21.1 Example Suppose G is *free with generation x and y except for the relation $xy = yx$*, which we may express as $xyx^{-1}y^{-1} = 1$. Note that the condition $xy = yx$ is exactly what is needed to make G abelian, even though $xyx^{-1}y^{-1}$ is just one of the many possible commutators of $F[\{x, y\}]$. Thus G is free abelian on two generators and is isomorphic to $F[\{x, y\}]$ modulo its commutator subgroup. This commutator subgroup of $F[\{x, y\}]$ is the smallest normal subgroup containing $xyx^{-1}y^{-1}$, since any normal subgroup containing $xyx^{-1}y^{-1}$ gives rise to a factor group that is abelian and thus contains the commutator subgroup by Theorem 13.22. ▲

The preceding example illustrates the general situation. Let $F[A]$ be a free group and suppose that we want to form a new group as much like $F[A]$ as it can be, subject to certain equations that we want satisfied. Any equation can be written in a form in which the right-hand side is 1. Thus we can consider the equations to be $r_i = 1$ for $i \in I$, where $r_i \in F[A]$. If we require that $r_i = 1$, then we will have to have

$$x\left({r_i}^n\right)x^{-1} = 1$$

for any $x \in F[A]$ and $n \in \mathbb{Z}$. Also any product of elements equal to 1 will again have to equal 1. Thus any finite product of the form

$$\prod_j x_j\left({r_{i_j}}^{n_j}\right)x_j^{-1},$$

where the r_{i_j} need not be distinct, will have to equal 1 in the new group. It is readily checked that the set of all these finite products is a normal subgroup R of $F[A]$. Thus any group looking as much as possible like $F[A]$, subject to the requirements $r_i = 1$, also has $r = 1$ for every $r \in R$. But $F[A]/R$ looks like $F[A]$ (remember that we multiply cosets by choosing representatives), except that R has been collapsed to form the identity 1. Hence the group we are after is (at least isomorphic to) $F[A]/R$. We can view this group as described by the generating set A and the set $\{r_i \mid i \in I\}$, which we will abbreviate $\{r_i\}$.

HISTORICAL NOTE

The idea of a group presentation already appears in Arthur Cayley's 1859 paper, "On the Theory of Groups as Depending on the Symbolic Equation $\theta^n = 1$. Third Part." In this article, Cayley gives a complete enumeration of the five groups of order 8, both by listing all the elements of each and by giving for each a presentation. For example, his third example is what is here called the *dihedral group D_4;* Cayley notes that this group is generated by the two elements α, β with the relations $\alpha^4 = 1, \beta^2 = 1, \alpha\beta = \beta\alpha^3$. He also shows more generally that a group of order mn is generated by α, β with the relations $\alpha^m = 1$, $\beta^n = 1, \alpha\beta = \beta\alpha^s$ if and only if $s^n \equiv 1 \pmod{m}$ (see Exercise 13).

In 1878, Cayley returned to the theory of groups and noted that a central problem in that theory is the determination of all groups of a given order n. In the early 1890s, Otto Hölder published several papers attempting to solve Cayley's problem. Using techniques similar to those discussed in Sections 17 and 21, Hölder determined all simple groups of order up to 200 and characterized all the groups of orders p^3, pq^2, pqr, and p^4, where p, q, r are distinct prime numbers. Furthermore, he developed techniques for determining the possible structures of a group G, if one is given the structure of a normal subgroup H and the structure of the factor group G/H. Interestingly, since the notion of an abstract group was still fairly new at this time, Hölder typically began his papers with the definition of a group and also emphasized that isomorphic groups are essentially one and the same object.

21.2 Definition Let A be a set and let $\{r_i\} \subseteq F[A]$. Let R be the least normal subgroup of $F[A]$ containing the r_i. An isomorphism ϕ of $F[A]/R$ onto a group G is a **presentation of** G. The sets A and $\{r_i\}$ give a **group presentation.** The set A is the set of **generators for the presentation** and each r_i is a **relator.** Each $r \in R$ is a **consequence of** $\{r_i\}$. An equation $r_i = 1$ is a **relation.** A **finite presentation** is one in which both A and $\{r_i\}$ are finite sets. ■

This definition may seem complicated, but it really is not. In Example 21.1, $\{x, y\}$ is our set of generators and $xyx^{-1}y^{-1}$ is the only relator. The equation $xyx^{-1}y^{-1} = 1$, or $xy = yx$, is a relation. This was an example of a finite presentation.

If a group presentation has generators x_j and relators r_i, we shall use the notations

$$(x_j : r_i) \qquad \text{or} \qquad (x_j : r_i = 1)$$

to denote the group presentation. We may refer to $F[\{x_j\}]/R$ as *the group with presentation* $(x_j : r_i)$.

Isomorphic Presentations

21.3 Example Consider the group presentation with

$$A = \{a\} \qquad \text{and} \qquad \{r_i\} = \{a^6\},$$

that is, the presentation

$$(a : a^6 = 1).$$

This group defined by one generator a, with the relation $a^6 = 1$, is isomorphic to $\mathbb{Z}_6$.

Now consider the group defined by two generators a and b, with $a^2 = 1, b^3 = 1$, and $ab = ba$, that is, the group with presentation

$$(a, b : a^2, b^3, aba^{-1}b^{-1}).$$

The condition $a^2 = 1$ gives $a^{-1} = a$. Also $b^3 = 1$ gives $b^{-1} = b^2$. Thus every element in this group can be written as a product of nonnegative powers of a and b. The relation $aba^{-1}b^{-1} = 1$, that is, $ab = ba$, allows us to write first all the factors involving a and then the factors involving b. Hence every element of the group is equal to some $a^m b^n$. But then $a^2 = 1$ and $b^3 = 1$ show that there are just six distinct elements,

$$1, b, b^2, a, ab, ab^2.$$

The subgroup $\langle ab \rangle$ contains the elements 1, ab, and the powers of ab:

$$\begin{aligned}(ab)^2 &= a^2b^2 = b^2\\ (ab)^3 &= abb^2 = a\\ (ab)^4 &= a(ab) = b\\ (ab)^5 &= (ab)b = ab^2.\end{aligned}$$

So this group is also a cyclic group of order 6 isomorphic with $\mathbb{Z}_6$. ▲

21.4 Example The dihedral group has presentation

$$D_n \quad : \quad (a, b \mid a^n, b^2, abab)$$

since if we let $a = \mu$ and $b = \rho$ the three relations are exactly the defining relations for D_n. (The last relation $abab = 1$ is equivalent to $ab = ba^{-1}$.) The element $abab$ is in R if and only if $b(abab)b^{-1}$ is in R since R is a normal subgroup. We have $b(abab)b^{-1} = baba$. So in any presentation with generators a and b and a relator $abab$, we can replace $abab$ with $baba$ and get the same subgroup R and therefore the same factor group. Hence the dihedral group also has presentation

$$D_n \quad : \quad (a, b \ : \ a^n, b^2, baba).$$

Setting the relators to 1 gives the equivalent presentation

$$(a, b \, : \, a^n = 1, b^2 = 1, ba = a^{-1}b^{-1})$$

which can also be rewritten as

$$(a, b \, : \, a^n = 1, b^2 = 1, ba = a^{n-1}b). \qquad \blacktriangle$$

The preceding examples illustrate that different presentations may give isomorphic groups. When this happens, we have **isomorphic presentations.** To determine whether two presentations are isomorphic may be very hard. It has been shown (see Rabin [22]) that a number of such problems connected with this theory are not generally solvable; that is, there is no *routine* and well-defined way of discovering a solution in all cases. These unsolvable problems include the problem of deciding whether two presentations are isomorphic, whether a group given by a presentation is finite, free, abelian, or trivial, and the famous *word problem* of determining whether a given word w is a consequence of a given set of relations $\{r_i\}$.

The importance of this material is indicated by our Theorem 20.13, which guarantees that *every group has a presentation*.

21.5 Example Let us show that

$$(x, y : y^2x = y, yx^2y = x)$$

is a presentation of the trivial group of one element. We need only show that x and y are consequences of the relators y^2xy^{-1} and yx^2yx^{-1}, or that $x = 1$ and $y = 1$ can be deduced from $y^2x = y$ and $yx^2y = x$. We illustrate both techniques.

As a consequence of y^2xy^{-1}, we get yx upon conjugation by y^{-1}. From yx we deduce $x^{-1}y^{-1}$, and then $(x^{-1}y^{-1})(yx^2yx^{-1})$ gives xyx^{-1}. Conjugating xyx^{-1} by x^{-1}, we get y. From y we get y^{-1}, and $y^{-1}(yx)$ is x.

Working with relations instead of relators, from $y^2x = y$ we deduce $yx = 1$ upon multiplication by y^{-1} on the left. Then substituting $yx = 1$ into $yx^2y = x$, that is, $(yx)(xy) = x$, we get $xy = x$. Then multiplying by x^{-1} on the left, we have $y = 1$. Substituting this in $yx = 1$, we get $x = 1$.

Both techniques amount to the same work, but it somehow seems more natural to most of us to work with relations. $\blacktriangle$

Applications

We conclude this chapter with two applications.

21.6 Example Let us determine all groups of order 10 up to isomorphism. We know from the Fundamental Theorem 9.12 that every abelian group of order 10 is isomorphic to $\mathbb{Z}_{10}$. Suppose that G is nonabelian of order 10. By Sylow theory, G contains a normal subgroup H of order 5, and H must be cyclic. Let a be a generator of H. Then G/H is of order 2 and thus isomorphic to Z_2. If $b \in G$ and $b \notin H$, we must then have $b^2 \in H$. Since every element of H except 1 has order 5, if b^2 were not equal to 1, then b^2 would have order 5, so b would have order 10. This would mean that G would be cyclic, contradicting our assumption that G is not abelian. Thus $b^2 = 1$. Finally, since H is a normal subgroup of $G, bHb^{-1} = H$, so in particular, $bab^{-1} \in H$. Since conjugation by b is an automorphism of H, bab^{-1} must be another element of H of order 5, hence bab^{-1} equals a, a^2, a^3, or a^4. But $bab^{-1} = a$ would give $ba = ab$, and then G would be abelian, since a and b generate G. Thus the possibilities for presentations of G are:

1. $(a, b : a^5 = 1, b^2 = 1, ba = a^2b)$,
2. $(a, b : a^5 = 1, b^2 = 1, ba = a^3b)$,
3. $(a, b : a^5 = 1, b^2 = 1, ba = a^4b)$.