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THESIS TITLE

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ABSTRACT OF THE DISSERTATION

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Dr. Peter Samuelson, Chairperson

The text of the abstract goes here. An abstract is required, and must not exceed 350 words. Further details can be found in UCR's formatting guidelines:

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Chapter 1

Introduction

Summarize the main background and new results here.

Chapter 2

Background

2.1 Skein Theory

Foundations and General Notions

In this work, we will be forced to discuss a couple different variants of skein modules. For this reason, it will be useful to first describe some general framework of skein theory so that each of these variants will be a special case. Unless otherwise stated, we will assume M is an * oriented 3-manifold with boundary ∂M (possibly empty), Σ is an * oriented surface, I is the real interval $[0, 1]$, R is a commutative and unital ring.

* Is it necessary that M is oriented? I forget why we had this assumption.

Definition 2.1.1. Let $T_1, T_2 : X \to M$ be smooth embeddings of a smooth manifold X into M. A **smooth ambient isotopy** $H : T_1 \Rightarrow T_2$ is a smooth homotopy of diffeomorphisms H_t such that $H_0 = \mathrm{id}_M$ and $H_1 \circ T_1 = T_2$. Furthermore, we demand that the boundary ∂M be fixed by the homotopy.

The relation

$$T_1 \sim T_2 \text{ if and only if there is a smooth ambient isotopy } H : T_1 \Rightarrow T_2$$

is an equivalence relation. The smoothness requirement is important when considering knots. Without it, all knots would fall into the same equivalence class.

Definition 2.1.2. Let N be a finite set of points contained in the boundary ∂M. An N**-tangle** in M (or just *tangle* for short) is the smooth ambient isotopy class of a smooth embedding

$$T : \underbrace{\bigsqcup_{j \in J} S^1}_{:=L} \sqcup \underbrace{\bigsqcup_{k \in K} I}_{:=B} \to M$$

for some finite sets J and K such that

1. the image of L lies in the interior of M,

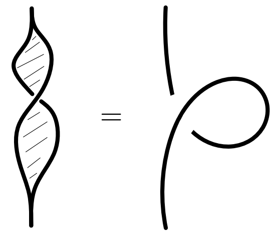
2. the image of the interior of B lies in the interior of M,

3. the image of the boundary of B equals N.

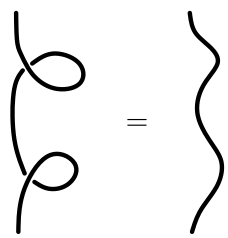
* Fix spacing of enumerations. If B is empty, then the result is called a **link** in M. Similarly, if L is empty, then it's called a **braid** in M.

One may also consider *oriented* or *framed* tangles by choosing an orientation or framing for each point in N and for each connected component of L and B such that the choices are compatible with each other with respect to the smooth embedding. For our purposes, the difference between framed and unframed tangles will be a matter of convention; we choose to work with framed tangles as is standard in the literature. If $M = \Sigma \times I$, then we will assume that the points in N are contained in $\Sigma \times \{\frac{1}{2}\}$ and that their framings are thought to be parallel to Σ.

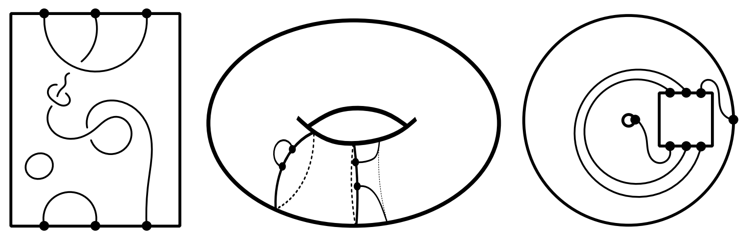
We say a framed tangle in $\Sigma \times I$ has **blackboard framing** if the framing is perpendicular to Σ. Every framed link in $\Sigma \times I$ is isotopic to one with a blackboard framing by turning each twist into a loop with a local blackboard framing:



This suggests that we may represent framed links in $\Sigma \times I$ as link diagrams on Σ fairly easily. Indeed, ambient isotopy is captured by the Reidemeister moves 2, 3, and a modified Reidemeister move 1:



* Add pictures of RII and RIII for good measure. Below are some examples of framed tangle diagrams.



* Fix spacing in image.

Definition 2.1.3. Let $\mathcal{T}(M, N)$ be the free R-module generated by the set of framed N-tangles in M. Analagously, we can define $\mathcal{T}^{or}(M, N)$ to be the free R-module generated by the set of oriented framed N-tangles in M. All definitions which are to follow in this subsection have an analagous definition using oriented tangles. Also, we will formally define $\mathcal{S}_R(\varnothing, \varnothing) := R$.

* Is it true that $\mathcal{T}(M, N)$ can be constructed as a quotient of $\mathcal{T}^{or}(M, N)$ by the relation that all possible orientations of a given link are equal? If so, there should be a natural

transformation of functors $\mathcal{T}^{or} \Rightarrow \mathcal{T}$. Check if this would descend to a natural transformation $\mathcal{H} \Rightarrow \mathcal{D}$, although I think I checked before that it doesn't.

The construction $\mathcal{T}(-,-)$ is actually a symmetric monoidal functor $\mathcal{T} : \mathsf{C} \to R$-Mod for a careful choice of category C which we now describe. The objects of C are pairs (M, N) of the same type as before. A morphism $(f, W) : (M', N') \to (M, N)$ and is given by a smooth embedding $f : M' \to M$ such that $M - f(M')$ is either a smooth 3-manifold or the empty set, and $W \in \mathcal{T}\big(M - f(M'), N \sqcup f(N')\big)$ (unless $M - f(M')$ is empty, in which case W is a formal symbol for the "empty link" in the empty set). Composition is given by $(g, W') \circ (f, W) = (g \circ f, W' \cup W)$, which is associative since $\circ$ and $\cup$ are associative.

* Give a picture of composition in C.

It is clear that $\mathcal{T}$ preserves composition and identity morphisms, and so $\mathcal{T}$ is functorial. The induced map denoted $W : \mathcal{T}(M', N') \to \mathcal{T}(M, N)$ is a linear map defined by $W(T) = W \cup T$, and we will refer to such a linear map W as a **wiring**. We are abusing notation by denoting this linear map by W, but it should be clear from the context what f is since it is technically encoded in the data of the element $W \in \mathcal{T}\big(M - f(M'), N \sqcup f(N')\big)$. Now, C can be equipped with a symmetric monoidal structure via disjoint union. It should be clear that

$$\mathcal{S}_R(M \sqcup M', N \sqcup N') \cong \mathcal{S}_R(M, N) \underset{R}{\otimes} \mathcal{S}_R(M', N')$$

for any sets of framed points $N \subset \partial M$ and $N' \subset \partial M'$. The unit is given by the object $(\varnothing, \varnothing)$ in C.

Definition 2.1.4. Let B be the smooth closed 3-ball, and let $X \subset \bigsqcup_{N_B} \mathcal{T}(B, N_B)$ be some set (usually finite), which we will call a set of **skein relations**. Now, given any tangle module $\mathcal{T}(M, N_M)$, there exists a submodule $\mathcal{I}(X)$ generated by the set

$$\{W(x) \mid x \in X \text{ and } W : \mathcal{T}(B, N_B) \to \mathcal{T}(M, N_M) \text{ is a wiring diagram}\}.$$

A quotient of the form $\mathcal{S}_X(M,N) := \mathcal{T}(M,N)/\mathcal{I}(X)$ is called a **skein module** of M relative to N. If $N = \varnothing$ is the empty set, we may use the notation $\mathcal{S}_X(M) := \mathcal{S}_X(M,\varnothing)$. Similar definitions may be given using oriented tangles instead.

The construction $\mathcal{S}_X(-,-)$ is a functor in the same way that $\mathcal{T}(-,-)$ is; a smooth embedding $f : M \to M'$ and an element $W \in \mathcal{S}_X\big(M - f(M'), N \sqcup f(N')\big)$ defines a linear map $W : \mathcal{S}_X(M,N) \to \mathcal{S}_X(M',N')$. In fact, the quotient maps $\alpha_{(M,N)} : \mathcal{T}(M,N) \to \mathcal{S}_X(M,N)$ yield a natural transformation. In other words, given a morphism $(M,N) \to (M',N')$ in C, the diagram

$$\begin{array}{ccc} \mathcal{T}(M,N) & \xrightarrow{W} & \mathcal{T}(M',N') \\ \big\downarrow{\scriptstyle \alpha_{(M,N)}} & & \big\downarrow{\scriptstyle \alpha_{(M',N')}} \\ \mathcal{S}_X(M,N) & \xrightarrow{W} & \mathcal{S}_X(M',N') \end{array}$$

commutes.

For any Σ, we can define a category $\mathsf{Skein}_X(\Sigma)$. The objects of this category are finite sets of framed points N in Σ, and the morphisms $N \to N'$ are elements of $\mathcal{S}_X\big(\Sigma \times I, (N \times \{0\}) \sqcup (N' \times \{1\})\big)$, so the category is R-linear. Write composition of morphisms by concatenation. If $y : N \to N'$ and $z : N' \to N''$ are morphisms, then their composite $yz : N \to N''$ is constucted by gluing z on y through N' and rescaling the interval coordinate appropriately.

* Picture of composition in $\mathsf{Skein}_X(\Sigma)$.

The endomorphism algebras in this category are called **skein algebras** and are denoted by $\mathcal{S}_X(\Sigma, N) := \mathcal{S}_X(\Sigma \times I, (N \times \{0\}) \sqcup (N \times \{1\})\big)$. If N is the empty set, then we reduce the notation to simply $\mathcal{S}_X(\Sigma)$.

If $f : \Sigma \to \Sigma'$ is a smooth, * orientation preserving embedding of surfaces, then there is an induced functor

$$f_* : \mathsf{Skein}_X(\Sigma') \to \mathsf{Skein}_X(\Sigma)$$

defined on objects by $f_*(N) = f(N)$ and on morphisms in the following way. First, extend f trivially to $f \times \mathrm{id}_I : \Sigma \times I \to \Sigma' \times I$. Then, in the skein algebra of the complement of the image of $f \times \mathrm{id}_I$, choose the multiplicative identity element $e \in \mathcal{S}_X\big(\Sigma' - \mathrm{Im}(f)\big)$ which is the empty tangle. The pair $(f \times \mathrm{id}_I, e)$ is an object in the category C, which gives rise to a wiring

$$e : \mathcal{S}_X\Big(\Sigma \times I, \big(N \times \{0\}\big) \sqcup \big(N' \times \{1\}\big)\Big) \to \mathcal{S}_X\Big(\Sigma' \times I, \big(f(N) \times \{0\}\big) \sqcup \big(f(N') \times \{1\}\big)\Big)$$

via the functor $\mathcal{S}_X$. Now we may define what f_* does to morphisms: $f_*(y) = e(y)$ for any $y \in \mathcal{S}_X\Big(\Sigma \times I, (N \times \{0\}) \sqcup (N' \times \{1\})\Big)$.

* Picture of how f_* works on morphisms.

It is clear that f_* preserves composition. In particular, f_* defines algebra homomorphisms on the skein algebras

$$e : \mathcal{S}_X\big(\Sigma, N\big) \to \mathcal{S}_X\big(\Sigma', f(N)\big).$$

* We use this type of algebra homomorphism when we embed the annulus into the torus.

The above homomorphisms are a special case of a more general type of map. If N is a set of framed points on Σ, then a smooth embedding $f : \Sigma \to \partial M$ induces a $\mathcal{S}_X(\Sigma, N)$-module structure on $\mathcal{S}_X\big(M, N'\big)$ for any N' with $f(N) \subseteq N'$. The action is given by "pushing tangles in through the boundary". In other words, the pre-composition of an embedding of a collar neighborhood $g : \partial M \times I \to M$ with the embedding $f \times \mathrm{id}_I : \Sigma \times I \to \partial M \times I$ induces a bilinear map

$$\mathcal{S}_X(\Sigma, N) \times \mathcal{S}_X(M, N') \to \mathcal{S}_X(M, N')$$

because M minus a collar neighborhood is diffeomorphic to itself. Alternatively, a choice of element in $\mathcal{S}_X(\Sigma, N')$ produces a wiring $\mathcal{S}_X(M, N') \to \mathcal{S}_X(M, N')$.

* Picture of action.

HOMFLYPT and Kauffman Skein Modules

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Chapter 4

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References