Operation: Foundational Repair
aka
Be More Careful with The Hammer

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Exercise I.3.10 (#1):

- *(a)* Show that the category $\mathfrak{Set}$ contains a subobject classifier.

  Proof. First of all, the obvious notion of "subobject" in $\mathfrak{Set}$ is "subset". So, what we need to do is exhibit a special set $\Omega \in \mathrm{Ob}\mathfrak{Set}$ and then exhibit morphisms in $\mathrm{Hom}_{\mathfrak{Set}}(A, \Omega)$ which are uniquely determined by subsets $X \subset A$ such that the following diagram commutes (1 is the final object in the category):

  $$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & 1 \arrow[d] \\ A \arrow[r] & \Omega \end{tikzcd}$$

  and such that X is universal with respect to this diagram.

  Well, let $\Omega = \{y, n\}$, let $\chi_X : A \to \Omega$ be defined by

  $$\chi_X(a) = \begin{cases} y & \text{if } a \in X \\ n & \text{if } a \notin X \end{cases},$$

  let $Y : 1 \to \Omega$ be the map $* \mapsto y$, and let $*_X : X \to 1 = \{*\}$ be the unique map from X to $\{*\}$. Then, the diagram

  $$\begin{tikzcd} X \arrow[r, "*_X"] \arrow[d, hook, "i_X"] & 1 = \{*\} \arrow[d, "Y"] \\ A \arrow[r, "\chi_X"] & \Omega = \{y, n\} \end{tikzcd}$$

  Manifestly commutes, as $\forall x \in X$ we have that $(Y \circ *_X)(x) = y = \chi_X \circ i_X(x)$ where $i_X : X \to A$ is the inclusion map.

Further, this makes X universal as if W is such that

$$\begin{tikzcd}
X \arrow[rr, "*_X"] \arrow[dd, hook, "i_X"'] & & 1 = \{*\} \arrow[dd, "Y"] \\
& W \arrow[ur, "*_W"] \arrow[dl, hook, "i_W"] & \\
A \arrow[rr, "\chi_X"] & & \Omega = \{y, n\}
\end{tikzcd}$$

commutes, then $\forall w \in W$ we have that $\chi_X(i_W(w)) = y = Y(*_W(w))$. But this can only be the case if $W \subset X$. Then, there is an inclusion $i_{WX} : W \to X$ such that

$$\begin{tikzcd}
X \arrow[rr, "*_X"] \arrow[dd, hook, "i_X"'] & & 1 = \{*\} \arrow[dd, "Y"] \\
& W \arrow[ul, hook, "i_{WX}"'] \arrow[ur, "*_W"] \arrow[dl, hook, "i_W"] & \\
A \arrow[rr, "\chi_X"] & & \Omega = \{y, n\}
\end{tikzcd}$$

commutes. This still preserves a bijective correspondence between subsets and morphisms from A to Ω, as $\chi_W = \chi_X \circ i_W$. So X is **final** with respect to making this diagram commute. ☐

- *(b)* Determine if the category $\mathfrak{Grp}$ contains a subobject classifier.
  Claim: $\mathfrak{Grp}$ does not have a subobject classifier.

  Proof. Firstly, the obvious notion of a subobject in $\mathfrak{Grp}$ is that of a subgroup, $X \leq A$. In $\mathfrak{Set}$ the final object is distinct from the initial object. Note this is not so in $\mathfrak{Grp}$, as the trivial group is both final and initial- it is a zero object. Now, any sub-group classification must also classify subsets. So, the previous construction must continue to hold. What we need is for the morphisms in the above construction to be group homomorphisms.

$$\begin{tikzcd}
X \arrow[r, "*_X"] \arrow[d, hook, "i_X"] & 1 = \{*\} \arrow[d, "Y"] \\
A \arrow[r, "\chi_X"] & \Omega = \{y, n\}
\end{tikzcd}$$

  However, now that $\{*\}$ is a zero object, the map from it to Ω must be unique, and it must take the identity (which must be $*$) to the identity. So, we have

$$\begin{tikzcd}
X \arrow[r, "*_X"] \arrow[d, hook, "i_X"] & 1 = \{* = e\} \arrow[d, "Y"] \\
A \arrow[r, "\chi_X"] & \Omega = \{e_\Omega, a\}
\end{tikzcd}$$

  with $Y(*) = e_\Omega$ a group? homomorphism into the two element group with $a^2 = e_\Omega$.

$*_X$ is likewise a group homomorphism, as is i_X. So, we need the morphism χ_X to be a group homomorphism. Well, we also need the diagram to commute so for all x in X, $\chi_X(i_X(x)) = Y(*_X(X)) = e_\Omega$ **This is not done!** $\square$

Exercise I.5.8 (#2) In every category with products, $A \,\Pi\, B$ is isomorphic to $B \,\Pi\, A$. Similarly, in every category with coproducts, $A \amalg B$ is isomorphic to $B \amalg A$.

Proof. Firstly, for products, note that each product "comes with" natural projection maps which we will denote by the following $\pi_{A1} : A \,\Pi\, B \to A$, $\pi_{B1} : A \,\Pi\, B \to B$, $\pi_{A2} : B \,\Pi\, A \to A$, and $\pi_{B2} : B \,\Pi\, A \to B$ (note that until we have proven $A \,\Pi\, B$, $B \,\Pi\, A$ isomorphic, these maps are technically distinct due to having different sources and targets). We thereby have the following diagram:

$$\begin{tikzcd}
& & & & A \\
A \,\Pi\, B \arrow[urrrr, bend left, "\pi_{A1}"] \arrow[drrrr, bend right, "\pi_{B1}"'] & B \,\Pi\, A \arrow[urrr, bend left, "\pi_{A2}"] \arrow[drrr, bend right, "\pi_{B2}"'] & A \,\Pi\, B \arrow[urr, bend left, "\pi_{A1}"] \arrow[drr, bend right, "\pi_{B1}"'] & B \,\Pi\, A \arrow[ur, "\pi_{A2}"] \arrow[dr, "\pi_{B2}"] & \\
& & & & B
\end{tikzcd}$$

Now, as $B \,\Pi\, A$ with its projection maps is a final object in $\mathscr{C}_{B,A}$, we have that for any other object in $\mathscr{C}$ with morphisms to B and A, there must be a unique morphism from this object to $B \,\Pi\, A$ such that the resulting diagram commutes. In particular, for $A \,\Pi\, B \in \mathrm{Ob}\mathscr{C}$ with its projection maps π_{B1} and π_{A1}, there is a unique morphism $\alpha : A \,\Pi\, B \to B \,\Pi\, A$ such that $\pi_{A1} = \pi_{A2} \circ \alpha$ and $\pi_{B1} = \pi_{B2} \circ \alpha$. By the same reasoning, there is also a unique morphism $\beta : B \,\Pi\, A \to A \,\Pi\, B$ such that $\pi_{A2} = \pi_{A1} \circ \beta$ and $\pi_{B2} = \pi_{B1} \circ \beta$.

Putting these into the previous diagram, we have that the following commutes:

$$\begin{tikzcd}
& & & & A \\
A \,\Pi\, B \arrow[r, "\alpha"] \arrow[urrrr, bend left, "\pi_{A1}"] \arrow[drrrr, bend right, "\pi_{B1}"'] & B \,\Pi\, A \arrow[r, "\beta"] \arrow[urrr, bend left, "\pi_{A2}"] \arrow[drrr, bend right, "\pi_{B2}"'] & A \,\Pi\, B \arrow[r, "\alpha"] \arrow[urr, bend left, "\pi_{A1}"] \arrow[drr, bend right, "\pi_{B1}"'] & B \,\Pi\, A \arrow[ur, "\pi_{A2}"] \arrow[dr, "\pi_{B2}"] & \\
& & & & B
\end{tikzcd}$$

Through composition, we have morphisms $\alpha\beta : B \,\Pi\, A \to B \,\Pi\, A$, and $\beta\alpha : A \,\Pi\, B \to A \,\Pi\, B$. However, by finality of $B\Pi A$ and $A \Pi B$ the only maps from these objects to themselves which commute under the projection maps are the identities. Thus, $\alpha\beta = \mathrm{id}_{B\Pi A}$ and $\beta\alpha = \mathrm{id}_{A\Pi B}$. Hence, α and β are isomorphisms between these objects, so $B \,\Pi\, A$ is isomorphic to $A \,\Pi\, B$.

Secondly, for coproducts the argument is the same. Let us "jump the gun" and observe the following diagram:

$$\begin{tikzcd}
 & & & & A \arrow[dl, "i_{A2}"'] \arrow[dll, bend right, "i_{A1}"'] \arrow[dlll, bend right, "i_{A2}"'] \arrow[dllll, bend right, "i_{A1}"'] \\
A \amalg B & B \amalg A \arrow[l, "\alpha"'] & A \amalg B \arrow[l, "\beta"'] & B \amalg A \arrow[l, "\alpha"'] & \\
 & & & & B \arrow[ul, "i_{B2}"] \arrow[ull, bend left, "i_{B1}"] \arrow[ulll, bend left, "i_{B2}"] \arrow[ullll, bend left, "i_{B1}"]
\end{tikzcd}$$

The morphisms α and β exist uniquely as $A \amalg B$ and $B \amalg A$ are initial objects with respect to any other object in the category which is a target of one morphism with source A and another morphism with source B. By this very initialness, there is only one morphism from $A \amalg B$ to itself, and only one morphism from $B \amalg A$ to itself- the identity morphisms. Since $\alpha\beta : A \amalg B \to A \amalg B$ and $\beta\alpha : B \amalg A \to B \amalg A$, these compositions are equal to these very identity morphisms, so α and β are isomorphisms, so $A \amalg B$ is isomorphic to $B \amalg A$, as was desired to be shown.

$\square$

Exercise II.3.1 (Lemma to #4): Let $\mathscr{C}$ be a category with products of pairs. Suppose $\varphi \in \mathrm{Hom}_{\mathscr{C}}(G, H)$. Then there must exist a unique map $(\varphi \,\Pi\, \varphi) : G \,\Pi\, G \to H \,\Pi\, H$ which is compatible with the natural projections.

Proof. As we have products and we have the map $\varphi : G \to H$, we have the following diagram:

$$\begin{tikzcd}
 & G \arrow[r, "\varphi"] & H \\
G \,\Pi\, G \arrow[ur, "\pi_{G1}"] \arrow[dr, "\pi_{G2}"'] & H \,\Pi\, H \arrow[ur, "\pi_{H1}"] \arrow[dr, "\pi_{H2}"] & \\
 & G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

Now, by composing $\varphi \circ \pi_{G_1}$ and $\varphi \circ \pi_{G_2}$ we have maps from $G \,\Pi\, G$ to each of the H's, i.e. we have:

$$\begin{tikzcd}
 & & H \\
G \,\Pi\, G \arrow[urr, bend left, "\varphi\circ\pi_{G1}"] \arrow[drr, bend right, "\varphi\circ\pi_{G2}"] & H \,\Pi\, H \arrow[ur, "\pi_{H1}"] \arrow[dr, "\pi_{H2}"] & \\
 & & H
\end{tikzcd}$$

Now, as $H \,\Pi\, H$ with the projections to each component is a final object in $\mathscr{C}_{H,H}$ (by definition of product), and we have exhibited another object in this category, there must exist a unique map
$\sigma : G \,\Pi\, G \to H \,\Pi\, H$ such that the following diagram commutes

$$\begin{tikzcd}
 & & H \\
G \,\Pi\, G \arrow[r, dashed, "\exists!\sigma"] \arrow[urr, bend left, "\varphi\circ\pi_{G1}"] \arrow[drr, bend right, "\varphi\circ\pi_{G2}"] & H \,\Pi\, H \arrow[ur, "\pi_{H1}"] \arrow[dr, "\pi_{H2}"] & \\
 & & H
\end{tikzcd}$$

That is to say, such that $\pi_{H_1} \circ \sigma = \varphi \circ \pi_{G_1}$ and $\pi_{H_2} \circ \sigma = \varphi \circ \pi_{G_2}$. As this map is uniquely determined by φ (and φ's source and target), we can unambiguously denote it by $\varphi \,\Pi\, \varphi$. $\square$

Exercise 3.2 (#4): Let $\mathscr{C}$ be a category with products of pairs.
Suppose $\varphi \in \mathrm{Hom}_{\mathscr{C}}(G, H)$ and $\psi \in \mathrm{Hom}_{\mathscr{C}}(H, K)$.
Then $(\psi\varphi) \amalg (\psi\varphi) = (\psi \amalg \psi) \circ (\varphi \amalg \varphi)$.

Proof. From Exercise II.3.1, the existence of $\varphi : G \to H$ and $\psi : H \to K$ immediately gives us the existence of the *unique* "product morphisms" $\varphi \amalg \varphi : G \amalg G \to H \amalg H$ and $\psi \amalg \psi : H \amalg H \to K \amalg K$ compatible with the natural projections. In other words, the following diagram commutes:

$$\begin{tikzcd}
 & G \arrow[r, "\varphi"] & H \arrow[r, "\psi"] & K \\
G \amalg G \arrow[ur, "\pi_{G1}"] \arrow[dr, "\pi_{G2}"'] \arrow[r, "\varphi\amalg\varphi"] & H \amalg H \arrow[ur, "\pi_{H1}"] \arrow[dr, "\pi_{H2}"'] \arrow[r, "\psi\amalg\psi"] & K \amalg K \arrow[ur, "\pi_{K1}"] \arrow[dr, "\pi_{K2}"'] & \\
 & G \arrow[r, "\varphi"] & H \arrow[r, "\psi"] & K
\end{tikzcd}$$

Now, composing morphisms yields:

$$\begin{tikzcd}
 & & K \\
G \amalg G \arrow[urr, bend left, "(\psi\varphi)\circ\pi_{G1}"] \arrow[drr, bend right, "(\psi\varphi)\circ\pi_{G2}"'] \arrow[r, "(\psi\amalg\psi)\circ(\varphi\amalg\varphi)"] & K \amalg K \arrow[ur, "\pi_{K1}"] \arrow[dr, "\pi_{K2}"'] & \\
 & & K
\end{tikzcd}$$

and this diagram still commutes, as it is merely the same diagram as before but with compositions of morphisms "pinching away the H's". However, $K \amalg K$ with the projection maps is final in $\mathscr{C}_{K,K}$, so there is a unique map from $G \amalg G$ to $K \amalg K$ which is entirely determined by $\psi\varphi$ as in Exercise 3.1. So, denote this unique map by $(\psi\varphi) \amalg (\psi\varphi)$ as in Exercise 3.1. This map is **unique!** But we also have the map $(\psi \amalg \psi) \circ (\varphi \amalg \varphi) : G \amalg G \to K \amalg K$, which **also** makes the diagram commute. So, these maps are actually the same, so we have that $(\psi \amalg \psi) \circ (\varphi \amalg \varphi) = (\psi\varphi) \amalg (\psi\varphi)$ as we desired to show. $\square$

Exercise II.7.12 (#3) (Written using notation given by Dr. Brennan)
Let $X \in \mathrm{Ob}\mathfrak{Set}$, $A \in \mathrm{Ob}\mathfrak{Ab}$ (an object in the category of Abelian groups- an abelian group), and $\varphi \in \mathrm{Hom}_{\mathfrak{Set}}(X, A)$. Then, there is a unique $\tilde{\varphi} \in \mathrm{Hom}_{\mathfrak{Ab}}(F(X)/[F(X), F(X)], A)$ such that

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "\varphi"'] & F(X)/[F(X),F(X)] \arrow[d, "\tilde{\varphi}"] \\
& A
\end{tikzcd}$$

Commutes. Furthermore, $F(X)/[F(X), F(X)]$ is isomorphic to $F^{\mathrm{ab}}(X) \cong \mathrm{Hom}_{\mathfrak{Set}}(X, \mathbb{Z})$, the free abelian group on X.

Proof. Firstly, let us think of φ as a set function to A with A considered as an object of $\mathfrak{Grp}$. There exists a map $j : X \to F(X)$, which is initial with respect to mapping X to a *group*. In other words, there exists a **unique** *group homomorphism* $\hat{\varphi} \in \mathrm{Hom}_{\mathfrak{Grp}}(F(X), A)$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "j"] \arrow[dr, "\varphi"] & \\
F(X) \arrow[r, "\hat{\varphi}"] & A
\end{tikzcd}$$

Now, the commutator subgroup of any group is normal in that group, and the quotient group formed by it is abelian.
In particular, $[F(X), F(X)] \trianglelefteq F(X)$ and $F(X)/[F(X), F(X)] \in \mathrm{Ob}\mathfrak{Ab}$.
We then have the surjective projection homomorphism $\pi : F(X) \to F(X)/[F(X), F(X)]$, shown in the following diagram:

$$\begin{tikzcd}
X \arrow[d, "j"] \arrow[dr, "\varphi"] & \\
F(X) \arrow[r, "\hat{\varphi}"] \arrow[d, "\pi"] & A \\
F(X)/[F(X),F(X)] &
\end{tikzcd}$$

Now, $[F(X), F(X)] \subset \ker\hat{\varphi}$, as all elements of $[F(X), F(X)]$ are of the form $fgf^{-1}g^{-1}$ for $f, g \in F(X)$ so $\hat{\varphi}(fgf^{-1}g^{-1}) = \hat{\varphi}(f)\hat{\varphi}(g)\hat{\varphi}(f)^{-1}\hat{\varphi}(g)^{-1}$ and as these are all in A, an abelian group, this gives $\hat{\varphi}(f)\hat{\varphi}(g)\hat{\varphi}(g)^{-1}\hat{\varphi}(f)^{-1} = \hat{\varphi}(f)e_A\hat{\varphi}(g)^{-1} = e_A$.

Thus, by Theorem 7.12 in Aluffi, there exists a **unique** homomorphism
$\tilde{\varphi} : F(X)/[F(X), F(X)] \to A$ such that

$$\begin{tikzcd}
X \arrow[d, "j"] \arrow[dr, "\varphi"] & \\
F(X) \arrow[r, "\hat{\varphi}"] \arrow[d, "\pi"] & A \\
F(X)/[F(X),F(X)] \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

Commutes. Note that $\tilde{\varphi}$ is actually an abelian group homomorphism.
Now, compose morphisms π and j to obtain

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\pi j"] & A \\
F(X)/[F(X),F(X)] \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

So, we have shown the existence of such a unique $\tilde{\varphi} \in \mathrm{Hom}_{\mathfrak{Ab}}(F(X)/[F(X),F(X)],A)$ as desired. All that remains is to show that $F(X)/[F(X),F(X)] \cong F^{\mathrm{ab}}(X)$.

Well, $J : X \to F^{\mathrm{ab}}(X)$ is in intial object in $\mathfrak{Set}^X_{\mathfrak{Ab}}$.
However, we have shown that $\pi j : X \to F(X)/[F(X),F(X)]$ is initial in this category also!
As initial objects in a category are isomorphic, it follows that

$$F(X)/[F(X),F(X)] \cong F^{\mathrm{ab}}(X) = \mathrm{Hom}_{\mathfrak{Set}}(X,\mathbb{Z})$$

as claimed.

$\square$

A combo of II.8.7 and II.9.14 (#6) Suppose G_1, G_2 are groups, and that $\varphi_1 : F(G_1) \to G_1$ and $\varphi_2 : F(G_2) \to G_2$ are the natural epimorphisms sending letters to their corresponding group elements.

- *(a)* $F(G_1 \dot{\cup} G_2)/ < \mathrm{ker}\varphi_1, \mathrm{ker}\varphi_2 >$ is the coproduct of G_1 and G_2 in $\mathfrak{Grp}$.

Exercise II.8.22 + A second part (#7)

- *(a)* Show that $\mathfrak{Grp}$ has cokernels.
- *(b)* Determine if $\mathfrak{Grp}$ has coequalizers.

Proof.

(a) $\mathfrak{Grp}$ does have cokernels. Given $\varphi \in \mathrm{Hom}_{\mathfrak{Grp}}(G, G')$,define $N := \bigcap_{H \trianglelefteq G' \text{ and im } \varphi \subset H} H$. Then, we will show that coker $\varphi = G'/N$. Now, for any $\alpha : G' \to L$ such that $\alpha\varphi$ is the trivial map, we must show that there exists a unique $\tilde{\alpha} : G'/N = \text{coker } \varphi \to L$. I.e. that such a unique morphism exists so that the diagram:

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & G' \arrow[rr, "\alpha"] \arrow[d, two heads, "\pi"] & & L \\ & \text{coker } \varphi = G'/N \arrow[urr, "\tilde{\alpha}"] & & \end{tikzcd}$$

Commutes. Well the only reasonable definition of $\tilde{\alpha}$ is given by $\tilde{\alpha}(gN) = \alpha(g)$. We must check that this is well defined. Well, if $\tilde{\alpha}(gN) = \tilde{\alpha}(hN)$ then we want $\alpha(g) = \alpha(h)$. If $\alpha(g) \neq \alpha(h)$, then $\alpha(g^{-1}h) \neq e_L$ so $g^{-1}h \notin \ker \alpha$. Note that as N is the smallest normal subgroup containing the image of φ, it must be that $N \subset \ker \alpha$ since $\alpha\varphi = \alpha 0 = 0$ and im $\varphi \subset \ker \alpha$. However, then as $gN = hN \implies N = g^{-1}hN \implies g^{-1}h \in N \subset \ker \alpha$, it must be that $\alpha(g^{-1}h) = e_L \implies \alpha(g) = \alpha(h)$ after all. So $\tilde{\alpha}$ is well defined.
Then, $\tilde{\alpha}\pi\varphi(a) = \tilde{\alpha}\pi(\varphi(a)) = \tilde{\alpha}(\varphi(a)N) = \tilde{\alpha}(N) = e_L$, so $\tilde{\alpha}\pi\varphi = 0$, and the given defintion of the cokernel satisfies the needed universal property.

(b) $\mathfrak{Grp}$ does have coequalizers. Given $\varphi, \psi \in \mathrm{Hom}_{\mathfrak{Grp}}(G, G')$, define $S = \{\varphi(a)\psi(a)^{-1} | a \in G\}$ and $N := \bigcap_{H \trianglelefteq G' \text{ and } S \subset H} H$. Then, we will show that a coequalizer C is given by $C = G'/N$. Now, for any $\alpha : G' \to L$ such that $\alpha\varphi = \alpha\psi$, we must show that there exists a unique $\tilde{\alpha} : G'/N = C \to L$. I.e. that such a unique morphism exists so that the diagram:

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & G' \arrow[rr, "\alpha"] \arrow[d, two heads, "\pi"] & & L \\ & G'/N = C \arrow[urr, "\tilde{\alpha}"] & & \end{tikzcd}$$

Commutes.
Firstly, if $s \in S$ then $\exists a \in G$ such that $s = \varphi(a)\psi(a)^{-1}$ and then $\alpha(s) = \alpha(\varphi(a)\psi(a)^{-1}) = \alpha(\varphi(a))\alpha(\psi(a)^{-1})) = \alpha\varphi(a)\alpha\psi(a)^{-1} = \alpha\psi(a)\alpha\psi(a)^{-1} = e_L$.
Now, note that as $S \subset \ker \alpha$ and N is the smallest normal subgroup containing S, we must have that $S \subset N \subset \ker \alpha$. Thus, we have that there must indeed exist a **unique** group homomorphism $\tilde{\alpha}$ from C to L such that the diagram commutes. I.e. $\tilde{\alpha}\pi = \alpha$ so $\alpha\varphi = \alpha\psi \implies \tilde{\alpha}\pi\varphi = \tilde{\alpha}\pi\psi$ $\square$

(#8) Show that if a category has (co)products of pairs of objects then it has (co)products of finitely indexed families of objects.

Proof. First, we will show that this holds in the case of triples. So, let $A, B, C \in \text{Ob}\mathscr{C}$. Then, what we want is an object $A \amalg B \amalg C$ with projection epimorphisms $\pi_A : A \amalg B \amalg C \to A$, $\pi_B : A \amalg B \amalg C \to B$, and $\pi_C : A \amalg B \amalg C \to C$ such that for any other object $W \in \text{Ob}\mathscr{C}$ with maps $h_A : W \to A$, $h_A B : W \to B$, and $h_C : W \to C$, there will exist a **unique** morphism $\sigma : W \to A \amalg B \amalg C$ such that $\pi_A \sigma = h_A$, $\pi_B \sigma = h_B$, and $\pi_C \sigma = h_C$. Our strategy will be to show that $(A \amalg B) \amalg C$ (with the projections $\pi_{A \amalg B} : (A \amalg B) \amalg C \to A \amalg B$, and $\pi_C : (A \amalg B) \amalg C \to C$) satisfies this property, and so is a final object in $\mathscr{C}_{A,B,C}$. Then, we will show that $A \amalg (B \amalg C)$ (with its projections) *also* satisfies this property, and is thereby isomorphic to $(A \amalg B) \amalg C$, and so we can denote "both" unambiguously by $A \amalg B \amalg C$. $\square$

Exercise II.9.13 (#5) Show that for all subgroups H of a group G and $\forall g \in G$, G/H and $G/(gHg^{-1})$ are isomorphic in $\mathfrak{G} - \mathfrak{Set}$ (each being acted upon by G via left-multiplication).

Proof. First of all, fix any $g \in G$. We want to show that there exists a G-equivariant (set) bijection $\varphi : \frac{G}{H} \to \frac{G}{gHg^{-1}}$ such that the following diagram commutes:

$$\begin{array}{ccc} G \amalg \frac{G}{H} & \xrightarrow{id_G \amalg \varphi} & G \amalg \frac{G}{gHg^{-1}} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ \frac{G}{H} & \xrightarrow{\varphi} & \frac{G}{gHg^{-1}} \end{array}$$

Where ρ and ρ' denote the left-multiplicative actions of G on the respective sets of left cosets.
We will take the easy way out, and explicitly define such a φ on the elements of these sets.

Let φ be defined by $aH \mapsto (ag^{-1})(gHg^{-1})$, for any $a \in G$.
We need to show, firstly, that this is well defined. well, if $aH = bH$ for $a, b \in G$, then

$$(aH)g^{-1} = (bH)g^{-1} \implies ag^{-1}(gHg^{-1}) = bg^{-1}(gHg^{-1}) \implies \varphi(aH) = \varphi(bH),$$

so φ is well defined. Next, we shall show injectivity, surjectivity, and G-equivariance.

- **Injective:**
  $\varphi(aH) = \varphi(bH) \implies ag^{-1}(gHg^{-1}) = bg^{-1}(gHg^{-1})$
  $\implies aHg^{-1} = bHg^{-1} \implies aH = bH.$

- **Surjective:** Given any $a(gHg^{-1}) \in G/(gHg^{-1})$, pick the element $(ag)H \in G/H$. Then we have that

  $$\varphi((ag)H) = (ag)g^{-1}(gHg^{-1}) = a(gHg^{-1}),$$

  so that φ is thereby surjective.

- **G-Equivariance:** For any $a, b \in G$, consider:
  $\varphi((ab)H) = (ab)g^{-1}(gHg^{-1}) = a(bg^{-1})(gHg^{-1}) = a\varphi(bH)$
  So φ is G-Equivariant.

Thus, φ is an isomorphism in $\mathfrak{G} - \mathfrak{Set}$ (based on any fixed $g \in G$), so G/H and $G/(gHg^{-1})$ are isomorphic for any $g \in G$ as claimed. $\square$