where $a_i \in R$, we get ourselves into a bit of trouble. For surely $0 + a_1x$ and $0 + a_1x + 0x^2$ are different as formal sums, but we want to regard them as the same polynomial. A practical solution to this problem is to define a polynomial as an *infinite formal sum*

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n + \cdots ,$$

where $a_i = 0$ for all but a finite number of values of i. Now there is no problem of having more than one finite formal sum represent what we wish to consider a single polynomial.

27.1 Definition Let R be a ring. A **polynomial** $f(x)$ **with coefficients in** R is an infinite formal sum

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n + \cdots ,$$

where $a_i \in R$ and for all but a finite number of values of i, $a_i = 0$. The a_i are **coefficients of** $f(x)$. If for some $i \geq 0$ it is true that $a_i \neq 0$, the largest such value of i is the **degree of** $f(x)$. If all $a_i = 0$, then the degree of $f(x)$ is undefined.[†] ■

To simplify working with polynomials, let us agree that if $f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots$ has $a_i = 0$ for $i > n$, then we may denote $f(x)$ by $a_0 + a_1x + \cdots + a_nx^n$. Also, if R has unity $1 \neq 0$, we will write a term $1x^k$ in such a sum as x^k. For example, in $\mathbb{Z}[x]$, we will write the polynomial $2 + 1x$ as $2 + x$. Finally, we shall agree that we may omit altogether from the formal sum any term $0x^i$, or a_0 if $a_0 = 0$ but not all $a_i = 0$. Thus 0, 2, x, and $2 + x^2$ are polynomials with coefficients in $\mathbb{Z}$. An element of R is a **constant polynomial.**

Addition and multiplication of polynomials with coefficients in a ring R are defined in a way familiar to us. If

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

and

$$g(x) = b_0 + b_1 x + \cdots + b_n x^n + \cdots ,$$

then for polynomial addition, we have

$$f(x) + g(x) = c_0 + c_1 x + \cdots + c_n x^n + \cdots \text{ where } c_n = a_n + b_n,$$

and for polynomial multiplication, we have

$$f(x)g(x) = d_0 + d_1 x + \cdots + d_n x^n + \cdots \text{ where } d_n = \sum_{i=0}^{n} a_i b_{n-i}$$

Observe that both c_i and d_i are 0 for all but a finite number of values of i, so these definitions make sense. Note that $\sum_{i=0}^{n} a_i b_{n-i}$ need not equal $\sum_{i=0}^{n} b_i a_{n-i}$ if R is not commutative. With these definitions of addition and multiplication, we have the following theorem.

27.2 Theorem The set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. If R is commutative, then so is $R[x]$, and if R has unity $1 \neq 0$, then 1 is also unity for $R[x]$.

Proof That $\langle R[x], +\rangle$ is an abelian group is apparent. The associative law for multiplication and the distributive laws are straightforward, but slightly cumbersome, computations. We illustrate by proving the associative law.

[†] The degree of the zero polynomial is sometimes defined to be -1, which is the first integer less than 0, or defined to be $-\infty$ so that the degree of $f(x)g(x)$ will be the sum of the degrees of $f(x)$ and $g(x)$ if one of them is zero.

Applying ring axioms to $a_i, b_j, c_k \in R$, we obtain

$$\begin{aligned}
\left[\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)\right]\left(\sum_{k=0}^{\infty} c_k x^k\right) &= \left[\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n\right]\left(\sum_{k=0}^{\infty} c_k x^k\right) \\
&= \sum_{s=0}^{\infty}\left[\sum_{n=0}^{s}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)c_{s-n}\right]x^s \\
&= \sum_{s=0}^{\infty}\left(\sum_{i+j+k=s} a_i b_j c_k\right)x^s \\
&= \sum_{s=0}^{\infty}\left[\sum_{m=0}^{s} a_{s-m}\left(\sum_{j=0}^{m} b_j c_{m-j}\right)\right]x^s \\
&= \left(\sum_{i=0}^{\infty} a_i x^i\right)\left[\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m} b_j c_{m-j}\right)x^m\right] \\
&= \left(\sum_{i=0}^{\infty} a_i x^i\right)\left[\left(\sum_{j=0}^{\infty} b_j x^j\right)\left(\sum_{k=0}^{\infty} c_k x^k\right)\right].
\end{aligned}$$

Whew! In this computation, the fourth expression, having just two summation signs, should be viewed as the value of the triple product $f(x)g(x)h(x)$ of these polynomials under this associative multiplication. (In a similar fashion, we view $f(g(h(x)))$ as the value of the associative composition $(f \circ g \circ h)(x)$ of three functions f, g, and h.)

The distributive laws are similarly proved. (See Exercise 26.)

The comments prior to the statement of the theorem show that $R[x]$ is a commutative ring if R is commutative, and a unity $1 \neq 0$ in R is also unity for $R[x]$, in view of the definition of multiplication in $R[x]$. ◆

Thus $\mathbb{Z}[x]$ is the ring of polynomials in the indeterminate x with integral coefficients, $\mathbb{Q}[x]$ the ring of polynomials in x with rational coefficients, and so on.

27.3 Example In $\mathbb{Z}_2[x]$, we have

$$(x+1)^2 = (x+1)(x+1) = x^2 + (1+1)x + 1 = x^2 + 1.$$

Still working in $\mathbb{Z}_2[x]$, we obtain

$$(x+1) + (x+1) = (1+1)x + (1+1) = 0x + 0 = 0. \qquad ▲$$

If R is a ring and x and y are two indeterminates, then we can form the ring $(R[x])[y]$, that is, the ring of polynomials in y with coefficients that are polynomials in x. Every polynomial in y with coefficients that are polynomials in x can be rewritten in a natural way as a polynomial in x with coefficients that are polynomials in y as illustrated by Exercise 20. This indicates that $(R[x])[y]$ is naturally isomorphic to $(R[y])[x]$, although a careful proof is tedious. We shall identify these rings by means of this natural isomorphism, and shall consider this ring $R[x, y]$ the **ring of polynomials in two indeterminates** x **and** y **with coefficients in** R. The **ring** $R[x_1, \cdots, x_n]$ **of polynomials in the** n **indeterminates** x_i **with coefficients in** R is similarly defined.

We leave as Exercise 24 the proof that if D is an integral domain, then so is $D[x]$. In particular, if F is a field, then $F[x]$ is an integral domain. Note that $F[x]$ is not a field, for x is not a unit in $F[x]$. That is, there is no polynomial $f(x) \in F[x]$ such that $xf(x) = 1$. By Theorem 26.6, one can construct the field of quotients $F(x)$ of $F[x]$. Any element in $F(x)$ can be represented as a quotient $f(x)/g(x)$ of two polynomials in $F[x]$ with