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ABSTRACT OF THE DISSERTATION

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Dr. Peter Samuelson, Chairperson

The text of the abstract goes here. An abstract is required, and must not exceed 350 words. Further details can be found in UCR's formatting guidelines:

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Introduction

Summarize the main background and new results here.

Chapter 2

Background

2.1 Skein Theory

Foundations and General Notions

In this work, we will be forced to discuss a few different variants of skein modules. For this reason, it will be useful to first describe some general framework of skein theory so that each of these variants will be a special case. Unless otherwise stated, we will assume M is an oriented 3-manifold with boundary ∂M (possibly empty), Σ is an oriented surface, I is the real interval $[0,1]$, R is a commutative and unital ring.

Definition 2.1.1. Let $T_1, T_2 : X \to M$ be smooth embeddings of a smooth manifold X into M. A **smooth ambient isotopy** $H : T_1 \Rightarrow T_2$ is a smooth homotopy of diffeomorphisms H_t such that $H_0 = \mathrm{id}_M$ and $H_1 \circ T_1 = T_2$. Furthermore, we demand that the boundary ∂M is fixed by the homotopy.

The relation

$$T_1 \sim T_2 \text{ if and only if there exists a smooth ambient isotopy } H : T_1 \Rightarrow T_2$$

is an equivalence relation. The smoothness requirement is important when considering knots. Without it, all knots would fall into the same equivalence class.

Definition 2.1.2. Let N be a finite set of points contained in the boundary ∂M. An N**-tangle** in M (or just *tangle* for short) is the smooth ambient isotopy class of a smooth embedding

$$T : \underbrace{\bigsqcup_{j \in J} S^1}_{:=L} \sqcup \underbrace{\bigsqcup_{k \in K} I}_{:=B} \to M$$

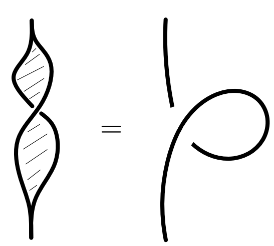
for some finite sets J and K such that

1. the image of L lies in the interior of M,
2. the image of the interior of B lies in the interior of M,
3. the image of the boundary of B equals N.

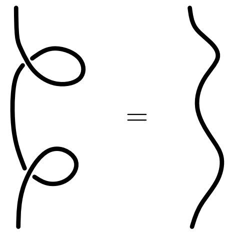
If B is empty, then the result is called a **link** in M. Similarly, if L is empty, then it's called a **braid** in M.

One may also consider *oriented* or *framed* tangles by choosing an orientation or framing for each point in N and for each connected component of L and B such that the choices are compatible with each other with respect to the smooth embedding. If $M = \Sigma \times I$, then we will assume that the points in N are contained in $\Sigma \times \{\frac{1}{2}\}$ and that their framings are thought to be embedded orthogonally to $\Sigma \times \{*\}$.

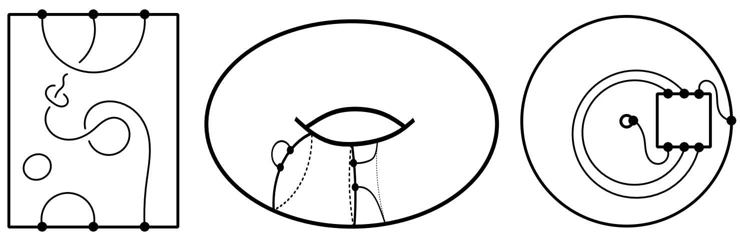
We say a framed tangle in $\Sigma \times I$ has **blackboard framing** if the entire framing is embedded orthogonally to Σ. Every framed link in $\Sigma \times I$ is isotopic to one with a blackboard framing by turning each twist into a loop with a local blackboard framing:



This suggests that we may represent framed links in $\Sigma \times I$ as link diagrams on Σ. Indeed, equivalence under ambient isotopy is captured by the Reidemeister moves 2, 3, and a modified Reidemeister move 1:



* Add pictures of RII and RIII for good measure. Below are some examples of framed tangle diagrams.



Define the **writhe** of a tangle diagram is the number of positive crossings minus the number of negative crossings. It is easy to see that the Reidemeister moves above preserve the writhe of a diagram, so the concept is well defined. Writhe should be thought of as a grading on the free R-module on the set of tangles in the given space, which provides a good reason to work with framed links over ordinary links. Such a module is a main ingredient of this theory, so let's honor it with a proper discussion.

Definition 2.1.3. Let $\mathcal{T}(M,N)$ be the free R-module generated by the set of framed N-tangles in M. Analagously, we can define $\mathcal{T}^{or}(M,N)$ to be the free R-module generated by the set of oriented framed N-tangles in M. All definitions which are to follow in this subsection have an analagous definition using oriented tangles. Also, we will formally define $\mathcal{S}_R(\varnothing,\varnothing) := R$.

The construction $\mathcal{T}(-,-)$ is actually a symmetric monoidal functor $\mathcal{T} : \mathsf{C} \to R\text{-}\mathsf{Mod}$ for a careful choice of category C which we now describe. The objects of C are pairs (M,N) of the same type as discussed previously. A morphism $(f,W) : (M',N') \to (M,N)$ is a pair of a smooth, orientation-preserving embedding $f : M' \to M$ such that $M - f(M')$, which is either a smooth 3-manifold or the empty set, and choice of $W \in \mathcal{T}\big(M - f(M'), N \sqcup f(N')\big)$

(unless $M - f(M')$ is empty, in which case W is a formal symbol for the "empty link" in the empty set). Composition is given by $(g, W') \circ (f, W) = (g \circ f, W' \cup W)$, which is associative since $\circ$ and $\cup$ are associative.

* Give a picture of composition in C.

The induced map denoted $W : \mathcal{T}(M', N') \to \mathcal{T}(M, N)$ is a linear map defined by $W(T) = W \cup T$, and we will refer to such a linear map W as a **wiring**. We are abusing notation by denoting this linear map by W, but it should be clear from the context what f is since it is technically encoded in the data of the element $W \in \mathcal{T}\big(M - f(M'), N \sqcup f(N')\big)$. It is a stimulating exercise in verifying definitions to check that $\mathcal{T}$ preserves composition and identity morphisms, making and so $\mathcal{T}$ is functorial. C can now be equipped with a symmetric monoidal structure via disjoint union. It is clear that

$$\mathcal{S}_R(M \sqcup M', N \sqcup N') \cong \mathcal{S}_R(M, N) \underset{R}{\otimes} \mathcal{S}_R(M', N')$$

for any sets of framed points $N \subset \partial M$ and $N' \subset \partial M'$. The unit is given by the object $(\varnothing, \varnothing) \in \mathsf{C}$ and define $\mathcal{T}(\varnothing, \varnothing) := R$, which makes $\mathcal{T}$ a symmetric monoidal functor.

Definition 2.1.4. Let B be the smooth closed 3-ball, N_i be some collection of $2i$ boundary points of B, and let $X \subset \overset{\bigsqcup}{i \in} \mathbb{N}\mathcal{T}(B, N_B)$ be some (typically finite) set, which we will call a set of **skein relations**. Given any tangle module $\mathcal{T}(M, N_M)$, there exists a submodule $\mathcal{I}(X)$ generated by the set

$$\{W(x) \mid x \in X \text{ and } W : \mathcal{T}(B, N_B) \to \mathcal{T}(M, N_M) \text{ is a wiring diagram}\}.$$

A quotient of the form $\mathcal{S}_X(M, N) := \mathcal{T}(M, N)/\mathcal{I}(X)$ is called a **skein module** of M relative to N. If $N = \varnothing$ is the empty set, we may use the notation $\mathcal{S}_X(M) := \mathcal{S}_X(M, \varnothing)$. Similar definitions may be given using oriented and/or unframed tangles instead.

The construction $\mathcal{S}_X(-,-)$ is a functor in the same way that $\mathcal{T}(-,-)$ is; a smooth embedding $f : M \to M'$ and an element $W \in \mathcal{S}_X\big(M - f(M'), N \sqcup f(N')\big)$ defines a linear map

$W : \mathcal{S}_X(M, N) \to \mathcal{S}_X(M', N')$. In fact, the quotient maps $\alpha_{(M,N)} : \mathcal{T}(M, N) \to \mathcal{S}_X(M, N)$ yield a natural transformation. In other words, given a morphism $(M, N) \to (M', N')$ in C, the diagram

$$\begin{tikzcd} \mathcal{T}(M,N) \arrow[r, "W"] \arrow[d, "\alpha_{(M,N)}"] & \mathcal{T}(M',N') \arrow[d, "\alpha_{(M',N')}"] \\ \mathcal{S}_X(M,N) \arrow[r, "W"] & \mathcal{S}_X(M',N') \end{tikzcd}$$

commutes. Such a functor will be called a **skein theory**.

For any oriented surface Σ, we can define a category $\mathsf{Skein}_X(\Sigma)$. The objects of this category are finite sets of framed points N in Σ, and the morphisms $N \to N'$ are elements of $\mathcal{S}_X\big(\Sigma \times I, (N \times \{0\}) \sqcup (N' \times \{1\})\big)$, so the category is R-linear. Write composition of morphisms by concatenation. If $y : N \to N'$ and $z : N' \to N''$ are morphisms, then their composite $yz : N \to N''$ is constucted by gluing z on y through N' and rescaling the interval coordinate appropriately.

* Picture of composition in $\mathsf{Skein}_X(\Sigma)$.

The endomorphism algebras in this category are called **skein algebras** and are denoted by $\mathcal{S}_X(\Sigma, N) := \mathcal{S}_X(\Sigma \times I, (N \times \{0\}) \sqcup (N \times \{1\})\big)$. If N is the empty set, then we reduce the notation to simply $\mathcal{S}_X(\Sigma)$.

If $f : \Sigma \to \Sigma'$ is a smooth, orientation-preserving embedding of surfaces, then there is an induced functor

$$f_* : \mathsf{Skein}_X(\Sigma') \to \mathsf{Skein}_X(\Sigma)$$

defined on objects by $f_*(N) = f(N)$ and on morphisms in the following way. First, extend f trivially to $f \times \mathrm{id}_I : \Sigma \times I \to \Sigma' \times I$. Then, in the skein algebra of the complement of the image of $f \times \mathrm{id}_I$, choose the multiplicative identity element $e \in \mathcal{S}_X\big(\Sigma' - \mathrm{Im}(f)\big)$ which is the empty tangle. The pair $(f \times \mathrm{id}_I, e)$ is an object in the category C, which gives rise to a wiring

$$e : \mathcal{S}_X\Big(\Sigma \times I, \big(N \times \{0\}\big) \sqcup \big(N' \times \{1\}\big)\Big) \to \mathcal{S}_X\Big(\Sigma' \times I, \big(f(N) \times \{0\}\big) \sqcup \big(f(N') \times \{1\}\big)\Big)$$

via the functor $\mathcal{S}_X$. Now we may define what f_* does to morphisms: $f_*(y) = e(y)$ for any $y \in \mathcal{S}_X\Big(\Sigma \times I, (N \times \{0\}) \sqcup (N' \times \{1\})\Big)$.

* Picture of how f_* works on morphisms.

It is clear that f_* preserves composition. In particular, f_* defines algebra homomorphisms on the skein algebras

$$e : \mathcal{S}_X\big(\Sigma, N\big) \to \mathcal{S}_X\big(\Sigma', f(N)\big).$$

* We use this type of algebra homomorphism when we embed the annulus into the torus.

The above homomorphisms are a special case of a more general type of map. If N is a set of framed points on Σ, then a smooth embedding $f : \Sigma \to \partial M$ induces a $\mathcal{S}_X(\Sigma, N)$-module structure on $\mathcal{S}_X\big(M, N'\big)$ for any N' with $f(N) \subseteq N'$. The action is given by "pushing tangles in through the boundary". In other words, the pre-composition of a smooth embedding of a collar neighborhood $g : \partial M \times I \to M$ with $f \times \mathrm{id}_I : \Sigma \times I \to \partial M \times I$ induces a bilinear map

$$\mathcal{S}_X(\Sigma, N) \times \mathcal{S}_X(M, N') \to \mathcal{S}_X(M, N')$$

because M minus a collar neighborhood is diffeomorphic to itself. Alternatively, a choice of element in $\mathcal{S}_X(\Sigma, N')$ produces a wiring $\mathcal{S}_X(M, N') \to \mathcal{S}_X(M, N')$.

* Picture of action.

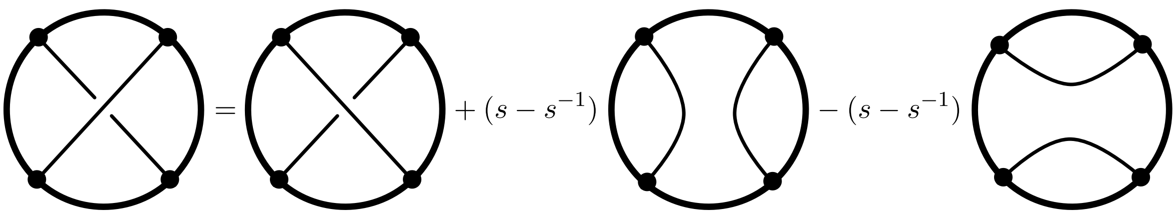
Examples of Skein Theories

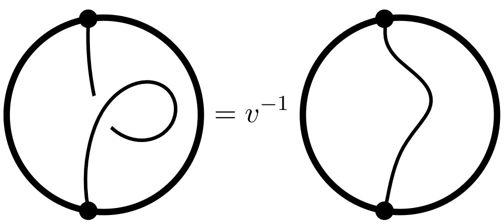
The last subsection leaves us with an important and unanswered question. Which sets of skein relations X produce interesting skein theories? One class of examples is found by examing sets of relations satisfied by morphisms in a linear ribbon category. Ribbon categories are braided monoidal categories which are rigid and equipped with a twist morphism for every object, satisfying some compatibility conditions. The axioms are such that the morphisms may be interpreted as framed braid diagrams. In particular, the morphisms satisfy the Reidemeister moves shown previously. We will discuss three examples of skein theories whose relations are meant to emulate linear relations satisfied by the braid and twist morphisms in certain ribbon categories coming from the representation theory of quantum groups.

Here, we are forced to fix a base ring. For our purposes, we choose our base ring to be $R := \mathbb{Q}(s, v)$, rational functions in two indeterminates over the rational numbers. This ring is quite large, and some of the analysis of the skein theories remain true under choices of smaller base rings, but we make this choice for simplicity. In particular, the theorem * *Cite BB is stated over this ring.

Kauffman (Dubrovnik) Skein Relations

Let X_1 denote the set of two unoriented skein relations

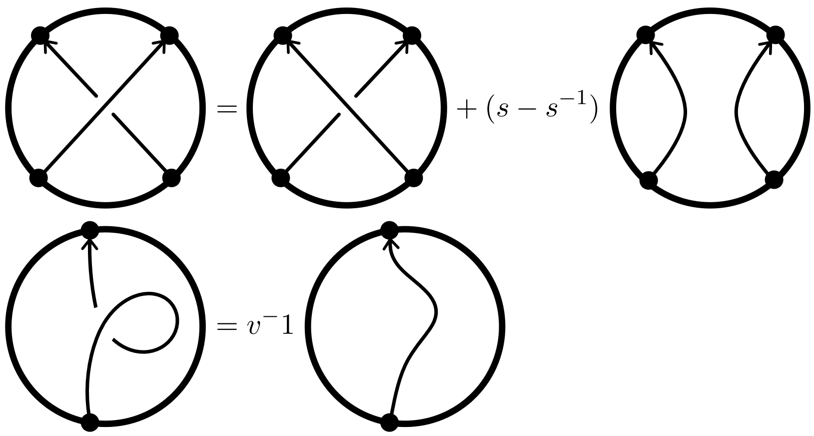




The functor $\mathcal{D}(-,-) := \mathcal{S}_{X_1}(-,-)$ is the Dubrovnik variant of the Kauffman skein theory. Using the Dubrovnik variant is important for us (see * universal enveloping algebra result). This theory is related to Dubrovnik polynomials in that the Dubrovnik polynomial of a link is a normalized value of the link in $\mathcal{D}(S^3)$. The normalization is often so that the Dubrovnik polynomial of the unknot is 1, whereas the value of the unknot in $\mathcal{D}(S^3)$ is $\delta_{\mathcal{D}} := 1 - \frac{v - v^{-1}}{s - s^{-1}}$, which can be deduced from the skein relations.

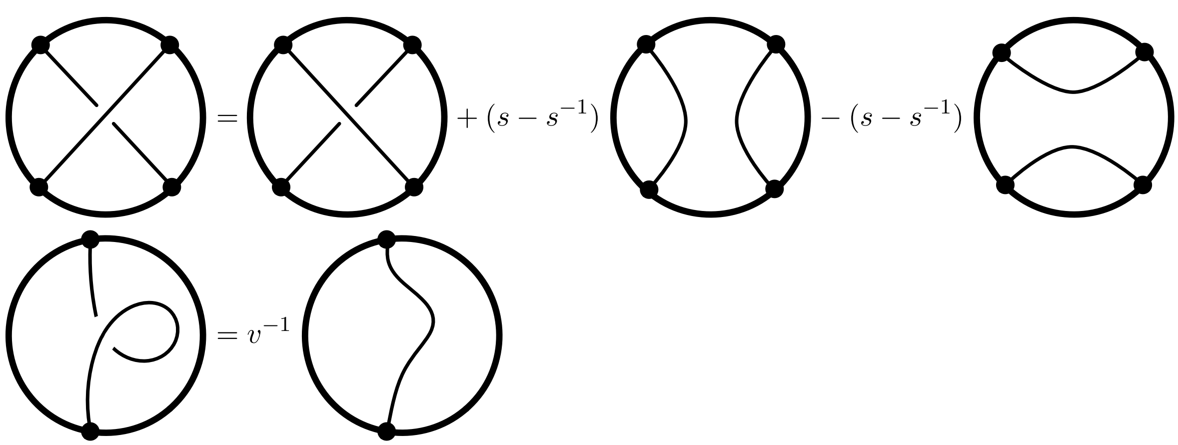
HOMFLYPT Skein Relations

Let X_2 denote the set of two oriented skein relations



The functor $\mathcal{H}(-,-) := \mathcal{S}_{X_2}(-,-)$ is the HOMFLYPT skein theory. As in the Dubrovnik case, this theory is related to HOMFLYPT polynomials so that the HOMFLYPT polynomial of a link is a normalized value of the link in $\mathcal{H}(S^3)$. Again, some often make the choice to normalize this polynomial so that the value of the unknot is 1, but the value of the unknot in $\mathcal{H}(S^3)$ is $\delta_{\mathcal{H}} := -\frac{v-v^{-1}}{s-s^{-1}}$.

Kauffman Bracket Skein Relations



HOMFLYPT and Kauffman Skein Modules

The Iwahori-Hecke Algebra and the BMW Algebra

Skein Algebras of the Annulus

Connections to Representation Theory

A Relative Skein Algebra of the Annulus

The HOMFLYPT Skein Algebra of the Torus

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Chapter 3

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Chapter 4

Closures of Minimal Idempotents in BMW_n

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References