By reversing the roles of H and K (as well as H^* and K^*), the proof given above proves Part 2 as well as the other half of Part 3 of the Zassenhaus Lemma. ◆

18.11 Theorem **(Schreier Theorem)** Two subnormal (normal) series of a group G have isomorphic refinements.

Proof Let G be a group and let

$$\{e\} = H_0 < H_1 < H_2 < \cdots < H_n = G \tag{1}$$

and

$$\{e\} = K_0 < K_1 < K_2 < \cdots < K_m = G \tag{2}$$

be two subnormal series for G. For i where $0 \leq i \leq n-1$, form the chain of groups

$$H_i = H_i(H_{i+1} \cap K_0) \leq H_i(H_{i+1} \cap K_1) \leq \cdots \leq H_i(H_{i+1} \cap K_m) = H_{i+1}.$$

This inserts $m-1$ not necessarily distinct groups between H_i and H_{i+1}. If we do this for each i where $0 \leq i \leq n-1$ and let $H_{i,j} = H_i(H_{i+1} \cap K_j)$, then we obtain the chain of groups

$$\begin{aligned}\{e\} = H_{0,0} &\leq H_{0,1} \leq H_{0,2} \leq \cdots \leq H_{0,m-1} \leq H_{1,0}\\ &\leq H_{1,1} \leq H_{1,2} \leq \cdots \leq H_{1,m-1} \leq H_{2,0}\\ &\leq H_{2,1} \leq H_{2,2} \leq \cdots \leq H_{2,m-1} \leq H_{3,0}\\ &\leq \cdots\\ &\leq H_{n-1,1} \leq H_{n-1,2} \leq \cdots \leq H_{n-1,m-1} \leq H_{n-1,m}\\ &= G.\end{aligned} \tag{3}$$

This chain (3) contains $nm+1$ not necessarily distinct groups, and $H_{i,0} = H_i$ for each i. By the Zassenhaus Lemma, chain (3) is a subnormal chain, that is, each group is normal in the following group. This chain refines the series (1).

In a symmetric fashion, we set $K_{j,i} = K_j(K_{j+1} \cap H_i)$ for $0 \leq j \leq m-1$ and $0 \leq i \leq n$. This gives a subnormal chain

$$\begin{aligned}\{e\} = K_{0,0} &\leq K_{0,1} \leq K_{0,2} \leq \cdots \leq K_{0,n-1} \leq K_{1,0}\\ &\leq K_{1,1} \leq K_{1,2} \leq \cdots \leq K_{1,n-1} \leq K_{2,0}\\ &\leq K_{2,1} \leq K_{2,2} \leq \cdots \leq K_{2,n-1} \leq K_{3,0}\\ &\leq \cdots\\ &\leq K_{m-1,1} \leq K_{m-1,2} \leq \cdots \leq K_{m-1,n-1} \leq K_{m-1,n}\\ &= G.\end{aligned} \tag{4}$$

This chain (4) contains $mn+1$ not necessarily distinct groups, and $K_{j,0} = K_j$ for each j. This chain refines the series (2).

By the Zassenhaus Lemma 18.10, we have

$$H_i(H_{i+1} \cap K_{j+1})/H_i(H_{i+1} \cap K_j) \simeq K_j(K_{j+1} \cap H_{i+1})/K_j(K_{j+1} \cap H_i),$$

or

$$H_{i,j+1}/H_{i,j} \simeq K_{j,i+1}/K_{j,i} \tag{5}$$

for $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$. The isomorphisms of relation (5) give a one-to-one correspondence of isomorphic factor groups between the subnormal chains (3) and (4). To verify this correspondence, note that $H_{i,0} = H_i$ and $H_{i,m} = H_{i+1}$, while $K_{j,0} = K_j$ and $K_{j,n} = K_{j+1}$. Each chain in (3) and (4) contains a rectangular array of mn symbols $\leq$. Each $\leq$ gives rise to a factor group. The factor groups arising from the rth *row* of $\leq$'s in chain (3) correspond to the factor groups arising from the rth *column* of $\leq$'s in chain (4). Deleting repeated groups from the chains in (3) and (4), we obtain subnormal series of distinct groups that are isomorphic refinements of chains (1) and (2). This establishes the theorem for subnormal series.

For normal series, where all H_i and K_j are normal in G, we merely observe that all the groups $H_{i,j}$ and $K_{j,i}$ formed above are also normal in G, so the same proof applies. This normality of $H_{i,j}$ and $K_{j,i}$ follows at once from the second assertion in Lemma 16.4 and from the fact that intersections of normal subgroups of a group yield normal subgroups. ◆

The Jordan–Hölder Theorem

We now come to the real meat of the theory.

18.12 Definition A subnormal series $\{H_i\}$ of a group G is a **composition series** if all the factor groups H_{i+1}/H_i are simple. A normal series $\{H_i\}$ of G is a **principal** or **chief series** if all the factor groups H_{i+1}/H_i are simple. ■

Note that for abelian groups the concepts of composition and principal series coincide. Also, since every normal series is subnormal, every principal series is a composition series for any group, abelian or not.

18.13 Example We claim that $\mathbb{Z}$ has no composition (and also no principal) series. For if

$$\{0\} = H_0 < H_1 < \cdots < H_{n-1} < H_n = \mathbb{Z}$$

is a subnormal series, H_1 must be of the form $r\mathbb{Z}$ for some $r \in \mathbb{Z}^+$. But then H_1/H_0 is isomorphic to $r\mathbb{Z}$, which is infinite cyclic with many nontrivial proper normal subgroups, for example, $2r\mathbb{Z}$. Thus $\mathbb{Z}$ has no composition (and also no principal) series. ▲

18.14 Example The series

$$\{e\} < A_n < S_n$$

for $n \geq 5$ is a composition series (and also a principal series) of S_n, because $A_n/\{e\}$ is isomorphic to A_n, which is simple for $n \geq 5$, and S_n/A_n is isomorphic to $\mathbb{Z}_2$, which is simple. Likewise, the two series given in Example 18.7 are composition series (and also principal series) of $\mathbb{Z}_{15}$. They are isomorphic, as shown in that example. This illustrates our main theorem, which will be stated shortly. ▲

Observe that by Theorem 13.20, H_{i+1}/H_i is simple if and only if H_i is a maximal normal subgroup of H_{i+1}. Thus for a composition series, each H_i must be a maximal normal subgroup of H_{i+1}. *To form a composition series of a group G, we just hunt for a maximal normal subgroup H_{n-1} of G, then for a maximal normal subgroup H_{n-2} of H_{n-1}, and so on. If this process terminates in a finite number of steps, we have a composition series*. Note that by Theorem 13.20, a composition series cannot have any further refinement. *To form a principal series, we have to hunt for a maximal normal subgroup H_{n-1} of G, then for a maximal normal subgroup H_{n-2} of H_{n-1} that is also normal in G, and so on*. The main theorem is as follows.

18.15 Theorem **(Jordan–Hölder Theorem)** Any two composition (principal) series of a group G are isomorphic.

Proof Let $\{H_i\}$ and $\{K_i\}$ be two composition (principal) series of G. By Theorem 18.11, they have isomorphic refinements. But since all factor groups are already simple, Theorem 13.20 shows that neither series has any further refinement. Thus $\{H_i\}$ and $\{K_i\}$ must already be isomorphic. ◆

For a finite group, we should regard a composition series as a type of factorization of the group into simple factor groups, analogous to the factorization of a positive integer into primes. In both cases, the factorization is unique, up to the order of the factors.

18.16 Example We illustrate the analogy between factoring integers and composition series with an example. Let $n \in \mathbb{Z}^+$. We factor n into its prime factors $n = p_1p_2p_3 \ldots p_k$, where the prime factors may be repeated and they are in any order. The series

$$\{0\} < \langle p_1p_2p_3 \cdots p_{k-1}\rangle < \langle p_1p_2p_3 \cdots p_{k-2}\rangle < \langle p_1p_2p_3 \cdots p_{k-3}\rangle < \cdots < \langle p_1\rangle < \mathbb{Z}_n$$

is a composition series since the factor groups are isomorphic with $\mathbb{Z}_{p_k}, \mathbb{Z}_{p_{k-1}}, \mathbb{Z}_{p_{k-2}}, \ldots, \mathbb{Z}_{p_1}$, which are all simple. For each choice of ordering the prime numbers $p_1, p_2, \ldots, p_k$ we get a different composition series, but they are all isomorphic since the factor groups are $\mathbb{Z}_{p_1}, \mathbb{Z}_{p_2}, \ldots, \mathbb{Z}_{p_k}$ in some order. ▲

Historical Note

This first appearance of what became the Jordan–Hölder theorem occurred in 1869 in a commentary on the work of Galois by the brilliant French algebraist Camille Jordan (1838–1922). The context of its appearance is the study of permutation groups associated with the roots of polynomial equations. Jordan asserted that even though the sequence of normal subgroups $G, I, J, \cdots$ of the group of the equation is not necessarily unique, nevertheless the sequence of indices of this composition series is unique. Jordan gave a proof in his monumental 1870 *Treatise on Substitutions and Algebraic Equations*. This latter work, though restricted to what we now call permutation groups, remained the standard treatise on group theory for many years.

The Hölder part of the theorem, that the sequence of factor groups in a composition series is unique up to order, was due to Otto Hölder (1859–1937), who played a very important role in the development of group theory once the completely abstract definition of a group had been given. Among his other contributions, he gave the first abstract definition of a "factor group" and determined the structure of all finite groups of square-free order.

18.17 Theorem If G has a composition (principal) series, and if N is a proper normal subgroup of G, then there exists a composition (principal) series containing N.

Proof The series

$$\{e\} < N < G$$

is both a subnormal and a normal series. Since G has a composition series $\{H_i\}$, then by Theorem 18.11 there is a refinement of $\{e\} < N < G$ to a subnormal series isomorphic to a refinement of $\{H_i\}$. But as a composition series, $\{H_i\}$ can have no further refinement. Thus $\{e\} < N < G$ can be refined to a subnormal series all of whose factor groups are simple, that is, to a composition series. A similar argument holds if we start with a principal series $\{K_j\}$ of G. ◆

18.18 Example A composition (and also a principal) series of $\mathbb{Z}_4 \times \mathbb{Z}_9$ containing $\langle(0, 1)\rangle$ is

$$\{(0, 0)\} < \langle(0, 3)\rangle < \langle(0, 1)\rangle < \langle 2\rangle \times \langle 1\rangle < \langle 1\rangle \times \langle 1\rangle = \mathbb{Z}_4 \times \mathbb{Z}_9. \quad ▲$$