segment in the shaded portion, representing a coset of N in G. It is crucial to remember that multiplication of cosets in G/N can be computed by multiplying in G, using any representative elements of the cosets as shown in the figure.

Additively, two elements of G will collapse into the same element of G/N if they differ by an element of N. Multiplicatively, a and b collapse together if ab^{-1} is in N. The degree of collapsing can vary from nonexistent to catastrophic. We illustrate the two extreme cases by examples.

13.2 Example The trivial subgroup $N = \{0\}$ of $\mathbb{Z}$ is, of course, a normal subgroup. Compute $\mathbb{Z}/\{0\}$.

Solution Since $N = \{0\}$ has only one element, every coset of N has only one element. That is, the cosets are of the form $\{m\}$ for $m \in \mathbb{Z}$. There is no collapsing at all, and consequently, $\mathbb{Z}/\{0\} \simeq \mathbb{Z}$. Each $m \in \mathbb{Z}$ is simply renamed $\{m\}$ in $\mathbb{Z}/\{0\}$. ▲

13.3 Example Let n be a positive integer. The set $n\mathbb{R} = \{nr \mid r \in \mathbb{R}\}$ is a subgroup of $\mathbb{R}$ under addition, and it is normal since $\mathbb{R}$ is abelian. Compute $\mathbb{R}/n\mathbb{R}$.

Solution A bit of thought shows that actually $n\mathbb{R} = \mathbb{R}$, because each $x \in \mathbb{R}$ is of the form $n(x/n)$ and $x/n \in \mathbb{R}$. Thus $\mathbb{R}/n\mathbb{R}$ has only one element, the subgroup $n\mathbb{R}$. The factor group is a trivial group consisting only of the identity element. ▲

As illustrated in Examples 13.2 and 13.3 for any group G, we have $G/\{e\} \simeq G$ and $G/G \simeq \{e\}$, where $\{e\}$ is the trivial group consisting only of the identity element e. These two extremes of factor groups are of little importance. We would like knowledge of a factor group G/N to give some information about the structure of G. If $N = \{e\}$, the factor group has the same structure as G and we might as well have tried to study G directly. If $N = G$, the factor group has no significant structure to supply information about G. If G is a finite group and $N \neq \{e\}$ is a normal subgroup of G, then G/N is a smaller group than G, and consequently may have a more simple structure than G. The multiplication of cosets in G/N reflects the multiplication in G, since products of cosets can be computed by multiplying in G representative elements of the cosets.

We give two examples showing that even when G/N has order 2, we may be able to deduce some useful results. If G is a finite group and G/N has just two elements, then we must have $|G| = 2|N|$. Note that every subgroup H containing just half the elements of a finite group G must be a normal subgroup, since for each element a in G but not in H, both the left coset aH and the right coset Ha must consist of all elements in G that are not in H. Thus the left and right cosets of H coincide and H is a normal subgroup of G.

13.4 Example Because $|S_n| = 2|A_n|$, we see that A_n is a normal subgroup of S_n, and S_n/A_n has order 2. Let σ be an odd permutation in S_n, so that $S_n/A_n = \{A_n, \sigma A_n\}$. Renaming the element A_n "even" and the element σA_n "odd," the multiplication in S_n/A_n shown in Table 13.5 becomes

13.5 Table

	A_n	σA_n
A_n	A_n	σA_n
σA_n	σA_n	A_n

$$\begin{array}{ll} \text{(even)(even)} = \text{even} & \text{(odd)(even)} = \text{odd} \\ \text{(even)(odd)} = \text{odd} & \text{(odd)(odd)} = \text{even.} \end{array}$$

Thus the factor group reflects these multiplicative properties for all the permutations in S_n. ▲

Example 13.4 illustrates that while knowing the product of two cosets in G/N does not tell us what the product of two elements of G is, it may tell us that the product in G of two *types* of elements is itself of a certain type.

13.6 Example **(The Converse of the Theorem of Lagrange is False)** Recall that the Theorem of Lagrange states that the order of a subgroup of a finite group G must divide the order of G. We are now in a position to demonstrate that although the group A_4 has 12 elements and 6 divides 12, A_4 has no subgroup of order 6.

Suppose that H were a subgroup of A_4 having order 6. As observed before in Example 13.4, it would follow that H would be a normal subgroup of A_4. Then A_4/H would have only two elements, H and σH for some $\sigma \in A_4$ not in H. Since in a group of order 2, the square of each element is the identity, we would have $HH = H$ and $(\sigma H)(\sigma H) = H$. Now computation in a factor group can be achieved by computing with representatives in the original group. Thus, computing in A_4, we find that for each $\alpha \in H$ we must have $\alpha^2 \in H$ and for each $\beta \in \sigma H$ we must have $\beta^2 \in H$. That is, the square of every element in A_4 must be in H. But in A_4, we have

$$(1, 2, 3) = (1, 3, 2)^2 \quad \text{and} \quad (1, 3, 2) = (1, 2, 3)^2$$

so (1, 2, 3) and (1, 3, 2) are in H. A similar computation shows that (1, 2, 4), (1, 4, 2), (1, 3, 4), (1, 4, 3), (2, 3, 4), and (2, 4, 3) are all in H. This shows that there must be at least 8 elements in H, contradicting the fact that H was supposed to have order 6. ▲

We now turn to several examples that *compute* factor groups. If the group we start with is finitely generated and abelian, then its factor group will be also. *Computing* such a factor group means classifying it according to the fundamental theorem (Theorem 9.12 or Theorem 9.14).

13.7 Example Let us compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0, 1)\rangle$. Here $\langle(0, 1)\rangle$ is the cyclic subgroup H of $\mathbb{Z}_4 \times \mathbb{Z}_6$ generated by (0, 1). Thus

$$H = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5)\}.$$

Since $\mathbb{Z}_4 \times \mathbb{Z}_6$ has 24 elements and H has 6 elements, all cosets of H must have 6 elements, and $(\mathbb{Z}_4 \times \mathbb{Z}_6)/H$ must have order 4. Since $\mathbb{Z}_4 \times \mathbb{Z}_6$ is abelian, so is $(\mathbb{Z}_4 \times \mathbb{Z}_6)/H$ (remember, we compute in a factor group by means of representatives from the original group). In additive notation, the cosets are

$$H = (0, 0) + H, \qquad (1, 0) + H, \qquad (2, 0) + H, \qquad (3, 0) + H.$$

Since we can compute by choosing the representatives (0, 0), (1, 0), (2, 0), and (3, 0), it is clear that $(\mathbb{Z}_4 \times \mathbb{Z}_6)/H$ is isomorphic to $\mathbb{Z}_4$. Note that this is what we would expect, since in a factor group modulo H, everything in H becomes the identity element; that is, we are essentially setting everything in H equal to zero. Thus the whole second factor $\mathbb{Z}_6$ of $\mathbb{Z}_4 \times \mathbb{Z}_6$ is collapsed, leaving just the first factor $\mathbb{Z}_4$. ▲

Example 13.7 is a special case of a general theorem that we now state and prove. We should acquire an intuitive feeling for this theorem in terms of *collapsing one of the factors to the identity element*.

13.8 Theorem Let $G = H \times K$ be the direct product of groups H and K. Then $\bar{H} = \{(h, e) \mid h \in H\}$ is a normal subgroup of G. Also $G/\bar{H}$ is isomorphic to K in a natural way. Similarly, $G/\bar{K} \simeq H$ in a natural way.

Proof Consider the homomorphism $\pi_2 : H \times K \to K$, where $\pi_2(h, k) = k$. Because $\text{Ker}(\pi_2) = \bar{H}$, we see that $\bar{H}$ is a normal subgroup of $H \times K$. Because π_2 is onto K, Theorem 12.14 tells us that $(H \times K)/\bar{H} \simeq K$. ◆

We continue with additional computations of abelian factor groups. To illustrate how easy it is to compute in a factor group if we can compute in the whole group, we prove the following theorem.

13.9 Theorem If G is a cyclic group and N is a subgroup of G, then G/N is cyclic.

Proof Let G be a cyclic group, so $\langle a\rangle = G$ for some $a \in G$. Let N be any subgroup of G. Since G is abelian, N is a normal subgroup of G. We compute the cyclic subgroup of G/N generated by aN.

$$\langle aN\rangle = \{(aN)^n \mid n \in \mathbb{Z}\} = \{a^n N \mid n \in \mathbb{Z}\}$$

Since $\{a^n \mid n \in \mathbb{Z}\} = G$,

$$\{a^n N \mid n \in \mathbb{Z}\} = \{gN \mid g \in G\}.$$

So $\langle aN\rangle$ contains every coset of G and we see that G/N is cyclic with generator $\langle aN\rangle$. ◆

13.10 Example Let us compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0,2)\rangle$. Now (0, 2) generates the subgroup

$$H = \{(0,0),(0,2),(0,4)\}$$

of $\mathbb{Z}_4 \times \mathbb{Z}_6$ of order 3. Here the first factor $\mathbb{Z}_4$ of $\mathbb{Z}_4 \times \mathbb{Z}_6$ is left alone. The $\mathbb{Z}_6$ factor, on the other hand, is essentially collapsed by a subgroup of order 3, giving a factor group in the second factor of order 2 that must be isomorphic to $\mathbb{Z}_2$. Thus $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0,2)\rangle$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.

We can verify that $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0,2)\rangle$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ by using Theorem 12.14. We need a homomorphism $\phi : \mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_4 \times \mathbb{Z}_2$ that is onto, with kernel $\langle(0,2)\rangle$. Defining ϕ by $\phi(a,b) = (0,r)$ where r is the remainder when b is divided by 2 does the trick. ▲

13.11 Example Let us compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2,3)\rangle$. *Be careful*! There is a great temptation to say that we are setting the 2 of $\mathbb{Z}_4$ and the 3 of $\mathbb{Z}_6$ both equal to zero, so that $\mathbb{Z}_4$ is collapsed to a factor group isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_6$ to one isomorphic to $\mathbb{Z}_3$, giving a total factor group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$. *This is wrong*! Note that

$$H = \langle(2,3)\rangle = \{(0,0),(2,3)\}$$

is of order 2, so $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2,3)\rangle$ has order 12, not 6. Setting (2, 3) equal to zero does not make (2, 0) and (0, 3) equal to zero individually, so the factors do not collapse separately.

The possible abelian groups of order 12 are $\mathbb{Z}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, and we must decide to which one our factor group is isomorphic. These two groups are most easily distinguished in that $\mathbb{Z}_4 \times \mathbb{Z}_3$ has an element of order 4, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ does not. We claim that the coset $(1,0) + H$ is of order 4 in the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/H$. To find the smallest power of a coset giving the identity in a factor group modulo H, we must, by choosing representatives, find the smallest power of a representative that is in the subgroup H. Now,

$$4(1,0) = (1,0) + (1,0) + (1,0) + (1,0) = (0,0)$$

is the first time that $(1,0)$ added to itself gives an element of H. Thus $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2,3)\rangle$ has an element of order 4 and is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_3$ or $\mathbb{Z}_{12}$.

We can use Theorem 12.14 to verify that $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2,3)\rangle$ is isomorphic to $\mathbb{Z}_{12}$, although it is a little challenging to see what the homomorphism $\phi : \mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_{12}$ should be. We define $\phi : \mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_{12}$ by setting $\phi(a,b) = 3a +_{12} (12 - 2b)$. Here we interpret $3a$ and $2b$ as integer multiplication, so $0 \le 3a < 12$ and $0 \le 2b < 12$. The map ϕ is a homomorphism, but this takes some checking, which we leave to the reader. Also, $\mathrm{Ker}(\phi) = \{(a,b) \in \mathbb{Z}_4 \times \mathbb{Z}_6 \mid 3a = 2b\} = \{(0,0),(2,3)\} = \langle(2,3)\rangle$. We also see that $\phi(1,1) = 1$, which implies that ϕ maps onto $\mathbb{Z}_{12}$. By the Fundamental Homomorphism Theorem, $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2,3)\rangle$ is isomorphic to $\mathbb{Z}_{12}$. ▲