Complex numbers were developed after the development of real numbers. The complex number i was *invented* to provide a solution to the quadratic equation $x^2 = -1$, so we require that

$$i^2 = -1. \quad \textbf{(1)}$$

Unfortunately, i has been called an **imaginary number,** and this terminology has led generations of students to view the complex numbers with more skepticism than the real numbers. Actually, *all* numbers, such as 1, 3, π, $-\sqrt{3}$, and i are inventions of our minds. There is no physical entity that *is* the number 1. If there were, it would surely be in a place of honor in some great scientific museum, and past it would file a steady stream of mathematicians, gazing at 1 in wonder and awe. A basic goal of this text is to show how we can invent solutions of polynomial equations when the coefficients of the polynomial may not even be real numbers!

Multiplication of Complex Numbers

The product $(a + bi)(c + di)$ is defined in the way it must be if we are to enjoy the familiar properties of real arithmetic and require that $i^2 = -1$, in accord with Eq. (1). Namely, we see that we want to have

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2\\ &= ac + adi + bci + bd(-1)\\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

Consequently, we define multiplication of $z_1 = a + bi$ and $z_2 = c + di$ as

$$z_1z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i, \quad \textbf{(2)}$$

which is of the form $r + si$ with $r = ac - bd$ and $s = ad + bc$. It is routine to check that the usual properties $z_1z_2 = z_2z_1$(commutative), $z_1(z_2z_3) = (z_1z_2)z_3$ (associative), and $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (distributive) all hold for all $z_1, z_2, z_3 \in \mathbb{C}$.

3.7 Example Compute $(2 - 5i)(8 + 3i)$.

Solution We don't memorize Eq. (2), but rather we compute the product as we did to motivate that equation. We have

$$(2 - 5i)(8 + 3i) = 16 + 6i - 40i + 15 = 31 - 34i. \quad \blacktriangle$$

To establish the geometric meaning of complex multiplication, we first define the **absolute value** $|a + bi|$ of $a + bi$ by

$$|a + bi| = \sqrt{a^2 + b^2}. \quad \textbf{(3)}$$

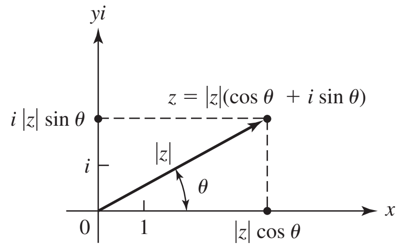
This absolute value is a nonnegative real number and is the distance from $a + bi$ to the origin in Fig. 3.6. We can now describe a complex number z in the polar-coordinate form

$$z = |z|(\cos\theta + i \sin\theta), \quad \textbf{(4)}$$

where θ is the angle measured counterclockwise from the positive x-axis to the vector from 0 to z, as shown in Fig. 3.8. A famous formula due to Leonard Euler states that

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Euler's Formula



3.8 Figure

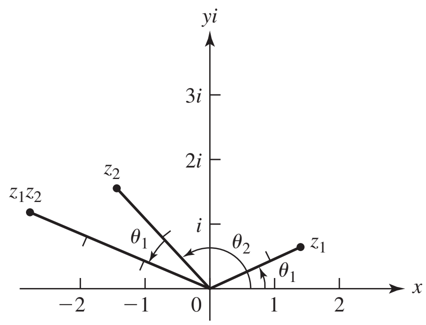
We ask you to derive Euler's formula formally from the power series expansions for $e^{\theta}, \cos\theta$, and $\sin\theta$ in Exercise 43. Using this formula, we can express z in Eq. (4) as $z = |z|e^{i\theta}$. Let us set

$$z_1 = |z_1|e^{i\theta_1} \qquad \text{and} \qquad z_2 = |z_2|e^{i\theta_2}$$

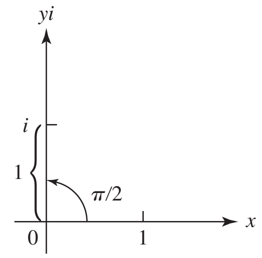
and compute their product in this form, assuming that the usual laws of exponentiation hold with complex number exponents. We obtain

$$\begin{aligned} z_1 z_2 &= |z_1|e^{i\theta_1}|z_2|e^{i\theta_2} = |z_1||z_2|e^{i(\theta_1+\theta_2)} \\ &= |z_1||z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned} \tag{5}$$

Note that Eq. 5 concludes in the polar form of Eq. 4 where $|z_1 z_2| = |z_1||z_2|$ and the polar angle θ for $z_1 z_2$ is the sum $\theta = \theta_1 + \theta_2$. Thus, geometrically, we multiply complex numbers by multiplying their absolute values and adding their polar angles, as shown in Fig. 3.9. Exercise 41 indicates how this can be derived via trigonometric identities without recourse to Euler's formula and assumptions about complex exponentiation.



3.9 Figure



3.10 Figure

Note that i has polar angle $\pi/2$ and absolute value 1, as shown in Fig. 3.10. Thus i^2 has polar angle $2(\pi/2) = \pi$ and $|1 \cdot 1| = 1$, so that $i^2 = -1$.

3.11 Example Find all solutions in $\mathbb{C}$ of the equation $z^2 = i$.

Solution Writing the equation $z^2 = i$ in polar form and using Eq. (5), we obtain

$$|z|^2(\cos 2\theta + i\sin 2\theta) = 1(0 + i).$$

Thus $|z|^2 = 1$, so $|z| = 1$. The angle θ for z must satisfy $\cos 2\theta = 0$ and $\sin 2\theta = 1$. Consequently, $2\theta = (\pi/2) + n(2\pi)$, so $\theta = (\pi/4) + n\pi$ for an integer n. The values of