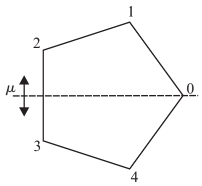
**4.20 Figure**



Let us check if $\mu\rho = \rho\mu$. We start by checking what each function does to 0.

$$\begin{aligned}\mu(\rho(0)) &= \mu(1)\\ &= n-1\end{aligned}$$

$$\begin{aligned}\rho(\mu(0)) &= \rho(0)\\ &= 1\end{aligned}$$

Since $n \geq 3$, $n-1 \neq 1$, which implies that $\mu\rho \neq \rho\mu$. Thus for all $n \geq 3$, $D_n$ is not abelian. ▲

**4.21 Theorem** Let $n \geq 3$. The order of the dihedral group $D_n$ is $2n$ and

$$D_n = \{\iota, \rho, \rho^2, \rho^3, \cdots, \rho^{n-1}, \mu, \mu\rho, \mu\rho^2, \mu\rho^3, \cdots, \mu\rho^{n-1}\}.$$

***Proof*** We first show there can be at most $2n$ elements of $D_n$. If we map the vertices $\mathbb{Z}_n$ to the vertices $\mathbb{Z}_n$, vertex 0 has $n$ possible images. Let $y$ be the image of vertex 0. Since $y$ is connected by an edge to just two vertices, 1 must map to one of these two vertices. So after the image of vertex 0 is determined, there are only two choices for the image of 1. After the images of vertices 0 and 1 are determined, the rest are fixed. This means that there are at most $2n$ elements of $D_n$.

To show that $|D_n| = 2n$ we only need to show that no two of the functions $\iota = \rho^0, \rho, \rho^2, \rho^3, \ldots, \rho^{n-1}, \mu, \mu\rho, \mu\rho^2, \mu\rho^3, \ldots, \mu\rho^{n-1}$ are the same. We first suppose that $\rho^k = \rho^r$ for some integers $0 \leq k \leq n-1$ and $0 \leq r \leq n-1$. Then:

$$\begin{aligned}\rho^k(0) &= \rho^r(0)\\ k +_n 0 &= r +_n 0\\ k &= r\end{aligned}$$

This shows that no two of $\iota = \rho^0, \rho, \rho^2, \rho^3, \ldots, \rho^{n-1}$ are the same.

We next show that no two of $\mu = \mu\rho^0, \mu\rho, \mu\rho^2, \mu\rho^3, \ldots, \mu\rho^{n-1}$ are the same. As before we assume that $\mu\rho^k = \mu\rho^r$ where $0 \leq k \leq n-1$ and $0 \leq r \leq n-1$ are integers. By cancellation, we have $\rho^k = \rho^r$. But then $k = r$ as shown above. Therefore no two of $\mu = \mu\rho^0, \mu\rho, \mu\rho^2, \mu\rho^3, \ldots, \mu\rho^{n-1}$ are the same.

It now only remains to show that there are no values for $k$ and $r$ with $\rho^k = \mu\rho^r$. Note that traversing the $n$-gon in the order

$$\rho^k(0), \rho^k(1), \rho^k(2), \cdots, \rho^k(n-1)$$

progresses in a counterclockwise manner regardless of which $k$ we use. On the other hand,

$$\mu\rho^k(0), \mu\rho^k(1), \mu\rho^k(3), \cdots, \mu\rho^k(n)$$

traverses the $n$-gon in a clockwise manner. This shows that there are no values of $k$ and $r$ for which $\rho^k = \mu\rho^r$. Therefore, $D_n$ has at least $2n$ elements. Combining this with the fact that $D_n$ has at most $2n$ elements shows that $|D_n| = 2n$ and

$$D_n = \{\iota, \rho, \rho^2, \rho^3, \cdots, \rho^{n-1}, \mu, \mu\rho, \mu\rho^2, \mu\rho^3, \cdots, \mu\rho^{n-1}\}.$$ ◆

Theorem 4.21 says that if $\phi \in D_n$, then there is an integer $0 \leq k \leq n-1$ such that either $\phi = \rho^k$ or else $\phi = \mu\rho^k$. We refer to this representation of $\phi$ as the **standard form**. We notice that each application of $\mu$ reverses the direction traversed by the images of $0, 1, 2, 3, \ldots, n$. We use this fact in the following example.

**4.22 Example** Let $n \geq 3$. We know $\rho\mu \neq \mu\rho$ from Example 4.18, so let us determine $\rho\mu \in D_n$ in standard form. Each time we apply $\mu$ we reverse the clock direction of the images of $0, 1, 2, 3, \ldots, n-1$. This means that $\mu\rho\mu$ reverses direction twice, so the rotation is back to counterclockwise. Thus $\mu\rho\mu = \rho^k$ for some $k$. We determine the value of $k$ by determining where 0 is sent:

$$k = \rho^k(0) = \mu\rho\mu(0) = \mu\rho(0) = \mu(1) = n - 1$$

Therefore,

$$\mu\rho\mu = \rho^{n-1}.$$

Multiplying both sides on the left by $\mu$ yields:

$$\mu\mu\rho\mu = \mu\rho^{n-1}$$

Since $\mu\mu = \iota$, we conclude that

$$\rho\mu = \mu\rho^{n-1}. \quad \blacktriangle$$

When computing products in $D_n$ we normally want our answer in standard form. This is not difficult if we keep in mind a few basic facts about the group $D_n$. We have shown some of the properties listed below, and the rest you will be asked to verify in the exercises.

1. $\rho^n = \iota$ (Rotation by $2\pi$ is the identity map.)
2. $(\rho^k)^{-1} = \rho^{n-k}$
3. $\mu^2 = \iota$, which implies $\mu^{-1} = \mu$ (Reflect across a line twice is the identity map.)
4. $\rho^k\mu = \mu\rho^{n-k}$ (Example 4.22 for $k = 1$ and Exercise 30 for any $k$.)

**4.23 Example** In the group $D_5$ compute $(\mu\rho^2)(\mu\rho)$. We see that

$$\begin{aligned}(\mu\rho^2)(\mu\rho) &= \mu\rho^2\mu\rho \\ &= \mu(\rho^2\mu)\rho \\ &= \mu(\mu\rho^{5-2})\rho \\ &= \mu^2\rho^4 \\ &= \rho^4\end{aligned} \quad \blacktriangle$$

**4.24 Example** In the dihedral group $D_n$ compute $(\mu\rho^k)^{-1}$.

$$\begin{aligned}(\mu\rho^k)^{-1} &= (\rho^k)^{-1}\mu^{-1} \\ &= \rho^{n-k}\mu \\ &= \mu\rho^{n-(n-k)} \\ &= \mu\rho^k\end{aligned} \quad \blacktriangle$$

In Example 4.24 we determined that the inverse of $\mu\rho^k$ is itself, which suggests that $\mu\rho^k$ could be reflection across a line of symmetry. In Exercise 37, you will be asked to show this is the case. Geometrically, we can see that each of the elements of the form $\mu\rho^k$ is reflection across a line. Placing one mirror along the line of reflection for $\mu$ and another mirror along the line of reflection for $\mu\rho$ is the basis for designing a kaleidoscope. Any element in $D_n$ can be written as a product using only the elements $\mu$ and $\mu\rho$ since we can write $\rho = \mu\mu\rho$. In a kaleidoscope successive reflections across the mirrors correspond to taking products involving $\mu$ and $\mu\rho$. So the image you see in the kaleidoscope has all the symmetries in $D_n$. That is, you can rotate the image by $\frac{360^\circ}{n}$ or reflect it across any one of the lines of reflection for the elements $\mu\rho^k$. Figure 4.25 is a typical image from a kaleidoscope with dihedral group $D_{16}$ symmetries.