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# Answers to Odd-Numbered Exercises Not Asking for Definitions or Proofs

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## SECTION 0

1.  $\{-\sqrt{3}, \sqrt{3}\}$
3.  $\{1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 10, -10, 12, -12, 15, -15, 20, -20, 30, -30, 60, -60\}$
5. Not a set (not well defined). A case can also be made for the empty set  $\emptyset$ .
7. The set  $\emptyset$
9. It is not a well-defined set.
11.  $(a, 1), (a, 2), (a, c), (b, 1), (b, 2), (b, c), (c, 1), (c, 2), (c, c)$
13. Draw the line through  $P$  and  $x$ , and let  $y$  be the point where it intersects the line segment  $CD$ .
17. Conjecture:  $n(\mathcal{P}(A)) = 2^s$ . (Proofs are usually omitted from answers.)
21.  $10^2, 10^5, 10^{80} = 12^{80} = 2^{80} = |\mathbb{R}|$ . (The numbers  $x$  where  $0 \leq x \leq 1$  can be written to base 12 and to base 2 as well as to base 10.)
23. 1                      25. 5                      27. 52
29. Not an equivalence relation
31. Not an equivalence relation.
33. An equivalence relation;  
 $\overline{1} = \{1, 2, \dots, 9\},$   
 $\overline{10} = \{10, 11, \dots, 99\},$   
 $\overline{100} = \{100, 101, \dots, 999\},$  and in general  
 $\overline{10^n} = \{10^n, 10^n + 1, \dots, 10^{n+1} - 1\}$
35. a.  $\{\dots, -3, 0, 3, \dots\}, \{\dots, -2, 1, 4, \dots\}, \{\dots, -5, -2, 1, \dots\}$   
b.  $\{\dots, -4, 0, 4, \dots\}, \{\dots, -3, 1, 4, \dots\}, \{\dots, -6, -2, 2, \dots\}, \{\dots, -5, -1, 3, \dots\}$   
c.  $\{\dots, -5, 0, 5, \dots\}, \{\dots, -4, 1, 6, \dots\}, \{\dots, -3, 2, 7, \dots\}, \{\dots, -2, 3, 8, \dots\}, \{\dots, -1, 4, 9, \dots\}$
37.  $\overline{1} = \{x \in \mathbb{Z} \mid x \div n \text{ has remainder } 1\}$  depends on the value of  $n$ .
41. The name *two-to-two function* suggests that such a function  $f$  should carry every pair of distinct points into two distinct points. Such a function is one to one in the conventional sense. (If the domain has only one element, a function cannot fail to be two to two, since the only way it can fail to be two to two is to carry two points into one point, and the set does not have two points.) Conversely, every function that is one to one in the conventional sense carries any pair of points into two distinct points. Thus the functions conventionally called one to one are precisely those that carry two points into two points, which is a much more intuitive unidirectional way of regarding them. Also, the standard way of trying to show a function is one to one is precisely to show that it does not carry two points into just one point. Thus, proving a function is one to one becomes more natural in the two-to-two terminology.

## SECTION 1

1.  $e, b, a$       3.  $a, c$ .  $*$  is not associative.
5. Top row:  $d$ ; second row:  $a$ ; fourth row:  $c, b$ .
7. Not commutative, not associative
9. Commutative, associative, has identity.
11. Not commutative, not associative
13. 8, 729,  $n^{[n(n+1)/2]}$
15.  $n^{(n-1)^2}$
19. An identity in the set  $S$  with operation  $*$  is an element  $e \in S$  such that for all  $a \in S$ ,  $a * e = e * a = a$ .
21. Yes      23. No. Condition 2 is violated.
25. No. Condition 1 is violated.
27. a. Yes.      b. Yes
29. Let  $S = \{?, \Delta\}$ . Define  $*$  and  $*'$  on  $S$  by  $a * b = ?$  and  $a *' b = \Delta$  for all  $a, b \in S$ . (Other answers are possible.)
31. True      33. True
35. False. Let  $f(x) = x^2$ ,  $g(x) = x$ , and  $h(x) = 2x + 1$ . Then  
 $(f(x) - g(x)) - h(x) = x^2 - 3x - 1$  but  
 $f(x) - (g(x) - h(x)) = x^2 - (-x - 1) = x^2 + x + 1$ .
37. True
39. True      41. False. Let  $*$  be  $+$  and let  $*'$  be on  $\mathbb{Z}$ .

## SECTION 2

1. No.  $\mathcal{S}_3$  fails.      3. No.  $\mathcal{S}_1$  fails.      5. No.  $\mathcal{S}_1$  fails.
7.  $\mathcal{S}_3$
9.  $\mathcal{S}_1$
11. Yes      13. Yes
15. No. The matrix with all entries 0 is upper triangular, but has no inverse.
17. Yes.
19. (Proofs are omitted.)      c.  $-1/3$
21. 2, 3. (It gets harder for 4 elements, where the answer is *not* 4.)
25. a.  $F$       c.  $T$       e.  $F$       g.  $T$       i.  $F$
35.  $b^2 a^{12}$

## SECTION 3

1.  $-i$       3.  $-1$       5.  $20 - 9i$
7.  $17 - 15i$       9.  $-4 + 4i$
11.  $\sqrt{\pi^2 + e^2}$       13.  $\sqrt{2} \left( -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right)$
15.  $\sqrt{34} \left( \frac{-3}{\sqrt{34}} + \frac{5}{\sqrt{34}}i \right)$
17.  $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$       19.  $3i, \pm \frac{3\sqrt{3}}{2} - \frac{3}{2}i$
21.  $\sqrt{3} \pm i, \pm 2i, -\sqrt{3} \pm i$       23. 7      25.  $\frac{3}{8}$       27.  $\sqrt{2}$
29.  $x = 6$       31. 5      33. 1, 7
37.  $\zeta^0 \leftrightarrow 0, \zeta^3 \leftrightarrow 7, \zeta^4 \leftrightarrow 4, \zeta^5 \leftrightarrow 1, \zeta^6 \leftrightarrow 6, \zeta^7 \leftrightarrow 3$
39. With  $\zeta \leftrightarrow 4$ , we must have  $\zeta^2 \leftrightarrow 2, \zeta^3 \leftrightarrow 0$ , and  $\zeta^4 \leftrightarrow 4$  again, which is impossible for a one-to-one correspondence.
41. Multiplying, we obtain

$$z_1 z_2 = |z_1| |z_2| [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)i]$$

and the desired result follows at once from Exercise 40 and the equation  $|z_1| |z_2| = |z_1 z_2|$ .

45. Let  $f: \mathbb{R}_b \rightarrow \mathbb{R}_c$  be given by  $f(x) = \frac{c}{b}x$ .

## SECTION 4

1.  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}$
3.  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 2 & 5 \end{pmatrix}$
5.  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}$
7.  $\iota$
9.  $\iota$
11. a.  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 5 & 1 & 6 & 7 & 8 \end{pmatrix}$
- b.  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 6 & 4 & 2 & 1 & 7 & 3 & 5 \end{pmatrix}$
- c.  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 5 & 4 & 7 & 8 & 6 \end{pmatrix}$
13. a.  $\rho^6$
- b.  $\rho$
- c.  $\mu\rho^{10}$
- d.  $\mu\rho^{10}$
15.  $\{1, 2, 3, 4, 5, 6\}$
17.  $\{1, 5\}$
19. a. These are “elementary permutation matrices,” resulting from permuting the rows of the identity matrix. When another matrix  $A$  is multiplied on the left by one of these matrices  $P$ , the rows of  $A$  are permuted in the same fashion that the rows of the  $3 \times 3$  identity matrix were permuted to obtain  $P$ . Because all 6 possible permutations of the three rows are present, we see they will act just like the elements of  $S_3$  in permuting the entries 1, 2, 3 of the given column vector. Thus they form a group because  $S_3$  is a group.
- b. The symmetric group  $S_3$ .
21. Need to add that  $\phi$  is one-to-one and onto.
23. This is a good definition.
25. Not a permutation
27. Not a permutation
29. a.  $T$
- c.  $T$
- e.  $F$
- g.  $F$

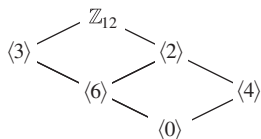
## SECTION 5

1. Yes
3. Yes
5. Yes
7.  $\mathbb{Q}^+$  and  $\{\pi^n \mid n \in \mathbb{Z}\}$
9. Yes
11. No. Not closed under multiplication.
13. Yes
15. a. Yes
- b. No. It is not even a subset of  $\tilde{F}$ .
17. a. No. Not closed under addition.
- b. Yes
19. a. Yes
- b. No. The zero constant function is not in  $\tilde{F}$ .
21. a.  $-50, -25, 0, 25, 50$
- b.  $4, 2, 1, 1/2, 1/4$
- c.  $1, \pi, \pi^2, 1/\pi, 1/\pi^2$
- d.  $\iota, \rho^3, \rho^6, \rho^9, \rho^{12}, \rho^{15}$
- e.  $\iota, (1, 2, 3)(5, 6), (1, 3, 2), (5, 6), (1, 2, 3), (1, 3, 2)(5, 6)$
23. All matrices  $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$  for  $n \in \mathbb{Z}$
25. All matrices of the form  $\begin{bmatrix} 4^n & 0 \\ 0 & 4^n \end{bmatrix}$  or  $\begin{bmatrix} 0 & -2^{2n+1} \\ -2^{2n+1} & 0 \end{bmatrix}$  for  $n \in \mathbb{Z}$
27. 4
29. 3
31. 4
33. 2
35. 3
39. a.  $T$
- c.  $T$
- e.  $F$
- g.  $F$
- i.  $T$
41. This is a subgroup:  $\iota$  is in the set, the set is closed under function composition, and if  $\sigma(b) = b$ , then  $\sigma^{-1}(b) = b$ .
43. This is not a subgroup. Let  $A = \mathbb{Z}$ ,  $B = \mathbb{Z}^+$ , and  $b = 1$ . The permutation  $\sigma(n) = n + 1$  mapping  $\mathbb{Z}$  to  $\mathbb{Z}$  is in the set, but  $\sigma^{-1}$  is not in the set.

## SECTION 6

1.  $q = 4, r = 6$
3.  $q = -5, r = 3$
5. 8
7. 60
9. 4
11. 24
13. 2
15. 2
17. 6
19. 4
21. An infinite cyclic group
23. 75
25. 6
27. 12
29. 30

31.



33.

41. a.  $T$  c.  $F$  e.  $T$  g.  $F$  i.  $T$ 43.  $< Q, + >$  45. There is none.47.  $i, -i$  49.  $\frac{\sqrt{2}}{2}(\pm 1 \pm i)$ 63.  $(p-1)p^{r-1}$ 

## SECTION 7

1. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 3.  $\mathbb{Z}_{25}$ 5.  $\dots, -24, -18, -12, -6, 0, 6, 12, 18, 24, \dots$ 7.  $\{i, \rho^2, \rho^4, \rho^6, \mu, \mu\rho^2, \mu\rho^4, \mu\rho^6\}$  9. a.  $c$  b.  $e$  c.  $d$ 

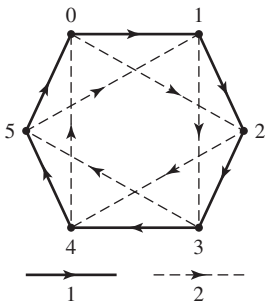
11.

	$e$	$a$	$b$	$c$	$d$	$f$
$e$	$e$	$a$	$b$	$c$	$d$	$f$
$a$	$a$	$e$	$c$	$b$	$f$	$d$
$b$	$b$	$d$	$e$	$f$	$a$	$c$
$c$	$c$	$f$	$a$	$d$	$e$	$b$
$d$	$d$	$b$	$f$	$e$	$c$	$a$
$f$	$f$	$c$	$d$	$a$	$b$	$e$

13. Choose a pair of generating directed arcs, call them  $arc1$  and  $arc2$ , start at any vertex of the digraph, and see if the sequences  $arc1, arc2$  and  $arc2, arc1$  lead to the same vertex. (This corresponds to asking if the two corresponding group generators commute.) The group is commutative if and only if these two sequences lead to the same vertex for every pair of generating directed arcs.

15. It is not obvious, since a digraph of a cyclic group might be formed using a generating set of two or more elements, no one of which generates the group.

17.



19. a. Starting from any vertex  $a$ , every path through the graph that terminates at that same vertex  $a$  represents a product of generators or their inverses that is equal to the identity and thus gives a relation.

b.  $a^4 = e, b^2 = e, (ab)^2 = e$