

**30.12 Definition** Let a map  $\phi : R \rightarrow R'$  be a homomorphism of rings. The subring

$$\phi^{-1}[0'] = \{r \in R \mid \phi(r) = 0'\}$$

is the **kernel** of  $\phi$ , denoted by  $\text{Ker}(\phi)$ . ■

If we forget about the multiplicative part of a ring, we see that the kernel of a ring homomorphism is the same as the kernel of the underlying group homomorphism. Any property of a group homomorphism must also hold for a ring homomorphism.

**30.13 Theorem** **Analogue of Theorem 10.17** Let  $\phi : R_1 \rightarrow R_2$  be a ring homomorphism. The elements  $a, b \in R_1$  are in the same additive coset of  $\text{Ker}(\phi)$  if and only if  $\phi(a) = \phi(b)$  ♦

**30.14 Theorem** **Analogue of Corollary 10.19** A ring homomorphism  $\phi : R_1 \rightarrow R_2$  is one-to-one if and only if  $\text{Ker}(\phi) = \{0\}$ . ♦

The kernel of a group homomorphism  $\phi : G_1 \rightarrow G_2$  is a normal subgroup of  $G_1$  and normality is what is needed in order to construct a factor group from a subgroup. The situation is similar in rings. We need a subring to be an ideal in order to construct a factor ring. The following theorem states that in fact the kernel of a ring homomorphism is an ideal.

**30.15 Theorem** Let  $\phi : R_1 \rightarrow R_2$  be a ring homomorphism. Then  $\text{Ker}(\phi)$  is an ideal in  $R_1$ .

**Proof** Since  $\{0\} \subset R_2$  is an ideal in  $R_2$ ,  $\text{Ker}(\phi) = \phi^{-1}[\{0\}]$  is an ideal in  $R_1$  by Property 7 of Theorem 30.11. ♦

### Fundamental Homomorphism Theorem

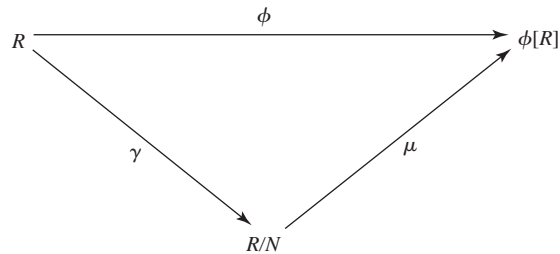
To complete our analogy with groups, we give the analogues of Theorems 12.12 and 12.14.

**30.16 Theorem** (**Analogue of Theorem 12.12**) Let  $N$  be an ideal of a ring  $R$ . Then  $\gamma : R \rightarrow R/N$  given by  $\gamma(x) = x + N$  is a ring homomorphism with kernel  $N$ .

**Proof** The additive part is done in Theorem 12.12. Turning to the multiplicative question, we see that

$$\gamma(xy) = (xy) + N = (x + N)(y + N) = \gamma(x)\gamma(y). \quad \blacklozenge$$

**30.17 Theorem** (**Fundamental Homomorphism Theorem; Analogue of Theorem 12.14**) Let  $\phi : R \rightarrow R'$  be a ring homomorphism with kernel  $N$ . Then  $\phi[R]$  is a ring, and the map  $\mu : R/N \rightarrow \phi[R]$  given by  $\mu(x + N) = \phi(x)$  is an isomorphism. If  $\gamma : R \rightarrow R/N$  is the homomorphism given by  $\gamma(x) = x + N$ , then for each  $x \in R$ , we have  $\phi(x) = \mu \circ \gamma(x)$ .



30.18 Figure

**Proof** This follows at once from Theorems 30.15 and 30.16. Figure 30.18 is the analogue of Fig. 12.15. ♦

**30.19 Example** Example 30.3 shows that  $n\mathbb{Z}$  is an ideal of  $\mathbb{Z}$ , so we can form the factor ring  $\mathbb{Z}/n\mathbb{Z}$ . Example 22.11 shows that  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_n$  where  $\phi(m)$  is the remainder of  $m$  modulo  $n$  is a homomorphism, and we see that  $\text{Ker}(\phi) = n\mathbb{Z}$ . Theorem 30.17 then shows that the map  $\mu : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}_n$  where  $\mu(m + n\mathbb{Z})$  is the remainder of  $m$  modulo  $n$  is well defined and is an isomorphism. ▲

**30.20 Example** Continuing Example 30.5, let  $F$  be the ring of all functions mapping  $\mathbb{R}$  into  $\mathbb{R}$  and let  $N$  be the subset of  $F$  consisting of all functions  $f$  with  $f(2) = 0$ . The set  $N$  is an ideal in  $F$ , so  $F/N$  is a ring. Furthermore,  $N$  is the kernel of the evaluation homomorphism  $\phi_2 : F \rightarrow \mathbb{R}$  defined by  $\phi_2(f) = f(2)$ . Since  $\phi_2$  maps onto  $\mathbb{R}$ ,  $F/N$  is isomorphic with  $\mathbb{R}$  by Theorem 30.17. The function  $\mu : F/N \rightarrow \mathbb{R}$  given by  $\mu(f) = f(2)$  is an isomorphism. ▲

In summary, every ring homomorphism with domain  $R$  gives rise to a factor ring  $R/N$ , and every factor ring  $R/N$  gives rise to a homomorphism mapping  $R$  into  $R/N$ . An *ideal* in ring theory is analogous to a *normal subgroup* in the group theory. Both are the type of substructure needed to form a factor structure.

## ■ EXERCISES 30

### Computations

1. Describe all ring homomorphisms of  $\mathbb{Z} \times \mathbb{Z}$  into  $\mathbb{Z} \times \mathbb{Z}$ . [Hint: Note that if  $\phi$  is such a homomorphism, then  $\phi((1, 0)) = \phi((1, 0))\phi((1, 0))$  and  $\phi((0, 1)) = \phi((0, 1))\phi((0, 1))$ . Consider also  $\phi((1, 0)(0, 1))$ .]
2. Find all positive integers  $n$  such that  $\mathbb{Z}_n$  contains a subring isomorphic to  $\mathbb{Z}_2$ .
3. Find all ideals  $N$  of  $\mathbb{Z}_{12}$ . In each case compute  $\mathbb{Z}_{12}/N$ ; that is, find a known ring to which the quotient ring is isomorphic.
4. Give addition and multiplication tables for  $2\mathbb{Z}/8\mathbb{Z}$ . Are  $2\mathbb{Z}/8\mathbb{Z}$  and  $\mathbb{Z}_4$  isomorphic rings?

### Concepts

In Exercises 5 through 7, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

5. An *isomorphism of a ring*  $R$  with a ring  $R'$  is a homomorphism  $\phi : R \rightarrow R'$  such that  $\text{Ker}(\phi) = \{0\}$ .
6. An *ideal*  $N$  of a ring  $R$  is an additive subgroup of  $\langle R, + \rangle$  such that for all  $r \in R$  and all  $n \in N$ , we have  $rn \in N$  and  $nr \in N$ .
7. The *kernel of a homomorphism*  $\phi$  mapping a ring  $R$  into a ring  $R'$  is  $\{\phi(r) = 0' \mid r \in R\}$ .
8. Let  $F$  be the ring of all functions mapping  $\mathbb{R}$  into  $\mathbb{R}$  and having derivatives of all orders. Differentiation gives a map  $\delta : F \rightarrow F$  where  $\delta(f(x)) = f'(x)$ . Is  $\delta$  a homomorphism? Why? Give the connection between this exercise and Example 30.4.
9. Give an example of a ring homomorphism  $\phi : R \rightarrow R'$  where  $R$  has unity 1 and  $\phi(1) \neq 0'$ , but  $\phi(1)$  is not unity for  $R'$ .
10. Determine whether each of the following is true or false.
  - a. The concept of a ring homomorphism is closely connected with the idea of a factor ring.
  - b. A ring homomorphism  $\phi : R \rightarrow R'$  carries ideals of  $R$  into ideals of  $R'$ .
  - c. A ring homomorphism is one-to-one if and only if the kernel is  $\{0\}$ .
  - d.  $\mathbb{Q}$  is an ideal in  $\mathbb{R}$ .

- e. Every ideal in a ring is a subring of the ring.
  - f. Every subring of every ring is an ideal of the ring.
  - g. Every quotient ring of every commutative ring is again a commutative ring.
  - h. The rings  $\mathbb{Z}/4\mathbb{Z}$  and  $\mathbb{Z}_4$  are isomorphic.
  - i. An ideal  $N$  in a ring  $R$  with unity 1 is all of  $R$  if and only if  $1 \in N$ .
  - j. The concept of an ideal is to the concept of a ring as the concept of a normal subgroup is to the concept of a group.
11. Let  $R$  be a ring. Observe that  $\{0\}$  and  $R$  are both ideals of  $R$ . Are the factor rings  $R/R$  and  $R/\{0\}$  of real interest? Why?
  12. Give an example to show that a factor ring of an integral domain may be a field.
  13. Give an example to show that a factor ring of an integral domain may have divisors of 0.
  14. Give an example to show that a factor ring of a ring with divisors of 0 may be an integral domain.
  15. Find a subring of the ring  $\mathbb{Z} \times \mathbb{Z}$  that is not an ideal of  $\mathbb{Z} \times \mathbb{Z}$ .
  16. A student is asked to prove that a quotient ring of a ring  $R$  modulo an ideal  $N$  is commutative if and only if  $(rs - sr) \in N$  for all  $r, s \in R$ . The student starts out:  
Assume  $R/N$  is commutative. Then  $rs = sr$  for all  $r, s \in R/N$ .
    - a. Why does the instructor reading this expect an incorrect proof?
    - b. What should the student have written?
    - c. Prove the assertion. (Note the “if and only if.”)

### Theory

17. Let  $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$  and let  $R'$  consist of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$  for  $a, b \in \mathbb{Z}$ . Show that  $R$  is a subring of  $\mathbb{R}$  and that  $R'$  is a subring of  $M_2(\mathbb{Z})$ . Then show that  $\phi : R \rightarrow R'$ , where  $\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$  is an isomorphism.
18. Show that each homomorphism from a field to a ring is either one-to-one or maps everything onto 0.
19. Show that if  $R, R'$ , and  $R''$  are rings, and if  $\phi : R \rightarrow R'$  and  $\psi : R' \rightarrow R''$  are homomorphisms, then the composite function  $\psi\phi : R \rightarrow R''$  is a homomorphism. (See Exercise 39 of Section 8.)
20. Let  $R$  be a commutative ring with unity of prime characteristic  $p$ . Show that the map  $\phi_p : R \rightarrow R$  given by  $\phi_p(a) = a^p$  is a homomorphism (the **Frobenius homomorphism**).
21. Let  $R$  and  $R'$  be rings and let  $\phi : R \rightarrow R'$  be a ring homomorphism such that  $\phi[R] \neq \{0'\}$ . Show that if  $R$  has unity 1 and  $R'$  has no 0 divisors, then  $\phi(1)$  is unity for  $R'$ .
22. Let  $\phi : R \rightarrow R'$  be a ring homomorphism and let  $N$  be an ideal of  $R$ .
  - a. Show that  $\phi[N]$  is an ideal of  $\phi[R]$ .
  - b. Give an example to show that  $\phi[N]$  need not be an ideal of  $R'$ .
  - c. Let  $N'$  be an ideal either of  $\phi[R]$  or of  $R'$ . Show that  $\phi^{-1}[N']$  is an ideal of  $R$ .
23. Let  $F$  be a field, and let  $S$  be any subset of  $F \times F \times \cdots \times F$  for  $n$  factors. Show that the set  $N_S$  of all  $f(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$  that have every element  $(a_1, \dots, a_n)$  of  $S$  as a zero (see Exercise 28 of Section 27) is an ideal in  $F[x_1, \dots, x_n]$ . This is of importance in algebraic geometry.
24. Show that a factor ring of a field is either the trivial (zero) ring of one element or is isomorphic to the field.
25. Show that if  $R$  is a ring with unity and  $N$  is an ideal of  $R$  such that  $N \neq R$ , then  $R/N$  is a ring with unity.
26. Let  $R$  be a commutative ring and let  $a \in R$ . Show that  $I_a = \{x \in R \mid ax = 0\}$  is an ideal of  $R$ .
27. Show that an intersection of ideals of a ring  $R$  is again an ideal of  $R$ .
28. Let  $R$  and  $R'$  be rings and let  $N$  and  $N'$  be ideals of  $R$  and  $R'$ , respectively. Let  $\phi$  be a homomorphism of  $R$  into  $R'$ . Show that  $\phi$  induces a natural homomorphism  $\phi_* : R/N \rightarrow R'/N'$  if  $\phi[N] \subseteq N'$ . (Use Exercise 41 of Section 12.)

29. Let  $\phi$  be a homomorphism of a ring  $R$  with unity onto a nonzero ring  $R'$ . Let  $u$  be a unit in  $R$ . Show that  $\phi(u)$  is a unit in  $R'$ .
30. An element  $a$  of a ring  $R$  is **nilpotent** if  $a^n = 0$  for some  $n \in \mathbb{Z}^+$ . Show that the collection of all nilpotent elements in a commutative ring  $R$  is an ideal, the **nilradical** of  $R$ .
31. Referring to the definition given in Exercise 30, find the nilradical of the ring  $\mathbb{Z}_{12}$  and observe that it is one of the ideals of  $\mathbb{Z}_{12}$  found in Exercise 3. What is the nilradical of  $\mathbb{Z}$ ? of  $\mathbb{Z}_{32}$ ?
32. Referring to Exercise 30, show that if  $N$  is the nilradical of a commutative ring  $R$ , then  $R/N$  has as nilradical the trivial ideal  $\{0 + N\}$ .
33. Let  $R$  be a commutative ring and  $N$  an ideal of  $R$ . Referring to Exercise 30, show that if every element of  $N$  is nilpotent and the nilradical of  $R/N$  is  $R/N$ , then the nilradical of  $R$  is  $R$ .
34. Let  $R$  be a commutative ring and  $N$  an ideal of  $R$ . Show that the set  $\sqrt{N}$  of all  $a \in R$ , such that  $a^n \in N$  for some  $n \in \mathbb{Z}^+$ , is an ideal of  $R$ , the **radical** of  $N$ .
35. Referring to Exercise 34, show by examples that for proper ideals  $N$  of a commutative ring  $R$ ,
- $\sqrt{N}$  need not equal  $N$
  - $\sqrt{N}$  may equal  $N$ .
36. What is the relationship of the ideal  $\sqrt{N}$  of Exercise 34 to the nilradical of  $R/N$  (see Exercise 30)? Word your answer carefully.
37. Show that  $\phi : \mathbb{C} \rightarrow M_2(\mathbb{R})$  given by

$$\phi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for  $a, b \in \mathbb{R}$  gives an isomorphism of  $\mathbb{C}$  with the subring  $\phi[\mathbb{C}]$  of  $M_2(\mathbb{R})$ .

## SECTION 31 PRIME AND MAXIMAL IDEALS

Exercises 12 through 14 of the preceding section asked us to provide examples of factor rings  $R/N$  where  $R$  and  $R/N$  have very different structural properties. We start with some examples of this situation, and in the process, provide solutions to those exercises.

**31.1 Example** As was shown in Corollary 23.5, the ring  $\mathbb{Z}_p$ , which is isomorphic to  $\mathbb{Z}/p\mathbb{Z}$ , is a field for  $p$  a prime. *Thus a factor ring of an integral domain may be a field.* ▲

**31.2 Example** The ring  $\mathbb{Z} \times \mathbb{Z}$  is not an integral domain, for

$$(0, 1)(1, 0) = (0, 0),$$

showing that  $(0, 1)$  and  $(1, 0)$  are 0 divisors. Let  $N = \{(0, n) \mid n \in \mathbb{Z}\}$ . Now  $N$  is an ideal of  $\mathbb{Z} \times \mathbb{Z}$ , and  $(\mathbb{Z} \times \mathbb{Z})/N$  is isomorphic to  $\mathbb{Z}$  under the correspondence  $[(m, 0) + N] \leftrightarrow m$ , where  $m \in \mathbb{Z}$ . Thus a *factor* ring of a ring may be an integral domain, even though the original ring is not. ▲

**31.3 Example** The subset  $N = \{0, 3\}$  of  $\mathbb{Z}_6$  is easily seen to be an ideal of  $\mathbb{Z}_6$ , and  $\mathbb{Z}_6/N$  has three elements,  $0 + N$ ,  $1 + N$ , and  $2 + N$ . These add and multiply in such a fashion as to show that  $\mathbb{Z}_6/N \simeq \mathbb{Z}_3$  under the correspondence

$$(0 + N) \leftrightarrow 0, \quad (1 + N) \leftrightarrow 1, \quad (2 + N) \leftrightarrow 2.$$

This example shows that *if  $R$  is not even an integral domain, that is, if  $R$  has zero divisors, it is still possible for  $R/N$  to be a field.* ▲

**31.4 Example** Note that  $\mathbb{Z}$  is an integral domain, but  $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}_6$  is not. The preceding examples showed that a factor ring may have a structure that seems *better* than the original ring.