UNIVERSITY OF CALIFORNIA
RIVERSIDE

THESIS TITLE

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Alexander Pokorny

MONTH 2021

Dissertation Committee:

Dr. Peter Samuelson, Chairperson
Dr. Jacob Greenstein
Dr. Stephano Vidussi

The Dissertation of Alexander Pokorny is approved:

Committee Chairperson

University of California, Riverside

Acknowledgements

ACKNOWLEDGEMENTS

This section may be omitted.

DEDICATION

This section may be omitted.

ABSTRACT OF THE DISSERTATION

THESIS TITLE

by

Alexander Pokorny

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, ***MONTH*** 2021
Dr. Peter Samuelson, Chairperson

The text of the abstract goes here. An abstract is required, and must not exceed 350 words. Further details can be found in UCR's formatting guidelines:

`https://graduate.ucr.edu/document/format-guide-201920`

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Summarize the main background and new results here.

Chapter 2

Background

2.1 Skein Theory

2.1.1 Foundations and General Notions

In this work, we will be forced to discuss a few different variants of skein modules. For this reason, it will be useful to first describe some general framework of skein theory so that each of these variants will be a special case. Unless otherwise stated, we will assume M is an oriented 3-manifold with boundary ∂M (possibly empty), Σ is an oriented surface, I is the real interval $[0,1]$, R is a commutative and unital ring.

Definition 2.1.1. Let $T_1, T_2 : X \to M$ be smooth embeddings of a smooth manifold X into M. A **smooth ambient isotopy** $H : T_1 \Rightarrow T_2$ is a smooth homotopy of diffeomorphisms $H_t : M \to M$ such that $H_0 = \mathrm{id}_M$ and $H_1 \circ T_1 = T_2$. Furthermore, we demand that the boundary ∂M is fixed by the homotopy.

The relation

$$T_1 \sim T_2 \text{ if and only if there exists a smooth ambient isotopy } H : T_1 \Rightarrow T_2$$

is an equivalence relation. The smoothness requirement of H is important when considering knots. Without it, all knots would be equivalent to the unknot by contracting all of the complexity of the knot to a point.

Definition 2.1.2. Let N be a finite set of points contained in the boundary ∂M. An N**-tangle** in M (or just *tangle* for short) is the smooth ambient isotopy class of a smooth embedding

$$T : \underbrace{\bigsqcup_{j\in J} S^1}_{:=L} \sqcup \underbrace{\bigsqcup_{k\in K} I}_{:=B} \to M$$

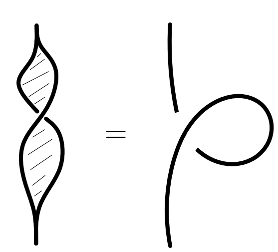
for some finite sets J and K such that

1. the image of L lies in the interior of M,
2. the image of the interior of B lies in the interior of M,
3. the image of the boundary of B equals N.

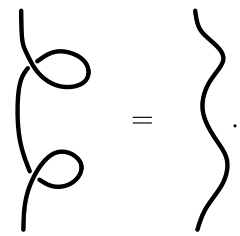
If B is empty, then the result is called a **link** in M. Similarly, if L is empty, then it's called a **braid** in M.

One may also consider *oriented* or *framed* tangles by choosing an orientation or framing for each point in N and for each connected component of L and B such that the choices are compatible with each other with respect to the smooth embedding. If $M = \Sigma \times I$, then we will assume that the points in N are contained in $\Sigma \times \{\frac{1}{2}\}$ and that their framings are thought to be embedded orthogonally to $\Sigma \times \{*\}$.

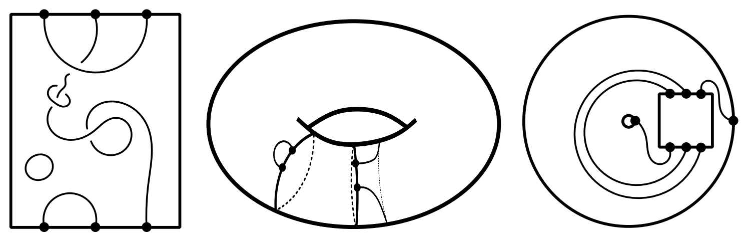
We say a framed tangle in $\Sigma \times I$ has **blackboard framing** if the entire framing is embedded orthogonally to Σ. Every framed link in $\Sigma \times I$ is isotopic to one with a blackboard framing by turning each twist into a loop with a local blackboard framing:



This suggests that we may represent framed links in $\Sigma \times I$ as link diagrams on Σ. Indeed, equivalence under ambient isotopy is captured by the Reidemeister moves 2, 3, and a modified Reidemeister move 1:



* Add pictures of RII and RIII for good measure. Below are some examples of framed tangle diagrams.



* fix torus picture by removing band.

Define the **writhe** of a tangle diagram is the number of positive crossings minus the number of negative crossings. It is easy to see that the Reidemeister moves above preserve the writhe of a diagram, so the concept is well defined. Writhe should be thought of as a grading on the free R-module on the set of tangles in the given space, which provides a good reason to work with framed links over ordinary links. Such a module is a main ingredient of this theory, so let's honor it with a proper discussion.

Definition 2.1.3. Let $\mathcal{T}(M, N)$ be the free R-module generated by the set of framed N-tangles in M. Analagously, we can define $\mathcal{T}^{or}(M, N)$ to be the free R-module generated by the set of oriented framed N-tangles in M. All definitions which are to follow in this subsection have an analagous definition using oriented tangles. Also, we will formally define $\mathcal{S}_R(\varnothing, \varnothing) := R$.

The construction $\mathcal{T}(-,-)$ is actually a symmetric monoidal functor $\mathcal{T} : \mathsf{C} \to R\text{-}\mathsf{Mod}$ for a careful choice of category C which we now describe. The objects of C are pairs (M, N) of the same type as discussed previously. A morphism $(f, W) : (M', N') \to (M, N)$ is a pair of a smooth, orientation-preserving embedding $f : M' \to M$ such that $M - f(M')$ is either a smooth 3-manifold or the empty set, and choice of $W \in \mathcal{T}\big(M - f(M'), N \sqcup f(N')\big)$ (unless $M - f(M')$ is empty, in which case W is a formal symbol for the "empty link" in the empty set). Composition is given by $(g, W') \circ (f, W) = (g \circ f, W' \cup W)$, which is associative since $\circ$ and $\cup$ are associative.

* Give a picture of composition in C.

The induced map denoted $W : \mathcal{T}(M', N') \to \mathcal{T}(M, N)$ is a linear map defined by $W(T) = W \cup T$, and we will refer to such a linear map W as a **wiring**. We are abusing notation by denoting this linear map by W, but it should be clear from the context what f is since it is technically encoded in the data of the element $W \in \mathcal{T}\big(M - f(M'), N \sqcup f(N')\big)$. It is true that $\mathcal{T}$ preserves composition and identity morphisms, making and so $\mathcal{T}$ is functorial. C can now be equipped with a symmetric monoidal structure via disjoint union. It is clear that

$$\mathcal{T}_R(M \sqcup M', N \sqcup N') \cong \mathcal{T}_R(M, N) \underset{R}{\otimes} \mathcal{T}_R(M', N')$$

for any sets of framed points $N \subset \partial M$ and $N' \subset \partial M'$. The unit is given by the object $(\varnothing, \varnothing) \in \mathsf{C}$ and define $\mathcal{T}(\varnothing, \varnothing) := R$, which makes $\mathcal{T}$ a symmetric monoidal functor.

Definition 2.1.4. Let B be the smooth closed 3-ball, N_i be a choice of $2i$ boundary points of B, and let $X \subset \bigsqcup_{i \in \mathbb{N}} \mathcal{T}(B, N_i)$ be some (typically finite) set, which we will call a set of **skein relations**. Given any tangle module $\mathcal{T}(M, N_M)$, there exists a submodule $\mathcal{I}(X)$ generated by the set

$$\{W(x) \mid x \in X \text{ and } W : \mathcal{T}(B, N_B) \to \mathcal{T}(M, N_M) \text{ is a wiring diagram}\}.$$

A quotient of the form $\mathcal{S}_X(M,N) := \mathcal{T}(M,N)/\mathcal{I}(X)$ is called a **skein module** of M relative to N. If $N = \varnothing$ is the empty set, we may use the notation $\mathcal{S}_X(M) := \mathcal{S}_X(M,\varnothing)$. Similar definitions may be given using oriented and/or unframed tangles instead.

The construction $\mathcal{S}_X(-,-)$ is a functor in the same way that $\mathcal{T}(-,-)$ is; a smooth, orientation-preserving embedding $f: M \to M'$ and an element $W \in \mathcal{S}_X\big(M - f(M'), N \sqcup f(N')\big)$ (assume N is disjoint from $\mathrm{Im}(f)$) defines a linear map $W : \mathcal{S}_X(M,N) \to \mathcal{S}_X(M',N')$. In fact, the quotient maps $\alpha_{(M,N)} : \mathcal{T}(M,N) \to \mathcal{S}_X(M,N)$ yield a natural transformation. In other words, given a morphism $(M,N) \to (M',N')$ in C, the diagram

$$\begin{tikzcd}
\mathcal{T}(M,N) \arrow[r, "W"] \arrow[d, "\alpha_{(M,N)}"] & \mathcal{T}(M',N') \arrow[d, "\alpha_{(M',N')}"] \\
\mathcal{S}_X(M,N) \arrow[r, "W"] & \mathcal{S}_X(M',N')
\end{tikzcd}$$

commutes. Such a functor will be called a **skein theory** (or *oriented* skein theory if the skein relations are based on oriented tangles).

For any oriented surface Σ, we can define a category $\mathsf{Skein}_X(\Sigma)$ which we will call a **skein category**. The objects of this category are finite sets of framed points (together with a choice of orientation if the skein theory is oriented) N in Σ, and the morphisms $N \to N'$ are elements of $\mathcal{S}_X\big(\Sigma \times I, (N \times \{0\}) \sqcup (N' \times \{1\})\big)$, so the category is R-linear. Write composition of morphisms by concatenation. If $y : N \to N'$ and $z : N' \to N''$ are morphisms, then their composite $yz : N \to N''$ is constucted by gluing y on top of z through N' and rescaling the interval coordinate appropriately.

* Picture of composition in $\mathsf{Skein}_X(\Sigma)$.

The endomorphism algebras in this category are called **skein algebras** and are denoted by $\mathcal{S}_X(\Sigma, N) := \mathcal{S}_X(\Sigma \times I, (N \times \{0\}) \sqcup (N \times \{1\})\big)$. If N is the empty set, then we reduce the notation to simply $\mathcal{S}_X(\Sigma)$.

If $f : \Sigma \to \Sigma'$ is a smooth embedding of surfaces, then there is an induced functor

$$\mathsf{Skein}_X(f) : \mathsf{Skein}_X(\Sigma') \to \mathsf{Skein}_X(\Sigma)$$

defined on objects by $\mathsf{Skein}_X(f)(N) = f(N)$ and on morphisms in the following way. First, extend f trivially to $f \times \mathrm{id}_I : \Sigma \times I \to \Sigma' \times I$. Then, in the skein algebra of the complement of the image of $f \times \mathrm{id}_I$, choose the multiplicative identity element $e \in \mathcal{S}_X\big(\Sigma' - \mathrm{Im}(f)\big)$ which is the empty tangle. The pair $(f \times \mathrm{id}_I, e)$ is an object in the category C, which gives rise to a wiring

$$e : \mathcal{S}_X\Big(\Sigma \times I, \big(N \times \{0\}\big) \sqcup \big(N' \times \{1\}\big)\Big) \to \mathcal{S}_X\Big(\Sigma' \times I, \big(f(N) \times \{0\}\big) \sqcup \big(f(N') \times \{1\}\big)\Big)$$

via the functor $\mathcal{S}_X$. Now we may define what $\mathsf{Skein}_X(f)$ does to morphisms: $\mathsf{Skein}_X(f)(y) = e(y)$ for any $y \in \mathcal{S}_X\Big(\Sigma \times I, (N \times \{0\}) \sqcup (N' \times \{1\})\Big)$.

* Picture of how $\mathsf{Skein}_X(f)$ works on morphisms.

It is clear that $\mathsf{Skein}_X(f)$ preserves composition and identity morphisms. Therefore, if we let Surf be the category of smooth embeddings between smooth oriented surfaces, we can summarize our last few points by saying we have a functor

$$\mathsf{Skein}_X : \mathsf{Surf} \to \mathsf{Cat}.$$

In particular, $\mathsf{Skein}_X(f)$ defines algebra homomorphisms on the skein algebras

$$e : \mathcal{S}_X\big(\Sigma, N\big) \to \mathcal{S}_X\big(\Sigma', f(N)\big).$$

* We use this type of algebra homomorphism when we embed the annulus into the torus.

Remark 2.1.5. The above homomorphisms are a special case of a more general type of map. If N is a set of framed points on Σ, then a smooth embedding $f : \Sigma \to \partial M$ induces a $\mathcal{S}_X(\Sigma, N)$-module structure on $\mathcal{S}_X\big(M, N'\big)$ for any N' with $f(N) \subseteq N'$. The action is given by "pushing tangles in through the boundary". In other words, the pre-composition of a smooth embedding of a collar neighborhood $g : \partial M \times I \to M$ with $f \times \mathrm{id}_I : \Sigma \times I \to \partial M \times I$

induces a bilinear map

$$\mathcal{S}_X(\Sigma, N) \times \mathcal{S}_X(M, N') \to \mathcal{S}_X(M, N')$$

because M minus a collar neighborhood is diffeomorphic to itself. Alternatively, a choice of element in $\mathcal{S}_X(\Sigma, N')$ produces a wiring $\mathcal{S}_X(M, N') \to \mathcal{S}_X(M, N')$.

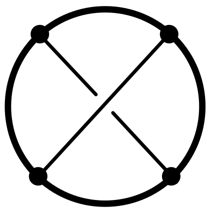
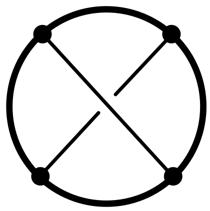
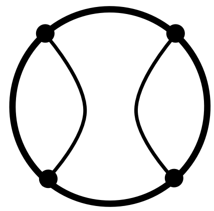
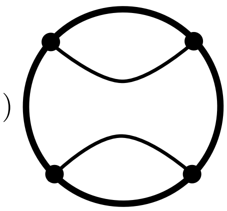
* Picture of action.

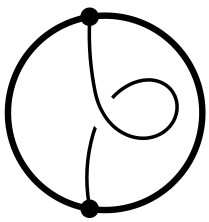
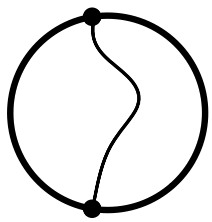
2.1.2 Examples of Skein Theories

The last subsection leaves us with an important and unanswered question. Which sets of skein relations X produce interesting skein theories? One class of examples is found by importing sets of relations satisfied by morphisms in a linear ribbon category as skein relations. Ribbon categories are braided monoidal categories which are rigid and equipped with a twist morphism for every object, satisfying some compatibility conditions. The axioms are such that the morphisms may be interpreted as framed braid diagrams. In particular, the morphisms satisfy the Reidemeister moves shown previously. We will discuss three examples of skein theories derived from skein relations which are meant to emulate linear relations satisfied by the braid and twist morphisms in certain ribbon categories coming from the representation theory of quantum groups, a topic which has generated a lot of interest from mathematicians since the 1980s. The categories of representations of quantum groups are ribbon categories with non-involutive braidings and the skein relations below capture how far off the braidings are from being involutive.

Here, we are forced to fix a base ring. For our purposes, R must be a commutative ring containing invertible elements s and v. Typical choices of R are $\mathbb{Z}[s^{\pm 1}, v^{\pm 1}], \mathbb{Q}(s, v)$, or some other ring in between these. In particular, the theorem * Cite BB is stated over this ring, a result we will depend heavily on later on.

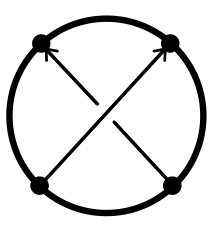
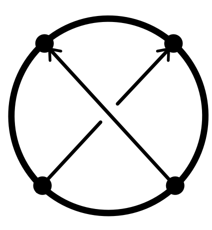
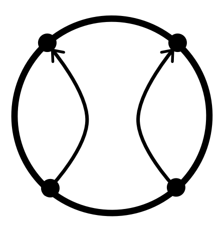
Example 2.1.6 (*Kauffman (Dubrovnik) Skein Relations*)**.** Let X_1 be the set of two unoriented skein relations

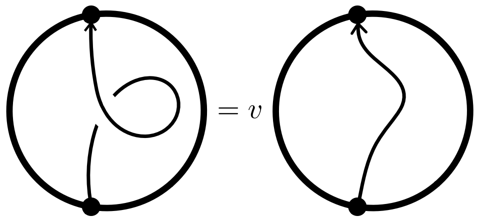
(1)  $=$  $+ (s - s^{-1})$  $- (s - s^{-1})$ 

(2)  $= v$  .

The functor $\mathcal{D}(-,-) := \mathcal{S}_{X_1}(-,-)$ is the Dubrovnik skein theory (sometimes just called the Kauffman skein theory). We will use the notation $\mathsf{D}(-) := \mathsf{Skein}_{X_1}(-)$ for the Dubrovnik skein categories. Using the Dubrovnik variant is important for us (see * universal enveloping algebra result). This theory is related to Dubrovnik polynomials in that the Dubrovnik polynomial of a link is a normalized value of the link in $\mathcal{D}(S^3)$. The normalization is often so that the Dubrovnik polynomial of the unknot is 1, whereas the value of the unknot in $\mathcal{D}(S^3)$ is $\delta_{\mathcal{D}} := 1 - \frac{v - v^{-1}}{s - s^{-1}}$, which can be deduced from the skein relations.

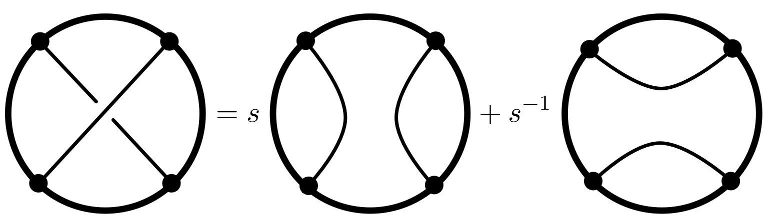
Example 2.1.7 (*HOMFLYPT Skein Relations*)**.** Next, let X_2 be the set of two oriented skein relations

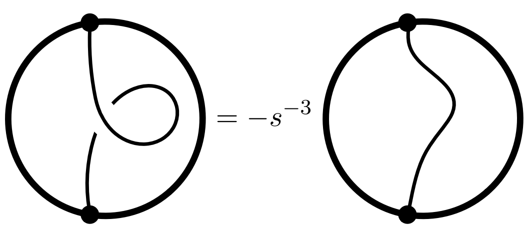
(1)  $=$  $+ (s - s^{-1})$ 

(2) 

The functor $\mathcal{H}(-,-) := \mathcal{S}_{X_2}(-,-)$ is the HOMFLYPT skein theory and we use the notation $\mathsf{H}(-) := \mathsf{Skein}_{X_2}(-)$ for the HOMFLYPT skein categories. As in the Dubrovnik case, this theory is related to HOMFLYPT polynomials so that the HOMFLYPT polynomial of a link is a normalized value of the link in $\mathcal{H}(S^3)$. Again, some often make the choice to normalize this polynomial so that the value of the unknot is 1, but the value of the unknot in $\mathcal{H}(S^3)$ is $\delta_{\mathcal{H}} := -\frac{v-v^{-1}}{s-s^{-1}}$.

Example 2.1.8 (*Kauffman Bracket Skein Relations*)**.** As a final example, let X_3 be the set of two unoriented skein relations

(1) 

(2) 

The functor $\mathcal{K}(-,-) := \mathcal{S}_{X_3}(-,-)$ is the Kauffman bracket skein theory (not be confused with the Kauffman skein theory) and we use $\mathsf{K}(-) := \mathsf{Skein}_{X_3}(-)$ to notate the Kauffman bracket skein categories. The value of a link in $\mathcal{K}(S^3)$ is equal to its bracket polynomial, which may be normalized as above to obtain its Jones polynomial. One could choose the framing parameter to be v instead of $-s^{-3}$ like we did before without any obvious consequences, but it is standard in the literature to make the choice we present here.

Remark 2.1.9. Any linear combination of tangles which satisfy the Dubrovnik skein relations will also satisfy the Kauffman bracket skein relations after making the specialization $v = -s^{-3}$. This can be seen by performing the calculation in the relative skein algebra of the ball with 4 points. Let $\sigma^{\pm}$ be the diagrams of positive and negative crossings, e the planar diagram with two vertical strands, and c the planar diagram with two horizontal strands. Then the Dubrovnik skein relation is

$$\sigma^{+}_{\mathcal{D}} - \sigma^{-}_{\mathcal{D}} - (s - s^{-1})e_{\mathcal{D}} + (s - s^{-1})c_{\mathcal{D}} = 0$$

and the Kauffman bracket skein relation implies

$$\begin{aligned} \sigma^{+}_{\mathcal{K}} &= se_{\mathcal{K}} + s^{-1}c_{\mathcal{K}} \\ \sigma^{-}_{\mathcal{K}} &= sc_{\mathcal{K}} + s^{-1}e_{\mathcal{K}} \end{aligned}$$

where the subscript indicates which skein module the diagrams are in. Consider the assignment $d_{\mathcal{D}} \mapsto d_{\mathcal{H}}$ for any tangle d in the ball relative to those 4 points. Then observe:

$$\begin{aligned} & \sigma^{+}_{\mathcal{K}} - \sigma^{-}_{\mathcal{K}} - (s - s^{-1})e_{\mathcal{K}} + (s - s^{-1})c_{\mathcal{K}} \\ = \quad & (se_{\mathcal{K}} + s^{-1}c_{\mathcal{K}}) - (sc_{\mathcal{K}} + s^{-1}e_{\mathcal{K}}) - (s - s^{-1})e_{\mathcal{K}} + (s - s^{-1})c_{\mathcal{K}} \\ = \quad & 0 \end{aligned}$$

Therefore, there is a natural transformation of skein theories

$$\eta : \mathcal{D}(-,-) \to \mathcal{K}(-,-)$$

whose components are essentially the identity map, using the same type of assignment as we did above. Note that any component of η corresponding to a skein algebra is an algebra homomorphism since the map preserves the (topological) product structure of the manifold.

In this work, we will be focused on generalizing existing results from the HOMFLYPT and Kauffman bracket skein theories to the Dubrovnik skein theory, but we will state a few facts regarding the HOMFLYPT and Kauffman bracket skein theories when it is valuable for us to do so.

* Maybe add a remark about Turaev's work on deformations of the Goldman Lie algebras? Could potentially say a lot to motivate the topic, but I'm not super familiar with a lot of it.

* Add comments about the Dubrovnik and HOMFLYPT skein algebras being graded by ther fundamental group of the surface.

2.1.3 Skein Algebras of Tangles in a Cube

In the case where $\Sigma = I \times I$, the endomorphism objects of $\mathsf{D}(I \times I), \mathsf{H}(I \times I)$, and $\mathsf{K}(I \times I)$ are known and provide the motivation for why the choices of skein relations are what they are. For an integer $n \geq 1$, let $[n]$ be a set of n points in $I \times I$, chosen to be evenly spaced along the line segment $\{1/2\} \times I$ (choose so that all points share the same orientation if in the context of an oriented skein theory). Then the endomorphism algebras

$$BMW_n := \mathrm{End}_{\mathsf{D}}(I \times I)([n]), \qquad H_n := \mathrm{End}_{\mathsf{H}}(I \times I)([n]), \qquad TL_n := \mathrm{End}_{\mathsf{K}}(I \times I)([n])$$

are known to be isomorphic to the Birman-Murakami-Wenzl, (Type A) Hecke, and Temperley-Lieb algebras, respectively (* *Add citations).

Let $q \in \mathbb{C}$ be not a root of unity, $U_q(\mathfrak{gl}_N)$ be the Drinfeld-Jimbo quantum group associated to the Lie algebra $\mathfrak{gl}_N$, and V be the natural representation of $U_q(\mathfrak{gl}_N)$ (* cite Chari-Pressley book). Then H_n acts on the n-fold tensor product $V^{\otimes n}$ by $U_q(\mathfrak{gl}_N)$-linear endomorphisms, and this action generates $\mathrm{End}_{U_q(\mathfrak{gl}_N)}(V^{\otimes n})$. This is actually part of the statement of quantum Frobenius-Schur-Weyl duality: $V^{\otimes n}$ is a $U_q(\mathfrak{gl}_N)$-H_n-bimodule, and the action of one algebra generates the linear endomorphisms with respect to the other. So a representation of one algebra determines a representation of the other via tensor product with $V^{\otimes n}$. The Birman-Murakami-Wenzl algebra plays the role of the Hecke algebra for

$U_q(\mathfrak{g}_N)$ in the case where $\mathfrak{g}$ is one of the orthogonal or symplectic Lie algebras. For this reason, from a Lie theoretic point of view, the HOMFLYPT skein theory is thought of as a "type A" theory, while the Dubrovnik skein theory is a "types B, C, D" skein theory.

As for the Temperley-Lieb algebra TL_n, it acts on the n-fold tensor power $V^{\otimes n}$ of the natural representation of $U_q(\mathfrak{gl}_2)$. Actually, there is a surjective algebra homomorphism $\pi_{TL} : H_n \to TL_n$ and the action $H_n \to \mathrm{End}_{U_q(\mathfrak{gl}_2)}(V^{\otimes n})$ factors through this homomorphism (see * cite Jimbo). One can conclude that the action of H_n on $V^{\otimes n}$ is not faithful, at least in the case when $N = 2$ and $n \geq 2$. * sl2 vs gl2?

Much is known about these algebras and some of the results surrounding them are very useful in our context. One of the main ideas is that each of the algebras discussed above has a family of idempotents which provide algebraically nice closures to links in the skein algebra of an annulus. Let's first describe the Hecke algebra in more detail before focusing on the BMW algebra.

For any partition λ of n, there exists an element $y_\lambda \in H_n$ which is idempotent, so that $y_\lambda^2 = y_\lambda$, and minimal in the sense that it generates a minimal left-ideal of H_n (see * cite Aiston-Morton). The elements $y_n := y_{(n)}$ corresponding to single row partitions are called **(Hecke) symmetrizers**. These elements have a certain absorption property which makes them unique, which we will now describe. Let us use $\sigma_i \in H_n$ be the positive crossing between the i^{th} and $(i+1)^{\text{th}}$ strands. * Picture of oriented $\sigma_i \in H_n$. The algebra H_n is generated by the σ_i and the symmetrizers are the unique idempotent elements satisfying $\sigma_i y_n = s y_n = y_n \sigma_i$. In this way, y_n corresponds to a 1-dimensional representation of H_n (it is a deformation of the trivial representation of $\mathbb{C}S_n$ where $s = 1$).

A similar story holds for the BMW algebra. Firstly, BMW_n is generated by positive crossing elements σ_i and cap-cup elements c_i. * pictures of σ_i and c_i The c_i generate a proper ideal I_n in BMW_n. In (* cite BB), the authors show that the complement of I_n in BMW_n is isomorphic to H_n, giving an isomorphism $BMW_n \cong H_n \oplus I_n$. Then they construct an additive and multiplicative (but non-unital) homomorphism

$$\Gamma_n : H_n \to BMW_n \tag{2.1}$$

which is a section of the natural projection $BMW_n \to H_n$ such that

$$\Gamma_n(x)y = 0 = y\Gamma_n(x) \quad \text{for } x \in H_n, y \in I_n. \tag{2.2}$$

Using this section, one can transport the minimal idempotents $y_\lambda \in H_n$ to minimal idempotents $\tilde{y}_\lambda := \Gamma_n(y_\lambda) \in BMW_n$. The elements $\tilde{y}_n := \tilde{y}_{(n)}$ are called the **(BMW) symmetrizers** and are the unique idempotent elements of BMW_n satisfying the properties of the Hecke symmetrizers and property (2.2). By * cite Shelly, these symmetrizers satisfy a very useful recurrence relation

$$* = * \tag{2.3}$$

* Change f_n to $\tilde{y}_n$ in symmetrizer pictures. * Define quantum integers and β_n For what it's worth, this relation descends to a well-known recurrence relation for the $y_n \in H_n$ via the projection map described above ($\tilde{y}_n$ gets sent to y_n and the last diagram on the right-hand side gets sent to 0).

* Come back and add braiding/twist identities from BB if we need them.

* Would it be wise to include the BB matrix algebra description of BMW_n? Might be useful to point out the central idempotents z_λ, which are distinct from the y_λ, even though I don't think I use them anywhere. Maybe a quick remark would suffice. Note that the annular closure of the y_λ is equal to the annular closure of z_λ.

2.1.4 Skein Algebras of the Annulus

Let's use the notation $A := S^1 \times [0, 1]$ for the annulus. The thickened annulus represents perhaps the simplest space with non-trivial topology that can harbor links. There are often many different smooth embeddings of the thickened annulus into a given 3-manifold M (one for every element of $\pi_1(M)$, at the very least). Keep in mind that if we have a smooth embedding $f : A \times I \hookrightarrow M$, then we have an induced linear map between skein modules $\mathcal{S}_X(f) : \mathcal{S}_X(A) \to \mathcal{S}_X(M)$, allowing us to prod for information about $\mathcal{S}_X(M)$. One useful type of embedding of A is into a tubular neighborhood of a knot, which is often called *decorating* or *threading* a knot.

A first observation is that any skein algebra of the form $\mathcal{S}_X(A)$ is commutative because the link algebra $\mathcal{S}_\varnothing(A)$ is a commutative algebra. To see this, consider the product of two links L_1 and L_2 in the link algebra and start by stretching L_2 towards the outer boundary past L_1, followed by moving it down below the furthest point of L_1, and finally contracting L_2 can to its original radial position.

We'll tackle describing the structure of $\mathcal{K}(A)$ before the others since it is the easiest. The Kauffman bracket skein relation allows one to resolve all crossings in any diagram on any surface Σ. It is a theorem of Pzytycki that the set of non-trivial mutli-curves (non-trivial meaning that no curve bounds a disk), together with the empty link in $A \times \{*\}$ forms a basis of $\mathcal{K}(A)$ (actually, the theorem is stated for any skein algebra, see * cite). There is only one non-trivial curve z and the set $\{z^k\}_{k\in\mathbb{N}}$ exhaust all of the multicurves. Therefore, $\mathcal{K}(A)$ is a polynomial algebra $R[z]$.

The HOMFLYPT and Dubrovnik cases are more complicated because the skein relations do not allow one to resolve all the crossings in a diagram. However, they do allow one to change a diagram with a negative crossing into a diagram with a positive crossing, plus diagrams with a lesser number of crossings. It follows that the skein algebras $\mathcal{H}(A)$ and $\mathcal{D}(A)$ are generated by knots with only positive crossings. Let z_i be a knot in A with winding number i around the annulus. Turaev shows in * cite that the z_i are algebraically independent by showing their HOMFLYPT and Dubrovnik polynomials are algebraically independent. Therefore, $\mathcal{H}(A)$ is a polynomial algebra $R[z_i, i \in \mathbb{Z}_{\neq 0}]$. $\mathcal{D}(A)$ has half as many generators since the z_i in that setting are unoriented, but $\mathcal{D}(A)$ is a polynomial algebra $R[z_i, i \in \mathbb{Z}_{>0}]$ by the same arguments.

* Picture of z_i

The skein algebras of tangles in a cube relate to the skein algebras of the annulus via a map $\mathrm{cl}(-)$, depicted as a wiring diagram below.

* Picture of closure

Let $\widetilde{Q}_\lambda := \mathrm{cl}(\tilde{y}_\lambda) \in \mathcal{D}(A)$. Zhong and Lu show in * cite that the set $\{\widetilde{Q}_\lambda\}_\lambda$ forms a basis of $\mathcal{D}(A)$ over the base ring $R = \mathbb{Q}(s, v)$, where λ ranges over all paritions. Actually, it is an eigenbasis with respect to the meridian map * picture of meridian map. where the eigenvalue of Q_λ is

$$c_\lambda = \delta_{\mathcal{D}} + (s - s^{-1}) \sum_{\square \in \lambda} v^{-1} s^{2\mathrm{cn}(\square)} - v s^{-2\mathrm{cn}(\square)}$$

and where $\mathrm{cn}(\square) := j - i$ is the *content* of the box in the $\square$ in i^{th} row and j^{th} column of the Ferrer's diagram of the partition λ. Two distinct partitions λ and λ' give rise to distinct values of c_λ and c'_λ. Therefore, each of the eigenspaces is 1-dimensional.

The behavior of $\widetilde{Q}_\lambda$ is similar to that of the $Q_\lambda := \mathrm{cl}^+(y_\lambda) \in \mathcal{H}(A)$ where $\mathrm{cl}^+(-)$ is defined by orienting the strands in wiring digram of $\mathrm{cl}(-)$ counter-clockwise. Using the description $\mathcal{H}(A) = R[z_i, i \in \mathbb{Z}_{\neq 0}]$, consider the subalgebras $\mathcal{H}(A)^+ := R[z_i, i \in \mathbb{Z}_{>0}]$ and $\mathcal{H}(A)^- := R[z_i, i \in \mathbb{Z}_{<0}]$. Then the two subalgebras are isomorphic via the linear involution defined by $z_i \mapsto z_{-i}$ and $\mathcal{H}(A) \cong \mathcal{H}(A)^+ \otimes \mathcal{H}(A)^-$. The $\{Q_\lambda\}_\lambda$ forms an eigenbasis of $\mathcal{H}(A)^+$ with resepect to the meridian map and the eigenspaces are 1-dimensional. As a side remark, the linear involution $z_i \mapsto z_{-i}$ may be realized topologically by rotating the thickened annulus π radians around an appropriately centered axis parallel to A and taking the induced skein map. We will call this map the *flip map*.

In * cite Lukac, it's shown how to interpret $\mathcal{H}(A)^+$ as the ring of symmetric functions Λ (also see * cite Morton Murphy Operators). There is an injective algebra homomorphism $\Lambda \to \mathcal{H}(A)^+$ which sends the Schur function s_λ to the minimal idempotent closure Q_λ. This theorem has plenty of implications. For example, the structure constants of Λ in the basis $\{s_\lambda\}_\lambda$ are the Littlewood-Richardson coefficients, which are then sent to structure constants of $\mathcal{H}(A)^+$ in the basis $\{Q_\lambda\}_\lambda$. In Chapter * reference, we discuss partial results surrounding the Dubrovnik analogue of this homomorphism. See Section * reference for a summary about the ring Λ.

Another application of the interpretation of $\mathcal{H}(A)^+$ as Λ is the definition of new special links in the skein algebra. There is a family of elements in Λ known as the power sum symmetric functions, whose counterparts in $\mathcal{H}(A)^+$ we will denote by P_k for integers $k \geq 1$. These elements are algebraically independent in $\Lambda_{\mathbb{Q}} := \mathbb{Q} \otimes \Lambda$. Therefore, ordered monomials in the P_k form a basis of $\mathcal{H}(A)^+$.

2.1.5 Skein Algebras of the Torus

The power sum elements P_k described above are known to behave wonderfully in skein theoretic computations. This allows for a simple description of the HOMFLYPT skein algebra of the torus $\mathcal{H}(T^2)$ in terms of generators and relations. First, let's define the generators. Given an r which is either a rational number, or $\pm\infty$, there is an oriented smooth embedding

$$\iota_r : A \hookrightarrow T^2$$

of the annulus into a tubular neighborhood of the line of slope r in the flat torus. Consider the embeddings ι_r and ι_{-r} to be the same embedding with opposite orientations. Now ι_r induces an algebra homomorphism

$$\mathcal{H}(\iota_r) : \mathcal{H}(A) \to \mathcal{H}(T^2)$$

on the level of skein algebras. The ι_r are distinct isotopically (even homotopically) and are exhaustive in the sense that any knot in the thickened torus is may be represented as being contained in the image of an ι_q. Therefore, any basis of $\mathcal{H}(A)^+$ defines a basis of $\mathcal{H}(T^2)$. Let's consider this with respect to the the power-sum monomial basis of $\mathcal{H}(A)^+$. More precisely, for any pair of integers $\mathbf{x} = (a, b)$ where $k = \gcd(\mathbf{x})$, define

$$P_{\mathbf{x}} := \mathcal{H}(\iota_{a/b})(P_k).$$

In * cite MS, Morton and Samuelson prove that the skein algebra $\mathcal{H}(T^2)$ admits a presentation with generators the elements of $\{P_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Z}^2\}$, subject to the single family of relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = \left(s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})}\right) P_{\mathbf{x}+\mathbf{y}}.$$

This presentation exhibits a relationship to the elliptic Hall algebra: a 2-parameter family of algebras obtained via Hall algebra decategorification of categories of coherent sheaves over certain elliptic curves (see * Burban-Schiffman). Along the diagonal of the parameter space, the Burban-Schiffman presentation of the elliptic Hall algebra matches the Morton-Samuelson presentation of the skein algebra of the torus. At the very least, this highlights two things. Firstly, the skein algebra of the torus admits a known deformation. Secondly, skein algebras are connected to surprising areas of mathematics and hence deserve our attention. * Is this paragraph correct?

Next we describe the Kauffman bracket skein algebra of the torus, although the order of exposition is opposite to the historical order of discovery. The story is actually quite similar to the HOMFLYPT case. Recall that $\mathcal{K}(A)$ is a polynomial algebra $R[z]$ where z is the simple closed curve around the hole of the annulus with winding number 1. There exist polynomials known as *Chebyshev polynomials* $T_k \in R[z]$ for all integer $k \geq 1$ which form a basis of $\mathcal{K}(A)$. For any pair of integers $\mathbf{x} = (a, b) \in \mathbb{Z}^2$, define elements $T_{\mathbf{x}} \in \mathcal{K}(T^2)$ as

$$T_{\mathbf{x}} := \mathcal{K}(\iota_{a/b})(T_k).$$

Note that $T_{\mathbf{x}} = T_{-\mathbf{x}}$ because the T_k are fixed under the flip map, which is possible since the Kauffman bracket skein theory is an unoriented skein theory. With this in mind, we may pass to the smaller indexing set $Z^2/\langle \mathbf{x} = -\mathbf{x} \rangle$. Using these elements, Frohman and Gelca * cite prove that the skein algebra $\mathcal{K}(T^2)$ admits a presentation with generators $T_{\mathbf{x}}$ subject to the relations

$$T_{\mathbf{x}} T_{\mathbf{y}} = s^{\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})} T_{\mathbf{x}-\mathbf{y}}$$

which the authors call the "product-to-sum" formulas. There is an algebra called the *noncommutative torus*, which is a one-parameter deformation of the algebra of continuous functions on the torus. The Frohman-Gelca presentation of $\mathcal{K}(T^2)$ matches a presentation of the invariant subalgebra of the noncommutative torus with respect to a certain involutive action, as shown in * cite FG. * Is this paragraph correct?

Given the similarity between the presentations of $\mathcal{H}(T^2)$ and $\mathcal{K}(T^2)$, one might suspect that there exists a similar description of the Dubrovnik skein algebra $\mathcal{D}(T^2)$. The answer to this question is affirmative, which is what we discuss in Chapter * How to reference a chapter?

2.1.6 A Relative Skein Algebra of the Annulus

At this point in the story we have introduced special links in the skein algebras of the annulus and described how we can transport these into other skein modules via threading knots. In order to say anything meaningful about these threadings, we ought to know how these special links will interact with other links once in the resulting space. For example, is there an easy way to describe how can we pass a special link through a single strand of another link? To answer this question universally, we should answer it in a certain relative skein algebra which is the endomorphism algebra of one point in the skein category of the annulus. Appropriate wirings from this skein algebra into other skein modules will give us our desired description. Here we will describe this algebra in more detail and summarize what is already known.

First we will restrict ourselves to the HOMFLYPT case, for which we summarize the results from * Morton, Murphy operators. The object we would like to discuss is $\mathcal{A}_{\mathcal{H}} := \mathcal{H}(A, [1])$, which is depicted diagramatically as an annulus with one point on each boundary component. This algebra is closely related to the affine Hecke algebra of type A, $\dot{H}_1$ (see * cite MS). Two of the most basic elements of $\mathcal{A}_{\mathcal{H}}$ are * Pictures of identity e and once around counter-clockwise a. The product in the algebra in this digrammatic notation is given by nesting annuli.

Let $\mathcal{C}_{\mathcal{H}} := \mathcal{H}(A)$. Then, $\mathcal{C}_{\mathcal{H}}$ admits both a left $\mathcal{C}_{\mathcal{H}}$-action by pushing links infront of tangles in $\mathcal{A}_{\mathcal{H}}$. * Picture of left action. It is known that there is an equality of algebras between $\mathcal{A}_{\mathcal{H}}$ and the Laurent polynomial algebra $\mathcal{C}_{\mathcal{H}}[a, a^{-1}]$. In particular, $\mathcal{A}_{\mathcal{H}}$ is commutative and the left action is determined by how it acts on e.

Analagously, there is a right $\mathcal{C}_{\mathcal{H}}$-action by pulling links in behind tangles in $\mathcal{A}_{\mathcal{H}}$. * picture of right action. Both actions are examples of those arising from embeddings into the boundary of the thickened annulus, as described in Remark 2.1.5. The left and right actions obviously commute, endowing $\mathcal{A}_{\mathcal{H}}$ with a $\mathcal{C}_{\mathcal{H}}$-$\mathcal{C}_{\mathcal{H}}$-bimodule structure. It is probably worth pointing out that the left action is not equal to the right action. A simple measurement of how far off the two actions are from being equal would answer our question posed above. The answer is given as a commutator relation

$$e \cdot P_k - P_k \cdot e = (s^k - s^{-k})a^k. \tag{2.4}$$

* Make sure this order is correct. The proof involves the definition of a certain wiring diagram, defining a linear map $H_n \to \mathcal{A}_{\mathcal{H}}$. The idempotents y_n satisfy a certain recurrence relation, and the commutator relation above follows from writing P_k in terms of the y_n and calculations involving the image of this recurrence relation.

Eventually we would like to prove a Dubrovnik analogue of Equation (2.4) (see Section **??**), but we still haven't defined any Dubrovnik analogue of the P_k (see * ref). Nevertheless, we can discuss some of what was previously known about the algebra $\mathcal{A}_{\mathcal{D}} := \mathcal{D}(A, [1])$ (see * Shelly for more details).

Let $\mathcal{C}_{\mathcal{D}} := \mathcal{D}(A)$ and let $a, e \in \mathcal{A}_{\mathcal{D}}$ be the unoriented versions of the elements of the same name above. As in the HOMFLYPT case, there is a left-$\mathcal{C}_{\mathcal{D}}$ action on $\mathcal{A}_{\mathcal{D}}$ and the algebra $\mathcal{A}_{\mathcal{D}}$ is equal to the Laurent polynomial algebra $\mathcal{C}_{\mathcal{D}}[a^{\pm 1}]$. There is also a right-$\mathcal{C}_{\mathcal{D}}$ action on $\mathcal{A}_{\mathcal{D}}$, endowing $\mathcal{A}_{\mathcal{D}}$ with a bimodule structure.

Consider the wiring diagrams * pictures of wirings. I would drop the subscript on W_n for brevity and rename $\widetilde{W}$ to W^* to not confuse with $\tilde{y}_n$. which define linear maps

$BMW_n \to \mathcal{A}_{\mathcal{D}}$. Let $W_n := W(\tilde{y}_{n+1})$ and $W_n^* := W^*(\tilde{y}_{n+1})$ where the subscript denotes the "winding number" around A. Taking the image of Equation (2.3) under W gives a relation

$$[n+1]W_n = e \cdot \tilde{h}_n + [n]s^{-1}aW_{n-1} + [n]s^{-1}\beta_n a^{-1}W_{n-1}^* \tag{2.5}$$

where $\tilde{h}_n := \tilde{Q}_{(n)}$ is the annular closure of the symmetrizer $\tilde{y}_n$.

There are maps

$$(-)^* : BMW_n \to BMW_n \qquad \overline{(-)} : BMW_n \to BMW_n$$

induced by the diffeomorphisms of the thickened square $(x, y, t) \mapsto (x, 1-y, 1-t)$ and $(x, y, t) \mapsto (x, y, 1-t)$, respectively. The map $\overline{(-)}$ is often called the *mirror map* and is an R-anti-linear involution, while $(-)^*$ will be called the *flip map* and is an R-linear involution. The symmetrizers $\tilde{y}_n$ are fixed under these maps because the mirror and flip maps preserve the properties which make $\tilde{y}_n$ unique. Using the quotient map defined by the equivalence relation $(x, 0, t) \sim (x, 1, t)$, we may analagously define maps

$$(-)^* : \mathcal{A}_{\mathcal{D}} \to \mathcal{A}_{\mathcal{D}}, \qquad \overline{(-)} : \mathcal{A}_{\mathcal{D}} \to \mathcal{A}_{\mathcal{D}}$$

which are linear and anti-linear involutions, respectively. We will also call these the flip map and the mirror map; it will be clear from the context which is being applied. These maps satisfy the relations

$$\begin{aligned} (W(x))^* &= W^*(x^*), & (y \cdot e)^* &= e \cdot y^*, \\ \overline{W(x)} &= W(\overline{x}), & \overline{(y \cdot e)} &= e \cdot \overline{y} \end{aligned}$$

for any $x \in BMW_n$ or $y \in \mathcal{C}_{\mathcal{D}}$. Apply the flip, the mirror, and the composite of the two separately to Equation (2.5) to obtain alternate versions of the original recurrence relation:

$$[n+1]W_n^* = \tilde{h}_n \cdot e + [n]s^{-1}aW_{n-1}^* + [n]s^{-1}\beta_n aW_{n-1}, \tag{2.6}$$

$$[n+1]W_n = \tilde{h}_n \cdot e + [n]saW_{n-1} + [n]s\bar{\beta}_n a^{-1}W_{n-1}^*, \tag{2.7}$$

$$[n+1]W_n^* = e \cdot \tilde{h}_n + [n]sa^{-1}W_{n-1}^* + [n]s^{-1}\bar{\beta}_n aW_{n-1}. \tag{2.8}$$

Rearranging the difference of Equations (2.5) and (2.6) gives a relation

$$\tilde{h}_n \cdot e - e \cdot \tilde{h}_n = \{n\}(a^{-1}W_{n-1}^* - aW_{n-1}) \tag{2.9}$$

which Shelly calls a *fundamental skein relation* in $\mathcal{A}_{\mathcal{D}}$ since it reduces to to usual Dubrovnik skein relation when $n = 1$. * This might also be true in another sense, since there might be some connection between the wiring W and something called the "tube algebra", as pointed out by Henry Tucker. In Chapter * ??, we will provide a similar relation which amounts to rewriting the right-hand side of the equation in terms of elements of the form $h_i \cdot a^k$ (alternatively $a^k \cdot h_i$).

2.2 The Ring of Symmetric Functions

As As described in the previous section, the subalgebra $\mathcal{H}^+(A)$ of the HOMFLYPT skein algebra of the annulus identifies with the ring of symmetric functions Λ via an injective ring homomorphism which sends Schur functions to closures of minimal idempotents in the Hecke algebra. From a Lie theoretic perspective, the ring of symmetric functions is the *universal character ring* of type A; the ring Λ is the graded inverse limit of the character rings $R(GL_n)$ by projection maps $R(GL_n) \to R(GL_{n-1})$. Since the HOMFLYPT skein relations are a model of relations satisfied by the braiding in the categories of representation of the quantum groups of type A, one might suspect that something similar holds for different types. The universal character rings in the types B and C case also identify with the ring of symmetric functions, although this identification is distinct from that in the type A case. Something similar holds for type D as well. If we expect some relationship between $\mathcal{D}(A)$ and Λ, then we should work to understand Λ.

2.2.1 Character Rings of Classical Groups

2.2.2 Bases of Λ and Identities

Chapter 3

The Skein Algebra of the Torus

* Add some introduction and remark about collaboration.

3.1 Power Sum Elements

Recall that there is a injective algebra homomorphism $\Lambda \to \mathcal{H}(A)^+$ which sends the Schur function s_λ to the minimal idempotent closure Q_λ. Use $h_n := Q_{(n)}$ to denote the image of the n^{th} complete homogeneous symmetric function under this homomorphism. In * MS the authors import power sum elements from Λ to $P_k \in \mathcal{H}(A)$. The power sum elements have a concrete definition in Λ, but alternatively they may be defined using an equation of formal power series in the ring $\mathcal{H}(A)[[t]]$ as

$$\sum_{k=1}^{\infty} \frac{P_k}{k} t^k = \ln \left(1 + \sum_{n=1}^{\infty} h_n t^n \right) \tag{3.1}$$

which writes each P_k in terms of the generators h_n.

Using the Beliakova-Blanchet section $\Gamma : H_n \to BMW_n$, we may emulate this definition to define "power sum" elements $\tilde{P}_k \in \mathcal{D}(A)$ by the formal power series equation

$$\sum_{k=1}^{\infty} \frac{\tilde{P}_k}{k} t^k = \ln \left(1 + \sum_{n=1}^{\infty} \tilde{h}_n t^n \right) \tag{3.2}$$

where $\tilde{h}_n := \tilde{Q}_{(n)}$ is the annular closure of the BMW symmetrizers $\tilde{y}_n = \Gamma(y_n)$.

Let's now continue our discussion of Section 2.1.6 with the following theorem.

Theorem 3.1.1. *For any $k \geq 1$, the relation*

$$e \cdot \tilde{P}_k - \tilde{P}_k \cdot e = (s^k - s^{-k})(a^k - a^{-k}) \tag{3.3}$$

holds. Equivalently,

$$a^i \cdot \tilde{P}_k - \tilde{P}_k \cdot a^i = (s^k - s^{-k})(a^{k+i} - a^{-k+i}) \tag{3.4}$$

for any integer i.

We will split the proof of this theorem into two technical lemmas.

Lemma 3.1.2. *The relations of Theorem 3.1.1 hold if and only if*

$$e \cdot (\tilde{h}_{n+2} + \tilde{h}_n) - (\tilde{h}_{n+2} + \tilde{h}_n) \cdot e = (sa + s^{-1}a^{-1})(e \cdot \tilde{h}_{n+1}) - (s^{-1}a + sa^{-1})(\tilde{h}_{n+1} \cdot e) \tag{3.5}$$

for all integers $n \geq -1$, where $\tilde{h}_0 := 1$ and $\tilde{h}_{-1} := 0$.

Proof. The relations of Theorem 3.1.1 may be organized into a single power series equation

$$\sum_{k=1}^{\infty} \frac{e \cdot \tilde{P}_k - \tilde{P}_k \cdot e}{k} t^k = \sum_{k=1}^{\infty} \frac{(s^k - s^{-k})(a^k - a^{-k})}{k} t^k \tag{3.6}$$

in $\mathcal{A}_{\mathcal{D}}[[t]]$. Rewrite this equation as

$$e \cdot \left(\sum_{k=1}^{\infty} \tilde{P}_k \right) - \left(\sum_{k=1}^{\infty} \tilde{P}_k \right) \cdot e = \sum_{k=1}^{\infty} \frac{(sat)^k}{k} + \sum_{k=1}^{\infty} \frac{(s^{-1}a^{-1}t)^k}{k} - \sum_{k=1}^{\infty} \frac{(s^{-1}at)^k}{k} - \sum_{k=1}^{\infty} \frac{(sa^{-1}t)^k}{k} \tag{3.7}$$

We can make sense of the left-hand side by extending the algebra homomorphism $x \mapsto e \cdot (x)$ to an algebra homomorphism of rings of formal power series

$$
\begin{tikzcd}
\mathcal{D}(A) \arrow[r, "e\cdot(-)"] \arrow[d, hook] & \mathcal{A}_{\mathcal{D}} \arrow[d, hook] \\
\mathcal{D}(A)[[t]] \arrow[r, "e\cdot(-)"] & \mathcal{A}_{\mathcal{D}}[[t]]
\end{tikzcd}
$$

and similarly for $(-) \cdot e$. Now for shorthand, define

$$H(t) := 1 + \sum_{n=1}^{\infty} \tilde{h}_n t^n$$

and recall the Taylor series expansion

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

which is a variation of the Newton-Mercator series. Then the Equation (3.6) becomes

$$e\cdot\Big(\ln\big(H(t)\big)\Big)-\Big(\ln\big(H(t)\big)\Big)\cdot e = -\ln(1-sat)-\ln(1-s^{-1}a^{-1}t)+\ln(1-s^{-1}at)+\ln(1-sa^{-1}t). \tag{3.8}$$

The maps $e\cdot(-)$ and $(-)\cdot e$ commute with the natural logarithm. Use this and other natural log properties to write

$$\ln\Big(e\cdot\big(H(t)\big)\big(1-(sa+s^{-1}a^{-1})t+t^2\big)\Big) = \ln\Big(\big(H(t)\big)\cdot e\big(1-(s^{-1}a+sa^{-1})t+t^2\big)\Big). \tag{3.9}$$

Exponentiating both sides and equating coefficients gives the system of equations defined in the statement of the lemma. Each step of the proof is invertible, and thus the two sets of relations are logically equivalent. $\square$

Lemma 3.1.3. *The relations of Lemma 3.1.2 hold.*

Proof. If $n = -1$, the relation we would like to show becomes

$$e\cdot\tilde{h}_1 - \tilde{h}_1\cdot e = \{1\}\left(a - a^{-1}\right)$$

which is just the Dubrovnik skein relation.

For general values of n, the proof is a technical computation using repeated applications of the recursive formula for the $\tilde{h}_n$. Since we will need them here, let's recall the formulas given in Section 2.1.6. There are recursive formulas

$$[n+1]W_n = e \cdot \tilde{h}_n + [n]s^{-1}aW_{n-1} + [n]s^{-1}\beta_n a^{-1}W^*_{n-1}, \tag{3.10}$$

$$[n+1]W^*_n = \tilde{h}_n \cdot e + [n]s^{-1}aW^*_{n-1} + [n]s^{-1}\beta_n aW_{n-1}, \tag{3.11}$$

$$[n+1]W_n = \tilde{h}_n \cdot e + [n]saW_{n-1} + [n]s\bar{\beta}_n a^{-1}W^*_{n-1}, \tag{3.12}$$

$$[n+1]W^*_n = e \cdot \tilde{h}_n + [n]sa^{-1}W^*_{n-1} + [n]s^{-1}\bar{\beta}_n aW_{n-1} \tag{3.13}$$

and the "fundamental relation"

$$e \cdot \tilde{h}_n - \tilde{h}_n \cdot e = \{n\}(aW_{n-1} - a^{-1}W^*_{n-1}). \tag{3.14}$$

Start by applying Equation (3.14) to the relation of Lemma 3.1.2 to obtain an equivalent relation

$$\begin{aligned} \{n+2\}\left(aW_{n+1} - a^{-1}W^*_{n+1}\right) = \left(sa + s^{-1}a^{-1}\right)\left(e \cdot \tilde{h}_{n+1}\right) - \left(sa^{-1} + s^{-1}a\right)\left(\tilde{h}_{n+1} \cdot e\right) \\ - \{n\}\left(aW_{n-1} - a^{-1}W^*_{n-1}\right). \end{aligned} \tag{3.15}$$

We will show that the left-hand side of this equation may be reduced to the right-hand side by a series of applications of the recursive formulas, which we will signify with an asterisk $*$.

$$\begin{aligned} & \{n+2\}\left(aW_{n+1} - a^{-1}W^*_{n+1}\right) \\ \stackrel{*}{=} \quad & \{n+2\}\Bigg(\frac{a}{[n+2]}\left(e \cdot \tilde{h}_{n+1} + [n+1]s^{-1}aW_n + [n+1]s^{-1}\beta_{n+1}a^{-1}W^*_n\right) - \\ & \quad \frac{a^{-1}}{[n+2]}\left(\tilde{h}_{n+1} \cdot e + [n+1]s^{-1}a^{-1}W^*_n + [n+1]s^{-1}\beta_{n+1}aW_n\right)\Bigg) \\ = \quad & \left(s - s^{-1}\right)\Big(\left(a \cdot \tilde{h}_{n+1} + [n+1]s^{-1}a^2W_n + [n+1]s^{-1}\beta_{n+1}W^*_n\right) \\ & \quad - \left(\tilde{h}_{n+1} \cdot a^{-1} + [n+1]s^{-1}a^{-2}W^*_n + [n+1]s^{-1}\beta_{n+1}W_n\right)\Big) \end{aligned}$$

$$
\begin{aligned}
\overset{*}{=} \quad & \left(s-s^{-1}\right)\Big(\Big(a\cdot\tilde{h}_{n+1}+s^{-2}a\Big([n+2]W_{n+1}-\tilde{h}_{n+1}\cdot e-[n+1]s\bar{\beta}_{n+1}a^{-1}W_n^*\Big) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +[n+1]s^{-1}\beta_{n+1}W_n^*\Big) \\
& \qquad -\Big(\tilde{h}_{n+1}\cdot a^{-1}+s^{-2}a^{-1}\Big([n+2]W_{n+1}^*-e\cdot\tilde{h}_{n+1}-[n+1]s\bar{\beta}_{n+1}aW_n\Big) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +[n+1]s^{-1}\beta_{n+1}W_n\Big)\Big) \\
= \quad & \left(sa+s^{-1}a^{-1}\right)\Big(e\cdot\tilde{h}_{n+1}\Big)-\left(sa^{-1}+s^{-1}a\right)\Big(\tilde{h}_{n+1}\cdot e\Big) \\
+ \quad & \left(s^{-1}a^{-1}+s^{-3}a\right)\Big(\tilde{h}_{n+1}\cdot e\Big)-\left(s^{-1}a+s^{-3}a^{-1}\right)\Big(e\cdot\tilde{h}_{n+1}\Big) \\
+ \quad & \{n+2\}s^{-2}\left(aW_{n+1}-a^{-1}W_{n+1}^*\right)+\{n+1\}s^{-1}\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(W_n-W_n^*\right).
\end{aligned}
$$

We break the computation here to note that the first two terms in the last line also appear on the right hand side of (3.15). Thus, we would like to prove the following equality:

$$
\begin{aligned}
- \quad & \{n\}\left(aW_{n-1}-a^{-1}W_{n-1}^*\right) \\
= \quad & \left(s^{-1}a^{-1}+s^{-3}a\right)\Big(\tilde{h}_{n+1}\cdot e\Big)-\left(s^{-1}a+s^{-3}a^{-1}\right)\Big(e\cdot\tilde{h}_{n+1}\Big) \\
+ \quad & \{n+2\}s^{-2}\left(aW_{n+1}-a^{-1}W_{n+1}^*\right)-\{n+1\}s^{-1}\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(W_n-W_n^*\right).
\end{aligned}
$$

We will work the right-hand side of this above equation down to the left-hand side by continuing to apply the same identities. A large number of terms cancel and what remains is the desired relation.

$$
\begin{aligned}
& \left(s^{-1}a^{-1}+s^{-3}a\right)\Big(\tilde{h}_{n+1}\cdot e\Big)-\left(s^{-1}a+s^{-3}a^{-1}\right)\Big(e\cdot\tilde{h}_{n+1}\Big) \\
+ \quad & \{n+2\}s^{-2}\left(aW_{n+1}-a^{-1}W_{n+1}^*\right)-\{n+1\}s^{-1}\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(W_n-W_n^*\right) \\
= \quad & \left(s^{-1}a^{-1}+s^{-3}a\right)\Big(\tilde{h}_{n+1}\cdot e\Big)-\left(s^{-1}a+s^{-3}a^{-1}\right)\Big(e\cdot\tilde{h}_{n+1}\Big) \\
+ \quad & [n+2]\left(s^{-1}-s^{-3}\right)\left(aW_{n+1}-a^{-1}W_{n+1}^*\right) \\
+ \quad & [n+1]\left(1-s^{-2}\right)\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(W_n-W_n^*\right) \\
= \quad & s^{-1}a\Big([n+2]W_{n+1}-e\cdot\tilde{h}_{n+1}\Big) \\
+ \quad & s^{-3}a^{-1}\Big([n+2]W_{n+1}^*-e\cdot\tilde{h}_{n+1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& - \quad s^{-3}a\left([n+2]W_{n+1} - \tilde{h}_{n+1}\cdot e\right) \\
& - \quad s^{-1}a^{-1}\left([n+2]W^*_{n+1} - \tilde{h}_{n+1}\cdot e\right) \\
& + \quad [n+1]\left(1-s^{-2}\right)\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(W_n - W^*_n\right) \\
\overset{*}{=} & \quad s^{-1}a\left([n+1]s^{-1}aW_n + [n+1]s^{-1}\beta_{n+1}a^{-1}W^*_n\right) \\
& + \quad s^{-3}a^{-1}\left([n+1]sa^{-1}W^*_n + [n+1]s\bar{\beta}_{n+1}aW_n\right) \\
& - \quad s^{-3}a\left([n+1]saW_n + [n+1]s\bar{\beta}_{n+1}a^{-1}W^*_n\right) \\
& - \quad s^{-1}a^{-1}\left([n+1]s^{-1}a^{-1}W^*_n + [n+1]s^{-1}\beta_{n+1}aW_n\right) \\
& + \quad [n+1]\left(1-s^{-2}\right)\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(W_n - W^*_n\right) \\
= & \quad [n+1]\left(s^{-2}a^2W_n + s^{-2}\beta_{n+1}W^*_n + s^{-2}a^{-2}W^*_n + s^{-2}\bar{\beta}_{n+1}W_n - s^{-2}a^2W_n\right. \\
& \qquad -s^{-2}\bar{\beta}_{n+1}W^*_n - s^{-2}a^{-2}W^*_n - s^{-2}\beta_{n+1}W_n + \bar{\beta}_{n+1}W_n - \bar{\beta}_{n+1}W^*_n - \beta_{n+1}W_n \\
& \qquad \left. +\beta_{n+1}W^*_n - s^{-2}\bar{\beta}_{n+1}W_n + s^{-2}\bar{\beta}_{n+1}W^*_n + s^{-2}\beta_{n+1}W_n - s^{-2}\beta_{n+1}W^*_n\right) \\
= & \quad [n+1]\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(W_n - W^*_n\right) \\
= & \quad \left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(([n+1]W_n) - ([n+1]W^*_n)\right) \\
\overset{*}{=} & \quad \left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(\left(e\cdot\tilde{h}_n + [n]s^{-1}aW_{n-1} + [n]s^{-1}\beta_n a^{-1}W^*_{n-1}\right)\right. \\
& \qquad\qquad \left. -\left(e\cdot\tilde{h}_n + [n]sa^{-1}W^*_{n-1} + [n]s\bar{\beta}_n aW_{n-1}\right)\right) \\
= & \quad \left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left([n]\left(s^{-1}-s\bar{\beta}_n\right)aW_{n-1} - [n]\left(s-s^{-1}\beta_n\right)a^{-1}W^*_{n-1}\right) \\
= & \quad [n]\left(\bar{\beta}_{n+1}-\beta_{n+1}\right)\left(s-s^{-1}\beta_n\right)\left(aW_{n-1} - a^{-1}W^*_{n-1}\right) \\
= & \; - \quad \{n\}\left(aW_{n-1} - a^{-1}W^*_{n-1}\right).
\end{aligned}
$$

Where the last equality follows from a quick computation in the base ring. This completes the proof. □

This next theorem follows directly from Equation (3.6), which makes it equivalent to Theorem 3.1.1 in some sense. This expresses the left $\mathcal{D}(A)$-action on $\mathcal{A}_{\mathcal{D}}$ in terms of the right action, and vice versa. This implies a commutation relation for the closures of the BMW symmetrizers in terms of either the elements of the set $\{\tilde{h}_j \cdot a^i\}_{j,i}$ or $\{a^i \cdot \tilde{h}_j\}_{j,i\geq 0}$ which are subsets of the bases $\{Q_\lambda \cdot a^i\}_{i\geq 0,\lambda}$ and $\{a^i \cdot Q_\lambda\}_{i\geq 0,\lambda}$ of $\mathcal{A}_\mathcal{D}$, respectively. These supersets are bases since $\mathcal{A}_\mathcal{D} = \mathcal{D}(A)[a, a^{-1}]$ as algebras in the category of left $\mathcal{D}(A)$-modules and because the map defined by $Q_\lambda \cdot a^i \mapsto a^i \cdot Q_\lambda$ is an invertible algebra homomorphism.

Theorem 3.1.4. *For any $n \geq 1$, the relations*

$$\tilde{h}_n \cdot e = \sum_{i=0}^{n} d_i(e \cdot \tilde{h}_{n-i}) \tag{3.16}$$

and

$$e \cdot \tilde{h}_n = \sum_{i=0}^{n} \bar{d}_i(\tilde{h}_{n-i} \cdot e) \tag{3.17}$$

hold in $\mathcal{A}_\mathcal{D}$, where

$$\begin{aligned}
d_0 &= 1,\\
d_i &= \sum_{l=0}^{i-1}(1-s^2)s^{2l-i}a^{i-2l} + (1-s^{-2})s^{i-2l}a^{2l-i} \qquad \forall i \geq 1,\\
\bar{d}_i &= \sum_{l=0}^{i-1}(1-s^{-2})s^{i-2l}a^{i-2l} + (1-s^2)s^{2l-i}a^{2l-i} \qquad \forall i \geq 1.
\end{aligned}$$

Equivalently,

$$e \cdot \tilde{h}_n - \tilde{h}_n \cdot e = \sum_{i=1}^{n} \bar{d}_i(\tilde{h}_{n-i} \cdot e) \tag{3.18}$$

or

$$\tilde{h}_n \cdot e - e \cdot \tilde{h}_n = \sum_{i=1}^{n} d_i(e \cdot \tilde{h}_{n-i}). \tag{3.19}$$

Proof. The formula for the d_i were discovered experimentally by coding a solver using the SymPy package in Python. The second equation is just the mirror map applied to the first equation, so we will just prove the first equation.

The idea of the proof depends on a reformulation of Equation (3.6) as

$$\tilde{h}_n \cdot e = e \cdot \tilde{h}_n - (sa + s^{-1}a^{-1})(e \cdot \tilde{h}_{n-1}) + e \cdot \tilde{h}_{n-2} + (s^{-1}a + sa^{-1})\tilde{h}_{n-1} \cdot e - \tilde{h}_{n-2} \cdot e$$

and a recursive application of this formula to its last two terms on the right-hand side of the equation.

The case of $n = 0$ is trivial. For $n = 1$, just apply the Kauffman skein relation. Now assume the induction hypothesis, that the formula in the statement is true for all $k \leq n-1$. Then apply this assumption to Equation (3.6):

$$\begin{aligned}
\tilde{h}_n \cdot e &= e \cdot \tilde{h}_n - (sa + s^{-1}a^{-1})(e \cdot \tilde{h}_{n-1}) + e \cdot \tilde{h}_{n-2} + (s^{-1}a + sa^{-1})(\tilde{h}_{n-1} \cdot e) - \tilde{h}_{n-2} \cdot e \\
&= e \cdot \tilde{h}_n - (sa + s^{-1}a^{-1})(e \cdot \tilde{h}_{n-1}) + e \cdot \tilde{h}_{n-2} + (s^{-1}a + sa^{-1}) \sum_{i=0}^{n-1} d_i (e \cdot \tilde{h}_{n-1-i}) \\
&\qquad - \sum_{i=0}^{n-2} d_i (e \cdot \tilde{h}_{n-2-i}) \\
&= e \cdot \tilde{h}_n + d_1 (e \cdot \tilde{h}_{n-1}) + (s^{-1}a + sa^{-1}) \sum_{i=1}^{n-1} d_i (e \cdot \tilde{h}_{n-1-i}) - \sum_{i=1}^{n-2} d_i (e \cdot \tilde{h}_{n-2-i}) \\
&= e \cdot \tilde{h}_n + d_1 (e \cdot \tilde{h}_{n-1}) + (s^{-1}a + sa^{-1}) \sum_{i=0}^{n-2} d_{i+1} (e \cdot \tilde{h}_{n-2-i}) - \sum_{i=1}^{n-2} d_i (e \cdot \tilde{h}_{n-2-i}) \\
&= e \cdot \tilde{h}_n + d_1 (e \cdot \tilde{h}_{n-1}) + (s^{-1}a + sa^{-1}) d_1 (e \cdot \tilde{h}_{n-2}) \\
&\qquad + \sum_{i=1}^{n-2} \big((s^{-1}a + sa^{-1}) d_{i+1} - d_i \big) (e \cdot \tilde{h}_{n-2-i}).
\end{aligned}$$

It is a straightforward computation to show that $(s^{-1}a + sa^{-1})d_1 = d_2$:

$$\begin{aligned}
(s^{-1}a + sa^{-1})d_1 &= (s^{-1}a + sa^{-1})\big((1-s^2)s^{-1}a + (1+s^{-2})sa^{-1}\big) \\
&= (1-s^2)s^{-2}a^2 + (1-s^{-2})s^0a^0 + (1-s^2)s^0a^0 + (1-s^{-2})s^2a^{-2} \\
&= d_2.
\end{aligned}$$

It's slightly more tedious to show that $(s^{-1}a + sa^{-1})d_{i+1} - d_i = d_{i+2}$ for all $i \geq 1$:

$$(s^{-1}a + sa^{-1})d_{i+1} - d_i$$

$$\begin{aligned}
&= && (s^{-1}a+sa^{-1})\sum_{l=0}^{i}(1-s^2)s^{2l-i}a^{i-2l}+(1-s^{-2})s^{i-2l}a^{2l-i} \\
& && -\ \sum_{l=0}^{i-1}(1-s^2)s^{2l-(i-1)}a^{(i-1)-2l}+(1-s^{-2})s^{(i-1)-2l}a^{2l-(i-1)} \\
&= && \sum_{l=0}^{i}(1-s^2)s^{2l-(i+1)}a^{(i+1)-2l}+(1-s^{-2})s^{(i+1)-2l}a^{2l-(i+1)} \\
& && +\ \sum_{l=0}^{i}(1-s^2)s^{2l-(i-1)}a^{(i-1)-2l}+(1-s^{-2})s^{(i-1)-2l}a^{2l-(i-1)} \\
& && -\ \sum_{l=0}^{i-1}(1-s^2)s^{2l-(i-1)}a^{(i-1)-2l}+(1-s^{-2})s^{(i-1)-2l}a^{2l-(i-1)} \\
&= && \sum_{l=0}^{i}(1-s^2)s^{2l-(i+1)}a^{(i+1)-2l}+(1-s^{-2})s^{(i+1)-2l}a^{2l-(i+1)} \\
& && +\ (1-s^2)s^{i+1}a^{-1-i}+(1-s^{-2})s^{-1-i}a^{i+1} \\
&= && d_{i+2}.
\end{aligned}$$

This completes the proof of the statement. □

Remark 3.1.5. There exists an algebra homomorphism from $\mathcal{D}(A)$ to the ring of symmetric functions Λ_R (see * chapter whatever). Conjecturally, this map is an isomorphism, which would imply that the sets $\{\tilde{h}_\lambda \cdot a^i\}_{\lambda,i}$ and $\{a^i \cdot \tilde{h}_\lambda\}_{\lambda,i}$ over integers i and partitions λ, where $\tilde{h}_\lambda := \tilde{h}_{\lambda_1}\cdots\tilde{h}_{\lambda_r}$, form bases of $\mathcal{A}_\mathcal{D} = \mathcal{D}(A)[a, a^{-1}]$ * or is this already known separately somehow?. If so, then Theorem 3.1.4 provides transition formulas between these two bases, and therefore giving a full description of $\mathcal{A}_\mathcal{D}$ as a $\mathcal{D}(A)$-$\mathcal{D}(A)$-bimodule.

As an aside, one might expect similar formulas to hold in the HOMFLYPT case. To our knowledge, there is no HOMFLYPT analogue of Theorem 3.1.4 written down in the literature. Let's do that here.

Lemma 3.1.6. *For all integers n, the following relation holds in $\mathcal{A}_\mathcal{H}$*

$$e \cdot h_n - h_n \cdot e = sa \cdot h_{n-1} - h_{n-1} \cdot s^{-1}a \tag{3.20}$$

where we use the convention $h_0 = 1$ and $h_n = 0$ if $n < 0$.

Proof. Recall the power sum elements P_k satisfy the power series equation

$$\sum_{k=1}^{\infty} \frac{P_k}{k} x^k = \ln\Big(\sum_{n=0}^{\infty} h_n x^n\Big) \tag{3.21}$$

By [?Mor02b, Theorem 4.2]* fix, the power sum elements satisfy a commutation relation in $\mathcal{A}_{\mathcal{H}}$

$$e \cdot P_k - P_k \cdot e = (s^k - s^{-k})a^k \tag{3.22}$$

which may be rephrased as a power series equation

$$e \cdot \Big(\sum_{k=1}^{\infty} P_k x^k\Big) - \Big(\sum_{k=1}^{\infty} P_k x^k\Big) \cdot e = \sum_{k=1}^{\infty} s^k a^k - \sum_{k=1}^{\infty} s^{-k} a^k.$$

On the left-hand side, use the defining equation (3.21). Use the power series formulation of natual log on the right-hand side. So we have

$$\ln\left(e \cdot \Big(\sum_{k=0}^{\infty} h_k x^k\Big)\right) - \ln\left(\Big(\sum_{k=0}^{\infty} h_k x^k\Big) \cdot e\right) = \ln(1 - sax) - \ln(1 - s^{-1}ax).$$

After moving terms around, using properties of natural log, and exponentiating both sides, we arrive at the equation

$$\Big(\sum_{n=0}^{\infty} (h_n \cdot e) x^n\Big)(1 - sax) = \Big(\sum_{n=0}^{\infty} (e \cdot h_n) x^k\Big)(1 - s^{-1}ax)$$

which implies the statement of the lemma. □

Recall that the algebra $\mathcal{A}_{\mathcal{H}}$ is equal to the Laurent polynomial ring $\mathcal{H}(A)^+[a, a^{-1}]$. Under the isomorphism between $\mathcal{H}(A)^+$ and the ring of symmetric functions Λ_R, the h_n identify with the complete homogeneous symmetric functions. It is well-known that ordered monomials in the complete homogeneous symmetric functions form a basis of Λ, hence the sets $\{h_\lambda \cdot a^i\}_{\lambda,i}$ and $\{a^i \cdot h_\lambda\}_{\lambda,i}$ over integers i and partitions λ, where $h_\lambda := h_{\lambda_1} \cdots h_{\lambda_r}$, form bases of $\mathcal{A}_{\mathcal{H}}$. The following theorem gives transition formulas between these two bases.

Theorem 3.1.7. *The Hecke symmetrizers h_n satisfy the equations*

$$h_n \cdot e = e \cdot h_n + (1 - s^2) \sum_{l=1}^{n} s^{-l}(a^l \cdot h_{n-l})$$

and

$$e \cdot h_n = h_n \cdot e + (1 - s^{-2}) \sum_{l=1}^{n} s^{l}(h_{n-l} \cdot a^l).$$

Proof. We will prove the first equality. The second is completely analagous. Proceed by induction. When $n = 1$, the statement follows from the HOMFLY skein relation.

We can rearrange the terms of Lemma 3.1.6 to get

$$h_n \cdot e = e \cdot h_n + s^{-1}a(h_{n-1} \cdot e) - sa(e \cdot h_{n-1}). \tag{3.23}$$

By the induction hypothesis,

$$\begin{aligned}
h_n \cdot e &= e \cdot h_n + s^{-1}a(h_{n-1} \cdot e) - sa(e \cdot h_{n-1}) \\
&= e \cdot h_n + s^{-1}a\Big(e \cdot h_{n-1} + (1 - s^2) \sum_{j=1}^{n-1} s^{-j}a^j(e \cdot h_{n-1-j})\Big) - sa(e \cdot h_{n-1}) \\
&= e \cdot h_n + (s^{-1} - s)a(e \cdot h_{n-1}) + (1 - s^2) \sum_{j=1}^{n-1} s^{-j-1}a^{j+1}(e \cdot h_{n-1-j}) \\
&= e \cdot h_n + (1 - s^2)s^{-1}a(e \cdot h_{n-1}) + (1 - s^2) \sum_{j=1}^{n-1} s^{-(j+1)}a^{j+1}(e \cdot h_{n-(j+1)}) \\
&= e \cdot h_n + (1 - s^2) \sum_{l=1}^{n} s^{-l}a^l(e \cdot h_{n-l})
\end{aligned}$$

where the last equality follows from the substitution $j = l + 1$. $\square$

3.2 A Presentation of $\mathcal{D}(T^2)$

In this section we will demonstrate the value of the elements $\tilde{P}_k$ by showing that the skein algebra $\mathcal{D}(T^2)$ admits a very simple presentation. First things first, let's define the generators. Recall that given an extended rational number r, there is a (homotopically) distinct oriented simple closed curve on the flat torus T^2 with rational slope r, and hence a smooth embedding of the annulus

$$\iota_r : A \hookrightarrow T^2$$

into tubular neighborhood of the curve. This induces an algebra homomorphism

$$\mathcal{D}(\iota_r) : \mathcal{D}(A) \to \mathcal{D}(T^2)$$

on the level of skein algebras. Any knot in $\mathcal{D}(T^2)$ is contained in the image of some $\mathcal{D}(\iota_r)$. Since $\mathcal{D}$ is an unoriented skein theory, let's only consider $r \geq 0$ to avoid redundancy due to the choice of orientations. Given any equivalence class $\mathbf{x} = (a, b)$ in $\mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x}\rangle$ with $k := \gcd(a, b)$, define the element

$$\tilde{P}_{\mathbf{x}} := \mathcal{D}\big(\iota_{a/b}\big)(\tilde{P}_k)$$

to be the embedding of the "power sum" element $\tilde{P}_k$ into this tubular neighborhood. Let's state the main theorem of the paper.

Theorem 3.2.1. *The skein algebra $\mathcal{D}(T^2)$ is presented by generators*

$$\{\tilde{P}_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x}\rangle\}$$

and relations

$$[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}}). \tag{3.24}$$

Corollary 3.2.2. *The linear span of the set* $\{\tilde{P}_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x}\rangle\}$ *is a Lie algebra, and* $\mathcal{D}(T^2)$ *is its universal enveloping algebra.*

The full proof of this theorem may be found in the collaboration * MPS. Here, we will be focused on showing that the generators satisfy the relations given in the presentation. We formulate this as a proposition.

Proposition 3.2.3. *The following special cases of Equation* (3.24) *hold*

$$[\tilde{P}_{1,0}, \tilde{P}_{0,n}] = (s^n - s^{-n})(\tilde{P}_{1,n} - \tilde{P}_{1,-n}) \tag{3.25}$$

$$[\tilde{P}_{1,0}, \tilde{P}_{1,n}] = (s^n - s^{-n})(\tilde{P}_{2,n} - \tilde{P}_{0,n}) \tag{3.26}$$

for any $n \geq 1$*. Furthermore, these relations generate all of the relations defined by Equation* (3.24).

Equation (3.25) is the image of Equation (3.3) under the wiring of $\mathcal{A}_{\mathcal{D}}$ into $\mathcal{D}(T^2)$ depicted below.

* Picture of wiring.

We will not give the proof of Equation (3.26) here * This part of the proof was finished before I started working on the project, but it may be found in * MPS. The idea given there is similar to the idea for the proof of Equation (3.25): first show a similar relation holds in the skein algebra of the annulus relative to two points $\mathcal{D}(A, [2])$ via a brute force computation, and then Equation (3.26) is the image of this relation under a simple wiring into $\mathcal{D}(T^2)$.

What's left for us to show here is the second statement of the proposition, that Equations (3.25) and (3.26) imply Equation (3.24). We will devote the rest of this section to doing so, starting with two technical lemmas and ending with Proposition 3.2.8.

We use the notation

$$d(\mathbf{x}, \mathbf{y}) := \det [\mathbf{x}\ \mathbf{y}] \qquad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2,$$

$$d(\mathbf{x}) := gcd(m,n) \qquad \text{when } \mathbf{x} = (m,n).$$

We will also use the following terminology:

$$(\mathbf{x},\mathbf{y}) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \text{ is } \textit{good} \text{ if } [\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = \{d(\mathbf{x},\mathbf{y})\}\left(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}}\right).$$

Remark 3.2.4. Note that because $\tilde{P}_{\mathbf{x}} = \tilde{P}_{-\mathbf{x}}$, if $(\mathbf{x},\mathbf{y})$ is good, then the pairs $(\pm\mathbf{x}, \pm\mathbf{y})$ are good as well.

The idea of the proof is to induct on the absolute value of the determinant of the matrix with columns $\mathbf{x}$ and $\mathbf{y}$. To induct, we write $\mathbf{x} = \mathbf{a}+\mathbf{b}$ for carefully chosen vectors $\mathbf{a}, \mathbf{b}$ and then use the following lemma.

Lemma 3.2.5. *Assume* $\mathbf{a}+\mathbf{b} = \mathbf{x}$ *and that* $(\mathbf{a},\mathbf{b})$ *is good. Further assume that the five pairs of vectors* $(\mathbf{y},\mathbf{a})$*,* $(\mathbf{y},\mathbf{b})$*,* $(\mathbf{y}+\mathbf{a},\mathbf{b})$*,* $(\mathbf{y}+\mathbf{b},\mathbf{a})$*, and* $(\mathbf{a}-\mathbf{b},\mathbf{y})$*, are good. Then the pair* $(\mathbf{x},\mathbf{y})$ *is good.*

Proof. We use the Jacobi identity and the goodness assumptions to compute

$$\begin{aligned}
& - && \{d(\mathbf{a},\mathbf{b})\}[\tilde{P}_{\mathbf{a}+\mathbf{b}}, \tilde{P}_{y}] + \{d(\mathbf{a},\mathbf{b})\}[\tilde{P}_{\mathbf{a}-\mathbf{b}}, \tilde{P}_{y}] \\
= & - && [[\tilde{P}_{\mathbf{a}}, \tilde{P}_{\mathbf{b}}], \tilde{P}_{\mathbf{y}}] \\
= & && [[\tilde{P}_{\mathbf{y}}, \tilde{P}_{\mathbf{a}}], \tilde{P}_{\mathbf{b}}] + [[\tilde{P}_{\mathbf{b}}, \tilde{P}_{y}], \tilde{P}_{\mathbf{a}}] \\
= & && \{d(\mathbf{y},\mathbf{a})\}[\tilde{P}_{\mathbf{y}+\mathbf{a}} - \tilde{P}_{\mathbf{y}-\mathbf{a}}, \tilde{P}_{\mathbf{b}}] + \{d(b,y)\}[\tilde{P}_{\mathbf{b}+\mathbf{y}} - \tilde{P}_{\mathbf{b}-\mathbf{y}}], \tilde{P}_{\mathbf{a}}] \\
= & && \{d(\mathbf{y},\mathbf{a})\}\left(\{d(\mathbf{y}+\mathbf{a},\mathbf{b})\}\left(\tilde{P}_{\mathbf{y}+\mathbf{a}+\mathbf{b}} - \tilde{P}_{\mathbf{y}+\mathbf{a}-\mathbf{b}}\right)\right. \\
& && \qquad \left. - \{d(\mathbf{y}-\mathbf{a},\mathbf{b})\}\left(\tilde{P}_{\mathbf{y}-\mathbf{a}+\mathbf{b}} - \tilde{P}_{\mathbf{y}-\mathbf{a}-\mathbf{b}}\right)\right) \\
& + && \{d(\mathbf{b},\mathbf{y})\}\left(\{d(\mathbf{b}+\mathbf{y},\mathbf{a})\}\left(\tilde{P}_{\mathbf{b}+\mathbf{y}+\mathbf{a}} - \tilde{P}_{\mathbf{b}+\mathbf{y}-\mathbf{a}}\right)\right. \\
& && \qquad \left. - \{d(\mathbf{b}-\mathbf{y},\mathbf{a})\}\left(\tilde{P}_{\mathbf{b}-\mathbf{y}+\mathbf{a}} - \tilde{P}_{\mathbf{b}-\mathbf{y}-\mathbf{a}}\right)\right) \\
= & && (\{d(\mathbf{y},\mathbf{a})\}\{d(\mathbf{y}+\mathbf{a},\mathbf{b})\} + \{d(\mathbf{b},\mathbf{y})\}\{d(\mathbf{b}+\mathbf{y},\mathbf{a})\})\,\tilde{P}_{\mathbf{a}+\mathbf{b}+\mathbf{y}} \\
& + && (\{d(\mathbf{y},\mathbf{a})\}\{d(\mathbf{y}-\mathbf{a},\mathbf{b})\} - \{d(\mathbf{b},\mathbf{y})\}\{d(\mathbf{b}-\mathbf{y},\mathbf{a})\})\,\tilde{P}_{\mathbf{a}+\mathbf{b}-\mathbf{y}} \\
& - && (\{d(\mathbf{y},\mathbf{a})\}\{d(\mathbf{y}+\mathbf{a},\mathbf{b})\} - \{d(\mathbf{b},\mathbf{y})\}\{d(\mathbf{b}-\mathbf{y},\mathbf{a})\})\,\tilde{P}_{\mathbf{a}-\mathbf{b}+\mathbf{y}}
\end{aligned}$$

$$
\begin{aligned}
& - \quad \left(\{d(\mathbf{y},\mathbf{a})\}\{d(\mathbf{y}-\mathbf{a},\mathbf{b})\} + \{d(\mathbf{b},\mathbf{y})\}\{d(\mathbf{b}+\mathbf{y},\mathbf{a})\}\right) \tilde{P}_{\mathbf{a}-\mathbf{b}-\mathbf{y}} \\
=: \quad & c_1 \tilde{P}_{\mathbf{a}+\mathbf{b}+\mathbf{y}} + c_2 \tilde{P}_{\mathbf{a}+\mathbf{b}-\mathbf{y}} - c_3 \tilde{P}_{\mathbf{a}-\mathbf{b}+\mathbf{y}} - c_4 \tilde{P}_{\mathbf{a}-\mathbf{b}-\mathbf{y}}.
\end{aligned}
$$

Using some simple algebra, we can show

$$
\begin{aligned}
c_1 &= \{d(\mathbf{y},\mathbf{a})\}\{d(\mathbf{y}+\mathbf{a},\mathbf{b})\} + \{d(\mathbf{b},\mathbf{y})\}\{d(\mathbf{b}+\mathbf{y},\mathbf{a})\} \\
&= \{d(\mathbf{y},\mathbf{a}) + d(\mathbf{y}+\mathbf{a},\mathbf{b})\}^+ - \{d(\mathbf{y},\mathbf{a}) - d(\mathbf{y}+\mathbf{a},\mathbf{b})\}^+ \\
&\quad + \{d(\mathbf{b},\mathbf{y}) + d(\mathbf{b}+\mathbf{y},\mathbf{a})\}^+ - \{d(\mathbf{b},\mathbf{y}) - d(\mathbf{b}+\mathbf{y},\mathbf{a})\}^+ \\
&= \{d(\mathbf{y},\mathbf{a}+\mathbf{b}) + d(\mathbf{a},\mathbf{b})\}^+ - \{d(\mathbf{y},\mathbf{a}-\mathbf{b}) - d(\mathbf{a},\mathbf{b})\}^+ \\
&\quad + \{d(\mathbf{y},\mathbf{a}-\mathbf{b}) - d(\mathbf{a},\mathbf{b})\}^+ - \{d(\mathbf{a},\mathbf{b}) - d(\mathbf{y},\mathbf{a}+\mathbf{b})\}^+ \\
&= \{d(\mathbf{a},\mathbf{b}) + d(\mathbf{y},\mathbf{a}+\mathbf{b})\}^+ - \{d(\mathbf{a},\mathbf{b}) - d(\mathbf{y},\mathbf{a}+\mathbf{b})\}^+ \\
&= \{d(\mathbf{a},\mathbf{b})\}\{d(\mathbf{y},\mathbf{a}+\mathbf{b})\} \\
&= -\{d(\mathbf{a},\mathbf{b})\}\{d(\mathbf{x},\mathbf{y})\}.
\end{aligned}
$$

Similar computations for the other c_i show that

$$
\begin{aligned}
\frac{1}{\{d(\mathbf{a},\mathbf{b})\}}[[\tilde{P}_{\mathbf{a}}, \tilde{P}_{\mathbf{b}}], \tilde{P}_{\mathbf{y}}] &= \frac{-1}{\{d(\mathbf{a},\mathbf{b})\}}\left(c_1 \tilde{P}_{\mathbf{a}+\mathbf{b}+\mathbf{y}} + c_2 \tilde{P}_{\mathbf{a}+\mathbf{b}-\mathbf{y}} - c_3 \tilde{P}_{\mathbf{a}-\mathbf{b}+\mathbf{y}} - c_4 \tilde{P}_{\mathbf{a}-\mathbf{b}-\mathbf{y}}\right) \\
&= \{d(\mathbf{x},\mathbf{y})\}\left(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}}\right) - \{d(\mathbf{a}-\mathbf{b},\mathbf{y})\}\left(\tilde{P}_{\mathbf{a}-\mathbf{b}+\mathbf{y}} - \tilde{P}_{\mathbf{a}-\mathbf{b}-\mathbf{y}}\right) \\
&= \{d(\mathbf{x},\mathbf{y})\}\left(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}}\right) - [\tilde{P}_{\mathbf{a}-\mathbf{b}}, \tilde{P}_{\mathbf{y}}]. \qquad (3.27)
\end{aligned}
$$

Since the pair $(\mathbf{a},\mathbf{b})$ is good, we have

$$
[\tilde{P}_{\mathbf{a}}, \tilde{P}_{\mathbf{b}}] = \{d(\mathbf{a},\mathbf{b})\}\big(\tilde{P}_{\mathbf{x}} - \tilde{P}_{\mathbf{a}-\mathbf{b}}\big). \qquad (3.28)
$$

Finally, combining equations (3.28) and (3.27) shows that the pair $(\mathbf{x},\mathbf{y})$ is good. □

Now we'll need to prove the following elementary lemma (which is a slight modification of [?FG00, Lemma 1] * fix). This is used to make a careful choice of vectors $\mathbf{a}, \mathbf{b}$ so that the previous lemma can be applied.

Lemma 3.2.6. *Suppose $p, q \in \mathbb{N}$ are relatively prime with $q < p$. Then there exist $u, v, w, z \in \mathbb{Z}$ such that the following conditions hold:*

$$\begin{aligned} 0 < u, w &< p, \\ \det \begin{bmatrix} u & w \\ v & z \end{bmatrix} &= 1, \\ \begin{bmatrix} u & w \\ v & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} p \\ q \end{bmatrix}. \end{aligned}$$

Proof. Since p and q are relatively prime, there exist $a, b \in \mathbb{Z}$ with $bq - ap = 1$. This solution can be modified to give another solution $a' = a + q$ and $b' = b + p$, so we may assume $0 \leq b < p$. We then define

$$u = b, \quad v = a, \quad w = p - b, \quad z = q - a.$$

By definition, u, v, w, z satisfy $u + w = p$ and $v + z = q$, and the inequalities $0 \leq b < p$ and $p > 1$ imply the condition $0 < u, w < p$. To finish the proof, we compute

$$uz - wv = b(q - a) - a(p - b) = bq - ap = 1.$$

□

Remark 3.2.7. The mapping class group of a (smooth) surface Σ is the group of isotopy classes of (smooth) orientation-preserving homeomorphisms $\Sigma \to \Sigma$. Any skein algebra $\mathcal{S}_X(\Sigma)$ is acted on by the mapping class group of Σ: the mapping class group is generated by Dehn twists on Σ and may be extended trivially onto $\Sigma \times I$, which induce algebra endomorphisms on the skein algebra due to functoriality of $\mathcal{S}_X(-)$. Thus, there is an $SL_2(\mathbb{Z})$-action on $\mathcal{D}(T^2)$ and is such that $g \cdot \tilde{P}_{\mathbf{x}} = \tilde{P}_{g\mathbf{x}}$ for any $g \in \mathrm{SL}_2(\mathbb{Z})$. One could choose to extend this to an action of the *extended* mapping class group of Σ (that is, to include orientation-reversing diffeomorphisms) so that $GL_2(Z)$ acts on $\mathcal{D}(T^2)$. If matrices of determinant -1 are to act on $\mathcal{D}(T^2)$ in the same way as before, then they will induce anti-linear algebra anti-homomorphisms on $\mathcal{D}(T^2)$. One could turn these into honest algebra homomorphisms by composing with the mirror map (note that $\tilde{P}_{\mathbf{x}}$ is fixed by the mirror map since the BMW symmetrizers are). It will be important that the orbits of this action on the $\tilde{P}_{\mathbf{x}}$ are the fibers of the assignment $\tilde{P}_{\mathbf{x}} \mapsto d(\mathbf{x})$ (which is essentially the statement of Lemma 3.2.6). In other words, up to the action of $GL_2(\mathbb{Z})$, vectors in $\mathbb{Z}^2$ are classified by the GCD of their entries.

Proposition 3.2.8. *Suppose A is an algebra with elements $\tilde{P}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^2/\langle -\mathbf{x} = \mathbf{x}\rangle$ that satisfy equations* (3.25) *and* (3.26). *Furthermore, suppose that there is a $\mathrm{GL}_2(\mathbb{Z})$-action on A so that $g \cdot \tilde{P}_{\mathbf{x}} = \tilde{P}_{g\mathbf{x}}$ for any $g \in \mathrm{GL}_2(\mathbb{Z})$. Then, any pair $(\mathbf{x}, \mathbf{y})$ is a good pair.*

Proof. The proof proceeds by induction on $|d(\mathbf{x}, \mathbf{y})|$, and the base case $|d(\mathbf{x}, \mathbf{y})| = 1$ is immediate from Remark 3.2.7 and the assumption (3.25) for $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (0, 1)$. We now make the following inductive assumption:

$$\text{For all } \mathbf{x}', \mathbf{y}' \in \mathbb{Z}^2 \text{ with } |d(\mathbf{x}', \mathbf{y}')| < |d(\mathbf{x}, \mathbf{y})|, \text{ the pair } (\mathbf{x}', \mathbf{y}') \text{ is good.} \tag{3.29}$$

We would like to show that $[\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}] = \{d(\mathbf{x}, \mathbf{y})\}\big(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}}\big)$. By Remark 3.2.7, we may assume

$$\mathbf{y} = \begin{bmatrix} 0 \\ r \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad d(\mathbf{x}) \le d(\mathbf{y}), \quad 0 \le q < p.$$

If $p = 1$, then this equation follows from Equation (3.26), so we may also assume $p > 1$. Furthermore, we may assume that $r > 0$ by Remark 3.2.4.

We will now show that if either $d(\mathbf{x}) = 1$ or $d(\mathbf{y}) = 1$, then $(\mathbf{x}, \mathbf{y})$ is good. By symmetry of the above construction of $\mathbf{x}$ and $\mathbf{y}$, we may assume $d(\mathbf{x}) = 1$, which immediately implies

$q > 0$. Furthermore, we may now assume that $r > 1$ by assuming Equation (3.26). We apply Lemma 3.2.6 to p, q to obtain $u, v, w, z \in \mathbb{Z}$ satisfying

$$uz - vw = 1, \quad uq - vp = 1, \quad u + w = p, \quad v + z = q, \quad 0 < u, w < p. \tag{3.30}$$

We then define vectors $\mathbf{a}$ and $\mathbf{b}$ as follows:

$$\mathbf{a} := \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} w \\ z \end{bmatrix}, \quad \mathbf{a} + \mathbf{b} = \mathbf{x}, \quad d(\mathbf{a}, \mathbf{b}) = 1. \tag{3.31}$$

Using Lemma 3.2.5 and Assumption (3.29), it is sufficient to show that each of $|d(\mathbf{a}, \mathbf{b})|$, $|d(\mathbf{y}, \mathbf{b})|$, $|d(\mathbf{y}, \mathbf{a})|$, $|d(\mathbf{y} + \mathbf{a}, \mathbf{b})|$, $|d(\mathbf{y} + \mathbf{b}, \mathbf{a})|$, and $|d(\mathbf{a} - \mathbf{b}, \mathbf{y})|$ are strictly less than $pr = |d(\mathbf{x}, \mathbf{y})|$. First, $|d(\mathbf{a}, \mathbf{b})| = 1$ is strictly less than pr since $p > 1$ and $r > 0$. Second, $|d(\mathbf{y}, \mathbf{a})| = ur$ and $|d(\mathbf{y}, \mathbf{b})| = wr$ are strictly less than pr by the inequalities in (3.30). Third, we compute

$$\begin{aligned} |d(\mathbf{y} + \mathbf{a}, \mathbf{b})| &= |-d(\mathbf{y} + \mathbf{a}, \mathbf{b})| \\ &= |-d(\mathbf{y}, \mathbf{b}) - d(\mathbf{a}, \mathbf{b})| \\ &= |wr - 1| \\ &= wr - 1 \\ &< wr \\ &< pr. \end{aligned}$$

Fourth, we compute

$$\begin{aligned} |-d(\mathbf{y} + \mathbf{b}, \mathbf{a})| &= |-d(\mathbf{y} + \mathbf{b}, \mathbf{a})| \\ &= |-d(\mathbf{y}, \mathbf{a}) - d(\mathbf{b}, \mathbf{a})| \\ &= |ur + 1| \\ &= ur + 1 \end{aligned}$$

$$\begin{aligned} &\leq (p-1)r+1 \\ &= pr-r+1 \\ &< pr. \end{aligned}$$

Finally, we compute

$$\begin{aligned} |d(\mathbf{a}-\mathbf{b},\mathbf{y})| &= |d(\mathbf{a},\mathbf{y})-d(\mathbf{b},\mathbf{y})| \\ &= |d(\mathbf{y},\mathbf{a})-d(\mathbf{y},\mathbf{b})| \\ &= |ur-wr| \\ &= |u-w|r \\ &< |u+w|r \\ &= pr. \end{aligned}$$

So we have shown that $(\mathbf{x},\mathbf{y})$ is good if $d(\mathbf{x})=1$ or $d(\mathbf{y})=1$. Let us now turn our attention to the more general case. We will immediately split this into cases depending on q.

Case 1: Assume $0<q$.

Let $p'=p/d(\mathbf{x})$ and $q'=q/d(\mathbf{x})$. By the assumption $0<q$, we see that $d(\mathbf{x})<p$, so $p'>1$. We can therefore apply Lemma 3.2.6 to p',q' to obtain $u,v,w,z\in\mathbb{Z}$ satisfying

$$uz-vw=1, \quad uq'-vp'=1, \quad u+w=p', \quad v+z=q', \quad 0<u,w<p'. \tag{3.32}$$

In a way similar to the above, we may pick vectors $\mathbf{a}$ and $\mathbf{b}$ like so:

$$\mathbf{a}:=\begin{bmatrix} d(\mathbf{x})u \\ d(\mathbf{x})v \end{bmatrix}, \quad \mathbf{b}:=\begin{bmatrix} d(\mathbf{x})w \\ d(\mathbf{x})z \end{bmatrix}, \quad \mathbf{a}+\mathbf{b}=\mathbf{x}, \quad d(\mathbf{a},\mathbf{b})=d(\mathbf{x})^2. \tag{3.33}$$

As before, it is sufficient to show that each of $|d(\mathbf{a},\mathbf{b})|$, $|d(\mathbf{y},\mathbf{b})|$, $|d(\mathbf{y},\mathbf{a})|$, $|d(\mathbf{y}+\mathbf{a},\mathbf{b})|$, $|d(\mathbf{y}+\mathbf{b},\mathbf{a})|$, and $|d(\mathbf{a}-\mathbf{b},\mathbf{y})|$ are strictly less than $pr = |d(\mathbf{x},\mathbf{y})|$. First,

$$|d(\mathbf{a},\mathbf{b})| = d(\mathbf{x})^2 \leq d(\mathbf{x})d(\mathbf{y}) = d(\mathbf{x})r < pr$$

where the last inequality follows from the assumption $0 < q < p$. Second, we can compute $|d(\mathbf{y},\mathbf{b})| = d(\mathbf{x})wr$ and $|d(\mathbf{y},\mathbf{a})| = d(\mathbf{x})ur$ are strictly less than pr by the inequalities in (3.32). Third, we compute

$$\begin{aligned}
|d(\mathbf{y}+\mathbf{a},\mathbf{b})| &= |-d(\mathbf{y}+\mathbf{a},\mathbf{b})| \\
&= |-d(\mathbf{y},\mathbf{b}) - d(\mathbf{a},\mathbf{b})| \\
&= |d(\mathbf{x})wr - d(\mathbf{x})^2| \\
&= d(\mathbf{x})wr - d(\mathbf{x})^2 \\
&< d(\mathbf{x})wr \\
&\leq pr.
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
|d(\mathbf{y}+\mathbf{b},\mathbf{a})| &= |-d(\mathbf{y}+\mathbf{b},\mathbf{a})| \\
&= |-d(\mathbf{y},\mathbf{a}) - d(\mathbf{b},\mathbf{a})| \\
&= d(\mathbf{x})ur + d(\mathbf{x})^2 \\
&\leq \left(d(\mathbf{x})u + d(\mathbf{x})\right) d(\mathbf{y}) \\
&= (u+1)d(\mathbf{x})r.
\end{aligned}$$

Therefore, we will be finished once we show that $(u+1)d(\mathbf{x})$ is strictly less than p. We now split into subcases:

Subcase 1a: If $u+1 < p'$, then $(u+1)d(\mathbf{x})r < p'd(\mathbf{x})r = pr$, and we are done.

Subcase 1b: Assume $u + 1 = p'$. By equation (3.32), we have

$$1 = uq' - vp' = (p' - 1)q' - vp' \implies p'(q' - v) = 1 + q' < 1 + p'.$$

Since $p' > 1$, the last inequality implies $q' - v = 1$, which implies $v = q' - 1$ and $z = 1$ since $v + z = q'$. Now the equation $uz - vw = 1$ implies $(p' - 1) - (q' - 1) = 1$, which implies $q' = p' - 1$. Writing $d = d(\mathbf{x})$ for short, we have

$$|d(\mathbf{y} + \mathbf{b}, \mathbf{a})| = \left|\det \begin{bmatrix} d & p - d \\ d + r & p - 2d \end{bmatrix}\right| = |d(p - 2d) - (p - d)(d + r)| = \left|rp + d\big(d - r\big)\right|$$

which is at most rp since we are assuming $d(\mathbf{x}) \leq r$. If this inequality is strict, then we are done. Otherwise, we move onto the next subcase.

Subcase 1c: In this subcase, we are reduced to showing the following pair of vectors is good:

$$\mathbf{y} = (0, r), \qquad \mathbf{x} = (rp', rp' - r).$$

If $r = 1$, then $d(\mathbf{y}) = 1$, which makes $(\mathbf{x}, \mathbf{y})$ good. Thus, we may assume that $r > 1$.

We must replace our previous choice of $\mathbf{a}$ and $\mathbf{b}$ with a choice which is better adapted to this particular subcase. We define

$$\mathbf{a} := \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} rp' - 1 \\ rp' - r + 1 \end{bmatrix}.$$

We know that the pairs $(\mathbf{a}, \mathbf{b}), (\mathbf{y}, \mathbf{a}), (\mathbf{y} + \mathbf{b}, \mathbf{a})$ are good since $d(\mathbf{a}) = 1$. Since $r > 1$ and $p' > 1$, we can compute that the determinants of the pairs $(\mathbf{y}, \mathbf{b}), (\mathbf{y} + \mathbf{a}, \mathbf{b}), (\mathbf{a} - \mathbf{b}, \mathbf{y})$ are strictly less than $r^2 p' = |d(\mathbf{x}, \mathbf{y})|$:

$$\begin{aligned} |d(\mathbf{a} - \mathbf{b}, \mathbf{y})| &= |r^2 p' - 2r| \\ &= r^2 p' - 2r \end{aligned}$$

$$< \quad r^2p'$$

$$\begin{aligned}
|d(\mathbf{y}, \mathbf{b})| &= |r^2p' - r| \\
&= r^2p' - r \\
&< r^2p'
\end{aligned}$$

$$\begin{aligned}
|d(\mathbf{y}+\mathbf{a}, \mathbf{b})| &= |r^2p' - (rp' + r - 1)| \\
&= r^2p' - (rp' + r - 1) \\
&< r^2p'.
\end{aligned}$$

This together with Assumption (3.29) and Lemma 3.2.5 shows that $(\mathbf{x}, \mathbf{y})$ is good, which finishes the proof of this subcase and finishes the proof of Case 1.

Case 2: In this case we assume $q = 0$. We define $\mathbf{a}, \mathbf{b}$ similarly to Subcase 1c, so we have

$$\mathbf{y} = \begin{bmatrix} 0 \\ r \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} p \\ 0 \end{bmatrix}, \quad \mathbf{a} := \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} p-1 \\ 1 \end{bmatrix}.$$

Since $d(\mathbf{a}) = d(\mathbf{b}) = 1$, the pairs $(\mathbf{a}, \mathbf{b}), (\mathbf{y}, \mathbf{a}), (\mathbf{y}, \mathbf{b}), (\mathbf{y}+\mathbf{a}, \mathbf{b}), (\mathbf{y}+\mathbf{b}, \mathbf{a})$ are all good. We must check that $|d(\mathbf{a}-\mathbf{b}, \mathbf{y})| < pr = |d(\mathbf{x}, \mathbf{y})|$. If $r = 1$, then the relation (3.25) implies that the pair $(\mathbf{x}, \mathbf{y})$ is good. Thus, we may assume that $r > 1$. We may also assume that $p > 1$. Finally, we check $|d(\mathbf{a}-\mathbf{b}, \mathbf{y})| = |rp - 2r| = rp - 2r < rp$. By using Assumption (3.29) and Lemma 3.2.5, this completes Case 2 which completes the proof. $\square$

3.3 Relationships With $\mathcal{K}(T^2)$ and $\mathcal{H}(T^2)$

3.3.1 The Kauffman Bracket Case

Recall that the Dubrovnik skein relations satisfy the Kauffman Bracket skein relations, and hence there is a natural transformation of functors

$$\eta : \mathcal{D} \to \mathcal{K}$$

whose components are essentially the identity map (more details are given in Section 2.1.2). In particular, there is an algebra homomorphism from the Birman-Murakami-Wenzl algebra to the Temperley-Lieb algebra.

Proposition 3.3.1. *Under the algebra homomorphism*

$$\eta_{(I\times I,[n])} : BMW_n \to TL_n$$

the BMW symmetrizer $\tilde{y}_n$ is sent to the Jones-Wenzl idempotent JW_n.

Proof. The element JW_n is the unique element of TL_n satisfying the properties

1. $JW_n \neq 0$,
2. $JW_n^2 = JW_n$,
3. $JW_n c_i = c_i JW_n = 0$ for all cap-cup generators c_i.

Since $\eta_{(I\times I,[n])}$ is an algebra map, we just need to show that $\eta_{(I\times I,[n])}(\tilde{y}_n)$ is non-zero. If $n = 1$, then the statement is true since $\tilde{y}_1$ and JW_1 are the simply the identity, a single strand. * Induction with recurrence relation? or...

□

Remark 3.3.2. In * Shelly, it is pointed out that, under the composition of algebra maps $BMW_n \to H_n \to TL_n$ (see Section 2.1.3), the BMW symmetrizer $\tilde{y}_n$ is sent to the Jones-Wenzl idempotent JW_n. However, these maps are defined abstractly on generators, and so *a priori* they are not induced by any natural transformations of skein theories. Furthermore, the diagram

$$\begin{tikzcd} & BMW_n \arrow[dl, two heads, "\pi_H"] \arrow[dd, two heads, "\eta_{(I\times I,[n])}"] \\ H_n \arrow[dr, two heads, "\pi_{TL}"] & \\ & TL_n \end{tikzcd}$$

does not commute since $\pi_H(c_i) = 0$. However, Proposition 3.3.1 shows that restricting the maps of the diagram on the relevant idempotents *does* commute.

It was mentioned in Section 2.1.2 that *Chebyshev polynomials* are imporant elements of the skein algebra $\mathcal{K}(A) = R[z]$. Let's now elaborate what that means. The **Chebyshev polynomials of the second kind** $C_n^S = C_n^S(z)$ are defined recursively as

$$C_0^U = 1, \qquad C_1^U = 2z, \qquad C_n^U = 2zC_{n-1}^U - C_{n-2}^U.$$

Let's consider slightly modified versions $U_n := C_n^U(z/2)$. These polynomials have amazing properties, most of which we won't discuss here. One such fact is that the annular closure of the Jones-Wenzl idempotent JW_n is equal to $U_n(z)$ * reference??.

There are also **Chebyshev polynomials of the first kind** $C_k^T = C_k^T(z)$ which are defined recursively as

$$C_0^T = 1, \qquad C_1^T = z, \qquad C_k^T = 2zC_{k-1}^T - C_{k-2}^T.$$

Again, we will need slightly modified versions of these, $T_k := 2C_k^T(z/2)$.

Corollary 3.3.3. *Under the algebra homomorphism*

$$\eta_A : \mathcal{D}(A) \to \mathcal{K}(A)$$

the image of the special element $\tilde{P}_k$ is equal to the Chebyshev polynomial T_k.

Proof. Since η is natural, the closure into the annulus induces a commuting square

$$\begin{tikzcd} BMW_n \arrow[r, "\mathrm{cl}"] \arrow[d, "\eta"] & \mathcal{D}(A) \arrow[d, "\eta"] \\ TL_n \arrow[r, "\mathrm{cl}"] & \mathcal{K}(A) \end{tikzcd}$$

Recall the $\tilde{P}_k$ are uniquely determined by the power series equation

$$\sum_{k=1}^{\infty} \frac{\tilde{P}_k}{k} t^k = \ln\left(1 + \sum_{n=1}^{\infty} \tilde{h}_n t^n\right).$$

If we show that the following equality holds

$$\sum_{k=1}^{\infty} \frac{T_k}{k} t^k = \ln\left(1 + \sum_{n=1}^{\infty} U_n t^n\right)$$

then the proof is complete since $\eta_A(\tilde{h}_n) = U_n$. The Chebyshev polynomials admit well-known generating functions (see Wikipedia)

$$\sum_{n\geq 0} C_n^U(z) t^n = \frac{1}{1 - 2tz + t^2}$$
$$\sum_{k\geq 1} C_k^T(z) \frac{t^k}{k} = \ln\left(\frac{1}{\sqrt{1 - 2tz + t^2}}\right).$$

These imply

$$\sum_{n\geq 0} C_n^U(z/2) t^n = \frac{1}{1 - tz + t^2}$$
$$\sum_{k\geq 1} C_k^T(z/2) \frac{t^k}{k} = \ln\left(\frac{1}{\sqrt{1 - tz + t^2}}\right).$$

Multiply the second equation by 2 to get

$$\sum_{k\geq 1} 2C_k^T(z/2) \frac{t^k}{k} = \ln\left(\frac{1}{1 - tz + t^2}\right)$$

which implies the desired equation since $T_k = 2C_k^T(z/2)$ and $U_n := C_n^U(z/2)$. $\square$

Recall once more the embedding of the annulus

$$\iota_r : A \hookrightarrow T^2$$

into a tubular neighborhood of the simple closed curve of rational slope $r = a/b$ for (a, b) considered as an element of $\mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x} \rangle$. These embeddings induce naturality squares

$$\begin{array}{ccc} \mathcal{D}(A) & \xrightarrow{\mathcal{D}(\iota_r)} & \mathcal{D}(T^2) \\ \downarrow \eta & & \downarrow \eta \\ \mathcal{K}(A) & \xrightarrow{\mathcal{K}(\iota_r)} & \mathcal{K}(T^2) \end{array}$$

which imply that $\eta_{T^2}(\tilde{P}_{\mathbf{x}}) = T_{\mathbf{x}}$ for any $\mathbf{x}$ where the $T_{\mathbf{x}}$ are the generators in the Frohman-Gelca presentation of $\mathcal{K}(T^2)$. Indeed, one can check that the T_k satisfy the relations Theorem 3.2.1:

$$\begin{aligned} [T_{\mathbf{x}}, T_{\mathbf{y}}] &= T_{\mathbf{x}}T_{\mathbf{y}} - T_{\mathbf{y}}T_{\mathbf{x}} \\ &= (s^{\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}-\mathbf{y}}) - (s^{\det(\mathbf{y},\mathbf{x})}T_{\mathbf{y}+\mathbf{x}} + s^{-\det(\mathbf{y},\mathbf{x})}T_{\mathbf{y}-\mathbf{x}}) \\ &= (s^{\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}+\mathbf{y}} + s^{-\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}-\mathbf{y}}) - (s^{-\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}+\mathbf{y}} + s^{\det(\mathbf{x},\mathbf{y})}T_{\mathbf{x}-\mathbf{y}}) \\ &= (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(T_{\mathbf{x}+\mathbf{y}} - T_{\mathbf{x}-\mathbf{y}}) \end{aligned}$$

Let's summarize this discussion with a corollary.

Corollary 3.3.4. *The presentation of $\mathcal{D}(T^2)$ is compatible with the Frohman-Gelca presentation of $\mathcal{K}(T^2)$ under the algebra homomorphism*

$$\eta_{T^2} : \mathcal{D}(T^2) \to \mathcal{K}(T^2).$$

It should probably be emphasized that this homomorphism is defined topologically on the level of diagrams, and not simply via an abstract assignment of generators. This is important because skein theoretic methods will translate through these types of homomorphisms.

3.3.2 The HOMFLYPT Case

Recall that $\mathcal{H}(T^2)$ is presented by generators

$$\{P_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Z}^2\}$$

and relations

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}}.$$

This implies that the span of the generators is a Lie algebra and $\mathcal{H}(T^2)$ is its universal enveloping algebra. Let's use the notation

$$\mathfrak{g}_{\mathcal{H}} := R\{P_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Z}^2\} \qquad \mathfrak{g}_{\mathcal{D}} := R\{\tilde{P}_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Z}^2/\langle \mathbf{x} = -\mathbf{x}\rangle\}.$$

Proposition 3.3.5. *There is an injective Lie algebra homomorphism*

$$\varphi : \mathfrak{g}_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{H}}$$

defined by $\varphi(\tilde{P}_{\mathbf{x}}) = P_{\mathbf{x}} + P_{-\mathbf{x}}$. This induces an algebra homomorphism

$$U(\varphi) : \mathcal{D}(T^2) \to \mathcal{H}(T^2)$$

where $U(-)$ is the universal enveloping algebra functor. Furthermore, the restrition of this homomorphism to an annulus provides an identification for $\mathcal{D}(A)$ as a subalgebra of $\mathcal{H}(A)$.

Proof. The assignment is linear since it's defined on basis elements. Let's check that

$$\varphi\big([\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}]\big) = [\varphi(\tilde{P}_{\mathbf{x}}), \varphi(\tilde{P}_{\mathbf{y}})]$$

for any $\mathbf{x}$ and $\mathbf{y}$. The left-hand side is

$$\varphi\big([\tilde{P}_{\mathbf{x}}, \tilde{P}_{\mathbf{y}}]\big)$$

$$
\begin{aligned}
&= \varphi\big((s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})(\tilde{P}_{\mathbf{x}+\mathbf{y}} - \tilde{P}_{\mathbf{x}-\mathbf{y}})\big) \\
&= (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})\big(\varphi(\tilde{P}_{\mathbf{x}+\mathbf{y}}) - \varphi(\tilde{P}_{\mathbf{x}-\mathbf{y}})\big) \\
&= (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})\big(P_{\mathbf{x}+\mathbf{y}} + P_{-\mathbf{x}-\mathbf{y}} - P_{\mathbf{x}-\mathbf{y}} - P_{-\mathbf{x}+\mathbf{y}}\big) \\
&= (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}+\mathbf{y}} + (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{-\mathbf{x}-\mathbf{y}} \\
&- (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}-\mathbf{y}} - (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{-\mathbf{x}+\mathbf{y}} \\
&= (s^{\det(\mathbf{x},\mathbf{y})} - s^{-\det(\mathbf{x},\mathbf{y})})P_{\mathbf{x}+\mathbf{y}} + (s^{\det(-\mathbf{x},-\mathbf{y})} - s^{-\det(-\mathbf{x},-\mathbf{y})})P_{-\mathbf{x}-\mathbf{y}} \\
&- (s^{-\det(\mathbf{x},-\mathbf{y})} - s^{\det(\mathbf{x},-\mathbf{y})})P_{\mathbf{x}-\mathbf{y}} - (s^{-\det(-\mathbf{x},\mathbf{y})} - s^{\det(-\mathbf{x},\mathbf{y})})P_{-\mathbf{x}+\mathbf{y}} \\
&= [P_{\mathbf{x}}, P_{\mathbf{y}}] + [P_{-\mathbf{x}}, P_{-\mathbf{y}}] + [P_{\mathbf{x}}, P_{-\mathbf{y}}] + [P_{-\mathbf{x}}, P_{\mathbf{y}}] \\
&= [P_{\mathbf{x}} + P_{-\mathbf{x}}, P_{\mathbf{y}} + P_{-\mathbf{y}}] \\
&= [\varphi(\tilde{P}_{\mathbf{x}}), \varphi(\tilde{P}_{\mathbf{y}})].
\end{aligned}
$$

Note that $\varphi(\tilde{P}_{\mathbf{x}}) = \varphi(\tilde{P}_{-\mathbf{x}})$, so the map is well-defined. This completes the proof. $\square$

Remark 3.3.6. It's worth pointing out that this homomorphism is defined abstractly on generators. It is not clear whether or not this map is induced by something topological in nature, as was the case for the map $\eta_{T^2} : \mathcal{D}(T^2) \to \mathcal{K}(T^2)$. After all, we are relating an unoriented skein theory with an oriented one. If this map were to arise from topology, it would have to involve a choice of orientations for each component of each link. All attempts to find such a choice has been unsuccessful.

Chapter 4

Annular Closures of Minimal Idempotents of BMW_n

4.0.1 ??

References