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PART VII Commutative Algebra

SECTION 33 VECTOR SPACES

The notions of a vector space, scalars, independent vectors, and bases may be familiar. In this section, we present these ideas where the scalars may be elements of any field. We use Greek letters like α and β for vectors. In our application, the vectors will be elements of a field E containing field F. The proofs are all identical with those often given in a first course in linear algebra.

Definition and Elementary Properties

The topic of vector spaces is the cornerstone of linear algebra. Since linear algebra is not the subject for study in this text, our treatment of vector spaces will be brief, designed to develop only the concepts of linear independence and dimension that we need for our development of field theory.

The terms *vector* and *scalar* are probably familiar from calculus. Here we allow scalars to be elements of any field, not just the real numbers, and develop the theory by axioms just as for the other algebraic structures we have studied.

33.1 Definition Let F be a field. A **vector space over** F (or F-**vector space**) consists of an abelian group V under addition together with an operation of scalar multiplication of each element of V by each element of F on the left, such that for all $a, b \in F$ and $\alpha, \beta \in V$ the following conditions are satisfied:

$\mathscr{V}_1$. $a\alpha \in V$.

$\mathscr{V}_2$. $a(b\alpha) = (ab)\alpha$.

$\mathscr{V}_3$. $(a + b)\alpha = (a\alpha) + (b\alpha)$.

$\mathscr{V}_4$. $a(\alpha + \beta) = (a\alpha) + (a\beta)$.

$\mathscr{V}_5$. $1\alpha = \alpha$.

The elements of V are **vectors** and the elements of F are **scalars.** When only one field F is under discussion, we drop the reference to F and refer to a *vector space*. ■

Note that scalar multiplication for a vector space is not a binary operation on one set in the sense we defined it in Section 1. It associates an element $a\alpha$ of V with each ordered pair (a, α), consisting of an element a of F and an element α of V. Thus scalar multiplication is a *function* mapping $F \times V$ into V. Both the additive identity for V, the 0-vector, and the additive identity for F, the 0-scalar, will be denoted by 0.

33.2 Example Consider the abelian group $\langle \mathbb{R}^n, + \rangle = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ for n factors, which consists of ordered n-tuples under addition by components. Define scalar multiplication for scalars in $\mathbb{R}$ by

$$r\alpha = (ra_1, \cdots, ra_n)$$

for $r \in \mathbb{R}$ and $\alpha = (a_1, \cdots, a_n) \in \mathbb{R}^n$. With these operations, $\mathbb{R}^n$ becomes a vector space over $\mathbb{R}$. The axioms for a vector space are readily checked. In particular, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as a vector space over $\mathbb{R}$ can be viewed as all "vectors whose starting points are the origin of the Euclidean plane" in the sense often studied in calculus courses. ▲

33.3 Example For any field F, $F[x]$ can be viewed as a vector space over F, where addition of vectors is ordinary addition of polynomials in $F[x]$ and scalar multiplication $a\alpha$ of an element of $F[x]$ by an element of F is ordinary multiplication in $F[x]$. The axioms $\mathscr{V}_1$ through $\mathscr{V}_5$ for a vector space then follow immediately from the fact that $F[x]$ is a ring with unity. ▲

33.4 Example Let F be a subfield of the field E. Then E can be regarded as a vector space over F, where addition of vectors is the usual addition in E and scalar multiplication $a\alpha$ is the usual field multiplication in E with $a \in F$ and $\alpha \in E$. The axioms follow at once from the field axioms for E. Here our field of scalars is actually a subset of our space of vectors. *It is this example that is the important one for us.* ▲

We are assuming nothing about vector spaces from previous work and shall prove everything we need from the definition, even though the results may be familiar from calculus.

Historical Note

The ideas behind the abstract notion of a vector space occurred in many concrete examples during the nineteenth century and earlier. For example, William Rowan Hamilton dealt with complex numbers explicitly as pairs of real numbers and, as noted in Section 32, also dealt with triples and eventually quadruples of real numbers in his invention of the quaternions. In these cases, the "vectors" turned out to be objects that could both be added and multiplied by scalars, using "reasonable" rules for both of these operations. Other examples of such objects included differential forms (expressions under integral signs) and algebraic integers.

Although Hermann Grassmann (1809–1877) succeeded in working out a detailed theory of n dimensional spaces in his *Die Lineale Ausdehnung slehre* of 1844 and 1862, the first mathematician to give an abstract definition of a vector space equivalent to Definition 33.1 was Giuseppe Peano (1858–1932) in his *Calcolo Geometrico* of 1888. Peano's aim in the book, as the title indicates, was to develop a geometric calculus. According to Peano, such a calculus "consists of a system of operations analogous to those of algebraic calculus, but in which the objects with which the calculations are performed are, instead of numbers, geometrical objects." Curiously, Peano's work had no immediate effect on the mathematical scene. Although Hermann Weyl (1885–1955) essentially repeated Peano's definition in his *Space-Time-Matter* of 1918, the definition of a vector space did not enter the mathematical mainstream until it was announced for a third time by Stefan Banach (1892–1945) in the 1922 publication of his dissertation dealing with what we now call *Banach spaces,* complete normed vector spaces.

33.5 Theorem If V is a vector space over F, then $0\alpha = 0, a0 = 0$, and $(-a)\alpha = a(-\alpha) = -(a\alpha)$ for all $a \in F$ and $\alpha \in V$.

Proof The equation $0\alpha = 0$ is to be read "(0-scalar)α = 0-vector." Likewise, $a0 = 0$ is to be read "a(0-vector) = 0-vector." The proofs here are very similar to those in Theorem 22.8 for a ring and again depend heavily on the distributive laws $\mathscr{V}_3$ and $\mathscr{V}_4$. Now

$$(0\alpha) = (0+0)\alpha = (0\alpha) + (0\alpha)$$

is an equation in the abelian group $\langle V, +\rangle$, so by the group cancellation law, $0 = 0\alpha$. Likewise, from

$$a0 = a(0+0) = a0 + a0,$$

we conclude that $a0 = 0$. Then

$$0 = 0\alpha = (a + (-a))\alpha = a\alpha + (-a)\alpha,$$

so $(-a)\alpha = -(a\alpha)$. Likewise, from

$$0 = a0 = a(\alpha + (-\alpha)) = a\alpha + a(-\alpha),$$

we conclude that $a(-\alpha) = -(a\alpha)$ also. ◆

Linear Independence and Bases

33.6 Definition Let V be a vector space over F. The vectors in a subset $S = \{\alpha_i \mid i \in I\}$ of V **span** (or **generate**) V if for every $\beta \in V$, we have

$$\beta = a_1\alpha_{i_1} + a_2\alpha_{i_2} + \cdots + a_n\alpha_{i_n}$$

for some $a_j \in F$ and $\alpha_{i_j} \in S, j = 1, \cdots, n$. A vector $\sum_{j=1}^{n} a_j\alpha_{i_j}$ is a **linear combination of the** α_{i_j}. ■

33.7 Example In the vector space $\mathbb{R}^n$ over $\mathbb{R}$ of Example 33.2, the vectors

$$(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)$$

clearly span $\mathbb{R}^n$, for

$$(a_1, a_2, \cdots, a_n) = a_1(1, 0, \cdots, 0) + a_2(0, 1, \cdots, 0) + \cdots + a_n(0, 0, \cdots, 1).$$

Also, the monomials x^m for $m \geq 0$ span $F[x]$ over F, the vector space of Example 33.3. ▲

33.8 Definition A vector space V over a field F is **finite dimensional** if there is a finite subset of V whose vectors span V. ■

33.9 Example Example 33.7 shows that $\mathbb{R}^n$ is finite dimensional. The vector space $F[x]$ over F is *not* finite dimensional, since polynomials of arbitrarily large degree could not be linear combinations of elements of any *finite* set of polynomials. ▲

The next definition contains the most important idea in this section.

33.10 Definition The vectors in a subset $S = \{\alpha_i \mid i \in I\}$ of a vector space V over a field F are **linearly independent over** F if, for any distinct vectors $\alpha_{i_j} \in S$, coefficients $a_j \in F$, and $n \in \mathbb{Z}^+$, we have $\sum_{j=1}^{n} a_j\alpha_{i_j} = 0$ in V only if $a_j = 0$ for $j = 1, \cdots, n$. If the vectors are not linearly independent over F, they are **linearly dependent over** F. ■