-4\text{cn}(\square)} \right) \right) \tilde{Q}_\lambda$$

$$\tilde{P}_{0,1} \cdot \tilde{Q}_\lambda = \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} \tilde{Q}_\mu + \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} \tilde{Q}_\nu$$

$$\tilde{P}_{1,1} \cdot \tilde{Q}_\lambda = v^{-1} \sum_{\substack{\lambda \subset \mu \\ \mu = \lambda + \square}} s^{2\text{cn}(\square)} \tilde{Q}_\mu + v \sum_{\substack{\nu \subset \lambda \\ \lambda = \nu + \square}} s^{-2\text{cn}(\square)} \tilde{Q}_\nu$$

SYMMETRIC FUNCTIONS AND $\mathcal{H}(A)^+$

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TYPE B/C/D SCHUR FUNCTIONS
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FRAME 1

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