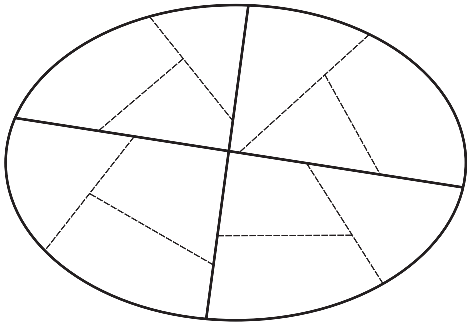

16.9 Figure

16.10 Example Let $G = \mathbb{Z}_8$, $H = \langle 2 \rangle = \{0, 2, 4, 6\} < G$, and $K = \langle 4 \rangle = \{0, 4\} < H$. We list the elements of each of the factor groups used in Theorem 16.8.

$$G/K = \{\{0, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}\}$$
$$G/H = \{\{0, 2, 4, 6\}, \{1, 3, 5, 7\}\}$$
$$H/K = \{\{0, 4\}, \{2, 6\}\}$$

Before listing the elements of $(G/K)/(H/K)$ we note that any element of this group would be a coset of the subgroup $H/K < G/K$. So each element is a set whose elements are sets.

$$(G/K)/(H/K) = \{\{\{0, 4\}, \{2, 6\}\}, \{\{1, 5\}, \{3, 7\}\}\}$$

Comparing G/H and $(G/K)/(H/K)$, we see that each element in G/H is the union of two cosets of K in G/K. Also, the isomorphism in Theorem 16.8 is the map $\phi : (G/K)/(H/K) \to G/H$ defined by

$$\phi(\{\{0, 4\}, \{2, 6\}\}) = \{0, 4, 2, 6\} \quad \text{and}$$
$$\phi(\{\{1, 5\}, \{3, 7\}\}) = \{1, 5, 3, 7\}.$$

So as illustrated in Figure 16.9, collapsing H to form G/H can be accomplished by first collapsing K and then collapsing H/K. ▲

EXERCISES 16

Computations

In using the three isomorphism theorems, it is often necessary to know the actual correspondence given by the isomorphism and not just the fact that the groups are isomorphic. The first six exercises give us training for this.

1. Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ be the homomorphism such that $\phi(1) = 2$.

- **a.** Find the kernel K of ϕ.
- **b.** List the cosets in $\mathbb{Z}_{12}/K$, showing the elements in each coset.
- **c.** Give the correspondence between $\mathbb{Z}_{12}/K$ and $\mathbb{Z}_3$ given by the map μ described in Theorem 16.2.

2. Let $\phi : \mathbb{Z}_{18} \to \mathbb{Z}_{12}$ be the homomorphism where $\phi(1) = 10$.

- **a.** Find the kernel K of ϕ.
- **b.** List the cosets in $\mathbb{Z}_{18}/K$, showing the elements in each coset.
- **c.** Find the group $\phi[\mathbb{Z}_{18}]$.
- **d.** Give the correspondence between $\mathbb{Z}_{18}/K$ and $\phi[\mathbb{Z}_{18}]$ given by the map μ described in Theorem 16.2.

3. In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.

a. List the elements in HN (which we might write $H + N$ for these additive groups) and in $H \cap N$.
b. List the cosets in HN/N, showing the elements in each coset.
c. List the cosets in $H/(H \cap N)$, showing the elements in each coset.
d. Give the correspondence between $H/(H \cap N)$ and HN/N described in the proof of Theorem 16.5.

4. Repeat Exercise 3 for the dihedral group D_6 with $N = \{\iota, \mu, \rho^2, \mu\rho^2, \rho^4, \mu\rho^4\}$ and $H = \langle \rho \rangle$.

5. In the group $G = \mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $K = \langle 8 \rangle$.

a. List the cosets in G/H, showing the elements in each coset.
b. List the cosets in G/K, showing the elements in each coset.
c. List the cosets in H/K, showing the elements in each coset.
d. List the cosets in $(G/K)/(H/K)$, showing the elements in each coset.
e. Give the correspondence between $(G/K)/(H/K)$ and G/H described in the proof of Theorem 16.8.

6. Repeat Exercise 5 for the dihedral group $G = D_8$, $H = \langle \rho^2 \rangle = \{\iota, \rho^2, \rho^4, \rho^6\}$, and $K = \langle \rho^4 \rangle = \{\iota, \rho^4\}$.

Theory

7. Show directly from the definition of a normal subgroup that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H.

8. Let H, K, and L be normal subgroups of G with $H < K < L$. Let $A = G/H$, $B = K/H$, and $C = L/H$.

a. Show that B and C are normal subgroups of A, and $B < C$.
b. To what factor group of G is $(A/B)/(C/B)$ isomorphic?

9. Let K and L be normal subgroups of G with $K \vee L = G$, and $K \cap L = \{e\}$. Show that $G/K \simeq L$ and $G/L \simeq K$.

10. Use one of the Isomorphism Theorems to prove that if $a, b \in \mathbb{Z}^+$ and a divides b, then $|a\mathbb{Z}/b\mathbb{Z}| = b/a$.

11. Let G be a group with subgroups $H^* \leq H \leq G$ and $K^* \leq K \leq G$. Prove that the sets $H^*(H \cap K^*) \cap (H \cap K)$ and $(H^* \cap K)(H \cap K^*)$ are equal.

SECTION 17 SYLOW THEOREMS

The Fundamental Theorem for Finitely Generated Abelian Groups (Theorems 9.12 and 9.14) give us complete information about all finite abelian groups. The study of finite nonabelian groups is much more complicated. The Sylow theorems give us some important information about them.

The Theorem of Lagrange says that if H is a subgroup of a finite group G, then the order of H divides the order of G. The Fundamental Theorem for Finitely Generated Abelian Groups implies that if k divides the order of a finite abelian group G, then G has a subgroup of order k. The situation is different for nonabelian groups. Example 13.6 shows that although A_4 has 12 elements, it has no subgroup of order 6. Furthermore, for $n \geq 5$, A_n can have no subgroup of index 2 since A_n is simple and any subgroup of index 2 is a normal subgroup. On the positive side, Cauchy's theorem (14.20) says that if the prime number p divides the order of a group G, then G has a subgroup of order p. The Sylow theorems generalize Cauchy's theorem to show that if p^n divides the order of G, then G has a subgroup of order p^n as long as p is a prime number. Furthermore, Sylow's theorems gives us information about relationships between these subgroups of G. As we shall see, this information is very useful in the study of finite nonabelian groups.

Proofs of the Sylow theorems give us another application of action of a group on a set described in Section 14. This time, the set itself is formed from the group; in some instances the set is the group itself, sometimes it is a collection of cosets of a subgroup, and sometimes it is a collection of subgroups.

The Sylow Theorems

Let G be a group, and let $\mathscr{S}$ be the collection of all subgroups of G. We make $\mathscr{S}$ into a G-set by letting G act on $\mathscr{S}$ by conjugation. That is, if $H \in \mathscr{S}$ so $H \leq G$ and $g \in G$, then g acting on H yields the conjugate subgroup gHg^{-1}. (To avoid confusion, we will never write this action as gH.) By Theorem 14.13 $G_H = \{g \in G \mid gHg^{-1} = H\}$ is a subgroup of G, which is called an isotropy subgroup. In Exercise 14 you will be asked to show directly that G_H is a subgroup of G. Since G_H consists of *all* elements of G that leave H invariant under conjugation, G_H is the largest subgroup of G having H as a normal subgroup.

17.1 Definition The subgroup G_H just discussed is the **normalizer of** H **in** G and will be denoted $N[H]$ from now on. ■

In the proof of the lemma that follows, we will use the fact that if H is a *finite* subgroup of a group G, then $g \in N[H]$ if $ghg^{-1} \in H$ for all $h \in H$. To see this, note that if $gh_1g^{-1} = gh_2g^{-1}$, then $h_1 = h_2$ by cancellation in the group G. Thus the conjugation map $i_g : H \to H$ given by $i_g(h) = ghg^{-1}$ is one-to-one. Because $|H|$ is finite, i_g must then map H *onto* H, so $gHg^{-1} = H$ and $g \in N[H]$.

17.2 Lemma Let H be a p-subgroup of a finite group G. Then

$$(N[H] : H) \equiv (G : H) \pmod{p}.$$

Proof Let $\mathscr{L}$ be the set of left cosets of H in G, and let H act on $\mathscr{L}$ by left translation, so that $h(xH) = (hx)H$. Then $\mathscr{L}$ becomes an H-set. Note that $|\mathscr{L}| = (G : H)$.

Let us determine $\mathscr{L}_H$, that is, those left cosets that are fixed under action by all elements of H. Now $xH = h(xH)$ if and only if $H = x^{-1}hxH$, or if and only if $x^{-1}hx \in H$. Thus $xH = h(xH)$ for all $h \in H$ if and only if $x^{-1}hx = x^{-1}h(x^{-1})^{-1} \in H$ for all $h \in H$, or if and only if $x^{-1} \in N[H]$ (see the comment before the lemma), or if and only if $x \in N[H]$. Thus the left cosets in $\mathscr{L}_H$ are those contained in $N[H]$. The number of such cosets is $(N[H] : H)$, so $|\mathscr{L}_H| = (N[H] : H)$.

Since H is a p-group, it has order a power of p. Theorem 14.9 then tells us that $|\mathscr{L}| \equiv |\mathscr{L}_H| \pmod{p}$, that is, that $(G : H) \equiv (N[H] : H) \pmod{p}$. ◆

17.3 Corollary Let H be a p-subgroup of a finite group G. If p divides $(G : H)$, then $N[H] \neq H$.

Proof It follows from Lemma 17.2 that p divides $(N[H] : H)$, which must then be different from 1. Thus $H \neq N[H]$. ◆

■ Historical Note

The Sylow theorems are due to the Norwegian mathematician Peter Ludvig Mejdell Sylow (1832–1918), who published them in a brief paper in 1872. Sylow stated the theorems in terms of permutation groups (since the abstract definition of a group had not yet been given). Georg Frobenius re-proved the theorems for abstract groups in 1887, even though he noted that in fact every group can be considered as a permutation group (Cayley's theorem [Theorem 8.11]). Sylow himself immediately applied the theorems to the question of solving algebraic equations and showed that any equation whose Galois group has order a power of a prime p is solvable by radicals.

Sylow spent most of his professional life as a high school teacher in Halden, Norway, and was only appointed to a position at Christiana University in 1898. He devoted eight years of his life to the project of editing the mathematical works of his countryman Niels Henrik Abel.