It can be shown that, conversely, every digraph satisfying these four properties is a Cayley digraph for some group. Due to the symmetry of such a digraph, we can choose labels like a, b, c for the various arc types, name any vertex e to represent the identity, and name each other vertex by a product of arc labels and their inverses that we can travel to attain that vertex starting from the one that we named e. Some finite groups were first constructed (found) using digraphs.

7.14 Example A digraph satisfying the four properties given above is shown in Fig. 7.13 (a). To obtain Fig. 7.13 (b), we selected the labels

$$\underset{a}{\longrightarrow} \quad \text{and} \quad \underset{b}{\text{- - - - - - - -}},$$

named a vertex e, and then named the other vertices as shown. We have a group

$$\{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$$

of eight elements. From the diagram we could compute any product. For example, to compute ba^2ba^3 we start at the vertex labeled ba^2, follow a dotted edge, and then follow three solid edges to arrive at a. Note that the way we labeled the vertices is not unique. For example, the vertex labeled ba^3 could have been labeled ab simply by going along a different path starting at e. This says that $ab = ba^3$. We also see that $a^4 = e$ and $b^2 = e$. We hope that this example is starting to look familiar. In fact, Figure 7.13 is a Cayley digraph of the dihedral group D_4. We simply relabel a with ρ and b with μ! ▲

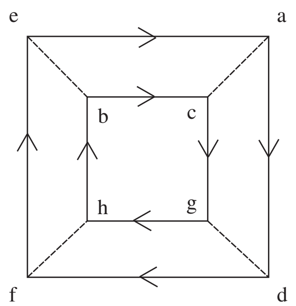
EXERCISES 7

Computations

In Exercises 1 through 8, list the elements of the subgroup generated by the given subset.

1. The subset {2, 3} of $\mathbb{Z}_{12}$
2. The subset {4, 6} of $\mathbb{Z}_{12}$
3. The subset $\{4, 6\}$ in $\mathbb{Z}_{25}$
4. The subset {12, 30} of $\mathbb{Z}_{36}$
5. The subset {12, 42} of $\mathbb{Z}$
6. The subset {18, 24, 39} of $\mathbb{Z}$
7. The subset $\{\mu, \mu\rho^2\}$ in D_8
8. The subset $\{\rho^8, \rho^{10}\}$ in D_{18}
9. Use the Cayley digraph in Figure 7.15 to compute these products. Note that the solid edges represent the generator a and the dashed lines represent b.

   a. $(ba^2)a^3$ **b.** $(ba)(ba^3)$ **c.** $b(a^2b)$



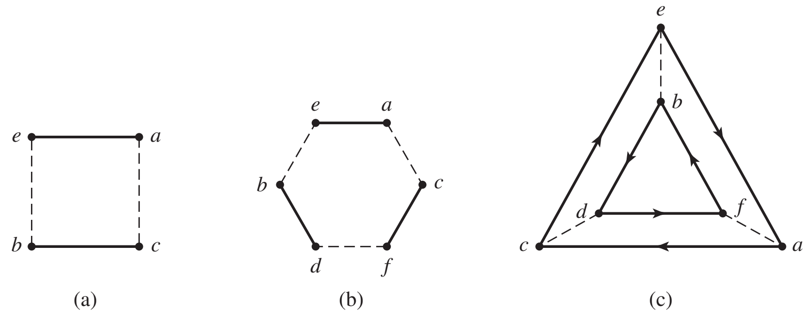
7.15 Figure

In Exercises 10 through 12, give the table for the group having the indicated digraph. In each digraph, take e as identity element. List the identity e first in your table, and list the remaining elements alphabetically, so that your answers will be easy to check.

10. The digraph in Fig. 7.16(a)

11. The digraph in Fig. 7.16(b)

12. The digraph in Fig. 7.16(c)



7.16 Figure

Concepts

13. How can we tell from a Cayley digraph whether or not the corresponding group is commutative?

14. Using the condition found in Exercise 13, show that the group corresponding to the Cayley digraph in Figure 7.13 is not commutative.

15. Is it obvious from a Cayley digraph of a group whether or not the group is cyclic? [*Hint:* Look at Fig. 7.9(b).]

16. The large outside triangle in Fig. 7.11(b) exhibits the cyclic subgroup {0, 2, 4} of $\mathbb{Z}_6$. Does the smaller inside triangle similarly exhibit a cyclic subgroup of $\mathbb{Z}_6$? Why or why not?

17. The generating set $S = \{1, 2\}$ for $\mathbb{Z}_6$ contains more generators than necessary, since 1 is a generator for the group. Nevertheless, we can draw a Cayley digraph for $\mathbb{Z}_6$ with this generating set S. Draw such a Cayley digraph.

18. Draw a Cayley digraph for $\mathbb{Z}_8$ with generating set $S = \{2, 5\}$.

19. A **relation** on a set S of generators of a group G is an equation that equates some product of generators and their inverses to the identity e of G. For example, if $S = \{a, b\}$ and G is commutative so that $ab = ba$, then one relation is $aba^{-1}b^{-1} = e$. If, moreover, b is its own inverse, then another relation is $b^2 = e$.

a. Explain how we can find some relations on S from a Cayley digraph of G.

b. Find three relations on the set $S = \{a, b\}$ of generators for the group described by Fig. 7.13(b).

20. Draw digraphs of the two possible structurally different groups of order 4, taking as small a generating set as possible in each case. You need not label vertices.

Theory

21. Use Cayley digraphs to show that for $n \geq 3$, there exists a nonabelian group with $2n$ elements that is generated by two elements of order 2.

22. Prove that there are at least three different abelian groups of order 8. [Hint: Find a Cayley digraph for a group of order 8 having one generator of order 4 and another of order 2. Find a second Cayley digraph for a group of order 8 having three generators each with order 2.]

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PART II

Structure of Groups

SECTION 8 GROUPS OF PERMUTATIONS

Let $\phi : G \to G'$ be a function mapping the group G to G'. Recall that the homomorphism property of an isomorphism states that for all $a, b \in G$, $\phi(ab) = \phi(a)\phi(b)$. Whenever a function has this property whether or not the function is one-to-one or onto, we say that ϕ is a **group homomorphism**. Of course any group isomorphism is a group homomorphism, but the reverse is not necessarily true.

8.1 Definition Let G and G' be groups with $\phi : G \to G'$. The map ϕ is a **homomorphism** if the homomorphism property

$$\phi(ab) = \phi(a)\phi(b)$$

holds for all $a, b \in G$. ■

8.2 Example Let $\phi : \mathbb{R} \to U$ (the circle group) be defined by the formula

$$\phi(x) = \cos(2\pi x) + i\sin(2\pi x) = e^{2\pi ix}.$$

Then

$$\phi(a + b) = \cos(2\pi(a + b)) + i\sin(2\pi(a + b)) = e^{2\pi i(a+b)}.$$

Using either the usual properties of the exponential function or the formulas from trigonometry involving the sum of two angles, we see that

$$\phi(a + b) = (\cos(2\pi a) + i\sin(2\pi a))(\cos(2\pi b) + i\sin(2\pi b)) = e^{2\pi ai}e^{2\pi bi},$$

so

$$\phi(a + b) = \phi(a)\phi(b),$$

which says that ϕ is a group homomorphism. Although ϕ maps onto U, it is not one-to-one, so ϕ is not an isomorphism.

The identity $0 \in \mathbb{R}$ maps to 1, the identity in U. Furthermore, for any $x \in \mathbb{R}$,

$$\phi(-x) = e^{-2\pi ix} = \frac{1}{e^{2\pi ix}} = (\phi(x))^{-1}.$$

▲