in $\bar{F}[x]$, with $\alpha_i \in \overline{F}$, and hence each α_i is algebraic over F. Then by Theorem 40.11 the field $F(\alpha_0, \cdots, \alpha_n)$ is a finite extension of F, and since β is algebraic over $F(\alpha_0, \cdots, \alpha_n)$, we also see that $F(\alpha_0, \cdots, \alpha_n, \beta)$ is a finite extension over $F(\alpha_0, \cdots, \alpha_n)$. Theorem 40.4 then shows that $F(\alpha_0, \cdots, \alpha_n, \beta)$ is a finite extension of F, so by Theorem 40.3, β is algebraic over F. Hence $\overline{F}(\omega) \in S$ and $\overline{F} < \overline{F}(\omega)$, which contradicts the choice of $\overline{F}$ as maximal in S. Thus $f(x)$ must have had a zero in $\overline{F}$, so $\overline{F}$ is algebraically closed. ◆

The mechanics of the preceding proof are routine to the professional mathematician. Since it may be the first proof that we have ever seen using Zorn's lemma, we wrote the proof out in detail.

EXERCISES 40

Computations

In Exercises 1 through 13, find the degree and a basis for the given field extension. Be prepared to justify your answers.

1. $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$
2. $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$
3. $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18})$ over $\mathbb{Q}$
4. $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$ over $\mathbb{Q}$
5. $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}$
6. $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}$
7. $\mathbb{Q}(\sqrt{2}\sqrt{3})$ over $\mathbb{Q}$
8. $\mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$ over $\mathbb{Q}$
9. $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}, \sqrt[3]{24})$ over $\mathbb{Q}$
10. $\mathbb{Q}(\sqrt{2}, \sqrt{6})$ over $\mathbb{Q}(\sqrt{3})$
11. $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}(\sqrt{3})$
12. $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{2} + \sqrt{3})$
13. $\mathbb{Q}(\sqrt{2}, \sqrt{6} + \sqrt{10})$ over $\mathbb{Q}(\sqrt{3} + \sqrt{5})$

Concepts

In Exercises 14 through 17, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

14. An *algebraic extension* of a field F is a field $F(\alpha_1, \alpha_2, \cdots, \alpha_n)$ where each α_i is a zero of some polynomial in $F[x]$.
15. A *finite extension field* of a field F is one that can be obtained by adjoining a finite number of elements to F.
16. The *algebraic closure $\bar{F}_E$ of a field F in an extension field E of F* is the field consisting of all elements of E that are algebraic over F.
17. A field F is *algebraically closed* if and only if every polynomial has a zero in F.
18. Show by an example that for a proper extension field E of a field F, the algebraic closure of F in E need not be algebraically closed.
19. Determine whether each of the following is true or false.
    - **a.** If a field E is a finite extension of a field F, then E is a finite field.
    - **b.** Every finite extension of a field is an algebraic extension.
    - **c.** Every algebraic extension of a field is a finite extension.
    - **d.** The top field of a finite tower of finite extensions of fields is a finite extension of the bottom field.
    - **e.** $\mathbb{Q}$ is its own algebraic closure in $\mathbb{R}$, that is, $\mathbb{Q}$ is **algebraically closed in** $\mathbb{R}$.
    - **f.** $\mathbb{C}$ is algebraically closed in $\mathbb{C}(x)$, where x is an indeterminate.
    - **g.** $\mathbb{C}(x)$ is algebraically closed, where x is an indeterminate.
    - **h.** The field $\mathbb{C}(x)$ has no algebraic closure, since $\mathbb{C}$ already contains all algebraic numbers.
    - **i.** An algebraically closed field must be of characteristic 0.
    - **j.** If E is an algebraically closed extension field of F, then E is an algebraic extension of F.

Proof Synopsis

20. Give a one-sentence synopsis of the proof of Theorem 40.3.

21. Give a one- or two-sentence synopsis of the proof of Theorem 40.4.

Theory

22. Let $(a + bi) \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $b \neq 0$. Show that $\mathbb{C} = \mathbb{R}(a + bi)$.

23. Show that if E is a finite extension of a field F and $[E : F]$ is a prime number, then E is a simple extension of F and, indeed, $E = F(\alpha)$ for every $\alpha \in E$ not in F.

24. Prove that $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt[3]{2})$.

25. What degree field extensions can we obtain by successively adjoining to a field F a square root of an element of F not a square in F, then a square root of some nonsquare in this new field, and so on? Argue from this that a zero of $x^{14} - 3x^2 + 12$ over $\mathbb{Q}$ can never be expressed as a rational function of square roots of rational functions of square roots, and so on, of elements of $\mathbb{Q}$.

26. Let E be a finite extension field of F. Let D be an integral domain such that $F \subseteq D \subseteq E$. Show that D is a field.

27. Prove in detail that $\mathbb{Q}(\sqrt{3} + \sqrt{7}) = \mathbb{Q}(\sqrt{3}, \sqrt{7})$.

28. Generalizing Exercise 27, show that if $\sqrt{a} + \sqrt{b} \neq 0$, then $\mathbb{Q}(\sqrt{a} + \sqrt{b}) = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ for all a and b in $\mathbb{Q}$. [*Hint:* Compute $(a - b)/(\sqrt{a} + \sqrt{b})$.]

29. Let E be a finite extension of a field F, and let $p(x) \in F[x]$ be irreducible over F and have degree that is not a divisor of $[E : F]$. Show that $p(x)$ has no zeros in E.

30. Let E be an extension field of F. Let $\alpha \in E$ be algebraic of odd degree over F. Show that α^2 is algebraic of odd degree over F, and $F(\alpha) = F(\alpha^2)$.

31. Show that if F, E, and K are fields with $F \leq E \leq K$, then K is algebraic over F if and only if E is algebraic over F, and K is algebraic over E. (You must *not* assume the extensions are finite.)

32. Let E be an extension field of a field F. Prove that every $\alpha \in E$ that is not in the algebraic closure $\bar{F}_E$ of F in E is transcendental over $\bar{F}_E$.

33. Let E be an algebraically closed extension field of a field F. Show that the algebraic closure $\bar{F}_E$ of F in E is algebraically closed. (Applying this exercise to $\mathbb{C}$ and $\mathbb{Q}$, we see that the field of all algebraic numbers is an algebraically closed field.)

34. Show that if E is an algebraic extension of a field F and contains all zeros in $\overline{F}$ of every $f(x) \in F[x]$, then E is an algebraically closed field.

35. Show that no finite field of odd characteristic is algebraically closed. (Actually, no finite field of characteristic 2 is algebraically closed either.) [*Hint*: By counting, show that for such a finite field F, some polynomial $x^2 - a$, for some $a \in F$, has no zero in F. See Exercise 32, Section 39.]

36. Prove that, as asserted in the text, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is not a finite extension of $\mathbb{Q}$.

37. Argue that every finite extension field of $\mathbb{R}$ is either $\mathbb{R}$ itself or is isomorphic to $\mathbb{C}$.

38. Use Zorn's lemma to show that every proper ideal of a ring R with unity is contained in some maximal ideal.

SECTION 41 †Geometric Constructions

In this section we digress briefly to give an application demonstrating the power of Theorem 40.4. For a more detailed study of geometric constructions, you are referred to Courant and Robbins [44, Chapter III].

In Euclid's *Elements*, geometry is approached from an axiomatic point of view. The first three axioms state that a line segment can be drawn between any two given points,

† This section is used only briefly in Section 48.

a line segment can be extended indefinitely to form a line, and a circle can be drawn centered at any given point with any given radius. A straightedge and a compass are objects that can accomplish the tasks set out in these axioms. A natural question asked by the ancient Greek geometers was exactly which points and configurations can be constructed using only a compass and a straightedge. They found many constructions that are well known today including bisecting angles, bisecting line segments, and trisecting line segments. Trisecting an arbitrary angle eluded mathematicians for well over 2000 years before field theory was developed and it was proven that there are some angles that are impossible to trisect using only a straightedge and compass. We shall discuss the impossibility of trisecting an angle and other classical questions.

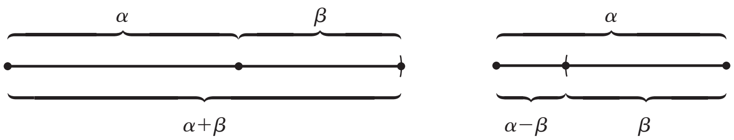
Constructible Numbers

Let us imagine that we are given only a single line segment that we shall define to be *one unit* in length. A real number α is **constructible** if we can construct a line segment of length $|\alpha|$ in a finite number of steps from this given segment of unit length by using a straightedge and a compass.

The rules of the game are pretty strict. We suppose that we are given just two points at the moment, the endpoints of our unit line segment, let us suppose that they correspond to the points (0, 0) and (1, 0) in the Euclidean plane. We are allowed to draw a line only with our straightedge through two points that we have already located. Thus we can start by using the straightedge and drawing the line through (0, 0) and (1, 0). We are allowed to open our compass only to a distance between points we have already found. Let us open our compass to the distance between (0, 0) and (1, 0). We can then place the point of the compass at (1, 0) and draw a circle of radius 1, which passes through the point (2, 0). Thus we now have located a third point, (2, 0). Continuing in this way, we can locate points (3, 0), (4, 0), $(-1, 0)$, $(-2, 0)$, and so on. Now open the compass the distance from (0, 0) to (0, 2), put the point at (1, 0), and draw a circle of radius 2. Do the same with the point at $(-1, 0)$. We have now found two new points, where these circles intersect, and we can put our straightedge on them to draw what we think of as the y-axis. Then opening our compass to the distance from (0, 0) to (1, 0), we draw a circle with center at (0, 0) and locate the point (0, 1) where the circle intersects the y-axis. Continuing in this fashion, we can locate all points (x, y) with integer coordinates in any rectangle containing the point (0, 0). Without going into more detail, it can be shown that it is possible, among other things, to erect a perpendicular to a given line at a known point on the line, and find a line passing through a known point and parallel to a given line. Our first result is the following theorem.

41.1 Theorem If α and β are constructible real numbers, then so are $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, and α/β, if $\beta \neq 0$.

Proof We are given that α and β are constructible, so there are line segments of lengths $|\alpha|$ and $|\beta|$ available to us. For $\alpha, \beta > 0$, extend a line segment of length α with the straightedge. Start at one end of the original segment of length α, and lay off on the extension the length β with the compass. This constructs a line segment of length $\alpha + \beta$; $\alpha - \beta$ is similiarly constructible (see Fig. 41.2). If α and β are not both positive, an obvious



41.2 Figure