Composition of functions mapping $\mathbb{R}$ into $\mathbb{R}$ was reviewed in Example 1.7. For any set S and any functions f and g mapping S into S, we similarly define the composition $f \circ g$ of g followed by f as the function mapping S into S such that $(f \circ g)(x) = f(g(x))$ for all $x \in S$. Some of the most important binary operations we consider are defined using composition of functions. It is important to know that function composition is always associative whenever it is defined.

1.13 Theorem **(Associativity of Composition)** Let S be a set and let f, g, and h be functions mapping S into S. Then $f \circ (g \circ h) = (f \circ g) \circ h$.

Proof To show these two functions are equal, we must show that they give the same assignment to each $x \in S$. Computing we find that

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x)))$$

and

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))),$$

so the same element $f(g(h(x)))$ of S is indeed obtained. ◆

As an example of using Theorem 1.13 to save work, recall that it is a fairly painful exercise in summation notation to show that multiplication of $n \times n$ matrices is an associative operation. If, in a linear algebra course, we first show that there is a one-to-one correspondence between matrices and linear transformations and that multiplication of matrices corresponds to the composition of the linear transformations (functions), we obtain this associativity at once from Theorem 1.13.

There is another property that an operation on a set may have that is of particular importance in algebra. The numbers 0 and 1 play special roles as real numbers because for any real number a, $a + 0 = a$ and $a \times 1 = a$. Because of these properties, 0 is called the *additive identity* in $\mathbb{R}$ and 1 is called the *multiplicative identity* in $\mathbb{R}$. In general we have the following definition of an identity.

1.14 Definition Let S be a set with binary operation $*$. If $e \in S$ has the property that for all $a \in S$, $a * e = e * a = a$, then e is called an **identity element for** $*$. ■

We included both conditions $a * e = a$ and $e * a = a$ in the definition of an identity because we are not assuming that the operation on S is commutative. Of course, if the operation is commutative, such as $+$ and $\times$ on the real numbers, then we would only have to check one of the conditions and the other follows from commutativity.

1.15 Theorem **(Uniqueness of Identity)** A set with binary operation $*$ has at most one identity element.

Proof We need to show that there cannot be two different identity elements. To do this, we assume that both e and e' are identities and show that $e = e'$. Consider the element $e * e'$. Since e is an identity, $e * e' = e'$. But $e * e' = e$ because e' is also an identity. Therefore $e = e'$. ◆

1.16 Example Continuing with Example 1.7, let F be the set of all functions mapping the real numbers to the real numbers. We verify that the function defined by $\iota(x) = x$ is the identity for the operation function composition. Let $f \in F$. Then $f \circ \iota(x) = f(\iota(x)) = f(x)$ and $\iota \circ f(x) = \iota(f(x)) = f(x)$.

The function $m(x) = 1$ is the identity for the operation function multiplication, $a(x) = 0$ is the identity for function addition, but function subtraction has no identity element. ▲

The last property that we consider in this section is the existance of inverse elements. For addition, the inverse of a real number a is $-a$. Using multiplication, the inverse of a nonzero real number a is $\frac{1}{a}$. We now give the formal definition of an inverse for an element x.

1.17 Definition If $*$ is an operation on the set S and S has an identity e, then for any $x \in S$, the inverse of x is an element x' such that $x * x' = x' * x = e$. ■

1.18 Example Continuing Example 1.16, let F be the set of functions mapping the real numbers to the real numbers with operation function composition. We have two definitions for the inverse of a function $f \in F$, the usual definition of an inverse function and Definition 1.17. The two definitions match since both say that an inverse for f is a function f' such that $f \circ f' = f' \circ f = \iota$. So $f \in F$ has an inverse if and only if f is one-to-one and onto. ▲

Tables

For a finite set, a binary operation on the set can be defined by means of a table in which the elements of the set are listed across the top as heads of columns and at the left side as heads of rows. We always require that the elements of the set be listed as heads across the top in the same order as heads down the left side. The next example illustrates the use of a table to define a binary operation.

1.19 Example Table 1.20 defines the binary operation $*$ on $S = \{a, b, c\}$ by the following rule:

$$(i\text{th } \textit{entry on the left}) * (j\text{th } \textit{entry on the top})$$
$$= (\textit{entry in the } i\text{th } \textit{row and } j\text{th } \textit{column of the table body}).$$

1.20 Table

$*$	a	b	c
a	b	c	b
b	a	c	b
c	c	b	a

Thus $a * b = c$ and $b * a = a$, so $*$ is not commutative. ▲

We can easily see that *a binary operation defined by a table is commutative if and only if the entries in the table are symmetric with respect to the diagonal that starts at the upper left corner of the table and terminates at the lower right corner*.

1.21 Example Complete Table 1.22 so that $*$ is a commutative operation on the set $S = \{a, b, c, d\}$.

Solution From Table 1.22, we see that $b * a = d$. For $*$ to be commutative, we must have $a * b = d$ also. Thus we place d in the appropriate square defining $a * b$, which is located symmetrically across the diagonal in Table 1.23 from the square defining $b * a$. We obtain the rest of Table 1.23 in this fashion to give our solution. ▲

1.22 Table

$*$	a	b	c	d
a	b			
b	d	a		
c	a	c	d	
d	a	b	b	c

1.23 Table

$*$	a	b	c	d
a	b	d	a	a
b	d	a	c	b
c	a	c	d	b
d	a	b	b	c

1.24 Example When an operation has an identity element, it is customary to put the identity first in the list of heads. This makes the first row and the first column match the head row and head column as seen in Table 1.25. ▲

1.25 Table

$*$	e	a	b	c
e	e	a	b	c
a	a	c	c	a
b	b	a	e	c
c	c	e	a	b

Some Words of Warning

Classroom experience shows the chaos that may result if a student is given a set and asked to define some binary operation on it. Remember that in an attempt to define a binary operation $*$ on a set S we must be sure that

1. *exactly one element is assigned to each possible ordered pair of elements of* S,
2. *for each ordered pair of elements of* S, *the element assigned to it is again in* S.

Regarding Condition 1, a student will often make an attempt that assigns an element of S to "most" ordered pairs, but for a few pairs, determines no element. In this event, $*$ is **not everywhere defined** on S. It may also happen that for some pairs, the attempt could assign any of several elements of S, that is, there is ambiguity. In any case of ambiguity, $*$ is **not well defined**. If Condition 2 is violated, then S is **not closed under** $*$.

1.26 Example On which of the sets $\mathbb{Q}$, $\mathbb{Q}^*$, and Z^+ does the formula $a * b = a/b$ define an operation? Note that this formula does not make sense in the case that $b = 0$. So for example, $2 * 0 = 2/0$ is not defined, which means Condition 1 is not satisfied. So $*$ is not an operation on $\mathbb{Q}$.

If we throw out 0, we do have an operation on $\mathbb{Q}^*$ since both Conditions 1 and 2 are satisfied. That is, for any $a, b \in \mathbb{Q}^*$, $a * b = a/b$ is a nonzero rational number.

The set $\mathbb{Z}^+$ does not include 0, but there is another issue. For example, $1 * 2 = 1/2 \notin \mathbb{Z}^+$, which means that Condition 2 is violated and $*$ is not an operation on $\mathbb{Z}^+$.▲

Following are several illustrations of attempts to define operations on sets. Some of them need some work! The symbol $*$ is used for the attempted operation in all these examples.

1.27 Example Let F be the set of all real-valued functions with domain $\mathbb{R}$ as in Example 1.7. Suppose we "define" $*$ to give the usual quotient of f by g, that is, $f * g = h$, where $h(x) = f(x)/g(x)$. Here Condition 2 is violated, for the functions in F are defined for *all* real numbers, and for some $g \in F, g(x)$ will be zero for some values of x in $\mathbb{R}$ and $h(x)$ would not be defined at those numbers in $\mathbb{R}$. For example, if $f(x) = \cos x$ and $g(x) = x^2$, then $h(0)$ is undefined, so $h \notin F$. ▲

1.28 Example Let F be as in Example 1.27 and let $f * g = h$, where h is the function greater than both f and g. This "definition" is extremely vague. In the first place, we have not defined what it means for one function to be greater than another. Even if we had, any sensible definition would result in there being many functions greater than both f and g, and $*$ would still be *not well defined*. ▲

1.29 Example Let S be a set consisting of 20 people, no two of whom are of the same height. Define $*$ by $a * b = c$, where c is the tallest person among the 20 in S. This is a perfectly good binary operation on the set, although not a particularly interesting one. ▲

1.30 Example Let S be as in Example 1.29 and let $a * b = c$, where c is the shortest person in S who is taller than both a and b. This $*$ is *not everywhere defined,* since if either a or b is the tallest person in the set, $a * b$ is not determined. ▲