12.16 Example Classify the group $(\mathbb{Z}_4 \times \mathbb{Z}_2)/(\{0\} \times \mathbb{Z}_2)$ according to the fundamental theorem of finitely generated abelian groups (Theorem 9.12).

Solution The projection map $\pi_1 : \mathbb{Z}_4 \times \mathbb{Z}_2 \to \mathbb{Z}_4$ given by $\pi_1(x, y) = x$ is a homomorphism of $\mathbb{Z}_4 \times \mathbb{Z}_2$ onto $\mathbb{Z}_4$ with kernel $\{0\} \times \mathbb{Z}_2$. By Theorem 12.14, we know that the given factor group is isomorphic to $\mathbb{Z}_4$. ▲

Normal Subgroups and Inner Automorphisms

We derive some alternative characterizations of normal subgroups, which often provide us with an easier way to check normality than finding both the left and the right coset decompositions.

Suppose that H is a subgroup of G such that $ghg^{-1} \in H$ for all $g \in G$ and all $h \in H$. Then $gHg^{-1} = \{ghg^{-1} \mid h \in H\} \subseteq H$ for all $g \in G$. We claim that actually $gHg^{-1} = H$. We must show that $H \subseteq gHg^{-1}$ for all $g \in G$. Let $h \in H$. Replacing g by g^{-1} in the relation $ghg^{-1} \in H$, we obtain $g^{-1}h(g^{-1})^{-1} = g^{-1}hg = h_1$ where $h_1 \in H$. Consequently, $h = gh_1g^{-1} \in gHg^{-1}$, and we are done.

Suppose that $gH = Hg$ for all $g \in G$. Then $gh = h_1g$, so $ghg^{-1} \in H$ for all $g \in G$ and all $h \in H$. By the preceding paragraph, this means that $gHg^{-1} = H$ for all $g \in G$. Conversely, if $gHg^{-1} = H$ for all $g \in G$, then $ghg^{-1} = h_1$ so $gh = h_1g \in Hg$, and $gH \subseteq Hg$. But also, $g^{-1}Hg = H$ giving $g^{-1}hg = h_2$, so that $hg = gh_2$ and $Hg \subseteq gH$.

The comments after Definition 12.3 show that the kernel of any homomorphism is a normal subgroup of the domain. Also, Theorem 12.12 says that any normal subgroup is the kernel of some homomorphism.

We summarize our work as a theorem.

12.17 Theorem The following are four equivalent conditions for a subgroup H of a group G to be a *normal* subgroup of G.

1. $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.
2. $gHg^{-1} = H$ for all $g \in G$.
3. There is a group homomorphism $\phi : G \to G'$ such that $\mathrm{Ker}(\phi) = H$.
4. $gH = Hg$ for all $g \in G$.

Condition (2) of Theorem 12.17 is often taken as the definition of a normal subgroup H of a group G. ◆

12.18 Example Every subgroup H of an abelian group G is normal. We need only note that $gh = hg$ for all $h \in H$ and all $g \in G$, so, of course, $ghg^{-1} = h \in H$ for all $g \in G$ and all $h \in H$. ▲

If G is a group and $g \in G$, then the map $i_g : G \to G$ defined by $i_g(x) = gxg^{-1}$ is a group homomorphism since $i_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = i_g(x)i_g(x)$. We see that $gag^{-1} = gbg^{-1}$ if and only if $a = b$, so i_g is one-to-one. Since $g(g^{-1}yg)g^{-1} = y$, we see that i_g is onto G, so it is an isomorphism of G with itself.

12.19 Definition An isomorphism $\phi : G \to G$ of a group G with itself is an **automorphism** of G. The automorphism $i_g : G \to G$, where $i_g(x) = gxg^{-1}$ for all $x \in G$, is the **inner automorphism of G by g**. Performing i_g on x is called **conjugation of x by g**. ■

The equivalence of conditions (1) and (2) in Theorem 12.17 shows that $gH = Hg$ for all $g \in G$ if and only if $i_g[H] = H$ for all $g \in G$, that is, if and only if H is **invariant** under all inner automorphisms of G. It is important to realize that $i_g[H] = H$ is an equation in *sets;* we need not have $i_g(h) = h$ for all $h \in H$. That is i_g may perform a nontrivial *permutation* of the set H. We see that the normal subgroups of a group G are precisely those that are invariant under all inner automorphisms. A subgroup K of G is a **conjugate subgroup** of H if $K = i_g[H] = gHg^{-1}$ for some $g \in G$.

EXERCISES 12

Computations

In Exercises 1 through 8, find the order of the given factor group.

1. $\mathbb{Z}_6/\langle 3\rangle$

2. $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle)$

3. $(\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle(2,1)\rangle$

4. $(\mathbb{Z}_3 \times \mathbb{Z}_5)/(\{0\} \times \mathbb{Z}_5)$

5. $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle(1,1)\rangle$

6. $(\mathbb{Z}_{50} \times \mathbb{Z}_{75})/\langle(15,15)\rangle$

7. $(\mathbb{Z}_{26} \times \mathbb{Z}_{15})/\langle(1,1)\rangle$

8. $(\mathbb{Z}_8 \times S_3)/\langle(2,(1,2,3))\rangle$

In Exercises 9 through 15, give the order of the element in the factor group.

9. $5 + \langle 4\rangle$ in $\mathbb{Z}_{12}/\langle 4\rangle$

10. $26 + \langle 12\rangle$ in $\mathbb{Z}_{60}/\langle 12\rangle$

11. $(2,1) + \langle(1,1)\rangle$ in $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle(1,1)\rangle$

12. $(3,1) + \langle(1,1)\rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_4)/\langle(1,1)\rangle$

13. $(2,3) + \langle(0,3)\rangle$ in $(\mathbb{Z}_{10} \times \mathbb{Z}_4)/\langle(0,3)\rangle$

14. $(2,5) + \langle(1,2)\rangle$ in $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle(1,2)\rangle$

15. $(2,0) + \langle(4,4)\rangle$ in $(\mathbb{Z}_6 \times \mathbb{Z}_8)/\langle(4,4)\rangle$

16. Compute $i_\rho[H]$ for the subgroup $H = \{\iota, \mu\}$ of the dihedral group D_3.

Concepts

In Exercises 17 through 19, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

17. A *normal subgroup* H of G is one satisfying $hG = Gh$ for all $h \in H$.

18. A *normal subgroup* H of G is one satisfying $g^{-1}hg \in H$ for all $h \in H$ and all $g \in G$.

19. An *automorphism* of a group G is a homomorphism mapping G into G.

20. What is the importance of a *normal* subgroup of a group G?

Students often write nonsense when first proving theorems about factor groups. The next two exercises are designed to call attention to one basic type of error.

21. A student is asked to show that if H is a normal subgroup of an abelian group G, then G/H is abelian. The student's proof starts as follows:

We must show that G/H is abelian. Let a and b be two elements of G/H.

a. Why does the instructor reading this proof expect to find nonsense from here on in the student's paper?

b. What should the student have written?

c. Complete the proof.

22. A **torsion group** is a group all of whose elements have finite order. A group is **torsion free** if the identity is the only element of finite order. A student is asked to prove that if G is a torsion group, then so is G/H for every normal subgroup H of G. The student writes

We must show that each element of G/H is of finite order. Let $x \in G/H$.

Answer the same questions as in Exercise 21.

23. Determine whether each of the following is true or false.

a. It makes sense to speak of the factor group G/N if and only if N is a normal subgroup of the group G.

b. Every subgroup of an abelian group G is a normal subgroup of G.

c. The only automorphism of an abelian group is the identity map.

d. Every factor group of a finite group is again of finite order.
e. Every factor group of a torsion group is a torsion group. (See Exercise 22.)
f. Every factor group of a torsion-free group is torsion free. (See Exercise 22.)
g. Every factor group of an abelian group is abelian.
h. Every factor group of a nonabelian group is nonabelian.
i. $\mathbb{Z}/n\mathbb{Z}$ is cyclic of order n.
j. $\mathbb{R}/n\mathbb{R}$ is cyclic of order n, where $n\mathbb{R} = \{nr \mid r \in \mathbb{R}\}$ and $\mathbb{R}$ is under addition.

Theory

24. Let G_1 and G_2 be groups and $\pi_1 : G_1 \times G_2 \to G_1$ be the function defined by $\pi_1(a, b) = a$. Prove that π_1 is a homomorphism, find $\text{Ker}(\pi_1)$, and prove $(G_1 \times G_2)/\text{Ker}(\pi_1)$ is isomorphic to G_1.

25. Let G_1 and G_2 be groups and $\phi : G_1 \times G_2 \to G_1 \times G_2$ be the function defined by $\phi(a, b) = (a, e_2)$ where e_2 is the identity in G_2. Prove that ϕ is a homomorphism, find $\text{Ker}(\phi)$, and prove $(G_1 \times G_2)/\text{Ker}(\phi)$ is isomorphic to G_1.

26. Complete the proof of Theorem 12.7 by showing that if H is a subgroup of a group G and if left coset multiplication $(aH)(bH) = (ab)H$ is well defined, then $Ha \subseteq aH$.

27. Prove that the torsion subgroup T of an abelian group G is a normal subgroup of G, and that G/T is torsion free. (See Exercise 22.)

28. A subgroup H is **conjugate to a subgroup** K of a group G if there exists an inner automorphism i_g of G such that $i_g[H] = K$. Show that conjugacy is an equivalence relation on the collection of subgroups of G.

29. Characterize the normal subgroups of a group G in terms of the cells where they appear in the partition given by the conjugacy relation in the preceding exercise.

30. Find all subgroups of D_3 that are conjugate to $H = \{\iota, \mu\}$. (See Exercise 28.)

31. **(Evaluation Homomorphism)** Let F be the set of all functions mapping the real numbers to the real numbers and let $c \in \mathbb{R}$. The sum of two functions $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$. Function addition makes F a group. Let $\phi_c : F \to \mathbb{R}$ be defined by $\phi_c(f) = f(c)$.

a. Show that ϕ_c is a group homomorphism.
b. Find $\text{Ker}(\phi_c)$.
c. Identify the coset of $\text{Ker}(\phi_c)$ that contains the constant function $f(x) = 1$.
d. Find a well-known group that is isomorphic with $F/\text{Ker}(\phi_c)$. Use the Fundamental Homomorphism Theorem to prove your answer.

32. Let H be a normal subgroup of a group G, and let $m = (G : H)$. Show that $a^m \in H$ for every $a \in G$.

33. Show that an intersection of normal subgroups of a group G is again a normal subgroup of G.

34. Given any subset S of a group G, show that it makes sense to speak of the smallest normal subgroup that contains S. [*Hint:* Use Exercise 33.]

35. Let G be a group. An element of G that can be expressed in the form $aba^{-1}b^{-1}$ for some $a, b \in G$ is a **commutator** in G. The preceding exercise shows that there is a smallest normal subgroup C of a group G containing all commutators in G; the subgroup C is the **commutator subgroup** of G. Show that G/C is an abelian group.

36. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

37. Show that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G.

38. Let G be a group containing at least one subgroup of a fixed finite order s. Show that the intersection of all subgroups of G of order s is a normal subgroup of G. [*Hint:* Use the fact that if H has order s, then so does $x^{-1}Hx$ for all $x \in G$.]

39. a. Show that all automorphisms of a group G form a group under function composition.

b. Show that the inner automorphisms of a group G form a normal subgroup of the group of all automorphisms of G under function composition. [*Warning:* Be sure to show that the inner automorphisms do form a subgroup.]

40. Show that the set of all $g \in G$ such that $i_g : G \to G$ is the identity inner automorphism i_e is a normal subgroup of a group G.

41. Let G and G' be groups, and let H and H' be normal subgroups of G and G', respectively. Let ϕ be a homomorphism of G into G'. Show that ϕ induces a natural homomorphism $\phi_* : (G/H) \to (G'/H')$ if $\phi[H] \subseteq H'$. (This fact is used constantly in algebraic topology.)

42. Use the properties $\det(AB) = \det(A) \cdot \det(B)$ and $\det(I_n) = 1$ for $n \times n$ matrices to show the $n \times n$ matrices with determinant ± 1 form a normal subgroup of $GL(n, \mathbb{R})$.

43. Let G be a group, and let $\mathscr{P}(G)$ be the set of all *subsets* of G. For any $A, B \in \mathscr{P}(G)$, let us define the product subset $AB = \{ab \mid a \in A, b \in B\}$.

a. Show that this multiplication of subsets is associative and has an identity element, but that $\mathscr{P}(G)$ is not a group under this operation.

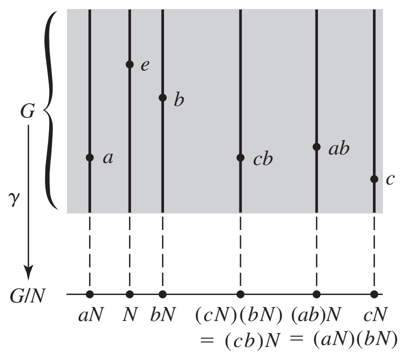
b. Show that if N is a normal subgroup of G, then the set of cosets of N is closed under the above operation on $\mathscr{P}(G)$, and that this operation agrees with the multiplication given by the formula in Corollary 12.8.

c. Show (without using Corollary 12.8) that the cosets of N in G form a group under the above operation. Is its identity element the same as the identity element of $\mathscr{P}(G)$?

SECTION 13 FACTOR-GROUP COMPUTATIONS AND SIMPLE GROUPS

Factor groups can be a tough topic for students to grasp. There is nothing like a bit of computation to strengthen understanding in mathematics. We start by attempting to improve our intuition concerning factor groups. Since we will be dealing with normal subgroups throughout this section, we often denote a subgroup of a group G by N rather than by H.

Let N be a normal subgroup of G. In the factor group G/N, the subgroup N acts as identity element. We may regard N as being *collapsed* to a single element, either to 0 in additive notation or to e in multiplicative notation. This collapsing of N together with the algebraic structure of G require that other subsets of G, namely, the cosets of N, also each collapse into a single element in the factor group. A visualization of this collapsing is provided by Fig. 13.1. Recall from Theorem 12.12 that $\gamma : G \to G/N$ defined by $\gamma(a) = aN$ for $a \in G$ is a homomorphism of G onto G/N. We can view the "line" G/N at the bottom of Figure 13.1 as obtained by collapsing to a point each coset of N in a copy of G. Each point of G/N thus corresponds to a whole vertical line



13.1 Figure