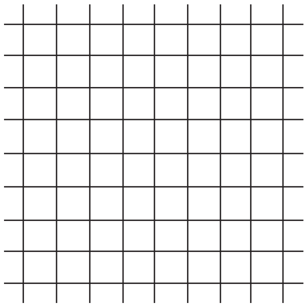
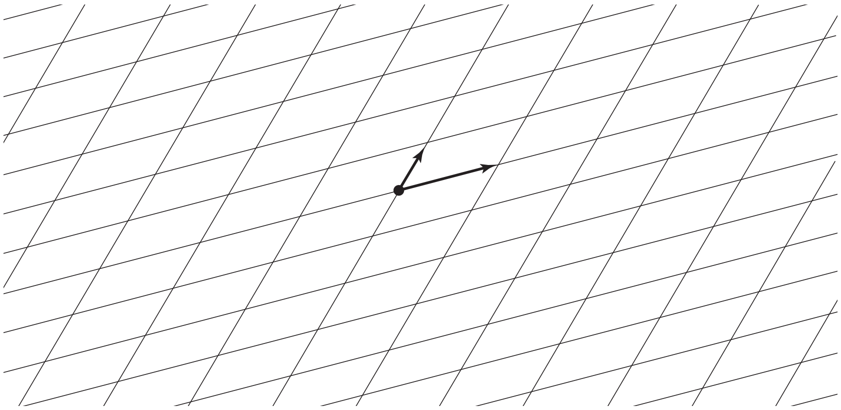


11.6 Figure



11.7 Figure

in space is more complicated; it can be shown that there are 230 three-dimensional crystallographic groups. The final exercise we give involves rotations in space.

M. C. Escher (1898–1973) was an artist whose work included plane-filling patterns. In the exercises you are asked to analyze two of his works of this type.

EXERCISES 11

1. This exercise shows that the group of symmetries of a certain type of geometric figure may depend on the dimension of the space in which we consider the figure to lie.

a. Describe all symmetries of a point in the real line $\mathbb{R}$; that is, describe all isometries of $\mathbb{R}$ that leave one point fixed.
b. Describe all symmetries (translations, reflections, etc.) of a point in the plane $\mathbb{R}^2$.
c. Describe all symmetries of a line segment in $\mathbb{R}$.
d. Describe all symmetries of a line segment in $\mathbb{R}^2$.
e. Describe some symmetries of a line segment in $\mathbb{R}^3$.

2. Let P stand for an orientation preserving plane isometry and R for an orientation reversing one. Fill in the table with P or R to denote the orientation preserving or reversing property of a product.

	P	R
P		
R		

3. Fill in the table to give *all* possible types of plane isometries given by a product of two types as indicated in Tables 11.1 through 11.4. For example, a product of two rotations may be a rotation, or it may be another type. Fill in the box corresponding to $\rho\rho$ with both letters. Use your answer to Exercise 2 to eliminate some types. Eliminate the identity from consideration.

	τ	ρ	μ	γ
τ				
ρ				
μ				
γ				

4. Draw a plane figure that has a one-element group as its group of symmetries in $\mathbb{R}^2$.

5. Draw a plane figure that has a two-element group as its group of symmetries in $\mathbb{R}^2$.

6. Draw a plane figure that has a three-element group as its group of symmetries in $\mathbb{R}^2$.

7. Draw a plane figure that has a four-element group isomorphic to $\mathbb{Z}_4$ as its group of symmetries in $\mathbb{R}^2$.

8. Draw a plane figure that has a four-element group isomorphic to the Klein 4-group V as its group of symmetries in $\mathbb{R}^2$.

9. For each of the four types of plane isometries (other than the identity), give the possibilities for the order of an isometry of that type in the group of plane isometries.

10. A plane isometry ϕ has a *fixed point* if there exists a point P in the plane such that $\phi(P) = P$. Which of the four types of plane isometries (other than the identity) can have a fixed point?

11. Referring to Exercise 10, which types of plane isometries, if any, have exactly one fixed point?

12. Referring to Exercise 10, which types of plane isometries, if any, have exactly two fixed points?

13. Referring to Exercise 10, which types of plane isometries, if any, have an infinite number of fixed points?

14. Argue geometrically that a plane isometry that leaves three noncolinear points fixed must be the identity map.

15. Using Exercise 14, show algebraically that if two plane isometries ϕ and ψ agree on three noncolinear points, that is, if $\phi(P_i) = \psi(P_i)$ for noncolinear points P_1, P_2, and P_3, then ϕ and ψ are the same map.

16. Do the rotations, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

17. Do the translations, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

18. Do the rotations about one particular point P, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

19. Does the reflection across one particular line L, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

20. Do the glide reflections, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

21. Which of the four types of plane isometries can be elements of a *finite* subgroup of the group of plane isometries?

22. Completing a detail of the proof of Theorem 11.5, let G be a finite group of plane isometries. Show that the rotations in G, together with the identity isometry, form a subgroup H of G, and that either $H = G$ or $|G| = 2|H|$. [*Hint:* Use the same method that we used to show that $|S_n| = 2|A_n|$.]

23. Completing a detail in the proof of Theorem 11.5, let G be a finite group consisting of the identity isometry and rotations about one point P in the plane. Show that G is cyclic, generated by the rotation in G that turns the plane counterclockwise about P through the smallest angle $\theta > 0$. [*Hint:* Follow the idea of the proof that a subgroup of a cyclic group is cyclic.]

Exercises 24 through 30 illustrate the seven different types of friezes when they are classified according to their symmetries. Imagine the figure shown to be continued infinitely to the right and left. The symmetry group of a frieze always contains translations. For each of these exercises answer these questions about the symmetry group of the frieze.

a. Does the group contain a rotation?

b. Does the group contain a reflection across a horizontal line?

c. Does the group contain a reflection across a vertical line?

d. Does the group contain a nontrivial glide reflection?

e. To which of the possible groups $\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}_2$, or $D_\infty \times \mathbb{Z}_2$ do you think the symmetry group of the frieze is isomorphic?

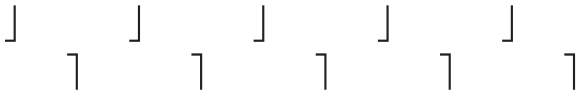
24. **F F F F F F F F F F F F F F F**

25. **T T T T T T T T T T**

26. **E E E E E E E E E E E E**

27. **Z Z Z Z Z Z Z Z Z Z Z Z**

28. **H H H H H H H H H H**

29. 

30. 

Exercises 31 through 37 describe a pattern to be used to fill the plane by translation in the two directions given by the specified vectors. Answer these questions in each case.

a. Does the symmetry group contain any rotations? If so, through what possible angles θ where $0 < \theta \leq 180°$?

b. Does the symmetry group contain any reflections?

c. Does the symmetry group contain any nontrivial glide reflections?

31. A square with horizontal and vertical edges using translation directions given by vectors (1, 0) and (0, 1).

32. A square as in Exercise 31 using translation directions given by vectors (1, 1/2) and (0, 1).

33. A square as in Exercise 31 with the letter L at its center using translation directions given by vectors (1, 0) and (0, 1).

34. A square as in Exercise 31 with the letter E at its center using translation directions given by vectors (1, 0) and (0, 1).

35. A square as in Exercise 31 with the letter H at its center using translation directions given by vectors (1, 0) and (0, 1).

36. A regular hexagon with a vertex at the top using translation directions given by vectors (1, 0) and $(1, \sqrt{3})$.

37. A regular hexagon with a vertex at the top containing an equilateral triangle with vertex at the top and centroid at the center of the hexagon, using translation directions given by vectors (1, 0) and $(1, \sqrt{3})$.

Exercises 38 and 39 are concerned with art works of M. C. Escher. Find images of the indicated art by searching on the internet. Neglect the shading and colors in the figures and assume the markings in each human figure, reptile, or horseman are the same, even though they may be invisible due to shading. Answer the same questions (a), (b), and (c) that were asked for Exercises 31 through 36, and also answer this part (d).

d. Assuming horizontal and vertical coordinate axes with equal scales as usual, give vectors in the two nonparallel directions of vectors that generate the translation subgroup. Do not concern yourself with the length of these vectors.

38. *The Study of Regular Division of the Plane with Horsemen.*

39. *The Study of Regular Division of the Plane with Reptiles.*

40. Let $\phi : \mathbb{R} \to U$ be given by $\phi(\theta) = \cos(\theta) + i\sin(\theta)$ and $S = \phi[\mathbb{Z}]$.

a. Show that any rotation mapping S to S is a rotation by an angle $n \in \mathbb{Z}$ where angles are measured in radians.

b. Show that reflection across the x-axis maps S to S.

c. What is the group of symmetries of S?

41. Show that the rotations of a cube in space form a group isomorphic to S_4. [*Hint:* A rotation of the cube permutes the diagonals through the center of the cube.]

PART III

Homomorphisms and Factor Groups

SECTION 12 FACTOR GROUPS

Recall from Section 10 that for some group tables we can arrange the head on top and on the left so that the elements are grouped into left cosets of a subgroup in such a way that the coset blocks form a group table. We start this section by looking more closely at why the cosets of $\{0, 3\} \leq \mathbb{Z}_6$ form a group and why the cosets of the subgroup $\{\iota, \mu\} \leq D_3$ do not. Table 12.1 is the group table for $\mathbb{Z}_6$ with the heads at the top and left sorted by cosets of $\{0, 3\}$.

12.1 Table

$+_6$	0	3	1	4	2	5
0	0	3	1	4	2	5
3	3	0	4	1	5	2
1	1	4	2	5	3	0
4	4	1	5	2	0	3
2	2	5	3	0	4	1
5	5	2	0	3	1	4

According to Table 12.1 the coset $\{1, 4\}$ plus the coset $\{2, 5\}$ is the coset $\{0, 3\}$. This means that if we add either 1 or 4 to either 2 or 5 in $\mathbb{Z}_6$, we should get either 0 or 3. This is easily checked by adding the four possibilities.

$$\begin{aligned} 1 +_6 2 &= 3 \\ 1 +_6 5 &= 0 \\ 4 +_6 2 &= 0 \\ 4 +_6 5 &= 3 \end{aligned}$$

We observe that if we wish to break up a group into its left cosets so the group table shows an operation on the left cosets, we need to be sure that if a_1, a_2 are in the same