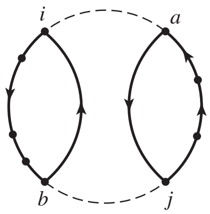

8.21 Figure

shown symbolically by the circle in Fig. 8.20. Computing the product of the first two cycles in $\tau\sigma = (i,j)\sigma$, we obtain

$$(i,j)(a,i,\times,\times,\times,b,j,\times,\times) = (a,j,\times,\times)(b,i,\times,\times,\times).$$

The original single orbit has been split into two as symbolized in Fig. 8.21.

We have shown that the number of orbits of $\tau\sigma$ differs from the number of orbits of σ by 1. The identity permutation ι has n orbits, because each element is the only member of its orbit. Now the number of orbits of a given permutation $\sigma \in S_n$ differs from n by either an even or an odd number, but not both. Thus it is impossible to write

$$\sigma = \tau_1\tau_2\tau_3 \cdots \tau_m\iota$$

where the τ_k are transpositions in two ways, once with m even and once with m odd. ◆

8.22 Definition A permutation of a finite set is **even** or **odd** according to whether it can be expressed as a product of an even number of transpositions or the product of an odd number of transpositions, respectively. ■

8.23 Example The identity permutation ι in S_n is an even permutation since we have $\iota = (1,2)(1,2)$. If $n = 1$ so that we cannot form this product, we define ι to be even. On the other hand, the permutation (1, 4, 5, 6) (2, 1, 5) in S_6 can be written as

$$(1,4,5,6)(2,1,5) = (1,6)(1,5)(1,4)(2,5)(2,1)$$

which has five transpositions, so this is an odd permutation. ▲

The Alternating Groups

We claim that for $n \geq 2$, the number of even permutations in S_n is the same as the number of odd permutations; that is, S_n is split equally and both numbers are $(n!)/2$. To show this, let A_n be the set of even permutations in S_n and let B_n be the set of odd permutations for $n \geq 2$. We proceed to define a one-to-one function from A_n onto B_n. This is exactly what is needed to show that A_n and B_n have the same number of elements.

Let τ be any fixed transposition in S_n; it exists since $n \geq 2$. We may as well suppose that $\tau = (1,2)$. We define a function

$$\lambda_\tau : A_n \to B_n$$

by

$$\lambda_\tau(\sigma) = \tau\sigma,$$

that is, $\sigma \in A_n$ is mapped into $(1,2)\sigma$ by λ_τ. Observe that since σ is even, the permutation $(1,2)\sigma$ can be expressed as a product of a (1 + even number), or odd number, of transpositions, so $(1,2)\sigma$ is indeed in B_n. If for σ and μ in A_n it is true that $\lambda_\tau(\sigma) = \lambda_\tau(\mu)$, then

$$(1,2)\sigma = (1,2)\mu,$$

and since S_n is a group, we have $\sigma = \mu$. Thus λ_τ is a one-to-one function. Finally,

$$\tau = (1,2) = \tau^{-1},$$

so if $\rho \in B_n$, then

$$\tau^{-1}\rho \in A_n,$$

and

$$\lambda_\tau(\tau^{-1}\rho) = \tau(\tau^{-1}\rho) = \rho.$$

Thus λ_τ maps onto B_n. Hence the number of elements in A_n is the same as the number in B_n since there is a one-to-one correspondence between the elements of the sets.

Note that the product of two even permutations is again even. Also since $n \geq 2, S_n$ has the transposition $(1, 2)$ and $\iota = (1, 2)(1, 2)$ is an even permutation. Finally, note that if σ is expressed as a product of transpositions, the product of the same transpositions taken in just the opposite order is σ^{-1}. Thus if σ is an even permutation, σ^{-1} must also be even. Referring to Theorem 5.12, we see that we have proved the following statement.

8.24 Theorem If $n \geq 2$, then the collection of all even permutations of $\{1, 2, 3, \cdots, n\}$ forms a subgroup of order $n!/2$ of the symmetric group S_n.

We can define a function called the **sign of a permutation**, sgn : $S_n \to \{1, -1\}$ by the formula

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Thinking of $\{1, -1\}$ as a group under multiplication, it is easy to see that sgn is a homomorphism. Since 1 is the identity in the group $\{1, -1\}$, Ker(sgn) = $\operatorname{sgn}^{-1}[\{1\}]$ is a subgroup of S_n consisting of all the even permutations. The homomorphism sgn is used in the standard way of defining the determinant of a square matrix. Exercise 52 asks you to prove some of the standard facts about determinants using this definition.

8.25 Definition The subgroup of S_n consisting of the even permutations of n letters is the **alternating group A_n on n letters**. ■

Both S_n and A_n are very important groups. Cayley's theorem shows that every finite group G is structurally identical to some subgroup of S_n for $n = |G|$. It can be shown that there are no formulas involving just radicals for solution of polynomial equations of degree n for $n \geq 5$. This fact is actually due to the structure of A_n, surprising as that may seem!

■ EXERCISES 8

Computations

In Exercises 1 through 10 determine whether the given map is a group homomorphism. [Hint: To verify that a map is a homomorphism, you must check the homomorphism property. To check that a map is not a homomorphism you could either find a and b such that $\phi(ab) \neq \phi(a)\phi(b)$, or else you could determine that any of the properties in Theorem 8.5 fail.]

1. Let $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_2$ be given by $\phi(x) =$ the remainder when x is divided by 2.
2. Let $\phi : \mathbb{Z}_9 \to \mathbb{Z}_2$ be given by $\phi(x) =$ the remainder when x is divided by 2.
3. Let $\phi : \mathbb{Q}^* \to \mathbb{Q}^*$ be given by $\phi(x) = |x|$.
4. Let $\phi : \mathbb{R} \to \mathbb{R}^+$ be given by $\phi(x) = 2^x$.
5. Let $\phi : D_4 \to \mathbb{Z}_4$ be given by $\phi(\rho^i) = \phi(\mu\rho^i) = i$ for $0 \leq i \leq 3$.

6. Let F be the additive group of all functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let $\phi : F \to F$ be given by $\phi(f) = g$ where $g(x) = f(x) + x$.

7. Let F be as in Exercise 6 and $\phi : F \to F$ be defined by $\phi(f) = 5f$.

8. Let F be the additive group of all continuous functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let $\phi : F \to \mathbb{R}$ be defined by $\phi(g) = \int_0^1 g(x)\, dx$.

9. Let M_n be the additive group of $n \times n$ matrices with real entries. Let $\phi : M_n \to \mathbb{R}$ be given by $\phi(A) = \det(A)$, the determinant of A.

10. Let M_n be as in Exercise 9 and $\phi : M_n \to \mathbb{R}$ be defined by $\phi(A) = \operatorname{tr}(A)$ where $\operatorname{tr}(A)$ is the trace of A, which is the sum of the entries on the diagonal.

In Exercises 11 through 16, compute the kernel for the given homomorphism ϕ.

11. $\phi : \mathbb{Z} \to \mathbb{Z}_8$ such that $\phi(1) = 6$.

12. $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $\phi(1) = 12$.

13. $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $\phi(1, 0) = 3$ and $\phi(0, 1) = -5$.

14. $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $\phi(1, 0) = 6$ and $\phi(0, 1) = 9$.

15. $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $\phi(1, 0) = (2, 5)$ and $\phi(0, 1) = (-3, 2)$.

16. Let D be the additive group of all differentiable functions mapping $\mathbb{R}$ to $\mathbb{R}$ and F the additive group of all functions from $\mathbb{R}$ to $\mathbb{R}$. $\phi : D \to F$ is given by $\phi(f) = f'$, the derivative of f.

In Exercises 17 through 22, find all orbits of the given permutation.

17. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$

18. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$

19. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$

20. $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 1$

21. $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 2$

22. $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n - 3$

In Exercises 23 through 25, express the permutation of {1, 2, 3, 4, 5, 6, 7, 8} as a product of disjoint cycles, and then as a product of transpositions.

23. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

24. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

25. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 2 & 8 & 4 & 7 & 6 & 1 \end{pmatrix}$

26. Figure 8.26 shows a Cayley digraph for the alternating group A_4 using the generating set $S = \{(1, 2, 3), (1, 2)(3, 4)\}$. Continue labeling the other nine vertices with the elements of A_4, expressed as a product of disjoint cycles.

Concepts

In Exercises 27 through 29, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

27. For a permutation σ of a set A, an *orbit* of σ is a nonempty minimal subset of A that is mapped onto itself by σ.

28. The left regular representation of a group G is the map of G into S_G whose value at $g \in G$ is the permutation of G that carries each $x \in G$ into gx.

29. The *alternating group* is the group of all even permutations.