4. $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
5. $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
6. $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$

Thus there are six different abelian groups (up to isomorphism) of order 360. ▲

There is another version of the Fundamental Theorem of Finitely Generated Abelian Groups. Each version can be proven from the other, so technically, if one version is used to prove something, the other version could also be used. However, it is sometimes more convenient to use one version rather than the other for a particular problem.

9.14 Theorem **(Invariant Factor Version of the Fundamental Theorem of Finitely Generated Abelian Groups)** Every finitely generated abelian group is isomorphic to a direct product of cyclic groups of the form

$$\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3} \times \cdots \times \mathbb{Z}_{d_k} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z},$$

where each of the $d_i \geq 2$ is an integer and d_i divides d_{i+1} for $1 \leq i \leq k-1$. Furthermore, the representation is unique. ♦

The Betti number of a group is the number of factors of $\mathbb{Z}$ in both Theorem 9.12 and 9.14. The numbers d_i are called the **invariant factors** or the **torsion coefficients**. Theorem 9.12 implies Theorem 9.14 and the other way around. Here we show with an example how to start with a finite group that is in the form specified in Theorem 9.12 and find its representation in the form of Theorem 9.14.

9.15 Example Let us find the invariant factor form of the abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_7$, which is in primary factor form. We make a table, one row for each prime number involved in G: 2, 3, and 7. We list the powers of each prime in the primary factor form starting with the highest power to the lowest power, filling the ends of the short rows with $1 = p^0$. Table 9.16 is the table for G. The group G is the direct product of cyclic groups of the orders listed in the table. The products of the entries in the columns give the invariant factors. For G, the invariant factors are $d_4 = 8 \cdot 9 \cdot 7 = 504$, $d_3 = 4 \cdot 3 \cdot 1 = 12$, $d_2 = 2 \cdot 1 \cdot 1 = 2$, and $d_1 = 2 \cdot 1 \cdot 1 = 2$. The construction of the table insures that d_1 divides d_2, d_2 divides d_3, d_3 divides d_4, and G is isomorphic with $\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3} \times \mathbb{Z}_{d_4} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{504}$. ▲

9.16 Table

8	4	2	2
9	3	1	1
7	1	1	1

Example 9.15 shows how to create a table from a finitely generated abelian group that is in primary factor form. From the table we can find the invariant form of the group. This process can easily be reversed by factoring the invariants to find the primary factors.

Applications

Because of Theorems 9.12 and 9.14, there is a plethora of theorems regarding finitely generated abelian groups that are fairly easily proven. We present a few examples.

9.17 Definition A group G is **decomposable** if it is isomorphic to a direct product of two proper nontrivial subgroups. Otherwise G is **indecomposable**. ■

9.18 Theorem The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

Proof Let G be a finite indecomposable abelian group. Then by Theorem 9.12, G is isomorphic to a direct product of cyclic groups of prime power order. Since G is indecomposable, this direct product must consist of just one cyclic group whose order is a power of a prime number.

Conversely, let p be a prime. Then $\mathbb{Z}_{p^r}$ is indecomposable, for if $\mathbb{Z}_{p^r}$ were isomorphic to $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$, where $i + j = r$, then every element would have an order at most $p^{\max(i,j)} < p^r$. ◆

9.19 Theorem If m divides the order of a finite abelian group G, then G has a subgroup of order m.

Proof By Theorem 9.12, we can think of G as being

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}},$$

where not all primes p_i need be distinct. Since $(p_1)^{r_1}(p_2)^{r_2} \cdots (p_n)^{r_n}$ is the order of G, then m must be of the form $(p_1)^{s_1}(p_2)^{s_2} \cdots (p_n)^{s_n}$, where $0 \leq s_i \leq r_i$. By Theorem 6.15, $(p_i)^{r_i - s_i}$ generates a cyclic subgroup of $\mathbb{Z}_{(p_i)^{r_i}}$ of order equal to the quotient of $(p_i)^{r_i}$ by the gcd of $(p_i)^{r_i}$ and $(p_i)^{r_i - s_i}$. But the gcd of $(p_i)^{r_i}$ and $(p_i)^{r_i - s_i}$ is $(p_i)^{r_i - s_i}$. Thus $(p_i)^{r_i - s_i}$ generates a cyclic subgroup of $\mathbb{Z}_{(p_i)^{r_i}}$ of order

$$[(p_i)^{r_i}]/[(p_i)^{r_i - s_i}] = (p_i)^{s_i}.$$

Recalling that $\langle a \rangle$ denotes the cyclic subgroup generated by a, we see that

$$\langle (p_1)^{r_1 - s_1} \rangle \times \langle (p_2)^{r_2 - s_2} \rangle \times \cdots \times \langle (p_n)^{r_n - s_n} \rangle$$

is the required subgroup of order m. ◆

9.20 Theorem If m is a square-free integer, that is, m is not divisible by the square of any integer $n \geq 2$ then every abelian group of order m is cyclic.

Proof Let G be a finite abelian group of square-free order m. Then by Theorem 9.14, G is isomorphic to

$$\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_k},$$

where each $d_i \geq 2$ divides d_{i+1} for $1 \leq i \leq k - 1$. The order of G is $m = d_1 \cdot d_2 \cdots d_k$. If $k \geq 2$, then d_1^2 divides m, which is a contradiction. Thus $k = 1$ and G is cyclic. ◆

EXERCISES 9

Computations

1. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each of the elements. Is this group cyclic?

2. Repeat Exercise 1 for the group $\mathbb{Z}_3 \times \mathbb{Z}_4$.

In Exercises 3 through 7, find the order of the given element of the direct product.

3. (2, 6) in $\mathbb{Z}_4 \times \mathbb{Z}_{12}$

4. $(3, 4)$ in $\mathbb{Z}_{21} \times \mathbb{Z}_{12}$

5. $(40, 12)$ in $\mathbb{Z}_{45} \times \mathbb{Z}_{18}$

6. (3, 10, 9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$

7. (3, 6, 12, 16) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$

8. What is the largest order among the orders of all the cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$? of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$?

9. Find all proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

10. Find all proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

11. Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$ of order 4.

12. Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ that are isomorphic to the Klein 4-group.

13. Disregarding the order of the factors, write direct products of two or more groups of the form $\mathbb{Z}_n$ so that the resulting product is isomorphic to $\mathbb{Z}_{60}$ in as many ways as possible.

14. Fill in the blanks.

a. The cyclic subgroup of $\mathbb{Z}_{24}$ generated by 18 has order___.

b. $\mathbb{Z}_3 \times \mathbb{Z}_4$ is of order___.

c. The element (4, 2) of $\mathbb{Z}_{12} \times \mathbb{Z}_8$ has order___.

d. The Klein 4-group is isomorphic to $\mathbb{Z}_{\_\_} \times \mathbb{Z}_{\_\_}$.

e. $\mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_4$ has___elements of finite order.

15. Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_6$.

16. Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not?

17. Find the maximum possible order for some element of $\mathbb{Z}_8 \times \mathbb{Z}_{28} \times \mathbb{Z}_{24}$.

18. Are the groups $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ isomorphic? Why or why not?

19. Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$.

20. Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Why or why not?

In Exercises 21 through 25, proceed as in Example 9.13 to find all abelian groups, up to isomorphism, of the given order. For each group, find the invariant factors and find an isomorphic group of the form indicated in Theorem 9.14.

21. Order 8 **22.** Order 16 **23.** Order 32

24. Order 720 **25.** Order 1089

26. How many abelian groups (up to isomorphism) are there of order 24? of order 25? of order (24)(25)?

27. Following the idea suggested in Exercise 26, let m and n be relatively prime positive integers. Show that if there are (up to isomorphism) r abelian groups of order m and s of order n, then there are (up to isomorphism) rs abelian groups of order mn.

28. Use Exercise 27 to determine the number of abelian groups (up to isomorphism) of order $(10)^5$.

29. a. Let p be a prime number. Fill in the second row of the table to give the number of abelian groups of order p^n, up to isomorphism.

n	2	3	4	5	6	7	8
number of groups							

b. Let $p, q,$ and r be distinct prime numbers. Use the table you created to find the number of abelian groups, up to isomorphism, of the given order.

i. $p^3q^4r^7$ **ii.** $(qr)^7$ **iii.** $q^5r^4q^3$

30. Indicate schematically a Cayley digraph for $\mathbb{Z}_m \times \mathbb{Z}_n$ for the generating set $S = \{(1, 0), (0, 1)\}$.

31. Consider Cayley digraphs with two arc types, a solid one with an arrow and a dashed one with no arrow, and consisting of two regular n-gons, for $n \geq 3$, with solid arc sides, one inside the other, with dashed arcs joining the vertices of the outer n-gon to the inner one. Figure 7.11(b) shows such a Cayley digraph with $n = 3$, and Figure 7.13(b) shows one with $n = 4$. The arrows on the outer n-gon may have the same (clockwise or counterclockwise) direction as those on the inner n-gon, or they may have the opposite direction. Let G be a group with such a Cayley digraph.

a. Under what circumstances will G be abelian?

b. If G is abelian, to what familiar group is it isomorphic?

c. If G is abelian, under what circumstances is it cyclic?

d. If G is not abelian, to what group we have discussed is it isomorphic?

Concepts

32. Determine whether each of the following is true or false.

a. If G_1 and G_2 are any groups, then $G_1 \times G_2$ is always isomorphic to $G_2 \times G_1$.

b. Computation in an external direct product of groups is easy if you know how to compute in each component group.

c. Groups of finite order must be used to form an external direct product.

d. A group of prime order could not be the internal direct product of two proper nontrivial subgroups.