When discussing a partition of a set S, we denote by $\bar{x}$ the cell containing the element x of S.

0.17 Example Splitting $\mathbb{Z}$ into the subset of even integers and the subset of odd integers, we obtain a partition of $\mathbb{Z}$ into the two cells listed below.

$$\overline{0} = \{\ldots, -8, -6, -4, -2, 0, 2, 4, \ldots\}$$
$$\overline{1} = \{\ldots, -7, -5, -3, -1, 1, 3, 5, \ldots\}$$

We can think of $\overline{0}$ as being the integers that are divisible by 2 and $\overline{1}$ as the integers that when divided by 2 yield a remainder of 1. This idea can be used for positive integers other than 2. For example, we can partition $\mathbb{Z}$ into three cells:

$$\overline{0} = \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 3\},$$
$$\overline{1} = \{x \in \mathbb{Z} \mid \text{the remainder of } x \text{ divided by 3 is } 1\}, \text{ and}$$
$$\overline{2} = \{x \in \mathbb{Z} \mid \text{the remainder of } x \text{ divided by 3 is } 2\}.$$

Note that when dividing a negative number by 3, we still obtain a non-negative remainder. For example, $-5 \div 3$ is -2 with remainder 1, which says that $\overline{-5} = \overline{1}$.

Generalizing, for each $n \in \mathbb{Z}^+$, we obtain a partition of $\mathbb{Z}$ consisting of n cells, $\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}$. For each $0 \leq r \leq n-1$, an integer x is in the cell $\overline{r}$ exactly when the remainder of $x \div n$ is r. These cells are the **residue classes modulo n** in $\mathbb{Z}$ and n is called the **modulus**. We define the set $\mathbb{Z}/n\mathbb{Z}$ as the set containing the cells in this partition. So, for example, $\mathbb{Z}/3\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}\}$. As we can see, $\mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$ has exactly n elements. ▲

Each partition of a set S yields a relation $\mathscr{R}$ on S in a natural way: namely, for $x, y \in S$, let $x \mathscr{R} y$ if and only if x and y are in the same cell of the partition. In set notation, we would write $x \mathscr{R} y$ as $(x, y) \in \mathscr{R}$ (see Definition 0.7). A bit of thought shows that this relation $\mathscr{R}$ on S satisfies the three properties of an *equivalence relation* in the following definition.

0.18 Definition An **equivalence relation** $\mathscr{R}$ on a set S is one that satisfies these three properties for all $x, y, z \in S$.

1. (Reflexive) $x \mathscr{R} x$.
2. (Symmetric) If $x \mathscr{R} y$, then $y \mathscr{R} x$.
3. (Transitive) If $x \mathscr{R} y$ and $y \mathscr{R} z$ then $x \mathscr{R} z$. ■

To illustrate why the relation $\mathscr{R}$ corresponding to a partition of S satisfies the symmetric condition in the definition, we need only observe that if y is in the same cell as x (that is, if $x \mathscr{R} y$), then x is in the same cell as y (that is, $y \mathscr{R} x$). We leave the similar observations to verify the reflexive and transitive properties to Exercise 28.

0.19 Example For any nonempty set S, the equality relation $=$ defined by the subset $\{(x, x) \mid x \in S\}$ of $S \times S$ is an equivalence relation. ▲

0.20 Example **(Congruence Modulo n)** Let $n \in \mathbb{Z}^+$. The equivalence relation on $\mathbb{Z}$ corresponding to the partition of $\mathbb{Z}$ into residue classes modulo n, discussed in Example 0.17, is **congruence modulo n**. It is sometimes denoted by $\equiv_n$. Rather than write $a \equiv_n b$, we usually write $a \equiv b \pmod{n}$, read, "a is congruent to b modulo n." For example, we have $15 \equiv 27 \pmod 4$ because both 15 and 27 have remainder 3 when divided by 4. ▲

0.21 Example Let a relation $\mathscr{R}$ on the set $\mathbb{Z}$ be defined by $n\,\mathscr{R}\,m$ if and only if $nm \geq 0$, and let us determine whether $\mathscr{R}$ is an equivalence relation.

Reflexive $a\,\mathscr{R}\,a$, because $a^2 \geq 0$ for all $a \in \mathbb{Z}$.

Symmetric If $a\,\mathscr{R}\,b$, then $ab \geq 0$, so $ba \geq 0$ and $b\,\mathscr{R}\,a$.

Transitive If $a\,\mathscr{R}\,b$ and $b\,\mathscr{R}\,c$, then $ab \geq 0$ and $bc \geq 0$. Thus $ab^2c = acb^2 \geq 0$.

If we knew $b^2 > 0$, we could deduce $ac \geq 0$ whence $a\,\mathscr{R}\,c$. We have to examine the case $b = 0$ separately. A moment of thought shows that $-3\,\mathscr{R}\,0$ and $0\,\mathscr{R}\,5$, but we do *not* have $-3\,\mathscr{R}\,5$. Thus the relation $\mathscr{R}$ is not transitive, and hence is not an equivalence relation. ▲

We observed above that a partition yields a natural equivalence relation. We now show that an equivalence relation on a set yields a natural partition of the set. The theorem that follows states both results for reference.

0.22 Theorem **(Equivalence Relations and Partitions)** Let S be a nonempty set and let $\sim$ be an equivalence relation on S. Then $\sim$ yields a partition of S, where

$$\bar{a} = \{x \in S \mid x \sim a\}.$$

Also, each partition of S gives rise to an equivalence relation $\sim$ on S where $a \sim b$ if and only if a and b are in the same cell of the partition.

Proof We must show that the different cells $\bar{a} = \{x \in S \mid x \sim a\}$ for $a \in S$ do give a partition of S, so that every element of S is in some cell and so that if $a \in \bar{b}$, then $\bar{a} = \bar{b}$. Let $a \in S$. Then $a \in \bar{a}$ by the reflexive condition (1), so a is in *at least one* cell.

Suppose now that $a \in \bar{b}$. We need to show that $\bar{a} = \bar{b}$ as sets; this will show that a cannot be in more than one cell. There is a standard way to show that two sets are the same:

Show that each set is a subset of the other.

We show that $\bar{a} \subseteq \bar{b}$. Let $x \in \bar{a}$. Then $x \sim a$. But $a \in \bar{b}$, so $a \sim b$. Then, by the transitive condition (3), $x \sim b$, so $x \in \bar{b}$. Thus $\bar{a} \subseteq \bar{b}$. Now we show that $\bar{b} \subseteq \bar{a}$. Let $y \in \bar{b}$. Then $y \sim b$. But $a \in \bar{b}$, so $a \sim b$ and, by symmetry (2), $b \sim a$. Then by transitivity (3), $y \sim a$, so $y \in \bar{a}$. Hence $\bar{b} \subseteq \bar{a}$ also, so $\bar{b} = \bar{a}$ and our proof is complete. ◆

Each cell in the partition arising from an equivalence relation is an **equivalence class**.

EXERCISES 0

In Exercises 1 through 4, describe the set by listing its elements.

1. $\{x \in \mathbb{R} \mid x^2 = 3\}$

2. $\{m \in \mathbb{Z} \mid m^2 + m = 6\}$

3. $\{m \in \mathbb{Z} \mid mn = 60 \text{ for some } n \in \mathbb{Z}\}$

4. $\{x \in \mathbb{Z} \mid x^2 - 10x + 16 \leq 0\}$

In Exercises 5 through 10, decide whether the object described is indeed a set (is well defined). Give an alternate description of each set.

5. $\{n \in \mathbb{Z}^+ \mid n \text{ is a large number}\}$

6. $\{n \in \mathbb{Z} \mid n^2 < 0\}$

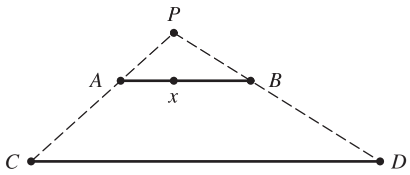
7. $\{n \in \mathbb{Z} \mid 39 < n^3 < 57\}$

8. $\{r \in \mathbb{Q} \mid$ When r is multiplied by a sufficiently large power of 2, one obtains a whole number.$\}$

9. $\{x \in \mathbb{Z}^+ \mid x \text{ is an easy number to factor}\}$

10. $\{x \in \mathbb{Q} \mid x$ may be written with positive denominator less than 4$\}$

11. List the elements in $\{a, b, c\} \times \{1, 2, c\}$.



0.23 Figure

12. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. For each relation between A and B given as a subset of $A \times B$, decide whether it is a function mapping A into B. If it is a function, decide whether it is one-to-one and whether it is onto B.

a. $\{\{1, 2\}, \{2, 6\}, \{3, 4\}\}$

b. [[1,3] and [5,7]]

c. $\{(1, 6), (1, 2), (1, 4)\}$

d. $\{\{2, 2\}, \{3, 6\}, \{1, 6\}\}$

e. $\{(1, 6), (2, 6), (3, 6)\}$

f. $\{\{1, 2\}, \{2, 6\}\}$

13. Illustrate geometrically that two line segments AB and CD of different lengths have the same number of points by indicating in Fig. 0.23 what point y of CD might be paired with point x of AB.

14. Recall that for $a, b \in \mathbb{R}$ and $a < b$, the **closed interval** $[a, b]$ in $\mathbb{R}$ is defined by $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. Show that the given intervals have the same cardinality by giving a formula for a one-to-one function f mapping the first interval onto the second.

a. [0, 1] and [0, 2]

b. [1, 3] and [5, 7]

c. $[a, b]$ and $[c, d]$

15. Show that $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has the same cardinality as $\mathbb{R}$. [*Hint:* Find an elementary function of calculus that maps an interval one-to-one onto $\mathbb{R}$, and then translate and scale appropriately to make the domain the set S.]

For any set A, we denote by $\mathscr{P}(A)$ the collection of all subsets of A. For example, if $A = \{a, b, c, d\}$, then $\{a, b, d\} \in \mathscr{P}(A)$. The set $\mathscr{P}(A)$ is the **power set** of A. Exercises 16 through 19 deal with the notion of the power set of a set A.

16. List the elements of the power set of the given set and give the cardinality of the power set.

a. $\varnothing$

b. $\{a\}$

c. $\{a, b\}$

d. $\{a, b, c\}$

17. Let A be a finite set, and let $|A| = s$. Based on the preceding exercise, make a conjecture about the value of $|\mathscr{P}(A)|$. Then try to prove your conjecture.

18. For any set A, finite or infinite, let B^A be the set of all functions mapping A into the set $B = \{0, 1\}$. Show that the cardinality of B^A is the same as the cardinality of the set $\mathscr{P}(A)$. [*Hint:* Each element of B^A determines a subset of A in a natural way.]

19. Show that the power set of a set A, finite or infinite, has too many elements to be able to be put in a one-to-one correspondence with A. Explain why this intuitively means that there are an infinite number of infinite cardinal numbers. [*Hint:* Imagine a one-to-one function ϕ mapping A into $\mathscr{P}(A)$ to be given. Show that ϕ cannot be onto $\mathscr{P}(A)$ by considering, for each $x \in A$, whether $x \in \phi(x)$ and using this idea to define a subset S of A that is not in the range of ϕ.] Is *the set of everything* a logically acceptable concept? Why or why not?

20. Let $A = \{1, 2\}$ and let $B = \{3, 4, 5\}$.

a. Illustrate, using A and B, why we consider that $2 + 3 = 5$. Use similar reasoning with sets of your own choice to decide what you would consider to be the value of

i. $3 + \aleph_0$,

ii. $\aleph_0 + \aleph_0$.

b. Illustrate why we consider that $2 \cdot 3 = 6$ by plotting the points of $A \times B$ in the plane $\mathbb{R} \times \mathbb{R}$. Use similar reasoning with a figure in the text to decide what you would consider to be the value of $\aleph_0 \cdot \aleph_0$.

21. How many numbers in the interval $0 \leq x \leq 1$ can be expressed in the form .##, where each # is a digit $0, 1, 2, 3, \cdots, 9$? How many are there of the form .#####? Following this idea, and Exercise 15, decide what you would consider to be the value of $10^{\aleph_0}$. How about $12^{\aleph_0}$ and $2^{\aleph_0}$?