n yielding values θ where $0 \le \theta < 2\pi$ are 0 and 1, yielding $\theta = \pi/4$ or $\theta = 5\pi/4$. Our solutions are

$$z_1 = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

or

$$z_1 = \frac{1}{\sqrt{2}}(1+i) \quad \text{and} \quad z_2 = \frac{-1}{\sqrt{2}}(1+i). \qquad \blacktriangle$$

3.12 Example Find all solutions of $z^4 = -16$.

Solution As in Example 3.11 we write the equation in polar form, obtaining

$$|z|^4(\cos 4\theta + i\sin 4\theta) = 16(-1+0i).$$

Consequently, $|z|^4 = 16$, so $|z| = 2$ while $\cos 4\theta = -1$ and $\sin 4\theta = 0$. We find that $4\theta = \pi + n(2\pi)$, so $\theta = (\pi/4) + n(\pi/2)$ for integers n. The different values of θ obtained where $0 \le \theta < 2\pi$ are $\pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$. Thus one solution of $z^4 = -16$ is

$$2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}(1+i).$$

In a similar way, we find three more solutions,

$$\sqrt{2}(-1+i), \quad \sqrt{2}(-1-i), \quad \text{and} \quad \sqrt{2}(1-i). \qquad \blacktriangle$$

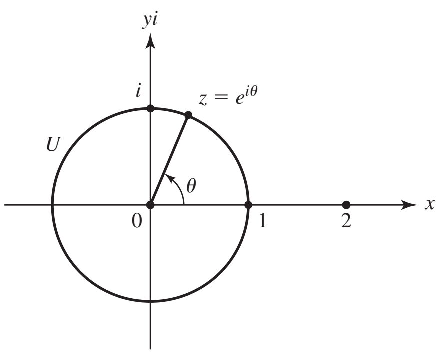
The last two examples illustrate that we can find solutions of an equation $z^n = a + bi$ by writing the equation in polar form. There will always be n solutions, provided that $a + bi \neq 0$. Exercises 16 through 21 ask you to solve equations of this type.

We will not use addition or division of complex numbers, but we probably should mention that addition is given by

$$(a+bi) + (c+di) = (a+c) + (b+d)i. \tag{6}$$

and division of $a + bi$ by nonzero $c + di$ can be performed using only division of real numbers as follows:

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.\end{aligned} \tag{7}$$



3.13 Figure

Algebra on the Unit Circle

Let $U = \{z \in \mathbb{C} \mid |z| = 1\}$, so that U is the circle in the Euclidean plane with center at the origin and radius 1, as shown in Fig. 3.13.

3.14 Theorem $\langle U, \cdot \rangle$ is an abelian group.

Proof We first check that U is closed under multiplication. Let $z_1, z_2 \in U$. Then $|z_1| = |z_2| = 1$, which implies that $|z_1 z_1| = 1$, showing $z_1 z_2 \in U$.

Since multiplication of complex numbers is associative and commutative in general, multiplication in U is also associative and commutative, which verifies $\mathscr{G}_1$ and the condition for abelian.

The number $1 \in U$ is the identity, verifying condition $\mathscr{G}_2$.

For each $a + bi \in U$,

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2 = |a + bi|^2 = 1.$$

So the inverse of $a + bi$ is $a - bi$, which verifies condition $\mathscr{G}_3$. Thus U is an abelian group under multiplication. ◆

Figure 3.13 gives us a way of relabeling points in U as points in $\mathbb{R}_{2\pi}$. We simply relabel z as θ where $0 \le \theta < 2\pi$. Let $f : U \to \mathbb{R}_{2\pi}$ be given by $f(z) = \theta$ according to this relabeling. Then for $z_1, z_2 \in U, f(z_1 z_2) = f(z_1) +_{2\pi} f(z_2)$ since multiplying in U simply adds the corresponding angles:

$$\text{if} \quad z_1 \leftrightarrow \theta_1 \quad \text{and} \quad z_2 \leftrightarrow \theta_2, \quad \text{then} \quad z_1 \cdot z_2 \leftrightarrow (\theta_1 +_{2\pi} \theta_2). \qquad \textbf{(8)}$$

Recall that all that remains to show that $\mathbb{R}_{2\pi}$ is a group is to show that $+_{2\pi}$ is associative. Since the operations of multiplication in U and addition modulo 2π in $\mathbb{R}_{2\pi}$ are the same using the above relabeling and multiplication in U is associative, addition modulo 2π is also associative. This completes the proof that $\langle \mathbb{R}_{2\pi}, +_{2\pi} \rangle$ is a group. Furthermore, the relabeling (8) shows that the two groups $\langle U, \cdot \rangle$ and $\langle \mathbb{R}_{2\pi}, +_{2\pi} \rangle$ are isomorphic. In Exercise 45, you will be asked to prove that for any $b > 0$ and $c > 0$, $\langle \mathbb{R}_b, +_b \rangle$ is an abelian group and $\langle \mathbb{R}_b, +_b \rangle \simeq \langle \mathbb{R}_c, +_c \rangle$. Since $\langle \mathbb{R}_{2\pi}, +_{2\pi} \rangle$ is isomorphic with $\langle U, \cdot \rangle$, for every $c > 0$, $\langle \mathbb{R}_c, +_c \rangle$ is also isomorphic with $\langle U, \cdot \rangle$, meaning they have the same algebraic properties.

3.15 Example The equation $z \cdot z \cdot z \cdot z = 1$ in U has exactly four solutions, namely, $1, i, -1$, and $-i$. Now $1 \in U$ and $0 \in \mathbb{R}_{2\pi}$ correspond, and the equation $x +_{2\pi} x +_{2\pi} x +_{2\pi} x = 0$ in $\mathbb{R}_{2\pi}$ has exactly four solutions, namely, $0, \pi/2, \pi$, and $3\pi/2$, which, of course, correspond to $1, i, -1$, and $-i$, respectively. ▲

Roots of Unity

The elements of the set $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$ are called the n^{th} **roots of unity**. In Exercise 46 you are asked to prove that U_n is a group under multiplication. Using the techniques from Examples 3.11 and 3.12, we see that the elements of this set are the numbers

$$e^{(m\frac{2\pi}{n})i} = \cos\left(m\frac{2\pi}{n}\right) + i \sin\left(m\frac{2\pi}{n}\right) \quad \text{for } m = 0, 1, 2, \ldots, n-1.$$

They all have absolute value 1, so $U_n \subset U$. If we let $\zeta = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$, then these n^{th} roots of unity can be written as

$$1 = \zeta^0, \zeta^1, \zeta^2, \zeta^3, \cdots, \zeta^{n-1}. \qquad \textbf{(9)}$$

Because $\zeta^n = 1$, these n powers of ζ are closed under multiplication. For example, with $n = 10$, we have

$$\zeta^6 \zeta^8 = \zeta^{14} = \zeta^{10} \zeta^4 = 1 \cdot \zeta^4 = \zeta^4.$$