PART IV

Advanced Group Theory

SECTION 16 ISOMORPHISM THEOREMS

There are several theorems concerning isomorphic factor groups that are known as the *isomorphism theorems* of group theory. The first of these is Theorem 12.14, which we restate for easy reference. The theorem is diagrammed in Fig. 16.1.

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\gamma_K"'] & & \phi[G] \\ & G/K \arrow[ur, dashed, "\mu \text{ (isomorphism)}"'] & \end{tikzcd}$$

16.1 Figure

16.2 Theorem **(First Isomorphism Theorem)** Let $\phi : G \to G'$ be a homomorphism with kernel K, and let $\gamma_K : G \to G/K$ be the canonical homomorphism. There is a unique isomorphism $\mu : G/K \to \phi[G]$ such that $\phi(x) = \mu(\gamma_K(x))$ for each $x \in G$. ◆

The lemma that follows will be of great aid in our proof and intuitive understanding of the other two isomorphism theorems.

16.3 Lemma Let N be a normal subgroup of a group G and let $\gamma : G \to G/N$ be the canonical homomorphism. Then the map ϕ from the set of normal subgroups of G containing N to the set of normal subgroups of G/N given by $\phi(L) = \gamma[L]$ is one-to-one and onto.

Proof Theorem 13.18 shows that if L is a normal subgroup of G containing N, then $\phi(L) = \gamma[L]$ is a normal subgroup of G/N. Because $N \leq L$, for each $x \in L$ the entire coset xN in G is contained in L. Thus by Theorem 10.17, $\gamma^{-1}[\phi(L)] = L$. Consequently, if L and M are normal subgroups of G, both containing N, and if $\phi(L) = \phi(M) = H$, then $L = \gamma^{-1}[H] = M$. Therefore ϕ is one-to-one.

If H is a normal subgroup of G/N, then $\gamma^{-1}[H]$ is a normal subgroup of G by Theorem 13.18. Because $N \in H$ and $\gamma^{-1}[\{N\}] = N$, we see that $N \subseteq \gamma^{-1}[H]$. Then $\phi(\gamma^{-1}[H]) = \gamma[\gamma^{-1}[H]] = H$. This shows that ϕ is onto the set of normal subgroups of G/N. ◆

If H and N are subgroups of a group G, then we let

$$HN = \{hn \mid h \in H, n \in N\}.$$

We define the **join** $H \vee N$ of H and N as the intersection of all subgroups of G that contain HN; thus $H \vee N$ is the smallest subgroup of G containing HN. Of course $H \vee N$ is also the smallest subgroup of G containing both H and N, since any such subgroup must contain HN. In general, HN need not be a subgroup of G. However, we have the following lemma.

16.4 Lemma If N is a normal subgroup of G, and if H is any subgroup of G, then $H \vee N = HN = NH$. Furthermore, if H is also normal in G, then HN is normal in G.

Proof We show that HN is a subgroup of G, from which $H \vee N = HN$ follows at once. Let $h_1, h_2 \in H$ and $n_1, n_2 \in N$. Since N is a normal subgroup, we have $n_1 h_2 = h_2 n_3$ for some $n_3 \in N$. Then $(h_1 n_1)(h_2 n_2) = h_1(n_1 h_2)n_2 = h_1(h_2 n_3)n_2 = (h_1 h_2)(n_3 n_2) \in HN$, so HN is closed under the induced operation in G. Clearly $e = ee$ is in HN. For $h \in H$ and $n \in N$, we have $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}n_4$ for some $n_4 \in N$, since N is a normal subgroup. Thus $(hn)^{-1} \in HN$, so $HN \leq G$. A similar argument shows that NH is a subgroup, so $NH = H \vee N = HN$.

Now suppose that H is also normal in G, and let $h \in H, n \in N$, and $g \in G$. Then $ghng^{-1} = (ghg^{-1})(gng^{-1}) \in HN$, so HN is indeed normal in G. ◆

We are now ready for the second isomorphism theorem.

16.5 Theorem **(Second Isomorphism Theorem)** Let H be a subgroup of G and let N be a normal subgroup of G. Then $(HN)/N \simeq H/(H \cap N)$.

Proof Since $N \leq HN \leq G$ and N is a normal subgroup of G, N is a normal subgroup of HN, which allows us to form the group HN/N. We define a map $\phi : H \to HN/N$ by $\phi(h) = hN$. The map ϕ is a homomorphism since for any $h_1, h_2 \in H$,

$$\phi(h_1 h_2) = (h_1 h_2)N = (h_1 N)(h_2 N) = \phi(h_1)\phi(h_2).$$

The map ϕ maps onto HN/N since any element of HN/N can be written as hnN for some $h \in H$ and $n \in N$ and $hnN = hN = \phi(h)$. We now compute the kernel of ϕ.

$$\text{Ker}(\phi) = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

By the First Isomorphism Theorem, Theorem 16.2, the map $\mu : H/(H \cap N) \to HN/N$ defined by $\mu(h(H \cap N)) = hN$ is an isomorphism. ◆

16.6 Example Let $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, H = \mathbb{Z} \times \mathbb{Z} \times \{0\}$, and $N = \{0\} \times \mathbb{Z} \times \mathbb{Z}$. Then clearly $HN = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $H \cap N = \{0\} \times \mathbb{Z} \times \{0\}$. We have $(HN)/N \simeq \mathbb{Z}$ and we also have $H/(H \cap N) \simeq \mathbb{Z}$. ▲

16.7 Example Let $G = \mathbb{Z}$, $N = \langle n \rangle$, and $H = \langle h \rangle$ where n and h are positive. The group $\mathbb{Z}$ is abelian, so N is a normal subgroup of Z. Since $\mathbb{Z}$ is an additive group, we write $N + H$ instead of NH to avoid confusion. The group $N + H = \langle \gcd(n, h) \rangle$ since anything in $N + H$ is a multiple of $\gcd(n, h)$ and $\gcd(n, h) = xn + yh \in N + H$ for some integers x and y. Also, $N \cap H = \langle \text{lcm}(n, h) \rangle$ since $a \in N \cap H$ if and only if a is a multiple of both n and h. The Second Isomorphism Theorem states that $(N + H)/N = \langle \gcd(n, h) \rangle / \langle n \rangle$

is isomorphic with $H/(N \cap H) = \langle h\rangle/\langle \mathrm{lcm}(n,h)\rangle$. In Exercise 10, you will be asked to prove that if $a, b \in \mathbb{Z}^+$ and a divides b, then $|\langle a\rangle/\langle b\rangle| = b/a$. Since $\langle \gcd(n,h)\rangle/\langle n\rangle \simeq \langle h\rangle/\langle \mathrm{lcm}(n,h)\rangle$, we have

$$\frac{n}{\gcd(n,h)} = \frac{\mathrm{lcm}(n,h)}{h}$$
$$nh = \gcd(n,h)\mathrm{lcm}(n,h),$$

which provides a complicated way of proving a basic number theory fact! We conclude that

$$\langle \gcd(n,h)\rangle/\langle n\rangle \simeq \langle h\rangle/\langle \mathrm{lcm}(n,h)\rangle \simeq \mathbb{Z}_{\frac{n}{\gcd(n,h)}}$$

since a factor group of a cyclic group is cyclic. ▲

If H and K are two normal subgroups of G and $K \leq H$, then H/K is a normal subgroup of G/K. The third isomorphism theorem concerns these groups.

16.8 Theorem **(Third Isomorphism Theorem)** Let H and K be normal subgroups of a group G with $K \leq H$. Then $G/H \simeq (G/K)/(H/K)$.

Proof Since K is a subgroup of H, for any $g \in G$, $gK \subseteq gH$. That is, each left coset of K is completely contained in one coset of H. We define $\phi : G/K \to G/H$ by $\phi(gK) = gH$. That is, we map a coset of K to the coset of H that contains it. Again, our strategy is to use the First Isomorphism Theorem. The map ϕ is a homomorphism since for any $g_1, g_2 \in G$,

$$\begin{aligned}\phi((g_1K)(g_2K)) &= \phi((g_1g_2)K) = (g_1g_2)H \\ &= (g_1H)(g_2H) \\ &= \phi(g_1K)\phi(g_2K).\end{aligned}$$

The map ϕ maps onto G/H since for any coset $gH \in G/H$, $\phi(gK) = gH$. We now compute the kernel of ϕ.

$$\begin{aligned}\mathrm{Ker}(\phi) &= \{gK \in G/K \mid gH = H\} \\ &= \{gK \in G/K \mid g \in H\} \\ &= H/K.\end{aligned}$$

By the First Isomorphism Theorem $(G/K)/(H/K)$ is isomorphic with G/H and a formula for an isomorphism $\mu : (G/K)/(H/K) \to G/H$ is $\mu((gK)H/K) = gH$. ◆

The formula for the isomorphism in the previous proof says that if we collapse the subgroup K in G to form G/K and then collapse all the K cosets inside of H we have the same group as collapsing the subgroup H. Figure 16.9 illustrates the situation. Think of the large ellipse as being the group G. The cosets of H are the sets bounded by the thick solid lines. The cosets of K are the smaller sets inside the cosets of H. The set G/H consists of the cosets of H, which are represented by the four larger areas, each of which is one point in G/H. On the other hand, G/K is represented by the twelve smaller sets each collapsed to a point. Then $(G/K)/(H/K)$ collapses each of the three small areas in the same H coset to their common H coset. Either way, we end up collapsing each H coset to a point.

It is sometimes difficult to think about what groups look like when they contain more than one factor group in their definition. For example, what does an element of $(G/K)/(H/K)$ really look like? Keep in mind that a factor group has cosets of the subgroup as elements. Example 16.10 is intended to clarify what the group $(G/K)/(H/K)$ looks like and to explicitly show what the isomorphism of Theorem 16.8 looks like.