To convince ourselves that two sets X and Y have the same cardinality, we try to exhibit a pairing of each x in X with only one y in Y in such a way that each element of Y is also used only once in this pairing. For the sets $X = \{2, 5, 7\}$ and $Y = \{?, !, \#\}$, the pairing

$$2 \leftrightarrow ?, \quad 5 \leftrightarrow \#, \quad 7 \leftrightarrow !$$

shows they have the same cardinality. Notice that we could also exhibit this pairing as {(2, ?), (5, #), (7, !)} which, as a subset of $X \times Y$, is a *relation* between X and Y. The pairing

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & -5 & \cdots \end{array}$$

shows that the sets $\mathbb{Z}$ and $\mathbb{Z}^+$ have the same cardinality. Such a pairing, showing that sets X and Y have the same cardinality, is a special type of relation $\leftrightarrow$ between X and Y called a **one-to-one correspondence.** Since each element x of X appears precisely once in this relation, we can regard this one-to-one correspondence as a *function* with domain X. The range of the function is Y because each y in Y also appears in some pairing $x \leftrightarrow y$. We formalize this discussion in a definition.

0.12 Definition *A function $\phi : X \to Y$ is **one-to-one** or **injective** if $\phi(x_1) = \phi(x_2)$ only when $x_1 = x_2$. The function ϕ is **onto** or **surjective** if the range of ϕ is Y. If ϕ is both injective and surjective, ϕ is said to be **bijective**. ■

If a subset of $X \times Y$ is a *one-to-one* function ϕ mapping X *onto* Y, then each $x \in X$ appears as the first member of exactly one ordered pair in ϕ and also each $y \in Y$ appears as the second member of exactly one ordered pair in ϕ. Thus if we interchange the first and second members of all ordered pairs (x, y) in ϕ to obtain a set of ordered pairs (y, x), we get a subset of $Y \times X$, which gives a one-to-one function mapping Y onto X. This function is called the **inverse function** of ϕ, and is denoted by ϕ^{-1}. Summarizing, if ϕ maps X one-to-one onto Y and $\phi(x) = y$, then ϕ^{-1} maps Y one-to-one onto X, and $\phi^{-1}(y) = x$.

0.13 Definition Two sets X and Y have the **same cardinality** if there exists a one-to-one function mapping X onto Y, that is, if there exists a one-to-one correspondence between X and Y. ■

0.14 Example The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is not one-to-one because $f(2) = f(-2) = 4$ but $2 \neq -2$. Also, it is not onto $\mathbb{R}$ because the range is the proper subset of all nonnegative numbers in $\mathbb{R}$. However, $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ is both one-to-one and onto $\mathbb{R}$. ▲

We showed that $\mathbb{Z}$ and $\mathbb{Z}^+$ have the same cardinality. We denote this cardinal number by $\aleph_0$, so that $|\mathbb{Z}| = |\mathbb{Z}^+| = \aleph_0$. It is fascinating that a proper subset of an infinite set may have the same number of elements as the whole set; an **infinite set** can be defined as a set having this property.

We naturally wonder whether all infinite sets have the same cardinality as the set $\mathbb{Z}$. A set has cardinality $\aleph_0$ if and only if *all* of its elements could be listed in an infinite row, so that we could "number them" using $\mathbb{Z}^+$. Figure 0.15 indicates that this is possible for the set $\mathbb{Q}$. The square array of fractions extends infinitely to the right and infinitely

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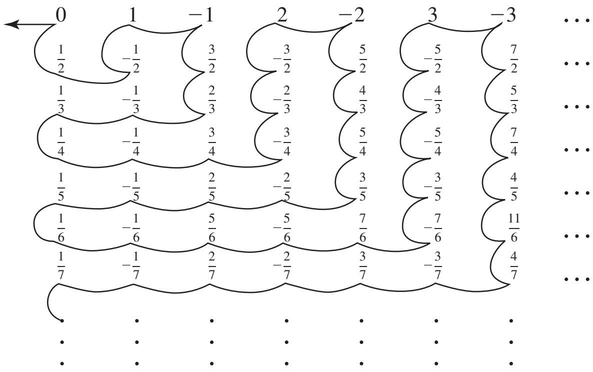
* We should mention another terminology, used by the disciples of N. Bourbaki, in case you encounter it elsewhere. In Bourbaki's terminology, a one-to-one map is an **injection,** an onto map is a **surjection,** and a map that is both one-to-one and onto is a **bijection.**



0.15 Figure

downward, and contains all members of $\mathbb{Q}$. We have shown a string winding its way through this array. Imagine the fractions to be glued to this string. Taking the beginning of the string and pulling to the left in the direction of the arrow, the string straightens out and all elements of $\mathbb{Q}$ appear on it in an infinite row as $0, \frac{1}{2}, -\frac{1}{2}, 1, -1, \frac{3}{2}, \cdots$. Thus $|\mathbb{Q}| = \aleph_0$ also.

If the set $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has cardinality $\aleph_0$, all its elements could be listed as unending decimals in a column extending infinitely downward, perhaps as

$$\begin{array}{c} 0.3659663426\cdots \\ 0.7103958453\cdots \\ 0.0358493553\cdots \\ 0.9968452214\cdots \\ \vdots \end{array}$$

We now argue that any such array must omit some number in S. Surely S contains a number r having as its nth digit after the decimal point a number different from 0, from 9, and from the nth digit of the nth number in this list. For example, r might start $.5637\cdots$. The 5 rather than 3 after the decimal point shows r cannot be the first number in S listed in the array shown. The 6 rather than 1 in the second digit shows r cannot be the second number listed, and so on. Because we could make this argument with *any list,* we see that S has too many elements to be paired with those in $\mathbb{Z}^+$. Exercise 15 indicates that $\mathbb{R}$ has the same number of elements as S. We just denote the cardinality of $\mathbb{R}$ by $|\mathbb{R}|$. Exercise 19 indicates that there are infinitely many different cardinal numbers even greater than $|\mathbb{R}|$.

Partitions and Equivalence Relations

Sets are **disjoint** if no two of them share a common element. In Example 0.17 we break up the integers into subsets. Eventually we will see how to define an algebraic structure on these subsets of $\mathbb{Z}$. That is, we will be able to "add" two of these subsets to get another subset. We will find that breaking a set into subsets is a valuable tool in a number of settings, so we conclude this section with a brief study of *partitions* of sets.

0.16 Definition A **partition** of a set S is a collection of nonempty subsets of S such that every element of S is in exactly one of the subsets. The subsets are the **cells** of the partition. ■