20.5 Definition If G is a group with a set $A = \{a_i\}$ of generators, and if G is isomorphic to $F[A]$ under a map $\phi : G \to F[A]$ such that $\phi(a_i) = a_i$, then G is **free on** A, and the a_i are **free generators of** G. A group is **free** if it is free on some nonempty set A. ■

20.6 Example The only example of a free group that has occurred before is $\mathbb{Z}$, which is free on one generator. Note that every free group is infinite, for it contains a subgroup isomorphic with $\mathbb{Z}$. ▲

Refer to the literature for proofs of the next three theorems. We will not use these results. They are stated simply to inform us of these interesting facts.

20.7 Theorem If a group G is free on A and also on B, then the sets A and B have the same number of elements; that is, any two sets of free generators of a free group have the same cardinality.

20.8 Definition If G is free on A, the number of elements in A is the **rank of the free group** G. ■

Actually, the next theorem is quite evident from Theorem 20.7.

20.9 Theorem Two free groups are isomorphic if and only if they have the same rank.

20.10 Theorem A nontrivial subgroup of a free group is free.

20.11 Example Let $F[\{x, y\}]$ be the free group on $\{x, y\}$. Let

$$y_k = x^k y x^{-k}$$

for $k \geq 0$. The y_k for $k \geq 0$ are free generators for the subgroup of $F[\{x, y\}]$ that they generate. This illustrates the bizarre fact that although a subgroup of a free group is free, the rank of the subgroup may be much greater than the rank of the whole group! ▲

Homomorphisms of Free Groups

Our work in this section will be concerned primarily with homomorphisms defined on a free group. The results here are simple and elegant.

20.12 Theorem Let G be generated by $A = \{a_i \mid i \in I\}$ and let G' be any group. If a_i' for $i \in I$ are any elements in G', not necessarily distinct, then there is at most one homomorphism $\phi : G \to G'$ such that $\phi(a_i) = a_i'$. If G is free on A, then there is exactly one such homomorphism.

Proof Let ϕ be a homomorphism from G into G' such that $\phi(a_i) = a_i'$. Now by Theorem 7.7, for any $x \in G$ we have

$$x = \prod_j {a_{i_j}}^{n_j}$$

for some finite product of the generators a_i, where the a_{i_j} appearing in the product need not be distinct. Then since ϕ is a homomorphism, we must have

$$\phi(x) = \prod_j \phi\left({a_{i_j}}^{n_j}\right) = \prod_j \left(\mathrm{a}_{i_j}{}'\right)^{n_j}.$$

Thus a homomorphism is completely determined by its values on elements of a generating set. This shows that there is at most one homomorphism such that $\phi(a_i) = a_i'$.

Now suppose G is free on A; that is, $G = F[A]$. For

$$x = \prod_j a_{i_j}{}^{n_j}$$

in G, define $\psi : G \to G'$ by

$$\psi(x) = \prod_j \left(a_{i_j}{}'\right)^{n_j}.$$

The map is well defined, since $F[A]$ consists precisely of reduced words; no two different formal products in $F[A]$ are equal. Since the rules for computation involving exponents in G' are formally the same as those involving exponents in G, it is clear that $\psi(xy) = \psi(x)\psi(y)$ for any elements x and y in G, so ψ is indeed a homomorphism. ◆

Perhaps we should have proved the first part of Theorem 20.12 earlier, rather than having relegated it to the exercises. Note that the theorem states that *a homomorphism of a group is completely determined if we know its value on each element of a generating set*. In particular, a homomorphism of a cyclic group is completely determined by its value on any single generator of the group.

20.13 Theorem Every group G' is a homomorphic image of a free group G.

Proof Let $G' = \{a_i' \mid i \in I\}$, and let $A = \{a_i \mid i \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$. Then by Theorem 20.12 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a_i'$. Clearly the image of G under ψ is all of G'. ◆

Another Look at Free Abelian Groups

It is important that we do not confuse the notion of a free group with the notion of a free abelian group. A free group on more than one generator is not abelian. In the preceding section, we defined a free abelian group as an abelian group that has a basis, that is, a generating set satisfying properties described in Theorem 19.1. There is another approach, via free groups, to free abelian groups. We now describe this approach.

Let $F[A]$ be the free group on the generating set A. We shall write F in place of $F[A]$ for the moment. Note that F is not abelian if A contains more than one element. Let C be the commutator subgroup of F. Then F/C is an abelian group, and it is not hard to show that F/C is free abelian with basis $\{aC \mid a \in A\}$. If aC is renamed a, we can view F/C as a free abelian group with basis A. This indicates how a free abelian group having a given set as basis can be constructed. Every free abelian group can be constructed in this fashion, up to isomorphism. That is, if G is free abelian with basis X, form the free group $F[X]$, form the factor group of $F[X]$ modulo its commutator subgroup, and we have a group isomorphic to G.

Theorems 20.7, 20.9, and 20.10 hold for free abelian groups as well as for free groups. In fact, the abelian version of Theorem 20.10 was proved for the finite rank case in Theorem 19.11. In contrast to Example 20.11 for free groups, it is true that for a free abelian group the rank of a subgroup is at most the rank of the entire group. Theorem 19.11 also showed this for the finite rank case.

EXERCISES 20

Computations

1. Find the reduced form and the inverse of the reduced form of each of the following words.

a. $a^2b^{-1}b^3a^3c^{-1}c^4b^{-2}$ **b.** $a^2a^{-3}b^3a^4c^4c^2a^{-1}$

2. Compute the products given in parts (a) and (b) of Exercise 1 in the case that $\{a, b, c\}$ is a set of generators forming a basis for a free abelian group. Find the inverse of these products.

3. How many different homomorphisms are there of a free group of rank 2 into

a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

4. How many different homomorphisms are there of a free group of rank 2 onto

a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

5. How many different homomorphisms are there of a free abelian group of rank 2 into

a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

6. How many different homomorphisms are there of a free abelian group of rank 2 onto

a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

Concepts

In Exercises 7 and 8, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

7. A *reduced word* is one in which there are no appearances in juxtaposition of two syllables having the same letter and also no appearances of a syllable with exponent 0.

8. The *rank of a free group* is the number of elements in a set of generators for the group.

9. Take one of the instances in this section in which the phrase “It would seem obvious that” was used and discuss your reaction in that instance.

10. Determine whether each of the following is true or false.

a. Every proper subgroup of a free group is a free group.
b. Every proper subgroup of every free abelian group is a free group.
c. A homomorphic image of a free group is a free group.
d. Every free abelian group has a basis.
e. The free abelian groups of finite rank are precisely the finitely generated abelian groups.
f. No free group is free.
g. No free abelian group is free.
h. No free abelian group of rank >1 is free.
i. Any two free groups are isomorphic.
j. Any two free abelian groups of the same rank are isomorphic.

Theory

11. Let G be a finitely generated abelian group with identity 0. A finite set $\{b_1, \cdots, b_n\}$, where $b_i \in G$, is a **basis for** G if $\{b_1, \cdots, b_n\}$ generates G and $\sum_{i=1}^{n} m_i b_i = 0$ if and only if each $m_i b_i = 0$, where $m_i \in \mathbb{Z}$.

a. Show that $\{2, 3\}$ is not a basis for $\mathbb{Z}_4$. Find a basis for $\mathbb{Z}_4$.

b. Show that both $\{1\}$ and $\{2, 3\}$ are bases for $\mathbb{Z}_6$. (This shows that for a finitely generated abelian group G with torsion, the number of elements in a basis may vary; that is, it need not be an *invariant* of the group G.)

c. Is a basis for a free abelian group as we defined it in Section 19 a basis in the sense in which it is used in this exercise?

d. Show that every finite abelian group has a basis $\{b_1, \cdots, b_n\}$, where the order of b_i divides the order of b_{i+1}.

In present-day expositions of algebra, a frequently used technique (particularly by the disciples of N. Bourbaki) for introducing a new algebraic entity is the following:

1. Describe algebraic properties that this algebraic entity is to possess.
2. Prove that any two algebraic entities with these properties are isomorphic, that is, that these properties characterize the entity.
3. Show that at least one such entity exists.

The next three exercises illustrate this technique for three algebraic entities, each of which we have met before. So that we do not give away their identities, we use fictitious names for them in the first two exercises. The last part of these first two exercises asks us to give the usual name for the entity.

12. Let G be any group. An abelian group G^* is a **blip group of** G if there exists a fixed homomorphism ϕ of G onto G^* such that each homomorphism ψ of G into an abelian group G' can be factored as $\psi = \theta\phi$, where θ is a homomorphism of G^* into G' (see Fig. 20.14).

a. Show that any two blip groups of G are isomorphic. [*Hint:* Let $G_1{}^*$ and $G_2{}^*$ be two blip groups of G. Then each of the fixed homomorphisms $\phi_1 : G \to G_1{}^*$ and $\phi_2 : G \to G_2{}^*$ can be factored via the other blip group according to the definition of a blip group; that is, $\phi_1 = \theta_1\phi_2$ and $\phi_2 = \theta_2\phi_1$. Show that θ_1 is an isomorphism of $G_2{}^*$ onto $G_1{}^*$ by showing that both $\theta_1\theta_2$ and $\theta_2\theta_1$ are identity maps.]

b. Show for every group G that a blip group G^* of G exists.

c. What concept that we have introduced before corresponds to this idea of a blip group of G?

$$\begin{tikzcd} G \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & G' \\ & G^* \arrow[ur, "\theta"'] & \end{tikzcd}$$

20.14 Figure

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "g"'] & & G' \\ & G \arrow[ur, "\phi_f"'] & \end{tikzcd}$$

20.15 Figure

13. Let S be any set. A group G together with a fixed function $g : S \to G$ constitutes a **blop group on** S if for each group G' and map $f : S \to G'$ there exists a *unique* homomorphism ϕ_f of G into G' such that $f = \phi_f g$ (see Fig. 20.15).

a. Let S be a fixed set. Show that if both G_1, together with $g_1 : S \to G_1$, and G_2, together with $g_2 : S \to G_2$, are blop groups on S, then G_1 and G_2 are isomorphic. [*Hint:* Show that g_1 and g_2 are one-to-one maps and that g_1S and g_2S generate G_1 and G_2, respectively. Then proceed in a way analogous to that given by the hint for Exercise 12.]

b. Let S be a set. Show that a blop group on S exists. You may use any theorems of the text.

c. What concept that we have introduced before corresponds to this idea of a blop group on S?

14. Characterize a free abelian group by properties in a fashion similar to that used in Exercise 13.

SECTION 21 GROUP PRESENTATIONS

Definition

Following most of the literature on group presentations, in this section we let 1 be the identity of a group. The idea of a *group presentation* is to form a group by giving a set of generators for the group and certain equations or relations that we want the generators to satisfy. We want the group to be as free as it possibly can be on the generators, subject to these relations.