We see that σ is a product of a 3-cycle and a 2-cycle. Sometimes we refer to a 2-cycle as a **transposition**.

A collection of cycles is said to be **disjoint** if no entry is in more than one cycle. Note that σ could also be written as $(3, 6, 1)(4, 2)$, $(2, 4)(1, 3, 6)$, or in a number of other ways. In general it doesn't matter which order we write the disjoint cycles, and inside each cycle we can start with any number as long as we keep the cyclic order the same. It is clear that any permutation in S_n can be written in disjoint cycle notation and that the representation is unique up to the order the cycles are written and the cyclic order within each cycle.

4.11 Example In disjoint cycle notation, $\sigma \in S_9$ is written as $(1, 5, 2, 7)(3, 4, 9)$. Let us rewrite σ in two-row notation. Reading off the disjoint cycle notation we see that $\sigma(1) = 5$, $\sigma(5) = 2$, $\sigma(2) = 7$, $\sigma(7) = 1$, $\sigma(3) = 4$, $\sigma(4) = 9$, and $\sigma(9) = 3$. Since 6 and 8 do not appear in either cycle, we know that $\sigma(6) = 6$ and $\sigma(8) = 8$. Therefore,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 7 & 4 & 9 & 2 & 6 & 1 & 8 & 3 \end{pmatrix}$$ ▲

The operation that makes S_n a group is composition of functions. Keeping this in mind, we can see how to multiply permutations written in disjoint cycle notation.

4.12 Example Let $\sigma = (1, 5, 3, 2, 6)$ and $\tau = (1, 2, 4, 3, 6)$ in S_6. Let us find $\sigma\tau$ in disjoint cycle notation without resorting to using two-row notation. So

$$\sigma\tau = (1, 5, 3, 2, 6)(1, 2, 4, 3, 6).$$

We need to rewrite this product in disjoint cycles. So we ask where 1 is mapped. Since the operation is function composition, we see that the cycle τ on the right sends 1 to 2 and then the cycle on the left sends 2 to 6. So $\sigma\tau(1) = 6$ and we start our cycle by writing

$$(1, 6$$

Now we see that τ maps 6 to 1 and σ maps 1 to 5, so we write

$$(1, 6, 5$$

We note that 5 is not in the cycle $(1, 2, 4, 3, 6)$, so $\tau(5) = 5$ and $\sigma\tau(5) = \sigma(5) = 3$. So we write

$$(1, 6, 5, 3$$

Continuing in the same manner, we see that 3 maps to 1 and we complete the first cycle:

$$(1, 6, 5, 3)$$

We are now ready to start the second cycle. We note that we have still not seen where 2 maps, so we start the next cycle with 2 and we write

$$\sigma\tau = (1, 5, 3, 2, 6)(1, 2, 4, 3, 6) = (1, 6, 5, 3)(2, 4)$$

using the same method we used for the first cycle. We know we are through since we have used every number 1 through 6. ▲

Example 4.12 illustrates the process of multiplying permutations in general. We move from right to left between the cycles, and within the cycles we move from left to right.

4.13 Example We compute the product of the permutations

$$\sigma = (1, 5)(2, 4)(1, 4, 3)(2, 5)(4, 2, 1)$$

using disjoint cycle notation.

We start by seeing where 1 is mapped. The first cycle on the right maps 1 to 4. We are using function composition, so we next check what $(2, 5)$ does to 4, which is nothing. So we move to the cycle $(1, 4, 3)$ and note that 4 is mapped to 3. Next, 3 is not in the cycle $(2, 4)$ and so $(2, 4)$ does not move 3 . Finally, $(1, 5)$ also does not move 3 and we conclude that $\sigma(1) = 3$. We next need to determine where 3 is mapped by σ and continue until we arrive at

$$\sigma = (1, 3, 5, 4)(2) = (1, 3, 5, 4). \qquad \blacktriangle$$

It is interesting to note that in Example 4.13 the group was never specified. The same calculation is valid whether the group is S_5, S_6, or S_n for any $n \geq 5$.

4.14 Example We compute the inverse of $\sigma = (1, 5, 7)(3, 8, 2, 4, 6)$. We first note that in general for a group $(ab)^{-1} = b^{-1}a^{-1}$, so

$$\sigma^{-1} = (3, 8, 2, 4, 6)^{-1}(1, 5, 7)^{-1}.$$

The inverse of a cycle is simply the cycle written backward:

$$\sigma^{-1} = (6, 4, 2, 8, 3)(7, 5, 1).$$

This is a perfectly good way of writing σ^{-1}, but since disjoint cycles commute and we can start each cycle with any entry in the cycle, we could write

$$\sigma^{-1} = (1, 7, 5)(2, 8, 3, 6, 4). \qquad \blacktriangle$$

With a little practice, computing products of permutations in disjoint cycle notation becomes routine. We give the table for S_3.

4.15 Table

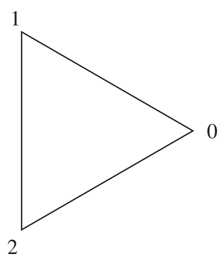
S_3 $\circ$	ι	$(1,2,3)$	$(1,3,2)$	$(1,2)$	$(1,3)$	$(2,3)$
ι	ι	$(1,2,3)$	$(1,3,2)$	$(1,2)$	$(1,3)$	$(2,3)$
$(1,2,3)$	$(1,2,3)$	$(1,3,2)$	ι	$(1,3)$	$(2,3)$	$(1,2)$
$(1,3,2)$	$(1,3,2)$	ι	$(1,2,3)$	$(2,3)$	$(1,2)$	$(1,3)$
$(1,2)$	$(1,2)$	$(2,3)$	$(1,3)$	ι	$(1,3,2)$	$(1,2,3)$
$(1,3)$	$(1,3)$	$(1,2)$	$(2,3)$	$(1,2,3)$	ι	$(1,3,2)$
$(2,3)$	$(2,3)$	$(1,3)$	$(1,2)$	$(1,3,2)$	$(1,2,3)$	ι

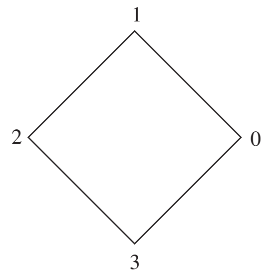
Again we can see that S_3 is not abelian since the table is not symmetric about the main diagonal. We also notice that although disjoint cycles commute, the same cannot be said for cycles that are not disjoint. For example we see in Table 4.15 that $(1, 2)(2, 3) = (1, 2, 3) \neq (1, 3, 2) = (2, 3)(1, 2)$.

The Dihedral Group

We next define a collection of finite groups based on the symmetries of regular n-gons. To be specific, we use as our standard regular n-gon the one whose points are U_n. Recall that U_n includes the point $(1, 0)$ and the other points are spaced uniformly around the unit circle to form the vertices of a regular n-gon, which we denote by P_n. We label the points starting at $(1, 0)$ with 0 and continue labeling them $1, 2, 3, \ldots, n - 1$ around the circle counterclockwise. Note that this is the same labeling as the isomorphism between U_n and $\mathbb{Z}_n$ that we saw in Section 3. When we refer to a vertex we will reference it by its label. So vertex 0 is the point $(1, 0)$. Note that the edges of P_n consist of the line segments between vertices k and $k +_n 1$ for $0 \leq k \leq n - 1$.

4.16 Definition Let $n \geq 3$. Then D_n is the set of all one-to-one functions $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ that map onto $\mathbb{Z}_n$ with the property that the line segment between vertices i and j is an edge in P_n if and only if the line segment between $\phi(i)$ and $\phi(j)$ is an edge of P_n. The n^{th} **dihedral group** is the set D_n with binary operation function composition. ■





We justify calling $\langle D_n, \circ\rangle$ a group with Theorem 4.17.

4.17 Theorem For any $n \geq 3$, $\langle D_n, \circ\rangle$ is a group.

Proof We first show that function composition is an operation on D_n. Let $\phi, \theta \in D_n$ and suppose that the line between vertices i and j is an edge in P_n. Since $\theta \in D_n$, the line between $\theta(i)$ and $\theta(j)$ is an edge of P_n. Because $\phi \in D_n$, and the line between $\theta(i)$ and $\theta(j)$ is an edge, the line between $\phi(\theta(i)) = \phi \circ \theta(i)$ and $\phi(\theta(j)) = \phi \circ \theta(j)$ is an edge of P_n.

We leave it to the reader to check that if the line segment between $\phi(\theta(i)) = \phi \circ \theta(i)$ and $\phi(\theta(j)) = \phi \circ \theta(j)$ is an edge of P_n, then the line segment between i and j is an edge of P_n.

We also know that the composition of one-to-one and onto functions is one-to-one and onto, so $\phi \circ \theta \in D_n$. Therefore, function composition is an operation on D_n.

The operation of composition of functions is associative, so $\mathscr{G}_1$ is satisfied. The function $\iota : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $\iota(k) = k$ is an identity in D_n, so $\mathscr{G}_2$ is satisfied. Finally, if $\phi \in D_n$, then $\phi^{-1} \in D_n$; the inverse function for f acts as the inverse in the group sense, so $\mathscr{G}_3$ is satisfied. Therefore, $\langle D_n, \circ\rangle$ is a group. ◆

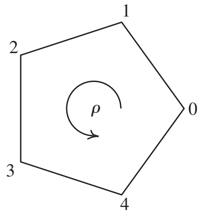
Following tradition, we will use multiplicative notation in the dihedral groups instead of using $\circ$. If the operation on D_n were abelian, we could use additive notation, but in Example 4.18 we find that D_n is not abelian.

4.18 Example Let $n \geq 3$ and $\rho : Z_n \to Z_n$ be given by rotating the n-gon P_n by $\frac{2\pi}{n}$, which just rotates each vertex to the next one. That is,

$$\rho(k) = k +_n 1$$

for each $k \in \mathbb{Z}_n$, as can be visualized in Figure 4.19. The function ρ matches edges to edges and it is one-to-one and onto. So $\rho \in D_n$.

4.19 Figure



A second element in D_n is reflection about the x-axis, which we call μ. By glancing at Figure 4.20 we see that in D_5, $\mu(0) = 0$, $\mu(1) = 4$, $\mu(2) = 3$, $\mu(3) = 2$, and $\mu(4) = 1$. For any $n \geq 3$ in general, if $k \in \mathbb{Z}_n$, then

$$\mu(k) = -k.$$

(Recall that in $\mathbb{Z}_n$, $-k$ is the additive inverse of k, which is $n - k$ for $k > 0$ and $-0 = 0$.)