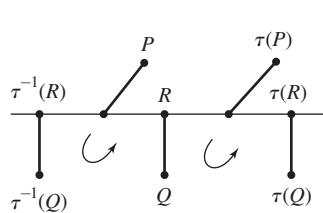
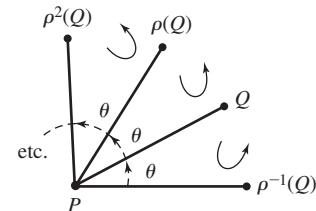
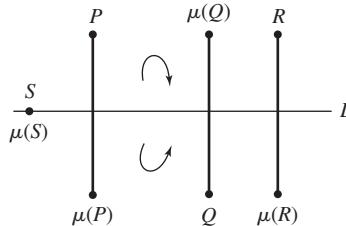
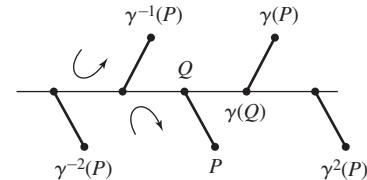


glide reflection γ : The product of a translation and a reflection across a line mapped into itself by the translation. See Fig. 11.4. (Example: $\gamma(x, y) = (x + 4, -y)$ is a glide reflection along the x -axis.)

Notice the little curved arrow that is carried into another curved arrow in each of Figs. 11.1 through 11.4. For the translation and rotation, the counterclockwise directions of the curved arrows remain the same, but for the reflection and glide reflection, the counterclockwise arrow is mapped into a clockwise arrow. We say that translations and rotations *preserve orientation*, while the reflection and glide reflection *reverse orientation*. We do not classify the identity isometry as any definite one of the four types listed; it could equally well be considered to be a translation by the zero vector or a rotation about any point through an angle of 0° . We always consider a glide reflection to be the product of a reflection and a translation that is different from the identity isometry.

11.1 Figure Translation τ .11.2 Figure Rotation ρ .11.3 Figure Reflection μ .11.4 Figure Glide reflection γ .

The theorem that follows describes the possible structures of finite subgroups of the full isometry group.

11.5 Theorem Every finite group G of isometries of the plane is isomorphic to either the Klein 4-group, \mathbb{Z}_n for $n \geq 1$, or D_n for some $n \geq 3$.

Proof (Outline) First we show that there is a point in the plane that is fixed by every element of G . We let $G = \{\phi_1, \phi_2, \phi_3, \dots, \phi_m\}$ and $(x_i, y_i) = \phi_i(0, 0)$. Then the point

$$P = (\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3 + \dots + x_m}{m}, \frac{y_1 + y_2 + y_3 + \dots + y_m}{m} \right)$$

is the center of mass of the set $S = \{(x_i, y_i) \mid 1 \leq i \leq m\}$ where each point is weighted by the number of ϕ_i that map $(0, 0)$ to that point. It is easy to see that the isometries in G permute the points in S since for each i and j , $\phi_i \circ \phi_j = \phi_k$ for some k . Thus $\phi_i(x_j, y_j) = (x_k, y_k)$. This implies the center of mass of $\phi(S)$ is the same as the center of mass of S . It can be shown that given the distances from the center of mass to the points of the set S , the center of mass is the only point having these distances from the points of S . This says that (\bar{x}, \bar{y}) is fixed by every isometry in G .

The orientation preserving isometries of G form a subgroup H of G which is either all of G or else of order $m/2$. You are asked to prove this in Exercise 22. Of course H consists of the identity and possibly rotations about the point (\bar{x}, \bar{y}) . If H has only one element, then G has one or two elements and is therefore isomorphic with \mathbb{Z}_1 or \mathbb{Z}_2 . If H has two elements, then G has two or four elements and is therefore isomorphic with either the Klein 4-group, \mathbb{Z}_4 , or \mathbb{Z}_2 . So we can assume that H has at least three elements.

If we choose a rotation ρ in H that rotates through the smallest positive angle θ among all the elements of H , ρ generates H . The proof of this fact is similar to the proof that a subgroup of a cyclic group is cyclic and you are asked to provide the details of the proof in Exercise 23. If $G = H$, then G is isomorphic with \mathbb{Z}_m . So we now assume that G contains a reflection, say μ . Then the coset μH contains only isometries of G that reverse orientation. Each coset H and μH contains half the elements of G , so $G = H \cup \mu H$.

Consider now a regular n -gon (recall that we are assuming that $n \geq 3$) with center the point (\bar{x}, \bar{y}) and having a vertex v_0 on the line fixed by μ . Each element of G permutes the vertices of the n -gon and preserves edges. Furthermore, no two elements of G permute the vertices in the same way. Thus G is isomorphic with a subgroup of the dihedral group D_n . Since $|G| = |D_n|$, G is isomorphic with D_n . \blacklozenge

In Theorem 11.5 the Klein 4-group, V , seems like an exception. However, V fits into the family of dihedral groups since V has two elements of order 2, a and b , with the property that $ab = ba^{-1}$. Sometimes V is denoted D_2 and considered a dihedral group. The isometries of the plane that map a line segment to itself are isomorphic with V .

The preceding theorem gives the complete story about finite plane isometry groups. We turn now to some infinite groups of plane isometries that arise naturally in decorating and art. Among these are the *discrete frieze groups*. A discrete frieze consists of a pattern of finite width and height that is repeated endlessly in both directions along its baseline to form a strip of infinite length but finite height; think of it as a decorative border strip that goes around a room next to the ceiling or wallpaper. We consider those isometries that carry each basic pattern onto itself or onto another instance of the pattern in the frieze. The set of all such isometries is called the “**frieze group**.” All discrete frieze groups are infinite and have a subgroup isomorphic to \mathbb{Z} generated by the translation that slides the frieze lengthwise until the basic pattern is superimposed on the position of its next neighbor pattern in that direction. As a simple example of a discrete frieze, consider integral signs spaced equal distances apart and continuing infinitely to the left and right, indicated schematically like this.

Let us consider the integral signs to be one unit apart. The symmetry group of this frieze is generated by a translation τ sliding the plane one unit to the right, and by a rotation ρ of 180° about a point in the center of some integral sign. There are no horizontal or vertical reflections, and no glide reflections. This frieze group is nonabelian; we can check that $\tau\rho = \rho\tau^{-1}$. This relation between τ and ρ looks very familiar. The dihedral group D_n is also generated by two elements ρ and μ that satisfy the relation $\rho\mu = \mu\rho^{-1}$. If τ and ρ in the frieze group are replaced by ρ and μ , respectively, we have the same relation. In D_n , μ has order 2, as does ρ in the frieze group, but the element ρ in D_n has order n while τ has infinite order. Thus it is natural to use the notation D_∞ for this nonabelian frieze group.

As another example, consider the frieze given by an infinite string of D's.

Its group is generated by a translation τ one step to the right and by a vertical reflection μ across a horizontal line cutting through the middle of all the D's. We can check that these group generators commute this time, that is, $\tau\mu = \mu\tau$, so this frieze group is abelian and isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$.

It can be shown that if we classify such discrete friezes only by whether or not their groups contain a

rotation	horizontal axis reflection
vertical axis reflection	nontrivial glide reflection

then there are a total of seven possibilities. A *nontrivial glide reflection* in a symmetry group is one that is not equal to a product of a translation in that group and a reflection in that group. The group for the string of D's above contains glide reflections across the horizontal line through the centers of the D's, but the translation component of each glide reflection is also in the group so they are all considered trivial glide reflections in that group. The frieze group for

$$\begin{array}{ccccccccccccc} \dots & \mathbf{D} & & \dots \\ \dots & \mathbf{D} & & \dots \end{array}$$

contains a nontrivial glide reflection whose translation component is not an element of the group. The exercises exhibit the seven possible cases, and ask you to tell, for each case, which of the four types of isometries displayed above appear in the symmetry group. We do not obtain seven different group structures. Each of the groups obtained can be shown to be isomorphic to one of

$$\mathbb{Z}, \quad D_\infty, \quad \mathbb{Z} \times \mathbb{Z}_2, \quad \text{or} \quad D_\infty \times \mathbb{Z}_2.$$

Equally interesting is the study of symmetries when a pattern in the shape of a square, parallelogram, rhombus, or hexagon is repeated by translations along *two non-parallel vector directions* to fill the entire plane, like patterns that appear on wallpaper. These groups are called the *wallpaper groups* or the *plane crystallographic groups*. While a frieze could not be carried into itself by a rotation through a positive angle less than 180° , it is possible to have rotations of 60° , 90° , 120° , and 180° for some of these plane-filling patterns. Figure 11.6 provides an illustration where the pattern consists of a square. We are interested in the group of plane isometries that carry this square onto itself or onto another square. Generators for this group are given by two translations (one sliding a square to the next neighbor to the right and one to the next above), by a rotation through 90° about the center of a square, and by a reflection in a vertical (or horizontal) line along the edges of the square. The one reflection is all that is needed to “turn the plane over”; a diagonal reflection can also be used. After being turned over, the translations and rotations can be used again. The isometry group for this *periodic pattern* in the plane surely contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ generated by the unit translations to the right and upward, and a subgroup isomorphic to D_4 generated by those isometries that carry one square (it can be any square) into itself.

If we consider the plane to be filled with parallelograms as in Fig. 11.7, we do not get all the types of isometries that we did for Fig. 11.6. The symmetry group this time is generated by the translations indicated by the arrows and a rotation through 180° about any vertex of a parallelogram.

It can be shown that there are 17 different types of wallpaper patterns when they are classified according to the types of rotations, reflections, and nontrivial glide reflections that they admit. We refer you to Gallian [8] for pictures of these 17 possibilities and a chart to help you identify them. The exercises illustrate a few of them. The situation