so $i(a+b) = i(a) + i(b)$. Furthermore,

$$i(ab) = [(ab, 1)],$$

while

$$i(a)i(b) = [(a, 1)][(b, 1)] = [(ab, 1)],$$

so $i(ab) = i(a)i(b)$.

It remains for us to show only that i is one-to-one. If $i(a) = i(b)$, then

$$[(a, 1)] = [(b, 1)],$$

so $(a, 1) \sim (b, 1)$ giving $a1 = 1b$; that is,

$$a = b.$$

Thus i is an isomorphism of D with $i[D] = D'$, and, of course, D' is then a subdomain of F. ◆

Since $[(a, b)] = [(a, 1)][(1, b)] = [(a, 1)]/[(b, 1)] = i(a)/i(b)$ clearly holds in F, we have now proved the following theorem.

26.6 Theorem Any integral domain D can be enlarged to (or embedded in) a field F such that every element of F can be expressed as a quotient of two elements of D. (Such a field F is a **field of quotients of** D.)

Uniqueness

The field F can be regarded as a minimal field containing D. This is intuitively evident, since every field containing D must contain all elements a/b for every $a, b \in D$ with $b \neq 0$. The next theorem will show that every field containing D contains a subfield that is a field of quotients of D, and that any two fields of quotients of D are isomorphic.

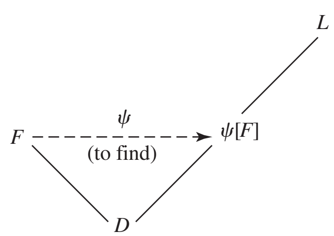
26.7 Theorem Let F be a field of quotients of D and let L be any field containing D. Then there exists a map $\psi : F \to L$ that gives an isomorphism of F with a subfield of L such that $\psi(a) = a$ for $a \in D$.

Proof The subring and mapping diagram in Fig. 26.8 may help you to visualize the situation for this theorem.

An element of F is of the form $a /_F b$ where $/_F$ denotes the quotient of $a \in D$ by $b \in D$ regarded as elements of F. We of course want to map $a /_F b$ onto $a /_L b$ where $/_L$ denotes the quotient of elements in L. The main job will be to show that such a map is well defined.

We must define $\psi : F \to L$, and we start by defining

$$\psi(a) = a \quad \text{for} \quad a \in D.$$



26.8 Figure

Every $x \in F$ is a quotient $a /_F b$ of some two elements a and $b, b \neq 0$, of D. Let us attempt to define ψ by

$$\psi(a /_F b) = \psi(a) /_L \psi(b).$$

We must first show that this map ψ is sensible and well-defined. Since ψ is the identity on D, for $b \neq 0$ we have $\psi(b) \neq 0$, so our definition of $\psi(a /_F b)$ as $\psi(a) /_L \psi(b)$ makes sense. If $a /_F b = c /_F d$ in F, then $ad = bc$ in D, so $\psi(ad) = \psi(bc)$. But since ψ is the identity on D,

$$\psi(ad) = \psi(a)\psi(d) \qquad \text{and} \qquad \psi(bc) = \psi(b)\psi(c).$$

Thus

$$\psi(a) /_L \psi(b) = \psi(c) /_L \psi(d)$$

in L, so ψ is well-defined.

The equations

$$\psi(xy) = \psi(x)\psi(y)$$

and

$$\psi(x + y) = \psi(x) + \psi(y)$$

follow easily from the definition of ψ on F and from the fact that ψ is the identity on D.

If $\psi(a /_F b) = \psi(c /_F d)$, we have

$$\psi(a) /_L \psi(b) = \psi(c) /_L \psi(d)$$

so

$$\psi(a)\psi(d) = \psi(b)\psi(c).$$

Since ψ is the identity on D, we then deduce that $ad = bc$, so $a /_F b = c /_F d$. Thus ψ is one-to-one.

By definition, $\psi(a) = a$ for $a \in D$. ◆

26.9 Corollary Every field L containing an integral domain D contains a field of quotients of D.

Proof In the proof of Theorem 26.7 every element of the subfield $\psi[F]$ of L is a quotient in L of elements of D. ◆

26.10 Corollary Any two fields of quotients of an integral domain D are isomorphic.

Proof Suppose in Theorem 26.7 that L is a field of quotients of D, so that every element x of L can be expressed in the form $a /_L b$ for $a, b \in D$. Then L is the field $\psi[F]$ of the proof of Theorem 26.7 and is thus isomorphic to F. ◆

EXERCISES 26

Computations

1. Describe the field F of quotients of the integral subdomain

$$D = \{n + mi \mid n, m \in \mathbb{Z}\}$$

of $\mathbb{C}$. "Describe" means give the elements of $\mathbb{C}$ that make up the field of quotients of D in $\mathbb{C}$. (The elements of D are the **Gaussian integers.**)

2. Describe (in the sense of Exercise 1) the field F of quotients of the integral subdomain $D = \{n + m\sqrt{3} \mid n, m \in \mathbb{Z}\}$ of $\mathbb{R}$.

Concepts

3. Correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

A *field of quotients* of an integral domain D is a field F in which D can be embedded so that every nonzero element of D is a unit in F.

4. Determine whether each of the following is true or false.

a. $\mathbb{Q}$ is a field of quotients of $\mathbb{Z}$.
b. $\mathbb{R}$ is a field of quotients of $\mathbb{Z}$.
c. $\mathbb{R}$ is a field of quotients of $\mathbb{R}$.
d. $\mathbb{C}$ is a field of quotients of $\mathbb{R}$.
e. If D is a field, then any field of quotients of D is isomorphic to D.
f. The fact that D has no divisors of 0 was used strongly several times in the construction of a field F of quotients of the integral domain D.
g. Every element of an integral domain D is a unit in a field F of quotients of D.
h. Every nonzero element of an integral domain D is a unit in a field F of quotients of D.
i. A field of quotients F' of a subdomain D' of an integral domain D can be regarded as a subfield of some field of quotients of D.
j. Every field of quotients of $\mathbb{Z}$ is isomorphic to $\mathbb{Q}$.

5. Show by an example that a field F' of quotients of a proper subdomain D' of an integral domain D may also be a field of quotients for D.

Theory

6. Prove Part 2 of Step 3. You may assume any preceding part of Step 3.

7. Prove Part 3 of Step 3. You may assume any preceding part of Step 3.

8. Prove Part 4 of Step 3. You may assume any preceding part of Step 3.

9. Prove Part 5 of Step 3. You may assume any preceding part of Step 3.

10. Prove Part 6 of Step 3. You may assume any preceding part of Step 3.

11. Prove Part 7 of Step 3. You may assume any preceding part of Step 3.

12. Let R be a nonzero commutative ring, and let T be a nonempty subset of R closed under multiplication and containing neither 0 nor divisors of 0. Starting with $R \times T$ and otherwise exactly following the construction in this section, we can show that the ring R can be enlarged to a *partial ring of quotients* $Q(R, T)$. Think about this for 15 minutes or so; look back over the construction and see why things still work. In particular, show the following:

a. $Q(R, T)$ has unity even if R does not.
b. In $Q(R, T)$, every nonzero element of T is a unit.

13. Prove from Exercise 12 that every nonzero commutative ring containing an element a that is not a divisor of 0 can be enlarged to a commutative ring with unity. Compare with Exercise 36 of Section 23.

14. With reference to Exercise 12, how many elements are there in the ring $Q(\mathbb{Z}_4, \{1, 3\})$?

15. With reference to Exercise 12, describe the ring $Q(\mathbb{Z}, \{2^n \mid n \in \mathbb{Z}^+\})$, by describing a subring of $\mathbb{R}$ to which it is isomorphic.

16. With reference to Exercise 12, describe the ring $Q(3\mathbb{Z}, \{6^n \mid n \in \mathbb{Z}^+\})$ by describing a subring of $\mathbb{R}$ to which it is isomorphic.

17. With reference to Exercise 12, suppose we drop the condition that T have no divisors of zero and just require that nonempty T not containing 0 be closed under multiplication. The attempt to enlarge R to a commutative ring with unity in which every nonzero element of T is a unit must fail if T contains an element a that is a divisor of 0, for a divisor of 0 cannot also be a unit. Try to discover where a construction parallel to that in the text but starting with $R \times T$ first runs into trouble. In particular, for $R = \mathbb{Z}_6$ and $T = \{1, 2, 4\}$, illustrate the first difficulty encountered. [*Hint:* It is in Step 1.]

SECTION 27 RINGS OF POLYNOMIALS

Polynomials in an Indeterminate

We all have a pretty workable idea of what constitutes a *polynomial in x with coefficients in a ring R*. We can guess how to add and multiply such polynomials and know what is meant by the *degree* of a polynomial. We expect that the set $R[x]$ of all polynomials with coefficients in the ring R is itself a ring with the usual operations of polynomial addition and multiplication, and that R is a subring of $R[x]$. However, we will be working with polynomials from a slightly different viewpoint than the approach in high school algebra or calculus, and there are a few things that we want to say.

In the first place, we will call x an **indeterminate** rather than a variable. Suppose, for example that our ring of coefficients is $\mathbb{Z}$. One of the polynomials in the ring $\mathbb{Z}[x]$ is $1x$, which we shall write simply as x. Now x is not 1 or 2 or any of the other elements of $\mathbb{Z}[x]$. Thus from now on we will never write such things as "$x = 1$" or "$x = 2$," as we have done in other courses. We call x an indeterminate rather than a variable to emphasize this change. Also, we will never write an expression such as "$x^2 - 4 = 0$," simply because $x^2 - 4$ is not the zero polynomial in our ring $\mathbb{Z}[x]$. We are accustomed to speaking of "solving a polynomial equation," and will be spending a lot of time in the remainder of our text discussing this, but we will always refer to it as "finding a zero of a polynomial." In summary, we try to be careful in our discussion of algebraic structures not to say in one context that things are equal and in another context that they are not equal.

If a person knows nothing about polynomials, it is not an easy task to describe precisely the nature of a polynomial in x with coefficients in a ring R. If we just define such a polynomial to be a *finite formal sum*

$$\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n,$$

HISTORICAL NOTE

The use of x and other letters near the end of the alphabet to represent an "indeterminate" is due to René Descartes (1596–1650). Earlier, François Viete (1540–1603) had used vowels for indeterminates and consonants for known quantities. Descartes is also responsible for the first publication of the factor theorem (Corollary 28.4) in his work *The Geometry,* which appeared as an appendix to his *Discourse on Method* (1637). This work also contained the first publication of the basic concepts of analytic geometry; Descartes showed how geometric curves can be described algebraically.

Descartes was born to a wealthy family in La Haye, France; since he was always of delicate health, he formed the habit of spending his mornings in bed. It was at these times that he accomplished his most productive work. The *Discourse on Method* was Descartes' attempt to show the proper procedures for "searching for truth in the sciences." The first step in this process was to reject as absolutely false everything of which he had the least doubt; but, since it was necessary that he who was thinking was "something," he conceived his first principle of philosophy: "I think, therefore I am." The most enlightening parts of the *Discourse on Method,* however, are the three appendices: *The Optics, The Geometry,* and *The Meteorology*. It was here that Descartes provided examples of how he actually applied his method. Among the important ideas Descartes discovered and published in these works were the sine law of refraction of light, the basics of the theory of equations, and a geometric explanation of the rainbow.

In 1649, Descartes was invited by Queen Christina of Sweden to come to Stockholm to tutor her. Unfortunately, the Queen required him, contrary to his long-established habits, to rise at an early hour. He soon contracted a lung disease and died in 1650.