$g(x) \neq 0$. We similarly define $F(x_1, \cdots, x_n)$ to be the field of quotients of $F[x_1, \cdots, x_n]$. This field $F(x_1, \cdots, x_n)$ is the **field of rational functions in n indeterminates over F**. These fields play a very important role in algebraic geometry.

The Evaluation Homomorphisms

We are now ready to proceed to show how homomorphisms can be used to study what we have always referred to as "solving a polynomial equation." Let E and F be fields, with F a subfield of E, that is, $F \leq E$. The next theorem asserts the existence of very important homomorphisms of $F[x]$ into E. *These homomorphisms will be the fundamental tools for much of the rest of our work.*

27.4 Theorem **(The Evaluation Homomorphisms for Field Theory)** Let F be a subfield of a field E, let α be any element of E, and let x be an indeterminate. The map $\phi_\alpha : F[x] \to E$ defined by

$$\phi_\alpha\left(a_0 + a_1 x + \cdots + a_n x^n\right) = a_0 + a_1\alpha + \cdots + a_n\alpha^n$$

for $(a_0 + a_1 x + \cdots + a_n x^n) \in F[x]$ is a homomorphism of $F[x]$ into E. Also, $\phi_\alpha(x) = \alpha$, and ϕ_α maps F isomorphically by the identity map; that is, $\phi_\alpha(a) = a$ for $a \in F$. The homomorphism ϕ_α is **evaluation at α**.

Proof The subfield and mapping diagram in Fig. 27.5 may help us to visualize this situation. The dashed lines indicate an element of the set. The theorem is really an immediate consequence of our definitions of addition and multiplication in $F[x]$. The map ϕ_α is well defined, that is, independent of our representation of $f(x) \in F[x]$ as a finite sum

$$a_0 + a_1 x + \cdots + a_n x^n,$$

since such a finite sum representing $f(x)$ can be changed only by insertion or deletion of terms $0x^i$, which does not affect the value of $\phi_\alpha(f(x))$.

If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m$, and $h(x) = f(x) + g(x) = c_0 + c_1 x + \cdots + c_r x^r$, then

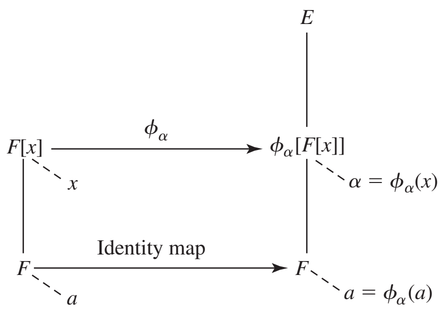
$$\phi_\alpha(f(x) + g(x)) = \phi_\alpha(h(x)) = c_0 + c_1\alpha + \cdots + c_r\alpha^r,$$

while

$$\phi_\alpha(f(x)) + \phi_\alpha(g(x)) = \left(a_0 + a_1\alpha + \cdots + a_n\alpha^n\right) + \left(b_0 + b_1\alpha + \cdots + b_m\alpha^m\right).$$

Since by definition of polynomial addition we have $c_i = a_i + b_i$, we see that

$$\phi_\alpha(f(x) + g(x)) = \phi_\alpha(f(x)) + \phi_\alpha(g(x)).$$



27.5 Figure

Turning to multiplication, we see that if

$$f(x)g(x) = d_0 + d_1x + \cdots + d_sx^s,$$

then

$$\phi_\alpha(f(x)g(x)) = d_0 + d_1\alpha + \cdots + d_s\alpha^s,$$

while

$$[\phi_\alpha(f(x))][\phi_\alpha(g(x))] = \big(a_0 + a_1\alpha + \cdots + \alpha_n\alpha^n\big)\big(b_0 + b_1\alpha + \cdots + b_m\alpha^m\big).$$

Since by definition of polynomial multiplication $d_j = \sum_{i=0}^{j} a_ib_{j-i}$, we see that

$$\phi_\alpha(f(x)g(x)) = [\phi_\alpha(f(x))][\phi_\alpha(g(x))].$$

Thus ϕ_α is a homomorphism.

The very definition of ϕ_α applied to a constant polynomial $a \in F[x]$, where $a \in F$, gives $\phi_\alpha(a) = a$, so ϕ_α maps F isomorphically by the identity map. Again by definition of ϕ_α, we have $\phi_\alpha(x) = \phi_\alpha(1x) = 1\alpha = \alpha$. ◆

We point out that this theorem is valid with the identical proof if F and E are merely commutative rings with unity rather than fields. However, we shall be interested primarily in the case in which they are fields.

It is hard to overemphasize the importance of this simple theorem for us. It is the very foundation for all of our further work in field theory. It is so simple that it could justifiably be called an *observation* rather than a theorem. It was perhaps a little misleading to write out the proof because the polynomial notation makes it look so complicated that you may be fooled into thinking it is a difficult theorem.

27.6 Example Let F be $\mathbb{Q}$ and E be $\mathbb{R}$ in Theorem 27.4, and consider the evaluation homomorphism $\phi_0 : \mathbb{Q}[x] \to \mathbb{R}$. Here

$$\phi_0\big(a_0 + a_1x + \cdots + a_nx^n\big) = a_0 + a_10 + \cdots + a_n0^n = a_0.$$

Thus every polynomial is mapped onto its constant term. ▲

27.7 Example Let F be $\mathbb{Q}$ and E be $\mathbb{R}$ in Theorem 27.4 and consider the evaluation homomorphism $\phi_2 : \mathbb{Q}[x] \to \mathbb{R}$. Here

$$\phi_2\big(a_0 + a_1x + \cdots + a_nx^n\big) = a_0 + a_12 + \cdots + a_n2^n.$$

Note that

$$\phi_2(x^2 + x - 6) = 2^2 + 2 - 6 = 0.$$

Thus $x^2 + x - 6$ is in the kernel N of ϕ_2. Of course,

$$x^2 + x - 6 = (x - 2)(x + 3),$$

and the reason that $\phi_2(x^2 + x - 6) = 0$ is that $\phi_2(x - 2) = 2 - 2 = 0$. ▲

27.8 Example Let F be $\mathbb{Q}$ and E be $\mathbb{C}$ in Theorem 27.4 and consider the evaluation homomorphism $\phi_i : \mathbb{Q}[x] \to \mathbb{C}$. Here

$$\phi_i\big(a_0 + a_1x + \cdots + a_nx^n\big) = a_0 + a_1i + \cdots + a_ni^n$$

and $\phi_i(x) = i$. Note that

$$\phi_i(x^2 + 1) = i^2 + 1 = 0,$$

so $x^2 + 1$ is in the kernel N of ϕ_i. ▲

27.9 Example Let F be $\mathbb{Q}$ and let E be $\mathbb{R}$ in Theorem 27.4 and consider the evaluation homomorphism $\phi_\pi : \mathbb{Q}[x] \to \mathbb{R}$. Here

$$\phi_\pi\left(a_0 + a_1x + \cdots + a_nx^n\right) = a_0 + a_1\pi + \cdots + a_n\pi^n.$$

It can be proved that $a_0 + a_1\pi + \cdots + a_n\pi^n = 0$ if and only if $a_i = 0$ for $i = 0, 1, \cdots, n$. Thus the kernel of ϕ_π is $\{0\}$, and ϕ_π is a one-to-one map. This shows that all *formal polynomials in* π *with rational coefficients* form a ring isomorphic to $\mathbb{Q}[x]$ in a natural way with $\phi_\pi(x) = \pi$. ▲

The New Approach

We now complete the connection between our new ideas and the classical concept of solving a polynomial equation. Rather than speak of *solving a polynomial equation,* we shall refer to *finding a zero of a polynomial*.

27.10 Definition Let F be a subfield of a field E, and let α be an element of E. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be in $F[x]$, and let $\phi_\alpha : F[x] \to E$ be the evaluation homomorphism of Theorem 27.4. Let $f(\alpha)$ denote

$$\phi_\alpha(f(x)) = a_0 + a_1\alpha + \cdots + a_n\alpha^n.$$

If $f(\alpha) = 0$, then α is a **zero of** $f(x)$. ■

In terms of this definition, we can rephrase the classical problem of finding all real numbers r such that $r^2 + r - 6 = 0$ by letting $F = \mathbb{Q}$ and $E = \mathbb{R}$ and *finding all* $\alpha \in \mathbb{R}$ *such that*

$$\phi_\alpha(x^2 + x - 6) = 0,$$

that is, finding all zeros of $x^2 + x - 6$ in $\mathbb{R}$. Both problems have the same answer, since

$$\{\alpha \in \mathbb{R} \mid \phi_\alpha(x^2 + x - 6) = 0\} = \{r \in \mathbb{R} \mid r^2 + r - 6 = 0\} = \{2, -3\}.$$

It may seem that we have merely succeeded in making a simple problem seem quite complicated. In fact, *what we have done is to phrase the problem in the language of mappings, and we can now use all the mapping machinery that we have developed and will continue to develop for its solution.*

Our Basic Goal

We continue to attempt to put our future work in perspective. Sections 30 and 31 are concerned with topics in ring theory that are analogous to the material on factor groups and homomorphisms for group theory. However, our aim in developing these analogous concepts for rings will be quite different from our aims in group theory. In group theory we used the concepts of factor groups and homomorphisms to study the structure of a given group and to determine the types of group structures of certain orders that could exist. We will be talking about homomorphisms and factor rings in Section 30 with an eye to finding zeros of polynomials, which is one of the oldest and most fundamental problems in algebra. Let us take a moment to talk about this aim in the light of mathematical history, using the language of "solving polynomial equations" to which we are accustomed.

We start with the Pythagorean school of mathematics of about 525 B.C. The Pythagoreans worked with the assumption that all distances are **commensurable**; that is, given distances a and b, there should exist a unit of distance u and integers n and m such that $a = (n)(u)$ and $b = (m)(u)$. In terms of numbers, then, thinking of u as being