SECTION 8

1. Homomorphism
3. Homomorphism
5. Not a homomorphism
7. Homomorphism
9. Not a homomorphism
11. $\mathrm{Ker}(\phi) = \langle 4 \rangle$.
13. $\mathrm{Ker}(\phi) = \langle (5, 3) \rangle$.
15. $\{0, 0\}$
17. $\{1, 2, 5\}, \{3\}, \{4, 6\}$
19. $\{1, 2, 3, 4, 5\}, \{6\}, \{7, 8\}$
21. $\{2n \mid n \in \mathbb{Z}\}, \{2n + 1 \mid n \in \mathbb{Z}\}$
23. $(1,\ 8)\,(3,\ 6,\ 4)\,(5,\ 7) = (1,\ 8)\,(3,\ 4)\,(3,\ 6)\,(5,\ 7)$
25. $(1,\ 5,\ 4,\ 8)\,(2,\ 3)\,(6,\ 7) = (1,\ 8)\,(1,\ 4)\,(1,\ 5)\,(2,\ 3)\,(6,\ 7)$
31. **a.** *F* **c.** *F* **e.** *F* **g.** *T* **i.** *T* **k.** *T*

SECTION 9

1.

Element	***Order***	***Element***	***Order***
(0, 0)	1	(0, 2)	2
(1, 0)	2	(1, 2)	2
(0, 1)	4	(0, 3)	4
(1, 1)	4	(1, 3)	4

The group is not cyclic

3. 2 **5.** 9 **7.** 60
9. $\{(0, 0), (0, 1)\}, \{(0, 0), (1, 0)\}, \{(0, 0), (1, 1)\}$
11. $\{(0, 0), (0, 1), (0, 2), (0, 3)\}$
$\{(0, 0), (0, 2), (1, 0), (1, 2)\}$
$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$
13. $\mathbb{Z}_{20} \times \mathbb{Z}_3, \mathbb{Z}_{15} \times \mathbb{Z}_4, \mathbb{Z}_{12} \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
15. 12
17. 168
19. 180
21. $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
23. $\mathbb{Z}_{32}, \mathbb{Z}_2 \times \mathbb{Z}_{16}, \mathbb{Z}_4 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4,$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
25. $\mathbb{Z}_9 \times \mathbb{Z}_{121}, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{121}, \mathbb{Z}_9 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$

29. **a.**

n	2	3	4	5	6	7	8
number of groups	2	3	5	7	11	15	22

b. **i)** 225 **ii)** 225 **iii)** 110
31. **a.** It is abelian when the arrows on both n-gons have the same (clockwise or counterclockwise) direction.
b. $\mathbb{Z}_2 \times \mathbb{Z}_n$
c. When n is odd.
d. The dihedral group D_n.
33. $\mathbb{Z}_2$ is an example.
35. S_3 is an example.
37. The numbers are the same. **41.** $\{-1, 1\}$

SECTION 10

1. $4\mathbb{Z} = \{\cdots, -8, -4, 0, 4, 8, \cdots\}$
$1 + 4\mathbb{Z} = \{\cdots, -7, -3, 1, 5, 9, \cdots\}$
$2 + 4\mathbb{Z} = \{\cdots, -6, -2, 2, 6, 10, \cdots\}$
$3 + 4\mathbb{Z} = \{\cdots, -5, -1, 3, 7, 11, \cdots\}$

3. $\{0, 3, 6, 9, 12, 15\}, \{1, 4, 7, 10, 13, 16\}, \{2, 5, 8, 11, 14, 17\}$

5. $\langle 18\rangle = \{0, 18\}, 1 + \langle 18\rangle = \{1, 19\}, 2 + \langle 18\rangle = \{2, 20\}, \cdots, 17 + \langle 18\rangle = \{17, 35\}$

7. $\{\iota, \mu\rho\}, \{\rho, \mu\rho^2\}, \{\rho^2, \mu\rho^3\}, \{\rho^3, \mu\}$ They are different.

9. $\{\iota, \rho^2\}, \{\rho, \rho^3\}, \{\mu, \mu\rho^2\}, \{\mu\rho, \mu\rho^3\}$

11. 4

13. 12 **15.** 24

21. $2\mathbb{Z} \leq \mathbb{Z}$ has only 2 cosets.

23. $G = \mathbb{Z}_2$, subgroup $H = \mathbb{Z}_2$.

25. Impossible. The number of cells must divide the order of the group, and 12 does not divide 6.

SECTION 11

1. **a.** The only isometries of $\mathbb{R}$ leaving a number c fixed are the reflection through c that carries $c + x$ to $c - x$ for all $x \in \mathbb{R}$, and the identity map.

b. The isometries of $\mathbb{R}^2$ that leave a point P fixed are the rotations about P through any angle θ where $0 \leq \theta < 360°$ and the reflections across any axis that passes through P.

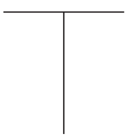
c. The only isometries of $\mathbb{R}$ that carry a line segment into itself are the reflection through the midpoint of the line segment (see the answer to part (a)) and the identity map.

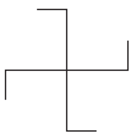
d. The isometries of $\mathbb{R}^2$ that carry a line segment into itself are a rotation of 180° about the midpoint of the line segment, a reflection in the axis containing the line segment, a reflection in the axis perpendicular to the line segment at its midpoint, and the identity map.

e. The isometries of $\mathbb{R}^3$ that carry a line segment into itself include rotations through any angle about an axis that contains the line segment, reflections across any plane that contains the line segment, and reflection across the plane perpendicular to the line segment at its midpoint.

3.

	τ	ρ	μ	γ
τ	τ	ρ	$\mu\gamma$	$\mu\gamma$
ρ	ρ	$\rho\tau$	$\mu\gamma$	$\mu\gamma$
μ	$\mu\gamma$	$\mu\gamma$	$\tau\rho$	$\tau\rho$
γ	$\mu\gamma$	$\mu\gamma$	$\tau\rho$	$\tau\rho$

5. 

7. 

9. *Translation:* order ∞
Rotation: order any $n \geq 2$ or ∞
Reflection: order 2
Glide reflection: order ∞

11. Rotations **13.** Only the identity and reflections.

17. Yes. The product of two translations is a translation and the inverse of a translation is a translation.

19. Yes. There is only one reflection μ across one particular line L, and μ^2 is the identity, so we have a group isomorphic to $\mathbb{Z}_2$.

21. Only reflections and rotations (and the identity) because translations and glide reflections do not have finite order in the group of all plane isometries.
25. **a.** No **b.** No **c.** Yes **d.** No **e.** D_∞
27. **a.** Yes **b.** No **c.** No **d.** No **e.** D_∞
29. **a.** No **b.** No **c.** No **d.** Yes **e.** $\mathbb{Z}$
31. **a.** Yes. 90°, 180° **b.** Yes **c.** No
33. **a.** No **b.** No **c.** No
35. **a.** Yes. 180° **b.** Yes **c.** No
37. **a.** Yes. 120° **b.** Yes **c.** No
39. **a.** Yes. 120° **b.** No **c.** No **d.** $(1, 0)$, $(1, \sqrt{3})$

SECTION 12

1. 3 **3.** 4 **5.** 3 **7.** 1
9. 4 **11.** 3 **13.** 5 **15.** 1
21. **a.** When working with a factor group G/H, you would let a and b be elements of G, not elements of G/H. The student probably does not understand what elements of G/H look like and can write nothing sensible concerning them.
b. We must show that G/H is abelian. Let aH and bH be two elements of G/H.
23. **a.** T **c.** T **e.** T **g.** T **i.** T
35. Example: Let $G = N = S_3$, and let $H = \{\rho_0, \mu_1\}$. Then N is normal in G, but $H \cap N = H$ is not normal in G.

SECTION 13

1. $\mathbb{Z}_2$ **3.** $\mathbb{Z}_4$ **5.** $\mathbb{Z}_4 \times \mathbb{Z}_8$ **7.** $\mathbb{Z} \times \mathbb{Z}_2$ **9.** $\mathbb{Z}_3 \times \mathbb{Z} \times \mathbb{Z}_4$
11. $\mathbb{Z}_2 \times \mathbb{Z}$ **13.** $\mathbb{Z} \times \mathbb{Z}_2$
15. $Z(D_4) = C = \{\iota, \rho^2\}$
17. $Z(S_3 \times D_4) = \{\{\iota, \iota\}, \{\iota, \rho\}\}, C = A_3 \times \{\iota, \rho\}$.
21. **a.** T **c.** F **e.** F **g.** F **i.** T
23. $\{f \in F^* \mid f(0) = 1\}$
25. Yes. Let $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$. Then $f(x) \cdot f(x) = 1$ for all x, so $f^2 \in K^*$ but f is not in K^*. Thus $f\,K^*$ has order 2 in F^*/K^*.
27. U
29. The multiplicative group U of complex numbers of absolute value 1
31. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $H = \langle(1, 0)\rangle$ is isomorphic to $K = \langle(0, 2)\rangle$, but G/H is isomorphic to $\mathbb{Z}_4$ while G/K is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
33. **a.** $\{e\}$ **b.** The whole group

SECTION 14

1. $X_\iota = X, X_\rho = \{C\}, X_\rho = \{m_1, m_2, d_1, d_2, C\}, X_{\rho^3} = \{C\}$,
$X_{\mu_\rho} = \{s_1, s_3, m_1, m_2, C, P_1, P_3\}, X_{\mu\rho^3} = \{s_2, s_0, m_1, m_2, C, P_2, P_0\}$,
$X_\mu = \{2, 0, d_1, d_2, C\}, X_{\mu\rho^2} = \{1, 3, d_1, d_2, C\}$.
3. $\{1, 2, 3, 0\}, \{s_1, s_2, s_3, s_0\}, \{m_1, m_2\}, \{d_1, d_2\}, \{C\}, \{P_1, P_2, P_3, P_0\}$
5. $\{\iota\}$
7. There are three orbits: $\{\langle\mu\rangle, \langle\mu\rho^2\rangle\}, \{\langle\mu\rho\rangle, \langle\mu\rho^3\rangle\}, \{\langle\rho^2\rangle\}$
9. 3, 3; 3, 1, 1, 1; 1, 1, 1, 1, 1, 1
11. 8, 2; 8, 1, 1; 4, 4, 2; 4, 4, 1, 1; 4, 2, 2, 2; 4, 2, 2, 1, 1; 4, 2, 1, 1, 1, 1; 4, 1, 1, 1, 1, 1, 1; 2, 2, 2, 2, 2; 2, 2, 2, 2, 1, 1; 2, 2, 2, 1, 1, 1, 1; 2, 2, 1, 1, 1, 1, 1, 1; 2, 1, 1, 1, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
15. A transitive G-set has just one orbit.
17. **a.** $\{s_1, s_2, s_3, s_0\}$ and $\{P_1, P_2, P_3, P_0\}$
21. **b.** The set of points on the circle with center at the origin and passing through P
c. The cyclic subgroup $\langle 2\pi\rangle$ of $G = \mathbb{R}$

25. **a.** $K = g_0 H g_0^{-1}$.
b. *Conjecture:* H and K should be conjugate subgroups of G.

27.

	X	Y		Z		
	a	a	b	a	b	c
0	a	a	b	a	b	c
1	a	b	a	b	c	a
2	a	a	b	c	a	b
3	a	b	a	a	b	c
4	a	a	b	b	c	a
5	a	b	a	c	a	b

There are four of them: X, Y, Z, and $\mathbb{Z}_6$.

SECTION 15

1. 5 **3.** 2 **5.** 11,712
7. **a.** 45 **b.** 231
9. **a.** 90 **b.** 6,426

SECTION 16

1. **a.** $K = \{0, 3, 6, 9\}$.
b. $0 + K = \{0, 3, 6, 9\}, 1 + K = \{1, 4, 7, 10\}, 2 + K = \{2, 5, 8, 11\}$.
c. $\mu(0 + K) = 0, \mu(1 + K) = 2, \mu(2 + K) = 1$.
3. **a.** $HN = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}, H \cap N = \{0, 12\}$.
b. $0 + N = \{0, 6, 12, 18\}, 2 + N = \{2, 8, 14, 20\}, 4 + N = \{4, 10, 16, 22\}$.
c. $0 + (H \cap N) = \{0, 12\}, 4 + (H \cap N) = \{4, 16\}, 8 + (H \cap N) = \{8, 20\}$.
d. $\mu(0 + (H \cap N)) = 0 + N = \{0, 6, 12, 18\}$, $\mu(4 + (H \cap N)) = 4 + N = \{4, 10, 16, 20\}$,
$\mu(8 + (H \cap N)) = 8 + N = \{2, 8, 13, 20\}$.
5. **a.** $0 + H = \{0, 4, 8, 12, 16, 20\}, 1 + H = \{1, 5, 9, 13, 17, 21\}$,
$2 + H = \{2, 6, 10, 14, 18, 22\}, 3 + H = \{3, 7, 11, 15, 19, 23\}$.
b. $0 + K = \{0, 8, 16\}, 1 + K = \{1, 9, 17\}, 2 + K = \{2, 10, 18\}$,
$3 + K = \{3, 11, 19\}$,
$4 + K = \{4, 12, 20\}, 5 + K = \{5, 13, 21\}, 6 + K = \{6, 14, 22\}$,
$7 + K = \{7, 15, 23\}$.
c. $0 + K = \{0, 8, 16\}, 4 + K = \{4, 12, 20\}$.
d. $(0 + K) + (H/K) = H/K = \{0 + K, 4 + K\} = \{\{0, 8, 16\}, 4, 12, 20\}\}$
$(1 + K) + (H/K) = \{1 + K, 5 + K\} = \{\{1, 9, 17\}, \{5, 13, 21\}\}$
$(2 + K) + (H/K) = \{2 + K, 6 + K\} = \{\{2, 10, 18\}, \{6, 14, 22\}\}$
$(3 + K) + (H/K) = \{3 + K, 7 + K\} = \{\{3, 11, 19\}, \{7, 15, 23\}\}$.
e. $\phi(\{\{0, 8, 16\}, \{4, 12, 20\}\}) = \{0, 4, 8, 12, 16, 20\}, \phi(\{\{1, 9, 17\}, \{5, 13, 21\}\})$
$= \{1, 5, 9, 13, 17, 21\}, \phi(\{\{2, 10, 18\}, \{6, 14, 22\}\}) = \{2, 6, 10, 14, 18, 22\}$,
$\phi(\{\{3, 11, 19\}, \{7, 15, 23\}\}) = \{3, 7, 11, 15, 19, 23\}$

SECTION 17

1. 3 **3.** 1, 3
5. The Sylow 3-subgroups are $\langle(1, 2, 3)\rangle$, $\langle(1, 2, 4)\rangle$, $\langle(1, 3, 4)\rangle$, and $\langle(2, 3, 4)\rangle$. Also $(3, 4)\langle(1, 2, 3)\rangle(3, 4) = \langle(1, 2, 4)\rangle$, etc.
7. 1, 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19.
13. **a.** T **c.** F **e.** T **g.** T **i.** F

SECTION 18

1. The refinements $\{0\} < 250\mathbb{Z} < 10\mathbb{Z} < \mathbb{Z}$ of $\{0\} < 10\mathbb{Z} < \mathbb{Z}$ and $\{0\} < 250\mathbb{Z} < 25\mathbb{Z} < \mathbb{Z}$ of $0 < 25\mathbb{Z} < \mathbb{Z}$ are isomorphic.

3. $\{0\} < \langle 27\rangle < \langle 9\rangle < \mathbb{Z}_{54}$ and $\{0\} < \langle 18\rangle < \langle 2\rangle < \mathbb{Z}_{54}$

5. The refinements
$\{(0,0)\} < (4800\mathbb{Z}) \times \mathbb{Z} < (240\mathbb{Z}) \times \mathbb{Z} < (60\mathbb{Z}) \times \mathbb{Z} < (10\mathbb{Z}) \times \mathbb{Z} < \mathbb{Z} \times \mathbb{Z}$ of the first series and
$\{(0,0)\} < \mathbb{Z} \times (4800\mathbb{Z}) < \mathbb{Z} \times (480\mathbb{Z}) < \mathbb{Z} \times (80\mathbb{Z}) < \mathbb{Z} \times (20\mathbb{Z}) < \mathbb{Z} \times \mathbb{Z}$ of the second series are isomorphic refinements.

7. $\{0\} < \langle 16\rangle < \langle 8\rangle < \langle 4\rangle < \langle 2\rangle < \mathbb{Z}_{48}$
$\{0\} < \langle 24\rangle < \langle 8\rangle < \langle 4\rangle < \langle 2\rangle < \mathbb{Z}_{48}$
$\{0\} < \langle 24\rangle < \langle 12\rangle < \langle 4\rangle < \langle 2\rangle < \mathbb{Z}_{48}$
$\{0\} < \langle 24\rangle < \langle 12\rangle < \langle 6\rangle < \langle 2\rangle < \mathbb{Z}_{48}$
$\{0\} < \langle 24\rangle < \langle 12\rangle < \langle 6\rangle < \langle 3\rangle < \mathbb{Z}_{48}$

9. $\{(\iota,0)\} < A_3 \times \{0\} < S_3 \times (0) < S_3 \times \mathbb{Z}_2$
$\{(\iota,0)\} < \{\iota\} \times \mathbb{Z}_2 < A_3 \times \mathbb{Z}_2 < S_3 \times \mathbb{Z}_2$
$\{(\iota,0)\} < A_3 \times \{0\} < A_3 \times \mathbb{Z}_2 < S_3 \times \mathbb{Z}_2$

11. $\{\iota\} \times \mathbb{Z}_4$ **13.** $\{\iota\} \times \mathbb{Z}_4 \leq \{\iota\} \times \mathbb{Z}_4 \leq \{\iota\} \times \mathbb{Z}_4 \leq \cdots$.

17. **a.** *T* **c.** *T* **e.** *F* **g.** *F* **i.** *T*
i. The Jordan-Hölder theorem applied to the groups $\mathbb{Z}_n$ implies the Fundamental Theorem of Arithmetic.

19. Yes. $\{\iota\} < \{\iota,\rho\} < \{\iota,\rho,\rho^2,\rho^3\} < D_4$ is a composition (actually a principal) series and all factor groups are isomorphic to $\mathbb{Z}_2$ and are thus abelian.

21. *Chain (3)*

$$\{0\} \leq \langle 12\rangle \leq \langle 12\rangle \leq \langle 12\rangle \leq \langle 12\rangle \leq \langle 12\rangle \leq \langle 4\rangle \leq \langle 2\rangle \leq \mathbb{Z}_{24} \leq \mathbb{Z}_{24}$$

Chain (4)

$$\{0\} \leq \langle 12\rangle < \langle 12\rangle \leq \langle 6\rangle \leq \langle 6\rangle \leq \langle 6\rangle \leq \langle 3\rangle \leq \langle 3\rangle \leq \mathbb{Z}_{24} \leq \mathbb{Z}_{24}$$

Isomorphisms

$$\begin{aligned}
&\langle 12\rangle/\{0\} \simeq \langle 12\rangle/\{0\} \simeq \mathbb{Z}_2, && \langle 12\rangle/\langle 12\rangle \simeq \langle 6\rangle/\langle 6\rangle \simeq \{0\},\\
&\langle 12\rangle/\langle 12\rangle \simeq \langle 3\rangle/\langle 3\rangle \simeq \{0\}, && \langle 12\rangle/\langle 12\rangle \simeq \langle 12\rangle/\langle 12\rangle \simeq \{0\},\\
&\langle 12\rangle/\langle 12\rangle \simeq \langle 6\rangle/\langle 6\rangle \simeq \{0\}, && \langle 4\rangle/\langle 12\rangle \simeq \mathbb{Z}_{24}/\langle 3\rangle \simeq \mathbb{Z}_3\\
&\langle 2\rangle/\langle 4\rangle \simeq \langle 6\rangle/\langle 12\rangle \simeq \mathbb{Z}_2, && \mathbb{Z}_{24}/\langle 2\rangle \simeq \langle 3\rangle/\langle 6\rangle \simeq \mathbb{Z}_2\\
&\mathbb{Z}_{24}/\mathbb{Z}_{24} \simeq \mathbb{Z}_{24}/\mathbb{Z}_{24} \simeq \{0\}
\end{aligned}$$

SECTION 19

1. $\{(1,1,1),(1,2,1),(1,1,2)\}$

3. No. $n(2,1) + m(4,1)$ can never yield an odd number for first coordinate.

7. $2\mathbb{Z} < \mathbb{Z}$, rank $r = 1$

SECTION 20

1. **a.** $a^2b^2a^3c^3b^{-2}, b^2c^{-3}a^{-3}b^{-2}a^{-2}$ **b.** $a^{-1}b^3a^4c^6a^{-1}, ac^{-6}a^{-4}b^{-3}a$

3. **a.** 16 **b.** 36 **c.** 36

5. **a.** 16 **b.** 36 **c.** 18

11. **a.** *Partial answer:* $\{1\}$ is a basis for $\mathbb{Z}_4$. **c.** Yes

13. **c.** A blop group on S is isomorphic to the *free group* $F[S]$ on S.