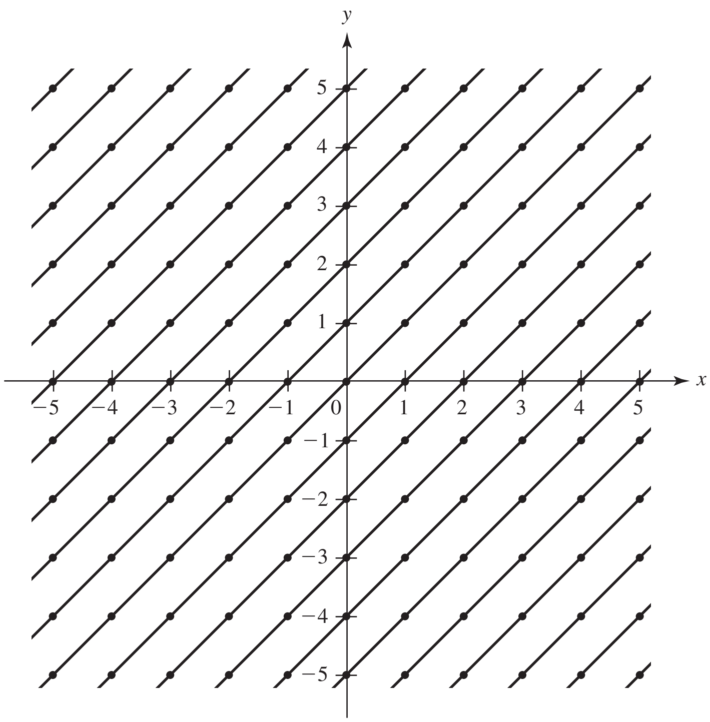
13.12 Example Let us compute (that is, classify as in Theorem 9.12) the group $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle$. We may visualize $\mathbb{Z} \times \mathbb{Z}$ as the points in the plane with both coordinates integers, as indicated by the dots in Fig. 13.13. The subgroup $\langle(1,1)\rangle$ consists of those points that lie on the 45° line through the origin, indicated in the figure. The coset $(1,0) + \langle(1,1)\rangle$ consists of those dots on the 45° line through the point (1, 0), also shown in the figure. Continuing, we see that each coset consists of those dots lying on one of the 45° lines in the figure. We may choose the representatives

$$\cdots, (-3,0), (-2,0), (-1,0), (0,0), (1,0), (2,0), (3,0), \cdots$$

of these cosets to compute in the factor group. Since these representatives correspond precisely to the points of $\mathbb{Z}$ on the x-axis, we see that the factor group $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle$ is isomorphic to $\mathbb{Z}$.

Again, we can use the Fundamental Homomorphism Theorem as another method of computing this group. We let $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(n,m) = n - m$. It is easy to verify that ϕ is a homomorphism, ϕ maps onto $\mathbb{Z}$, and $\text{Ker}(\phi) = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid n = m\} = \langle(1,1)\rangle$. So by the Fundamental Homomorphism Theorem, $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle$ is isomorphic to $\mathbb{Z}$. Furthermore, an isomorphism is given by $\mu((n,m) + \langle(1,1)\rangle) = n-m$. This is the same isomorphism that we saw above. ▲



13.13 Figure

13.14 Example We now compute $(\mathbb{Z} \times \mathbb{Z})/\langle(2,4)\rangle$. This is similar to Example 13.12, but there is a little twist to this one. In this example, we know that the factor group has an element with order 2, since $(1,2) \notin \langle(2,4)\rangle$, but $(1,2) + (1,2) \in \langle(2,4)\rangle$. Furthermore, $(\mathbb{Z} \times \mathbb{Z})/\langle(2,4)\rangle$ has an element $(1,0) + \langle(2,4)\rangle$ with infinite order since $(n,0) \notin \langle(2,4)\rangle$ for any $n \in \mathbb{Z}^+$. Figure 13.15 illustrates the situation. Along the line $y = 2x$ only every other lattice point is in $\langle(2,4)\rangle$. These points are filled dots in the figure. Each line with slope two contains two cosets, one indicated with solid dots and one with hollow dots. Adding $(1, 2)$ moves the solid dot cosets to the hollow dot cosets and the hollow dot cosets to the solid dot cosets while staying on the same line. Adding $(0, 1)$ moves a coset from one line to the next. We may choose coset representatives

$$\ldots, (0, -3), (0, -2), (0, -1), (0, 0), (0, 1), (0, 2), (0, 3), \ldots$$

for the solid dot cosets and

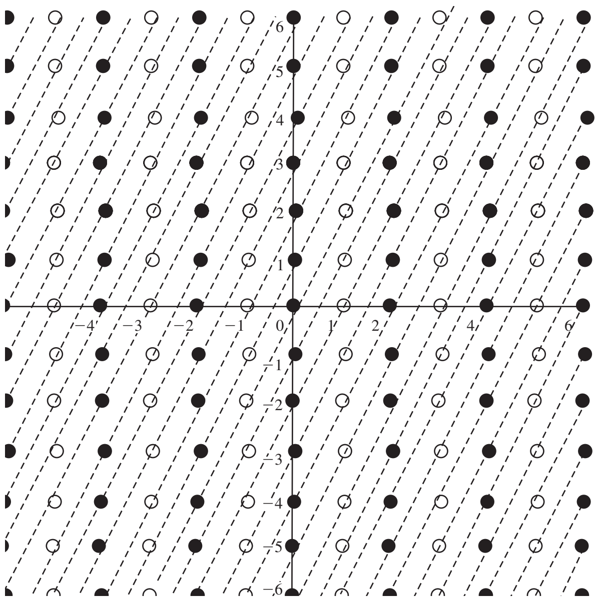
$$\ldots, (1, -3), (1, -2), (1, -1), (1, 0), (1, 1), (1, 2), (1, 3), \ldots$$

for the hollow dot cosets. So it seems that we have two copies of the integers, one with a zero in the first coordinate and one with a one in the first coordinate. This leads us to guess that $(\mathbb{Z} \times \mathbb{Z})/\langle(2, 4)\rangle$ is isomorphic with $\mathbb{Z}_2 \times \mathbb{Z}$.

To verify that our guess is correct, we seek a homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}$ that maps onto $\mathbb{Z}_2 \times \mathbb{Z}$ and whose kernel is $\langle(2, 4)\rangle$. We let $\phi(a, b) = (r, 2a - b)$ where r is the remainder when a is divided by 2. It is easy to check that ϕ is a homomorphism. Furthermore, $\phi(0, -1) = (0, 1)$ and $\phi(1, 2) = (1, 0)$, which implies that ϕ maps onto $\mathbb{Z} \times \mathbb{Z}_2$. It remains to compute $\mathrm{Ker}(\phi)$.

$$\mathrm{Ker}(\phi) = \{(a, b) \mid b = 2a \text{ and } a \text{ is even}\} = \{(2n, 4n) \mid n \in \mathbb{Z}\} = \langle(2, 4)\rangle.$$

Thus $(\mathbb{Z} \times \mathbb{Z})/\langle(2, 4)\rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ by the Fundamental Homomorphism Theorem. Furthermore, an isomorphism $\mu : (\mathbb{Z} \times \mathbb{Z})/\langle(2, 4)\rangle \to \mathbb{Z}_2 \times \mathbb{Z}$ is defined by the formula $\mu((a, b) + \langle(2, 4)\rangle) = (r, 2a - b)$ where r is the remainder when a is divided by 2. ▲



13.15 Figure

Simple Groups

As we mentioned in the preceding section, one feature of a factor group is that it gives crude information about the structure of the whole group. Of course, sometimes there may be no nontrivial proper normal subgroup. For example, Lagrange's Theorem shows that a group of prime order can have no nontrivial proper subgroup of any sort.