SECTION 39

1. $x^2 - 2x - 1$ **3.** $x^2 - 2x + 2$
5. $x^{12} + 3x^8 - 4x^6 + 3x^4 + 12x^2 + 5$
7. $\mathrm{Irr}(\alpha, \mathbb{Q}) = x^4 - \frac{2}{3}x^2 - \frac{62}{9}$; $\deg(\alpha, \mathbb{Q}) = 4$
9. Algebraic, $\deg(\alpha, F) = 2$
11. Transcendental
13. Algebraic, $\deg(\alpha, F) = 2$
15. Algebraic, $\deg(\alpha, F) = 1$
17. $x^2 + x + 1 = (x - \alpha)(x + 1 + \alpha)$
23. **a.** T **c.** T **e.** F **g.** F **i.** F
25. **b.** $x^3 + x^2 + 1 = (x - \alpha)(x - \alpha^2)[x - (1 + \alpha + \alpha^2)]$
27. The polynomial $\mathrm{irr}(\alpha, F)$ is a generator of the principal ideal of all polynomials in $F[x]$ that have α as a zero. Therefore, irr is the monic polynomial of **minimum degree** that has α as a zero. Also, $\mathrm{irr}(\alpha, F)$ is the only **irreducible** monic polynomial that has α as a zero.

SECTION 40

1. 2, $\{1, \sqrt{2}\}$ **3.** 4, $\{1, \sqrt{3}, \sqrt{2}, \sqrt{6}\}$
5. 6, $\{1, \sqrt{2}, \sqrt[3]{2}, \sqrt{2}(\sqrt[3]{2}), (\sqrt[3]{2})^2, \sqrt{2}(\sqrt[3]{2})^2\}$ **7.** 2, $\{1, \sqrt{6}\}$
9. 9, $\{1, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt[3]{3}, \sqrt[3]{6}, \sqrt[3]{12}, \sqrt[3]{9}, \sqrt[3]{18}, \sqrt[3]{36}\}$
11. 2, $\{1, \sqrt{2}\}$ **13.** 2, $\{1, \sqrt{2}\}$
19. **a.** F **c.** F **e.** F **g.** F **i.** F
23. *Partial answer:* Extensions of degree 2^n for $n \in \mathbb{Z}^+$ are obtained.

SECTION 41

All odd-numbered answers require proofs and are not listed here.

SECTION 42

1. Yes **3.** Yes **5.** 6 **7.** 0

SECTION 43

1. $\sqrt{2}, -\sqrt{2}$ **3.** $3 + \sqrt{2}, 3 - \sqrt{2}$ **5.** $\sqrt{2} + i, \sqrt{2} - i, -\sqrt{2} + i, -\sqrt{2} - i$
7. $\sqrt{1 + \sqrt{2}}, -\sqrt{1 + \sqrt{2}}, \sqrt{1 - \sqrt{2}}, -\sqrt{1 - \sqrt{2}}$ **9.** $\sqrt{3}$
11. $-\sqrt{2} + 3\sqrt{5}$ **13.** $-\sqrt{2} + \sqrt{45}$
15. $\sqrt{3} + \sqrt{5}$
17. $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ **19.** $\mathbb{Q}(\sqrt{3}, \sqrt{10})$ **21.** $\mathbb{Q}$
25. **a.** $3 - \sqrt{2}$ **b.** They are the same maps.
39. Yes

SECTION 44

1. 2 **3.** 4 **5.** 2 **7.** 1 **9.** 2
11. $\sqrt{2} \to \sqrt{2}, \sqrt{3} \to \sqrt{3}, \sqrt{5} \to \sqrt{5}$; and $\sqrt{2} \to \sqrt{2}, \sqrt{3} \to -\sqrt{3}, \sqrt{5} \to -\sqrt{5}$
13. $\sqrt{2} \to \sqrt{2}, \sqrt{3} \to \sqrt{3}, \sqrt{5} \to -\sqrt{5}$; $\sqrt{2} \to \sqrt{2}, \sqrt{3} \to -\sqrt{3}, \sqrt{5} \to \sqrt{5}$;
$\sqrt{2} \to -\sqrt{2}, \sqrt{3} \to \sqrt{3}, \sqrt{5} \to \sqrt{5}$; $\sqrt{2} \to -\sqrt{2}, \sqrt{3} \to -\sqrt{3}, \sqrt{5} \to -\sqrt{5}$
15. There are six extensions. One for each of the combinations where $\sqrt{3}i$ maps to $\pm\sqrt{3}i$ and $\sqrt[3]{2}$ maps to one of $\alpha_1, \alpha_2, \alpha_3$.

17. **a.** $\mathbb{Q}(\pi^2)$ **b.** $\sqrt{\pi}$ can map to either $\pm\sqrt{\pi}i$.

19. $1 \leq [E : F] \leq n!$

21. Let $F = \mathbb{Q}$ and $E = \mathbb{Q}(\sqrt{2})$. Then

$$f(x) = x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$$

has a zero in E, but does not split in E.

SECTION 45

1. $\alpha = \sqrt[6]{2} = 2/(\sqrt[3]{2}\sqrt{2}).\sqrt{2} = (\sqrt[6]{2})^3$, $\sqrt[3]{2} = (\sqrt[6]{2})^2$. (Other answers are possible.)

3. $\alpha = \sqrt{2} + \sqrt{5}$. $\sqrt{2} = \dfrac{1}{6}\alpha^3 - \dfrac{11}{6}\alpha$, $\sqrt{5} = \dfrac{17}{6}\alpha - \dfrac{1}{6}\alpha^3$. (Other answers are possible.)

7. $f(x) = x^4 - 4x^2 + 4 = (x^2 - 2)^2$. Here $f(x)$ is not an irreducible polynomial. Every irreducible factor of $f(x)$ has zeros of multiplicity 1 only.

SECTION 46

1. 4 **3.** 8 **5.** 4 **7.** 2

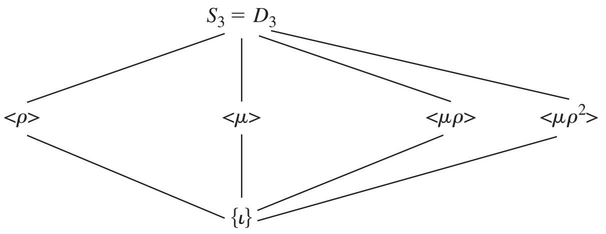
9. The group has two elements, the identity automorphism ι of $\mathbb{Q}(i)$ and σ such that $\sigma(i) = -i$.

11. **b.** Let $\alpha_1 = \sqrt[3]{5}$, $\alpha_2 = \sqrt[3]{5}\dfrac{-1 + i\sqrt{3}}{2}$, and $\alpha_3 = \sqrt[3]{5}\dfrac{-1 - i\sqrt{3}}{2}$.

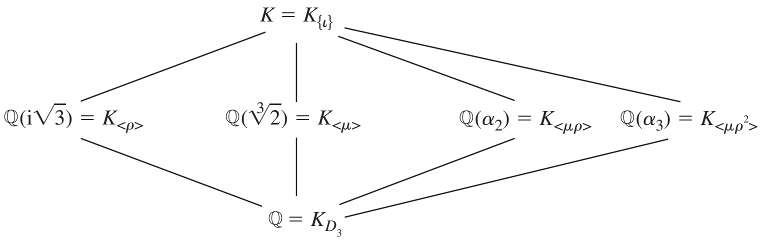
The maps are
ι, where ι is the identity map;
ρ, where $\rho(\alpha_1) = \alpha_2$ and $\rho(i\sqrt{3}) = i\sqrt{3}$;
ρ^2, where $\rho^2(\alpha_1) = \alpha_3$ and $\rho^2(i\sqrt{3}) = i\sqrt{3}$;
μ, where $\mu(\alpha_1) = \alpha_1$ and $\mu(i\sqrt{3}) = -i\sqrt{3}$;
$\mu\rho$, where $\mu\rho(\alpha_1) = \alpha_3$ and $\mu\rho(i\sqrt{3}) = -i\sqrt{3}$;
$\mu\rho^2$, where $\mu\rho^2(\alpha_1) = \alpha_2$ and $\mu\rho^2(i\sqrt{3}) = -i\sqrt{3}$.

c. S_3. The notation in (a) was chosen to coincide with the standard notation for $D_3 \simeq S_3$.

d.



Group diagram


$K = K_{\{\iota\}}$
$\mathbb{Q}(\mathrm{i}\sqrt{3}) = K_{<\rho>}$
$\mathbb{Q}(\sqrt[3]{2}) = K_{<\mu>}$
$\mathbb{Q}(\alpha_2) = K_{<\mu\rho>}$
$\mathbb{Q}(\alpha_3) = K_{<\mu\rho^2>}$
$\mathbb{Q} = K_{D_3}$

Field diagram

13. The splitting field of $(x^3 - 1) \in \mathbb{Q}[x]$ is $\mathbb{Q}(i\sqrt{3})$, and the group is cyclic of order 2 with elements: ι, where ι is the identity map of $\mathbb{Q}(i\sqrt{3})$, and σ, where $\sigma(i\sqrt{3}) = -i\sqrt{3}$.

15. **a.** *F* **c.** *T* **e.** *T* **g.** *F* **i.** *F*

25. *Partial answer:* $G(K/(E \vee L)) = G(K/E) \cap G(K/L)$

SECTION 47

3. $\mathbb{Q}(\sqrt[4]{2}, i)$: $\sqrt[4]{2} + i, x^8 + 4x^6 + 2x^4 + 28x^2 + 1$;
$\mathbb{Q}(\sqrt[4]{2})$: $\sqrt[4]{2}, x^4 - 2$;
$\mathbb{Q}(i\sqrt[4]{2}))$: $i(\sqrt[4]{2}), x^4 - 2$;
$\mathbb{Q}(\sqrt{2}, i)$: $\sqrt{2} + i, x^4 - 2x^2 + 9$;
$\mathbb{Q}(\sqrt[4]{2} + i(\sqrt[4]{2}))$: $\sqrt[4]{2} + i(\sqrt[4]{2}), x^4 + 8$;
$\mathbb{Q}(\sqrt[4]{2} - i(\sqrt[4]{2}))$: $\sqrt[4]{2} - i(\sqrt[4]{2}), x^4 + 8$;
$\mathbb{Q}(\sqrt{2})$: $\sqrt{2}, x^2 - 2$;
$\mathbb{Q}(i)$: $i, x^2 + 1$;
$\mathbb{Q}(i\sqrt{2})$: $i\sqrt{2}, x^2 + 2$;
$\mathbb{Q}$: $1, x - 1$

5. The group is cyclic of order 5, and its elements are

	ι	σ_1	σ_2	σ_3	σ_4
$\sqrt[5]{2} \rightarrow$	$\sqrt[5]{2}$	$\zeta(\sqrt[5]{2})$	$\zeta^2(\sqrt[5]{2})$	$\zeta^3(\sqrt[5]{2})$	$\zeta^4(\sqrt[5]{2})$

where $\sqrt[5]{2}$ is the real 5th root of 2.

7. The splitting field of $x^8 - 1$ over $\mathbb{Q}$ is the same as the splitting field of $x^4 + 1$ over $\mathbb{Q}$, so a complete description is contained in Example 47.7. (This is the easiest way to answer the problem.)

9. **a.** ${s_1}^2 - 2s_2$ **b.** $\dfrac{s_1 s_2 - 3s_3}{s_3}$

SECTION 48

3. **a.** 16 **b.** 400 **c.** 2160

5. 3^0

7. **a.** *T* **c.** *F* **e.** *T* **g.** *T* **i.** *F*

9. $\Phi_1(x) = x - 1$
$\Phi_2(x) = x + 1$
$\Phi_3(x) = x^2 + x + 1$
$\Phi_4(x) = x^2 + 1$
$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$
$\Phi_6(x) = x^2 - x + 1$

SECTION 49

1. No. Yes, *K* is an extension of $\mathbb{Z}_2$ by radicals.

3. **a.** *T* **c.** *T* **e.** *T* **g.** *T* **i.** *F* ($x^3 - 2x$ over $\mathbb{Q}$ gives a counterexample.)

APPENDIX

1. $\begin{bmatrix} 2 & 1 \\ 2 & 7 \end{bmatrix}$ **3.** $\begin{bmatrix} -3 + 2i & -1 - 4i \\ 2 & -i \\ 0 & -i \end{bmatrix}$

5. $\begin{bmatrix} 5 & 16 & -3 \\ 0 & -18 & 24 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & -i \\ 4 - 6i & -2 - 2i \end{bmatrix}$

9. $\begin{bmatrix} 8 & -8i \\ 8i & 8 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **13.** -48

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